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$K3$ surfaces with maximal complex multiplication

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Abstract. Let X be a complex projective $K3$ surface having complex multiplication by a CM field E, and let T_X be its transcendental lattice. We say that X has *maximal complex multiplication* if $\mathrm{End}_{\mathrm{Hdg}}(T_X)$ is the ring of integers of E.

For which CM fields E does such a $K3$ surface exist? What are the possibilities for the transcendental lattices, Picard lattices of these surfaces? The aim of this paper is to study these questions and give some examples.

1. Introduction

Let E be a CM field, and let O_E be its ring of integers. A complex projective $K3$ surface X is said to have *complex multiplication by* E (or CM by E, for short) if $\mathrm{End}_{\mathrm{Hdg}}(T_X \otimes_{\mathbf{Z}} \mathbf{Q}) \simeq E$ and $\mathrm{rank}(T_X) = [E : \mathbf{Q}]$, where T_X is the transcendental lattice of X. This implies that $[E : \mathbf{Q}] \leqslant 20$, and Taelman proved that if $[E : \mathbf{Q}] \leqslant 20$ then there exists a $K3$ surface with CM by E (cf. [T 16, Theorem 3]).

Following Valloni [V 21], we say that a $K3$ surface X has *complex multiplication by* O_E if X has CM by E and moreover $\mathrm{End}_{\mathrm{Hdg}}(T_X) \simeq O_E$; we also say that X has then *maximal complex multiplication* (in [V 21] this is called "principal complex multiplication").

If X is a $K3$ surface, set $L_X = H^2(X, \mathbf{Z})$; the intersection form makes L_X into a (unimodular) lattice, and T_X is a sublattice of L_X. If moreover X has maximal complex multiplication, then T_X has a structure of O_E-module, hence so has its dual $T_X^\sharp$; therefore the quotient $T_X^\sharp / T_X$ is isomorphic to $O_E / \mathfrak{D}_X$, where $\mathfrak{D}_X \subset O_E$ is an O_E-ideal, called the *discriminant ideal* of X; note that the norm of $\mathfrak{D}_X$ is the determinant of T_X, hence also the absolute value of the determinant of the Picard lattice of X.

Question 1. *What are the possibilities for the ideal $\mathfrak{D}_X$?*

This is the subject matter of Section 9 (see Corollary 9.2), based on results of Sections 4 and 7.

The next issue is to classify up to isomorphism the $K3$ surfaces with a given discriminant ideal.

Question 2. *What are the possibilities for the $K3$ surfaces X with a given discriminant ideal $\mathfrak{D}_X$?*

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It is well known that if E is an imaginary quadratic field, then the isomorphism classes of elliptic curves with complex multiplication by O_E are in bijection with the ideal class group of E.

For an arbitrary CM field E, we define a finite group $\mathcal{C}$ having a similar property for the isomorphism classes of $K3$ surfaces with complex multiplication by O_E with the same discriminant ideal (see Corollary 9.3). We show the following theorem.

Theorem 1. *There are only finitely many isomorphism classes of $K3$ surfaces having maximal complex multiplication by the same CM field and the same discriminant ideal.*

The next sections contain some applications of the previous results. In [V 21], Valloni raised the following question.

Question 3. *For which CM fields E do there exist $K3$ surfaces with maximal CM by E?*

Valloni proved that if $[E : \mathbf{Q}] \leqslant 10$, then there exist infinitely many non-isomorphic $K3$ surfaces having CM by O_E. The aim of Section 8 is to give a sufficient criterion for the existence of infinitely many non-isomorphic $K3$ surfaces with CM by O_E in terms of ramification properties of E (see Theorem 8.2). One of the applications is a generalization of Valloni's result (cf. Corollary 8.4).

Theorem 2. *If $[E : \mathbf{Q}] \leqslant 14$, then there exist infinitely many non-isomorphic complex projective $K3$ surfaces with complex multiplication by O_E.*

This is no longer true in general if $[E : \mathbf{Q}] = 16, 18$ or 20 (cf. Proposition 8.5 and Example 8.6); but it does hold for *cyclotomic fields*.

Theorem 3. *If E is a cyclotomic field with $2 \leqslant [E : \mathbf{Q}] \leqslant 20$, then there exist infinitely many non-isomorphic complex projective $K3$ surfaces with complex multiplication by O_E.*

In another direction, maximal complex multiplication on the transcendental lattice implies some properties of the Picard lattice, such as possible degrees of polarization, existence of elliptic fibrations.

A start on this is made in Section 10; the following example illustrates the results of this section (see Example 10.7; here U denotes the 2-dimensional hyperbolic lattice, and for all integers N, the lattice $U(N)$ denotes U multiplied by N).

Example 1. *Let $E = \mathbf{Q}(\zeta_m)$ with $m = 44$ or 66. There exists a $K3$ surface with maximal complex multiplication by E with Picard lattice $L \Leftrightarrow L \simeq U(N)$ where $N \geqslant 1$ is an integer $\equiv 1 \pmod{m}$ such that all the prime divisors of N are $\equiv \pm 1 \pmod{m}$.*

With this strategy in mind, the second part of the paper concerns $K3$ surfaces having maximal complex multiplication by cyclotomic fields. One of the results is the following.

Theorem 4. *Let p be an odd prime number with $3 \leqslant p \leqslant 11$, and set $E = \mathbf{Q}(\zeta_p)$. Let $a \geqslant 1$ be an odd integer. There exists a unique* (up to isomorphism) *complex projective $K3$ surface X_a with maximal complex multiplication by E such that* $\det(T_{X_a}) = p^a$.

Moreover, the surfaces X_a are isogeneous for all $a \geqslant 1$.

If $a = 1$, these surfaces are isomorphic to Vorontsov's $K3$ surfaces (see [V 83, K 92]).

If $p = 3, 7$ or 11, then for all $a \geqslant 1$ the $K3$ surfaces X_a above and their twists (cf. Definition 7.3) have automorphisms of order p inducing the complex multiplication by O_E (see Theorem 18.1), and the corresponding pairs ($K3$ surface, automorphism of order p) are the CM points by O_E of the moduli spaces $\mathcal{M}^p_{K3}$ defined by Artebani, Sarti and Taki in [AST 11] (see Section 19); see also [AS 08] for $p = 3$ and [OZ 10, ACV 21] for $p = 11$.

The above observation suggests two approaches to fields of definition of the surfaces X_a and their twists: one using class field theory (see Valloni, [V 21, V 23]), the other using the geometry of the moduli space $\mathcal{M}^p_{K3}$ and elliptic fibrations. This is illustrated by an example due to Brandhorst and Elkies in [BE 23]; their example turns out to be isomorphic to a twist of $X_1(7)$ by a prime O_E-ideal above 13 (see Example 20.1).

The existence of the $K3$ surfaces is proved by transcendental methods, using the surjectivity of the period map. However, the method of Brandhorst and Elkies can be used to obtain explicit equations for this family of surfaces. This is done in [BE 23] for the above mentioned twist of $X_1(7)$; I thank Simon Brandhorst for sending me similar results for two other surfaces in this family, obtained by the method of [BE 23] (see Examples 21.1 and 21.2).

2. Lattices, discriminant forms and embeddings

A *lattice* is a pair (L, q), where L is a free $\mathbf{Z}$-module of finite rank, and $q : L \times L \to \mathbf{Z}$ is a symmetric bilinear form such that $\det(q) \neq 0$; it is *unimodular* if $\det(1) = \pm 1$, and *even* if $q(x, x)$ is an even integer for all $x \in L$. Set

$$L^\sharp = \{x \in L \otimes_{\mathbf{Z}} \mathbf{Q} \mid q(x, y) \in \mathbf{Z} \text{ for all } y \in L\},$$

and $G_L = L^\sharp / L$. The form q induces $G_L \times G_L \to \mathbf{Q}/\mathbf{Z}$, called the *discriminant form* of L, and G_L the *discriminant group* of L; note that the absolute value of $\det(q)$ is the order of G_L. The discriminant form is denoted by (G_L, q_L).

The Witt group of symmetric bilinear forms on finite abelian groups with values in $\mathbf{Q}/\mathbf{Z}$ is denoted by $W(\mathbf{Q}/\mathbf{Z})$; see [Sch 85, Chapter V, Section 1].

An embedding of lattices $L \to L'$ is called *primitive* if its cokernel is free.

Definition 2.1. Let L and L' be two lattices. We say that L *embeds uniquely into* L' if there exists a primitive embedding $f : L \to L'$, and if $g : L \to L'$ is another primitive embedding, then there exists $\varphi \in \mathrm{O}(L')$ such that $g = \varphi \circ f$.

3. $K3$ surfaces

The aim of this section is to recall some basic facts concerning $K3$ surfaces; see [H 16] or [K 20] for details. If X is a complex projective $K3$ surface, set $L_X = H^2(X, \mathbf{Z})$, and let

$$H^2(X, \mathbf{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

be its Hodge decomposition; we have $\dim(H^{2,0}) = \dim(H^{0,2}) = 1$, and $\dim(H^{1,1}) = 20$.

Let $S_X = L_X \cap H^{1,1}$ be the Picard lattice of X, and set $\rho_X = \mathrm{rank}_{\mathbf{Z}}(S_X)$. The intersection form of X makes L_X into an even unimodular lattice of signature $(3, 19)$; since X is projective, the signature of S_X is $(1, \rho_X - 1)$. Let T_X be the orthogonal complement of S_X in L_X. The lattice T_X has signature $(2, 20 - \rho_X)$, and is called the *transcendental lattice* of X.

Theorem 3.1. *Let X and Y be two complex projective $K3$ surfaces, and let $f : L_X \to L_Y$ be an isometry of lattices whose* **C***-linear extension maps $H^{2,0}(X)$ to $H^{2,0}(Y)$. Then the surfaces X and Y are isomorphic.*

Proof. This is the weak Torelli theorem, see for instance [H 16, Chapter 7, Theorem 5.3]. ■

Definition 3.2. Let X and Y be two complex projective $K3$ surfaces, and let $f : T_X \to T_Y$ be an isometry of lattices. We say that f is a *Hodge isometry* if its **C**-linear extension maps $H^{2,0}(X)$ to $H^{2,0}(Y)$.

Let us fix an even unimodular lattice Λ of signature $(3, 19)$.

Theorem 3.3. *Let X and Y be two complex projective $K3$ surfaces. Suppose that the lattice T_X embeds uniquely into Λ, and that there exists a Hodge isometry $f : T_X \to T_Y$. Then the surfaces X and Y are isomorphic.*

Proof. Let us choose isometries $\varphi_X : L_X \to \Lambda$ and $\varphi_Y : L_Y \to \Lambda$; note that $\varphi_X : T_X \to \Lambda$ and $\varphi_Y \circ f : T_X \to \Lambda$ are two primitive embeddings of the lattice T_X into Λ. Therefore, there exists an isometry $g : L_X \to L_X$ such that $g \circ \varphi_X = \varphi_Y \circ f$; note that the **C**-linear extension of $\varphi_Y^{-1} \circ g \circ \varphi_X : L_X \to L_Y$ sends $H^{2,0}(X)$ to $H^{2,0}(Y)$. Hence, by Theorem 3.1, the $K3$ surfaces X and Y are isomorphic. ■

4. O_E-lattices, discriminant ideals and discriminant modules

Let E be an algebraic number field with a non-trivial involution $x \mapsto \bar{x}$, and let F be the fixed field of this involution; let n be an integer such that $[E : \mathbf{Q}] = 2n$, and let $\theta \in F^\times$ be such that $E = F(\sqrt{\theta})$. Let O_E be the ring of integers of E, and let D_E be the different of E.

An *O_E-lattice* is by definition a pair (I, q), where I is a fractional O_E-ideal and $q : I \times I \to \mathbf{Z}$ is given by $q(x, y) = \mathrm{Tr}_{E/\mathbf{Q}}(\alpha x \bar{y})$, for some $\alpha \in F^\times$; we also use the notation (I, α) for this lattice.

If L is an O_E-lattice, then so is its dual $L^\sharp$, and the quotient $G_L = L^\sharp/L$ is also an O_E-module, called the *discriminant module* of L; it is isomorphic to $O_E/\mathcal{D}$ for some ideal $\mathcal{D} \subset O_E$.

Definition 4.1. The *discriminant ideal* of an O_E-lattice L is by definition the ideal $\mathcal{D} \subset O_E$ such that $L^\sharp/L$ is isomorphic to $O_E/\mathcal{D}$. The discriminant ideal of L is denoted by $\mathcal{D}(L)$.

The aim of this section is to characterize the discriminant ideals (equivalently, the discriminant modules) of O_E-lattices.

Let Ram be the set of finite places of F that ramify in E, and let Int be the set of finite places of F that are inert in E. If $v \in$ Ram or Int, we denote by P_v the prime O_E-ideal corresponding to the unique place v_E of E above v. Let Ram_E be the set of places w of E such that $w = v_E$ for some $v \in$ Ram.

Let $\mathrm{Ram}_{\mathrm{odd}}$ be the set of places v of Ram such that $v_E(D_E)$ is odd, and let $\mathrm{Int}_{\mathrm{odd}}$ be the set of $v \in$ Int such that $v_E(D_E)$ is odd; let t be the cardinality of $\mathrm{Int}_{\mathrm{odd}}$.

Let us denote by s the number of real embeddings of F that extend to imaginary embeddings of E.

Theorem 4.2. *Let σ_1, σ_2 be integers $\geqslant 0$ such that $\sigma_1 + \sigma_2 = 2n$. Let $L = (I, q)$ be an O_E-lattice of signature (σ_1, σ_2). Then we have*

(i) $\sigma_1 \geqslant n - s$, $\sigma_2 \geqslant n - s$, $\sigma_1 \equiv \sigma_2 \equiv n - s \pmod 2$.

Let $\mathfrak{D}(L)$ be the discriminant ideal of L. Then $\mathfrak{D}(L) = \prod P^{e_P}$ where the product is taken over the prime ideals of O_E such that the following conditions hold:

(ii) *We have $e_P = 0$ for almost all P.*

(iii) *For all P, we have $e_{\bar{P}} = e_P$.*

(iv) *If $P = P_v$ with $v \in \mathrm{Ram}_{\mathrm{odd}}$, then e_P is odd.*

(v) *Let m be the number of $v \in Int$ such that e_{P_v} is odd. If Ram $= \varnothing$, then*

$$\sigma_1 - \sigma_2 \equiv 4m \pmod 8.$$

Conversely, let σ_1, σ_2 be integers $\geqslant 0$ such that $\sigma_1 + \sigma_2 = 2n$ and such that (i) *holds, and let $e_P \geqslant 0$ be integers such that $\mathfrak{D} = \prod P^{e_P}$ satisfies conditions* (ii)–(v)*. Then there exists an O_E-lattice L of signature (σ_1, σ_2) and discriminant ideal $\mathfrak{D}$.*

Moreover, if a is an integer with $0 \leqslant a \leqslant s$ and if A is a set of real places of F of cardinality a, then we can choose L such that $L = (I, \alpha)$ with α negative at the places in A and positive at all the other places of F.

Proof. (i) follows from [B 99, Theorem 1 (i)]. Let $\alpha \in F^\times$ be such that

$$q(x, y) = \mathrm{Tr}_{E/\mathbf{Q}}(\alpha x \bar{y}) \quad \text{for all } x, y \in I.$$

We have $I^\sharp = D_E^{-1} \alpha^{-1} \bar{I}^{-1}$, hence $\mathfrak{D}(L) = \alpha I \bar{I} D_E$. This implies that $\mathfrak{D}(L)$ is of the required form, and that (ii) holds. Note that $\overline{D_E} = D_E$, hence condition (iii) is satisfied. If $w = v_E$ for some $v \in \mathrm{Ram}_{\mathrm{odd}}$, then

$$w(\alpha I \bar{I} D_E) \equiv w(D_E) \equiv 1 \pmod 2;$$

this implies (iv). Finally, let us prove (v). Let a be the number of infinite places v of F such that $(\alpha, \theta)_v = -1$, and let a' be the number of $v \in$ Int such that $v(\alpha)$ is odd; note that $v(\alpha)$ is odd if and only if $(\alpha, \theta)_v = -1$. Assume that Ram $= \varnothing$; then the product formula implies that $a' \equiv a \pmod 2$. We have $t \equiv a' + m \pmod 2$ by definition, hence $t \equiv a + m \pmod 2$.

We have $\sigma_1 - \sigma_2 = 2s - 4a$ (see for instance [B 99, Proposition 2.2]), and $s \equiv 2t \pmod 4$ (cf. [BM 94, Theorem 1.6]). Hence,

$$\sigma_1 - \sigma_2 = 2s - 4a \equiv 4t - 4a \equiv 4(a+m) - 4a \equiv 4m \pmod 8,$$

as claimed.

Conversely, suppose that conditions (i)–(v) hold. Let a be an integer such that $0 \leqslant a \leqslant s$ and that $\sigma_1 - \sigma_2 = 2s - 4a$; such an integer exists by (i). Let A be a set of real places of F of cardinality a that extend to imaginary places of K; this is possible since $a \leqslant s$.

Let M be the set of $v \in \mathrm{Int}$ such that e_{P_v} is odd; recall that the cardinality of M is denoted by m. Let A' be the symmetric difference of M and $\mathrm{Int}_{\mathrm{odd}}$, and let a' be the cardinality of A'. We define $\varepsilon_v = \pm 1$ for all the places of v as follows. Set $\varepsilon_v = -1$ if $v \in A \cup A'$. If $\mathrm{Ram} = \varnothing$, then we set $\varepsilon_v = 1$ for all the other places of v. Otherwise, let us choose a finite place w of F that ramifies in E, and set $\varepsilon_w = (-1)^{a+a'}$; set $\varepsilon_v = 1$ for all the other places of F.

We have $\prod_v \varepsilon_v = 1$. This is clear if $\mathrm{Ram} \neq \varnothing$. Suppose that $\mathrm{Ram} = \varnothing$; then $\prod_v \varepsilon_v = (-1)^{a+a'}$. Condition (v) implies that $\sigma_1 - \sigma_2 \equiv 4m \pmod 8$; since $\sigma_1 - \sigma_2 = 2s - 4a$, this implies that $s - 2a \equiv 2m \pmod 4$. We have $s \equiv 2t \pmod 4$ by [BM 94, Theorem 1.6] and $m \equiv t + a' \pmod 2$ by construction; this implies that $a' \equiv a \pmod 2$, hence $\prod_v \varepsilon_v = 1$.

There exists $\alpha \in F^{\times}$ such that $(\alpha, \theta)_v = \varepsilon_v$ for all places v of F (see for instance [O'M 73, Theorem 71.19]). We have $v_E(\alpha D_E) \equiv 1 \pmod 2$ if $v \in \mathrm{Ram}_{\mathrm{odd}}$ or $v \in M$. Let I be an O_E-ideal such that $w(\alpha I \bar{I} D_E) = e_P$ for all places w of E, where P is such that $w(P) = 1$.

The O_E-lattice (I, q) given by $q(x,y) = \mathrm{Tr}_{E/\mathbf{Q}}(\alpha x \bar{y})$ has signature (σ_1, σ_2) and discriminant ideal $\mathcal{D}$. Moreover, α is negative at the places in A and positive at all the other places of F. This completes the proof of the theorem. ∎

Corollary 4.3. *Let σ_1, σ_2 be integers $\geqslant 0$ such that $\sigma_1 + \sigma_2 = 2n$. Let $L = (I, q)$ be an O_E-lattice of signature (σ_1, σ_2), and let G_L be the discriminant module of L. Then $G_L \simeq \bigoplus O_E/P^{e_P}$ where the sum is taken over the prime ideals of O_E such that conditions* (ii)–(v) *above hold.*

Conversely, let σ_1, σ_2 be integers $\geqslant 0$ such that $\sigma_1 + \sigma_2 = 2n$ and such that (i) *holds, and let $e_P \geqslant 0$ be integers such that $G = \bigoplus O_E/P^{e_P}$ satisfies conditions* (ii)–(v). *Then there exists an O_E-lattice L of signature (σ_1, σ_2) and discriminant module G.*

Moreover, if a is an integer with $0 \leqslant a \leqslant s$ and if A is a set of real places of F of cardinality a, then we can choose L such that $L = (I, \alpha)$ with α negative at the places in A and positive at all the other places of F.

Proof. This is an immediate consequence of Theorem 4.2. ∎

The following results will be used in the next sections.

Lemma 4.4. *If no dyadic place of F ramifies in E, then every O_E-lattice is even.*

Proof. See for instance [B 99, Proposition 1]. ■

Lemma 4.5. *Let I be an ideal of O_E, let $\alpha \in F^\times$ and let $L = (I, q)$ with $q(x, y) = \mathrm{Tr}_{E/\mathbf{Q}}(\alpha x \bar{y})$. Suppose that for all dyadic places w of E we have $w(\alpha I \bar{I}) \geqslant 0$. Then L is an even lattice.*

Proof. Let w be a dyadic place of E. Since $w(\alpha I \bar{I}) \geqslant 0$, we have $\alpha x \bar{y} \in O_w$ for all $x, y \in I$. On the other hand, $\bar{\alpha} = \alpha$, therefore $\mathrm{Tr}_{K_w/\mathbf{Q}_2}(\alpha x \bar{x})$ is divisible by 2. This implies that L is even.

Recall that s is the number of real places of F that extend to imaginary places of E. ■

Proposition 4.6. *Suppose that no finite place of F ramifies in E. Then $s \equiv 0 \pmod 2$.*

Proof. Let S be the set of real places of F that extend to imaginary places of E; if v is an infinite place of F, then $(-1, \theta)_v = -1$ if and only if $v \in S$. If v is a finite place of F, then v is unramified in E by hypothesis, hence $(-1, \theta)_v = 1$. Therefore, the product formula implies that $\prod_{v \in S} (-1, \theta)_v = 1$, hence s is even, as claimed. ■

Corollary 4.7. *Assume that E is a CM field with maximal totally real subfield F, and that no finite place of F ramifies in E. Then n is even.*

Proof. This follows from the previous proposition, since $s = n$. ■

Definition 4.8. Let (L, q) and (L', q') be two O_E-lattices. We say that L and L' are isomorphic (as O_E-lattices) if there exists an isomorphism of O_E-modules $f : L \to L'$ such that $q'(f(x), f(y)) = q(x, y)$ for all $x, y \in L$.

Set $(V, q) = (L, q) \otimes_{\mathbf{Z}} \mathbf{Q}$ and $(V', q) = (L', q') \otimes_{\mathbf{Z}} \mathbf{Q}$. We say that L and L' *become isomorphic over* $\mathbf{Q}$ if there exists an isomorphism of E-vector spaces $f : V \to V'$ such that $q'(f(x), f(y)) = q(x, y)$ for all $x, y \in V$.

If (L, q) is an O_E-lattice, then there exists an O_E-ideal I and $\alpha \in F^\times$ such that $q(x, y) = \mathrm{Tr}_{E/\mathbf{Q}}(\alpha x \bar{y})$. If L is given by (I, α) and L' by (J, β) as above, then the O_E-lattices L and L' are isomorphic if and only if there exists $e \in E^\times$ such that $J = eI$ and that $\alpha = e\bar{e}\beta$.

Definition 4.9. Let $L = (I, \alpha)$ and $L' = (J, \beta)$ be two O_E-lattices. We say that L and L' have the same signature (as O_E-lattices) if and only if $\tau(\alpha\beta) > 0$ for all embeddings $\tau : F \to \mathbf{R}$ that extend to imaginary embeddings of E.

Proposition 4.10. *Let $L = (I, \alpha)$ and $L' = (J, \beta)$ be two O_E-lattices of the same signature, and let $G_L = \bigoplus O_E/P^{e_P}$, $G_{L'} = \bigoplus O_E/P^{e'_P}$ be their discriminant modules. The O_E-lattices L and L' become isomorphic over $\mathbf{Q}$ if and only if*

$$e_P \equiv e'_P \pmod 2$$

for all prime O_E-ideals P such that $\bar{P} = P$.

Proof. Recall that $E = F(\sqrt{\theta})$. The O_E-lattices L and L' become isomorphic over $\mathbf{Q}$ if and only if $\alpha\beta^{-1}$ belongs to $\mathrm{N}_{E/F}(E^\times)$; equivalently, if $(\alpha,\theta)_v = (\beta,\theta)_v$ for all places v of F. If v is a real place of F, then $(\alpha,\theta)_v = (\beta,\theta)_v$ if and only if $\alpha\beta$ is positive at v, i.e., if $\tau_v(\alpha\beta) > 0$ where $\tau_v : F \to \mathbf{R}$ is the embedding corresponding to v; since the lattices have the same signature, the condition holds at v. The condition trivially holds if v is an imaginary place of F, hence it holds for all infinite places. Suppose now that v is a finite place of F. If v is split in E, then $(\alpha,\theta)_v = (\beta,\theta)_v = 1$, hence the condition also holds at split places. If v is inert or ramified in E, then $(\alpha,\theta)_v = (\beta,\theta)_v$ if and only if $e_{P_v} \equiv e'_{P_v} \pmod 2$, where P_v is the unique prime ideal of O_E that is above the prime O_F-ideal corresponding to v. This completes the proof of the proposition. ∎

Notation 4.11. We denote by h_E the class number of E.

Proposition 4.12. *Suppose that E is a CM field with maximal totally real subfield F, and that $h_E = 1$. Then two O_E-lattices of the same signature are isomorphic if and only if their O_E-discriminant modules are isomorphic.*

Proof. Let L and L' be two O_E-lattices of the same signature. It is clear that if L and L' are isomorphic O_E-lattices, then their O_E-discriminant modules are isomorphic.

Conversely, suppose that the O_E-discriminant modules of L and L' are isomorphic. Since $h_E = 1$, we have $L \simeq (O_E, \alpha)$ and $L' \simeq O_E, \beta$ for some $\alpha, \beta \in E_0^\times$. Since $G_L \simeq G_{L'}$ as O_E-modules, the discriminant ideals of L and L' are equal, and hence $\alpha O_E = \beta O_E$, therefore $\beta\alpha^{-1}$ is a unit of O_F. Moreover, it is a totally positive unit, because L and L' have the same signature. Since $h_E = 1$, every totally positive unit of E_0 is a norm of a unit of E (see [Sh 77, Proposition A2], [B 84, Lemma 3.2]). This implies that the O_E-lattices L and L' are isomorphic. ∎

5. Classification

We keep the notation of Section 4; in particular, E is a number field of degree $2n$ with a non-trivial involution and fixed field F. To complement the results of Section 4, in this section we describe the isomorphism classes of O_E-lattices having the same discriminant ideal and the same signature (see Definition 4.9).

Let $C(E)$ be the set of pairs (I,α), where I is an O_E-ideal and $\alpha \in F^\times$ such that $\alpha I\bar{I} = O_E$, and let $C^+(E)$ be the subset of pairs with $\sigma(\alpha) > 0$ for all real embeddings σ of F that extend to imaginary embeddings of E. Let us consider the equivalence relation on $C(E)$ (respectively $C^+(E)$) given by

$$(I,\alpha) \equiv (I',\alpha') \iff \text{there exists } \gamma \in E^\times \text{ with } I' = \gamma I \text{ and } \alpha' = \left(\frac{1}{\gamma\bar{\gamma}}\right)\alpha.$$

In both cases, the set of equivalence classes is a finite abelian group, the multiplication being induced by $(I,\alpha)(I',\alpha') = (II',\alpha\alpha')$; we denote this group by $\mathcal{C}(E)$ (respectively $\mathcal{C}^+(E)$).

Let $L = (J, \beta)$ be an O_E-lattice, and let $a = (I, \alpha) \in C(E)$. Setting $a.L = (IJ, \alpha\beta)$ induces an action of $\mathcal{C}(E)$ on the set isomorphism classes of O_E-lattices with the same discriminant ideal as L; if moreover, $a \in C^+(E)$, then $a.L$ has the same signature as L, hence we obtain an action of the group $\mathcal{C}(E)^+$ on the set of isomorphism classes of O_E-lattices with the same discriminant ideal and the same signature as L.

Proposition 5.1. (i) *The set of isomorphism classes of O_E-lattices with the same discriminant ideal is a principal homogeneous space over the group $\mathcal{C}(E)$.*

(ii) *The set of isomorphism classes of O_E-lattices with the same discriminant ideal and the same signature is a principal homogeneous space over the group $\mathcal{C}^+(E)$.*

Proof. Let $L = (J, \beta)$ and $L' = (J', \beta')$ are O_K-lattices. We have $a.L = L'$ for $a = (J'J^{-1}, \beta'\beta^{-1})$; if L and L' have the same discriminant ideal, then $a \in C(E)$; if moreover L and L' also have the same signature, then $a \in C(E)^+$.

Suppose that E is a CM field, and that F is the maximal totally real subfield of E. ∎

Notation 5.2. Let $\mathcal{C}l(E/F)$ be the relative class group of E/F, and let $\mathcal{C}l^+(E/F)$ be the strict relative class group. Let $O_F^{\times +}$ be the group of totally positive units of O_F, and let $\mathrm{N} : E^\times \to F^\times$ be the norm map.

Proposition 5.3. *We have the exact sequences*

$$1 \to O_F^\times / \mathrm{N}(O_E^\times) \to \mathcal{C}(E) \to \mathcal{C}l(E/F) \to 1,$$
$$1 \to O_F^{\times +} / \mathrm{N}(O_E^\times) \to \mathcal{C}(E) \to \mathcal{C}l^+(E/F) \to 1.$$

Proof. The maps $\mathcal{C}(E) \to \mathcal{C}l(E/F)$ and $\mathcal{C}(E) \to \mathcal{C}l^+(E/F)$ are induced by $(I, \alpha) \mapsto I$. It is easy to check that this gives rise to the above exact sequences. ∎

6. Twisting

We keep the notation of Section 4.

Definition 6.1. Let L be an O_E-lattice, and let $J \subset O_E$ be an O_E-ideal prime to the discriminant ideal $\mathcal{D}(L)$ such that $\bar{J} = J$. We say that an O_E-lattice L' is a *twist of L by J* if L and L' have the same signature (see Definition 4.9), and if $\mathcal{D}(L') = \mathcal{D}(L)J$.

We first examine the conditions under which an O_E-lattice L has a twist by an ideal J.

Proposition 6.2. *An O_E-lattice L has a twist by an ideal J if and only if $\mathcal{D}(L)J$ satisfies condition* (v) *of Theorem* 4.2.

Proof. Let (σ_1, σ_2) the signature of L; since L is an O_E-lattice, condition (i) of Theorem 4.2 holds. Condition (ii) obviously holds for $\mathcal{D}(L)J$, and condition (iii) is also satisfied, since $\bar{J} = J$; condition (iv) also holds since J is supposed to be prime to $\mathcal{D}(L')$, hence J does not have any factor P_v with $v \in \mathrm{Ram}_{\mathrm{odd}}$. Therefore, Theorem 4.2 implies that there exists an O_E-lattice L' with $\mathcal{D}(L') = \mathcal{D}(L)J$ if and only if condition (v) hold for $\mathcal{D}(L)J$. Moreover, Theorem 4.2 shows that we can choose L' with the same signature (as O_E-lattice) as L. This concludes the proof of the proposition. ∎

Corollary 6.3. *Suppose that there exists a finite prime of F that ramifies in E. Let L be an O_E-lattice, and let $J \subset O_E$ be an O_E-ideal prime to the discriminant ideal $\mathcal{D}(L)$ such that $\bar{J} = J$. Then L has a twist by J.*

Proof. This follows from Proposition 6.2; indeed, condition (v) is trivially satisfied since $\mathrm{Ram} \neq \varnothing$. ∎

Such a twist is not unique in general; however, we have the following.

Proposition 6.4. *Suppose that E is a CM field. Let L be an O_K-lattice and let $J \subset O_E$ be an ideal prime to the discriminant ideal $\mathcal{D}(L)$ such that $\bar{J} = J$. Let L_1 and L_2 be two twists of L by J. If $h_E = 1$, then the O_E-lattices L_1 and L_2 are isomorphic.*

Proof. This follows from Proposition 4.12. ∎

Example 6.5. Let $L = (I, \alpha)$ be an O_E-lattice, and let P be a prime O_E-ideal such that $\bar{P} \neq P$. Suppose that P and $\bar{P}$ are prime to I. Then $L' = (PI, \alpha)$ is a twist of L by $P\bar{P}$. Moreover, Proposition 4.10 implies that L and L' become isomorphic over $\mathbf{Q}$.

Proposition 6.6. *Let $J \subset O_E$ be an ideal with $\bar{J} = J$. If all prime factors P of J are such that $P \neq \bar{P}$, then every twist of O_E-lattice L by J becomes isomorphic to L over $\mathbf{Q}$.*

Proof. This is a consequence of Proposition 4.10. ∎

Example 6.7. Let L be an O_E-lattice, and let P be a prime O_E-ideal prime to $\mathcal{D}(L)$ such that $\bar{P} = P$. Theorem 4.2 and Proposition 6.2 imply that L has a twist by P if and only if $\mathrm{Ram} \neq \varnothing$; such a twist does not necessarily become isomorphic to L over $\mathbf{Q}$.

7. O_E-lattices and $K3$ surfaces

We keep the notation of the previous sections, and set $[E : \mathbf{Q}] = 2n$. We assume in addition that E is a CM field with $[E : \mathbf{Q}] \leqslant 20$, i.e., E is totally imaginary and F is totally real with $1 \leqslant n \leqslant 10$. If X is a $K3$ surface, we denote by T_X its transcendental lattice.

Let us fix an even unimodular lattice Λ of signature $(3, 19)$.

Proposition 7.1. *Let L be an O_E-lattice of signature $(2, 2n-2)$ and assume that L embeds primitively into Λ. Then there exists a complex projective $K3$ surface X such that the transcendental lattice T_X of X has a structure of O_E-lattice isomorphic to L, and* $\mathrm{End}_{\mathrm{Hdg}}(T_X) = O_E$.

Proof. Let L be given by (I, α), where I is an O_E-ideal and $\alpha \in F^\times$. Let $\sigma' : F \to \mathbf{R}$ be the real embedding of F such that $\sigma'(\alpha) > 0$; note that α is negative at all the other real embeddings of F, since the signature of L is $(2, 2n-2)$. Let $\sigma : E \to \mathbf{C}$ be an extension of σ' to E.

Let $f : L \to \Lambda$ be a primitive embedding, and let us also denote by f its extension to

$$f : L \otimes_{\mathbf{Z}} \mathbf{C} \to \Lambda \otimes_{\mathbf{Z}} \mathbf{C}.$$

We have $L \otimes_{\mathbf{Z}} \mathbf{C} = \bigoplus_{\tau : E \to \mathbf{C}} \mathbf{C}_\tau$. Set $\Lambda^{2,0} = \mathbf{C}_\sigma$.

We obtain the desired $K3$ surface by surjectivity of the period map. Indeed, the choice of $\Lambda^{2,0}$ induces on Λ a Hodge structure. Let X be the corresponding $K3$ surface. By construction, we have $T_X = L$ and $\mathrm{End}_{\mathrm{Hdg}}(T_X) = O_E$. Since the signature of L is $(2, 2n-2)$, the surface X is projective. ∎

Notation 7.2. If X is a $K3$ surface with complex multiplication by O_E, we denote by G_X the discriminant O_E-module $T_X^\sharp / T_X$. The minimal number of generators (as an abelian group) of G_X is denoted by $\ell(X)$, and is called the *length* of X.

Definition 7.3. Let X and Y be two complex projective $K3$ surfaces with complex multiplication by O_E. Let $J \subset O_E$ be an O_E-ideal such that $\bar{J} = J$. We say that Y is a *twist of X by J* if $G_Y \simeq G_X \oplus O_E/J$.

Let us consider E embedded in $\mathbf{C}$. This implies that if X and Y are two $K3$ surfaces with CM by O_E, then the O_E-lattices T_X and T_Y have the same signature (as O_E-lattices).

Proposition 7.4. *Let L be an O_E-lattice, and assume that L embeds uniquely into Λ. Let X and Y be two complex $K3$ surfaces with CM by O_E, and suppose that the O_E-lattices T_X and T_Y are isomorphic to L. Then the surfaces X and Y are isomorphic.*

Proof. This follows from Theorem 3.3. ∎

We next note that $h_E = 1$, then $K3$ surfaces with maximal complex multiplication by E of length $\leqslant 20 - 2n$ are determined by their discriminant modules.

Proposition 7.5. *Suppose that $h_E = 1$, and let X and Y be two $K3$ surfaces with maximal complex multiplication by E of length $\leqslant 20 - 2n$. Then X and Y are isomorphic if and only if the discriminant O_E-modules of T_X and T_Y are isomorphic.*

Proof. If X and Y are isomorphic, then the O_E-lattices T_X, T_Y are isomorphic, and hence so are their discriminant modules. Let us prove the converse. Since $h_E = 1$, Proposition 4.12 implies that the O_E-lattices T_X and T_Y are isomorphic.

We fix an even unimodular lattice Λ of signature $(3, 19)$. Since by hypothesis

$$\ell(X), \ell(Y) \leqslant 20 - 2n,$$

the lattices T_X, T_Y are uniquely embedded in Λ (see Nikulin [N 79, Theorem 1.14.4]). Therefore, by Proposition 7.4 the $K3$ surfaces X and Y are isomorphic. ∎

Corollary 7.6. *Let X be a complex $K3$ surface with CM by O_E, and suppose that $h_E = 1$. Let $J \subset O_E$ be an O_E-ideal such that $\bar{J} = J$, and let Y_1, Y_2 be two twists of X by J of length $\leqslant 20 - 2n$. Then Y_1 and Y_2 are isomorphic.*

Proof. This follows from Proposition 7.5. ∎

8. Existence of $K3$ surfaces with maximal complex multiplication

We keep the notation of the previous sections; in particular, E is a CM field of degree $\leqslant 20$. The aim of this section is to give a criterion for the existence of infinitely many isomorphism classes of $K3$ surfaces with complex multiplication by O_E. Valloni proved that this is always the case if the degree of E is $\leqslant 10$ (cf. [V 21, Proposition 6.11]); as we will see, this result extends to fields of degree $\leqslant 14$ (see Corollary 8.4).

We start by introducing some notation. If w is a place of E, we denote by f_w its residual degree.

Notation 8.1. If p is a prime number such that $p \neq 2$, we denote by $\mathrm{Ram}(p)$ the set of places of Ram_E above p. Let $\mathrm{Ram}(2)$ be the set of dyadic places of E such that $w(D_E) > 0$. For all prime numbers p, set

$$f(p) = \sum_{w \in \mathrm{Ram}(p)} f_w.$$

Note that for almost all p, we have $\mathrm{Ram}(p) = \varnothing$, hence $f(p) = 0$.

Theorem 8.2. *Suppose that $f(p) < 22 - 2n$ for all prime numbers p such that $\mathrm{Ram}(p) \neq \varnothing$. Then there exist infinitely many non-isomorphic complex projective $K3$ surfaces having complex multiplication by O_E.*

Moreover, there exist infinitely many such surfaces in the same isogeny class.

Proof. Suppose first that $\mathrm{Ram} \neq \varnothing$. Let P be a prime ideal of O_E of degree 1; there exist infinitely many such ideals by Chebotarev's density theorem. Assume that P is not dyadic, and that $\mathrm{N}(P)$ is relatively prime to $\mathrm{N}(P_v)$ for all $v \in \mathrm{Ram}$, where N is the norm map.

By Corollary 4.3, there exists an O_E-lattice with signature $(2, 2n-2)$ and discriminant module $\bigoplus_{v \in \mathrm{Ram}_{\mathrm{odd}}} O_E/P_v \oplus O_E/P \oplus O_E/\bar{P}$; let I be an O_E-ideal and let $\alpha \in F^\times$ such that the lattice $q : I \times I \to \mathbf{Z}$ with $q(x,y) = \mathrm{Tr}_{E/\mathbf{Q}}(\alpha x \bar{y})$ for all $x, y \in I$ is such a lattice.

If there exist dyadic places of F that ramify in E, let $J = I \prod_{w \in \mathrm{Ram}(2)} P_w^{e_w}$ with $e_w \in \mathbf{Z}$ such that $w(\alpha J \bar{J}) \geqslant 0$ for all dyadic places w of E, where P_w is the prime O_E-ideal such that $w(P_w) = 1$. Let L be the O_E-lattice given by $q : J \times J \to \mathbf{Z}$ such that $q(x,y) = \mathrm{Tr}_{E/\mathbf{Q}}(\alpha x \bar{y})$ for all $x, y \in J$. By Lemmas 4.4 and 4.5, the lattice L is even.

Assume now that $\mathrm{Ram} = \varnothing$, i.e., no finite place of F ramifies in E; by Lemma 4.7 this implies that n is even. As before, let P be a non-dyadic prime ideal of O_E of degree 1. If $n \equiv 2 \pmod 4$, set $G = O_E/P \oplus O_E/\bar{P}$. Suppose that $n \equiv 0 \pmod 4$. Since E is a CM field, Chebotarev's density theorem implies that there exist infinitely many prime O_E-ideals Q such that $Q \cap O_F$ is inert in E/F, and that the residual degree of Q is 2. Let Q be such an ideal, and set $G = O_E/P \oplus O_E/\bar{P} \oplus O_E/Q$ in this case. By Corollary 4.3, there exists an O_E-lattice L of signature $(2, 2n-2)$ and discriminant module G; by Lemma 4.4, this lattice is even.

In all the above cases, we denote by G the discriminant module of L, and let $\ell(G)$ be the number of generators of G as an abelian group. Let f be the maximum of the integers $f(p)$ such that $\mathrm{Ram}(p) \neq \varnothing$ if there exists such a p with $f(p) > 1$; otherwise, set $f = 2$. We have $\ell(G) \leqslant f$. If $n \leqslant 9$, then the hypothesis implies that $\ell(G) < 22 - 2n$; by Nikulin's result [N 79, Corollary 1.12.3] this implies that L can be primitively embedded in an even, unimodular lattice Λ of signature $(3, 19)$. Suppose that $n = 10$, and let $p = N(P)$. The p-component of G is $(\mathbf{F}_p)^2$, and the p-component of the discriminant form has determinant $-p^2$. By [N 79, Theorem 1.12.2], the lattice L can be primitively embedded in an even, unimodular lattice Λ of signature $(3, 19)$ in this case as well.

Let $\sigma' : F \to \mathbf{R}$ be the unique embedding of F such that $\sigma'(\alpha) > 0$, and let $\sigma : E \to \mathbf{C}$ be one of the two extensions of σ' to E. Let $J \otimes_{\mathbf{Z}} \mathbf{C} = \bigoplus_{\tau : E \to \mathbf{C}} \mathbf{C}_\tau$ and set $\Lambda^{2,0} = \mathbf{C}_\sigma$, where we consider $J \otimes_{\mathbf{Z}} \mathbf{C}$ contained in $\Lambda \otimes_{\mathbf{Z}} \mathbf{C}$. This endows the lattice Λ with a Hodge structure. Let X be the corresponding $K3$ surface: such a surface exists by the surjectivity of the period map. The transcendental lattice of X is isomorphic to L, a lattice of signature $(2, 2n - 2)$, hence the surface X is projective. It has complex multiplication by O_E by construction. Varying the ideal P gives rise to infinitely many non-isomorphic projective $K3$ surfaces having complex multiplication by O_E. By Proposition 4.10 the O_E-lattices become isomorphic over $\mathbf{Q}$, hence the $K3$ surfaces are all isogeneous (cf. [Mu 87, N 87, Bu 19]). This completes the proof of the theorem. ∎

We now state some consequences of this result.

Corollary 8.3. *If no finite place of F ramifies in E, then there exist infinitely many non-isomorphic complex projective $K3$ surfaces with complex multiplication by O_E.*

Proof. $\mathrm{Ram} = \varnothing$ in this case, hence $f(p) = 0$ for all prime numbers p. ∎

Corollary 8.4. *If $[E : \mathbf{Q}] \leqslant 14$, then there exist infinitely many non-isomorphic complex projective $K3$ surfaces with complex multiplication by O_E.*

Proof. Recall that $[E : \mathbf{Q}] = 2n$. It is easy to see that for all prime numbers p, we have $f(p) < n$. If $[E : \mathbf{Q}] \leqslant 14$, then $f(p) < 7$, and hence $f(p) < 22 - 2n \leqslant 8$. Therefore, by Theorem 8.2 there exist infinitely many non-isomorphic complex projective $K3$ surfaces with complex multiplication by O_E. ∎

Note that this no longer holds in general when $[E : \mathbf{Q}] \geqslant 16$, as shown by the following proposition and example.

Proposition 8.5. *Let p be a prime number, $p \neq 2$. Suppose that there exists a prime O_F-ideal P that ramifies in E such that $N(P) = p^f$ with $f > 22 - 2n$. Then there does not exist any complex projective $K3$ surfaces with complex multiplication by O_E.*

Proof. Let X be a complex projective $K3$ surface, let T_X be the transcendental lattice and let S_X be the Picard lattice of X. If X has complex multiplication by E, then $\mathrm{rank}(T_X) = 2n$ and hence $\mathrm{rank}(S_X) = 22 - 2n$. Let G be the discriminant group of the lattice T_X; then G is also the discriminant group of the lattice S_X. Theorem 4.2 implies that O_E/P

is a subgroup of G; the hypothesis on P implies that the minimal number of generators of G is $> 22 - 2n$. This contradicts the fact that G is the discriminant group of the lattice S_X, of rank $22 - 2n$. ∎

Example 8.6. Let F be the maximal totally real subfield of the cyclotomic field $\mathbf{Q}(\zeta_{17})$, and set $E = F(\sqrt{-3})$. Note that E is a CM field of degree 16. There exists a unique prime ideal P above 3 in F; this ideal ramifies in E, and its residual degree is 8, i.e., $N(P) = 3^8$; by Proposition 8.5 this implies that there does not exist any complex projective $K3$ surfaces with complex multiplication by O_E. The same method gives rise to infinitely many examples in degrees 16, 18 and 20.

Corollary 8.7. *If E is a cyclotomic field with $2 \leqslant [E : \mathbf{Q}] \leqslant 20$, then there exist infinitely many non-isomorphic complex projective $K3$ surfaces with complex multiplication by O_E.*

Proof. If no finite prime of F ramifies in E, then this follows from Corollary 10.1. Suppose now that there exist finite primes of F that ramify in E; this implies that $E = \mathbf{Q}(\zeta_{p^r})$, where p is a prime number and $r \geqslant 1$ is an integer. Let P be the unique ramified ideal of O_E. Then the residual degree of P is 1, and P is the only prime ideal of O_E above p, hence $f(p) = 1$. We have $f(q) = 0$ for all prime numbers $q \neq p$, hence by Theorem 8.2, this implies that there exist infinitely many non-isomorphic complex projective $K3$ surfaces with complex multiplication by O_E. ∎

9. $K3$ surfaces with a given discriminant ideal

We keep the notation of the previous sections: E is a CM field of degree $2n$, with maximal totally real subfield F, and $2n \leqslant 20$. We now apply the results of Sections 4–7 to the existence and classification of $K3$ surfaces with a given discriminant ideal.

Definition 9.1. Let X be a $K3$ surface with CM by O_E. The *discriminant ideal* of X, denoted by $\mathfrak{D}_X$, is the integral ideal of O_E such that the O_E-modules $T_X^\sharp / T_X$ and $O_E / \mathfrak{D}_X$ are isomorphic.

Recall that $G_X = T_X^\sharp / T_X$ is called the discriminant module of X, and that the length of X, denoted by $\ell(X)$, is by definition the minimal number of generators of G_X, as an abelian group.

If X is a $K3$ surface with CM by O_E, then T_X is an even O_E-lattice of signature $(2, 2n-2)$, hence the discriminant ideal $\mathfrak{D}_X$ satisfies the conditions of Theorem 4.2 for $\sigma_1 = 2$ and $\sigma_2 = 2n - 2$; moreover, $\ell(X) \leqslant 22 - 2n$.

Corollary 9.2. *Suppose that no dyadic place of F ramifies in E, and let $\mathfrak{D} \subset O_E$ be an ideal satisfying conditions* (ii)–(v) *of Theorem 4.2 for* $(\sigma_1, \sigma_2) = (2, 2n-2)$*, and suppose that $\ell(O_E/\mathfrak{D}) < 22 - 2n$. Then there exists a $K3$ surface X with $\mathfrak{D}_X = \mathfrak{D}$.*

Proof. Theorem 4.2 implies that there exists an O_E-lattice L with discriminant ideal $\mathfrak{D}$ and signature $(2, 2n-2)$. Since no dyadic place of F ramifies in E, the lattice is even (see

Lemma 4.4). The lattice L embeds primitively into the $K3$-lattice Λ (see [N 79, Corollary 1.12.3]), hence by Proposition 7.1 there exists a complex projective $K3$ surface X with complex multiplication by O_E such that $T_X \simeq L$.

Let us consider E embedded in $\mathbf{C}$; hence all transcendental lattices of $K3$ surfaces with maximal complex multiplication by E have the same signature (as O_E-lattices).

Set $\mathcal{C} = \mathcal{C}^+(E)$, with the notation of Section 5. Let $\mathcal{D} \subset O_E$ be an ideal such that $\ell(O_E/\mathcal{D}) < 22 - 2n$. ■

Corollary 9.3. *Suppose that no dyadic place of F ramifies in E. Then the set of complex projective $K3$ surfaces with complex multiplication by O_E and discriminant ideal $\mathcal{D}$ is a principal homogeneous space over $\mathcal{C}$.*

Proof. The hypotheses imply there exists a $K3$ surface with $\mathcal{D}_X = \mathcal{D}$ (see Corollary 9.2), and that the transcendental lattice T_X embeds uniquely into the $K3$-lattice. The corollary now follows from Proposition 5.1 (ii), the fact that by Lemma 4.4 every O_E-lattice is even, combined with Proposition 7.1 and Proposition 7.4. ■

Corollary 9.4. *There exist only finitely many isomorphism classes of $K3$ surfaces with CM by O_E and discriminant ideal $\mathcal{D}$.*

Proof. Recall that X is a $K3$ surface with CM by O_E and discriminant ideal $\mathcal{D}$. Let Y be another $K3$ surface with these properties. Both T_X and T_Y are even O_E-lattices of the same signature, and their discriminant ideals are equal by hypothesis. Proposition 5.1 (ii) implies that there exists $a \in \mathcal{C}$ such that $a.T_X = T_Y$. The group $\mathcal{C}$ is finite, hence there are only finitely many possibilities for the isomorphism class of the O_E-lattice T_Y. By Proposition 7.4, this implies that there are only finitely many isomorphism classes of $K3$ surfaces Y as above. ■

Remark 9.5. Let $|\mathcal{C}|$ be the order of the group $\mathcal{C}$. The number of isomorphism classes of $K3$ surfaces with CM by O_E and discriminant ideal $\mathcal{D}$ is $\leqslant |\mathcal{C}|$, and equality holds if no dyadic prime of F ramifies in E.

10. Picard lattices and complex multiplication

The aim of this section is to discuss the relationship between complex multiplication by a ring of integers, and properties of the Picard and transcendental lattices. We keep the notation of the previous sections; in particular, E is a CM field, F is its maximal totally real subfield, $\deg(E) = 2n$, with $2n \leqslant 20$. In this section, we assume that no dyadic prime of F ramifies in E.

Proposition 10.1. *Let X be a complex projective $K3$ surface with maximal complex multiplication by E. Suppose that the Picard lattice S_X is unimodular* (equivalently, T_X is unimodular). *Then no finite prime of F ramifies in E, and $2n \equiv 4 \pmod 8$.*

Conversely, if no finite prime of F ramifies in E and $2n \equiv 4 \pmod 8$, then there exists a complex projective $K3$ surface X such that S_X and T_X are unimodular.

Proof. The hypothesis implies that T_X is a unimodular O_E-lattice; recall that this lattice is even. With the notation of Section 4, this implies that $e_{P_v} = 0$ for all places v of E, therefore by Theorem 4.2 (iv) we have Ram $= \varnothing$. We have $(\sigma_1, \sigma_2) = (2, 2n-2)$; since T_X is unimodular, $m = 0$. Therefore, Theorem 4.2 (v) implies that $2n - 4 \equiv 0 \pmod 8$, hence $2n \equiv 4 \pmod 8$, as claimed.

Conversely, assume that no finite prime of F ramifies in E and that $2n \equiv 4 \pmod 8$. Then by Theorem 4.2 there exists a unimodular O_E-lattice L of signature $(2, 2n-2)$. Since no finite prime of F ramifies in E, this lattice is even (cf. Lemma 4.4). By Proposition 7.1 there exists a complex projective $K3$ surface X with CM by O_E and $T_X \simeq L$. Since S_X is the orthogonal complement of T_X in $H^2(X, \mathbf{Z})$, the lattice S_X is also unimodular.

Suppose now that $2n = 20$; in this case, the Picard lattice is of rank 2. We denote by U the rank 2 hyperbolic lattice, and if N is an integer, we denote by $U(N)$ the lattice U with values multiplied by N. ∎

Notation 10.2. Let S_1 be the set of prime numbers p such that there exists a prime O_E-ideal P with $\overline{P} \neq P$ and $p = \mathrm{N}_{E/\mathbf{Q}}(P)$, let S_2 be the set of prime numbers p such that there exists a prime O_E-ideal P with $\overline{P} = P$ and $p^2 = \mathrm{N}_{E/\mathbf{Q}}(P)$, and let S_3 be the set of prime numbers p such that there exists a prime ideal P of E such that $\overline{P} = P$ and $p = \mathrm{N}_{E/\mathbf{Q}}(P)$.

Lemma 10.3. (i) *The set S_3 is finite.* (ii) *If no finite prime of F ramifies in E, then $S_3 = \varnothing$.*

Proof. If $p \in S_3$, then there exists a prime O_F-ideal above p that ramifies in E; this proves both (i) and (ii). ∎

Notation 10.4. Let $\mathcal{N}_E$ be the set of integers $N \geqslant 1$ such that $N = \prod_{i \in I} p_i^{n_i}$, where for all $i \in I$ we have $p_i \in S_1$ or $p_i \in S_2$, and $n_i \geqslant 0$ is an integer such that if no finite prime of F ramifies in E, then $\sum_{p_i \in S_2} n_i$ is even.

Recall that if P is a prime ideal of O_E above the prime number p, we denote by f_P the residual degree of P, i.e. $f_P = [O_E/P : \mathbf{F}_p]$.

Proposition 10.5. *Suppose that $[E : \mathbf{Q}] = 20$, and that $S_3 = \varnothing$. Let X be a complex projective $K3$ surface with maximal complex multiplication by E.*

Then the Picard lattice S_X is isomorphic to $U(N)$ with $N \in \mathcal{N}_E$.

Conversely, if $N \in \mathcal{N}_E$, then there exists a complex projective $K3$ surface with maximal complex multiplication by E with Picard lattice isomorphic to $U(N)$.

Proof. The discriminant module G_X is isomorphic to $\bigoplus_P O_E/P^{e_P}$ for some prime O_E ideals P and integers $e_P \geqslant 0$ with $e_P = e_{\overline{P}}$. Since $S_3 = \varnothing$, the order of G_X is a square, and therefore $\det(S_X)$ is a square. This implies that $S_X \simeq U(N)$ for some integer N, and $G_X \simeq (\mathbf{Z}/N\mathbf{Z})^2$; therefore $\ell(X) = 2$.

If P is a prime O_E-ideal such that $e_P \neq 0$ and $\overline{P} \neq P$, then this implies that $f_P = 1$. Set $p = \mathrm{N}_{E/\mathbf{Q}}(P)$; we have $p \in S_1$.

If P is a prime O_E-ideal such that $e_P \neq 0$ and $\bar{P} = P$, then we have $p = \mathrm{N}_{E/\mathbf{Q}}(P)$ or $p^2 = \mathrm{N}_{E/\mathbf{Q}}(P)$, and this implies $p \in S_3$ in the first case and $p \in S_2$ in the second one. But $S_3 = \varnothing$ by hypothesis, hence we have $p^2 = \mathrm{N}_{E/\mathbf{Q}}(P)$ and $p \in S_2$.

Note that if a prime number p divides N, then there exists a prime O_E-ideal above p with $e_P \neq 0$, hence we proved that N is a product of primes in $S_1 \cup S_2$.

Set $N = \prod_{i \in I} p_i^{n_i}$; it remains to prove that if no finite prime of F ramifies in E, then $\sum_{p_i \in S_2} n_i$ is even.

Suppose that no finite prime of F ramifies in E, and note that under this hypothesis, if P is a prime ideal with $\bar{P} = P$, then $e_P > 0 \Leftrightarrow \mathrm{N}(P) = p_i^2$ for some $p_i \in S_2$. Moreover, $e_P = n_i$. With the notation of Theorem 4.2, we have $\sigma_1 = 2$ and $\sigma_2 = 18$, hence $\sigma_1 - \sigma_2 = -16$; by Theorem 4.2 (v) this implies that m is even. Therefore, e_P is odd for an even number of prime O_E-ideals with $\bar{P} = P$; this implies that the sum $\sum_{p_i \in S_2} n_i$ is even, as claimed.

Conversely, let $p = p_i$ be a divisor of N, and let P be a prime O_E-ideal with $p = \mathrm{N}_{E/\mathbf{Q}}(P)$ if $p \in S_1$, and $p^2 = \mathrm{N}_{E/\mathbf{Q}}(P)$ if $p \in S_2$. Set $e_P = n_i$ and $G = \bigoplus_P O_E/P^{e_P}$. If no finite prime of F ramifies in E, then $\sum_{p_i \in S_2} n_i$ is even, hence the number of prime ideals P with $\bar{P} = P$ and $e_P > 0$ is even; with the notation of Theorem 4.2, this implies that m is even. By Theorem 4.2 there exists an O_E-lattice T of signature $(2, 18)$ and discriminant module G. The lattice T embeds primitively into the $K3$-lattice Λ, hence by Proposition 7.1 there exists a complex projective $K3$ surface X with transcendental lattice T and maximal complex multiplication by O_E. The orthogonal complement of T in Λ is isomorphic to $U(N)$ by construction, and this completes the proof of the proposition. ■

Remark 10.6. If E is a Galois extension of $\mathbf{Q}$ and if no finite prime of F ramifies in E, then S_1 is the set of prime numbers that split completely in E, and S_2 is the set of those that split completely in F, but not in E.

Example 10.7. Let $E = \mathbf{Q}(\zeta_m)$ with $m = 44$ or 66, and let p be a prime number. No finite prime of F ramifies in E, hence $S_3 = \varnothing$. We have

- $p \in S_1 \Leftrightarrow p \equiv 1 \pmod{m}$;
- $p \in S_2 \Leftrightarrow p^2 \equiv 1 \pmod{m}$.

Proposition 10.5 implies that there exists a $K3$ surface with maximal complex multiplication by E with Picard lattice $L \Leftrightarrow L \simeq U(N)$ where $N \geqslant 1$ is an integer $\equiv 1 \pmod{m}$ such that all the prime divisors of N are $\equiv \pm 1 \pmod{m}$.

Note that for $N = 1$ we recover Kondo's $K3$ surfaces, see [K 92, LSY 10].

Still supposing that $2n = 20$, we now deal with the case where $S_3 \neq \varnothing$. We start by introducing some notation.

Notation 10.8. Let K be a real quadratic field, and let $\sigma : K \to K$ be the unique non-trivial element of the Galois group of K over $\mathbf{Q}$. Let O be an order of K and let $I \subset O$ be a projective ideal of O. Let $\mathrm{N} : K \to \mathbf{Q}$ be the norm map. We denote by q_I the quadratic form $q_I : I \times I \to \mathbf{Z}$ defined by $q_I(x, y) = \frac{1}{\mathrm{N}(I)} \mathrm{Tr}_{K/\mathbf{Q}}\, x\sigma(y)$.

The *conductor* of an order O is by definition the index of O in the ring of integers of K; we denote by $\mathrm{cond}(O)$ the conductor of O.

Notation 10.9. Let $\mathcal{M}_E$ be the set of integers $N \geqslant 1$ such that $N = \prod_{i \in I} p_i^{n_i}$, where for all $i \in I$ we have $p_i \in S_1$, $p_i \in S_2$ or $p_i \in S_3$, and $n_i \geqslant 0$ is an integer such that n_i is even if $p_i \in S_3$.

Proposition 10.10. *Suppose that* $[E : \mathbf{Q}] = 20$ *and that* $S_3 \neq \varnothing$. *Let* X *be a complex projective* $K3$ *surface with maximal complex multiplication by* E.

(i) *If* $\det(T_X)$ *is a square, then* $S_X \simeq U(N)$ *for some* $N \in \mathcal{M}_E$.

(ii) *Suppose that* $\det(T_X)$ *is not a square, and let* $\det(T_X) = dc^2$, *where* d *is a squarefree integer. Set* $K = \mathbf{Q}(\sqrt{d})$. *Then*

$$S_X \simeq q_I$$

where I *is an* O*-ideal of an order* O *of* K *of conductor* c*; moreover,* $c \in \mathcal{M}_E$.

Proof. (i) Since $\det(T_X)$ is a square, $|\det(S_X)|$ is also a square, hence $S_X \simeq U(N)$ for some integer N. We have $\ell(X) = 2$. The discriminant module G_X is isomorphic to $\bigoplus_P O_E/P^{e_P}$ for some prime O_E ideals P and integers $e_P \geqslant 0$ with $e_P = e_{\overline{P}}$. If a prime number p divides N, then there exists a prime O_E-ideal P above p such that $e_P \neq 0$.

Suppose that p is a prime divisor of N and that P is a prime O_E-ideal above p with $e_P \neq 0$ such that $\overline{P} \neq P$. Since $\ell(X) = 2$, this implies that $f_P = 1$ and $p = \mathrm{N}_{E/\mathbf{Q}}(P)$, hence $p \in S_1$.

Let p be a prime divisor of N such that there exists a prime O_E-ideal above p with $e_P \neq 0$ and $\overline{P} = P$. If $f_P = 2$, then $p^2 = \mathrm{N}_{E/\mathbf{Q}}(P)$, and $p \in S_2$. Suppose that $f_P = 1$. Then we have $p = \mathrm{N}_{E/\mathbf{Q}}(P)$, hence $p \in S_3$. Since $\det(T_X)$ is a square, p^2 divides N, and this implies that $N \in \mathcal{M}_E$.

(ii) We have $|\det(S_X)| = dc^2$, and this implies that $S_X \simeq q_I$ where I is an O-ideal of an order O of conductor c of K. We have $\ell(X) \leqslant 2$, hence $c \in \mathcal{M}_E$. ■

Proposition 10.11. *Suppose that* $[E : \mathbf{Q}] = 20$ *and that* $S_3 \neq \varnothing$. *Set* $d_E = \prod_{p \in S_3} p$ *and* $K_E = \mathbf{Q}(\sqrt{d_E})$. *If* $c \in \mathcal{M}_E$, *then there exists a complex projective* $K3$ *surface with maximal complex multiplication by* E *with Picard lattice isomorphic to* q_I *for some projective ideal* I *of an order of conductor* c *of* K_E.

Proof. Let $p = p_i$ be a divisor of c, and let P be a prime O_E-ideal with $p = \mathrm{N}_{E/\mathbf{Q}}(P)$ if $p \in S_1$ or S_3, and $p^2 = \mathrm{N}_{E/\mathbf{Q}}(P)$ if $p \in S_2$. Set $e_P = n_i$, and $G_1 = \bigoplus_P O_E/P^{e_P}$. For all $p_i \in S_3$, let P_i be a prime O_E-ideal above p_i, and set $G_2 = \bigoplus_{p_i \in S_3} O_E/P_i$; let $G = G_1 \oplus G_2$. By Theorem 4.2 there exists an O_E-lattice T of signature $(2, 18)$ and discriminant group G. The lattice T embeds primitively into the $K3$-lattice Λ, hence by Proposition 7.1 there exists a complex projective $K3$ surface X with transcendental lattice T and maximal complex multiplication by O_E. The orthogonal complement of T in Λ is an indefinite binary quadratic form of determinant $-d_E c^2$, hence it is of the form q_I for some projective ideal I of an order of conductor c of K_E. ■

Example 10.12. Let $E = \mathbf{Q}(\zeta_{25})$, and let P be the unique ramified prime O_E-ideal. We have $f_P = 1$, hence with the above notation we have $d_E = 5$ and $K_E = \mathbf{Q}(\sqrt{5})$.

- $p \in S_1 \Leftrightarrow p \equiv 1 \pmod{25}$.
- $p \in S_2 \Leftrightarrow p^2 \equiv 1 \pmod{25}$.
- $S_3 = \{5\}$.

Proposition 10.10 implies that if X is a $K3$ surface with maximal complex multiplication by E, then $S_X \simeq q_I$, where I is a projective ideal of an order O of K_E. Moreover, if c is the conductor of O, then we have $c = 5^{2r}N$ where $r \geqslant 0$ is an integer, and if p is a prime divisor of N, then $p \equiv \pm 1 \pmod{25}$.

For $N = 1$, we recover one of Vorontsov's $K3$ surfaces, see [V 83, LSY 10].

11. $K3$ surfaces with maximal complex multiplication by cyclotomic fields

We keep the notation of the previous sections, and suppose that E is a cyclotomic field. We consider E embedded in $\mathbf{C}$, with $E = \mathbf{Q}(\zeta_m)$, where $m \geqslant 3$ is an integer and ζ_m is a primitive m-th root of unity. As in the previous sections, the degree of E is denoted by $2n$; note that $2n = \varphi(m)$, and that by hypothesis $2n \leqslant 20$.

Recall that if X is a $K3$ surface with complex multiplication by O_E, we denote by G_X the discriminant O_E-module $T_X^\sharp/T_X$, and that the minimal number of generators (as an abelian group) of G_X is denoted by $\ell(X)$; it is called the length of X.

We start by observing that $K3$ surfaces with maximal complex multiplication by E of length $\leqslant 20 - 2n$ are determined by their discriminant modules.

Proposition 11.1. *Let X and Y be two $K3$ surfaces with maximal complex multiplication by E of length $\leqslant 20 - 2n$. Then X and Y are isomorphic if and only if the discriminant O_E-modules of T_X and T_Y are isomorphic.*

Proof. We have $h_E = 1$, since $2n \leqslant 20$ (see for instance [W 97, Tables, Sections 3 and 4]), therefore the proposition follows from Proposition 7.5. ∎

12. Discriminant forms of Craig-like lattices

Let p be a prime number, $p \neq 2$. Craig's lattices are positive definite lattices associated to the cyclotomic field $\mathbf{Q}(\zeta_p)$; see for instance [BB 92, Section 4] or [CS 99, Section 5.4]. In this section and the next one, we define (definite and indefinite) analogs of these lattices.

Let $r \geqslant 1$ be an integer, and set $E = \mathbf{Q}(\zeta_{p^r})$; we have $[E : \mathbf{Q}] = 2n = (p-1)p^{r-1}$. Let P be the unique prime ideal of O_E above p, let us write $D_E = P^\delta$, and let d be an integer such that $\delta = 1 - 2d$.

We consider the quadratic space (E, q) with $q : E \times E \to \mathbf{Q}$ defined by $q(x, y) = \mathrm{Tr}_{E/\mathbf{Q}}(x\bar{y})$. For all integers k such that $k \geqslant d$, let us denote by C_k the lattice of (E, q) given by $C_k = P^k$.

Notation 12.1. If $b \neq 0$ is an integer, we denote by $(\mathbf{Z}/p\mathbf{Z}, \frac{b}{p}xy)$ the symmetric bilinear form

$$\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z} \to \mathbf{Q}/\mathbf{Z}$$

sending (x, y) to $\frac{b}{p}xy$.

Let $W(\mathbf{Q}/\mathbf{Z})$ be the Witt group of symmetric bilinear forms on finite abelian groups (see for instance [Sch 85, Chapter 5, Section 1]), and let $[G, q]$ be the Witt class of (G, q) in $W(\mathbf{Q}/\mathbf{Z})$.

Theorem 12.2. *Let $L = C_k$ with k as above, set $a = \delta + 2k$, and let $e \in \{\pm 1\}$ be such that $p^{r-1} \equiv e \pmod 4$. We have $G_L \simeq O_E/P^a$, and*

$$[(G_L, q_L)] = \left[\left(\mathbf{Z}/p\mathbf{Z}, \frac{-e}{p}xy\right)\right]$$

in $W(\mathbf{Q}/\mathbf{Z})$.

Lemma 12.3. *The Witt class of (G_{C_k}, q_{C_k}) is independent of k.*

Proof. Let k, ℓ be such that $d \leqslant k \leqslant \ell$. We have $C_\ell \subset C_k$, hence $C_k^\sharp \subset C_\ell^\sharp$; therefore, C_k/C_ℓ is totally isotropic in $C_\ell^\sharp/C_\ell$, and $(C_k/C_\ell)^\perp = C_k^\sharp/C_\ell$. By [Sch 85, Lemma 5.1.3] this implies that the Witt classes of (G_{C_k}, q_{C_k}) and of (G_{C_ℓ}, q_{C_ℓ}) are equal. ∎

Lemma 12.4. *We have $C_k^\sharp = C_{-\delta-k}$ and $C_{k+2n} = pC_k$.*

Proof. Indeed, $C_k^\sharp = D_E^{-1}P^{-k} = C_{-\delta-k}$, and $C_{k+2n} = P^k P^{2n} = pC_k$. ∎

Lemma 12.5. *Suppose that $r = 1$. Then the lattice C_d is isomorphic to the root lattice A_{p-1}, and C_{-d+1} is isomorphic to $pA_{p-1}^\sharp$.*

Proof. We have $C_d \simeq A_{p-1}$ by [E 94, Lemma 5.4]. The second assertion follows from this, and the previous lemma. ∎

Proposition 12.6. *Assume that $r = 1$, and let $L = C_d$. Then (G_L, q_L) is isomorphic to $(\mathbf{Z}/p\mathbf{Z}, \frac{-1}{p}xy)$.*

First proof of Proposition 12.6. It is clear that $G_L \simeq \mathbf{Z}/p\mathbf{Z}$. To show that $q_L(x, y) = \frac{-1}{p}xy$, apply [BT 20, Proposition 6.3] with $\lambda = 1$. Note that (with the notation of [BT 20]) we have $\pi_E^{-2d+2} = \pi_E^{p-1} = p$, and that $\overline{\pi}_E = -\pi_E$. This implies that the invariant unit u of Proposition 6.3 is $\mathrm{Tr}_{E/\mathbf{Q}}(\frac{1}{p})$ if d is even, and $\mathrm{Tr}_{E/\mathbf{Q}}(\frac{-1}{p})$ if d is odd; hence $u = \frac{p-1}{2}$ if $p \equiv 3 \pmod 4$ and $u = \frac{1-p}{2}$ if $p \equiv 1 \pmod 4$. Suppose first that $p \equiv 3 \pmod 4$. Then $q_L(x, y) = \frac{-1}{p}xy$, as claimed. If $p \equiv 1 \pmod 4$, then -1 is a square $\pmod 4$, hence $q_L(x, y) = \frac{-1}{p}xy$ in this case as well. ∎

Second proof of Proposition 12.6. By Lemma 12.5, the lattice $L = C_d$ is isomorphic to the root lattice A_{p-1}, and hence (G_L, q_L) is isomorphic to $(\mathbf{Z}/p\mathbf{Z}, \frac{-1}{p}xy)$; see for instance [McM 11, Proposition 3.5]. ∎

Remark 12.7. Assume that $r = 1$. With the notation of [BB 92], we have $C_k = A^{\ell}_{p-1}$, where $\ell = k + \frac{p-1}{2}$.

Proof of Theorem 12.2. We have $L \simeq P^k$ and $L^\sharp \simeq P^{-k} D_E^{-1} = P^{-k} P^{-\delta}$, hence $G_L \simeq O_E/P^{\delta+2k} = O_E/P^a$. To prove that $[(G_L, q_L)] = [(\mathbf{Z}/p\mathbf{Z}, \frac{-e}{p}xy)]$ in $W(\mathbf{Q}/\mathbf{Z})$. we may assume that $L = O_E$ (cf. Lemma 12.3). By [B 06, Proposition 9.1], the lattice (O_E, q) is isomorphic to the orthogonal sum of p^{r-1} copies of $p^r A^\sharp_{p-1}$. Set $M = p^r A^\sharp_{p-1}$ and $T = (\mathbf{Z}/p\mathbf{Z}, \frac{-1}{p}xy)$. The Witt class of (G_M, q_M) in $W(\mathbf{Q}/\mathbf{Z})$ is equal to T; indeed, Lemma 12.5 implies that $pA^\sharp_{p-1}$ is isomorphic to C_{-d+1} for $r = 1$, and hence by Lemma 12.4 the lattice $p^r A^\sharp_{p-1}$ is also of the form C_ℓ for some ℓ and $r = 1$. By Proposition 12.6 and Lemma 12.3, this implies that (G_M, q_M) and T are in the same class in $W(\mathbf{Q}/\mathbf{Z})$; hence (G_L, q_L) is Witt equivalent to the orthogonal sum of p^{r-1} copies of T. In $W(\mathbf{Q}/\mathbf{Z})$, the orthogonal sum of 4 copies of T is always 0, the sum of two copies of T is 0 if and only if $p \equiv 1 \pmod 4$. This implies that (G_L, q_L) is Witt equivalent to $T = (\mathbf{Z}/p\mathbf{Z}, \frac{-e}{p}xy)$, as claimed. ∎

Example 12.8. Let $p = 3$ and $r = 2$; then $d = -4$, and the lattice C_d is isomorphic to the root lattice E_6 (see [B 99, Section 3]).

13. Indefinite Craig-like lattices

We keep the notation of the previous section. Let F be the maximal totally real subfield of E, and let $\sigma_0 : F \to \mathbf{R}$ be a real embedding of F. In this section, we assume that *there exists a unit* $u \in O_F^\times$ *such that* $\sigma_0(u) > 0$ *and that* $\sigma(u) < 0$ *for all the other real embeddings* σ *of* F.

Let h^- be the relative class number of E (i.e., the class number of E divided by the class number of F). If h^- is odd, then there exists a unit u as above, and its image in $O_F^\times / \mathrm{N}_{E/F}(O_E^\times)$ is unique (see for instance [B 84, Lemma 3.2]).

We consider the quadratic space (E, q_0), where $q_0 : E \times E \to \mathbf{Q}$ is given by $q_0(x, y) = \mathrm{Tr}_{E/\mathbf{Q}}(ux\bar{y})$; the signature of (E, q_0) is $(2, 2n-2)$.

If k is an integer with $k \geqslant d$, we denote by L_k the lattice of (E, q_0) given by $L_k = P^k$.

Lemma 13.1. *The Witt class of the discriminant form of* L_k *in* $W(\mathbf{Q}/\mathbf{Z})$ *is independent of* k.

Proof. This follows from the same argument as Lemma 12.3. ∎

Let $\bar{\varepsilon} \in \mathbf{F}_p^\times/\mathbf{F}_p^{\times 2}$ be the unique non-trivial element, and let $\varepsilon \in \mathbf{Z}$ be such that the image of ε in $\mathbf{F}_p^\times/\mathbf{F}_p^{\times 2}$ is equal to $\bar{\varepsilon}$.

Theorem 13.2. *Let* $L = L_k$ *be as above, set* $a = \delta + 2k$ *and let* $e \in \{\pm 1\}$ *be such that* $p^{r-1} \equiv e \pmod 4$. *We have* $G_L \simeq O_E/P^a$, *and*

$$[(G_L, q_L)] = \left[\left(\mathbf{Z}/p\mathbf{Z}, \frac{\varepsilon e}{p}xy\right)\right]$$

in $W(\mathbf{Q}/\mathbf{Z})$.

Proof. We have $G_L \simeq O_E/P^a$ as in Theorem 12.2. Let us show that

$$[(G_L, q_L)] = \left[\left(\mathbf{Z}/p\mathbf{Z}, \frac{\varepsilon e}{p}xy\right)\right]$$

in $W(\mathbf{Q}/\mathbf{Z})$. By Lemma 13.1 it is enough to consider the case $k = d$, hence we can assume that $G_L = \mathbf{Z}/p\mathbf{Z}$. By Theorem 12.2, the discriminant form of the positive definite lattice C_d is $(\mathbf{Z}/p\mathbf{Z}, \frac{-e}{p}xy)$; let us show that the discriminant form of L_d is $(\mathbf{Z}/p\mathbf{Z}, \frac{\varepsilon e}{p}xy)$.

Recall that $n = [F : \mathbf{Q}]$. Let v_p be the unique finite place of F above p. Let us write $E = F(\sqrt{\theta})$, with $\theta \in F^\times$. We have $(u, \theta)_v = 1$ for all finite places of v of F with $v \neq v_p$. Suppose that n is odd. Then $(u, \theta)_v = 1$ at an even number of infinite places v of F, hence the product formula implies that $(u, \theta)_{v_p} = 1$. This implies that the discriminant forms of L_d and C_d are isomorphic, hence they are isomorphic to $(\mathbf{Z}/p\mathbf{Z}, \frac{-e}{p}xy)$. Note that n is odd if and only if $p \equiv 3 \pmod 4$, and in this case -1 is not a square $\pmod p$, hence we can take $\varepsilon = -1$.

Suppose that n is even. Then the above argument shows that $(u, \theta)_{v_p} = -1$. Applying [BT 20, Proposition 6.6], we see that the discriminant form of L_d is isomorphic to $(\mathbf{Z}/p\mathbf{Z}, \frac{-\varepsilon e}{p}xy)$ in this case. Note that n is even if and only if $p \equiv 1 \pmod 4$, and -1 is a square $\pmod 4$ in this case. This implies that the discriminant form of L_d is isomorphic to $(\mathbf{Z}/p\mathbf{Z}, \frac{\varepsilon e}{p}xy)$ in this case as well. ■

Remark 13.3. The lattices L_k and C_k are defined for $k \geqslant d$, and their determinant is p^a, with $a = \delta + 2k$. The condition $k \geqslant d$ is equivalent to $a \geqslant 1$. This motivates the following notation.

Notation 13.4. If $a \geqslant 1$ is an integer, set $\Lambda_a = L_k$ with $k = \frac{a-\delta}{2}$.

Example 13.5. Let $E = \mathbf{Q}(\zeta_p)$, i.e. $r = 1$. Then $e = 1$, hence the discriminant form of Λ_1 is $(\mathbf{Z}/p\mathbf{Z}, \frac{\varepsilon}{p}xy)$. If $p \equiv 3 \pmod 4$, then we can take $\varepsilon = -1$, and this implies that the discriminant form is $(\mathbf{Z}/p\mathbf{Z}, \frac{-1}{p}xy)$. Suppose that $p \equiv 1 \pmod 4$. If $p \equiv 5 \pmod 8$, then 2 is not a square $\pmod p$, therefore we can take $\varepsilon = 2$, and the discriminant form is then isomorphic to $(\mathbf{Z}/p\mathbf{Z}, \frac{2}{p}xy)$. This covers all the prime numbers p needed for the applications to $K3$ surfaces, except for $p = 17$. In this case, we can take $\varepsilon = 3$, hence the discriminant form is isomorphic to $(\mathbf{Z}/17\mathbf{Z}, \frac{3}{17}xy)$.

Example 13.6. Let $E = \mathbf{Q}(\zeta_m)$, with $m = 9, 25$ or 27. We have $e = -1$ if $m = 9$ and $e = 1$ for $m = 25$ or 27. We can take $\varepsilon = -1$ for $m = 9$ or 27, and $\varepsilon = 2$ if $m = 25$ (cf. Example 13.5). Hence the discriminant form of Λ_1 is

$$\left(\mathbf{Z}/p\mathbf{Z}, \frac{1}{3}xy\right) \text{ if } m = 9; \ \left(\mathbf{Z}/p\mathbf{Z}, \frac{-1}{3}xy\right) \text{ if } m = 27; \ \left(\mathbf{Z}/p\mathbf{Z}, \frac{2}{5}xy\right) \text{ if } m = 25.$$

If L is a lattice and n is an integer, we denote by $L(n)$ the lattice with values multiplied by n.

Notation 13.7. If $a \geqslant 1$ is an integer, set $\Delta_a = C_k(-1)$ with $k = \frac{a-\delta}{2}$.

Lemma 13.8. *Suppose that $r = 1$, and let $a \geqslant 1$ be an integer. Set $L = \Delta_a$. Then we have*

$$[(G_L, q_L)] = \Big[\Big(\mathbf{Z}/p\mathbf{Z}, \frac{1}{p}xy\Big)\Big]$$

in $W(\mathbf{Q}/\mathbf{Z})$.

Proof. This follows from Theorem 12.2, noting that since $r = 1$, we have $e = 1$, and that Δ_a is negative definite. ■

14. Twisted lattices

We keep the notation of the previous section, and assume that $r = 1$, hence $E = \mathbf{Q}(\zeta_p)$, and $F = \mathbf{Q}(\zeta_p + \zeta_p^{-1})$. We fix a real embedding $\sigma_0 : F \to \mathbf{R}$, and we assume that there exists a unit $u \in O_F^\times$ such that $\sigma_0(u) > 0$ and that $\sigma(u) < 0$ for all the other real embeddings σ of F. For all odd integers $a \geq 1$, we define the lattices Λ_a and Δ_a as in the previous section. We denote by P the unique ramified ideal of O_E.

Definition 14.1. Let (L, q) be a negative definite even lattice. A *root* of L is an element $x \in L$ such that $q(x, x) = -2$. We say that L is a root lattice if it has an integral basis of roots.

Proposition 14.2. *Let $J \subset O_E$ be an ideal, and let $a \geqslant 1$ be an odd integer; let L be a twist of Δ_a by J. If $a > 1$ or if $J \neq O_E$, then the lattice L does not contain any roots.*

Proof. The lattice L has an isometry $t : L \to L$ with characteristic polynomial Φ_p. By [BM 94, Appendix], this shows that L does not contain any root sublattice if $a > 1$ or if $J \neq O_E$. ■

Proposition 14.3. *Let p be a prime number with $p \equiv 3 \pmod 4$, and let ζ be a primitive p-th root of unity. Let $a \geqslant 1$ be an integer, and let $J \subset O_E$ be an ideal prime to P. Let Λ be a twist of Λ_a by J, and let Δ be a twist of Δ_a by J.*

$t_\Lambda : \Lambda \to \Lambda$ and $t_\Delta : \Delta \to \Delta$ be the isometries induced by multiplication by ζ. Then there exists an even, unimodular lattice L containing $\Lambda \oplus \Delta$ as a sublattice of finite index, and an isometry $t : L \to L$ such that $t|\Lambda = t_\Lambda$, $t|\Delta = t_\Delta$.

Proof. The proof uses gluing of lattices and isometries as in McMullen's papers [McM 11, Section 2] or [McM 16, Section 4]. We check that the conditions of [McM 11, p. 5] (gluing of a pair of lattices, extending isometries) hold for Λ, t_Λ and Δ, t_Δ. If suffices to check these conditions for all prime numbers q dividing the orders of the discriminant groups (glue groups) of Λ and Δ, because of the primary decomposition of these groups (see [McM 11, p. 4]).

We start with the p-primary components. The discriminant modules of Λ_a and Δ_a are isomorphic, and their discriminant forms have opposite signs, since their Witt classes are $[(\mathbf{Z}/p\mathbf{Z}, \frac{-1}{p}xy)]$, respectively $[(\mathbf{Z}/p\mathbf{Z}, \frac{1}{p}xy)]$; hence McMullen's conditions hold at p.

Let q be a prime number with $q \neq p$. If J is squarefree, then the discriminant groups are vector spaces over $\mathbf{F}_q$, hence the conditions of [McM 11, Section 3] are fulfilled; if J is not squarefree, then Milnor's argument in [M 69, Section 3, Theorem 3.4], shows that it is enough to consider the case where the discriminant groups are vector spaces over $\mathbf{F}_q$.

Hence McMullen's conditions hold at every prime number, and therefore there exists an even, unimodular lattice L containing $\Lambda \oplus \Delta$ with finite index, and an isometry $t : L \to L$ such that $t|\Lambda = t_\Lambda$, $t|\Delta = t_\Delta$. ■

Remark 14.4. Let $a = 1$ and $J = O_E$. The previous proposition holds with $t_\Delta = \mathrm{id}$; we obtain an even, unimodular lattice L and an isometry $t : L \to L$ such that $t|\Lambda$ is the multiplication by ζ and $t|\Delta$ is the identity.

15. A family of $K3$ surfaces

Let p be a prime number, $3 \leqslant p \leqslant 11$. We keep the notation of the previous sections; in particular, $\varepsilon \in \mathbf{Z}$ is such that the image of ε in $\mathbf{F}_p^\times/\mathbf{F}_p^{\times 2}$ is the unique non-trivial element. Set $E = \mathbf{Q}(\zeta_p)$, and let P be the unique ramified prime ideal of O_E.

If X is a $K3$ surface, we denote by T_X its transcendental lattice and by S_X its Picard lattice.

Theorem 15.1. *Let $a \geqslant 1$ be an odd integer. There exists a unique* (up to isomorphism) *complex projective $K3$ surface $X_a = X_a(p)$ with maximal complex multiplication by E such that the following equivalent conditions hold:*

(i) *The discriminant module of T_{X_a} is isomorphic to P/P^a, and the Witt class of its discriminant form (G_{X_a}, q_{X_a}) is $[(\mathbf{Z}/p\mathbf{Z}, \frac{\varepsilon}{p}xy)]$.*

(ii) $G_{X_a} \simeq P/P^a$.

(iii) $\det(S_{X_a}) = \det(T_{X_a}) = p^a$.

Moreover, the surfaces X_a are isogeneous for all $a \geqslant 1$.

Recall that for all integers $a \geqslant 1$, the O_E-lattice Λ_a is defined in Section 13, see Notation 13.4.

Theorem 15.2. *Let $a \geqslant 1$ be an odd integer. There exists a unique* (up to isomorphism) *complex projective $K3$ surface X_a with maximal complex multiplication by E such that the transcendental lattice T_{X_a} is Hodge isomorphic to the O_E-lattice Λ_a.*

Proof. Since $p \leqslant 11$, we have $\ell(G_{\Lambda_a}) \leqslant 10$, hence Λ_a embeds uniquely into Λ. By Proposition 7.1, there exists a complex projective $K3$ surface X_a having CM by E such that the O_E-lattice T_{X_a} is Hodge isometric to Λ_a; the surface X_a is unique up to isomorphism (cf. Proposition 7.4). By construction, we have

$$\Lambda_a \otimes_{\mathbf{Z}} \mathbf{Q} \simeq \Lambda_b \otimes_{\mathbf{Z}} \mathbf{Q} \quad \text{for all } a, b \geqslant 1;$$

hence, these surfaces are all isogeneous (see [Mu 87, N 87, Bu 19]). ■

Proof of Theorem 15.1. The existence and the uniqueness of the $K3$ surfaces X_a with property (i) results from Theorem 15.2 and Theorem 13.2; Theorem 15.2 also implies that the surfaces X_a are all isogeneous. It is clear that (i)⇒(ii)⇒(iii), and by Proposition 7.5 we have (ii)⇒(i). It remains to show that (iii)⇒(ii). The only prime ideal of O_E above p is the unique ramified ideal P, and we have $\mathrm{N}(P) = p$. Therefore, $\det(T_X) = p^a$ implies that $G_{T_X} \simeq O_E/P^a$ as O_E-modules, and therefore (ii) holds. ■

Example 15.3. For $a = 1$, we recover Vorontsov's examples of $K3$ surfaces (see [V 83, K 92]) arising in connection with automorphisms acting trivially on the Picard lattice; these surfaces are elliptic with a section and defining Weierstrass equations over $\mathbf{Q}$ are given in [K 92]. The discriminant forms of these surfaces are computed using an elliptic fibration in [LSY 10, Table 5].

16. More $K3$ surfaces

Let $E = \mathbf{Q}(\zeta_m)$ with $m = 9, 13, 17, 19, 25$ or 27, and let us denote by P the unique ramified ideal of O_E.

Theorem 16.1. *Let $a \geqslant 1$ be an odd integer, and suppose that $a \leqslant 7$ if $m = 13$, $a \leqslant 5$ if $m = 17$, $a = 1$ if $m = 19, 25$ or 27.*

Then there exists a unique (up to isomorphism) *complex projective $K3$ surface $X_a(m)$ with maximal complex multiplication by E such that the following equivalent conditions hold:*

- (i) *The discriminant module of T_{X_a} is isomorphic to P/P^a, and the Witt class of its discriminant form (G_{X_a}, q_{X_a}) is $[(\mathbf{Z}/p\mathbf{Z}, \frac{\varepsilon}{p}xy)]$ if $m = p = 13, 17$ or 19; $[(\mathbf{Z}/3\mathbf{Z}, \frac{1}{3}xy)]$ if $m = 9$; $[(\mathbf{Z}/3\mathbf{Z}, \frac{-1}{3}xy)]$ if $m = 27$ and $[(\mathbf{Z}/5\mathbf{Z}, \frac{2}{5}xy)]$ if $m = 25$.*
- (ii) $G_{X_a(m)} \simeq P/P^a$.
- (iii) $\det(S_{X_a}(m)) = \det(T_{X_a}(m)) = p^a$ *if m is a power of p.*

Moreover, the surfaces $X_a(m)$ are isogeneous for all $a \geqslant 1$.

Proof. Suppose first that $m \neq 25$. Then the proof goes along the same lines as the proof of Theorem 15.1; the conditions on a ensure that Λ_a embeds uniquely in Λ (see [N 79, Corollary 1.12.3 and Theorem 1.14.4]). By Propositions 7.1 and 7.4, there exists a unique (up to isomorphism) complex projective $K3$ surface X_a having CM by E such that the O_E-lattice T_{X_a} is Hodge isometric to Λ_a of Section 13, cf. Notation 13.4.

If $m = 25$, then Λ_a embeds primitively in Λ (see [N 79, Corollary 1.12.3]), hence there exists a complex projective $K3$ surface X_a having CM by E such that the O_E-lattice T_{X_a} is Hodge isometric to Λ_a. The surface X_a has an automorphism that is the identity on the Picard lattice and induces complex multiplication on the transcendental lattice; this can be checked by noting that the action of the complex multiplication on T_{X_a} induces multiplication by -1 on G_{X_a}.

The discriminant form of $X_a(m)$ can be deduced from Theorem 13.2, see also Example 13.6; the proof is completed as in the proof of Theorem 15.1. ∎

Example 16.2. As in Example 15.3, for $a = 1$ we recover Vorontsov's examples of $K3$ surfaces (see [V 83, K 92]). These surfaces are elliptic with a section, except if $m = 25$; see [K 92] for defining equations over $\mathbf{Q}$.

17. Automorphisms

Let $p = 3, 7$ or 11 and let $a \geqslant 1$ be an odd integer; let $X_a = X_a(p)$ be the $K3$ surface of Section 15. Let ζ be a primitive p-th root of unity, and let $E = \mathbf{Q}(\zeta)$, considered as a subfield of $\mathbf{C}$.

Definition 17.1. Let X be a complex projective $K3$ surface with complex multiplication by E. Let $T : X \to X$ be an automorphism, and let $t : T_X \to T_X$ be the isometry induced by T. We say that t *induces the complex multiplication by E* if $t(\sigma) = \zeta\sigma$ where σ is a non-zero 2-form in $T_X \otimes_{\mathbf{Q}} \mathbf{C}$.

Theorem 17.2. (i) *For all $a \geqslant 1$, the $K3$ surface X_a has an automorphism of order p inducing the complex multiplication by E.*

(ii) *For all $a \geqslant 1$, the surface X_a is elliptic with a section.*

(iii) *If $a > 1$, then the Mordell–Weil lattice of X_a is isomorphic to Δ_a.*

Proof. If $a = 1$, then this is well known: the surfaces X_1 are isomorphic to Vorontsov's $K3$ surfaces, see Example 15.3.

Suppose that $a > 1$. By Proposition 14.3 there exists an even, unimodular lattice L containing $\Lambda_a \oplus \Delta_a$ as a sublattice of finite index, and an isometry $t_L : L \to L$ such that $t_L|\Lambda_a = t_\Lambda$, $t_L|\Delta_a = t_\Delta$.

Set $r = 24 - 2p$, and let M be an even, unimodular lattice of signature $(1, r - 1)$; such a lattice exists (and is unique up to isomorphism) since $r - 2$ is divisible by 8. Let $t_M : M \to M$ be the identity.

Set $N = M \oplus L$, and let $t : N \to N$ be such that $t|L = t_L$, $t|M = t_M = \mathrm{id}$. The lattice N is even, unimodular of signature $(3, 19)$. Set $S = M \oplus \Delta_a$ and $T = \Lambda_a$. Since $a > 1$, the lattice Δ_a does not contain any roots (cf. Lemma 14.2). The isometry t is the identity on N, hence t_S satisfies the conditions of McMullen in [McM 16, Section 6]. Therefore, by [McM 16, Theorem 6.1] there exists a complex projective $K3$ surface X with $S_X \simeq S$, $T_X \simeq T$, and an automorphism $T : X \to X$ such that $T^* = t$. This $K3$ surface is isomorphic to X_a. This shows (i).

The lattice M is even, unimodular and of signature $(1, r)$, hence it has an orthogonal factor isomorphic to the 2-dimensional hyperbolic lattice U. This implies that X_a is elliptic with a section (see [SS 19, Theorem 11.24]) hence (ii) holds. Note that the orthogonal complement of U in M is a (negative) root lattice.

If $a > 1$, then Δ_a has no roots, hence the trivial lattice of the fibration is isomorphic to M, and the Mordell–Weil lattice to Δ_a; this implies (iii). ∎

18. Twisted $K3$ surfaces

The aim of this section is to extend the results of Sections 15 and 17 to certain twisted $K3$ surfaces. Let p be a prime number, $3 \leqslant p \leqslant 11$, and set $E = \mathbf{Q}(\zeta_p)$. We keep the notation of the previous sections; in particular, P is the unique ramified prime ideal of E.

If $a \geqslant 1$ is an odd integer, the lattices Λ_a and Δ_a are defined in Section 15. Let $J \subset O_E$ be an O_E-ideal prime to P such that $\bar{J} = J$. Since $h_E = 1$, the twist of an O_E-lattice by J is uniquely defined (up to isomorphism; see Proposition 6.4). We denote by $\Lambda_{a,J}$ and $\Delta_{a,J}$ the twists of Λ_a and Δ_a by J.

Theorem 18.1. *Let $a \geqslant 1$ be an odd integer, and let $J \subset O_E$ be an O_E-ideal prime to P such that $\bar{J} = J$. Then*

(i) *There exists a unique* (up to isomorphism) *complex projective $K3$ surface $X_{a,J}$ with maximal complex multiplication by E such that the transcendental lattice $T_{X_{a,J}}$ is Hodge isomorphic to the O_E-lattice $\Lambda_{a,J}$.*

Suppose that $p = 3$, 7 or 11.

(ii) *The $K3$ surface $X_{a,J}$ has an automorphism of order p inducing the complex multiplication by E.*

(iii) *The surface $X_{a,J}$ is elliptic with a section.*

(iv) *If $a > 1$ or $J \neq O_K$, then the Mordell–Weil lattice of $X_{a,J}$ is isomorphic to $\Delta_{a,J}$.*

Proof. (i) We have $[E : \mathbf{Q}] = p - 1$ and $p \leqslant 11$, hence the lattice $\Lambda_{a,J}$ embeds uniquely into Λ. Proposition 7.1 implies that there exists a complex projective $K3$ surface $X_{a,J}$ having CM by E such that the O_E-lattice $T_{X_{a,J}}$ is Hodge isometric to $\Lambda_{a,J}$; the surface $X_{a,J}$ is unique up to isomorphism (cf. Proposition 7.4).

(ii) and (iii) If $a = 1$, then this follows from Theorem 17.2 (i) and (ii). Set $\Lambda = \Lambda_{a,J}$ and $\Delta = \Delta_{a,J}$. By Proposition 14.3 there exists an even, unimodular lattice L containing $\Lambda \oplus \Delta$ as a sublattice of finite index, and an isometry $t_L : L \to L$ such that $t_L|\Lambda = t_\Lambda$, $t_L|\Delta = t_\Delta$. Set $r = 24 - 2p$, and let M be an even, unimodular lattice of signature $(1, r-1)$; such a lattice exists (and is unique up to isomorphism) since $r - 2$ is divisible by 8. Let $t_M : M \to M$ be the identity. Set $N = M \oplus L$, and let $t : N \to N$ be such that $t|L = t_L$, $t|M = t_M = \mathrm{id}$. The lattice N is even, unimodular of signature $(3, 19)$. Set $S = M \oplus \Delta$ and $T = \Lambda$.

Suppose that $a > 1$. Then the lattice Δ does not contain any roots (cf. Lemma 14.2). The isometry t is the identity on N, hence t_S satisfies the conditions of McMullen in [McM 16, Section 6]. Therefore, by [McM 16, Theorem 6.1], there exists a complex projective $K3$ surface X with $S_X \simeq S$, $T_X \simeq T$, and an automorphism $T : X \to X$ such that $T^* = t$. This $K3$ surface is isomorphic to $X_{a,J}$, and this implies (ii).

The lattice M has an orthogonal factor isomorphic to the 2-dimensional hyperbolic lattice U, therefore $X_{a,J}$ is elliptic with section (see [SS 19, Theorem 11.24]), hence (iii) holds.

(iv) If $a > 1$ or $J \neq O_E$, then by Lemma 14.2 the lattice Δ does not contain any roots. Therefore, the trivial lattice of the fibration is isomorphic to M, and the Mordell–Weil lattice is isomorphic to Δ. ∎

Theorem 18.2. *Let $p = 3, 7$ or 11, and let X be a $K3$ surface having an automorphism of order p inducing the complex multiplication by E. Then there exists an integer $a \geqslant 1$ and an ideal $J \subset O_E$ such that $X \simeq X_{a,J}$.*

Proof. The O_E-module $G_X = T_X^{\sharp}/T_X$ is isomorphic to $O_E/P^a \oplus O_E/J$ for some integer $a \geqslant 1$ and some O_E-ideal J. The discriminant module of $X_{a,J}$ is also isomorphic to $O_E/P^a \oplus O_E/J$, and the length of this abelian group is $\leqslant 10$, hence by Proposition 7.5 we have $X \simeq X_{a,J}$. ∎

19. Moduli spaces

An automorphism of a $K3$ surface is said to be *symplectic* if it induces the identity on the transcendental lattice (hence on the symplectic form), and *non-symplectic* otherwise. Artebani, Sarti and Taki identified the irreducible components of the moduli space of $K3$ surfaces with a non-symplectic automorphism of prime order (see [AST 11], see also [AS 08] for $p = 3$ and [OZ 10, ACV 21] for $p = 11$).

Let $\Lambda = \Lambda_{3,19}$ be the $K3$-lattice. If X is a $K3$-surface, we denote by ω_X a nowhere vanishing holomorphic 2-form on X. Let p be a prime number, and let ζ_p be a primitive p-th root of unity. Let $\rho : \Lambda \to \Lambda$ be an isometry of order p, and let us denote by $[\rho]$ its conjugacy class in $O(\Lambda)$. A $[\rho]$-polarized $K3$ surface is a pair (X, t) where X is a $K3$ surface and t a non-symplectic automorphism of X of order p such that $t^*(\omega_X) = \zeta_p \omega_X$ and $t^* = \Phi \circ \rho \circ \Phi^{-1}$, for some (fixed) isometry $\Phi : \Lambda \to H^2(X, \mathbf{Z})$, which is called a marking; the moduli space of such polarized $K3$ surfaces is denoted by $\mathcal{M}^p$ (see [AST 11, Section 9]). We say that a point of this moduli space is a *CM point* if the $K3$ surface X has complex multiplication.

Let $p = 3, 7$ or 11, and let $E = \mathbf{Q}(\zeta_p)$. We keep the notation of the previous sections: in particular, P denotes the unique ramified ideal of E.

Let $a \geqslant 1$ be an odd integer and let $J \subset O_E$ be an ideal prime to P. Let $X_{a,J}$ be the $K3$ surface defined in Section 18. The following corollary is an immediate consequence of Theorem 18.1 (ii).

Corollary 19.1. *The $K3$ surface $X_{a,J}$ has an automorphism of order p inducing the complex conjugation on the transcendental lattice $T_{X_{a,T}}$, and determines a CM point on the moduli space $\mathcal{M}^p$.*

Moreover, Theorem 18.2 implies that all CM points with maximal complex multiplication by E arise in this way.

20. Fields of definition, class fields and elliptic fibrations

Piatetski-Shapiro and Shafarevich proved that a $K3$ surface with complex multiplication can be defined over a number field (see [PS 73, Theorem 4]). If moreover the complex multiplication is *maximal*, Valloni obtained more precise results in [V 21, V 23]; if E is a

CM field and $I \subset O_E$ an ideal with $\overline{I} = I$, he defined a finite abelian extension $F_I(E)$ such that every $K3$ surface with CM by O_E with discriminant ideal I can be defined over $F_I(E)$.

Let $p = 3, 7$ or 11, let ζ_p be a primitive p-th root of unity and let $E = \mathbf{Q}(\zeta_p)$. Let $a \geqslant 1$ be an integer, let $J \subset O_E$ be an ideal relatively prime to the unique ramified ideal P of E such that $\overline{J} = J$. Let $X_{a,J}$ be the $K3$ surface defined in Section 18. As we have seen in Section 19, this gives rise to a CM point on the moduli space $\mathcal{M}^p$; the description of Artebani, Sarti and Taki of the moduli space can be used to obtain a field of definition of $X_{a,J}$. This is illustrated by the following example, due to Brandhorst and Elkies.

Example 20.1. McMullen proved the existence of a $K3$ surface with an automorphism of entropy equal to the logarithm of the Lehmer number (see [McM 16]), and raised the question of constructing this surface and the automorphism explicitly. This was achieved by Brandhorst and Elkies in [BE 23].

Set $E = \mathbf{Q}(\zeta_7)$, $F = \mathbf{Q}(\zeta_7 + \zeta_7^{-1})$ and let J be one of the O_E-ideals above 13. The $K3$ surface S constructed in [BE 23] has a non-symplectic automorphism of order 7, and the construction shows that it is isomorphic to $X_{1,J}$. In [BE 23, Sections 3 and 4], the authors use the description of $\mathcal{M}^7$ by [AST 11] to obtain an equation for the surface S, with coefficients in a quadratic extension K of the field F; set $K = \mathbf{Q}(w)$, where w is such that

$$w^6 - 2w^5 + 2w^4 - 3w^3 + 2w^2 - 2w + 1 = 0.$$

The field K has discriminant $7^4 13$, and contains the field F of discriminant 49.

Let P be the unique ramified prime ideal of E. The composite field KE is isomorphic to Valloni's number field $F_{PJ}(E)$, i.e., $F_{PJ}(E) \simeq \mathbf{Q}(\zeta_7, w)$. This computation was done with the help of PARI/GP.

21. Some equations

Let $p = 3, 7$ or 11, let ζ_p be a primitive p-th root of unity, and set $E = \mathbf{Q}(\zeta_p)$. Let a be an odd integer, let $J \subset O_E$ be an ideal relatively prime to the unique ramified ideal P of E such that $\overline{J} = J$. We denote by $X_{a,J}(p)$ the $K3$ surface defined in Section 18; if $J = O_E$, then we use the notation $X_a(p)$, as in Section 15. As mentioned in Example 20.1, Brandhorst and Elkies gave an explicit equation for the surface $X_{1,J}(7)$, where J is one of the prime O_E-ideals above 13 (see [BE 23, Section 4]); their method can be used for other choices of a and J.

I thank Simon Brandhorst for the following examples.

Example 21.1. Let $p = 7$, $E = \mathbf{Q}(\zeta_7)$, set $w = \zeta_7 + \zeta_7^{-1}$, and let J be one of the prime O_E-ideals above 2. An equation of the surface $X_{1,J}(7)$ is given by

$$y^2 = x^3 + bx + ct^7 + d,$$

where

$$b = (-3403/16)(w^2+4w+4), \quad c = 14(w^2+2w+1), \quad d = (293419/32)(w^2+2w+1).$$

Example 21.2. Let $p = 7$, $E = \mathbf{Q}(\zeta_7)$, and set $w = \zeta_7 + \zeta_7^{-1}$. An equation of the surface $X_3(7)$ is given by

$$y^2 = x^3 + bx + ct^7 + d,$$

where

$$b = (-230578777287775/2)w^2 + (127961567541885/2)w + 4144846476936445/16,$$
$$c = -5842669785012830924w^2 + 3242437110294043228w + 13128359838180149367,$$
$$d = 1514618874530843832470 79/32.$$

The computations are due to Simon Brandhorst, and they were done by Sage.

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