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Geometric rigidity in variable domains and derivation of linearized models for elastic materials with free surfaces

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Abstract. We present a quantitative geometric rigidity estimate in dimensions $d = 2, 3$ generalizing the celebrated result by Friesecke, James, and Müller [Comm. Pure Appl. Math. 55 (2002), 1461–1506] to the setting of variable domains. Loosely speaking, we show that for each $y \in H^1(U; \mathbb{R}^d)$ and for each connected component of an open, bounded set $U \subset \mathbb{R}^d$, the L^2-distance of ∇y from a single rotation can be controlled up to a constant by its L^2-distance from the group $\mathrm{SO}(d)$, with the constant not depending on the precise shape of U, but only on an integral curvature functional related to ∂U. We further show that for linear strains the estimate can be refined, leading to a uniform control independent of the set U. The estimate can be used to establish compactness in the space of generalized special functions of bounded deformation (GSBD) for sequences of displacements related to deformations with uniformly bounded elastic energy. As an application, we rigorously derive linearized models for nonlinearly elastic materials with free surfaces by means of Γ-convergence. In particular, we study energies related to epitaxially strained crystalline films and to the formation of material voids inside elastically stressed solids.

1. Introduction

Rigidity estimates have a long history dating back to Liouville's fundamental result which states that smooth mappings are necessarily affine if their gradient is a rotation everywhere. After various generalizations of this classical theorem over recent decades [58, 60, 77], a fundamental breakthrough was achieved by Friesecke, James, and Müller [48] with their celebrated quantitative geometric rigidity result in nonlinear elasticity theory. In its basic form, the estimate states that in any dimension $d \geq 2$, for a mapping $y \in H^1(\Omega; \mathbb{R}^d)$ there exists a corresponding rotation $R \in \mathrm{SO}(d)$ such that

$$\int_\Omega |\nabla y - R|^2 \,\mathrm{d}x \leq C \int_\Omega \mathrm{dist}^2(\nabla y, \mathrm{SO}(d)) \,\mathrm{d}x \tag{1.1}$$

for a constant $C > 0$ only depending on the (sufficiently regular) bounded domain Ω. This result is fundamental in the analysis of variational models in nonlinear elasticity, as it

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provides compactness for sequences of deformations and corresponding displacements with uniformly bounded elastic energy in a sharp quantitative fashion. In fact, it has proved to be the cornerstone for rigorous derivations of lower-dimensional theories for plates, shells, and rods in various scaling regimes [47–49, 64, 70, 71], and for providing relations between geometrically nonlinear and linear models in elasticity [30]. The estimate (1.1) was generalized in various directions to analyze variational models for materials with elastic and plastic behavior. Among others, we mention results for mixed growth conditions [22], incompatible fields [24, 62, 72], and settings involving multiple energy wells [20, 21, 25, 31, 34, 56, 65].

Background and motivation. In this paper, we are interested in rigidity estimates for nonlinearly elastic energies involving free surfaces. Our motivation lies in studying models in the framework of *stress driven rearrangement instabilities* (SDRI), i.e., morphological instabilities of interfaces between elastic phases generated by the competition between elastic bulk and surface energies, including many different phenomena such as brittle fracture, formation of material voids inside elastically stressed solids, or hetero-epitaxial growth of elastic thin films. We refer to [8, 50–52, 59, 83, 85] for an overview of some mathematical and physical literature. From a variational viewpoint, the common feature of functionals describing SDRI is the presence of both stored elastic energies in the bulk and surface energies. This can be formulated in the language of *free discontinuity problems* [33], where the set of discontinuities is not preassigned, but determined from an energy minimization principle.

In this context, a major challenge in obtaining rigidity results lies in the fact that the functional setting goes beyond Sobolev spaces and requires functions allowing for jump discontinuities, more precisely *(special) functions of bounded variation* (SBV), see [4, Section 4], or *(special) functions of bounded deformation* (SBD), see [3, 29]. Moreover, the formulation is genuinely more involved compared to (1.1), as the domain may be disconnected by the jump set into various components, and therefore at most *piecewise rigidity* results can be expected, i.e., on each connected component of the domain without the jump set the deformation is close to a possibly different rigid motion.

Recent years have witnessed tremendous progress for rigidity results in the linearly elastic setting [13–15, 23, 42, 43], suitably generalizing the classical Korn inequality to SBD, and also controlling the surface contributions of the energy. The situation in the geometrically nonlinear setting, however, is far less well understood. A first key step in this direction was achieved by Chambolle, Giacomini, and Ponsiglione [18], showing a Liouville-type result for brittle materials storing no elastic energy. To the best of our knowledge, to date counterparts of the quantitative estimate (1.1) are limited to dimension two [46] or, in general dimensions, to a model for *nonsimple materials* [44], where the elastic energy depends additionally on the second gradient of the deformation; cf. [87]. The latter results have been employed successfully to identify linearized models in the small-strain limit [41, 44], and to perform dimension reduction [82].

In this paper, we prove a novel *quantitative geometric rigidity result for variable domains* in dimensions $d = 2, 3$; see Theorem 2.1. While our proof strategy in principle allows us to establish the result in higher dimensions as well, there is a single missing point, namely a specific geometric estimate of possible independent interest; see Remark 2.22. In the physically relevant dimensions $d = 2, 3$, we believe that our result may be applicable in a variety of different contexts, in particular to study problems on dimension reduction. In the present paper, as a first application, we employ the estimate to rigorously derive linearized models for elastic materials with free surfaces.

The rigidity estimate. Loosely speaking, given a fixed open, bounded set $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, our main result states the following: for every regular open set $E \subset \Omega$, we can find a *thickened set* $E \subset E^* \subset \Omega$ such that

$$\text{(i)}\ \ \mathcal{L}^d(E^* \setminus E) \ll 1, \quad \text{(ii)}\ \ |\mathcal{H}^{d-1}(\partial E^* \cap \Omega) - \mathcal{H}^{d-1}(\partial E \cap \Omega)| \ll 1, \tag{1.2}$$

where $\mathcal{L}^d$ and $\mathcal{H}^{d-1}$ denote the d-dimensional Lebesgue and $(d-1)$-dimensional Hausdorff measures, respectively, and for each $y \in H^1(\Omega \setminus \overline{E}; \mathbb{R}^d)$ with *elastic energy* $\varepsilon := \int_{\Omega \setminus \overline{E}} \mathrm{dist}^2(\nabla y, SO(d))\, \mathrm{d}x$, and in the case $\Omega \setminus \overline{E^*}$ is connected, there exists a proper rotation $R \in SO(d)$ such that

$$\begin{aligned} &\text{(i)}\ \int_{\Omega \setminus \overline{E^*}} |\,\mathrm{sym}(R^{\mathsf{T}} \nabla y) - \mathrm{Id}|^2\, \mathrm{d}x \le C(1 + C^{\mathrm{curv}}_{\partial E}\varepsilon)\varepsilon, \\ &\text{(ii)}\ \int_{\Omega \setminus \overline{E^*}} |\nabla y - R|^2\, \mathrm{d}x \le C^{\mathrm{curv}}_{\partial E}\varepsilon, \end{aligned} \tag{1.3}$$

where $\mathrm{sym}(F) := \frac{1}{2}(F + F^{\mathsf{T}})$ for $F \in \mathbb{R}^{d\times d}$, $\mathrm{Id} \in \mathbb{R}^{d\times d}$ denotes the identity matrix, and $C > 0$ is a constant depending on Ω but not on E. Eventually, $C^{\mathrm{curv}}_{\partial E} > 0$ is a constant depending on a suitable integral curvature functional of ∂E and can possibly become large as the curvature of ∂E becomes large. If $\Omega \setminus \overline{E^*}$ consists of different connected components, the rotation R may be different for each connected component; cf. also the piecewise estimate [18, Theorem 1.1].

Here, the role played by the unknown (i.e., variable) set E depends on the application, e.g., it may model material voids inside an elastic material with reference domain Ω. As E is regular, an estimate of the form (1.3) would in general follow directly from (1.1) for a *constant depending on* E. We therefore emphasize that the essential point of our estimate is that the constant C is *independent* of E and $C^{\mathrm{curv}}_{\partial E}$ does not depend on the precise shape of E, but only on

$$\int_{\partial E \cap \Omega} |\boldsymbol{A}|^q\, \mathrm{d}\mathcal{H}^{d-1} \tag{1.4}$$

for some fixed $q \ge d - 1$, where $\boldsymbol{A}$ denotes the second fundamental form of ∂E. (The choice $q \ge d - 1$ is essential for the proof; see Lemma 2.12 and Example 2.13.)

Given a uniform control on the above curvature term, (1.3) (ii) yields the exact counterpart of estimate (1.1), generalized to the setting of variable domains. Moreover, (1.3) (i), say for simplicity for $R = \mathrm{Id}$, shows that the L^2-norm of the symmetric part of $\nabla y - \mathrm{Id}$ can be controlled by the nonlinear elastic energy independently of $C^{\mathrm{curv}}_{\partial E}$, provided that ε is small compared to the inverse of $C^{\mathrm{curv}}_{\partial E}$. The latter property will allow us to obtain a uniform control on linear strains $e(u) := \frac{1}{2}(\nabla u + \nabla u^{\mathsf{T}})$ for displacements $u = y - \mathrm{id}$, where id denotes the identity mapping. This naturally leads to effective descriptions in the realm of SBD functions [29], for which only symmetrized gradients are controlled.

Proof strategy and discussion. The core of the proof consists of a geometric construction to modify the set E, along with the proof strategy for (1.1) devised in [48]. More specifically, we find a thickened set $E^* \supset E$ consisting essentially of a union of cubes of a specific sidelength $\rho > 0$, which depends only on the size of the curvature term in (1.4). As already observed in [48], the rigidity constant of $\Omega \setminus \overline{E^*}$ only depends on Ω and ρ, which implies (1.3) (ii). To derive (1.3) (i), we use (1.3) (ii) and the fact that the tangent space of the smooth manifold $\mathrm{SO}(d)$ at the identity matrix is given by the linear space of all skew-symmetric matrices, which in particular implies that

$$|(F^{\mathsf{T}} + F)/2 - \mathrm{Id}| = \mathrm{dist}(F, \mathrm{SO}(d)) + \mathrm{O}(|F - \mathrm{Id}|^2).$$

Here, as in [48], we also reduce the problem to harmonic mappings in order to control higher-order terms through an L^2-L^∞ estimate obtained by the mean value property. After controlling the symmetric part of the gradient, the last step in the proof of (1.1) in [48] consists in applying Korn's inequality to obtain (1.1). This, however, is not possible in our setting as the constant in Korn's inequality again depends on the shape of the domain $\Omega \setminus \overline{E^*}$ which would only give back an estimate of the form (1.3) (ii). In conclusion, even in the regime where the elastic energy is sufficiently small with respect to the curvature energy term in (1.4), uniform bounds independent of E can only be obtained for symmetrized gradients but not for full gradients. Simple examples show that estimate (1.3) (ii) is indeed sharp; see Example 2.7.

Whereas (1.3) can be derived by adapting the original strategy devised in [48], the real novelty of our work lies in the construction of the *thickened set* $E^* \supset E$. In the application to variational models for SDRI presented below, estimate (1.2) is essential to ensure that the thickening of the set does not affect E asymptotically in volume and surface measure. In a first auxiliary step, in order to ensure that $\Omega \setminus \overline{E^*}$ is essentially a union of cubes with equal sidelength, we tessellate $\mathbb{R}^d$ with cubes of sidelength $\rho > 0$ and add to E all cubes intersecting ∂E, the so-called *boundary cubes*. In order to verify (1.2) (i), one needs to control the number of boundary cubes. This is highly nontrivial as the boundary ∂E might become extremely complex, exhibiting *thin spikes* or microscopically small components with small surface measure on different length scales; see Figure 1. The key ingredient is Lemma 2.12 which, in rough terms, states that for a specific choice of the sidelength ρ, in each boundary cube Q_ρ we get that $\mathcal{H}^{d-1}(\partial E \cap Q_\rho)$ or $\int_{\partial E \cap Q_\rho} |\boldsymbol{A}|^q \, \mathrm{d}\mathcal{H}^{d-1}$ is at least of order ρ^{d-1}.

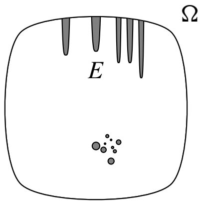

Figure 1. A possible void set E, depicted in gray, that contains thin spikes or small components that may prevent rigidity for deformations defined on the set $\Omega \setminus \overline{E}$.

Loosely speaking, this means that spikes or microscopic components of ∂E accumulating on scales smaller than ρ induce too high curvature energy, and can therefore be excluded. Let us emphasize here that establishing the higher-dimensional version of the last assertion for closed hypersurfaces is exactly the missing ingredient to generalize our result to any space dimension.

Subsequently, the construction of E^* needs to be refined in order to also satisfy (1.2) (ii). To this end, we use the property that under a specific area and curvature bound in a boundary cube Q_ρ, the surface $\partial E \cap Q_\rho$ inside a smaller cube is essentially a finite union of graphs of Lipschitz functions with appropriate a priori estimates. Based on this, a direct geometric construction can be performed to thicken the sets. Whereas this local graphical approximation of ∂E is elementary in dimension $d = 2$ (see Lemma 2.15), in dimension $d = 3$ and for $q = 2$, it is a deep ε-regularity result in geometric analysis due to Simon [84]; see Lemma 2.16 and also Remark 2.22.

We note that the passage to a thickened set E^* is not due to our specific proof strategy, but is indeed necessary for a uniform rigidity estimate. Simple examples, where $\Omega \setminus \overline{E}$ is connected but only through a *thin tunnel*, show that (1.1) (with a uniform constant) can be violated for deformations concentrating elastic energy in the tunnel; see Example 2.6.

Our result appears to address an immediate situation between the result in the Sobolev setting [48] and the above-mentioned results [13, 18, 43, 46] in the function spaces SBV and SBD, where additional difficulties are present due to the lack of regularity of deformations. Indeed, in our setting, deformations are still Sobolev, yet defined on sets with free boundary. By approximation results in SBV and SBD [16, 26] however, jump sets can be regularized and can be covered by regular sets E. In this sense, our estimate is in spirit closer to results in SBV and SBD, and throughout the proof we encounter many intricacies present in these function spaces concerning the topology and geometry of jump sets.

As a final comment on the rigidity result, let us emphasize that the idea of deriving uniform estimates for variable domains under certain assumptions on the sets E (or assumptions on the geometry of the jump set) is not new but has been used in a variety of free discontinuity problems; see e.g., [63, 73, 78]. These models, however, are based on considering very specific classes of discontinuity sets with certain geometric features such as well-separateness. Our approach, instead, readily relies on a curvature control of the form (1.4) which can be implemented easily in a variational model. Indeed, curvature regularizations are widely used in the mathematical and physical literature of SDRI models, including the description of (the evolution of) elastically stressed thin films or material voids; see [5, 12, 35, 39, 40, 53, 54, 75, 76, 83].

Applications to linearization of variational SDRI models. We employ the rigidity result to derive a rigorous connection between models for hyperelastic materials in nonlinear (finite) elasticity and their linear (infinitesimal) counterparts. Although a classical topic in elasticity theory, this relation has been derived rigorously via Γ-convergence [9,28] only comparatively recently by Dal Maso, Negri, and Percivale [30]. The authors performed a nonlinear-to-linear analysis in terms of suitably rescaled displacement fields and proved the convergence of minimizers for corresponding boundary value problems. Their study has been extended in various directions, ranging from models for incompressible materials [57, 67], from atomistic models [11, 81], to multiwell energies [1, 80], plasticity [69], viscoelasticity [45], or fracture [41, 44]. In all of these results, the rigidity estimate (1.1) or one of its variants plays a key role in establishing compactness.

Despite the huge body of literature on variational SDRI models, in particular on epitaxially strained elastic thin films (see e.g., [6, 7, 19, 27, 32, 38]) and material voids [10, 27, 37, 79], results on rigorous relations between nonlinear and linear theories are scarce. To the best of our knowledge, the only available result is the recent work [61] on two-dimensional elastic thin films. In this setting, one can resort to the Hausdorff topology for sets, which in turn allows one to apply the rigidity estimate (1.1). Yet the situation in higher dimensions and in the case of a possibly unbounded number of surface components (as in the case of material voids) is much more intricate, and a more general rigidity result of the form (1.3) is indispensable.

We consider functionals defined on function-set pairs featuring nonlinear elastic bulk and surface contributions of the form

$$F_\delta(y,E) := \frac{1}{\delta^2}\int_{\Omega\setminus\bar{E}} W(\nabla y)\,\mathrm{d}x + \int_{\partial E\cap\Omega} \varphi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} + \gamma_\delta \int_{\partial E\cap\Omega} |\boldsymbol{A}|^q\,\mathrm{d}\mathcal{H}^{d-1},$$

where $E\subset\Omega$ is open and regular, $q\geq d-1$, $y\in H^1(\Omega\setminus\bar{E};\mathbb{R}^d)$, and $\gamma_\delta\to 0$ as $\delta\to 0$. The first part of the functional represents the elastic energy, where W is a frame-indifferent stored energy density and $\delta>0$ represents the scaling of the strain. The surface energy consists of a perimeter term depending on a (possibly anisotropic) density φ evaluated at the outer unit normal ν_E to ∂E, and a curvature regularization term. In the case $d=3$, $q=2$, we will also discuss variants where $|\boldsymbol{A}|^2$ is replaced by a mean curvature regularization corresponding to the *Willmore energy*. The setting is complemented with prescribed Dirichlet boundary conditions which induce a stress in the solid.

This energy and its relaxation were studied in [10, 19] without the curvature regularization term, where, depending on the application, E describes material voids in elastically stressed solids or the complement of an elastic thin film. In this paper, we are interested in deriving an effective description in the small-strain limit $\delta\to 0$, in terms of displacement fields $u = \frac{1}{\delta}(y - \mathrm{id})$. We prove that the Γ-limit of the functionals $(F_\delta)_{\delta>0}$ is of the form

$$\mathcal{F}_0(u, E) := \frac{1}{2}\int_{\Omega\setminus E} \mathcal{Q}(e(u))\,\mathrm{d}x + \int_{\partial^* E\cap\Omega} \varphi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{J_u\setminus\partial^* E} 2\varphi(\nu_u)\,\mathrm{d}\mathcal{H}^{d-1},$$

i.e., coincides with the relaxation of the models studied in [27]. Here, the map u lies in $GSBD^2(\Omega)$ (see Appendix A.4), where $e(u)$ denotes the approximate symmetrized gradient and J_u is the jump set with corresponding measure-theoretical unit normal ν_u. Moreover, E is a set of finite perimeter with essential boundary $\partial^* E$ and outward-pointing measure-theoretical unit normal ν_E. The elastic energy depends on the linear strain $e(u)$ in terms of the quadratic form $\mathcal{Q} = D^2W(\mathrm{Id})$. Besides the linearization of the elastic term, a further relaxation occurs in the surface energy: parts of the set E may collapse into a discontinuity J_u of the displacement u, and are counted twice in the energy. Eventually, our assumption $\gamma_\delta \to 0$ as $\delta \to 0$ implies that the curvature regularization of the nonlinear energy does not affect the linearized limit.

Organization of the paper and notation. The paper is organized as follows: Section 2 is devoted to the rigidity estimate. We give an exact statement of our result along with several extensions in Section 2.1. The proof is contained in Sections 2.2–2.5. In Section 3 we present our applications to the linearization of SDRI models. Sections 3.1–3.2 address the case of material voids in elastically stressed solids and epitaxially strained thin films, respectively. The proofs are given in Sections 3.3–3.4. Finally, in Appendix A we prove some elementary lemmata used in the proofs of our main results, and collect basic properties of the space $GSBD^2$.

We close the introduction with some basic notation. Given $\Omega \subset \mathbb{R}^d$ open, $d = 2, 3$, we denote by $\mathfrak{M}(\Omega)$ the collection of all measurable subsets of Ω. By $\mathcal{A}_{\mathrm{reg}}(\Omega)$ we indicate the collection of all open subsets $E \subset \Omega$ such that $\partial E \cap \Omega$ is a $(d-1)$-dimensional C^2-submanifold of $\mathbb{R}^d$. Manifolds and functions of C^2-regularity will be called *regular* in the following. Given $A \in \mathfrak{M}(\Omega)$, we denote by $\mathrm{int}(A)$ its interior and by $A^c = \mathbb{R}^d \setminus A$ its complement. The diameter of A is denoted by $\mathrm{diam}(A)$. Moreover, for $r > 0$ we let

$$(A)_r := \{x \in \mathbb{R}^d : \mathrm{dist}(x, A) < r\}. \tag{1.5}$$

Given $A, B \in \mathfrak{M}(\Omega)$, we write $A \subset\subset B$ if $\bar{A} \subset B$. The Hausdorff distance of A and B is denoted by $\mathrm{dist}_{\mathcal{H}}(A, B)$ and we write $A \triangle B = (A \setminus B) \cup (B \setminus A)$ for the symmetric difference. By id we denote the identity mapping on $\mathbb{R}^d$ and by $\mathrm{Id} \in \mathbb{R}^{d\times d}$ the identity matrix. For each $F \in \mathbb{R}^{d\times d}$ we let $\mathrm{sym}(F) = \frac{1}{2}(F + F^\top)$, and we define $SO(d) := \{F \in \mathbb{R}^{d\times d} \colon F^\top F = \mathrm{Id},\ \det F = 1\}$. Moreover, we denote by $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $\mathbb{R}^{d\times d}_{\mathrm{skew}}$ the set of symmetric and skew-symmetric matrices, respectively. We further write $\mathbb{S}^{d-1} := \{\nu \in \mathbb{R}^d : |\nu| = 1\}$.

By $Q_r(x)$ we denote the half-open-cube $Q_r(x) := x + r[-\frac{1}{2}, \frac{1}{2})^d$ of sidelength $r > 0$ centered at $x \in \mathbb{R}^d$. We introduce a tessellation of $\mathbb{R}^d$ by

$$\mathcal{Q}_r := \{Q_r(x) \colon x \in r\mathbb{Z}^d\}. \tag{1.6}$$

In the following, we often omit the center (x) and simply write $Q_r \in \mathcal{Q}_r$ if no confusion arises. In a similar fashion, by $Q_{\mu r}$ we indicate the cube with the same center, but side-length μr for $\mu > 0$. We will use the following elementary fact several times: for each $Q_r \in \mathcal{Q}_r$ and each $k \in \mathbb{N}$ it holds that

$$\#\{Q'_r \in \mathcal{Q}_r \colon Q_{kr} \cap Q'_{kr} \neq \emptyset\} \leq (2k-1)^d, \tag{1.7}$$

where # indicates the cardinality of a set. Finally, by $B_\rho \subset \mathbb{R}^d$ we denote the open ball with radius ρ centered in 0.

2. A geometric rigidity result in variable domains

In this section we present a geometric rigidity result generalizing the celebrated result in [48, Theorem 3.1] to the setting of variable domains with C^2-boundary. Here, with *variable domains* we intend sets of the form $\Omega \setminus \overline{E}$, where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a fixed bounded, open set and $E \in \mathcal{A}_{\mathrm{reg}}(\Omega)$ is arbitrary. The main feature of the result lies in the fact that the rigidity constant is *independent* of the choice of E, provided that a certain curvature regularization for ∂E is assumed. In Section 2.1 we state our main result and present the proof in Sections 2.2–2.5.

2.1. Statement of the rigidity result

Given $E \in \mathcal{A}_{\mathrm{reg}}(\Omega)$, we denote by $\boldsymbol{A}$ the second fundamental form of $\partial E \cap \Omega$. In particular, for $d = 3$, we have $|\boldsymbol{A}| = \sqrt{\kappa_1^2 + \kappa_2^2}$, where κ_1 and κ_2 are the principal curvatures of $\partial E \cap \Omega$. For $d = 2$, we simply have $|\boldsymbol{A}| = \kappa$, where κ denotes the curvature of the boundary, which is one-dimensional in this case. Given $q \in [d-1, +\infty)$ and $\gamma \in (0,1)$, we will assume a curvature regularization for ∂E of the form $\gamma \int_{\partial E \cap \Omega} |\boldsymbol{A}|^q \, \mathrm{d}\mathcal{H}^{d-1}$. Given also a norm φ on $\mathbb{R}^d$, we introduce the *local surface energy*, consisting of a perimeter term with respect to φ and the curvature regularization, defined for every $K \in \mathfrak{M}(\Omega)$, by

$$\mathcal{F}^{\varphi,\gamma,q}_{\mathrm{surf}}(E; K) := \int_{\partial E \cap K} \varphi(\nu_E) \, \mathrm{d}\mathcal{H}^{d-1} + \gamma \int_{\partial E \cap K} |\boldsymbol{A}|^q \, \mathrm{d}\mathcal{H}^{d-1}, \tag{2.1}$$

where ν_E denotes the unit outer normal to $\partial E \cap \Omega$. When $K = \Omega$, we omit the dependence of the surface energy on the second argument. We now formulate the main result of this paper.

Theorem 2.1 (Geometric rigidity in variable domains). *Let $d = 2, 3$, $q \in [d-1, +\infty)$, $\gamma \in (0,1)$, and φ be a norm on $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be open and bounded and let $\widetilde{\Omega} \subset\subset \Omega$ be an open subset. Then there exist constants $C_0 = C_0(\varphi) > 0$, $\eta_0 = \eta_0(\mathrm{dist}(\partial\Omega, \widetilde{\Omega}), \varphi) \in (0,1)$ and for each $\eta \in (0, \eta_0]$ there exists $C_\eta = C_\eta(\eta, \Omega, \widetilde{\Omega}) > 0$ such that the following holds:*

For every $E \in \mathcal{A}_{\rm reg}(\Omega)$ there exists an open set $E_{\eta,\gamma}$ such that $E \subset E_{\eta,\gamma} \subset \Omega$, $\partial E_{\eta,\gamma} \cap \Omega$ is a union of finitely many regular submanifolds, and

$$\begin{aligned} &\text{(i)}\ \ \mathcal{L}^d(E_{\eta,\gamma} \setminus E) \le \eta\gamma^{1/q}\mathcal{F}^{\varphi,\gamma,q}_{\rm surf}(E), \quad \mathrm{dist}_{\mathcal{H}}(E, E_{\eta,\gamma}) \le \eta\gamma^{1/q}, \\ &\text{(ii)}\ \ \int_{\partial E_{\eta,\gamma}\cap\Omega} \varphi(\nu_{E_{\eta,\gamma}})\,\mathrm{d}\mathcal{H}^{d-1} \le (1 + C_0\eta)\mathcal{F}^{\varphi,\gamma,q}_{\rm surf}(E), \end{aligned} \tag{2.2}$$

such that for the connected components $(\widetilde{\Omega}^{\eta,\gamma}_j)_j$ of $\widetilde{\Omega} \setminus \overline{E_{\eta,\gamma}}$ and for every $y \in H^1(\Omega \setminus \overline{E}; \mathbb{R}^d)$ there exist corresponding rotations $(R^{\eta,\gamma}_j)_j \subset \mathrm{SO}(d)$ and vectors $(b^{\eta,\gamma}_j)_j \subset \mathbb{R}^d$ such that

$$\begin{aligned} &\text{(i)}\ \ \sum_j \int_{\widetilde{\Omega}^{\eta,\gamma}_j} |\,\mathrm{sym}((R^{\eta,\gamma}_j)^{\mathsf{T}}\nabla y - \mathrm{Id})|^2 \\ &\qquad\qquad \le C_0(1 + C_\eta\gamma^{-5d/q}\varepsilon) \int_{\Omega\setminus\overline{E}} \mathrm{dist}^2(\nabla y, \mathrm{SO}(d)), \\ &\text{(ii)}\ \ \sum_j \int_{\widetilde{\Omega}^{\eta,\gamma}_j} |(R^{\eta,\gamma}_j)^{\mathsf{T}}\nabla y - \mathrm{Id}|^2 \le C_\eta\gamma^{-2d/q} \int_{\Omega\setminus\overline{E}} \mathrm{dist}^2(\nabla y, \mathrm{SO}(d)), \\ &\text{(iii)}\ \ \sum_j \int_{\widetilde{\Omega}^{\eta,\gamma}_j} |y - (R^{\eta,\gamma}_j x + b^{\eta,\gamma}_j)|^2 \le C_\eta\gamma^{(2-4d)/q} \int_{\Omega\setminus\overline{E}} \mathrm{dist}^2(\nabla y, \mathrm{SO}(d)), \end{aligned} \tag{2.3}$$

where for brevity $\varepsilon := \int_{\Omega\setminus\overline{E}} \mathrm{dist}^2(\nabla y(x), \mathrm{SO}(d))$.

We note that Theorem 2.1, in particular (2.3), provides a *piecewise geometric rigidity* result in the spirit of [18,43,46]. In fact, global rigidity may fail if the domain Ω (or more precisely $\widetilde{\Omega}$) is disconnected by E into several parts on each of which y is close to a different rigid motion. A separation of the domain into the sets $(\widetilde{\Omega}^{\eta,\gamma}_j)_j$ might still be necessary *even if $\Omega \setminus \overline{E}$ is connected*. In fact, this is indispensable if the domain is connected only through a *thin tunnel*, as explained in Example 2.6. Such phenomena are accounted for in our result by defining the components $(\widetilde{\Omega}^{\eta,\gamma}_j)_j$ with respect to an appropriate *thickened set* $E_{\eta,\gamma}$ containing E. Note that (2.2) (i) ensures that we obtain a rigidity result outside the *small set* $E_{\eta,\gamma} \setminus E$, which vanishes for $\eta, \gamma \to 0$. In addition, (2.2) (ii) provides a sharp control on the (anisotropic) perimeter of $E_{\eta,\gamma}$ as $\eta, \gamma \to 0$, which will be essential for our applications to models involving surface energies; see Section 3.

When comparing our result to [48], the constant in (2.3) depends on the small parameter η and the curvature regularization parameter γ, with $C_\eta \to +\infty$ as $\eta \to 0$. We emphasize, however, that for configurations with gradient close to the set of rotations, in the sense of

$$\int_{\Omega\setminus\overline{E}} \mathrm{dist}^2(\nabla y, \mathrm{SO}(d))\,\mathrm{d}x \le C_\eta^{-1}\gamma^{5d/q}, \tag{2.4}$$

we obtain a uniform control on symmetrized gradients; see (2.3) (i). (The subspace $\mathbb{R}^{d\times d}_{\rm sym}$ corresponds to the orthogonal space to $\mathrm{SO}(d)$ at the identity matrix. Since different rotations appear in our statement, $\mathbb{R}^{d\times d}_{\rm sym}$ has to be replaced accordingly.) Indeed, in our application to linearization, since the elastic energy is of order $\varepsilon \sim \delta^2$, we can fix an a priori rate on the curvature regularization parameter $\gamma = \gamma_\delta$ in such a way that (2.4) holds.

In our applications, this uniform control will be essential to obtain compactness for rescaled displacement fields; see (3.2) and Propositions 3.1 and 3.6 below. The estimate (2.3) (ii) is needed to control higher-order terms in the passage to linearized elastic energies; see Lemma 3.12. Note that even under the assumption (2.4), a uniform control on the gradients independently of the set E cannot be expected, as in Example 2.7 we show that the estimate is actually sharp. This is related to the fact that the constant in Korn's inequality (see e.g., [74]) is not uniform for variable domains $\Omega \setminus \overline{E}$. In the proof, we will first establish (2.3) (ii), (iii) and then derive (2.3) (i) from (2.3) (ii).

We also emphasize that the choice $q \geq d-1$ for the curvature regularization is essential for the proof; see Lemma 2.12 and Example 2.13. We proceed with several slightly modified versions of the statement which will be convenient for our applications.

Corollary 2.2 (Version with Dirichlet conditions). *Suppose that $\Omega = U \cup U_D$ for two bounded sets $U, U_D \subset \mathbb{R}^d$ with Lipschitz boundary. Then, for every $E \in \mathcal{A}_{\mathrm{reg}}(\Omega)$ and every $y \in H^1(\Omega \setminus \overline{E}; \mathbb{R}^d)$ with $y = \mathrm{id}$ on U_D, the statement of Theorem 2.1 holds with the additional property that*

$$\text{if for some } j \text{ it holds that } \mathcal{L}^d(\widetilde{\Omega}_j^{\eta,\gamma} \cap U_D) > 0, \text{ then we can take } R_j^{\eta,\gamma} = \mathrm{Id},$$

where the constant C_η additionally depends on U_D.

In the applications, Dirichlet conditions will indeed be imposed on a set of positive $\mathcal{L}^d$-measure, as is customary in free discontinuity problems.

Corollary 2.3 (Version for graphs). *Consider $\Omega = \omega \times (-1, M+1)$ for some open and bounded $\omega \subset \mathbb{R}^{d-1}$ and $M > 0$. Suppose that $E = \{(x', x_d) \in \Omega : x' \in \omega,\ x_d > h(x')\}$ for a regular function $h\colon \omega \to [0, M]$, i.e., $\partial E \cap \Omega$ is the graph of the function h. Then, in Theorem 2.1, we find another set $E'_{\eta,\gamma} \supset E_{\eta,\gamma}$, which is the supergraph of a smooth function $h_{\eta,\gamma}\colon \omega \to [0, M]$ with $h_{\eta,\gamma} \leq h$, i.e., we have $E'_{\eta,\gamma} = \{(x', x_d) \in \Omega : x' \in \omega,\ x_d > h_{\eta,\gamma}(x')\}$ such that*

$$\begin{aligned} &\text{(i)}\ \ \mathcal{L}^d(E'_{\eta,\gamma} \setminus E) \leq \eta \gamma^{1/q} \mathcal{F}^{\varphi,\gamma,q}_{\mathrm{surf}}(E), \\ &\text{(ii)}\ \int_{\partial E'_{\eta,\gamma} \cap \Omega} \varphi(\nu_{E'_{\eta,\gamma}})\, \mathrm{d}\mathcal{H}^{d-1} \leq C_0 \mathcal{F}^{\varphi,\gamma,q}_{\mathrm{surf}}(E). \end{aligned} \tag{2.5}$$

In particular, the thickened set can be chosen as a supergraph, at the expense of a coarser estimate in (2.2) (ii). Corollaries 2.2 and 2.3 will be proved in Section 2.2 and Section 2.3, respectively. We proceed with some further comments on the result.

Remark 2.4 (Version with mean curvature). For $d = 3$, $q = 2$, and a regular domain $\Omega \subset \mathbb{R}^3$, there are situations where in estimate (2.2) we can replace the second fundamental form $\boldsymbol{A}$ by the mean curvature $\boldsymbol{H}\colon \partial E \cap \Omega \to \mathbb{R}$, i.e., $\boldsymbol{H} := \kappa_1 + \kappa_2$, where again κ_1 and κ_2 are the principal curvatures of $\partial E \cap \Omega$. In fact, denote by $\boldsymbol{G} := \kappa_1\kappa_2$ the Gaussian curvature of $\partial E \cap \Omega$, by $\chi(\partial E \cap \Omega)$ the *Euler characteristic* of $\partial E \cap \Omega$, and by κ_g the *geodesic curvature* of $\partial(\partial E \cap \Omega) \subset \partial\Omega$. (The outermost ∂ is meant here to denote the boundary of the two-dimensional surface $\partial E \cap \Omega$ in the differential geometric sense and we assume for simplicity that $\partial(\partial E \cap \Omega)$ is C^2.) Then the Gauss–Bonnet theorem yields

$$\begin{aligned}\int_{\partial E\cap\Omega} |\boldsymbol{A}|^2 \,\mathrm{d}\mathcal{H}^2 &= \int_{\partial E\cap\Omega} |\boldsymbol{H}|^2 \,\mathrm{d}\mathcal{H}^2 - 2\int_{\partial E\cap\Omega} \boldsymbol{G} \,\mathrm{d}\mathcal{H}^2 \\ &= \int_{\partial E\cap\Omega} |\boldsymbol{H}|^2 \,\mathrm{d}\mathcal{H}^2 - 4\pi\chi(\partial E\cap\Omega) + 2\int_{\partial(\partial E\cap\Omega)} \kappa_g \,\mathrm{d}\mathcal{H}^1.\end{aligned}$$

Exemplarily, we address two special cases:

(a) If $E \subset\subset \Omega$, i.e., $\partial E \cap \Omega = \partial E$ has no boundary, and if one has

$$-4\pi\gamma\chi(\partial E) \le C_0\eta, \tag{2.6}$$

then one can replace $\gamma \int_{\partial E} |\boldsymbol{A}|^2 \,\mathrm{d}\mathcal{H}^2$ by $\gamma \int_{\partial E} |\boldsymbol{H}|^2 \,\mathrm{d}\mathcal{H}^2$ without essentially affecting estimate (2.2) (ii) (and similarly (2.2) (i)), which in this case would be

$$\begin{aligned}&\int_{\partial E_{\eta,\gamma}\cap\Omega} \varphi(\nu_{E_{\eta,\gamma}}) \,\mathrm{d}\mathcal{H}^2 \\ &\qquad \le (1 + C_0\eta)\Big(\int_{\partial E} \varphi(\nu_E) \,\mathrm{d}\mathcal{H}^2 + \gamma \int_{\partial E} |\boldsymbol{H}|^2 \,\mathrm{d}\mathcal{H}^2 + C_0\eta\Big).\end{aligned} \tag{2.7}$$

For instance, in this case, (2.6) holds true if $\partial E \cap \Omega = \partial E$ consists of m connected components which are all topologically equivalent to the sphere $\mathbb{S}^2$, and since in this case $\chi(\partial E) = 2m > 0$, the second $C_0\eta$-term on the right-hand side of (2.7) is actually obsolete.

(b) In a similar manner, if $\partial E \cap \Omega$ consists of a single connected component topologically equivalent to the flat disk and $2\gamma \int_{\partial(\partial E\cap\Omega)} \kappa_g \,\mathrm{d}\mathcal{H}^1 \le C_0\eta$, we can again replace (2.2) (ii) by (2.7).

Remark 2.5 (Set $\widetilde{\Omega}$). Due to our proof strategy based on cubic sets, see (2.12) below, the rigidity estimate is only local, given in terms of $\widetilde{\Omega}$. Yet under certain regularity assumptions on Ω and E, one can replace $\widetilde{\Omega}$ by Ω, provided that we replace (2.2) (ii) by

$$\begin{aligned}&\int_{\partial E_{\eta,\gamma}\cap\Omega} \varphi(\nu_{E_{\eta,\gamma}}) \,\mathrm{d}\mathcal{H}^{d-1} \\ &\qquad \le (1 + C_0\eta)\Big(\int_{\partial E\cap\Omega} \varphi(\nu_E) \,\mathrm{d}\mathcal{H}^{d-1} + \gamma \int_{\partial E\cap\Omega} |\boldsymbol{A}|^q \,\mathrm{d}\mathcal{H}^{d-1} + C_{\Omega,\varphi,\gamma,q}\Big)\end{aligned}$$

for a suitable constant $C_{\Omega,\varphi,\gamma,q} > 0$ independent of E. In fact, this follows by selecting $\Omega_* \supset\supset \Omega$ and applying Theorem 2.1 for Ω_* in place of Ω, the set Ω in place of $\widetilde{\Omega}$, and for $E_* = E \cup (\Omega_* \setminus \overline{\Omega})$ in place of E, whenever $\partial E_* \cap \Omega_*$ is regular (e.g., if $E \subset\subset \Omega$). More specifically, the result would then yield a set $E_* \subset E^*_{\eta,\gamma} \subset \Omega_*$, and then we define $E_{\eta,\gamma} := E^*_{\eta,\gamma} \cap \Omega$.

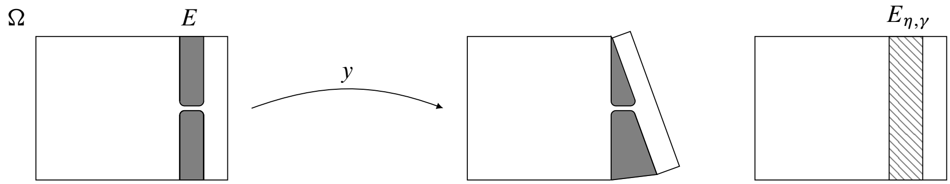

Figure 2. A thin tunnel that leads to failure of uniform rigidity on the unique connected component of $\Omega \setminus \overline{E}$, depicted schematically. On the left: The set $\Omega \setminus \overline{E}$, where E is depicted in gray. In the middle: The set $y(\Omega \setminus \overline{E})$. On the right: The set $\Omega \setminus \overline{E_{\eta,\gamma}}$, where $E_{\eta,\gamma}$ is the hatched set.

Example 2.6 (Thin tunnel). We give an example for the necessity of thickening the set and refer to the schematic Figure 2 for an illustration. For $\delta > 0$ we suppose that, up to a negligible set, $\Omega \setminus \overline{E}$ is given by the three sets $U_1 = (-1, 0) \times (0, 1)$, $U_2^\delta = (0, 1) \times (\frac{1}{2}, \frac{1}{2} + \delta)$, and $U_3 = (1, 2) \times (0, 1)$. (Strictly speaking, smooth approximations of U_1, U_2^δ, and U_3 need to be considered.) For $\sigma \in (0, \pi/2)$ we define

$$y_{\delta,\sigma}(x) = \begin{cases} x + \tau_1^\sigma, & x \in U_1, \\ \Big(\Big(x_2 - \frac{1}{2}\Big) + \frac{1}{\sigma}\Big)(\sin(\sigma x_1), \cos(\sigma x_1)), & x \in U_2^\delta, \\ R_\sigma x + \tau_3^\sigma, & x \in U_3, \end{cases} \tag{2.8}$$

where $R_\sigma \in SO(2)$ denotes the rotation around the origin by the angle σ and τ_1^σ, τ_3^σ are suitable translations such that $y_{\delta,\sigma}$ is continuous. Then $\nabla y_{\delta,\sigma} \in SO(2)$ on $U_1 \cup U_3$ and on U_2^δ we have

$$\nabla y_{\delta,\sigma}(x_1, x_2) = \begin{pmatrix} (1 + \sigma(x_2 - \frac{1}{2}))\cos(\sigma x_1) & \sin(\sigma x_1) \\ -(1 + \sigma(x_2 - \frac{1}{2}))\sin(\sigma x_1) & \cos(\sigma x_1) \end{pmatrix}.$$

This yields $\mathrm{dist}^2(\nabla y_{\delta,\sigma}, SO(2)) = |\sqrt{\nabla y_{\delta,\sigma}^T \nabla y_{\delta,\sigma}} - \mathrm{Id}|^2 = \sigma^2(x_2 - \frac{1}{2})^2$ on U_2^δ, and therefore

$$\int_{\Omega\setminus\overline{E}} \mathrm{dist}^2(\nabla y_{\delta,\sigma}, SO(2))\,\mathrm{d}x = \sigma^2\delta^3/3. \tag{2.9}$$

It is also easy to see that for all $R \in SO(2)$ one has

$$\int_{\Omega\setminus\overline{E}} |\nabla y_{\delta,\sigma} - R|^2\,\mathrm{d}x \geq c\sigma^2$$

for a universal constant $c > 0$. Therefore, neither (2.3) (i) nor (2.3) (ii) can hold true on $\Omega \setminus \overline{E}$ with a constant independent of E.

Example 2.7 (Sharpness of constant). The constant in (2.3) (ii) is sharp. To this end, consider Ω and E in dimension $d = 2$ as depicted in Figure 3, and note that the thickening of the set E will not disconnect $\Omega \setminus \overline{E}$; see (2.2) (i). The set $\Omega \setminus \overline{E}$ consists essentially

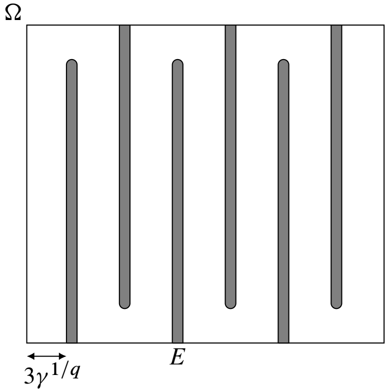

Figure 3. A set $\Omega \setminus \bar{E}$ that shows that the constant in (2.3) (ii) is sharp, for instance for $\Omega = (0,1)^2$. The set E is depicted in gray.

of $m \sim \gamma^{-1/q}$ "vertical stripes" depicted in white in the picture. We define a deformation y on $\Omega \setminus \bar{E}$ which on each of the stripes bends by an angle of $\sigma := \gamma^{1/q}$ as indicated in (2.8) (for $\delta := 3\gamma^{1/q}$) such that between the first and the last stripe a macroscopic rotation is performed. Repeating the argument in (2.9), we get $\int_{\Omega\setminus\bar{E}} \mathrm{dist}^2(\nabla y, SO(2))\,dx \lesssim m(\gamma^{1/q})^2(3\gamma^{1/q})^3/3 \sim \gamma^{4/q}$ and on the other hand $\int_{\Omega\setminus\bar{E}} |\nabla y - R|^2\,dx \geq c$ for all $R \in SO(2)$.

2.2. Proof of Theorem 2.1

This subsection is devoted to the proof of Theorem 2.1. We start with a short outline of the proof collecting the main intermediate steps. The core of our proof is the construction of the thickened set $E_{\eta,\gamma}$ with the properties in (2.2). We formulate this in a separate auxiliary result, and for this purpose we recall the definition of $\mathcal{F}_{\mathrm{surf}}^{\varphi,\gamma,q}$ in (2.1).

Proposition 2.8 (Thickening of sets). *Let $d = 2, 3$, $q \in [d-1, +\infty)$, $\gamma \in (0,1)$, and φ be a norm on $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be open and bounded and let $\tilde{\Omega} \subset\subset \Omega$ be an open subset. Then there exist a constant $C_0 = C_0(\varphi) > 0$, $\eta_0 \in (0,1)$ depending only on* $\mathrm{dist}(\partial\Omega, \tilde{\Omega})$ *and φ, and for each $\eta \in (0, \eta_0]$ there exists $c_\eta \in (0,1)$, with $c_\eta \to 0$ as $\eta \to 0$, such that the following holds:*

Given $E \in \mathcal{A}_{\mathrm{reg}}(\Omega)$, we can find an open set $E_{\eta,\gamma}$ such that $E \subset E_{\eta,\gamma} \subset \Omega$, $\partial E_{\eta,\gamma} \cap \Omega$ is a union of finitely many regular submanifolds, and

$$\begin{aligned} &\text{(i)}\ \ \mathrm{dist}(x, E) \geq c_\eta \gamma^{1/q} \quad \textit{for all } x \in \big\{y \in \Omega \setminus \overline{E_{\eta,\gamma}} \colon \mathrm{dist}(y, \tilde{\Omega}) < c_\eta\gamma^{1/q}\big\}, \\ &\text{(ii)}\ \ \mathcal{L}^d(E_{\eta,\gamma} \setminus E) \leq \eta\gamma^{1/q}\mathcal{F}_{\mathrm{surf}}^{\varphi,\gamma,q}(E), \quad \mathrm{dist}_{\mathcal{H}}(E, E_{\eta,\gamma}) \leq \eta\gamma^{1/q}, \\ &\text{(iii)}\ \int_{\partial E_{\eta,\gamma}\cap\Omega} \varphi(\nu_{E_{\eta,\gamma}})\,d\mathcal{H}^{d-1} \leq (1 + C_0\eta)\mathcal{F}_{\mathrm{surf}}^{\varphi,\gamma,q}(E). \end{aligned} \tag{2.10}$$

We defer the proof to Section 2.3 below. Note that (2.10) (ii), (iii) are exactly the properties stated in the main result; see (2.2). The additional property (2.10) (i) is essential for the proof of (2.3) as it allows one to cover

$$\widetilde{\Omega}^E_{\eta,\gamma} := \widetilde{\Omega} \setminus \overline{E_{\eta,\gamma}} \tag{2.11}$$

with cubes which are all contained in $\Omega \setminus \bar{E}$. More precisely, for $r > 0$ and $U \subset \mathbb{R}^d$ open and bounded, recalling the definition in (1.6), we define the *r-cubic set* corresponding to U by

$$(U)^r := \mathrm{int}\Big(\bigcup_{Q_r \in \mathcal{Q}_r(U)} Q_r \Big), \tag{2.12}$$

where $\mathcal{Q}_r(U) := \{Q_r \in \mathcal{Q}_r : Q_r \cap U \neq \emptyset\}$. We define

$$r_{\eta,\gamma} := \frac{c_\eta \gamma^{1/q}}{2\sqrt{d}}, \tag{2.13}$$

where c_η is the constant of Proposition 2.8. Now, by using (2.10) (i) and $c_\eta \to 0$ as $\eta \to 0$, by possibly passing to a smaller constant η_0 depending on $\mathrm{dist}(\partial\Omega, \widetilde{\Omega})$ one can check that

$$Q_{r_{\eta,\gamma}} \in \mathcal{Q}_{r_{\eta,\gamma}}(\widetilde{\Omega}^E_{\eta,\gamma}) \quad \Rightarrow \quad Q_{2r_{\eta,\gamma}} \subset \Omega \setminus \bar{E}. \tag{2.14}$$

For general r-cubic sets the following rigidity result holds.

Proposition 2.9 (Rigidity on r-cubic sets). *Let $d \geq 2$, $U \subset \mathbb{R}^d$ be open and bounded, let $r > 0$, and suppose that the r-cubic set $(U)^r$ defined in (2.12) is connected. Then there exists an absolute constant $C > 0$ independent of U and r such that for all $y \in H^1((U)^r;\mathbb{R}^d)$ there exist $R \in \mathrm{SO}(d)$ and $b \in \mathbb{R}^d$ such that*

$$\begin{aligned} &\text{(i)} \quad \int_{(U)^r} |\nabla y - R|^2 \,\mathrm{d}x \leq C(\#\mathcal{Q}_r(U))^2 \int_{(U)^r} \mathrm{dist}^2(\nabla y, \mathrm{SO}(d))\,\mathrm{d}x, \\ &\text{(ii)} \quad r^{-2} \int_{(U)^r} |y(x) - (Rx+b)|^2 \,\mathrm{d}x \\ &\qquad\qquad \leq C(\#\mathcal{Q}_r(U))^4 \int_{(U)^r} \mathrm{dist}^2(\nabla y, \mathrm{SO}(d))\,\mathrm{d}x. \end{aligned} \tag{2.15}$$

Additionally, if there exists $Q \in \mathcal{Q}_r(U)$ with $\mathcal{L}^d(Q \cap \{\nabla y = \mathrm{Id}\}) \geq c r^d$ for some absolute constant $c \in (0,1)$, then (2.15) holds for $R = \mathrm{Id}$, for a constant $C > 0$ depending on c.

The result is a direct consequence of the rigidity estimate (1.1) proved by Friesecke, James, and Müller [48], applied on a cube, along with estimating the variation of the rotations on different cubes. Although the latter argument is well known and has been performed, e.g., in [48, Section 4], we include a short proof in Appendix A.3 for the convenience of the reader.

Observe that typically one has $\#\mathcal{Q}_r(U) \sim \mathcal{L}^d(U) r^{-d}$, which along with (2.13) explains the scaling in (2.3) (ii), (iii). The proof of (2.3) (i) instead will rely on Proposition 2.9 along with the linearization formula [48, equation (3.20)],

$$|\operatorname{sym}(R^\top F - \mathrm{Id})| = \operatorname{dist}(F, SO(d)) + O(|F-R|^2) \tag{2.16}$$

for $F \in \mathbb{R}^{d\times d}$ and $R \in SO(d)$. In fact, the latter shows that it suffices to have a good bound on $\int |\nabla y - R|^4 \,\mathrm{d}x$ in order to control the symmetrized gradient in L^2. We are now ready to give the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $q \in [d-1, +\infty)$, $\gamma \in (0,1)$, $\tilde{\Omega} \subset\subset \Omega$ be an open subset, and let φ be a norm on $\mathbb{R}^d$. Without restriction we can assume that $\tilde{\Omega}$ is smooth. We let η_0 be as in Proposition 2.8 and $\eta \in (0, \eta_0]$. We can assume that for c_η given in Proposition 2.8, (2.14) also holds, where $r_{\eta,\gamma}$ is defined as in (2.13). From now on, we write r in place of $r_{\eta,\gamma}$ for notational simplicity.

We let $E_{\eta,\gamma}$ be the set obtained from Proposition 2.8. In particular, (2.2) holds by (2.10) (ii), (iii). Let $\tilde{\Omega}^E_{\eta,\gamma}$ be the set in (2.11), and denote by $(\tilde{\Omega}^{\eta,\gamma}_j)_j$ the connected components of $\tilde{\Omega}^E_{\eta,\gamma}$. Note that these are finitely many due to the regularity of $E_{\eta,\gamma}$ and $\tilde{\Omega}$. The main part of the proof now consists in deriving (2.3). To this end, similarly to the proof of [48, Proposition 3.4] (cf. Step 1 therein), a crucial step is to reduce the problem to harmonic mappings, see Steps 1–2 below. In Steps 3–4 we then provide the rigidity estimate (2.3) (i), (ii), and briefly indicate the Poincaré-type estimate (2.3) (iii) in Step 5. In the following, $C > 0$ denotes a generic constant only depending Ω, which may change from line to line. Without restriction, we suppose that the sets

$$\hat{\Omega}^{\eta,\gamma}_j := \operatorname{int}\Big(\bigcup_{Q_r \in \mathcal{Q}_r(\tilde{\Omega}^{\eta,\gamma}_j)} \overline{Q_{2r}}\Big) \quad \text{are pairwise disjoint.} \tag{2.17}$$

Indeed, whenever $\hat{\Omega}^{\eta,\gamma}_i \cap \hat{\Omega}^{\eta,\gamma}_j \neq \emptyset$, one can replace $\tilde{\Omega}^{\eta,\gamma}_i$ and $\tilde{\Omega}^{\eta,\gamma}_j$ in the reasoning below by $\tilde{\Omega}^{\eta,\gamma}_i \cup \tilde{\Omega}^{\eta,\gamma}_j$ and can derive (2.3) for a single rotation on $\tilde{\Omega}^{\eta,\gamma}_i \cup \tilde{\Omega}^{\eta,\gamma}_j$.

Step 1 (Reduction to Lipschitz mappings on cubes). For every cube $Q_r \in \mathcal{Q}_r(\tilde{\Omega}^E_{\eta,\gamma})$ we have $Q_{2r} \subset \Omega \setminus \bar{E}$ by (2.14). By a variant of [36, Theorem 6.15], see also [48, Proposition A.1], we let $y_Q \in W^{1,\infty}(Q_{2r}; \mathbb{R}^d)$ be a Lipschitz truncation obtained from y satisfying

$$\begin{aligned} &\text{(i)}\ \ \|\nabla y_Q\|_{L^\infty(Q_{2r})} \le C, \\ &\text{(ii)}\ \int_{Q_{2r}} |\nabla y - \nabla y_Q|^2 \,\mathrm{d}x \le C \int_{Q_{2r}\cap\{|\nabla y| > 2\sqrt{d}\}} |\nabla y|^2 \,\mathrm{d}x. \end{aligned} \tag{2.18}$$

Here, with a slight abuse of notation, we write y_Q instead of y_{Q_r}. We now claim that it suffices to prove that there exist $(R_j^{\eta,\gamma})_j \subset \mathrm{SO}(d)$ such that

$$\sum_j \sum_{Q_r \in \mathcal{Q}_r(\tilde{\Omega}_j^{\eta,\gamma})} \int_{Q_r} |\operatorname{sym}((R_j^{\eta,\gamma})^{\mathsf{T}} \nabla y_Q - \mathrm{Id})|^2 \\ \leq C(1 + r^{-5d}\varepsilon) \int_{\Omega \setminus \bar{E}} \operatorname{dist}^2(\nabla y, \mathrm{SO}(d)), \tag{2.19}$$

where here and below we use the shorthand notation $\varepsilon := \int_{\Omega \setminus \bar{E}} \operatorname{dist}^2(\nabla y, \mathrm{SO}(d))\,\mathrm{d}x$, and

$$\sum_j \sum_{Q_r \in \mathcal{Q}_r(\tilde{\Omega}_j^{\eta,\gamma})} \int_{Q_r} |(R_j^{\eta,\gamma})^{\mathsf{T}} \nabla y_Q - \mathrm{Id}|^2 \,\mathrm{d}x \\ \leq C r^{-2d} \int_{\Omega \setminus \bar{E}} \operatorname{dist}^2(\nabla y, \mathrm{SO}(d))\,\mathrm{d}x. \tag{2.20}$$

Indeed, let us note that

$$\int_{Q_{2r} \cap \{|\nabla y| > 2\sqrt{d}\}} |\nabla y|^2 \,\mathrm{d}x \leq C \int_{Q_{2r}} \operatorname{dist}^2(\nabla y, \mathrm{SO}(d))\,\mathrm{d}x, \tag{2.21}$$

since $|F| \leq 2 \operatorname{dist}(F, \mathrm{SO}(d))$ for all $F \in \mathbb{R}^{d \times d}$ with $|F| > 2\sqrt{d}$. This along with (1.7), (2.14), (2.17), and (2.18) (ii) shows that

$$\sum_j \sum_{Q_r \in \mathcal{Q}_r(\tilde{\Omega}_j^{\eta,\gamma})} \int_{Q_{2r}} |\nabla y - \nabla y_Q|^2 \,\mathrm{d}x \leq C \int_{\Omega \setminus \bar{E}} \operatorname{dist}^2(\nabla y, \mathrm{SO}(d))\,\mathrm{d}x. \tag{2.22}$$

Then (2.3) (i),(ii) for a constant $C_\eta = C_\eta(\eta, \Omega, \tilde{\Omega}) > 0$ and $C_0 > 0$ (depending on Ω) clearly follows from (2.19)–(2.20), (2.22), the triangle inequality, the definition of $r = r_{\eta,\gamma}$ in (2.13), and the definition of $\mathcal{Q}_r(\tilde{\Omega}_j^{\eta,\gamma})$ below (2.12). Therefore, it suffices to prove (2.19)–(2.20).

Step 2 (Reduction to harmonic mappings). For every $Q_r \in \mathcal{Q}_r(\tilde{\Omega}^E_{\eta,\gamma})$, we consider $y_Q = w_Q + z_Q$, where, in the sense of distributions,

$$\begin{cases} \Delta w_Q = 0 & \text{on } Q_{2r}, \\ w_Q = y_Q & \text{on } \partial Q_{2r}, \end{cases} \quad \text{and} \quad \begin{cases} \Delta z_Q = \operatorname{div}(\nabla y_Q - \operatorname{cof} \nabla y_Q) & \text{on } Q_{2r}, \\ z_Q = 0 & \text{on } \partial Q_{2r}. \end{cases}$$

It holds that

$$\int_{Q_{2r}} |\nabla z_Q|^2 \,\mathrm{d}x \leq \int_{Q_{2r}} |\operatorname{cof} \nabla y_Q - \nabla y_Q|^2 \,\mathrm{d}x \leq C \int_{Q_{2r}} \operatorname{dist}^2(\nabla y_Q, \mathrm{SO}(d))\,\mathrm{d}x. \tag{2.23}$$

In fact, this follows from the arguments in [48, Proof of Theorem 3.1, Step 1], in particular using that $|\operatorname{cof} F - F| \leq c \operatorname{dist}(F, \mathrm{SO}(d))$ for all $F \in \mathbb{R}^{d \times d}$ with $|F| \leq C$ for some

$c = c(C) > 0$, where here C denotes the constant of (2.18) (i). In view of (2.18) (ii) and (2.21), (2.23) implies

$$\int_{Q_{2r}} |\nabla y_Q - \nabla w_Q|^2 \, dx \le C \int_{Q_{2r}} \mathrm{dist}^2(\nabla y_Q, SO(d)) \, dx$$
$$\le C \int_{Q_{2r}} \mathrm{dist}^2(\nabla y, SO(d)) \, dx. \tag{2.24}$$

This along with (1.7), (2.14), and (2.17) shows that, in order to establish (2.19)–(2.20), it suffices to show that there exist $(R_j^{\eta,\gamma})_j \subset SO(d)$ such that

$$\sum_j \sum_{Q_r \in \mathcal{Q}_r(\widetilde{\Omega}_j^{\eta,\gamma})} \int_{Q_r} |\mathrm{sym}((R_j^{\eta,\gamma})^\top \nabla w_Q - \mathrm{Id})|^2$$
$$\le C(1 + r^{-5d}\varepsilon) \int_{\Omega \setminus \bar{E}} \mathrm{dist}^2(\nabla y, SO(d)) \tag{2.25}$$

and

$$\sum_j \sum_{Q_r \in \mathcal{Q}_r(\widetilde{\Omega}_j^{\eta,\gamma})} \int_{Q_r} |(R_j^{\eta,\gamma})^\top \nabla w_Q - \mathrm{Id}|^2 \, dx$$
$$\le C r^{-2d} \int_{\Omega \setminus \bar{E}} \mathrm{dist}^2(\nabla y, SO(d)) \, dx. \tag{2.26}$$

Step 3 (Local (L^2-L^∞)-estimate for harmonic mappings). In this step we show that for each $\widetilde{\Omega}_j^{\eta,\gamma}$ there exists $R_j^{\eta,\gamma} \in SO(d)$ such that

$$\sum_j \sum_{Q_r \in \mathcal{Q}_r(\widetilde{\Omega}_j^{\eta,\gamma})} \int_{Q_{2r}} |\nabla w_Q - R_j^{\eta,\gamma}|^2 \, dx \le C r^{-2d} \int_{\Omega \setminus \bar{E}} \mathrm{dist}^2(\nabla y, SO(d)) \, dx, \tag{2.27}$$

and for each $Q_r \in \mathcal{Q}_r(\widetilde{\Omega}_j^{\eta,\gamma})$ it holds that

$$\|\nabla w_Q - R_j^{\eta,\gamma}\|_{L^\infty(Q_r)} \le C r^{-3d/2} \Big(\int_{\Omega \setminus \bar{E}} \mathrm{dist}^2(\nabla y, SO(d)) \, dx \Big)^{1/2}$$
$$= C r^{-3d/2} \sqrt{\varepsilon}, \tag{2.28}$$

where we recall the notation for ε below (2.19). To see this, we apply Proposition 2.9 for $2r/3$ in place of r on the function y and on the sets $\widehat{\Omega}_j^{\eta,\gamma}$ introduced in (2.17) in place of U. In view of the fact that $\widehat{\Omega}_j^{\eta,\gamma} = (\widehat{\Omega}_j^{\eta,\gamma})^{2r/3}$, we find $(R_j^{\eta,\gamma})_j \subset SO(d)$ such that

$$\int_{\widehat{\Omega}_j^{\eta,\gamma}} |\nabla y - R_j^{\eta,\gamma}|^2 \, dx \le C r^{-2d} \int_{\widehat{\Omega}_j^{\eta,\gamma}} \mathrm{dist}^2(\nabla y, SO(d)) \, dx, \tag{2.29}$$

where we used that $\#\mathcal{Q}_{2r/3}(\widehat{\Omega}_j^{\eta,\gamma}) \le \mathcal{L}^d(\Omega)(2r/3)^{-d}$, i.e., C in (2.29) also depends on Ω. By (1.7) this yields

$$\sum_j \sum_{Q_r \in \mathcal{Q}_r(\widetilde{\Omega}_j^{\eta,\gamma})} \int_{Q_{2r}} |(R_j^{\eta,\gamma})^\top \nabla y - \mathrm{Id}|^2 \,\mathrm{d}x \le C r^{-2d} \int_{\Omega\setminus\bar{E}} \mathrm{dist}^2(\nabla y, SO(d)) \,\mathrm{d}x, \quad (2.30)$$

where as before we also employed (2.14) and (2.17). In view of (1.7), (2.14), (2.17), (2.22), (2.24), and the triangle inequality we get

$$\sum_j \sum_{Q_r \in \mathcal{Q}_r(\widetilde{\Omega}_j^{\eta,\gamma})} \int_{Q_{2r}} |\nabla w_Q - \nabla y|^2 \,\mathrm{d}x \le C \int_{\Omega\setminus\bar{E}} \mathrm{dist}^2(\nabla y, SO(d)) \,\mathrm{d}x. \quad (2.31)$$

Consequently, by (2.30) we finally obtain (2.27).

We now address (2.28). For every j and every $Q_r \in \mathcal{Q}_r(\widetilde{\Omega}_j^{\eta,\gamma})$, due to (2.27), the fact that w_Q is a harmonic mapping on Q_{2r} and $Q_{2r} \subset \Omega \setminus \bar{E}$, as a consequence of the mean value property and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \|\nabla w_Q - R_j^{\eta,\gamma}\|_{L^\infty(Q_r)} &\le \frac{C}{r^{d/2}} \Big(\int_{Q_{2r}} |\nabla w_Q - R_j^{\eta,\gamma}|^2 \Big)^{\frac{1}{2}} \\ &\le \frac{C}{r^{3d/2}} \Big(\int_{\Omega\setminus\bar{E}} \mathrm{dist}^2(\nabla y, SO(d)) \Big)^{\frac{1}{2}}. \end{aligned} \quad (2.32)$$

This yields (2.28), and Step 3 of the proof is concluded.

Step 4 (Global estimates). In this step we finally prove (2.3) (i), (ii). In view of Step 2, it suffices to check (2.25)–(2.26). First, (2.26) follows directly from (2.27). By the linearization formula (2.16), (2.28), (2.31), and Young's inequality we have

$$\begin{aligned} &\sum_j \sum_{Q_r \in \mathcal{Q}_r(\widetilde{\Omega}_j^{\eta,\gamma})} \int_{Q_r} |\mathrm{sym}((R_j^{\eta,\gamma})^\top \nabla w_Q - \mathrm{Id})|^2 \,\mathrm{d}x \\ &\quad \le C \sum_j \sum_{Q_r \in \mathcal{Q}_r(\widetilde{\Omega}_j^{\eta,\gamma})} \Big(\int_{Q_r} \mathrm{dist}^2(\nabla w_Q, SO(d)) \,\mathrm{d}x + \int_{Q_r} |\nabla w_Q - R_j^{\eta,\gamma}|^4 \,\mathrm{d}x \Big) \\ &\quad \le C \int_{\Omega\setminus\bar{E}} \mathrm{dist}^2(\nabla y, SO(d)) \,\mathrm{d}x \\ &\qquad + C r^{-3d} \varepsilon \sum_j \sum_{Q_r \in \mathcal{Q}_r(\widetilde{\Omega}_j^{\eta,\gamma})} \int_{Q_r} |\nabla w_Q - R_j^{\eta,\gamma}|^2 \,\mathrm{d}x. \end{aligned} \quad (2.33)$$

Then, by using (2.27) we get

$$\begin{aligned} &\sum_j \sum_{Q_r \in \mathcal{Q}_r(\widetilde{\Omega}_j^{\eta,\gamma})} \int_{Q_r} |\mathrm{sym}((R_j^{\eta,\gamma})^\top \nabla w_Q - \mathrm{Id})|^2 \,\mathrm{d}x \\ &\qquad \le C(1 + r^{-5d}\varepsilon) \int_{\Omega\setminus\bar{E}} \mathrm{dist}^2(\nabla y, SO(d)) \,\mathrm{d}x. \end{aligned}$$

This yields (2.25) and concludes the proof of (2.3) (i), (ii).

Step 5 (Poincaré estimate). We briefly indicate how to derive (2.3) (iii). By applying (2.15) (ii) of Proposition 2.9 for $2r/3$ in place of r on the function $y(x) - R_j^{\eta,\gamma} x$ and on $\widehat{\Omega}_j^{\eta,\gamma}$ in place of U, using again that $\#\mathcal{Q}_{2r/3}(\widehat{\Omega}_j^{\eta,\gamma}) \le \mathcal{L}^d(\Omega)(2r/3)^{-d}$, we also find $(b_j^{\eta,\gamma})_j \subset \mathbb{R}^d$ such that

$$\sum_j \sum_{Q_r \in \mathcal{Q}_r(\widetilde{\Omega}_j^{\eta,\gamma})} \int_{Q_{2r}} |y(x) - (R_j^{\eta,\gamma} x + b_j^{\eta,\gamma})|^2 \,\mathrm{d}x \le C r^{2-4d} \int_{\Omega \setminus \bar{E}} \mathrm{dist}^2(\nabla y, SO(d)) \,\mathrm{d}x.$$

Recalling the definition of $r = r_{\eta,\gamma}$ in (2.13), and the definition of $\mathcal{Q}_r(\widetilde{\Omega}_j^{\eta,\gamma})$ below (2.12) we conclude (2.3) (iii). ∎

Remark 2.10. A closer inspection of the proof shows that Theorem 2.1 can be *localized*, in the sense that in the rigidity estimates (2.3) the sets Ω, $\widetilde{\Omega}$ can be replaced by open sets U, $\widetilde{U}$ respectively, with $\widetilde{U} \subset\subset U \subset \Omega$, $\widetilde{U} \subset \widetilde{\Omega}$, and $\mathrm{dist}(\partial U, \widetilde{U}) \ge \mathrm{dist}(\partial\Omega, \widetilde{\Omega})$, where the connected components of $\widetilde{U} \setminus \bar{E}_{\eta,\gamma}$ would then be denoted by $(\widetilde{U}_j^{\eta,\gamma})_j$. In that case, the constant C_η does not depend on Ω, $\widetilde{\Omega}$, but only on η and on $(\mathcal{L}^3(U))^2$.

Indeed, the above arguments essentially rely on (2.14) and the estimates on cubic sets given in Proposition 2.9. The estimate $\mathrm{dist}(\partial U, \widetilde{U}) \ge \mathrm{dist}(\partial\Omega, \widetilde{\Omega})$ guarantees (2.14) for U, $\widetilde{U}$. The fact that the constant $C > 0$ appearing in (2.29) depends quadratically on $\mathcal{L}^3(U)$ follows by the comment just below it, while by scaling invariance, $C > 0$ appearing in (2.32) can be chosen to be an absolute constant. Therefore, the estimates in (2.33) yield this precise dependence.

We close this subsection with a short proof of Corollary 2.2.

Proof of Corollary 2.2. A careful inspection of the previous proof shows that we only need to check that, whenever $\mathcal{L}^d(\widetilde{\Omega}_j^{\eta,\gamma} \cap U_D) > 0$ holds, then in (2.29) we can choose $R_j^{\eta,\gamma} = \mathrm{Id}$. To this end, when $\mathcal{L}^d(\widetilde{\Omega}_j^{\eta,\gamma} \cap U_D) > 0$, we find $Q_r \in \mathcal{Q}_r(\widetilde{\Omega}_j^{\eta,\gamma})$ such that $Q_r \cap U_D \ne \emptyset$. Then we can select $Q'_{2r/3} \in \mathcal{Q}_{2r/3}(\widehat{\Omega}_j^{\eta,\gamma})$, $Q'_{2r/3} \subset Q_{2r} \subset \widehat{\Omega}_j^{\eta,\gamma}$, see (2.17), such that by (2.13), the fact that $\gamma \in (0,1)$, and by the fact that U_D has Lipschitz boundary we get $\mathcal{L}^d(Q'_{2r/3} \cap U_D) \ge c r^d$ for a small absolute constant $c \in (0,1)$, provided that c_η is sufficiently small also depending on U_D. Then the desired property follows from the additional statement in Proposition 2.9 and the fact that $y = \mathrm{id}$ on U_D. In this context, note that the constant C_η in (2.3) depends on c_η and therefore C_η also depends on U_D. ∎

2.3. Thickening of sets

In this subsection we prove Proposition 2.8. Without restriction we will assume from now on that $\varphi_{\min} := \min_{\mathbb{S}^{d-1}} \varphi = 1$. Indeed, we can simply perform the proof for $\varphi_{\min}^{-1}\varphi$ in place of φ and $\varphi_{\min}^{-1}\gamma$ in place of γ to see that (2.10) (iii) holds. The proof essentially relies on a local construction to thicken the set E in a suitable way. To formulate the local statement, we introduce some further notation. Given $\rho > 0$ and a cube $Q_\rho \in \mathcal{Q}_\rho$, see (1.6), we denote the set of *neighboring cubes* by

$$\mathcal{N}(Q_\rho) := \big\{Q'_\rho \in \mathcal{Q}_\rho \colon \mathcal{H}^{d-1}(\partial Q_\rho \cap \partial Q'_\rho) > 0\big\}. \tag{2.34}$$

Note that $\#\mathcal{N}(Q_\rho) = 2d$. We also recall the definition of $\mathcal{F}^{\varphi,\gamma,q}_{\mathrm{surf}}$ in (2.1). Moreover, for notational convenience, we denote the anisotropic $(d-1)$-dimensional Hausdorff measure by

$$\mathcal{H}^{d-1}_\varphi(\Gamma) := \int_\Gamma \varphi(\nu_\Gamma)\,\mathrm{d}\mathcal{H}^{d-1} \tag{2.35}$$

for a norm φ on $\mathbb{R}^d$ and for a $(d-1)$-rectifiable set Γ, where ν_Γ denotes a measure-theoretical unit normal to Γ. Note that the integral is invariant under changing the orientation of ν_Γ as φ is a norm. The proof of Proposition 2.8 will make use of the following lemma, whose proof will be given later in Section 2.5.

Lemma 2.11 (Local thickening of sets). *Let $d = 2, 3$, $q \in [d-1, +\infty)$, $\gamma \in (0,1)$, and φ be a norm on $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $\tilde{\Omega} \subset\subset \Omega$ be an open subset, and let $\Lambda > 0$. Then there exist constants $C = C(\varphi, \Lambda) > 0$ and $\eta_0 = \eta_0(\Lambda) \in (0,1)$ such that for all $\eta \in (0, \eta_0]$ the following holds:*

For every $0 < \rho \le \eta^7 \gamma^{1/q}$ and for each $Q_\rho \in \mathcal{Q}_\rho$ such that $\overline{Q_{12\rho}} \subset \Omega$, and

$$\mathcal{F}^{\varphi,\gamma,q}_{\mathrm{surf}}(E; Q_{8\rho}) \le \Lambda \rho^{d-1}, \quad \mathcal{F}^{\varphi,\gamma,q}_{\mathrm{surf}}(E; Q'_{8\rho}) \le \Lambda \rho^{d-1} \quad \forall\, Q'_\rho \in \mathcal{N}(Q_\rho), \tag{2.36}$$

we can find pairwise disjoint sets $(\Gamma_i)_{i=1}^I$ in $\partial E \cap Q_{3\rho}$ with $I \le C$, corresponding closed sets $(T_i)_{i=1}^I \subset Q_{8\rho}$, with ∂T_i being a union of finitely many regular submanifolds and a disjoint decomposition $\{1, \dots, I\} = \mathcal{I}_{\mathrm{good}} \cup \bigcup_{Q'_\rho \in \mathcal{N}(Q_\rho)} \mathcal{I}_{\mathrm{bad}}(Q'_\rho)$ such that

$$\begin{aligned} &\text{(i)}\ \ \mathcal{H}^{d-1}_\varphi(\partial T_i \setminus \bar{E}) \le \mathcal{H}^{d-1}_\varphi(\Gamma_i \cap Q_\rho) + C\eta\rho^{d-1} \quad \forall i \in \mathcal{I}_{\mathrm{good}},\\ &\text{(ii)}\ \ \mathcal{H}^{d-1}_\varphi(\partial T_i \setminus \bar{E}) \le \mathcal{H}^{d-1}_\varphi(\Gamma_i \cap (Q_\rho \cup Q'_\rho))\\ &\qquad\qquad + C\eta\rho^{d-1} \quad \forall Q'_\rho \in \mathcal{N}(Q_\rho),\ \forall i \in \mathcal{I}_{\mathrm{bad}}(Q'_\rho), \end{aligned} \tag{2.37}$$

and

$$\mathrm{dist}\Big(\partial E \cap Q_\rho, \Big(E \cup \bigcup_{i=1}^I T_i\Big)^c\Big) \ge \eta\rho. \tag{2.38}$$

Moreover, fixing $Q'_\rho \in \mathcal{N}(Q_\rho)$, introducing the notation $\mathcal{I}'_{\mathrm{bad}}(Q_\rho)$ as above with respect to the cube Q'_ρ, and letting Γ'_i and T'_i be the corresponding sets, we have

$$i \in \mathcal{I}_{\mathrm{bad}}(Q'_\rho) \quad \Rightarrow \quad \exists j \in \mathcal{I}'_{\mathrm{bad}}(Q_\rho) \text{ such that } \Gamma_i = \Gamma'_j \text{ and } T_i = T'_j. \tag{2.39}$$

Properties (2.37) (i) and (2.38) are the fundamental points of the lemma: essentially, in the proof we show that the connected components of $\partial E \cap Q_\rho$ can be covered with thin polyhedra, leading to the definition of the sets $(T_i)_i$. The case (2.37) (ii) is only of technical nature, as additional care is needed if a component of $\partial E \cap Q_\rho$ is close to a neighboring cube; see Figure 4.

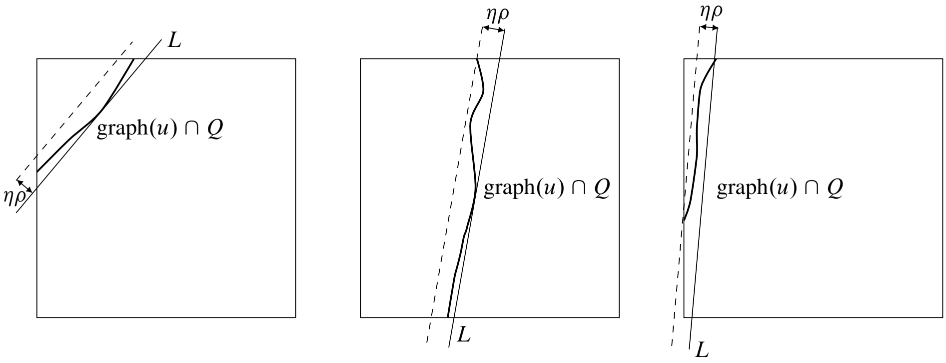

Figure 4. Different positions of planes inside a cube. In the left and in the middle cube, the two different cases of good planes are depicted, whereas the figure on the right shows a bad plane. The thick surfaces illustrate $\mathrm{graph}(u) \cap Q_\rho$ and the dashed planes are at distance $\eta\rho$ to L, i.e., at the maximal distance of $\mathrm{graph}(u)$ from the plane L inside Q_ρ. In the two pictures on the left, the area of the dashed plane inside Q_ρ and of the plane L inside Q_ρ are comparable up to an error of order $\eta\rho^2$. This is the key observation for the proof of (2.64)–(2.65). In the case of a bad plane, this is in general not true.

The construction of $E_{\eta,\gamma}$ in Proposition 2.8 will rely on suitably modifying E by applying Lemma 2.11 on cubes intersecting ∂E. To this end, we consider the tessellation of $\mathbb{R}^d$ with the family of cubes $\mathcal{Q}_\rho$ for $\rho = \eta^7\gamma^{1/q}$, so that Lemma 2.11 is applicable. In this context, it is important to control the number of *boundary cubes*, given by

$$\big\{Q_\rho \in \mathcal{Q}_\rho : \partial E \cap Q_\rho \neq \emptyset,\ \overline{Q_{12\rho}} \subset \Omega\big\}. \tag{2.40}$$

This will be achieved by the following lemma, whose proof will be given in the next subsection.

Lemma 2.12 (Small area implies large curvature). *Let $d = 2, 3$, $\Lambda > 0$, $q \in [d-1, +\infty)$, and $\gamma \in (0, 1)$. Then there exist an absolute constant $c_0 > 0$ and a constant $c_\Lambda > 0$ only depending on Λ such that for all $0 < \rho \le c_\Lambda\gamma^{1/q}$, $E \in \mathcal{A}_{\mathrm{reg}}(\Omega)$, and $Q_\rho \in \mathcal{Q}_\rho$ such that $\overline{Q_{8\rho}} \subset \Omega$ and $\partial E \cap Q_{3\rho} \neq \emptyset$, the following implication holds true:*

$$\mathcal{H}^{d-1}(\partial E \cap Q_{8\rho}) < c_0\rho^{d-1} \quad \Rightarrow \quad \gamma \int_{\partial E \cap Q_{8\rho}} |\boldsymbol{A}|^q \,\mathrm{d}\mathcal{H}^{d-1} > \Lambda\rho^{d-1}.$$

Indeed, the implication shows that whenever the surface ∂E inside a cube has small but nonzero area, then necessarily the curvature contribution is high. This will allow us to control the number of boundary cubes; see particularly (2.51) and (2.57) in the proof below. The result is a consequence of [84, Corollary 1.3] and we present its proof in Section 2.4 below. Let us mention that the analog of Lemma 2.12 is the main obstacle to generalize our result to higher dimensions; see Remark 2.22 for more details in this direction.

Example 2.13. The statement of Lemma 2.12 is false for $q < d-1$. In fact, let $E = B_\sigma \subset Q_{8\rho}$ be a ball of radius σ for $\sigma > 0$ small. Then clearly $\mathcal{H}^{d-1}(\partial E \cap Q_{8\rho}) < c_0\rho^{d-1}$ for σ small enough. On the other hand, $\int_{\partial E \cap Q_{8\rho}} |A|^q \, \mathrm{d}\mathcal{H}^{d-1}$ coincides up to a constant with $\sigma^{d-1}\sigma^{-q}$.

We are now in a position to give the proof of Proposition 2.8.

Proof of Proposition 2.8. Recall that without restriction we have assumed that $\min_{\mathbb{S}^{d-1}} \varphi = 1$. In the following proof we will write $\varphi_{\max} = \max_{\mathbb{S}^{d-1}} \varphi$ for brevity. First of all, we define the constant $\Lambda := 2d12^{d-1}15^d\varphi_{\max}$, whose role will become clear in (2.54) below. For this Λ, we apply Lemma 2.11 to obtain η_0, and from now on we fix $\eta \in (0, \eta_0]$. We consider the tessellation of $\mathbb{R}^d$ with the collection of cubes $\mathcal{Q}_\rho$, where

$$\rho := \eta^7\gamma^{1/q} \tag{2.41}$$

is chosen in such a way that Lemma 2.11 is applicable. Here, without restriction, up to passing to a smaller constant η_0, we can assume that $\eta_0 \le c_\Lambda^{1/7}$, and therefore also Lemma 2.12 is applicable. Moreover, we can further choose $\eta_0 > 0$ also depending on Ω, $\tilde{\Omega}$ such that for all $\eta \in (0, \eta_0]$ we have $20\sqrt{d}\rho \le \eta\,\mathrm{dist}(\tilde{\Omega}, \partial\Omega)$. Then, with a standard layering argument and recalling (1.5), we can find an open set Ω' with $\tilde{\Omega} \subset\subset \Omega' \subset\subset \Omega$, and

$$(\partial\Omega')_{15\sqrt{d}\rho} \subset \Omega \setminus \tilde{\Omega}, \tag{2.42}$$

such that for a constant $C > 0$ only depending on φ it holds that

$$\begin{aligned} \mathcal{H}^{d-1}_\varphi(\partial E \cap (\partial\Omega')_{3\sqrt{d}\rho}) &\le C\rho(\mathrm{dist}(\tilde{\Omega}, \partial\Omega))^{-1}\mathcal{H}^{d-1}_\varphi(\partial E \cap \Omega) \\ &\le C\eta\mathcal{H}^{d-1}_\varphi(\partial E \cap \Omega). \end{aligned} \tag{2.43}$$

Note that the choice of η_0 ensures that η_0 depends only on $\mathrm{dist}(\partial\Omega, \tilde{\Omega})$ and φ. Roughly speaking, the construction of the *thickened set* $E_{\eta,\gamma}$ will be performed according to the following procedure:

(i) We choose a ρ-grid of cubes $\mathcal{Q}_\rho$ as in (1.6), and perform a local set modification with respect to each cube $Q_\rho \in \mathcal{Q}_\rho$.

(ii) We distinguish between cubes where a lot of *surface energy is concentrated* in the sense that they violate hypothesis (2.36) of Lemma 2.11, and cubes for which Lemma 2.11 is applicable.

(iii) For cubes where *surface energy is concentrated* we can further distinguish two subcases: Either the energy bound in (2.36) is violated with respect to the cube Q_ρ (the collection of all such will be denoted by $\mathcal{Q}_\rho^{\mathrm{acc}}$ in what follows), or just by a neighboring cube (the latter collection will be denoted by $\mathcal{Q}_\rho^{\mathrm{neigh}}$). In such cubes, Lemma 2.11 is not applicable.

First, if $Q_\rho \in \mathcal{Q}_\rho^{\mathrm{acc}}$, we include in the void a sufficient enlargement of the cube, namely $Q_{12\rho}$. Secondly, if $Q_\rho \in \mathcal{Q}_\rho^{\mathrm{neigh}}$, by the previous choice, the cube is well contained in the enlargements coming from the cubes in $\mathcal{Q}_\rho^{\mathrm{acc}}$ and thus no set modification needs to be made due to these cubes. By a proper choice of Λ (see directly above (2.41)), the collective error in volume and perimeter made by this modification will be small.

(iv) For cubes where not too much surface energy is concentrated (whose collection will be called $\mathcal{Q}_\rho^{\mathrm{flat}}$), a more careful thickening of the void set needs to be performed by means of Lemma 2.11. This will imply again that the total error in volume and perimeter is small.

We now perform the construction in detail.

Step 1 (Boundary cubes). We define the collection of *boundary cubes* by

$$\mathcal{Q}_\rho^\partial := \big\{Q_\rho \in \mathcal{Q}_\rho : \overline{Q_{12\rho}} \subset \Omega \text{ and: } \partial E \cap Q_\rho \neq \emptyset \text{ or } \mathcal{F}_{\mathrm{surf}}^{\varphi,\gamma,q}(E; Q_{8\rho}) > \Lambda\rho^{d-1}\big\}. \tag{2.44}$$

(For technical reasons, the definition slightly differs from (2.40) mentioned above.) We decompose $\mathcal{Q}_\rho^\partial$ as follows: first, we let $\mathcal{Q}_\rho^{\mathrm{acc}}$ be the collection of the cubes $Q_\rho \in \mathcal{Q}_\rho^\partial$ satisfying

$$\mathcal{F}_{\mathrm{surf}}^{\varphi,\gamma,q}(E; Q_{8\rho}) > \Lambda\rho^{d-1}. \tag{2.45}$$

This definition collects the cubes whose 8-times enlargement *accumulates* a lot of surface energy. We further let $\mathcal{Q}_\rho^{\mathrm{neigh}} \subset \mathcal{Q}_\rho^\partial \setminus \mathcal{Q}_\rho^{\mathrm{acc}}$ be the collection of cubes Q_ρ in a *neighborhood* of $\mathcal{Q}_\rho^{\mathrm{acc}}$, i.e.,

$$\text{there exists } Q'_\rho \in \mathcal{Q}_\rho^{\mathrm{acc}} \text{ such that } Q_\rho \subset Q'_{12\rho}. \tag{2.46}$$

Eventually, we set $\mathcal{Q}_\rho^{\mathrm{flat}} := \mathcal{Q}_\rho^\partial \setminus (\mathcal{Q}_\rho^{\mathrm{acc}} \cup \mathcal{Q}_\rho^{\mathrm{neigh}})$. As we will see in the statement of Lemmata 2.15–2.16 below, the latter collection corresponds to the cubes where the surface ∂E is approximately *flat*. For later purposes, we observe that by applying Lemma 2.12 we find that

$$\mathcal{H}^{d-1}(\partial E \cap Q_{8\rho}) \geq c_0\rho^{d-1} \text{ and } \mathcal{F}_{\mathrm{surf}}^{\varphi,\gamma,q}(E; Q_{8\rho}) \leq \Lambda\rho^{d-1} \quad \forall Q_\rho \in \mathcal{Q}_\rho^{\mathrm{neigh}} \cup \mathcal{Q}_\rho^{\mathrm{flat}}. \tag{2.47}$$

The set $E_{\eta,\gamma}$ be will defined by

$$E_{\eta,\gamma} := \mathrm{int}\Big(E \cup \bigcup_{Q_\rho \in \mathcal{Q}_\rho^\partial} E_{\eta,\gamma}(Q_\rho)\Big), \tag{2.48}$$

where the definition of the sets $E_{\eta,\gamma}(Q_\rho)$ for $Q_\rho \in \mathcal{Q}_\rho^\partial$ is given in Step 2 of the proof. In Step 3 we address (2.10) (i), (ii), and eventually Step 4 is devoted to the proof of (2.10) (iii).

Step 2 (Definition of the sets $E_{\eta,\gamma}(Q_\rho)$). We address the three cases $\mathcal{Q}_\rho^{\mathrm{acc}}$, $\mathcal{Q}_\rho^{\mathrm{neigh}}$, and $\mathcal{Q}_\rho^{\mathrm{flat}}$ separately:

(a) First, if $Q_\rho \in \mathcal{Q}_\rho^{\mathrm{acc}}$, we set $E_{\eta,\gamma}(Q_\rho) := \overline{Q_{12\rho}}$.

(b) If $Q_\rho \in \mathcal{Q}_\rho^{\mathrm{neigh}}$, we set $E_{\eta,\gamma}(Q_\rho) := \emptyset$.

(c) Finally, we address the case of $Q_\rho \in \mathcal{Q}_\rho^{\mathrm{flat}}$. If $Q_\rho \cap \Omega' = \emptyset$, we let $E_{\eta,\gamma}(Q_\rho) := \emptyset$. Otherwise, by (2.42) we have $\overline{Q_{14\rho}} \subset \Omega$ and, in view of (2.44) and (2.46), for every cube $Q' \in \mathcal{N}(Q_\rho)$ we have that $\mathcal{F}_{\mathrm{surf}}^{\varphi,\gamma,q}(E; Q'_{8\rho}) \le \Lambda \rho^{d-1}$. This along with (2.47) allows one to apply Lemma 2.11 for $Q_\rho \in \mathcal{Q}_\rho^{\mathrm{flat}}$. We obtain finitely many corresponding pairwise disjoint sets $(\Gamma_i^Q)_{i=1}^I$ in $\partial E \cap Q_{3\rho}$ and closed sets $(T_i^Q)_{i=1}^I$, with $T_i^Q \subset Q_{8\rho}$ and ∂T_i^Q being a union of finitely many regular submanifolds, such that (2.37)–(2.39) hold. In this case, we define

$$E_{\eta,\gamma}(Q_\rho) = \bigcup_i T_i^Q. \tag{2.49}$$

By definition it is clear that $E_{\eta,\gamma} \subset \Omega$ and that $\partial E_{\eta,\gamma} \cap \Omega$ is a union of finitely many regular submanifolds. We now confirm (2.10).

Step 3 (Proof of (2.10) (i), (ii)). We start with the proof of (2.10) (i). To this end, it suffices to check that

$$\mathrm{dist}(y, \Omega \setminus \overline{E_{\eta,\gamma}}) \ge \eta\rho \quad \text{for all } y \in \partial E \cap \Omega'. \tag{2.50}$$

Indeed, let us assume for a moment that we have (2.50), and let us set $c_\eta := \eta^8$. Consider an arbitrary $x \in \Omega \setminus \overline{E_{\eta,\gamma}}$ with $\mathrm{dist}(x, \widetilde{\Omega}) < \eta\rho = c_\eta \gamma^{1/q}$, where the last equality follows from the choice of ρ in (2.41). Since $E \subset E_{\eta,\gamma}$ we have that $\mathrm{dist}(x, E) = \mathrm{dist}(x, \partial E)$. In view of (2.50) it remains to check that for every $y \in \partial E \setminus \Omega'$ we have that $|y - x| \ge \eta\rho$. This is trivial by the fact that $\mathrm{dist}(x, \widetilde{\Omega}) < \eta\rho$ and (2.42).

To verify (2.50), we first observe that each $y \in \partial E \cap \Omega'$ is contained in some cube of $\mathcal{Q}_\rho^\partial$; see (2.42) and (2.44). Therefore, let $y \in Q_\rho$ for some $Q_\rho \in \mathcal{Q}_\rho^\partial$ with $Q_\rho \cap \Omega' \ne \emptyset$. If $Q_\rho \in \mathcal{Q}_\rho^{\mathrm{acc}}$, then $\mathrm{dist}(y, \Omega \setminus E_{\eta,\gamma}(Q_\rho)) \ge 11\rho/2$, and (2.50) follows in view of (2.48). If $Q_\rho \in \mathcal{Q}_\rho^{\mathrm{neigh}}$, by (2.46) we find some $Q'_\rho \in \mathcal{Q}_\rho^{\mathrm{acc}}$ such that $Q_\rho \subset \overline{Q'_{12\rho}} = E_{\eta,\gamma}(Q'_\rho)$. As $\mathrm{dist}(\partial Q_\rho, \partial Q'_{12\rho}) \ge \rho/2$ by the definition of $\mathcal{Q}_\rho$, we get $\mathrm{dist}(y, \Omega \setminus E_{\eta,\gamma}(Q'_\rho)) \ge \rho/2$, and as before, as long as $0 < \eta \le \eta_0 \le 1/2$, (2.50) follows from (2.48). Eventually, we suppose that $Q_\rho \in \mathcal{Q}_\rho^{\mathrm{flat}}$. Then by (2.38) along with (2.49) we get $\mathrm{dist}(y, \Omega \setminus E_{\eta,\gamma}(Q_\rho)) \ge \eta\rho$ and we conclude as before.

We now show (2.10) (ii). The estimate $\mathrm{dist}_{\mathcal{H}}(E, E_{\eta,\gamma}) \le \eta\gamma^{1/q}$ follows immediately from (2.48) and the fact that each $E_{\eta,\gamma}(Q_\rho)$, $Q_\rho \in \mathcal{Q}_\rho^\partial$, is contained in $\overline{Q_{12\rho}}$, thus having diameter controlled by $12\sqrt{d}\rho \le \eta\gamma^{1/q}$, for η_0 sufficiently small; see the definitions in Step 2 and (2.41). In a similar fashion, as $E_{\eta,\gamma}(Q_\rho) \subset \overline{Q_{12\rho}}$ for all $Q_\rho \in \mathcal{Q}_\rho^\partial$, and $E_{\eta,\gamma}(Q_\rho) = \emptyset$ for $Q_\rho \in \mathcal{Q}_\rho^{\mathrm{neigh}}$, we obtain

$$\mathcal{L}^d(E_{\eta,\gamma} \setminus E) \le \sum_{Q_\rho \in \mathcal{Q}_\rho^{\mathrm{acc}} \cup \mathcal{Q}_\rho^{\mathrm{flat}}} \mathcal{L}^d(E_{\eta,\gamma}(Q_\rho)) \le (12\rho)^d \#(\mathcal{Q}_\rho^{\mathrm{acc}} \cup \mathcal{Q}_\rho^{\mathrm{flat}}).$$

In view of (2.45) and (2.47) we have $\mathcal{F}_{\mathrm{surf}}^{\varphi,\gamma,q}(E; Q_{8\rho}) \ge \min\{\Lambda, c_0\}\rho^{d-1}$ for all $Q_\rho \in \mathcal{Q}_\rho^{\mathrm{acc}} \cup \mathcal{Q}_\rho^{\mathrm{flat}}$, where we used the fact that we assumed $\varphi_{\min} = 1$. Now, by (1.7) we conclude

$$\mathcal{L}^d(E_{\eta,\gamma} \setminus E) \le C\rho \mathcal{F}_{\mathrm{surf}}^{\varphi,\gamma,q}(E) \tag{2.51}$$

for $C > 0$ depending on φ. In view of (2.41), for η_0 sufficiently small this concludes the proof of (2.10) (ii).

Step 4 (Proof of (2.10) (iii)). First, the construction of $E_{\eta,\gamma}$ in (2.48) and (2.10) (i) imply that

$$\partial E_{\eta,\gamma} \cap \Omega \subset \bigcup_{Q_\rho \in \mathcal{Q}_\rho^\partial} \big(\partial(E_{\eta,\gamma}(Q_\rho)) \setminus \bar{E}\big) \cup \partial^{\rm rest} E,$$

where

$$\partial^{\rm rest} E := (\partial E \cap \Omega) \setminus \bigcup_{Q_\rho \in \mathcal{Q}_\rho^\partial} E_{\eta,\gamma}(Q_\rho).$$

Hence, as $E_{\eta,\gamma}(Q_\rho) = \emptyset$ for $Q_\rho \in \mathcal{Q}_\rho^{\rm neigh}$, recalling the notation in (2.35), we find

$$\begin{aligned}\mathcal{H}^{d-1}_\varphi(\partial E_{\eta,\gamma} \cap \Omega) \le & \sum_{Q_\rho \in \mathcal{Q}_\rho^{\rm acc}} \mathcal{H}^{d-1}_\varphi\big(\partial(E_{\eta,\gamma}(Q_\rho)) \setminus \bar{E}\big) \\ & + \sum_{Q_\rho \in \mathcal{Q}_\rho^{\rm flat}} \mathcal{H}^{d-1}_\varphi\big(\partial(E_{\eta,\gamma}(Q_\rho)) \setminus \bar{E}\big) + \mathcal{H}^{d-1}_\varphi(\partial^{\rm rest} E). \end{aligned} \tag{2.52}$$

We now estimate the terms on the right-hand side of (2.52) separately. Let $\mathcal{Q}^{\rm flat}_{\rho,\Omega'}$ be the subset of cubes in $\mathcal{Q}^{\rm flat}_\rho$ intersecting Ω'. First, by construction, in particular by the fact that $\partial E \cap Q_\rho \subset \partial E \cap E_{\eta,\gamma}(Q_\rho)$ for $Q_\rho \in \mathcal{Q}^{\rm flat}_{\rho,\Omega'}$ (recall (2.38) and the construction in Step 2), we have

$$\mathcal{H}^{d-1}_\varphi(\partial^{\rm rest} E) \le \mathcal{H}^{d-1}_\varphi\Bigg((\partial E \cap \Omega) \setminus \Bigg(\bigcup_{Q_\rho \in \mathcal{Q}_\rho^{\rm acc}} \overline{Q_{12\rho}} \cup \bigcup_{Q_\rho \in \mathcal{Q}^{\rm flat}_{\rho,\Omega'}} Q_\rho\Bigg)\Bigg). \tag{2.53}$$

We continue with the first term. Since $\mathcal{H}^{d-1}_\varphi(\partial(E_{\eta,\gamma}(Q_\rho))) \le \varphi_{\max} 2d(12\rho)^{d-1}$ for $Q_\rho \in \mathcal{Q}_\rho^{\rm acc}$, in view of (2.45), we calculate

$$\begin{aligned}\sum_{Q_\rho \in \mathcal{Q}_\rho^{\rm acc}} \mathcal{H}^{d-1}_\varphi\big(\partial(E_{\eta,\gamma}(Q_\rho))\big) &\le \frac{\varphi_{\max} 2d(12\rho)^{d-1}}{\Lambda \rho^{d-1}} \sum_{Q_\rho \in \mathcal{Q}_\rho^{\rm acc}} \mathcal{F}^{\varphi,\gamma,q}_{\rm surf}(E; Q_{8\rho}) \\ &\le \frac{\varphi_{\max} 2d12^{d-1}15^d}{\Lambda} \mathcal{F}^{\varphi,\gamma,q}_{\rm surf}\Bigg(E; \bigcup_{Q_\rho \in \mathcal{Q}_\rho^{\rm acc}} Q_{8\rho}\Bigg),\end{aligned}$$

where in the second step we used that each point in $\bigcup_{Q_\rho \in \mathcal{Q}_\rho^{\rm acc}} Q_{8\rho}$ is contained in at most 15^d different cubes $Q_{8\rho}$; see (1.7). By the definition of $\Lambda = \varphi_{\max} 2d12^{d-1}15^d$ at the beginning of the proof, this exactly gives

$$\sum_{Q_\rho \in \mathcal{Q}_\rho^{\rm acc}} \mathcal{H}^{d-1}_\varphi\big(\partial(E_{\eta,\gamma}(Q_\rho))\big) \le \mathcal{F}^{\varphi,\gamma,q}_{\rm surf}\Bigg(E; \bigcup_{Q_\rho \in \mathcal{Q}_\rho^{\rm acc}} Q_{8\rho}\Bigg). \tag{2.54}$$

Finally, for the second term on the right-hand side of (2.52) we will prove that

$$\sum_{Q_\rho \in \mathcal{Q}_\rho^{\mathrm{flat}}} \mathcal{H}_\varphi^{d-1}\big(\partial(E_{\eta,\gamma}(Q_\rho)) \setminus \bar{E}\big) \le \mathcal{H}_\varphi^{d-1}\Bigg(\partial E \cap \bigcup_{Q_\rho \in \mathcal{Q}_{\rho,\Omega'}^{\mathrm{flat}}} Q_{3\rho}\Bigg) + C_0 \eta \mathcal{H}_\varphi^{d-1}(\partial E \cap \Omega), \tag{2.55}$$

for $C_0 > 0$ depending only on φ. To this end, we enumerate the cubes in $\mathcal{Q}_{\rho,\Omega'}^{\mathrm{flat}}$ by $\{Q_\rho^1, \dots, Q_\rho^N\}$, and for each Q_ρ^n, $n = 1, \dots, N$, we denote by $(\Gamma_i^n)_{i=1}^{I_n}$ the pairwise disjoint sets in $\partial E \cap Q_{3\rho}^n$ and by $(T_i^n)_{i=1}^{I_n}$ the sets obtained by Lemma 2.11. Accordingly, we denote the set of indices by $\mathcal{I}_{\mathrm{good}}^n$ and $\mathcal{I}_{\mathrm{bad}}^n$. In particular, in view of (2.49), we have that

$$\bigcup_{Q_\rho \in \mathcal{Q}_{\rho,\Omega'}^{\mathrm{flat}}} \partial(E_{\eta,\gamma}(Q_\rho)) \setminus \bar{E} \subset \bigcup_{i \in \mathcal{I}_{\mathrm{good}}^n \cup \mathcal{I}_{\mathrm{bad}}^n} \partial T_i^n \setminus \bar{E}.$$

In order to avoid multiple counting of indices in the estimate, since a "bad index" for a certain cube may as well be a "bad index" with respect to an adjacent cube, we make a more careful distinction between the indices according to the following inductive procedure:

- Starting from the enumeration $\mathcal{Q}_{\rho,\Omega'}^{\mathrm{flat}} = \{Q_\rho^1, \dots, Q_\rho^N\}$, we define the first set of indices as $\mathcal{J}_1 = \{1, \dots, I_1\}$.
- For every $n = 2, \dots, N$, and having defined the set of indices $\mathcal{J}_1, \dots, \mathcal{J}_{n-1}$, we define $\mathcal{J}_n$ as the subset of $\{1, \dots, I_n\}$ which does not contain the indices $\mathcal{I}_{\mathrm{bad}}^n(Q'_\rho)$ for $Q'_\rho \in \mathcal{N}(Q_\rho^n) \cap \{Q_\rho^1, \dots, Q_\rho^{n-1}\}$, i.e., the indices related to parts of ∂E which have already been covered by the procedure, related to one of the previous cubes $\{Q_\rho^1, \dots, Q_\rho^{n-1}\}$.
- With this re-enumeration, every index in $\bigcup_{n=1}^N I_n = \bigcup_{n=1}^N (\mathcal{I}_{\mathrm{good}}^n \cup \mathcal{I}_{\mathrm{bad}}^n)$ is counted exactly once in the family $\bigcup_{n=1}^N \mathcal{J}_n$. We also note that $\#\mathcal{J}_n \le I_n \le C = C(\varphi, \Lambda)$; see Lemma 2.11.

Note that, as a consequence of (2.37) and (2.39), for each $n \in \{1, \dots, N\}$ and each $i \in \mathcal{J}_n$ we find sets $\Psi_i^n \subset \partial E \cap Q_{3\rho}^n$ such that $(\Psi_i^n)_{n,i}$ are pairwise disjoint and

$$\mathcal{H}_\varphi^{d-1}(\partial T_i^n \setminus \bar{E}) \le \mathcal{H}_\varphi^{d-1}(\Psi_i^n) + C\eta\rho^{d-1}. \tag{2.56}$$

Indeed, if $i \in \mathcal{I}_{\mathrm{good}}^n$, one takes $\Psi_i^n = \Gamma_i^n \cap Q_\rho^n$. If $i \in \mathcal{I}_{\mathrm{bad}}^n(Q'_\rho)$, we set $\Psi_i^n = \Gamma_i^n \cap (Q_\rho^n \cup Q'_\rho)$. The construction along with (2.56) shows that

$$\begin{aligned}
\sum_{n=1}^N \mathcal{H}_\varphi^{d-1}(\partial(E_{\eta,\gamma}(Q_\rho^n)) \setminus \bar{E}) &\le \sum_{n=1}^N \sum_{i \in J_n} \mathcal{H}_\varphi^{d-1}(\partial T_i^n \setminus \bar{E}) \\
&\le \sum_{n=1}^N \sum_{i \in J_n} (\mathcal{H}_\varphi^{d-1}(\Psi_i^n) + C\eta\rho^{d-1}) \\
&\le \mathcal{H}_\varphi^{d-1}\Bigg(\partial E \cap \bigcup_{Q_\rho \in \mathcal{Q}_{\rho,\Omega'}^{\mathrm{flat}}} Q_{3\rho}\Bigg) + C\eta\rho^{d-1}\#\mathcal{Q}_\rho^{\mathrm{flat}},
\end{aligned}$$

where in the last step we used the fact that $(\Psi_i^n)_{n,i}$ are pairwise disjoint and their union is contained in $\partial E \cap \bigcup_{Q_\rho \in \mathcal{Q}^{\rm flat}_{\rho,\Omega'}} Q_{3\rho}$. This along with (2.47) and the fact that $E_{\eta,\gamma}(Q_\rho) = \emptyset$ whenever $Q_\rho \cap \Omega' = \emptyset$ shows that

$$
\begin{aligned}
\sum_{Q_\rho \in \mathcal{Q}_\rho^{\rm flat}} & \mathcal{H}^{d-1}_\varphi(\partial(E_{\eta,\gamma}(Q_\rho)) \setminus \bar{E}) \\
&\le \mathcal{H}^{d-1}_\varphi\Big(\partial E \cap \bigcup_{Q_\rho \in \mathcal{Q}^{\rm flat}_{\rho,\Omega'}} Q_{3\rho}\Big) + C\eta \sum_{Q_\rho \in \mathcal{Q}_\rho^{\rm flat}} \mathcal{H}^{d-1}(\partial E \cap Q_{8\rho}) \\
&\le \mathcal{H}^{d-1}_\varphi\Big(\partial E \cap \bigcup_{Q_\rho \in \mathcal{Q}^{\rm flat}_{\rho,\Omega'}} Q_{3\rho}\Big) + C\eta \mathcal{H}^{d-1}(\partial E \cap \Omega), \qquad (2.57)
\end{aligned}
$$

where in the last step we again used that each point in $\bigcup_{Q_\rho \in \mathcal{Q}_\rho^{\rm flat}} Q_{8\rho}$ is contained in at most 15^d different cubes $Q_{8\rho}$, see (1.7), and $Q_{8\rho} \subset \Omega$, see (2.42). As Λ itself is a constant depending only on φ, we obtain (2.55).

We now conclude the proof as follows: note that $Q_\rho \in \mathcal{Q}_\rho^{\rm flat}$ implies $Q_{3\rho} \cap Q'_{8\rho} = \emptyset$ for all $Q'_\rho \in \mathcal{Q}_\rho^{\rm acc}$; see (2.46). Moreover, we get that

$$
\Big(\partial E \cap \bigcup_{Q_\rho \in \mathcal{Q}^{\rm flat}_{\rho,\Omega'}} Q_{3\rho}\Big) \setminus \bigcup_{Q_\rho \in \mathcal{Q}^{\rm flat}_{\rho,\Omega'}} Q_\rho \subset \bigcup_{Q_\rho \in \mathcal{Q}^{\rm acc}_{\rho}} \overline{Q_{12\rho}} \cup (\partial E \cap (\partial \Omega')_{3\sqrt{d}\rho}).
$$

Then, combining (2.52) and (2.53)–(2.55), and using (2.42)–(2.43), we obtain (2.10) (iii), where $C_0 > 0$ indeed only depends on φ. ∎

We close this subsection with the version for graphs.

Proof of Corollary 2.3. Consider $\Omega = \omega \times (-1, M+1)$ for some open and bounded $\omega \subset \mathbb{R}^{d-1}$ and $M > 0$. Suppose that $E = \{(x', x_d) \in \Omega : x' \in \omega,\ x_d > h(x')\}$ for a regular function $h \colon \omega \to [0, M]$. We start by introducing the set

$$
E^*_{\eta,\gamma} = \mathrm{int}\Big(E \cup \bigcup_{Q_\rho \in \mathcal{Q}^\partial_\rho} \overline{Q_{12\rho}}\Big).
$$

Clearly, by construction, $E^*_{\eta,\gamma} \supset E_{\eta,\gamma}$. Moreover, by Lemma 2.12 we find that

$$
\mathcal{L}^d(E^*_{\eta,\gamma} \setminus E) \le C_0 \rho \mathcal{F}^{\varphi,\gamma,q}_{\rm surf}(E), \quad \mathcal{H}^{d-1}(\partial E^*_{\eta,\gamma} \cap \Omega) \le C_0 \mathcal{F}^{\varphi,\gamma,q}_{\rm surf}(E),
$$

for an absolute constant $C_0 > 0$, where we use the definition of ρ in (2.41). We note that the set $\Omega \setminus E^*_{\eta,\gamma}$ can be seen as the subgraph of a suitable BV-function with $\mathcal{H}^{d-1}(\overline{\partial^* E^*_{\eta,\gamma}} \setminus \partial^* E^*_{\eta,\gamma}) = 0$. The desired set $E'_{\eta,\gamma} \supset E^*_{\eta,\gamma}$ is then obtained by approximating the set $\Omega \setminus E^*_{\eta,\gamma}$ from below with a suitable smooth graph so that (2.5) holds true; see [27, Lemma 6.3]. ∎

2.4. Small area implies large curvature: Proof of Lemma 2.12

This subsection is devoted to the proof of Lemma 2.12. We start with a lemma due to L. Simon, whose original statement and proof can be found in [84, Corollary 1.3]. In the next statement, by $\partial\Sigma$ we intend the boundary of a regular surface Σ in the differential-geometric sense.

Lemma 2.14. *Given $R > 0$ and $\mu \in (0,1)$, there exist $\alpha_0 = \alpha_0(\mu) > 0$ and $c_0 = c_0(\mu) > 0$ such that the following holds: Consider a connected, regular, two-dimensional surface Σ in $\mathbb{R}^3$ with $\mathcal{H}^1(\partial\Sigma \cap \overline{B_R}) = 0$ such that*

$$\int_{\Sigma\cap B_R} |\boldsymbol{A}|\,\mathrm{d}\mathcal{H}^2 < \alpha_0 R, \quad \Sigma \cap \partial B_R \neq \emptyset, \quad \textit{and} \quad \Sigma \cap \partial B_{\mu R} \neq \emptyset.$$

Then we have

$$\mathcal{H}^2(\Sigma \cap B_R) \geq c_0 R^2.$$

We proceed now with the proof of Lemma 2.12.

Proof of Lemma 2.12. We first treat the elementary case $d = 2$, and then we address the case $d = 3$ by using Lemma 2.14.

Step 1 $(d = 2)$. Let $c_0 = 1$ and $c_\Lambda = (\Lambda + 1)^{-1/q} \in (0,1)$. Consider $Q_\rho \in \mathcal{Q}_\rho$ such that $\overline{Q_{8\rho}} \subset \Omega$, $\partial E \cap Q_{3\rho} \neq \emptyset$, and $\mathcal{H}^1(\partial E \cap Q_{8\rho}) < \rho$. Let $\boldsymbol{\gamma}$ be a connected component of $\partial E \cap Q_{8\rho}$ intersecting $Q_{3\rho}$. Clearly, $\boldsymbol{\gamma}$ is a regular planar curve and we have

$$\mathrm{diam}(\boldsymbol{\gamma}) \leq \mathcal{H}^1(\boldsymbol{\gamma}) \leq \mathcal{H}^1(\partial E \cap Q_{8\rho}) < \rho.$$

Therefore, $\boldsymbol{\gamma}$ is a closed curve inside the cube $Q_{8\rho}$. Hence, for all $0 < \rho \leq c_\Lambda \gamma^{1/q}$, recalling that $c_\Lambda = (\Lambda + 1)^{-1/q}$ and $q \geq 1$, Lemma A.1 yields

$$\begin{aligned}\gamma \int_{\partial E\cap Q_{8\rho}} |\boldsymbol{A}|^q\,\mathrm{d}\mathcal{H}^1 \geq c_\Lambda^{-q}\rho^q \int_{\boldsymbol{\gamma}} |\kappa_{\boldsymbol{\gamma}}|^q\,\mathrm{d}\mathcal{H}^1 &\geq (\Lambda+1)\rho^q(\mathrm{diam}(\boldsymbol{\gamma}))^{1-q}\\ &\geq (\Lambda+1)\rho^q\rho^{1-q} > \Lambda\rho.\end{aligned}$$

Step 2 $(d = 3)$. Let $c_0 = c_0(3\sqrt{3}/8)$ and $\alpha_0 = \alpha_0(3\sqrt{3}/8)$ be the constants in Lemma 2.14, applied for $R = 4\rho$ and $\mu = 3\sqrt{3}/8$. We define

$$c_\Lambda := \min\Big\{c_0^{\frac{1-q}{q}}\, 4\alpha_0(\Lambda+1)^{-1/q}, (4\pi)^{1/2} c_0^{1/q-1/2}(\Lambda+1)^{-1/q}\Big\}. \tag{2.58}$$

Consider $Q_\rho \in \mathcal{Q}_\rho$ such that $\overline{Q_{8\rho}} \subset \Omega$, $\partial E \cap Q_{3\rho} \neq \emptyset$, and

$$\mathcal{H}^2(\partial E \cap Q_{8\rho}) < c_0\rho^2 < c_0(4\rho)^2. \tag{2.59}$$

Let K be a connected component of $\partial E \cap \Omega$ such that $K \cap Q_{3\rho} \neq \emptyset$. As $\partial E \cap \Omega$ is a regular surface, we note that K is a regular surface as well, with $\partial K \subset \partial\Omega$. We first suppose that $K \cap \partial Q_{8\rho} \neq \emptyset$. Then the connectedness of the regular surface K and the fact that $Q_{3\rho} \subset B_{3\sqrt{3}\rho/2} \subset B_{4\rho} \subset Q_{8\rho} \subset \Omega$ imply that

$$K \cap \partial B_{3\sqrt{3}\rho/2} \neq \emptyset, \quad K \cap \partial B_{4\rho} \neq \emptyset.$$

Moreover, since $\partial K \subset \partial\Omega$ and $\overline{B_{4\rho}} \subset \overline{Q_{8\rho}} \subset \Omega$, we have that $\mathcal{H}^1(\partial K \cap \overline{B_{4\rho}}) = 0$. Therefore, in view of (2.59), by applying Lemma 2.14 (or more precisely its negation) for $R = 4\rho > 0$, $\mu = 3\sqrt{3}/8 \in (0,1)$, and $\Sigma = K$, we deduce that

$$\int_{K\cap B_{4\rho}} |\boldsymbol{A}| \,\mathrm{d}\mathcal{H}^2 \geq 4\alpha_0\rho. \tag{2.60}$$

Using Hölder's inequality, (2.59), (2.60), and the fact that $q \geq 2 > 1$, we obtain for all $0 < \rho \leq c_\Lambda \gamma^{1/q}$,

$$\begin{aligned}\gamma \int_{\partial E\cap Q_{8\rho}} |\boldsymbol{A}|^q \,\mathrm{d}\mathcal{H}^2 &\geq c_\Lambda^{-q}\rho^q \int_{K\cap B_{4\rho}} |\boldsymbol{A}|^q \,\mathrm{d}\mathcal{H}^2 \\ &\geq c_\Lambda^{-q}\rho^q (\mathcal{H}^2(K\cap B_{4\rho}))^{1-q} \Big(\int_{K\cap B_{4\rho}} |\boldsymbol{A}| \,\mathrm{d}\mathcal{H}^2\Big)^q \\ &\geq (c_\Lambda^{-q}\rho^q)(c_0\rho^2)^{1-q}(4\alpha_0\rho)^q = (4^q\alpha_0^q c_0^{1-q} c_\Lambda^{-q})\rho^2 > \Lambda\rho^2,\end{aligned}$$

where the last step follows from the definition of c_Λ in (2.58).

If instead we have $K \cap \partial Q_{8\rho} = \emptyset$, then K is closed inside the cube $Q_{8\rho}$, i.e., $\partial(K \cap Q_{8\rho}) = \emptyset$. By a classical topological-differential geometric fact regarding a lower bound on the Willmore energy of closed surfaces, we then have that

$$\int_{K\cap Q_{8\rho}} |\boldsymbol{A}|^2 \,\mathrm{d}\mathcal{H}^2 \geq 4\pi;$$

see e.g., [84, formula (0.2)] and the references therein for its simple proof. By using Hölder's inequality again, the fact that $q \geq 2$, and (2.59), as before we estimate

$$\begin{aligned}\gamma \int_{\partial E\cap Q_{8\rho}} |\boldsymbol{A}|^q \,\mathrm{d}\mathcal{H}^2 &\geq c_\Lambda^{-q}\rho^q \int_{K\cap Q_{8\rho}} |\boldsymbol{A}|^q \,\mathrm{d}\mathcal{H}^2 \\ &\geq c_\Lambda^{-q}\rho^q (\mathcal{H}^2(K\cap Q_{8\rho}))^{1-q/2} \Big(\int_{K\cap Q_{8\rho}} |\boldsymbol{A}|^2 \,\mathrm{d}\mathcal{H}^2\Big)^{q/2} \\ &\geq (c_\Lambda^{-q}\rho^q)(c_0\rho^2)^{1-q/2}(4\pi)^{q/2} \geq (4\pi)^{q/2} c_0^{1-q/2} c_\Lambda^{-q}\rho^2 > \Lambda\rho^2,\end{aligned}$$

where the last step again follows from the definition of c_Λ in (2.58). This concludes the proof. ∎

2.5. Local thickening of sets: Proof of Lemma 2.11

This subsection is devoted to the proof of Lemma 2.11. We start with a preliminary observation: given $\eta, \gamma > 0$ and $Q_\rho \in \mathcal{Q}_\rho$ for some $0 < \rho \leq \eta^7\gamma^{1/q}$ such that $\mathcal{F}_{\mathrm{surf}}^{\varphi,\gamma,q}(E; Q_{8\rho}) \leq$

$\Lambda\rho^{d-1}$, see (2.36), then by (2.1), Hölder's inequality, and by $\min_{\mathbb{S}^{d-1}} \varphi = 1$ (which was assumed without loss of generality), we obtain

$$\begin{aligned}\int_{\partial E \cap Q_{8\rho}} |\boldsymbol{A}| \, \mathrm{d}\mathcal{H}^{d-1} &\leq (\mathcal{H}^{d-1}(\partial E \cap Q_{8\rho}))^{\frac{q-1}{q}} \Big(\int_{\partial E \cap Q_{8\rho}} |\boldsymbol{A}|^q \Big)^{\frac{1}{q}} \\ &\leq (\Lambda \rho^{d-1})^{\frac{q-1}{q}} \Big(\frac{\Lambda}{\gamma} \rho^{d-1} \Big)^{\frac{1}{q}} \\ &\leq \Lambda \gamma^{-1/q} \rho^{d-1} \leq \Lambda \eta^7 \rho^{d-2}. \end{aligned} \tag{2.61}$$

Therefore, we can ensure that the L^1-norm of $|\boldsymbol{A}|$ in $\partial E \cap Q_{8\rho}$ is small compared to ρ^{d-2}.

Our proof fundamentally relies on the fact that, under the above bound on the curvature, $\partial E \cap Q_\rho$ is essentially a finite union of graphs of regular functions with suitable a priori C^1-estimates. To state this result, we introduce the following notation: given an affine subspace $L \subset \mathbb{R}^d$ of codimension 1 (i.e., a line in $\mathbb{R}^2$ or a plane in $\mathbb{R}^3$), we denote by $L^\perp$ the one-dimensional subspace spanned by a unit normal vector ν_L to L. Accordingly, for $U \subset L$ and $u \colon U \to L^\perp$, we define $\mathrm{graph}(u) := \{x + u(x) \colon x \in U\} \subset \mathbb{R}^d$. We first state the result for $d = 2$ separately, since its proof is significantly easier than for $d = 3$. Note that the parameter ε which appears in the next lemmata should not be confused with the one used below (2.3), since it serves a totally different purpose.

Lemma 2.15 (Almost straight curves). *There exist $\varepsilon_0 > 0$ and an absolute constant $C_1 \geq 1$ such that for every $\Lambda > 0$ the following holds: for every $\varepsilon \in (0, \varepsilon_0]$, every square $Q_\rho \subset \mathbb{R}^2$, $\rho > 0$, and every $E \in \mathcal{A}_{\mathrm{reg}}(\mathbb{R}^2)$ satisfying*

$$\partial E \cap Q_{3\rho} \neq \emptyset, \quad \mathcal{H}^1(\partial E \cap Q_{8\rho}) \leq \Lambda\rho, \quad \textit{and} \quad \int_{\partial E \cap Q_{8\rho}} |\boldsymbol{A}| \, \mathrm{d}\mathcal{H}^1 \leq \varepsilon,$$

there exist regular curves $(\boldsymbol{\gamma}_i)_{i=1}^M$ with $M \leq \Lambda$ such that

$$\partial E \cap Q_{3\rho} = \bigcup_{i=1}^M \boldsymbol{\gamma}_i \cap Q_{3\rho},$$

corresponding lines L_i and functions $u_i \colon \overline{U_i} \to L_i^\perp$, where $U_i \subset L_i$ are open segments with $\mathrm{diam}(U_i) \leq C_1\rho$, *such that* $\mathrm{graph}(u_i) = \boldsymbol{\gamma}_i$ *for $i = 1, \dots, M$, and*

$$\|u_i\|_{L^\infty(U_i)} \leq C_1 \varepsilon \rho, \quad \|u_i'\|_{L^\infty(U_i)} \leq C_1 \varepsilon.$$

The proof is elementary, and we refer to Appendix A.1. The analogous statement in dimension $d = 3$ is more involved: it is known as the *approximate graphical decomposition lemma*, proved by L. Simon; see [84, Lemma 2.1].

Lemma 2.16 (Simon's approximate graphical decomposition lemma). *For any $\Lambda > 0$ and $\mu \in (0, 1)$ there exist $\varepsilon_S = \varepsilon_S(\Lambda, \mu) \in (0, 1)$ and a constant $C_1 = C_1(\Lambda, \mu) \geq 1$ such* *that for every* $\varepsilon \in (0, \varepsilon_S]$, *every* $\rho > 0$, *and every* $E \in \mathcal{A}_{\mathrm{reg}}(\mathbb{R}^3)$ *satisfying* $\partial E \cap B_{\mu\rho} \neq \emptyset$, *and*

$$\mathcal{H}^2(\partial E \cap B_\rho) \leq \Lambda\rho^2 \quad \textit{and} \quad \int_{\partial E \cap B_\rho} |\boldsymbol{A}| \,\mathrm{d}\mathcal{H}^2 \leq \varepsilon\rho,$$

the following holds true: there exist pairwise disjoint, closed sets $P_1, \ldots, P_N \subset \partial E$ *with*

$$\sum_{j=1}^{N} \mathrm{diam}(P_j) \leq C_1\sqrt{\varepsilon}\rho$$

and functions $u_i \in C^2(\overline{U_i}; L_i^\perp)$ *for* $i = 1, \ldots, M$, *with* $M \leq C_1$, *such that*

$$(\partial E \cap B_{\mu\rho}) \setminus \bigcup_{j=1}^{N} P_j = \Big(\bigcup_{i=1}^{M} \mathrm{graph}(u_i)\Big) \cap B_{\mu\rho}.$$

Here, for every $i = 1, \ldots, M$, L_i *is a two-dimensional plane in* $\mathbb{R}^3$, $U_i \subset L_i$ *is a smooth bounded domain with*

$$\mathrm{diam}(U_i) \leq C_1\rho \tag{2.62}$$

of the form $U_i = U_i^0 \setminus (\bigcup_{k=1}^{R_i} d_{i,k})$, *where* U_i^0 *is a simply connected subdomain of* L_i *and* $(d_{i,k})_{k=1}^{R_i}$ *are pairwise disjoint closed disks in* L_i, *which do not intersect* ∂U_i^0. *Moreover,* $\mathrm{graph}(u_i)$ *is connected and* u_i *also satisfies the estimates*

$$\sup_{x\in U_i} |u_i(x)| \leq C_1\varepsilon^{1/6}\rho, \qquad \sup_{x\in U_i} |\nabla u_i(x)| \leq C_1\varepsilon^{1/6}.$$

Here, ∂U_i^0 has to be understood with respect to the relative topology of L_i. Roughly speaking, the result states that, apart from sets $(P_j)_{j=1,\ldots,N}$ of small diameter, so-called *pimples*, $\partial E \cap B_{\mu\rho}$ can be written as the union of finitely many graphs of regular functions with small heights and small gradients. Compare the result to the easier statement of Lemma 2.15.

Remark 2.17 (Adaptions to the original statement). We have phrased the result slightly differently compared to the original statement in [84, Lemma 2.1], where the lemma was stated only for $\mu = 1/2$ but for general smooth, closed two-dimensional manifolds Σ. However, it is easy to verify through the proof that it is an interior *ε-regularity* result, valid for every $\partial E \in C^2$ and for every $\mu \in (0, 1)$, up to adapting the constants. The estimate (2.62) is implicitly mentioned in the original statement, being a simple geometric observation; see also the proof of Lemma 2.15 in Appendix A.1 for the analogous fact in two dimensions. Finally, the original statement has the assumption $0 \in \partial E$ which, however, can readily be generalized to requiring that $\partial E \cap B_{\mu\rho} \neq \emptyset$.

Remark 2.18 (Result on cubes). As in [84], the lemma is phrased as an interior statement for balls in $\mathbb{R}^3$. In the application below, we will apply it on cubes, by using $Q_{8\rho}$ in place of B_ρ and $Q_{3\rho}$ in place of $B_{\mu\rho}$. Indeed, to this end, it suffices to note that $B_{4\rho} \subset Q_{8\rho}$ and $Q_{3\rho} \subset B_{4\mu\rho}$ for $\mu \in (3\sqrt{3}/8, 1)$.

Both statements above involve functions which are defined with respect to suitable lines or planes, respectively. As a second preparation, we need to distinguish between *good and bad lines and planes* for a cube Q_ρ. We discuss the following definitions and properties only for planes in $\mathbb{R}^3$ as the analogous definitions for lines in $\mathbb{R}^2$ can be simply obtained by identifying lines in $\mathbb{R}^2$ with planes in $\mathbb{R}^3$ with one tangent vector given by e_3.

Without restriction we suppose for the following arguments that Q_ρ is centered in 0, as this can always be achieved by a translation.

Definition 2.19. Let $\theta \in (0, 1/\sqrt{3})$ and let L be a plane with normal $\nu_L = (\nu_1, \nu_2, \nu_3) \in \mathbb{S}^2$ such that $(L)_{3\eta\rho} \cap Q_\rho \neq \emptyset$; see (1.5). We say that L is a θ-*good plane* for Q_ρ if and only if one of the following two properties holds true:

(1) There exist $i, j \in \{1, 2, 3\}$, $i \neq j$, such that $|\nu_i|, |\nu_j| \geq \theta$.

(2) There exists $k \in \{1, 2, 3\}$ such that $|\nu_k| \geq \theta$ and

$$\operatorname{dist}(L \cap Q_{3\rho}, \{x_k = -\rho/2\} \cup \{x_k = \rho/2\}) \geq 20\theta\rho. \tag{2.63}$$

If L is not a θ-good plane for Q_ρ, we say that it is a θ-*bad plane* for Q_ρ. In the statement of Lemma 2.11, the two different possibilities, namely θ-good or θ-bad planes, are reflected in the two cases described in (2.37) (i) and (2.37) (ii), respectively. As stated after Lemma 2.11, the case (2.37) (ii) and thus the case of θ-bad planes is only of a technical nature and the main idea is already reflected in the construction for θ-good planes. The different cases of good and bad planes are depicted in Figure 4. In the following, we denote by ν_L a unit vector normal to L whose orientation will be specified in the proof below. Recall also the shorthand notation for the anisotropic perimeter in (2.35).

Lemma 2.20 (Surface estimate for θ-good planes). *There exist $\theta \in (0, 1/\sqrt{3})$ small enough and a constant $C_\theta > 0$ such that for any $\rho > 0$ and any θ-good plane L for Q_ρ the following holds:*

(i) *By letting $S_L := Q_{(1+6\eta)\rho} \cap (L)_{3\eta\rho}$, we obtain*

$$\mathcal{H}^2_\varphi(\partial^- S_L) \leq \mathcal{H}^2_\varphi(L \cap Q_\rho) + C_\theta \varphi_{\max} \eta \rho^2, \tag{2.64}$$

where $\partial^- S_L := \partial S_L \setminus (-3\eta\rho\nu_L + L)$.

(ii) *Let $u \in L^\infty(U; L^\perp)$ for some bounded domain $L \cap Q_\rho \subset U \subset L$ with $\|u\|_{L^\infty(U)} \leq 2\eta\rho$. Define $\omega_u^\rho := \Pi_L(\operatorname{graph}(u) \cap Q_\rho)$, where Π_L denotes the orthogonal projection onto the plane L. Then*

$$\mathcal{H}^2(\omega_u^\rho \triangle (L \cap Q_\rho)) \leq C_\theta \eta \rho^2. \tag{2.65}$$

Lemma 2.21 (θ-bad planes). *There exists $\theta \in (0, 1/\sqrt{3})$ small enough such that for any $\rho > 0$ and any θ-bad plane L for Q_ρ the following holds: Let $k \in \{1, 2, 3\}$ be the unique component such that $|\nu_k| \geq \theta$. Then we have*

$$\text{either } -\frac{3\rho}{4} < x \cdot e_k < -\frac{\rho}{4}\ \forall x \in L \cap Q_{3\rho}, \quad \text{or } \frac{\rho}{4} < x \cdot e_k < \frac{3\rho}{4}\ \forall x \in L \cap Q_{3\rho}. \tag{2.66}$$

The proofs of the above lemmata are elementary but tedious. They are deferred to Appendix A.2. We are now in a position to give the proof of Lemma 2.11.

Proof of Lemma 2.11. Let $\gamma \in (0,1)$ and without restriction $\Lambda \geq 1$. Consider Q_ρ centered without restriction at 0 such that $\overline{Q_{12\rho}} \subset \Omega$ and (2.36) holds. In the case $d = 2$, we choose $\eta_0 = \eta_0(\Lambda)$ such that $\Lambda \eta_0^7 \leq \varepsilon_0$, where $\varepsilon_0 > 0$ is the constant of Lemma 2.15. Then, by (2.36) and (2.61), it is possible to apply Lemma 2.15. In the case $d = 3$, we choose $\eta_0 = \eta_0(\Lambda)$ such that $\Lambda \eta_0 \leq \min\{C_1 \varepsilon_S^{1/6}, C_1^{-6}, 2^{-(1+q/2)} c_0\}$, where ε_S and $C_1 \geq 1$ are the constants in Lemma 2.16, and c_0 is the constant in Lemma 2.12. Consequently, in view of (2.61), we have

$$\int_{\partial E \cap Q_{8\rho}} |\boldsymbol{A}| \, \mathrm{d}\mathcal{H}^2 \leq \Lambda \eta_0 \eta^6 \rho^{d-2} \leq (\eta/C_1)^6 \rho.$$

In particular, as $\eta \leq \eta_0 \leq C_1 \varepsilon_S^{1/6}$, we get $\int_{\partial E \cap Q_{8\rho}} |\boldsymbol{A}| \, \mathrm{d}\mathcal{H}^2 \leq \varepsilon_S \rho$. This along with (2.36) allows one to apply Lemma 2.16 in the version of Remark 2.18. From now on, we only treat the case $d = 3$ since the case $d = 2$ is simpler (in the latter case, the sets $(P_j)_j$ below in (2.68) can be chosen empty). In the following, $C > 0$ again denotes a generic absolute constant, whose value is allowed to vary from line to line.

Step 1 (Application of Lemma 2.16). By Lemma 2.16 in the version of Remark 2.18, applied for $\varepsilon := (\eta/C_1)^6 \leq \varepsilon_S$, there exist planes $L_i \subset \mathbb{R}^3$ and functions $u_i : \overline{U_i} \to L_i^\perp$ for $i = 1, \ldots, M$, with $M \leq C_1$, where $U_i = U_i^0 \setminus \bigcup_{k=1}^{R_i} d_{i,k}$ for (two-dimensional) disks $(d_{i,k})_{i,k}$ in the planes L_i, as well as pairwise disjoint closed subsets $(P_j)_{j=1}^N \subset \partial E$ such that

$$\partial E \cap Q_{3\rho} = \left(\bigcup_{i=1}^M \mathrm{graph}(u_i) \cup \bigcup_{j=1}^N P_j \right) \cap Q_{3\rho}. \tag{2.67}$$

Moreover,

$$\sum_{j=1}^N \mathrm{diam}(P_j) \leq C_1 (\eta/C_1)^3 \rho \leq \eta \rho, \tag{2.68}$$

and for the functions $(u_i)_{i=1,\ldots,M}$ we have the C^1-estimates

$$\sup_{x \in U_i} |u_i(x)| \leq C_1 (\eta/C_1) \rho = \eta \rho \leq 2\eta\rho, \qquad \sup_{x \in U_i} |\nabla u_i(x)| \leq C_1 (\eta/C_1) = \eta. \tag{2.69}$$

Here, in the estimates (2.68)–(2.69) we used that $\varepsilon = (\eta/C_1)^6$ and the fact that we can choose $\eta \leq \eta_0 \leq C_1$. The nonoptimal estimate with $2\eta\rho$ is introduced for later purposes in Step 4 below. To simplify the exposition, we assume for the moment that there are no pimples in $\partial E \cap Q_{3\rho}$, i.e., by (2.67), that we have

$$\partial E \cap Q_{3\rho} = \bigcup_{i=1}^M \mathrm{graph}(u_i) \cap Q_{3\rho}. \tag{2.70}$$

In particular, this further implies that $\bigcup_{k=1}^{R_i} d_{i,k} = \emptyset$, i.e., $U_i = U_i^0$ for the domain of definition of the functions u_i. We defer the analysis of the case with pimples to Step 4. We fix $\theta > 0$ sufficiently small such that Lemmata 2.20–2.21 are applicable. We distinguish the two cases

$$\text{(i)}\ \ L_i \text{ is a } \theta\text{-good plane for } Q_\rho, \quad \text{(ii)}\ \ L_i \text{ is a } \theta\text{-bad plane for } Q_\rho.$$

Let us note that $I := M \leq C_1$ by the statement of Lemma 2.16.

Step 2 (Good planes). First, let L_i be a θ-good plane for Q_ρ and consider $u_i\colon \overline{U_i} \subset L_i \to L_i^\perp$. In this case, we will define $\Gamma_i := \mathrm{graph}(u_i) \cap Q_\rho$ and thus it is not restrictive to assume that $\mathrm{graph}(u_i)$ intersects Q_ρ. In the following, for notational convenience, we drop the subscript i and simply write L for the plane, ν_L for a unit normal to L, u for the function, and U for its corresponding domain.

We will first verify that $L \cap Q_{2\rho} \subset U$. Indeed, by (2.69) and the fact that $\mathrm{graph}(u) \cap Q_\rho \neq \emptyset$ we get that $U \cap Q_{2\rho} \neq \emptyset$ for η sufficiently small. Moreover, by (2.69) and by taking η smaller if necessary, we get $|x + u(x)|_\infty < \frac{3}{2}\rho$ for all $x \in L \cap Q_{2\rho} \cap \partial U$. Since $\partial(\partial E \cap Q_{3\rho}) \subset \partial Q_{3\rho}$, (2.70) implies that $\partial U \cap Q_{2\rho} = \emptyset$. As $U \cap Q_{2\rho} \neq \emptyset$, we conclude $U \supset L \cap Q_{2\rho}$, as desired.

We choose the following orientation for ν_L, which is important for the definition of $\partial^- S_L$ in Lemma 2.20 (i): we denote by $\boldsymbol{n}(x)$ the outer unit normal to $\partial E \cap \Omega$ at x and choose the orientation ν_L as well as an orthonormal basis (τ_1, τ_2) of L such that the normal vector $\tilde{\boldsymbol{n}}(x) = -(\partial_{\tau_1} u)\tau_1 - (\partial_{\tau_2} u)\tau_2 + \nu_L$ to $\mathrm{graph}(u)$ at the point $x + u(x)$ satisfies $\boldsymbol{n}(x) = \tilde{\boldsymbol{n}}(x)/|\tilde{\boldsymbol{n}}(x)|$. Then, in view of (2.69), we have

$$\|\boldsymbol{n} - \nu_L\|_{L^\infty(U)} \leq C\eta. \tag{2.71}$$

As in Lemma 2.20, we introduce the *stripes* $S_L := Q_{(1+6\eta)\rho} \cap (L)_{3\eta\rho}$. We claim that

$$\mathrm{dist}(\mathrm{graph}(u) \cap Q_\rho, S_L^c) \geq \eta\rho \tag{2.72}$$

and

$$\mathcal{H}^2_\varphi(\mathrm{graph}(u) \cap Q_\rho) \geq \mathcal{H}^2_\varphi(L \cap Q_\rho) - C_\theta \eta\rho^2 \geq \mathcal{H}^2_\varphi(\partial^- S_L) - C_\theta \eta\rho^2, \tag{2.73}$$

where $C_\theta := C(\theta, \varphi) > 0$ is a constant depending only on θ and φ. Here, recall the notation in (2.35) and the definition of $\partial^- S_L$ in Lemma 2.20 (i). To obtain (2.72), it suffices to check that

$$\text{(a)}\ \ \mathrm{dist}(\mathrm{graph}(u) \cap Q_\rho, Q^c_{(1+6\eta)\rho}) \geq \eta\rho \quad \text{and} \quad \text{(b)}\ \ \mathrm{dist}(\mathrm{graph}(u) \cap Q_\rho, (L)^c_{3\eta\rho}) \geq \eta\rho.$$

Item (a) is clear. To see (b), we first note that $\mathrm{dist}(L, (L)^c_{3\eta\rho}) = 3\eta\rho$. Then, in view of (2.69), for each $y \in \mathrm{graph}(u) \cap Q_\rho$ we have $\mathrm{dist}(y, L) \leq 2\eta\rho$. Consequently,

$$\mathrm{dist}(\mathrm{graph}(u) \cap Q_\rho, (L)^c_{3\eta\rho}) \geq \mathrm{dist}(L, (L)^c_{3\eta\rho}) - 2\eta\rho \geq 3\eta\rho - 2\eta\rho = \eta\rho.$$

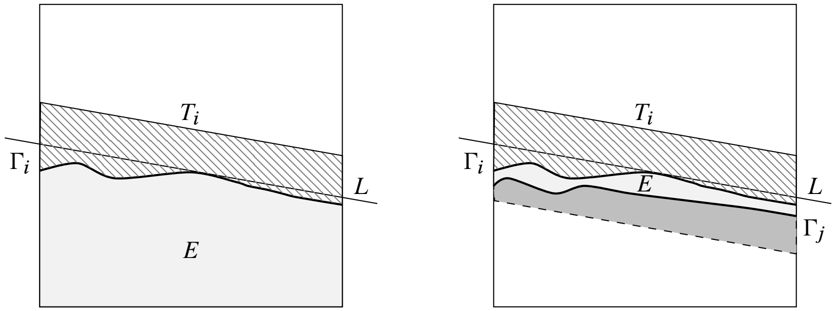

Figure 5. The two fundamental cases for the selection of the sets T_i (hatched). In the right-hand figure, the dark gray set is another connected component of $\overline{S_L} \setminus E$ that is not selected.

Regarding (2.73) we argue as follows: set as before $\omega_u^\rho = \Pi_L(\mathrm{graph}(u) \cap Q_\rho)$, where Π_L denotes the orthogonal projection onto the plane L. By Lemma 2.20 (ii) and the fact that $L \cap Q_\rho \subset U$, we have

$$\mathcal{H}^2(\omega_u^\rho \triangle (L \cap Q_\rho)) \le C_\theta \eta \rho^2. \tag{2.74}$$

Due to (2.71) and the fact that φ is Lipschitz, being a norm, we get $\|\varphi(\boldsymbol{n}) - \varphi(\nu_L)\|_{L^\infty(U)} \le C'\eta$, for a constant C' depending additionally on φ. Therefore, by (2.74), and the fact that we have assumed without restriction that $\min_{\mathbb{S}^2} \varphi = 1$, we obtain

$$\begin{aligned}\mathcal{H}^2_\varphi(\mathrm{graph}(u) \cap Q_\rho) &= \int_{\omega_u^\rho} \varphi(\boldsymbol{n}(x))\sqrt{1 + |\nabla u(x)|^2}\, \mathrm{d}\mathcal{H}^2(x) \ge \mathcal{H}^2(\omega_u^\rho)(\varphi(\nu_L) - C'\eta) \\ &\ge \big(\mathcal{H}^2(L \cap Q_\rho) - \mathcal{H}^2(\omega_u^\rho \triangle (L \cap Q_\rho))\big)(\varphi(\nu_L) - C'\eta) \\ &\ge \mathcal{H}^2_\varphi(L \cap Q_\rho) - C_\theta \eta \rho^2,\end{aligned}$$

where in the last step we also used the obvious bound $\mathcal{H}^2(L \cap Q_\rho) \le 3\rho^2$. Then, by Lemma 2.20 (i),

$$\mathcal{H}^2_\varphi(\mathrm{graph}(u) \cap Q_\rho) \ge \mathcal{H}^2_\varphi(L \cap Q_\rho) - C_\theta \eta \rho^2 \ge \mathcal{H}^2_\varphi(\partial^- S_L) - C_\theta \eta \rho^2.$$

This concludes the proof of (2.73).

We now set $\Gamma_i = \mathrm{graph}(u_i) \cap Q_\rho$ and let T_i be the connected component of $\overline{S_{L_i}} \setminus E$ which contains Γ_i; see Figure 5. (From now on, for clarification we again add the index i as the definition depends on $u_i : U_i \to L_i^\perp$.) Let us verify (2.37) (i). To this end, we observe by (2.69) that

$$\partial T_i \setminus \overline{E} \subset \partial S_{L_i} \setminus (-3\eta\rho\nu_{L_i} + L_i) = \partial^- S_{L_i},$$

where the last identity is the definition of $\partial^- S_{L_i}$. Thus, (2.73) implies (2.37) (i). We close this step with the observation that for $x \notin E$,

$$\mathrm{dist}(x, \Gamma_i) \le \mathrm{dist}\Big(x, \bigcup_{j=1}^{M} \mathrm{graph}(u_j) \cap Q_\rho\Big) < \eta\rho \quad \Rightarrow \quad x \in T_i. \tag{2.75}$$

In fact, by using (2.72) and by assuming that $\mathrm{dist}(x, \Gamma_i) < \eta\rho$, we get $x \in S_{L_i}$, in particular $x \in S_{L_i} \setminus E$. If we had $x \in S_{L_i} \setminus T_i$, then we would necessarily find Γ_j, $j \neq i$, such that $\mathrm{dist}(x, \Gamma_j) < \mathrm{dist}(x, \Gamma_i)$; see also Figure 5. This is a contradiction.

Step 3 (Bad planes). Now we suppose that L_i is a θ-bad plane for Q_ρ. Then there exists exactly one $k \in \{1,2,3\}$ such that $|\nu_k| \geq \theta$ and $|\nu_j| < \theta$ for $j \neq k$, and by Lemma 2.21 we find that (2.66) holds. Without restriction we suppose that $k = 1$ and that $x \in L_i \cap Q_{3\rho}$ implies that $-\frac{3\rho}{4} \leq x \cdot e_1 < -\frac{\rho}{4}$. In fact, the other cases can be treated along the very same lines. We let $Q'_\rho := Q_\rho - \rho e_1 \in \mathcal{N}(Q_\rho)$ be the neighboring cube of Q_ρ to the left of it; recall notation (2.34).

Due to (2.66), we have that L_i is a θ-good plane for the shifted cube $\widetilde{Q}_\rho := Q_\rho - \frac{\rho}{2} e_1$. In fact, case (2) of Definition 2.19 is satisfied, provided that $\theta > 0$ is chosen small enough. Consequently, given (2.36) and the fact that $\widetilde{Q}_{8\rho} \subset \Omega$, we can repeat the above reasoning for $\widetilde{Q}_\rho$ in place of Q_ρ. Accordingly, we define $\Gamma_i := \mathrm{graph}(u_i) \cap \widetilde{Q}_\rho$ and T_i as the connected component of $\overline{S_{L_i}} \setminus E$ containing Γ_i, where now $S_{L_i} = (\widetilde{Q}_\rho)_{(1+6\eta)\rho} \cap (L_i)_{3\eta\rho}$. Then (2.37) (ii) can be proved along similar lines to (2.37) (i) above, by using $\widetilde{Q}_\rho \subset Q_\rho \cup Q'_\rho$. In the same way, we obtain (2.75) in this case. We now observe that (2.75) for good and bad planes yields (2.38). In particular, we also note that (2.66) and (2.73) imply

$$\mathcal{H}^2_\varphi(\mathrm{graph}(u_i) \cap (Q_\rho \cup Q'_\rho)) \geq \frac{1}{C}\rho^2, \tag{2.76}$$

provided that $\eta_0 > 0$ is chosen sufficiently small.

Next we confirm (2.39). To this end, we exemplarily apply the construction for the neighboring cube $Q'_\rho = Q_\rho - \rho e_1$. By Lemma 2.16 and Remark 2.18 (which are applicable by (2.36) and the fact that $Q'_{8\rho} \subset \Omega$) we find planes $L'_j \subset \mathbb{R}^3$, open sets U'_j in L'_j, and functions u'_j such that (2.67)–(2.69) hold. Given the θ-bad plane L_i with corresponding $\mathrm{graph}(u_i)$ for the original cube Q_ρ considered above, in view of (2.67) applied for both Q_ρ and Q'_ρ, and by using (2.76), we observe that there exists a unique function u'_j such that $\mathrm{graph}(u'_j) \cap \mathrm{graph}(u_i) \cap (Q_\rho \cup Q'_\rho) \neq \emptyset$. (In fact, since $\partial E \cap \Omega$ is a regular manifold with boundary only in $\partial\Omega$ and $\overline{Q_{12\rho}} \subset \Omega$, different graphs cannot intersect and the graphs of the functions in the above representation are unique.) Then we observe that one could replace $\mathrm{graph}(u_i)$ and $\mathrm{graph}(u'_j)$ in Lemma 2.16 applied on Q_ρ and Q'_ρ, respectively, by the union $\mathrm{graph}(u_i) \cup \mathrm{graph}(u'_j)$ which can again be understood as the graph of a function defined on the plane L_i. This shows that the objects Γ'_j and T'_j for L'_j can be chosen identical to Γ_i and T_i, i.e., the sets can indeed be constructed such that (2.39) is ensured.

Step 4 (Presence of pimples). Now we argue how to reduce the case of existence of pimples to the case of nonexistence of pimples. As a preparation, we first show that for every pimple $P_j \subset \partial E$ such that $P_j \cap Q_{3\rho} \neq \emptyset$ there exists $i \in \{1, \dots, M\}$ such that $P_j \cap \mathrm{graph}(u_i) \neq \emptyset$. In fact, suppose by contradiction that this was not the case. Due to the fact that Lemma 2.16 guarantees that $P_k \cap P_j = \emptyset$ for all $k \neq j$, we would get that

P_j is a compact manifold without boundary. Thus, applying [84, Lemma 1.1] we get

$$\mathcal{H}^2(P_j) \le (\mathrm{diam}(P_j))^2 \int_{P_j} |\boldsymbol{H}|^2 \,\mathrm{d}\mathcal{H}^2,$$

where $\boldsymbol{H}$ denotes the mean curvature. As the estimate clearly still holds with $\boldsymbol{A}$ in place of $\boldsymbol{H}$ up to a factor of 2, we get, along with Hölder's inequality for $q/2 \ge 1$, (2.1), (2.36), and (2.68), that

$$\begin{aligned}\mathcal{H}^2(P_j) &\le 2(\mathrm{diam}(P_j))^2 \int_{P_j} |\boldsymbol{A}|^2 \,\mathrm{d}\mathcal{H}^2 \le 2\eta^2\rho^2(\mathcal{H}^2(P_j))^{1-2/q}\Big(\int_{P_j} |\boldsymbol{A}|^q \,\mathrm{d}\mathcal{H}^2\Big)^{2/q}\\ &\le 2\eta^2\rho^2(\mathcal{H}^2(P_j))^{1-2/q}(\Lambda\gamma^{-1}\rho^2)^{2/q}.\end{aligned}$$

Simplifying the above formula and using the assumption $\rho \le \eta^7\gamma^{1/q}$, we have

$$\mathcal{H}^2(P_j) \le 2^{q/2}\Lambda\gamma^{-1}\rho^2\rho^q\eta^q \le 2^{q/2}\Lambda\eta^{8q}\rho^2 < c_0\rho^2,$$

where the last step follows from the fact that $2^{q/2}\Lambda\eta_0^{8q} \le 2^{q/2}\Lambda\eta_0 < c_0$; see the beginning of the proof for our choice of η_0. By Lemma 2.12 applied to P_j we would then obtain the estimate $\mathcal{F}^{\varphi,\gamma,q}_{\mathrm{surf}}(E; Q_{8\rho}) \ge \mathcal{F}^{\varphi,\gamma,q}_{\mathrm{surf}}(P_j) > \Lambda\rho^2$, where we used that $P_j \subset \partial E \cap Q_{8\rho}$, which follows from (2.68). This is a contradiction. Therefore, for all $j \in \{1,\dots,N\}$ there exists an index $i \in \{1,\dots,M\}$ such that $P_j \cap \mathrm{graph}(u_i) \ne \emptyset$.

Now, again omitting the indices for simplicity, we consider a plane L and a function $u\colon \overline{U} \subset L \to L^\perp$ satisfying (2.69), in particular $\|u\|_\infty \le \eta\rho$. Here, U is of the form $U = U^0 \setminus \bigcup_k d_k$, where U^0 is a simply connected subdomain of L and $(d_k)_k$ are pairwise disjoint closed disks in L, which do not intersect ∂U^0. If a pimple P touches $\mathrm{graph}(u)$, it can be covered by a cube that also touches $\mathrm{graph}(u)$, has normal ν_L to one of its faces (the orientation of the others being irrelevant), and sidelength $\mathrm{diam}(P)$. Due to (2.68), performing this construction for every pimple, the additional surface introduced by the cubes is bounded by $C\eta^2\rho^2$ for an absolute constant $C > 0$. Furthermore, by this procedure we obtain a piecewise smooth function $\tilde{u}\colon U^0 \subset L \to L^\perp$ such that $\|\tilde{u}\|_\infty \le \|u\|_\infty + \max_{j=1}^N \mathrm{diam}(P_j) \le 2\eta\rho$, i.e., (2.69) holds true, where (the classical gradient) $\nabla\tilde{u}$ is well defined up to a set of $\mathcal{H}^1$-measure zero. Additionally, due to the diameter bound on the cubes, see (2.68), we have

$$\mathcal{H}^2(\mathrm{graph}(\tilde{u}) \cap Q_\rho) \le \mathcal{H}^2(\mathrm{graph}(u) \cap Q_\rho) + C\eta^2\rho^2.$$

Now Steps 2 and 3 can be performed for the function $\tilde{u}$ instead of the function u in order to conclude the proof. ■

Remark 2.22 (Obstacles in higher dimensions). We close this section by commenting on the current obstacles to generalizing our results to higher dimensions. The two essential ingredients depending crucially on the dimension are Lemmata 2.12 and 2.16, whereas the rest of our proof strategy can be carried out with very minor modifications. Lemma 2.16

can in some sense be generalized to any dimension $d \geq 2$ in the spirit of *ε-regularity results*, with respect to the L^q-norm of the second fundamental form, but for $q > d - 1$. The result is due to Hutchinson, see [2, pp. 281–306], in particular Theorem 3.7 on page 295, as well as [55], and it is a *graphical representation* rather than an *approximation* result, i.e., the condition $q > d - 1$ excludes the presence of pimples. As we saw in Lemma 2.15, for $d = 2$ this graphical representation can easily be obtained for every $q \geq 1$, while for $d = 3$ Simon's lemma also handles the case $1 \leq q \leq 2$, modulo the presence of small pimples. For $d > 3$, it would be interesting to investigate to what extent Simon's lemma can be generalized for $q = d - 1$.

The other obstacle to generalizing our result to higher dimensions, especially for the critical case $q = d - 1$, is Lemma 2.12. As in the statement of Lemma 2.14, the main question is the validity of the implication that

$$\mathcal{H}^{d-1}(\Sigma \cap B_R) \geq c_0 R^{d-1}$$

for every connected, regular $(d-1)$-dimensional hypersurface Σ in $\mathbb{R}^d$ with $\mathcal{H}^{d-2}(\partial\Sigma \cap \overline{B_R}) = 0$, and

$$\int_{\Sigma \cap B_R} |\boldsymbol{A}|^{d-2} \, \mathrm{d}\mathcal{H}^{d-1} < \alpha_0 R, \quad \Sigma \cap \partial B_R \neq \emptyset, \quad \text{and} \quad \Sigma \cap \partial B_{\mu R} \neq \emptyset,$$

for suitable $\alpha_0 = \alpha_0(d, \mu) > 0$ and $c_0 = c_0(d, \mu) > 0$, $\mu \in (0, 1)$. In fact, this would allow us to repeat the proof of Lemma 2.12 for $q \geq d - 1$. Whereas the above implication holds true in $d = 2$ and $d = 3$, to the best of our knowledge it is an open question for $d > 3$. For related results in higher dimensions, although not sufficient for our purposes, we refer to [86, Theorem 1.1] and [68, Theorem A].

3. Applications

This section is devoted to applications of our rigidity result. We identify effective linearized models of nonlinear elastic energies in the small-strain limit in two settings, namely for a model with material voids in elastically stressed solids and for epitaxially strained elastic thin films. In the following, for $d = 2, 3$ we let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and $W : \mathbb{R}^{d \times d} \to [0, +\infty)$ be a frame-indifferent stored elastic energy density with the usual assumptions in nonlinear elasticity. Altogether, we suppose that W satisfies the following assumptions:

(i) frame indifference: $W(RF) = W(F)$ for all $R \in SO(d)$, $F \in \mathbb{R}^{d \times d}$,

(ii) single energy-well structure: $\{W = 0\} = SO(d)$,

(iii) regularity: $W \in C^3$ in a neighborhood of $SO(d)$, (3.1)

(iv) coercivity: there exists $c > 0$ such that for all $F \in \mathbb{R}^{d \times d}$ it holds that

$$W(F) \geq c \, \mathrm{dist}^2(F, SO(d)).$$

Notice that the above assumptions particularly imply that $DW(\mathrm{Id}) = 0$. The general approach in linearization results in many different settings (see, e.g., [1, 11, 30, 45, 46, 73, 80, 81]) is to consider sequences of deformations $(y_\delta)_{\delta>0}$ with small elastic energy, more precisely,

$$\sup_{\delta>0} \delta^{-2} \int_\Omega W(\nabla y_\delta)\,\mathrm{d}x < +\infty,$$

and to pass to the small-strain limit as $\delta \to 0$, in terms of *rescaled displacement fields*, i.e., mappings

$$u_\delta = \frac{1}{\delta}(y_\delta - \mathrm{id}). \tag{3.2}$$

These maps measure the distance of the deformations from the identity, rescaled by the typical strain $\delta > 0$. This yields a linearization of the elastic energy, which can be expressed in terms of the quadratic form $\mathcal{Q}\colon \mathbb{R}^{d\times d} \to [0, +\infty)$ defined by

$$\mathcal{Q}(F) := D^2W(\mathrm{Id})F : F \quad \text{for all } F \in \mathbb{R}^{d\times d}. \tag{3.3}$$

In view of (3.1), $\mathcal{Q}$ is positive definite on $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ and vanishes on $\mathbb{R}^{d\times d}_{\mathrm{skew}}$. We will consider models containing surface energies with an additional curvature regularization as indicated in (2.1), where we choose a sequence of scaling parameters $(\gamma_\delta)_{\delta>0} \subset (0, +\infty)$ for which we require

$$\gamma_\delta \to 0 \quad \text{and} \quad \liminf_{\delta\to 0}(\delta^{-\frac{q}{3d}}\gamma_\delta) = +\infty. \tag{3.4}$$

In fact, this allows us to define a sequence $(\kappa_\delta)_{\delta>0} \subset (0, +\infty)$ satisfying

$$\delta\kappa_\delta^3 \to 0, \quad \gamma_\delta^{d/q}\kappa_\delta \to \infty \quad \text{as } \delta \to 0, \tag{3.5}$$

which will play a pivotal role in the linearization procedure. In the following, we will focus on a curvature regularization in terms of the second fundamental form $\boldsymbol{A}$. Under certain assumptions however, in the case $d = 3$, $q = 2$, $\boldsymbol{A}$ can be replaced by the mean curvature $\boldsymbol{H}$. We refer to Corollaries 3.5 and 3.9 for details in this direction.

In our applications, it will turn out that limiting mappings lie in the space of generalized special functions of bounded deformation $GSBD^2(\Omega)$. For basic properties of $GSBD^2(\Omega)$, we refer to [29] and Appendix A.4 below. In particular, for $u \in GSBD^2(\Omega)$, we will denote by $e(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ the approximate symmetric differential and by J_u the jump set of u with measure-theoretical normal ν_u. Moreover, by $L^0(\Omega; \mathbb{R}^d)$ we denote the space of $\mathcal{L}^d$-measurable mappings $v\colon \Omega \to \mathbb{R}^d$, endowed with the topology of the convergence in measure. For any $s \in [0, 1]$ and any $E \in \mathfrak{M}(\Omega)$, E^s denotes the set of points with d-dimensional density s with respect to E. By $\partial^* E$ we indicate the essential boundary of E; see [4, Definition 3.60].

We now present our two applications in Sections 3.1–3.2. The proofs of the results are deferred to Sections 3.3–3.4.

3.1. Material voids in elastically stressed solids

We study boundary value problems for elastically stressed solids with voids. We suppose that the boundary data are imposed on an open subset $\partial_D\Omega \subset \partial\Omega$ and are close to the identity. To this end, let $u_0 \in W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$, $d = 2, 3$, and for $\delta > 0$ define $y_0^\delta :=$ id $+ \delta u_0$. Further, let φ be a norm, $q \in [d-1, +\infty)$, and $(\gamma_\delta)_{\delta>0}$ as in (3.4). Then for the density $W: \mathbb{R}^{d\times d} \to [0,\infty)$ introduced in (3.1), we let $F_\delta: L^0(\Omega;\mathbb{R}^d) \times \mathfrak{M}(\Omega) \to [0,+\infty]$ be the functional defined by

$$F_\delta(y,E) := \frac{1}{\delta^2}\int_{\Omega\setminus\bar{E}} W(\nabla y)\,\mathrm{d}x + \int_{\partial E\cap\Omega} \varphi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} + \gamma_\delta \int_{\partial E\cap\Omega} |\boldsymbol{A}|^q\,\mathrm{d}\mathcal{H}^{d-1}, \quad (3.6)$$

if $E \in \mathcal{A}_{\mathrm{reg}}(\Omega)$, $\bar{E} \cap \partial_D\Omega = \emptyset$, $y|_{\Omega\setminus\bar{E}} \in H^1(\Omega\setminus\bar{E};\mathbb{R}^d)$, $y|_E =$ id, and $\mathrm{tr}(y) = \mathrm{tr}(y_0^\delta)$ on $\partial_D\Omega$, and $F_\delta(y,E) = +\infty$ otherwise. Here, ν_E again denotes the outer unit normal to ∂E. We emphasize that the energy is determined by E and the values of y on $\Omega\setminus\bar{E}$. The condition $y|_E =$ id is for definiteness only. The relaxation of this model without the curvature regularization term has been studied in [10, 79]. Here, instead, we are interested in an effective description in the small-strain limit $\delta \to 0$, in terms of displacement fields defined in (3.2). From now on, we write

$$\mathcal{F}_\delta(u,E) := F_\delta(\mathrm{id} + \delta u, E)$$

for notational convenience. We start with a compactness result which fundamentally relies on Theorem 2.1. Note that in what follows, the sets ω_u^δ, ω_u serve a totally different purpose, and should not be confused with the set ω_u^ρ in Section 2; see for instance (2.65).

Proposition 3.1 (Compactness, void case). *For every sequence of pairs $(u_\delta, E_\delta)_{\delta>0}$ with*

$$M := \sup_{\delta>0} \mathcal{F}_\delta(u_\delta, E_\delta) < +\infty, \quad (3.7)$$

there exist a subsequence (not relabeled), $u \in \mathrm{GSBD}^2(\Omega)$, sets of finite perimeter $E \in \mathfrak{M}(\Omega)$, $(E_\delta^*)_{\delta>0} \subset \mathfrak{M}(\mathbb{R}^d)$ with $E_\delta \subset E_\delta^*$, and sets $\omega_u, (\omega_u^\delta)_{\delta>0} \subset \mathfrak{M}(\Omega)$ such that $u \equiv 0$ on $E \cup \omega_u$,

$$\mathcal{H}^{d-1}(\partial^*\omega_u) + \sup_{\delta>0} \mathcal{H}^{d-1}(\partial^*\omega_u^\delta) \le C_M$$

for a constant $C_M > 0$ depending only on M, and as $\delta \to 0$,

$$\begin{aligned} &\text{(i)} \quad u_\delta \to u \text{ in measure on } \Omega\setminus\omega_u,\\ &\text{(ii)} \quad \chi_{\Omega\setminus(E_\delta^*\cup\omega_u^\delta)} e(u_\delta) \rightharpoonup \chi_{\Omega\setminus(E\cup\omega_u)} e(u) \text{ weakly in } L^2_{\mathrm{loc}}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}),\\ &\text{(iii)} \quad \mathcal{L}^d(\{|\nabla u_\delta| > \kappa_\delta\}\setminus\omega_u) \to 0,\\ &\text{(iv)} \quad \liminf_{\delta\to0} \int_{\partial E_\delta^*\cap\Omega} \varphi(\nu_{E_\delta^*})\,\mathrm{d}\mathcal{H}^{d-1} \le \liminf_{\delta\to0} \mathcal{F}_{\mathrm{surf}}^{\varphi,\gamma_\delta,q}(E_\delta),\\ &\text{(v)} \quad \lim_{\delta\to0} \mathcal{L}^d(\omega_u^\delta \triangle \omega_u) = \lim_{\delta\to0} \mathcal{L}^d(E_\delta^*\setminus E_\delta) = \lim_{\delta\to0} \mathcal{L}^d(E_\delta \triangle E) = 0, \end{aligned} \quad (3.8)$$

where κ_δ is defined in (3.5) and $\mathcal{F}_{\mathrm{surf}}^{\varphi,\gamma_\delta,q}$ in (2.1).

In view of the above compactness result, we introduce the following notion of convergence on function-set pairs.

Definition 3.2. We say that a sequence $(u_\delta, E_\delta)_{\delta>0} \subset L^0(\Omega;\mathbb{R}^d) \times \mathfrak{M}(\Omega)$ converges to a pair $(u, E) \in L^0(\Omega;\mathbb{R}^d) \times \mathfrak{M}(\Omega)$ in the τ-sense and write $(u_\delta, E_\delta) \overset{\tau}{\to} (u, E)$ if and only if there exists a set $\omega_u \in \mathfrak{M}(\Omega)$ such that $\chi_{E_\delta} \to \chi_E$ in $L^1(\Omega)$, $u_\delta \to u$ in measure on $\Omega \setminus \omega_u$, and $u \equiv 0$ on $E \cup \omega_u$.

The compactness result is nonstandard in the sense that the behavior of the sequence $(u_\delta)_{\delta>0}$ on ω_u cannot be controlled. This set is related to the fact that $\Omega \setminus \overline{E_\delta}$ might be disconnected into various connected components $(P_j^\delta)_j$ by E_δ, and on the sets not intersecting $\partial_D\Omega$ the corresponding rotations R_j^δ, obtained from (2.3), cannot be controlled. It is however essential that $|R_j^\delta - \mathrm{Id}|$ is at most of order δ, as otherwise u_δ defined in (3.2) blows up on P_j^δ. In this sense, roughly speaking, ω_u^δ consists of the components $(P_j^\delta)_j$ not intersecting $\partial_D\Omega$. Moreover, the sets E_δ need to be replaced by the slightly larger sets E_δ^* corresponding to the sets in (2.2).

We now introduce the linearized model studied in [27]. Given $u \in \mathrm{GSBD}^2(\Omega)$ and $E \in \mathfrak{M}(\Omega)$ with $\mathcal{H}^{d-1}(\partial^* E) < +\infty$, we first define the *boundary energy term* by

$$\mathcal{F}^{\mathrm{bdry}}(u, E) := \int_{\partial^* E \cap \partial_D \Omega} \varphi(\nu_E)\, \mathrm{d}\mathcal{H}^{d-1} + \int_{\substack{\{\mathrm{tr}(u) \neq \mathrm{tr}(u_0)\} \\ \cap (\partial_D\Omega \setminus \partial^* E)}} 2\varphi(\nu_\Omega)\, \mathrm{d}\mathcal{H}^{d-1}, \tag{3.9}$$

which is nontrivial if the void goes up to the Dirichlet part of the boundary or the mapping u does not satisfy the imposed boundary conditions. Here, ν_Ω denotes the outer unit normal to $\partial\Omega$, and $\mathrm{tr}(u)$ indicates the trace of u at $\partial\Omega$, which is well defined for functions in $\mathrm{GSBD}^2(\Omega)$; see Appendix A.4. Recalling the definition of $\mathcal{Q}$ in (3.3), we introduce the *effective limiting energy* $\mathcal{F}_0\colon L^0(\Omega;\mathbb{R}^d) \times \mathfrak{M}(\Omega) \to [0, +\infty]$ as

$$\begin{aligned}\mathcal{F}_0(u, E) &:= \frac{1}{2}\int_{\Omega\setminus E} \mathcal{Q}(e(u))\, \mathrm{d}x + \int_{\partial^* E \cap \Omega} \varphi(\nu_E)\, \mathrm{d}\mathcal{H}^{d-1} \\ &\quad + \int_{J_u \setminus \partial^* E} 2\varphi(\nu_u)\, \mathrm{d}\mathcal{H}^{d-1} + \mathcal{F}^{\mathrm{bdry}}(u, E)\end{aligned} \tag{3.10}$$

if $\mathcal{H}^{d-1}(\partial^* E) < +\infty$ and $u = \chi_{\Omega\setminus E} u \in \mathrm{GSBD}^2(\Omega)$, and $\mathcal{F}_0(u, E) = +\infty$ otherwise.

We now address that (3.10) can be identified as the Γ-limit of (3.6) for $\delta \to 0$. In fact, the functional (3.10) is effective in two respects: first, in the small-strain limit the density of nonlinear elasticity is replaced by its linearized version $\mathcal{Q}$. Secondly, the fact that $\mathcal{F}_\delta$ is not lower semicontinuous in the variable E with respect to L^1-convergence of sets is remedied by a suitable relaxation. Indeed, in the limiting process, the voids E may collapse into a discontinuity of the displacement u. In particular, this phenomenon is taken into account in the relaxed functional since collapsed surfaces are counted twice in the surface energy. Eventually, we point out that, due to the fact that $\gamma_\delta \to 0$ as $\delta \to 0$, the curvature regularization of the nonlinear energy $\mathcal{F}_\delta$ does not affect the linearized limit.

For the Γ-limsup inequality, more precisely for the application of a density result in $GSBD^2$, see [27, Lemma 5.7], we make the following geometrical assumption on the Dirichlet boundary $\partial_D\Omega$: there exists a decomposition $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega \cup N$ with

$$\partial_D\Omega, \partial_N\Omega \text{ relatively open}, \quad \mathcal{H}^{d-1}(N) = 0, \quad \partial_D\Omega \cap \partial_N\Omega = \emptyset, \quad \partial(\partial_D\Omega) = \partial(\partial_N\Omega),$$

where the outermost boundary has to be understood in the relative sense, and there exist $\bar{\sigma} > 0$ small enough and $x_0 \in \mathbb{R}^d$ such that for all $\sigma \in (0, \bar{\sigma})$ it holds that

$$O_{\sigma,x_0}(\partial_D\Omega) \subset \Omega,$$

where $O_{\sigma,x_0}(x) := x_0 + (1-\sigma)(x - x_0)$. Recall the convergence τ in Definition 3.2.

Theorem 3.3 (Γ-convergence, void case). *Under the above assumptions, as $\delta \to 0$, we have that the sequence of functionals $(\mathcal{F}_\delta)_{\delta>0}$ Γ-converges to $\mathcal{F}_0$ with respect to the convergence τ.*

Remark 3.4 (Volume of voids). We proceed with two comments on the result:
(i) In the previous result, if $\mathcal{L}^d(E) > 0$, then for any $(u, E) \in L^0(\Omega;\mathbb{R}^d)\times\mathfrak{M}(\Omega)$ there exists a recovery sequence $(u_\delta, E_\delta)_{\delta>0} \subset L^0(\Omega;\mathbb{R}^d)\times\mathfrak{M}(\Omega)$ such that $\mathcal{L}^d(E_\delta) = \mathcal{L}^d(E)$ for all $\delta > 0$. This shows that it is possible to incorporate a volume constraint on E in the Γ-convergence result.
(ii) If we impose the condition $\mathcal{L}^d(E_\delta) \to 0$ along the sequence, we obtain $E = \emptyset$, and the limiting model corresponds to an (anisotropic) *Griffith energy of brittle fracture*.

We address an alternative formulation with the mean curvature in place of the second fundamental form, in the case $d = 3$, $q = 2$.

Corollary 3.5 (Mean curvature regularization). *We consider (3.6) with $|\boldsymbol{H}|^2$ in place of $|\boldsymbol{A}|^2$ when $d = 3$, $q = 2$. We suppose that for $\mathcal{F}_\delta$, only sets E satisfying $E \subset\subset \Omega$ and $-4\pi\chi(\partial E) \le \lambda_\delta\gamma_\delta^{-1}$ for some $\lambda_\delta \to 0$ are admissible, where $\chi(\partial E)$ indicates the Euler characteristic of ∂E. (For instance, this holds if ∂E consists of connected components topologically equivalent to the sphere $\mathbb{S}^2$.) Then the statements of Proposition 3.1 and Theorem 3.3 hold.*

3.2. Energies on domains with a subgraph constraint: Epitaxially strained films

We now address a second application, namely deformations of an elastic material in a domain which is the subgraph of an unknown nonnegative function h. Assuming that h is defined on a smooth bounded domain $\omega \subset \mathbb{R}^{d-1}$, $d = 2, 3$, deformations y will be defined on the subgraph

$$\Omega_h^+ := \{x \in \omega \times \mathbb{R} : 0 < x_d < h(x')\},$$

where here and in the following we use the notation $x = (x', x_d)$ for $x \in \mathbb{R}^d$. To model Dirichlet boundary data on the flat surface $\omega \times \{0\}$, we will suppose that mappings are extended to the set $\Omega_h := \{x \in \omega \times \mathbb{R} : -1 < x_d < h(x')\}$ and satisfy $y = y_0^\delta := \mathrm{id} +$

δu_0 on $\omega\times(-1,0]$ for a given function $u_0 \in W^{1,\infty}(\omega\times(-1,0];\mathbb{R}^d)$. In the application to epitaxially strained films, y_0^δ represents the interaction with the substrate and h indicates the profile of the free surface of the film. We refer to [7, 19, 27] for a thorough description of the model and a detailed account of the available literature.

For convenience, we introduce the reference domain $\Omega := \omega\times(-1, M+1)$ for some $M > 0$. For $q \in [d-1, +\infty)$, γ_δ as in (3.4), and the density $W\colon \mathbb{R}^{d\times d} \to [0,\infty)$ introduced in (3.1), we define the energy $G_\delta\colon L^0(\Omega;\mathbb{R}^d) \times L^1(\omega;[0,M]) \to [0,+\infty]$ by

$$G_\delta(y,h) := \frac{1}{\delta^2}\int_{\Omega_h^+} W(\nabla y(x))\,\mathrm{d}x + \mathcal{H}^{d-1}(\partial\Omega_h \cap \Omega)$$

$$+\,\gamma_\delta \int_{\partial\Omega_h\cap\Omega} |\boldsymbol{A}|^q\,\mathrm{d}\mathcal{H}^{d-1}, \qquad (3.11)$$

if $h \in C^2(\omega;[0,M])$, $y|_{\Omega_h} \in H^1(\Omega_h;\mathbb{R}^d)$, $y = \mathrm{id}$ in $\Omega \setminus \overline{\Omega_h}$, $y = y_0^\delta$ in $\omega\times(-1,0]$, and $G_\delta(y,h) := +\infty$ otherwise. We emphasize that the two surface terms only contribute in terms of the upper surface $\partial\Omega_h \cap \Omega$ of the film, which exactly corresponds to the graph of h. In other words, the first surface term is exactly $\int_\omega \sqrt{1+|\nabla h(x')|^2}\,\mathrm{d}x'$. On the other hand, the curvature term can be written as $\int_\omega |\nabla^2 h(x')|^q (1+|\nabla h(x')|^2)^{\frac{1-q}{2}}\,\mathrm{d}x'$. Note that this model can be seen as a special case of (3.6) when we choose $E = \Omega\setminus\overline{\Omega_h}$. As in Section 3.1, the assumption $y = \mathrm{id}$ in $\Omega\setminus\overline{\Omega_h}$ is for definiteness only.

The relaxation of this model has been studied in [19]. Notice that, in contrast to [7, 19], here we assume that the functions h are equibounded by a value M: this is for technical reasons only and is justified from a mechanical point of view, as indeed other physical effects come into play for very high crystal profiles. In the present work, we address the effective behavior of the model in the small-strain limit $\delta \to 0$, again in terms of displacement fields as defined in (3.2). From now on, we write

$$\mathcal{G}_\delta(u,h) := G_\delta(\mathrm{id} + \delta u, h)$$

for notational convenience. Based on Theorem 2.1, we obtain the following compactness result.

Proposition 3.6 (Compactness, graph case). *For any sequence of pairs* $(u_\delta, h_\delta)_{\delta>0}$ *with*

$$K := \sup_{\delta>0} \mathcal{G}_\delta(u_\delta, h_\delta) < +\infty,$$

there exist a subsequence (not relabeled), sets of finite perimeter $(E_\delta^*)_{\delta>0} \subset \mathfrak{M}(\Omega)$ *with* $\Omega\setminus\overline{\Omega_{h_\delta}} \subset E_\delta^*$, *as well as* $(\omega_u^\delta)_{\delta>0} \subset \mathfrak{M}(\Omega)$, *and functions* $u \in \mathrm{GSBD}^2(\Omega)$, $h \in \mathrm{BV}(\omega;[0,M])$ *with* $u = \chi_{\Omega_h} u$ *and* $u = u_0$ *on* $\omega\times(-1,0]$ *such that*

$$\sup_{\delta>0} \mathcal{H}^{d-1}(\partial^*\omega_u^\delta) \le C_K$$

for a constant $C_K > 0$ depending only on K, and as $\delta \to 0$,

$$
\begin{aligned}
&\text{(i)}\ \ u_\delta \to u \text{ in measure on } \Omega,\\
&\text{(ii)}\ \ \chi_{\Omega\setminus(E_\delta^*\cup\omega_u^\delta)}e(u_\delta) \rightharpoonup e(u) = \chi_{\Omega_h}e(u) \text{ weakly in } L^2_{\mathrm{loc}}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}),\\
&\text{(iii)}\ \ \mathcal{L}^d(\{|\nabla u_\delta| > \kappa_\delta\}) \to 0,\\
&\text{(iv)}\ \ \liminf_{\delta\to 0}\mathcal{H}^{d-1}(\partial E_\delta^* \cap \Omega) \le \liminf_{\delta\to 0}\mathcal{F}^{q,\delta}_{\mathrm{surf}}(E_\delta),\\
&\text{(v)}\ \ \lim_{\delta\to 0}\|h_\delta - h\|_{L^1(\omega)} = \lim_{\delta\to 0}\mathcal{L}^d(\omega_u^\delta) = \lim_{\delta\to 0}\mathcal{L}^d(E_\delta^*\cap\Omega_{h_\delta}) = 0,
\end{aligned}
\tag{3.12}
$$

where κ_δ is defined in (3.5), and $\mathcal{F}^{q,\delta}_{\mathrm{surf}}$ in (2.1) for $\varphi \equiv 1$ and $\gamma = \gamma_\delta$.

We note that in contrast to Proposition 3.1, no exceptional set ω_u is needed here. This is due to the imposed graph constraint $\partial\Omega^+_{h_\delta} \cap \Omega$ which excludes the creation of components disconnected from the substrate $\omega \times (-1, 0]$. Indeed, in this setting a stronger compactness result holds; see [27, Theorems 2.4 and 2.5], and in particular Step 3 in the proof of the lower inequality for Theorem 2.4, which is given in Section 6 therein. Note that nevertheless the sets $(\omega_u^\delta)_{\delta>0}$ obtained by an application of Proposition 3.1 are still present in the statement.

We now introduce the effective model studied in [27]. Recalling the definition of $\mathcal{Q}$ in (3.3), we introduce $\mathcal{G}_0\colon L^0(\Omega;\mathbb{R}^d) \times L^1(\omega;[0,M]) \to [0,+\infty]$ by

$$
\mathcal{G}_0(u,h) := \frac{1}{2}\int_{\Omega_h^+}\mathcal{Q}(e(u))\,\mathrm{d}x + \mathcal{H}^{d-1}(\partial^*\Omega_h \cap \Omega) + 2\mathcal{H}^{d-1}(J_u' \cap \Omega_h^1) \tag{3.13}
$$

if $u = \chi_{\Omega_h}u \in GSBD^2(\Omega)$, $u = u_0$ in $\omega\times(-1,0]$, $h \in BV(\omega;[0,M])$, and $\mathcal{G}_0(u,h) = +\infty$ otherwise. Here, $e(u) = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}})$ again denotes the symmetric part of the (approximate) gradient of $u \in GSBD^2(\Omega)$, Ω_h^1 denotes the set of points with density 1, and

$$
J_u' := \big\{(x', x_d + t) : x \in J_u,\ t \ge 0\big\}. \tag{3.14}
$$

As for the functional (3.10), the energy (3.13) is effective in the sense that the elastic energy density W is replaced by the linearized density $\mathcal{Q}$ and the model accounts for “vertical cuts” $J_u' \cap \Omega_h^1$ (see [38]) which may appear throughout the relaxation process. Similarly to the corresponding term in (3.10), this part is counted twice in the energy. The set $(\partial^*\Omega_h \cap \Omega) \cup (J_u' \cap \Omega_h^1)$ can be interpreted as a “generalized interface”; cf. Figure 6 for a two-dimensional section of a possible limiting Ω_h. As before, due to the fact that $\gamma_\delta \to 0$ as $\delta \to 0$, the curvature regularization of the nonlinear energy $\mathcal{G}_\delta$ does not affect the linearized limit.

We work under the additional assumption that $\omega \subset \mathbb{R}^{d-1}$ is uniformly star shaped with respect to the origin, i.e.,

$$
tx \in \omega \quad \text{for all } t \in [0,1),\ x \in \partial\omega.
$$

This condition, however, is only of a technical nature and could be dropped at the expense of more elaborate estimates; see also [19, 27]. We obtain the following result.

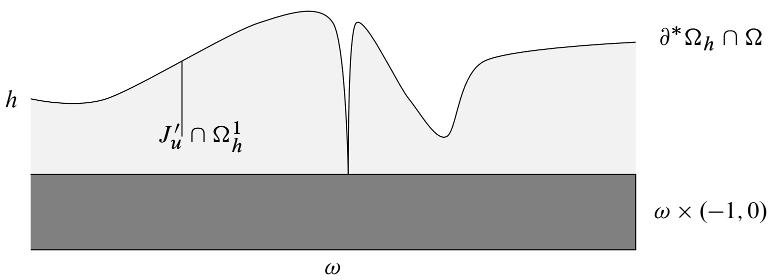

Figure 6. Possible limiting set Ω_h.

Theorem 3.7 (Γ-convergence, graph case). *Under the above assumptions, as $\delta \to 0$, we have that the sequence of functionals $(\mathcal{G}_\delta)_{\delta>0}$ Γ-converges to the functional $\mathcal{G}_0$ with respect to the topology of $L^0(\Omega;\mathbb{R}^d)\times L^1(\omega;[0,M])$.*

Remark 3.8 (Volume constraint). We note that throughout the linearization process one could consider an additional volume constraint on the film, i.e., $\mathcal{L}^d(\Omega_h^+) = \int_\omega h(x')\,\mathrm{d}x'$ is fixed.

We close this section with a result for an alternative setting where in (3.11) the second fundamental form is replaced by the mean curvature, again in the case $d = 3$, $q = 2$.

Corollary 3.9 (Mean curvature regularization). *We consider (3.11) with $|\boldsymbol{H}|^2$ in place of $|\boldsymbol{A}|^2$ when $d = 3$, $q = 2$. We suppose that for $\mathcal{G}_\delta$ only functions h are admissible such that $\Gamma_h := \partial\Omega_h \cap \Omega$ satisfies that $\partial\Gamma_h$ is C^2 and that*

$$\int_{\partial\Gamma_h} \kappa_{h,g}\,\mathrm{d}\mathcal{H}^1 \le \lambda_\delta \gamma_\delta^{-1}$$

for some $\lambda_\delta \to 0$ as $\delta \to 0$, where $\kappa_{h,g}$ denotes the geodesic curvature of $\partial\Gamma_h$. Then the statements of Proposition 3.6 and Theorem 3.7 hold.

The next subsections are devoted to the proofs announced in this section. As the proofs for both applications are similar, we proceed simultaneously. We first address the compactness statements in Section 3.3, and afterwards the Γ-convergence results in Section 3.4.

3.3. Compactness results

We start with the proof of Proposition 3.1. Afterwards, we present the small adaptions necessary for the proof of Proposition 3.6.

Proof of Proposition 3.1. Consider a sequence $(u_\delta, E_\delta)_{\delta>0}$ satisfying (3.7), i.e.,

$$\sup_{\delta>0} \mathcal{F}_\delta(u_\delta, E_\delta) \le M < +\infty.$$

Hence, $\partial E_\delta \cap \partial_D \Omega = \emptyset$ and, as $\min_{\mathbb{S}^{d-1}} \varphi > 0$, it holds that $\sup_{\delta>0} \mathcal{H}^{d-1}(\partial E_\delta) < +\infty$. Thus, a compactness result for sets of finite perimeter (see [4, Theorem 3.39]) implies that there exists a set of finite perimeter $E \subset \Omega$ with $\mathcal{H}^{d-1}(\partial^* E) < +\infty$ such that $\chi_{E_\delta} \to \chi_E$ in $L^1(\Omega)$, up to a subsequence (not relabeled). This shows the last part of (3.8) (v).

We now proceed with the compactness for the deformations. We start by introducing sets for a suitable formulation of the Dirichlet boundary conditions: choose an open set $V \supset \Omega$ such that V and $V \setminus \overline{\Omega}$ are Lipschitz sets and $V \cap \partial\Omega = \partial_D \Omega$. (Such a choice of V is possible due to the fact that $\partial_D \Omega$ is also Lipschitz, being a relatively open subset of $\partial\Omega$; recall its definition before Theorem 3.3.) Our goal is to apply Theorem 2.1 in the version of Corollary 2.2 for $U := \Omega$ and $U_D := V \setminus \Omega$. To this end, we introduce the functions $\hat{y}_\delta$ as

$$\hat{y}_\delta = \begin{cases} \mathrm{id} + \delta(u_\delta - u_0) & \text{on } U = \Omega, \\ \mathrm{id} & \text{on } U_D = V \setminus \Omega. \end{cases} \tag{3.15}$$

Note that $\hat{y}_\delta$ are Sobolev functions when restricted to $V \setminus \overline{E_\delta}$ since $V \cap \partial\Omega = \partial_D \Omega$ and $\mathrm{tr}(\hat{y}_\delta) = \mathrm{tr}(y_0^\delta - \delta u_0) = \mathrm{id}$ on $\partial_D \Omega$, by the fact that $\mathcal{F}_\delta(u_\delta, E_\delta) < +\infty$. Then, by the triangle inequality, (3.1), and the fact that $\mathcal{F}_\delta(u_\delta, E_\delta) \le M$, we get

$$\int_{V\setminus \bar{E}} \mathrm{dist}^2(\nabla \hat{y}_\delta(x), SO(d))\, \mathrm{d}x \le C'\delta^2 \tag{3.16}$$

for a constant $C' > 0$ depending on M and also on u_0. We want to apply Theorem 2.1 on $(\hat{y}_\delta, E_\delta)$. To this end, in view of the fact that $\gamma_\delta \to 0$ as $\delta \to 0$, see (3.4), and the definition of κ_δ in (3.5), by a suitable diagonal argument we can find a sequence $(\eta_\delta)_{\delta>0}$ with $\eta_\delta \to 0$ and smooth sets $\widetilde{\Omega}_\delta \subset\subset V$ such that, as $\delta \to 0$,

$$\text{(i)}\ \ C_{\eta_\delta} \kappa_\delta^{-2} \gamma_\delta^{-2d/q} \to 0, \qquad \text{(ii)}\ \ \sup_{\delta>0} C_{\eta_\delta} \delta^{1/3} < +\infty, \tag{3.17}$$

$$\text{(i)}\ \ \mathcal{L}^d(V \setminus \widetilde{\Omega}_\delta) \to 0, \qquad \text{(ii)}\ \ \sup_{\delta>0} \mathcal{H}^{d-1}(\partial \widetilde{\Omega}_\delta) < +\infty, \tag{3.18}$$

where C_{η_δ} is the constant in (2.3). We then apply Theorem 2.1 for η_δ and γ_δ, for V in place of Ω, and for $\widetilde{\Omega}_\delta$ in place of $\widetilde{\Omega}$. We use the notation $\mathcal{F}_{\mathrm{surf}}^{\varphi,\gamma_\delta,q}$ introduced in (2.1). Since $E_\delta \subset \Omega$, $V \cap \partial\Omega = \partial_D \Omega$, and $\overline{E_\delta} \cap \partial_D \Omega = \emptyset$, we have $E_\delta \in \mathcal{A}_{\mathrm{reg}}(V)$ and $\mathcal{F}_{\mathrm{surf}}^{\varphi,\gamma_\delta,q}(E_\delta, V) = \mathcal{F}_{\mathrm{surf}}^{\varphi,\gamma_\delta,q}(E_\delta) \le C'$ for every $\delta > 0$. Now, by applying (2.2)–(2.3) and using that $\gamma_\delta \to 0$, $\eta_\delta \to 0$ as $\delta \to 0$, we get that there exist sets $(E_\delta^*)_{\delta>0}$ with $E_\delta \subset E_\delta^* \subset V$, $\partial E_\delta^* \cap V$ is a union of finitely many regular submanifolds for every $\delta > 0$, and

$$\begin{aligned} &\text{(i)}\ \ \lim_{\delta\to 0} \mathcal{L}^d(E_\delta^* \setminus E_\delta) = 0, \\ &\text{(ii)}\ \ \liminf_{\delta\to 0} \int_{\partial E_\delta^* \cap V} \varphi(\nu_{E_\delta^*})\, \mathrm{d}\mathcal{H}^{d-1} \le \liminf_{\delta\to 0} \mathcal{F}_{\mathrm{surf}}^{\varphi,\gamma_\delta,q}(E_\delta), \end{aligned} \tag{3.19}$$

such that for the finitely many connected components of $\widetilde{\Omega}_\delta \setminus E^*_\delta$, denoted by $(\widetilde{\Omega}_j^{\eta_\delta,\gamma_\delta})_j$, there exist corresponding rotations $(R_j^{\eta_\delta,\gamma_\delta})_j \subset \mathrm{SO}(d)$ such that by (3.16),

$$\begin{aligned} &\text{(i)} \quad \sum_j \int_{\widetilde{\Omega}_j^{\eta_\delta,\gamma_\delta}} |\mathrm{sym}((R_j^{\eta_\delta,\gamma_\delta})^{\mathsf{T}}\nabla \hat{y}_\delta - \mathrm{Id})|^2 \,\mathrm{d}x \le C_0 C' \delta^2, \\ &\text{(ii)} \quad \sum_j \int_{\widetilde{\Omega}_j^{\eta_\delta,\gamma_\delta}} |(R_j^{\eta_\delta,\gamma_\delta})^{\mathsf{T}}\nabla \hat{y}_\delta - \mathrm{Id}|^2 \,\mathrm{d}x \le C' C_{\eta_\delta} \gamma_\delta^{-2d/q} \delta^2. \end{aligned} \tag{3.20}$$

In fact, for (3.20) (i) we used that $C_{\eta_\delta}\gamma_\delta^{-5d/q}\delta^2 = (\delta^{-q/3d}\gamma_\delta)^{-5d/q} \cdot C_{\eta_\delta}\delta^{1/3} \to 0$ by (3.4) and (3.17) (ii). In view of Corollary 2.2 and (3.15), we can choose $R_j^{\eta_\delta,\gamma_\delta} = \mathrm{Id}$ whenever we have $\mathcal{L}^d(U_D \cap \widetilde{\Omega}_j^{\eta_\delta,\gamma_\delta}) > 0$. We denote the union of the components with this property by $\Omega_\delta^{\mathrm{good}}$. Note that by its definition, the set $\Omega_\delta^{\mathrm{good}}$ satisfies

$$(U_D \cap \widetilde{\Omega}_\delta) \setminus E^*_\delta \subset \Omega_\delta^{\mathrm{good}} \quad \text{and} \quad \partial \Omega_\delta^{\mathrm{good}} \subset (\partial E^*_\delta \cap V) \cup \partial \widetilde{\Omega}_\delta. \tag{3.21}$$

We introduce the mappings $(v_\delta)_{\delta>0} \in \mathrm{GSBD}^2(V)$ as

$$v_\delta = \begin{cases} u_\delta & \text{on } \Omega_\delta^{\mathrm{good}} \cap \Omega, \\ u_0 & \text{on } \Omega_\delta^{\mathrm{good}} \cap (V \setminus \Omega), \\ 0 & \text{on } E^*_\delta \cup (V \setminus \widetilde{\Omega}_\delta), \\ \dfrac{1}{\delta} e_1 & \text{on } \widetilde{\Omega}_\delta \setminus (\Omega_\delta^{\mathrm{good}} \cup E^*_\delta), \end{cases} \tag{3.22}$$

where e_1 denotes the first coordinate vector; see Figure 7 for the different regions in the definition of v_δ. By (3.15), (3.20), (3.22), the definition of $\Omega_\delta^{\mathrm{good}}$, and the triangle inequality, we find for all $\delta > 0$ that

$$\text{(i)} \quad \|e(v_\delta)\|^2_{L^2(V)} \le C', \qquad \text{(ii)} \quad \|\nabla v_\delta\|^2_{L^2(V)} \le C' C_{\eta_\delta}\gamma_\delta^{-2d/q}, \tag{3.23}$$

where C' depends additionally on u_0. As $J_{v_\delta} \subset (\partial E^*_\delta \cap V) \cup \partial \widetilde{\Omega}_\delta$, (3.18) (ii), (3.19), (3.23), and the fact that $\min_{\mathbb{S}^{d-1}} \varphi > 0$ imply that

$$\sup_{\delta>0} (\|e(v_\delta)\|^2_{L^2(V)} + \mathcal{H}^{d-1}(J_{v_\delta})) < +\infty.$$

By a compactness result in GSBD^2, see Theorem A.3 in Section A.4, letting

$$\omega_u := \big\{x \in V \colon |v_\delta(x)| \to \infty \text{ as } \delta \to 0\big\}, \tag{3.24}$$

by (A.21) (iii), we get that there exists $C_M > 0$ (recall $M > 0$ in (3.7)) such that

$$\mathcal{H}^{d-1}(\partial^* \omega_u \cap V) \le C_M, \tag{3.25}$$

and we find $v \in \mathrm{GSBD}^2(V)$ with $v = 0$ on ω_u such that (again up to a subsequence, not relabeled) v_δ converges in measure to v on $V \setminus \omega_u$. (In the language of [27, Section 3.4],

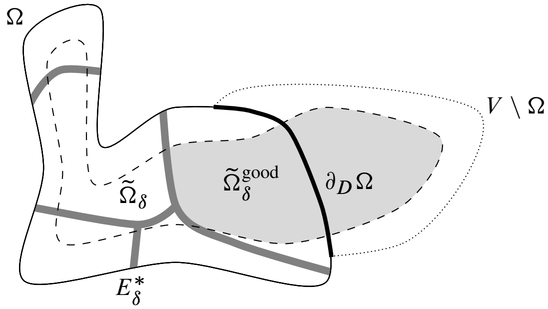

Figure 7. The sets relevant for the definition of v_δ: the thick curve indicates the set $\partial_D \Omega$ and E_δ^* is depicted in gray. The region delimited by the dashed curve is $\widetilde{\Omega}_\delta$. The region enclosed by the dotted curve is $V \setminus \Omega$. The set $\widetilde{\Omega}_\delta^{\mathrm{good}}$ is depicted in light gray.

we say that $v_\delta \to v$ weakly in $\mathrm{GSBD}^2_\infty(V)$.) Moreover, we note that $v = 0$ a.e. on E, which follows from the convergence in measure, the fact that $\chi_{E_\delta} \to \chi_E$, (3.19) (i), and (3.22). Thus, $v = 0$ a.e. on $E \cup \omega_u$. We also find that

$$v = u_0 \quad \text{a.e. on } U_D = V \setminus \Omega \tag{3.26}$$

by (3.18) (i), (3.19) (i), (3.21), (3.22), and the fact that $E \subset \Omega$. (Here and in the following, set inclusions will be intended in the measure-theoretical sense, i.e., up to sets of $\mathcal{L}^d$-measure zero.) Therefore, we get $\omega_u \subset U = \Omega$. We denote the restriction of v to Ω by u, and note that then $u = 0$ on $E \cup \omega_u$. We also observe that

$$\mathcal{L}^d(\Omega \setminus (\omega_u \cup \Omega_\delta^{\mathrm{good}} \cup E_\delta)) \to 0 \quad \text{as } \delta \to 0. \tag{3.27}$$

In fact, $\mathcal{L}^d(\Omega \setminus \widetilde{\Omega}_\delta) \to 0$ by (3.18) (i), $\mathcal{L}^d(E_\delta^* \setminus E_\delta) \to 0$ by (3.19) (i), and $\mathcal{L}^d((\widetilde{\Omega}_\delta \setminus (\Omega_\delta^{\mathrm{good}} \cup E_\delta^*)) \setminus \omega_u) \to 0$ by (3.22) and (3.24).

We now show properties (3.8). First of all, (3.8) (iv) follows directly from (3.19) (ii). Since $\mathcal{F}_\delta(u_\delta, E_\delta) < +\infty$ for all $\delta > 0$, we have $u_\delta = \chi_{\Omega \setminus E_\delta} u_\delta$. Then, using (3.22) as well as (3.27), we get that $\mathcal{L}^d((\Omega \setminus \omega_u) \cap \{v_\delta \neq u_\delta\}) \to 0$ as $\delta \to 0$, and thus $u_\delta \to v = u$ in measure on $\Omega \setminus \omega_u$. This shows (3.8) (i). To see (3.8) (iii), we again use that $\mathcal{L}^d((\Omega \setminus \omega_u) \cap \{v_\delta \neq u_\delta\}) \to 0$ as $\delta \to 0$, as well as (3.17) (i) and (3.23) (ii) to calculate

$$\begin{aligned}
\limsup_{\delta \to 0} \mathcal{L}^d(\{|\nabla u_\delta| > \kappa_\delta\} \setminus \omega_u) &\le \limsup_{\delta \to 0} \mathcal{L}^d(\{|\nabla v_\delta| > \kappa_\delta\} \setminus \omega_u) \\
&\le \limsup_{\delta \to 0} \kappa_\delta^{-2} \int_V |\nabla v_\delta|^2 \, \mathrm{d}x \\
&\le C' \limsup_{\delta \to 0} C_{\eta_\delta} \kappa_\delta^{-2} \gamma_\delta^{-2d/q} = 0.
\end{aligned}$$

It therefore remains to define the sets $(\omega_u^\delta)_{\delta > 0} \subset \mathfrak{M}(\Omega)$ and to prove (3.8) (ii), (v). Let $V' \subset\subset V$. Since $\chi_{E_\delta} \to \chi_E$ in $L^1(\Omega)$, (3.19) (i) also implies that $\chi_{E_\delta^*} \to \chi_E$ in $L^1(V)$.

Moreover, for $\delta > 0$ small, depending also on V', we have $v_\delta = 0$ on E^*_δ and $v_\delta|_{V'\setminus \overline{E^*_\delta}} \in H^1(V' \setminus \overline{E^*_\delta}; \mathbb{R}^d)$; see (3.18) (i) and (3.22). This, along with the fact that $(v_\delta)_{\delta>0}$ converges weakly to v in $\mathrm{GSBD}^2_\infty(V)$, means that we can apply [27, Theorem 5.1] on the set V' for $(v_\delta)_{\delta>0}$ and $(E^*_\delta)_{\delta>0}$ to find

$$\chi_{V'\setminus(E^*_\delta\cup\omega_u)} e(v_\delta) \rightharpoonup \chi_{V'\setminus(E\cup\omega_u)} e(v) \quad \text{weakly in } L^2(V'; \mathbb{R}^{d\times d}_{\mathrm{sym}}), \tag{3.28}$$

as well as

$$\begin{aligned}\int_{J_v\cap E^0\cap V'} 2\varphi(\nu_v)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\partial^* E\cap V'} \varphi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} \\ \le \liminf_{\delta\to 0} \int_{\partial E^*_\delta \cap V'} \varphi(\nu_{E^*_\delta})\,\mathrm{d}\mathcal{H}^{d-1},\end{aligned} \tag{3.29}$$

where E^0 denotes the set of points with density zero for E and ν_v is a measure-theoretical unit normal to J_v. Define $\omega^\delta_u := \omega_u \cup ((\Omega \cap \widetilde{\Omega}_\delta) \setminus (\Omega^{\mathrm{good}}_\delta \cup E^*_\delta))$. Note that $\mathcal{L}^d(\omega^\delta_u \triangle \omega_u) \to 0$ by (3.19) (i) and (3.27), which finishes the verification of (3.8) (v). Moreover, we have

$$\mathcal{H}^{d-1}(\partial^*\omega_u) + \sup_{\delta>0} \mathcal{H}^{d-1}(\partial^*\omega^\delta_u) \le C_M$$

for a constant $C_M > 0$ (with M as defined in (3.7)). The last inequality follows from the set inclusion

$$\partial^*\omega^\delta_u \subset \partial^*\omega_u \cap \cup(\partial E^*_\delta \cap V) \cup \partial\widetilde{\Omega}_\delta,$$

being a consequence of (3.21), together with the estimates (3.18) (ii), (3.19) (ii), (3.25), and that by the choice of V actually $\mathcal{H}^{d-1}(\partial V) < +\infty$. As $e(v_\delta) = 0$ on $\Omega \setminus \Omega^{\mathrm{good}}_\delta$ and $u_\delta = v_\delta$ on $\Omega \cap \Omega^{\mathrm{good}}_\delta$, see (3.22), by recalling (3.18) (i), for $\delta > 0$ small enough we get a.e. on V that

$$\chi_{(V'\cap\Omega)\setminus(E^*_\delta\cup\omega_u)} e(v_\delta) = \chi_{(V'\cap\Omega^{\mathrm{good}}_\delta\cap\Omega)\setminus(E^*_\delta\cup\omega_u)} e(u_\delta) = \chi_{(V'\cap\Omega)\setminus(E^*_\delta\cup\omega^\delta_u)} e(u_\delta).$$

By using (3.28) and recalling that by definition $u = v$ on Ω, we obtain (3.8) (ii) as $V' \subset\subset V$ was arbitrary. This concludes the proof of (3.8) (ii). For later purposes, we also directly discuss the implications of the estimate (3.29) in the subsequent remark. ∎

Remark 3.10. In the setting of the previous result, we also have

$$\begin{aligned}\int_{J_u\setminus\partial^* E} 2\varphi(\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\partial^* E\cap\Omega} \varphi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} \\ + F^{\mathrm{bdry}}(u, E) \le \liminf_{\delta\to 0} F^{\varphi,\gamma_\delta,q}_{\mathrm{surf}}(E_\delta),\end{aligned} \tag{3.30}$$

where F^{bdry} is defined in (3.9) and $F^{\varphi,\gamma_\delta,q}_{\mathrm{surf}}$ in (2.1). Indeed, note that $J_u \cap E^0 = J_u \setminus \partial^* E$ since $u = 0$ on E. Then, by the fact that $u = v$ on Ω, $E \subset \Omega$, $V \cap \partial\Omega = \partial_D\Omega$, and (3.26), we observe that

$$\int_{J_u\setminus\partial^* E} 2\varphi(\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\partial^* E\cap\Omega} \varphi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} + F^{\mathrm{bdry}}(u,E)$$
$$= \int_{J_v\cap E^0} 2\varphi(\nu_v)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\partial^* E\cap V} \varphi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1}.$$

Then, in view of (3.19) (ii) and (3.29) for a sequence $(V_n)_{n\in\mathbb{N}} \subset\subset V$ with $\mathcal{L}^d(V\setminus V_n)\to 0$ as $n\to\infty$ we get (3.30).

Remark 3.11. A closer inspection of the previous proof reveals that the compactness result in Proposition 3.1 remains valid, even if we impose *thickened boundary conditions* for the sequence $(u_\delta)_{\delta>0}$, for instance in the following way.

As before, let $u_0\in W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$, $d=2,3$. Similarly to the argument in the previous proof, we introduce an open set $V\supset\Omega$ such that V and $V\setminus\overline{\Omega}$ are Lipschitz sets and $V\cap\partial\Omega=\partial_D\Omega$. For $\delta>0$ define $y_0^\delta := \mathrm{id}+\delta u_{0,\delta}$, where $(u_{0,\delta})_{\delta>0}\subset W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ is such that $u_{0,\delta}\to u_0$ locally uniformly in $\mathbb{R}^d$ as $\delta\to 0$. Consider also a sequence of open Lipschitz sets $(V_\delta)_{\delta>0}$, with $V_\delta\subset\subset V\setminus\overline{\Omega}$ such that $\chi_{V_\delta}\to\chi_{V\setminus\Omega}$ locally uniformly. Again we let $F_\delta(y,E)$ be defined by (3.6) if $E\in\mathcal{A}_{\mathrm{reg}}(\Omega)$, $\overline{E}\cap\partial_D\Omega=\emptyset$, $y|_{V\setminus\overline{E}}\in H^1(V\setminus\overline{E};\mathbb{R}^d)$, $y|_E=\mathrm{id}$, and now $y|_{V_\delta}=y_0^\delta|_{V_\delta}$, and $F_\delta(y,E)=+\infty$, otherwise. Then the conclusion of Proposition 3.1 still holds.

Proof of Proposition 3.6. Consider $(u_\delta,h_\delta)_{\delta>0}$ with $K:=\sup_{\delta>0}\mathcal{G}_\delta(u_\delta,h_\delta)<+\infty$. First, by this energy bound, (3.11), and a standard compactness argument, we find $h\in\mathrm{BV}(\omega;[0,M])$ such that $h_\delta\to h$ in $L^1(\omega)$, up to a subsequence (not relabeled). For the compactness of $(u_\delta)_{\delta>0}$, we proceed as in the proof of Proposition 3.1, applied for $V:=\omega\times(-2,M+1)$, i.e., $U_D:=\omega\times(-2,-1]$, and $E_\delta:=\Omega\setminus\overline{\Omega_{h_\delta}}$. The only point to prove is that ω_u given in (3.24) satisfies $\mathcal{L}^d(\omega_u)=0$. In fact, then (3.12) for a limit $u\in\mathrm{GSBD}^2(\Omega)$ follows from (3.8). Eventually, since $u_\delta=\chi_{\Omega_{h_\delta}}u_\delta$ and $u_\delta=u_0$ on $\omega\times(-1,0]$, by the fact that $\mathcal{G}_\delta(u_\delta,h_\delta)<+\infty$ (see (3.11)), (3.12) (i) shows $u=\chi_{\Omega_h}u$ and $u=u_0$ on $\omega\times(-1,0]$.

Let us now check that $\mathcal{L}^d(\omega_u)=0$. To this end, we apply Corollary 2.2 once again, now in the version for graphs; see Corollary 2.3. We denote the corresponding set $E'_{\eta_\delta,\gamma_\delta}$ by E'_δ for simplicity and we let $h'_\delta\colon\omega\to\mathbb{R}$ be such that $\Omega_{h'_\delta}=\Omega\setminus\overline{E'_\delta}$. We note that $E'_\delta\supset E^*_\delta$ and thus $h'_\delta\le h_\delta$. This along with (2.5) (i) implies $h'_\delta\to h$ in $L^1(\omega)$ since $\eta_\delta,\gamma_\delta\to 0$. In view of (3.23), (2.5) (ii) applied for $\varphi\equiv 1$, and the fact that $E'_\delta\supset E^*_\delta$, we get $v_\delta|_{\Omega_{h'_\delta}}\in H^1(\Omega_{h'_\delta};\mathbb{R}^d)$ and

$$\sup_{\delta>0}\Big(\int_{\Omega^+_{h'_\delta}}|e(v_\delta)|^2\,\mathrm{d}x+\int_\omega\sqrt{1+|\nabla h'_\delta|^2}\,\mathrm{d}x'\Big)<+\infty.$$

Therefore, by [27, Theorem 2.5] we find that $u=\chi_{\Omega_h}u\in\mathrm{GSBD}^2(\Omega)$ is such that $\chi_{\Omega_{h'_\delta}}v_\delta\to u$ in measure. (Indeed, u coincides with the limiting function identified above.) As

$v_\delta = 0$ on E_δ and $\mathcal{L}^d(E'_\delta \setminus E_\delta) \to 0$ by (2.5) (i), we conclude that ω_u defined in (3.24) satisfies $\mathcal{L}^d(\omega_u) = 0$. ■

3.4. Derivation of effective linearized limits by Γ-convergence

We start with two results on the linearization of nonlinear elastic energies which are by now classical; see e.g., [1,11,30,45,46,73,80,81]. For completeness, however, we include short proofs, in particular due to the fact that our setting, involving varying sets $(E_\delta)_{\delta>0}$, is slightly different compared to the above mentioned works. Recall the quadratic form $\mathcal{Q}$ defined in (3.3).

Lemma 3.12. *Let $(u_\delta)_{\delta>0} \subset GSBD^2(\Omega)$, $u \in GSBD^2(\Omega)$, and let $(\Theta_\delta)_{\delta>0}, \Theta \in \mathfrak{M}(\Omega)$ be such that $\chi_{\Theta_\delta} e(u_\delta) \rightharpoonup \chi_\Theta e(u)$ weakly in $L^2(\Omega, \mathbb{R}^{d\times d}_{\rm sym})$, $\mathcal{L}^d(\{|\nabla u_\delta| > \kappa_\delta\} \cap \Theta) \to 0$, and $u = 0$ on $\Omega \setminus \Theta$, where κ_δ is defined in (3.5). Then*

$$\liminf_{\delta\to 0} \frac{1}{\delta^2}\int_\Omega W(\mathrm{Id} + \delta\nabla u_\delta)\,\mathrm{d}x \geq \frac{1}{2}\int_\Omega \mathcal{Q}(e(u))\,\mathrm{d}x.$$

Proof. We define $\vartheta_\delta \in L^\infty(\Omega)$ by $\vartheta_\delta(x) = \chi_{[0,\kappa_\delta]}(|\nabla u_\delta(x)|)$, and note that $\mathcal{L}^d(\{|\nabla u_\delta| > \kappa_\delta\} \cap \Theta) \to 0$ implies $\vartheta_\delta \to 1$ boundedly in measure on Θ, as $\delta \to 0$. By the regularity and the structural hypotheses of W we get $W(\mathrm{Id} + F) = \frac{1}{2}\mathcal{Q}(\mathrm{sym}(F)) + \Phi(F)$, where $\Phi \colon \mathbb{R}^{d\times d} \to \mathbb{R}$ is a function satisfying $|\Phi(F)| \leq C|F|^3$ for all $F \in \mathbb{R}^{d\times d}$ with $|F| \leq 1$. Then the fact that (3.5) implies $\delta\kappa_\delta \to 0$ and hence $0 < \delta\kappa_\delta \leq 1$ for $\delta > 0$ sufficiently small, together with the fact that $W \geq 0$, implies that

$$\begin{aligned}
&\liminf_{\delta\to 0} \frac{1}{\delta^2}\int_\Omega W(\mathrm{Id} + \delta\nabla u_\delta)\,\mathrm{d}x \\
&\quad\geq \liminf_{\delta\to 0} \frac{1}{\delta^2}\int_\Theta \vartheta_\delta W(\mathrm{Id} + \delta\nabla u_\delta)\,\mathrm{d}x \\
&\quad= \liminf_{\delta\to 0} \int_\Theta \vartheta_\delta\Big(\frac{1}{2}\mathcal{Q}(e(u_\delta)) + \frac{1}{\delta^2}\Phi(\delta\nabla u_\delta)\Big)\,\mathrm{d}x \\
&\quad\geq \liminf_{\delta\to 0}\Big(\frac{1}{2}\int_\Theta \vartheta_\delta Q(\chi_{\Theta_\delta} e(u_\delta))\,\mathrm{d}x - C\int_\Theta \vartheta_\delta\delta|\nabla u_\delta|^3\Big).
\end{aligned}$$

The second term converges to zero since $\vartheta_\delta\delta|\nabla u_\delta|^3$ is uniformly controlled from above by $\delta\kappa_\delta^3$, where $\delta\kappa_\delta^3 \to 0$ by (3.5). As $\chi_{\Theta_\delta} e(u_\delta) \rightharpoonup \chi_\Theta e(u)$ weakly in $L^2(\Omega, \mathbb{R}^{d\times d}_{\rm sym})$, by the convexity of $\mathcal{Q}$, and the fact that ϑ_δ converges to 1 boundedly in measure on Θ, we conclude that

$$\liminf_{\delta\to 0} \frac{1}{\delta^2}\int_\Omega W(\mathrm{Id} + \delta\nabla u_\delta)\,\mathrm{d}x \geq \frac{1}{2}\int_\Theta \mathcal{Q}(e(u))\,\mathrm{d}x = \int_\Omega \frac{1}{2}\mathcal{Q}(e(u))\,\mathrm{d}x,$$

where the last step follows from the fact that $u = 0$ on $\Omega \setminus \Theta$. This concludes the proof. ■

Lemma 3.13. *Let $\tilde{\Omega} \subset \Omega$ be open, and let $u \in W^{1,\infty}(\tilde{\Omega}; \mathbb{R}^3)$. Then, as $\delta \to 0$, we have for $y_\delta := \mathrm{id} + \delta u$ that*

$$\lim_{\delta\to 0}\left|\frac{1}{\delta^2}\int_{\tilde{\Omega}} W(\nabla y_\delta)\,\mathrm{d}x - \frac{1}{2}\int_{\tilde{\Omega}} \mathcal{Q}(e(u))\,\mathrm{d}x\right| = 0.$$

Proof. As in the previous proof, we use that $W(\mathrm{Id}+F) = \frac{1}{2}\mathcal{Q}(\mathrm{sym}(F)) + \Phi(F)$ with $|\Phi(F)| \le C|F|^3$ for $|F| \le 1$. Then, for $y_\delta := \mathrm{id} + \delta u$, we compute

$$\begin{aligned}\frac{1}{\delta^2}\int_{\tilde\Omega} W(\nabla y_\delta)\,\mathrm{d}x &= \frac{1}{\delta^2}\int_{\tilde\Omega} W(\mathrm{Id}+\delta\nabla u)\,\mathrm{d}x \\ &= \int_{\tilde\Omega}\Big(\frac{1}{2}\mathcal{Q}(e(u)) + \frac{1}{\delta^2}\Phi(\delta\nabla u)\Big)\,\mathrm{d}x \\ &= \frac{1}{2}\int_{\tilde\Omega}\mathcal{Q}(e(u))\,\mathrm{d}x + \int_{\tilde\Omega} \mathrm{O}(\delta|\nabla u|^3) \\ &= \frac{1}{2}\int_{\tilde\Omega}\mathcal{Q}(e(u))\,\mathrm{d}x + \|\nabla u\|^3_{L^\infty}\,\mathrm{O}(\delta),\end{aligned}$$

and the result now follows by taking the limit as $\delta \to 0$. ∎

We now proceed with the Γ-convergence results. The proofs essentially rely on the above preparation, the estimates in Section 3.3, and the results in the linearized setting obtained in [27, Section 2]. We start with Theorem 3.3.

Proof of Theorem 3.3. We first address the lower bound and afterwards the upper bound.

Step 1 (Lower bound). Suppose that $(u_\delta, E_\delta) \xrightarrow{\tau} (u, E)$, i.e., there exists a set of finite perimeter $\omega_u \in \mathfrak{M}(\Omega)$ such that $\chi_{E_\delta} \to \chi_E$ in $L^1(\Omega)$, $u_\delta \to u$ in measure on $\Omega \setminus \omega_u$, and $u \equiv 0$ on $E \cup \omega_u$. Without restriction, we can assume that $\sup_{\delta>0} \mathcal{F}_\delta(u_\delta, E_\delta) < +\infty$. In view of Proposition 3.1, this yields $u = \chi_{\Omega\setminus E} u \in GSBD^2(\Omega)$, $\mathcal{H}^{d-1}(\partial^* E) < +\infty$, and that (3.8) holds. Therefore, we obtain $\mathcal{F}_0(u, E) < +\infty$. Now the lower bound for the surface energy follows directly from Remark 3.10. For the elastic part, we use (3.8) (ii), (iii) and apply Lemma 3.12 for an arbitrary $\Omega' \subset\subset \Omega$, for $\Theta_\delta = \Omega' \setminus (E^*_\delta \cup \omega^\delta_u)$ and $\Theta = \Omega' \setminus (E \cup \omega_u)$.

Step 2 (The Γ-lim sup inequality). For each $E \in \mathfrak{M}(\Omega)$ with $\mathcal{H}^{d-1}(\partial^* E) < +\infty$ and each $u = \chi_{\Omega\setminus E} u \in GSBD^2(\Omega)$, recalling the definition of τ-convergence in (3.2), we set

$$\begin{aligned}\mathcal{F}_0'(u,E) &:= \Gamma - \liminf_{\delta\to 0} \mathcal{F}_\delta(u,E) \\ &:= \inf\Big\{\liminf_{\delta\to 0} \mathcal{F}_\delta(u_\delta,E_\delta) \colon (u_\delta,E_\delta) \xrightarrow{\tau} (u,E)\Big\}, \\ \mathcal{F}_0''(u,E) &:= \Gamma - \limsup_{\delta\to 0} \mathcal{F}_\delta(u,E) \\ &:= \inf\Big\{\limsup_{\delta\to 0} \mathcal{F}_\delta(u_\delta,E_\delta) \colon (u_\delta,E_\delta) \xrightarrow{\tau} (u,E)\Big\}.\end{aligned}$$

By Step 1 we have that $\mathcal{F}_0'(u,E) \ge \mathcal{F}_0(u,E)$, so it suffices to show that $\mathcal{F}_0''(u,E) \le \mathcal{F}_0(u,E)$. Assume first that $E \subset\subset \Omega$ with ∂E of class C^∞, and that $u \in W^{1,\infty}(\Omega \setminus \overline{E};\mathbb{R}^3)$ with $u = u_0$ on $\partial_D\Omega$. Choosing the constant approximating sequence $(u_\delta, E_\delta) := (u, E)$

for every $\delta > 0$ and recalling the definition of the energy in (3.6), we obtain

$$\begin{aligned}\mathcal{F}_0''(u,E) &\leq \limsup_{\delta\to 0}\mathcal{F}_\delta(u_\delta,E_\delta)\\ &= \limsup_{\delta\to 0}\Big(\frac{1}{\delta^2}\int_{\Omega\setminus\bar{E}} W(\mathrm{id}+\delta u)\Big) + \int_{\partial E}\varphi(\nu_{\partial E})\,\mathrm{d}\mathcal{H}^{d-1}\\ &\quad + \big(\lim_{\delta\to 0}\gamma_\delta\big)\int_{\partial E}|\boldsymbol{A}|^q\,\mathrm{d}\mathcal{H}^{d-1}\\ &= \frac{1}{2}\int_{\Omega\setminus\bar{E}}\mathcal{Q}(e(u))\,\mathrm{d}x + \int_{\partial E}\varphi(\nu_{\partial E})\,\mathrm{d}\mathcal{H}^{d-1} = \mathcal{F}_0(u,E),\end{aligned}$$

where we used Lemma 3.13 (applied to the set $\tilde{\Omega} := \Omega\setminus\bar{E}$), that $\gamma_\delta\to 0$ as $\delta\to 0$, and that ∂E is of class C^∞ (so that $\|\boldsymbol{A}\|_{L^\infty(\partial E)} < +\infty$).

In the general case, by [27, Theorem 2.2], for each $E\in\mathfrak{M}(\Omega)$ with $\mathcal{H}^{d-1}(\partial^* E) < +\infty$ and each $u = \chi_{\Omega\setminus E}u\in GSBD^2(\Omega)$, there exists a sequence of sets $(E_n)_{n\in\mathbb{N}}$ with $E_n\subset\subset\Omega$, $\partial E_n\in C^\infty$, $\chi_{E_n}\to\chi_E$ in $L^1(\Omega)$, and a sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n|_{\Omega\setminus\overline{E_n}}\in H^1(\Omega\setminus\overline{E_n};\mathbb{R}^d)$, $u_n|_{E_n} = 0$, and $\mathrm{tr}(u_n) = \mathrm{tr}(u_0)$ on $\partial_D\Omega$ such that $u_n\to u$ in $L^0(\Omega;\mathbb{R}^d)$ and

$$\begin{aligned}\lim_{n\to\infty}\mathcal{F}_0(u_n,E_n) &= \lim_{n\to\infty}\Big(\frac{1}{2}\int_{\Omega\setminus\overline{E_n}}\mathcal{Q}(e(u_n))\,\mathrm{d}x + \int_{\partial E_n}\varphi(\nu_{E_n})\,\mathrm{d}\mathcal{H}^{d-1}\Big)\\ &= \mathcal{F}_0(u,E).\end{aligned}\tag{3.31}$$

Strictly speaking, [27, Theorem 2.2] only ensures that ∂E_n is Lipschitz, see [27, equation (2.2)], but in the proof it is shown that E_n can be chosen compactly contained in Ω and ∂E_n of class C^∞; see [27, Proposition 5.4]. By a density argument and the fact that Ω is Lipschitz, without relabeling of functions and sets, it is not restrictive to further assume that each $u_n\in W^{1,\infty}(\Omega\setminus\overline{E_n};\mathbb{R}^d)$ so that, arguing as before, we obtain

$$\mathcal{F}_0''(u_n,E_n) \leq \frac{1}{2}\int_{\Omega\setminus\overline{E_n}}\mathcal{Q}(e(u_n))\,\mathrm{d}x + \int_{\partial E_n}\varphi(\nu_{E_n})\,\mathrm{d}\mathcal{H}^{d-1} = \mathcal{F}_0(u_n,E_n).$$

Using the lower semicontinuity of the Γ-lim sup, the above inequality, and (3.31) we arrive at

$$\mathcal{F}_0''(u,E) \leq \liminf_{n\to\infty}\mathcal{F}_0''(u_n,E_n) \leq \liminf_{n\to\infty}\mathcal{F}_0(u_n,E_n) = \mathcal{F}_0(u,E),$$

which concludes the proof of the Γ-lim sup inequality. Finally, [27, Theorem 2.2] also shows that a volume constraint can be incorporated, which yields Remark 3.4 (i). ■

We now proceed with the proof of Theorem 3.7. To this end, we recall the notion of σ^2_{sym}-convergence introduced in [27, Section 4], in a slightly simplified version. In the following, we use the notation $A\tilde{\subset} B$ if $\mathcal{H}^{d-1}(A\setminus B) = 0$ and $A\tilde{=} B$ if $A\tilde{\subset} B$ and $B\tilde{\subset} A$.

Definition 3.14 ($\sigma^2_{\rm sym}$-convergence). Let $U \subset \mathbb{R}^d$ be open, $U' \supset U$ be open with $\mathcal{L}^d(U' \setminus U) > 0$. Consider a sequence $(\Gamma_n)_{n\in\mathbb{N}} \subset \overline{U} \cap U'$ with $\sup_{n\in\mathbb{N}} \mathcal{H}^{d-1}(\Gamma_n) < +\infty$. We suppose that for each $C > 0$ the sets

$$\mathcal{X}_{C,n} := \left\{ v \in \mathrm{GSBD}^2(U') : v = 0 \text{ in } U' \setminus U,\ \|e(v)\|_{L^2(U')} \le C,\ J_v \,\tilde{\subset}\, \Gamma_n \right\} \tag{3.32}$$

are equi-precompact in $L^0(U';\mathbb{R}^d)$, in the sense that every sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n \in \mathcal{X}_{C,n}$ admits a convergent subsequence in $L^0(U';\mathbb{R}^d)$. Then we say that $(\Gamma_n)_{n\in\mathbb{N}}$ $\sigma^2_{\rm sym}$-converges to Γ satisfying $\Gamma \subset \overline{U} \cap U'$ and $\mathcal{H}^{d-1}(\Gamma) < +\infty$, if the following hold:

(i) for any $C > 0$ and any sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n \in \mathcal{X}_{C,n}$, if a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ converges in measure to $v \in \mathrm{GSBD}^2(U')$, then $J_v \,\tilde{\subset}\, \Gamma$;

(ii) there exist a function $v \in \mathrm{GSBD}^2(U')$ and a sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n \in \mathcal{X}_{C,n}$ for some $C > 0$ such that $v_n \to v$ in measure on U' and $J_v \,\tilde{=}\, \Gamma$.

Our definition is simplified compared to [27, Section 4] as we assume a compactness property for the sets in (3.32). Indeed, all involved sequences converge in measure on U', and therefore we can neglect the set G_∞ appearing in [27, Definition 4.1], which is related to the set where a sequence $(v_n)_{n\in\mathbb{N}}$ as in (i) may converge to infinity. In a similar fashion, the space GSBD^2_∞ introduced in [27, Section 3.4] is not needed. Note that imposing boundary conditions in (3.32) is fundamental for compactness, by excluding nonzero constant functions. We refer to [27, Section 4] for a more general discussion on this notion and mention here only the fundamental compactness result; see [27, Theorem 4.2].

Theorem 3.15 (Compactness of $\sigma^2_{\rm sym}$-convergence). *Let $U \subset \mathbb{R}^d$ be open, $U' \supset U$ be open with $\mathcal{L}^d(U' \setminus U) > 0$. Then every sequence $(\Gamma_n)_{n\in\mathbb{N}} \subset \overline{U} \cap U'$ satisfying the assumptions in Definition 3.14 has a $\sigma^2_{\rm sym}$-convergent subsequence (not relabeled) with limit Γ satisfying the inequality $\mathcal{H}^{d-1}(\Gamma) \le \liminf_{n\to\infty} \mathcal{H}^{d-1}(\Gamma_n)$.*

Moreover, the following lower-semicontinuity result can be shown.

Lemma 3.16 (Lower semicontinuity of surfaces). *Let $\Omega = \omega \times (-1, M+1)$. Let $(D_\delta)_{\delta>0}$ be a sequence of Lipschitz sets such that $\Gamma_\delta := \partial D_\delta \cap \Omega$ are $\sigma^2_{\rm sym}$-converging to Γ in the sense of Definition 3.14 with respect to the sets $U = \omega \times (-\frac{1}{2}, M)$ and $U' = \Omega$. Suppose that there exists a function $h \in \mathrm{BV}(\omega; [0, M])$ such that $\mathcal{L}^d((\Omega \setminus D_\delta) \triangle \Omega_h) \to 0$ as $\delta \to 0$. Then, we have*

$$\mathcal{H}^{d-1}(\partial^* \Omega_h \cap \Omega) + 2\mathcal{H}^{d-1}(\Gamma \cap \Omega^1_h) \le \liminf_{\delta\to 0} \mathcal{H}^{d-1}(\Gamma_\delta).$$

Proof. For the proof we refer to [27, Section 6.1], in particular to [27, equations (6.4), (6.6)]. Note that there the proof was only performed in the case that $\partial D_\delta \cap \Omega$ are graphs, but this assumption is not needed since the argument relies on the lower-semicontinuity result in [27, Theorem 5.1]. ■

We are now in a position to give the proof of Theorem 3.7.

Proof of Theorem 3.7. We first address the lower bound and afterwards the upper bound.

Step 1 (Lower bound). Suppose that $u_\delta \to u$ in $L^0(\Omega;\mathbb{R}^d)$ and that $h_\delta \to h$ in $L^1(\omega)$. Without restriction, we can assume that $\sup_{\delta>0} \mathcal{G}_\delta(u_\delta, h_\delta) < +\infty$. By Proposition 3.6 this implies that $h \in BV(\omega;[0,M])$, $u = \chi_{\Omega_h} u \in GSBD^2(\Omega)$, and $u = u_0$ on $\omega \times (-1,0]$. Therefore, $\mathcal{G}_0(u,h) < +\infty$. Moreover, (3.12) holds. The lower bound for the elastic energy follows by (3.12) (ii), (iii) and by Lemma 3.12 applied for $\Theta_\delta := \Omega' \setminus (E^*_\delta \cup \omega^\delta_u)$ and $\Theta := \Omega'$ for arbitrary $\Omega' \subset\subset \Omega$. Therefore, it remains to prove that

$$\begin{aligned}&\mathcal{H}^{d-1}(\partial^*\Omega_h \cap \Omega) + 2\mathcal{H}^{d-1}(J'_u \cap \Omega^1_h)\\ &\quad\le \liminf_{\delta\to 0}\Big(\mathcal{H}^{d-1}(\partial\Omega_{h_\delta} \cap \Omega) + \gamma_\delta \int_{\partial\Omega_{h_\delta}\cap\Omega} |A_\delta|^q \, \mathrm{d}\mathcal{H}^{d-1}\Big),\end{aligned}$$

where A_δ denotes the second fundamental form corresponding to $\partial\Omega_{h_\delta} \cap \Omega$, and J'_u is defined in (3.14). To this end, we define

$$\Gamma_\delta := \partial E^*_\delta \cap \Omega,$$

where $(E^*_\delta)_{\delta>0}$ are given in (3.12), and note that $\sup_{\delta>0} \mathcal{H}^{d-1}(\Gamma_\delta) < +\infty$ by (3.12) (iv) (up to a subsequence, not relabeled). We let $U = \omega \times (-\frac{1}{2}, M)$ and $U' = \Omega = \omega \times (-1, M+1)$. By [27, Theorem 2.5] and Corollary 2.3 for $\varphi \equiv 1$, we now observe that the sets given in (3.32) are equi-precompact in $L^0(U';\mathbb{R}^d)$. In fact, given $v_\delta \in \mathcal{X}_{C,\delta}$, we define $w_\delta := \chi_{\Omega\setminus \overline{E'_\delta}} v_\delta$, where $\partial E'_\delta \cap \Omega$ is the graph of a function; see Corollary 2.3. By (2.5) (ii) and the fact that $\sup_{\delta>0} \|e(w_\delta)\|_{L^2(\Omega)} < +\infty$, we can apply [27, Theorem 2.5] to find that $(w_\delta)_{\delta>0}$ converges in measure on Ω to some $w \in L^0(\Omega;\mathbb{R}^d)$. By (2.5) (i) we conclude $v_\delta \to w$ in measure, as well. Therefore, as $(\mathcal{X}_{C,\delta})_{\delta>0}$ are equi-precompact, we can apply Theorem 3.15 to deduce that $(\Gamma_\delta)_{\delta>0}$ σ^2_{sym}-converges (up to a subsequence) to some $\Gamma \subset \overline{U} \cap U'$. By combining (3.12) (iv) and Lemma 3.16 for $D_\delta = E^*_\delta$ (note that indeed $\mathcal{L}^d((\Omega \setminus E^*_\delta)\triangle\Omega_h) \to 0$ as $\delta \to 0$ by Proposition 3.6) we get

$$\begin{aligned}&\mathcal{H}^{d-1}(\partial^*\Omega_h \cap \Omega) + 2\mathcal{H}^{d-1}(\Gamma \cap \Omega^1_h)\\ &\quad\le \liminf_{\delta\to 0}\Big(\mathcal{H}^{d-1}(\partial\Omega_{h_\delta} \cap \Omega) + \gamma_\delta \int_{\partial\Omega_{h_\delta}\cap\Omega} |A_\delta|^q \, \mathrm{d}\mathcal{H}^{d-1}\Big).\end{aligned} \tag{3.33}$$

Thus, to conclude the proof, it remains to check that $J'_u \cap \Omega^1_h \subset \Gamma \cap \Omega^1_h$ up to an $\mathcal{H}^{d-1}$-negligible set. To this end, we follow [27, Section 6.1]: Consider the sequence of mappings $v_\delta := \psi\chi_{\Omega\setminus E^*_\delta} u_\delta$, where $\psi \in C^\infty(\Omega)$ with $\psi = 1$ in a neighborhood of $\Omega^+ = \Omega \cap \{x_d > 0\}$ and $\psi = 0$ on $\omega \times (-1, -\frac{1}{2})$. Moreover, for $t > 0$, we let $v'_\delta(x) := \chi_{\Omega\setminus E^*_\delta}(x) v_\delta(x', x_d - t)$, extended by zero in $\omega\times(-1,-1+t)$. Defining $v'(x) = \psi\chi_{\Omega_h}(x)u(x', x_d - t)$, we observe that v'_δ converge to v' in measure on U' since $u_\delta \to u$ in measure on U'; see (3.12) (i). We also observe that $v'_\delta = 0$ on $U' \setminus U = \omega \times ((-1, -\frac{1}{2}] \cup [M, M+1))$. Thus, applying Definition 3.14 (i) on the sequence $(v'_\delta)_{\delta>0}$, which clearly satisfies $J_{v'_\delta} \,\tilde{\subset}\, \Gamma_\delta$, we obtain $J_{v'} \,\tilde{\subset}\, \Gamma$. This shows

$$(J_u + te_d) \cap \Omega_h^1 = J_{v'} \cap \Omega_h^1 \subset \Gamma \cap \Omega_h^1.$$

Since $t \geq 0$ was arbitrary, recalling the definition of $J'_u = \{(x', x_d + t) : x \in J_u,\ t \geq 0\}$, see (3.14), we indeed find $J'_u \cap \Omega_h^1 \subset \Gamma \cap \Omega_h^1$. In view of (3.33), this concludes the proof of the lower bound.

Step 2 (Proof of the Γ-lim sup inequality). Adopting similar notation to Step 2 of the proof of Theorem 3.3, in exactly the same manner we can prove that

$$\mathscr{G}_0''(u,h) \leq \mathscr{G}_0(u,h),$$

whenever $h \in C^\infty(\omega; [0, M])$ and $u \in W^{1,\infty}(\Omega_h^+; \mathbb{R}^3)$ with $u = u_0$ on $\omega \times (-1, 0]$.

For a general $h \in \mathrm{BV}(\omega; [0, M])$ and $u = \chi_{\Omega_h} u \in \mathrm{GSBD}^2(\Omega)$ with $u = u_0$ on $\omega \times (-1, 0]$, by applying [27, Theorem 2.4], there exist a sequence $(h_n)_{n\in\mathbb{N}} \subset C^\infty(\omega; [0, M])$ and mappings $(u_n)_{n\in\mathbb{N}}$ with $u_n|_{\Omega_{h_n}} \in H^1(\Omega_{h_n}; \mathbb{R}^d)$, $u_n = 0$ on $\Omega \setminus \overline{\Omega_{h_n}}$, and $u_n = u_0$ on $\omega \times (-1, 0]$ such that $h_n \to h$ in $L^1(\omega)$, $u_n \to u$ in $L^0(\Omega; \mathbb{R}^d)$, and

$$\begin{aligned}\lim_{n\to\infty} \mathscr{G}_0(u_n, h_n) &= \lim_{n\to\infty} \left(\frac{1}{2} \int_{\Omega_{h_n}^+} \mathcal{Q}(e(u_n))\,\mathrm{d}x + \mathcal{H}^{d-1}(\partial\Omega_{h_n} \cap \Omega) \right) \\ &= \mathscr{G}_0(u, h). \end{aligned} \tag{3.34}$$

Strictly speaking, [27, Theorem 2.4] only ensures that h_n is a C^1-function, but in the proof recovery sequences are constructed for profiles of regularity C^∞; see [27, Lemma 6.4]. Moreover, by a density argument we can assume that each u_n is Lipschitz on $\Omega_{h_n}^+$. Arguing exactly as in Step 2 of the proof of Theorem 3.3, based on the lower semicontinuity of the Γ-lim sup, we again obtain

$$\mathscr{G}_0''(u,h) \leq \liminf_{n\to\infty} \mathscr{G}_0''(u_n, h_n) \leq \liminf_{n\to\infty} \mathscr{G}_0(u_n, h_n) = \mathscr{G}_0(u,h),$$

where in the last step we used (3.34). This concludes the proof of the Γ-lim sup inequality. Finally, [27, Remark 6.8] also shows that a volume constraint on the film can be taken into account, as mentioned in Remark 3.8. ∎

We close with short proofs of Corollaries 3.5 and 3.9:

Proof. In view of the above proofs, we observe that replacing $|\boldsymbol{A}_\delta|^2$ by $|\boldsymbol{H}_\delta|^2$ does not affect the Γ-limit, but is only relevant for the compactness results in Propositions 3.1 and 3.6, respectively. To proceed as above, in particular in order to obtain (3.8) (iv) and (3.12) (iv), it suffices to check that, under the assumptions given in Corollaries 3.5 and 3.9, it holds that

$$\liminf_{\delta\to 0} \gamma_\delta \int_{\partial E_\delta \cap \Omega} |\boldsymbol{A}_\delta|^2 \,\mathrm{d}\mathcal{H}^{d-1} \leq \liminf_{\delta\to 0} \gamma_\delta \int_{\partial E_\delta \cap \Omega} |\boldsymbol{H}_\delta|^2 \,\mathrm{d}\mathcal{H}^{d-1}.$$

We refer to the cases (a) and (b) discussed in Remark 2.4. ∎

A. Some auxiliary lemmata

A.1. Two elementary lemmata on planar curves

Lemma A.1. *Let $q \geq 1$. For every closed, planar C^2-curve $\boldsymbol{\gamma}$ it holds that*

$$\int_{\boldsymbol{\gamma}} |\kappa_{\boldsymbol{\gamma}}|^q \,\mathrm{d}\mathcal{H}^1 \geq (\operatorname{diam} \boldsymbol{\gamma})^{1-q},$$

where $\kappa_{\boldsymbol{\gamma}}$ denotes the curvature of the curve.

Proof. Let $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)\colon [0, L_{\boldsymbol{\gamma}}] \mapsto \mathbb{R}^2$ be an arc-length parametrization of $\boldsymbol{\gamma}$, where $L_{\boldsymbol{\gamma}}$ denotes the length of the curve. Without restriction, after a possible translation, we assume that $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(L_{\boldsymbol{\gamma}}) = 0$. Also let $s_0 \in [0, L_{\boldsymbol{\gamma}}]$ be such that $|\boldsymbol{\gamma}(s_0)| = \|\boldsymbol{\gamma}\|_{L^\infty}$. Since $|\dot{\boldsymbol{\gamma}}| \equiv 1$ in this parametrization, by integration by parts and Hölder's inequality, we get

$$\begin{aligned} L_{\boldsymbol{\gamma}} &= \int_0^{L_{\boldsymbol{\gamma}}} |\dot{\boldsymbol{\gamma}}|^2 \,\mathrm{d}s = -\int_0^{L_{\boldsymbol{\gamma}}} \boldsymbol{\gamma} \cdot \ddot{\boldsymbol{\gamma}} \,\mathrm{d}s \leq \|\boldsymbol{\gamma}\|_{L^\infty} \int_0^{L_{\boldsymbol{\gamma}}} |\ddot{\boldsymbol{\gamma}}| \,\mathrm{d}s \\ &= |\boldsymbol{\gamma}(s_0) - \boldsymbol{\gamma}(0)| \int_0^{L_{\boldsymbol{\gamma}}} |\kappa_{\boldsymbol{\gamma}}| \,\mathrm{d}s \leq \operatorname{diam} \boldsymbol{\gamma} \cdot L_{\boldsymbol{\gamma}}^{1-1/q} \Big(\int_0^{L_{\boldsymbol{\gamma}}} |\kappa_{\boldsymbol{\gamma}}|^q \,\mathrm{d}s \Big)^{1/q}. \end{aligned}$$

This, along with the obvious fact that $L_{\boldsymbol{\gamma}} \geq \operatorname{diam} \boldsymbol{\gamma}$ for every closed curve $\boldsymbol{\gamma}$, concludes the proof. ∎

We proceed with the proof of Lemma 2.15.

Proof of Lemma 2.15. Clearly, $\partial E \cap Q_{8\rho}$ can be written as a finite union of pairwise disjoint curves $(\boldsymbol{\gamma}_i)_{i=1}^N$. We denote by $(\boldsymbol{\gamma}_i)_{i=1}^M$ the subset of those curves intersecting $Q_{3\rho}$. It suffices to establish the desired properties for one curve only, denoted by $\boldsymbol{\gamma}$ for simplicity. Additionally, we show that

$$\mathcal{H}^1(\boldsymbol{\gamma} \cap Q_{8\rho}) \geq \rho. \tag{A.1}$$

The latter, along with the assumption that $\mathcal{H}^1(\partial E \cap Q_{8\rho}) \leq \Lambda\rho$, shows that $M \leq \Lambda$.

Without restriction, we let $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)\colon [0, L_{\boldsymbol{\gamma}}] \mapsto \mathbb{R}^2$ be an arc-length parametrization of $\boldsymbol{\gamma}$, where $L_{\boldsymbol{\gamma}}$ denotes the length of the curve. Let $L := \boldsymbol{\gamma}(0) + \mathbb{R}\dot{\boldsymbol{\gamma}}(0)$. Without restriction, up to an isometry we suppose that $L = \mathbb{R}(1, 0)$, i.e., $\boldsymbol{\gamma}(0) = 0$, $\dot{\boldsymbol{\gamma}}(0) = (1, 0)$, for notational convenience. As $\int_{\partial E \cap Q_{8\rho}} |\boldsymbol{A}| \,\mathrm{d}\mathcal{H}^1 \leq \varepsilon$, we get

$$\begin{aligned} |\dot{\boldsymbol{\gamma}}(s) - \dot{\boldsymbol{\gamma}}(0)| = \Big| \int_0^s \ddot{\boldsymbol{\gamma}}(t) \,\mathrm{d}t \Big| &\leq \int_0^{L_{\boldsymbol{\gamma}}} |\ddot{\gamma}(t)| \,\mathrm{d}t \\ &= \int_{\boldsymbol{\gamma}} |\kappa_{\boldsymbol{\gamma}}| \,\mathrm{d}\mathcal{H}^1 \leq \varepsilon \leq \varepsilon_0 \quad \text{for all } s \in [0, L_{\boldsymbol{\gamma}}]. \end{aligned}$$

Thus, provided that ε_0 is chosen sufficiently small, we get $\dot{\boldsymbol{\gamma}}_1 \geq 1/2$ and $|\dot{\boldsymbol{\gamma}}_2| \leq \varepsilon$. Consequently, $\boldsymbol{\gamma}$ is the graph of a regular function $u\colon \overline{U} \to L^\perp$ for an open segment

$U \subset L$ containing $\boldsymbol{\gamma}(0) = 0$ satisfying $u(\boldsymbol{\gamma}(0)) = u'(\boldsymbol{\gamma}(0)) = 0$, more precisely $u(x) = \boldsymbol{\gamma}_2(\boldsymbol{\gamma}_1^{-1}(x))e_2$. This implies $u' = \dot{\boldsymbol{\gamma}}_2/\dot{\boldsymbol{\gamma}}_1 e_2$ and thus $\|u'\|_\infty \le 2\varepsilon$.

Then, switching back to a general line L in $\mathbb{R}^2$, the fundamental theorem of calculus along with the fact that $u(\boldsymbol{\gamma}(0)) = 0$ and that $\mathcal{H}^1(U) = \operatorname{diam}(U) \le 8\sqrt{2}\rho$ yields $\|u\|_\infty \le \|u'\|_\infty \mathcal{H}^1(U) \le C_1\varepsilon\rho$. It remains to show (A.1). In fact, by $\gamma \cap Q_{3\rho} \neq \emptyset$ and $\|u\|_\infty \le C_1\varepsilon\rho$, provided that $\varepsilon_0 > 0$ is small enough, we have that $L \cap Q_{4\rho} \neq \emptyset$. Therefore, $\mathcal{H}^1(L \cap Q_{6\rho}) \ge \rho$, which along with $\|u\|_\infty \le C_1\varepsilon\rho$ implies the estimate. ■

A.2. Lemmata on good and bad planes

In this subsection we give the proofs of Lemmata 2.20–2.21. Let $0 < \theta < 1/\sqrt{3}$. Without restriction, let Q_ρ be the cube centered at 0, and let L be a plane with normal $\nu_L := \nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{S}^2$ such that $(L)_{3\eta\rho} \cap Q_\rho \neq \emptyset$; see (1.5). Before we start with the proofs, we observe the following elementary property: suppose that there exists $k \in \{1, 2, 3\}$ such that $|\nu_j| \le \theta$ for both $j \neq k$. Then we get

$$|(x - y) \cdot e_k| \le 18\theta\rho \quad \text{for all } x, y \in L \cap Q_{3\rho}. \tag{A.2}$$

Indeed, suppose without restriction (up to an appropriate reflection if necessary) that $\nu_1 > \theta$ and $|\nu_2|, |\nu_3| \le \theta$. Thus, we have $\nu_1 \ge \sqrt{1 - 2\theta^2}$, and an elementary computation yields

$$|e_1 - \nu|^2 \le 2\theta^2 + \big(1 - \sqrt{1 - 2\theta^2}\big)^2 \le 4\theta^2$$

as $0 < \theta \le 1/\sqrt{2}$. Then there exists $R_\nu \in \mathrm{SO}(3)$ with $R_\nu \nu = e_1$ such that $|R_\nu - \mathrm{Id}|^2 = 3|e_1 - \nu|^2 \le 12\theta^2$, i.e., $|R_\nu - \mathrm{Id}| \le 2\sqrt{3}\theta$. We fix two arbitrary points $x, y \in L \cap Q_{3\rho}$, and observe that $(x - y) \cdot \nu = 0$. Therefore, we compute

$$|(x-y)\cdot e_1| = |(x-y)\cdot R_\nu \nu| \le |(x-y)\cdot \nu| + |x-y|\,|R_\nu - \mathrm{Id}| \le 2\sqrt{3}\theta|x-y| \le 18\theta\rho,$$

where in the last step we used that $x, y \in Q_{3\rho}$, and therefore $|x - y| \le 3\sqrt{3}\rho$.

Proof of Lemma 2.20. The main step of the proof consists in showing the following statement: There exist $\theta \in (0, 1/\sqrt{3})$ small enough and a constant $C_\theta > 0$ such that for any Q_ρ and any θ-good plane L for Q_ρ the following holds: given a function $v \in L^\infty(V; L^\perp)$ for some bounded domain $L \cap Q_\rho \subset V \subset L$ and $\|v\|_{L^\infty(V)} \le 3\eta\rho$, for all $\rho \le r \le (1 + 6\eta)\rho$ we get that

$$\mathcal{H}^2(\omega_v^r \triangle (L \cap Q_\rho)) \le C_\theta \eta \rho^2, \tag{A.3}$$

where $\omega_v^r := \Pi_L(\operatorname{graph}(v) \cap Q_r)$ and Π_L denotes the orthogonal projection onto the plane L.

Step 1 (Reduction to (A.3)). In fact, once (A.3) has been shown, the statement can be derived as follows: (2.65) is immediate from (A.3). For (2.64), observe that $(\partial^- S_L)_{\mathrm{int}} := \partial^- S_L \cap \operatorname{int}(Q_{(1+6\eta)\rho})$ can be expressed as the graph of the constant function $z \colon L \to L^\perp$

given by $z \equiv 3\eta\rho\nu$, i.e., $(\partial^- S_L)_{\text{int}} = \text{graph}(z) \cap \text{int}(Q_{(1+6\eta)\rho}) = \omega_z^{(1+6\eta)\rho} + 3\eta\rho\nu$. Then, by (A.3), we obtain

$$\begin{aligned}\mathcal{H}^2((\partial^- S_L)_{\text{int}}) = \mathcal{H}^2(\omega_z^{(1+6\eta)\rho}) &\le \mathcal{H}^2(\omega_z^{(1+6\eta)\rho} \triangle (L \cap Q_\rho)) + \mathcal{H}^2(L \cap Q_\rho) \\ &\le \mathcal{H}^2(L \cap Q_\rho) + C_\theta \eta \rho^2.\end{aligned}$$

As the normal vector is a constant equal to $\pm\nu$ on both $(\partial^- S_L)_{\text{int}}$ and $L \cap Q_\rho$, we also get

$$\mathcal{H}^2_\varphi((\partial^- S_L)_{\text{int}}) \le \mathcal{H}^2_\varphi(L \cap Q_\rho) + C_\theta \varphi_{\max} \eta \rho^2,$$

where we recall the notation in (2.35). Consequently, to conclude the argument, we need to check that

$$\mathcal{H}^2(\partial^- S_L \setminus (\partial^- S_L)_{\text{int}}) \le C_\theta \eta \rho^2. \tag{A.4}$$

Then (2.64) indeed follows. To this end, note that the set $\partial^- S_L \setminus (\partial^- S_L)_{\text{int}}$ consists of the six (possibly empty) sets $(L)_{3\eta\rho} \cap \partial Q_{(1+6\eta)\rho} \cap \{\pm x_k = \frac{1}{2}(1+6\eta)\rho\}$, for $k \in \{1,2,3\}$. We derive the estimate for only one of these sets. Without restriction let $W := (L)_{3\eta\rho} \cap \partial Q_{(1+6\eta)\rho} \cap \{x_3 = \frac{1}{2}(1+6\eta)\rho\}$ and suppose that $W \neq \emptyset$. First, provided that η_0 is chosen small with respect to θ, we note that this set is nonempty only if (2.63) for $k = 3$ does not hold. Therefore, as L is a θ-good plane, we necessarily have $|\nu_1| \ge \theta$ or $|\nu_2| \ge \theta$ and thus $|\nu_3| \le \sqrt{1-\theta^2}$.

Let $W_t := W \cap \{x : (x - x_0) \cdot \nu = t\}$ for some arbitrary $x_0 \in L$. Note that $\mathcal{H}^1(W_t) = 0$ for $|t| > 3\eta\rho$ and $\mathcal{H}^1(W_t) \le \sqrt{2}(1+6\eta)\rho$. Then, by the coarea formula (see [66], formula (18.25), applied with slicing direction ν in place of e_n and e_3 as unit normal to the surface W) we get

$$\begin{aligned}\sqrt{1-(\nu\cdot e_3)^2}\,\mathcal{H}^2(W) = \int_W \sqrt{1-(\nu\cdot e_3)^2}\,\mathrm{d}\mathcal{H}^2 &= \int_{\mathbb{R}} \mathcal{H}^1(W_t)\,\mathrm{d}t \\ &\le 6\eta\rho \cdot \sqrt{2}(1+6\eta)\rho \le C\eta\rho^2.\end{aligned}$$

By using the fact that $\sqrt{1-(\nu\cdot e_3)^2} \ge \theta$ and by repeating the estimate for all six sets, we indeed get (A.4). A similar argument shows that

$$\mathcal{H}^2(L \cap (\overline{Q_{(1+12\eta)\rho}} \setminus Q_\rho)) \le C_\theta \eta \rho^2. \tag{A.5}$$

We omit the details.

Step 2 (Proof of (A.3), preparation). Let us now show (A.3). For convenience, we extend v to a function w defined on $L \cap \overline{Q_{(1+12\eta)\rho}}$ satisfying $\|w\|_\infty \le 3\eta\rho$. It suffices to show that for all $\rho \le r \le (1+6\eta)\rho$,

$$\mathcal{H}^2(\omega_w^r \triangle (L \cap Q_\rho)) \le C_\theta \eta \rho^2, \tag{A.6}$$

as then the statement readily follows from the fact that

$$\omega_v^r \triangle (L \cap Q_\rho) \subset (\omega_w^r \triangle (L \cap Q_\rho)) \cup ((L \cap \overline{Q_{(1+12\eta)\rho}}) \setminus V)$$

and that by (A.5) and $L \cap Q_\rho \subset V$ we have

$$\mathcal{H}^2((L \cap \overline{Q_{(1+12\eta)\rho}}) \setminus V) \le C_\theta \eta \rho^2.$$

We start with the observation that, in view of $\|w\|_\infty \le 3\eta\rho$, for all $\rho \le r \le (1+6\eta)\rho$ it holds that

$$L \cap \overline{Q_{(1-6\eta)\rho}} \subset \overline{\omega_w^r} \subset L \cap \overline{Q_{(1+12\eta)\rho}}. \tag{A.7}$$

Indeed, to see the left inclusion, for each $x \in L \cap \overline{Q_{(1-6\eta)\rho}}$ and every $i \in \{1,2,3\}$ we estimate

$$|(x + w(x)) \cdot e_i| \le |x \cdot e_i| + \|w\|_\infty \le \frac{(1-6\eta)\rho}{2} + 3\eta\rho = \frac{\rho}{2} \le \frac{r}{2}.$$

To see the right inclusion, for every $\rho \le r \le (1+6\eta)\rho$ and $x \in \overline{\omega_w^r}$, we estimate for $i \in \{1,2,3\}$,

$$|x_i| \le |(x + w(x)) \cdot e_i| + \|w\|_{L^\infty} \le \frac{r}{2} + 3\eta\rho \le \frac{(1+12\eta)\rho}{2}.$$

For notational convenience, we let $\Sigma_w^r := \omega_w^r \triangle (L \cap Q_\rho)$. We treat the two possible cases in the definition of θ-good planes separately.

Step 3 (Proof of (A.6), case (1)). Let L be a θ-good plane belonging to case (1) in Definition 2.19. Without restriction we suppose that $\mathrm{argmin}_{i=1,2,3} |\nu_i| = 3$. This implies that $|\nu_3| \le 1/\sqrt{3}$ and $|\nu_1|, |\nu_2| \ge \theta$. For $t \in \mathbb{R}$ and $\rho \le r \le (1+6\eta)\rho$, we introduce the sets

$$Q_r^t := Q_r \cap \{x_3 = t\}, \quad \omega^{r,t} := \overline{\omega_w^r} \cap \{x_3 = t\} \quad \text{and} \quad L^t := (L \cap \overline{Q_\rho}) \cap \{x_3 = t\}.$$

By the fact that $|\nu_3| \le 1/\sqrt{3}$, the second inclusion in (A.7), and by the coarea formula we have for $\rho_\eta := (1+12\eta)\rho$ that

$$\begin{aligned}\sqrt{\frac{2}{3}}\mathcal{H}^2(\Sigma_w^r) \le \int_{\Sigma_w^r} \chi_{Q_{\rho_\eta}} \sqrt{1 - (\nu \cdot e_3)^2}\, \mathrm{d}\mathcal{H}^2 &= \int_{\mathbb{R}} \mathcal{H}^1(\Sigma_w^r \cap Q_{\rho_\eta}^t)\, \mathrm{d}t \\ &\le \int_{-\rho_\eta/2}^{\rho_\eta/2} \mathcal{H}^1(\omega^{r,t} \triangle L^t)\, \mathrm{d}t,\end{aligned} \tag{A.8}$$

where we use that ν is a unit normal to Σ_w^r. We now proceed to estimate $\mathcal{H}^1(\omega^{r,t} \triangle L^t)$ for $|t| \le \rho_\eta/2$. To this end, fixing some $z \in L$, we first introduce a parametrization of the one-dimensional sets $\omega^{r,t}$ and L^t. First, for $s \in \mathbb{R}$ we introduce

$$X^t(s) := \Big(s, -\frac{\nu_1}{\nu_2}s + b_{t,\nu}, t\Big), \quad \text{where } b_{t,\nu} := \frac{z \cdot \nu - t\nu_3}{\nu_2},$$

and we observe that $L \cap \{x_3 = t\} = \{X^t(s) : s \in \mathbb{R}\}$ since $X^t(s) \cdot \nu = z \cdot \nu$. Thus, for $|t| \le \rho/2$ it holds that

$$L^t = \big\{X^t(s) : s \in I_L^t\big\}, \quad \text{where } I_L^t := \Big[-\frac{\rho}{2}, \frac{\rho}{2}\Big] \cap \Big[-\frac{|\nu_2|}{|\nu_1|}\frac{\rho}{2} + \frac{\nu_2}{\nu_1}b_{t,\nu}, \frac{|\nu_2|}{|\nu_1|}\frac{\rho}{2} + \frac{\nu_2}{\nu_1}b_{t,\nu}\Big]$$

and $L^t = \emptyset$ for $|t| > \rho/2$. In a similar fashion, we obtain $\omega^{r,t} = \emptyset$ for $|t| > \rho_\eta/2$ and for $|t| \le \rho_\eta/2$ we get $\omega^{r,t} = \{X^t(s)\colon s \in I_\omega^{r,t}\}$, where

$$I_\omega^{r,t} := \Big\{s : |s + w_1(X^t(s))| \le \tfrac{r}{2},\ \Big|\Big(-\tfrac{\nu_1}{\nu_2}s + b_{t,\nu}\Big) + w_2(X^t(s))\Big| \le \tfrac{r}{2},\\ |t + w_3(X^t(s))| \le \tfrac{r}{2}\Big\}$$

where w_k denotes the k-th component of w. Here, we have again used the second inclusion in (A.7). By the area formula we get

$$\mathcal{H}^1(\omega^{r,t} \triangle L^t) \le \sqrt{1 + \Big(\frac{\nu_1}{\nu_2}\Big)^2}\, \mathcal{H}^1(I_\omega^{r,t} \triangle I_L^t) \quad \text{for all } |t| \le \rho_\eta/2.$$

Then, from (A.8) and the facts that $|\nu_1| \le 1$, $|\nu_2| \ge \theta$, we derive

$$\mathcal{H}^2(\Sigma_w^r) \le \sqrt{\frac{3}{2}} \int_{-\frac{\rho_\eta}{2}}^{\frac{\rho_\eta}{2}} \mathcal{H}^1(\omega^{r,t} \triangle L^t)\, \mathrm{d}t \le \frac{\sqrt{3}}{\theta} \int_{-\frac{\rho_\eta}{2}}^{\frac{\rho_\eta}{2}} \mathcal{H}^1(I_\omega^{r,t} \triangle I_L^t)\, \mathrm{d}t. \tag{A.9}$$

A careful inspection of the definition of $I_\omega^{r,t}$ and I_L^t implies that

$$\mathcal{H}^1(I_\omega^{r,t} \setminus I_L^t) \le \begin{cases} (r-\rho) + 2\|w\|_\infty \\ \quad + \dfrac{|\nu_2|}{|\nu_1|}((r-\rho) + 2\|w\|_\infty) & \text{if } |t| \le \dfrac{\rho}{2}, \\ r + 2\|w\|_\infty & \text{if } \dfrac{\rho}{2} < |t| \le \dfrac{\rho_\eta}{2}, \end{cases} \tag{A.10}$$

as well as

$$\mathcal{H}^1(I_L^t \setminus I_\omega^{r,t}) \le \begin{cases} 2\|w\|_\infty + \dfrac{|\nu_2|}{|\nu_1|} 2\|w\|_\infty & \text{if } |t| \le \dfrac{\rho}{2} - \|w\|_\infty, \\ \rho & \text{if } \dfrac{\rho}{2} - \|w\|_\infty < |t| \le \dfrac{\rho}{2}. \end{cases} \tag{A.11}$$

Combining (A.10)–(A.11) and using the facts that $|\nu_1| \ge \theta$, $|\nu_2| \le 1$, and $r \le \rho_\eta$, we get

$$\int_{-\frac{\rho_\eta}{2}}^{\frac{\rho_\eta}{2}} \mathcal{H}^1(I_\omega^{r,t} \triangle I_L^t)\, \mathrm{d}t \le (2\rho - 2\|w\|_\infty)\Big((\rho_\eta - \rho) + 2\|w\|_\infty + \frac{1}{\theta}((\rho_\eta - \rho) + 2\|w\|_\infty)\Big)\\ + (\rho_\eta - \rho)(\rho_\eta + 2\|w\|_\infty) + 2\|w\|_\infty \rho,$$

and, since $\|w\|_\infty \le 3\eta\rho$ and $\rho_\eta = (1 + 12\eta)\rho$, we conclude by recalling (A.9) that

$$\mathcal{H}^2(\omega_w^r \triangle (L \cap Q_\rho)) = \mathcal{H}^2(\Sigma_w^r) \le C_\theta \eta \rho^2$$

for a constant $C_\theta > 0$. This concludes the proof of (A.6) in case (1).

Step 4 (Proof of (A.6), case (2)). Now let L be a θ-good plane for Q_ρ belonging to case (2) in Definition 2.19, i.e., there exists $k \in \{1,2,3\}$ such that $|\nu_k| \geq \theta$ and

$$\mathrm{dist}(L \cap Q_{3\rho}, \{x_k = -\rho/2\} \cup \{x_k = \rho/2\}) \geq 20\theta\rho. \tag{A.12}$$

Without restriction, we suppose that $k = 3$ and that $|\nu_1|, |\nu_2| < \theta$ as otherwise case (1) of the definition applies. We start by observing that (A.7) yields the estimate

$$\begin{aligned}\mathcal{H}^2(\omega_w^r \triangle (L \cap Q_\rho)) &\leq \mathcal{H}^2((L \cap \overline{Q_{(1+12\eta)\rho}}) \setminus (L \cap Q_\rho)) \\ &\quad + \mathcal{H}^2((L \cap Q_\rho) \setminus (L \cap \overline{Q_{(1-6\eta)\rho}})) \\ &= \mathcal{H}^2(L \cap \overline{Q_{(1+12\eta)\rho}}) - \mathcal{H}^2(L \cap \overline{Q_{(1-6\eta)\rho}}),\end{aligned} \tag{A.13}$$

for $\rho \leq r \leq (1+6\eta)\rho$. In view of (A.12) and the fact that $\mathrm{dist}(L, Q_\rho) \leq 3\eta\rho$ by definition, (A.2) implies for η_0 small with respect to θ that

$$L \cap ([-r/2, r/2]^2 \times \mathbb{R}) \subset \overline{Q_r} \quad \text{for all } (1-6\eta)\rho \leq r \leq 3\rho. \tag{A.14}$$

Let us denote by $h_L : \mathbb{R}^2 \to \mathbb{R}$ the affine function with $\mathrm{graph}(h_L) = L$. Observe that $\nabla h_L \equiv (-\nu_1/\nu_3, -\nu_2/\nu_3)$ and therefore

$$\sqrt{1 + |\nabla h_L|^2} = 1/|\nu_3|. \tag{A.15}$$

Now, by the area formula, (A.14), and (A.15) we find

$$\mathcal{H}^2(L \cap \overline{Q_{(1+12\eta)\rho}}) = \int_{(-\rho/2-6\eta\rho, \rho/2+6\eta\rho)^2} \sqrt{1+|\nabla h_L|^2}\, \mathrm{d}\mathcal{H}^2 = \frac{(1+12\eta)^2}{|\nu_3|}\rho^2,$$

and in a similar fashion

$$\mathcal{H}^2(L \cap \overline{Q_{(1-6\eta)\rho}}) = \frac{(1-6\eta)^2}{|\nu_3|}\rho^2.$$

Combining the previous two equalities with (A.13), we conclude

$$\mathcal{H}^2(\omega_w^r \triangle (L \cap Q_\rho)) \leq ((1+12\eta)^2\rho^2 - (1-6\eta)^2\rho^2)/|\nu_3| \leq C_\theta \eta \rho^2$$

for a constant $C_\theta > 0$ depending only on θ, where in the last step we used that $|\nu_3| \geq \sqrt{1-2\theta^2}$. This concludes the proof of (A.6). ■

Proof of Lemma 2.21. Let L be a θ-bad plane for Q_ρ. Let $k \in \{1,2,3\}$ be such that $|\nu_k| \geq \theta$ and $|\nu_j| \leq \theta$ for $j \neq k$. Since (2.63) does not hold, we get $\mathrm{dist}(L \cap Q_{3\rho}, \{x_k = \pm\rho/2\}) < 20\theta\rho$, where $\pm$ is a placeholder for $+$ or $-$. Thus, we find $x_0 \in L \cap Q_{3\rho}$ such that $|(x_0 \pm \rho/2) \cdot e_k| < 20\theta\rho$. This along with (A.2) shows that $|(x \pm \rho/2) \cdot e_k| < 38\theta\rho$ for all $x \in L \cap Q_{3\rho}$. For $\theta < 1/152$, we obtain the statement. ■

A.3. Rigidity estimate on cubic sets

Here, we give the proof of Proposition 2.9. Recall the notation introduced in (2.12).

Proof of Proposition 2.9. We give the argument in detail for (2.15) (i), and only sketch the proof for (2.15) (ii), which can be derived along similar lines. For convenience, we drop the index r and simply write Q for cubes $Q \in \mathcal{Q}_r$. Let us fix $Q, Q' \in \mathcal{Q}_r(U)$ with $\mathcal{H}^{d-1}(\partial Q \cap \partial Q') > 0$. By applying [48, Theorem 3.1] for y and $\mathrm{int}(Q)$ or $\mathrm{int}(Q \cup Q')$, respectively, there exist $R_Q, R_{Q,Q'} \in \mathrm{SO}(d)$ such that

$$\int_Q |\nabla y - R_Q|^2 \,\mathrm{d}x \le C \int_Q \mathrm{dist}^2(\nabla y, \mathrm{SO}(d)) \,\mathrm{d}x, \tag{A.16}$$

$$\int_{Q\cup Q'} |\nabla y - R_{Q,Q'}|^2 \,\mathrm{d}x \le C \int_{Q\cup Q'} \mathrm{dist}^2(\nabla y, \mathrm{SO}(d)) \,\mathrm{d}x \tag{A.17}$$

for an absolute constant $C > 0$. Then, due to (A.16) and (A.17), we have

$$\begin{aligned} r^d |R_Q - R_{Q,Q'}|^2 &= \int_Q |R_Q - R_{Q,Q'}|^2 \,\mathrm{d}x \\ &\le 2\Big(\int_Q |R_Q - \nabla y|^2 \,\mathrm{d}x + \int_{Q\cup Q'} |R_{Q,Q'} - \nabla y|^2 \,\mathrm{d}x\Big) \\ &\le C \int_{Q\cup Q'} \mathrm{dist}^2(\nabla y, \mathrm{SO}(d)) \,\mathrm{d}x. \end{aligned}$$

The same argument can be repeated with Q' in place of Q for a corresponding $R_{Q'} \in \mathrm{SO}(d)$ to obtain an estimate on $|R_{Q'} - R_{Q,Q'}|^2$. Then we obtain

$$r^d |R_Q - R_{Q'}|^2 \le C \int_{Q\cup Q'} \mathrm{dist}^2(\nabla y, \mathrm{SO}(d)) \,\mathrm{d}x. \tag{A.18}$$

Based on this, we compare R_Q and $R_{Q'}$ for arbitrary $Q, Q' \in \mathcal{Q}_r(U)$, $Q \neq Q'$. We show that

$$r^d \max_{Q,Q'} |R_Q - R_{Q'}|^2 \le CN \int_{(U)^r} \mathrm{dist}^2(\nabla y, \mathrm{SO}(d)) \,\mathrm{d}x, \tag{A.19}$$

where for notational convenience we have set $N := \#\mathcal{Q}_r(U)$. To this end, we consider $Q, Q' \in \mathcal{Q}_r(U)$, $Q \neq Q'$, and let $\{Q_0, \dots, Q_M\} \subset \mathcal{Q}_r(U)$ be a simple path, i.e., $Q_0 = Q$, $Q_M = Q'$, $Q_i \neq Q_j$ for all $i \neq j$, and $\mathcal{H}^{d-1}(\partial Q_i \cap \partial Q_{i+1}) > 0$ for all $i = 0, \dots, M-1$. Here, we use that $(U)^r$ is connected. Clearly, we have $M \le N$. Then, due to (A.18) and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} r^d |R_Q - R_{Q'}|^2 &= r^d \Big|\sum_{i=0}^{M-1} (R_{Q_{i+1}} - R_{Q_i})\Big|^2 \le r^d M \sum_{i=0}^{M-1} |R_{Q_{i+1}} - R_{Q_i}|^2 \\ &\le CN \sum_{i=0}^{M-1} \int_{Q_i \cup Q_{i+1}} \mathrm{dist}^2(\nabla y, \mathrm{SO}(d)) \,\mathrm{d}x \\ &\le CN \int_{(U)^r} \mathrm{dist}^2(\nabla y, \mathrm{SO}(d)) \,\mathrm{d}x. \end{aligned}$$

As the choice of the cubes $Q, Q' \in \mathcal{Q}_r(U)$ was arbitrary, we indeed get (A.19). We are now in a position to prove the statement for $R = R_{Q^*} \in \mathrm{SO}(d)$ for some arbitrary $Q^* \in \mathcal{Q}_r(U)$. Indeed, by using (A.16) and (A.19) we have

$$\begin{aligned}
\int_{(U)^r} |\nabla y - R|^2 \,\mathrm{d}x &= \sum_{Q \in \mathcal{Q}_r(U)} \int_Q |\nabla y - R|^2 \,\mathrm{d}x \\
&\le 2 \sum_{Q \in \mathcal{Q}_r(U)} \Big(\int_Q |\nabla y - R_Q|^2 \,\mathrm{d}x + r^d \max_{Q,Q'} |R_Q - R_{Q'}|^2 \Big) \\
&\le 2C \sum_{Q \in \mathcal{Q}_r(U)} \int_Q \mathrm{dist}^2(\nabla y, \mathrm{SO}(d)) \,\mathrm{d}x + 2Nr^d \max_{Q,Q'} |R_Q - R_{Q'}|^2 \\
&\le C \int_{(U)^r} \mathrm{dist}^2(\nabla y, \mathrm{SO}(d)) \,\mathrm{d}x + CN^2 \int_{(U)^r} \mathrm{dist}^2(\nabla y, \mathrm{SO}(d)) \,\mathrm{d}x \\
&\le CN^2 \int_{(U)^r} \mathrm{dist}^2(\nabla y, \mathrm{SO}(d)) \,\mathrm{d}x.
\end{aligned}$$

In view of $N = \#\mathcal{Q}_r(U)$, this concludes the proof of (2.15). It remains to observe that one can choose $R = \mathrm{Id}$ if there exists $Q \in \mathcal{Q}_r(U)$ with $\mathcal{L}^d(Q \cap \{\nabla y = \mathrm{Id}\}) \ge cr^d$. Indeed, by (A.16) one gets $\mathcal{L}^d(Q \cap \{\nabla y = \mathrm{Id}\})|R_Q - \mathrm{Id}|^2 \le C \int_Q \mathrm{dist}^2(\nabla y, \mathrm{SO}(d)) \,\mathrm{d}x$ and therefore (A.16) holds for Id in place of R_Q, for C also depending on c. This, along with the fact that $R = R_{Q^*} \in \mathrm{SO}(d)$ can be chosen for an arbitrary $Q^* \in \mathcal{Q}_r(U)$, concludes the proof of (2.15) (i).

The proof of (2.15) (ii) follows analogously, as a direct consequence of the following version of the Poincaré inequality on the cubic set $(U)^r$:

In the setting of Proposition 2.9, there exists an absolute constant $C > 0$ (independent of U and r) such that for every $v \in H^1((U)^r;\mathbb{R}^d)$ there exists a vector $b_v \in \mathbb{R}^d$ such that

$$r^{-2} \int_{(U)^r} |v(x) - b_v|^2 \le C(\#\mathcal{Q}_r(U))^2 \int_{(U)^r} |\nabla v|^2 \,\mathrm{d}x. \tag{A.20}$$

Once (A.20) is established, its application for $v(x) := y(x) - Rx$ along with (2.15) (i) implies (2.15) (ii). For the proof of (A.20), note that for every $Q \in \mathcal{Q}_r$, Poincaré's inequality in Q gives a vector $b_Q \in \mathbb{R}^d$ for which

$$r^{-2} \int_Q |v - b_Q|^2 \,\mathrm{d}x \le C \int_Q |\nabla v|^2 \,\mathrm{d}x,$$

where $C > 0$ is an absolute constant. The proof can then be performed following the same steps as the proof of (2.15) (i) above, with the obvious adaptions. ■

A.4. Generalized special functions of bounded deformation

Let $U \subset \mathbb{R}^d$ be open. A function $v \in L^1(U;\mathbb{R}^d)$ belongs to the space of *functions of bounded deformation*, denoted by $\mathrm{BD}(U)$, if the distribution $\mathrm{E}v := \frac{1}{2}(\mathrm{D}v + (\mathrm{D}v)^{\mathsf{T}})$ is a bounded $\mathbb{R}^{d\times d}_{\mathrm{sym}}$-valued Radon measure on U, where $\mathrm{D}v = (\mathrm{D}_1 v, \dots, \mathrm{D}_d v)$ is the distributional differential. For $v \in \mathrm{BD}(U)$, the jump set J_v is countably $\mathcal{H}^{d-1}$-rectifiable (in the sense of [4, Definition 2.57]) and it holds that $\mathrm{E}v = \mathrm{E}^a v + \mathrm{E}^c v + \mathrm{E}^j v$, where $\mathrm{E}^a v$ is absolutely continuous with respect to $\mathcal{L}^d$, $\mathrm{E}^c v$ is singular with respect to $\mathcal{L}^d$ and such that $|\mathrm{E}^c v|(B) = 0$ if $\mathcal{H}^{d-1}(B) < \infty$, while $\mathrm{E}^j v$ is concentrated on J_v. The density of $\mathrm{E}^a v$ with respect to $\mathcal{L}^d$ is denoted by $e(v)$. The space $\mathrm{SBD}(U)$ is the subspace of all functions $v \in \mathrm{BD}(U)$ such that $\mathrm{E}^c v = 0$.

We now come to the definition of the space of *generalized functions of bounded deformation* $\mathrm{GBD}(U)$ and of *generalized special functions of bounded deformation* $\mathrm{GSBD}(U) \subset \mathrm{GBD}(U)$. These spaces were introduced and investigated in [29]. We first state the definition; see [29, Definitions 4.1 and 4.2].

Definition A.2. Let $U \subset \mathbb{R}^d$ be a bounded open set, and let $v\colon U \to \mathbb{R}^d$ be measurable. We introduce the notation

$$\Pi^\xi := \{y \in \mathbb{R}^d \colon y \cdot \xi = 0\}, \quad B^\xi_y := \{t \in \mathbb{R} \colon y + t\xi \in B\}$$

for any $B \subset \mathbb{R}^d$, $\xi \in \mathbb{S}^{d-1}$, $y \in \Pi^\xi$, and for every $t \in B^\xi_y$ we let

$$v^\xi_y(t) := v(y + t\xi), \quad \hat{v}^\xi_y(t) := v^\xi_y(t) \cdot \xi.$$

Then $v \in \mathrm{GBD}(U)$ if and only if there exists a nonnegative bounded Radon measure λ_v on U such that $\hat{v}^\xi_y \in \mathrm{BV}_{\mathrm{loc}}(U^\xi_y)$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$, and for every Borel set $B \subset U$,

$$\int_{\Pi^\xi} \big(|\mathrm{D}\hat{v}^\xi_y|\big(B^\xi_y \setminus J^1_{\hat{v}^\xi_y}\big) + \mathcal{H}^0\big(B^\xi_y \cap J^1_{\hat{v}^\xi_y}\big)\big)\, \mathrm{d}\mathcal{H}^{d-1}(y) \le \lambda_v(B),$$

where $J^1_{\hat{v}^\xi_y} := \{t \in J_{\hat{v}^\xi_y} : |[\hat{v}^\xi_y]|(t) \ge 1\}$. Moreover, v belongs to $\mathrm{GSBD}(U)$ if and only if $v \in \mathrm{GBD}(U)$ and $\hat{v}^\xi_y \in \mathrm{SBV}_{\mathrm{loc}}(U^\xi_y)$ for every $\xi \in \mathbb{S}^{d-1}$ and for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$.

Every $v \in \mathrm{GBD}(U)$ has an *approximate symmetric gradient* $e(v) \in L^1(U; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and an *approximate jump set* J_v which is still countably $\mathcal{H}^{d-1}$-rectifiable (cf. [29, Theorems 9.1, 6.2]). The notation for $e(v)$ and J_v, which is the same as in the SBD case, is consistent: in fact, if v lies in $\mathrm{SBD}(U)$, the objects coincide, up to negligible sets of points with respect to $\mathcal{L}^d$ and $\mathcal{H}^{d-1}$, respectively. The subspace $\mathrm{GSBD}^2(U)$ is given by

$$\mathrm{GSBD}^2(U) := \big\{v \in \mathrm{GSBD}(U) \colon e(v) \in L^2(U; \mathbb{R}^{d\times d}_{\mathrm{sym}}),\ \mathcal{H}^{d-1}(J_v) < \infty\big\}.$$

If U has a Lipschitz boundary, for each $v \in \mathrm{GBD}(U)$ the traces on ∂U are well defined, see [29, Theorem 5.5], in the sense that for $\mathcal{H}^{d-1}$-a.e. $x \in \partial U$ there exists $\mathrm{tr}(v)(x) \in \mathbb{R}^d$ such that

$$\lim_{\varepsilon \to 0} \varepsilon^{-d} \mathcal{L}^d(U \cap B_\varepsilon(x) \cap \{|v - \mathrm{tr}(v)(x)| > \varrho\}) = 0 \quad \text{for all } \varrho > 0.$$

We close this short subsection with a compactness result in $\mathrm{GSBD}^2(U)$; see [17, Theorem 1.1].

Theorem A.3 (GSBD^2 compactness). *Let $U \subset \mathbb{R}^d$ be an open, bounded set, and let $(u_n)_{n\in\mathbb{N}} \subset \mathrm{GSBD}^2(U)$ be a sequence satisfying*

$$\sup_{n\in\mathbb{N}} (\|e(u_n)\|_{L^2(U)} + \mathcal{H}^{d-1}(J_{u_n})) < +\infty.$$

Then there exists a subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$, such that the set

$$\omega_u := \{x \in U : |u_n(x)| \to \infty\}$$

has finite perimeter, and there exists $u \in \mathrm{GSBD}^2(U)$ with $u = 0$ on ω_u such that

$$\begin{aligned} &\text{(i)}\ \ u_n \to u \quad \text{in } L^0(U \setminus \omega_u; \mathbb{R}^d), \\ &\text{(ii)}\ \ e(u_n) \rightharpoonup e(u) \quad \text{weakly in } L^2(U \setminus \omega_u; \mathbb{R}^{d\times d}_{\mathrm{sym}}), \\ &\text{(iii)}\ \ \liminf_{n\to\infty} \mathcal{H}^{d-1}(J_{u_n}) \geq \mathcal{H}^{d-1}(J_u \cup (\partial^* \omega_u \cap U)). \end{aligned} \tag{A.21}$$

In the language of [27, Section 3.4], *we say that $u_n \to u$ weakly in $\mathrm{GSBD}^2_\infty(U)$.*

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