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s-stability for $W^{s,n/s}$-harmonic maps in homotopy groups

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Abstract. We study s-dependence for minimizing $W^{s,n/s}$-harmonic maps $u\colon \mathbb{S}^n \to \mathbb{S}^\ell$ in homotopy classes. Sacks–Uhlenbeck theory shows that, for each s, minimizers exist in a generating subset of $\pi_n(\mathbb{S}^\ell)$. We show that this generating subset can be chosen locally constant in s. We also show that as s varies, the minimal $W^{s,n/s}$-energy in each homotopy class changes continuously. In particular, we provide progress on a question raised by Mironescu [in: Perspectives in nonlinear partial differential equations (2007), 413–436] and Brezis–Mironescu [Sobolev maps to the circle (2021)].

1. Introduction

We study minimizing $W^{s,p}$-harmonic maps between spheres in homotopy classes, which are defined as maps $u \in W^{s,p}(\mathbb{S}^n, \mathbb{S}^\ell)$ with least energy

$$\mathcal{E}_{s,p}(u) := \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y = [u]^p_{W^{s,p}(\mathbb{S}^n,\mathbb{S}^\ell)} \tag{1.1}$$

among maps of the same homotopy. Here, $s \in (0,1)$, $p > 1$. In (1.1) we take the $\mathbb{R}^{n+1}$-Euclidean distance and $\mathbb{R}^{\ell+1}$-Euclidean distance in the numerator and in the denominator, respectively.

A natural question arises: Given $\alpha \in \pi_n(\mathbb{S}^\ell)$, is the infimum of $\mathcal{E}_{s,p}$ attained in α? In other words, can we find a map $u \in W^{s,p}(\mathbb{S}^n, \mathbb{S}^\ell)$, $u \in \alpha$ such that

$$\mathcal{E}_{s,p}(u) \le \mathcal{E}_{s,p}(v) \quad \forall v \in W^{s,p}(\mathbb{S}^n,\mathbb{S}^\ell),\ v \in \alpha.$$

If $p > \frac{n}{s}$ the answer is yes, by standard methods of calculus of variation, due to the compact embedding of $W^{s,p}(\mathbb{S}^n,\mathbb{S}^\ell)$ into $C^0(\mathbb{S}^n,\mathbb{S}^\ell)$. If $p < \frac{n}{s}$ it was shown by Brezis–Nirenberg [4] that no homotopy theory can be defined for maps in $W^{s,p}(\mathbb{S}^n,\mathbb{S}^\ell)$ and the infimum energy of $\mathcal{E}_{s,p}$ in any homotopy class is zero. Thus, throughout this work we will focus on the critical, conformally invariant case $p = \frac{n}{s}$. By [4], it is known that the standard notions of homotopy can be extended to $W^{s,\frac{n}{s}}$-Sobolev maps; see also [15, Section 2] for an overview.

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For $s \in (0,1)$ set

$$\#_s\alpha := \inf_{\substack{u \in W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^\ell),\\ u\in\alpha}} \mathcal{E}_{s,\frac{n}{s}}(u), \quad \alpha \in \pi_n(\mathbb{S}^\ell).$$

In the case of maps between spheres of the same dimension $u\colon \mathbb{S}^n \to \mathbb{S}^n$, i.e., when the homotopy classes are given by their degree, we instead write

$$\#_s d := \inf_{\substack{u \in W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^\ell),\\ \deg(u)=d}} \mathcal{E}_{s,\frac{n}{s}}(u), \quad d \in \mathbb{Z}.$$

In general the question whether $\#_s\alpha$ is attained is rather involved even in the local case $s = 1$; see, e.g., [8, 20, 22] or, for $n = 1$ and $s \in (0,1)$, see [3, Chapter 12], as well as [1, 18]. In [15] we showed the following theorem.[1]

Theorem 1. *For any $\ell, n \geq 1$, with either $(\ell,n) = (1,1)$ or $\ell \geq 2$, $s \in (0,1)$. There exists a generating set $X_s \subset \pi_n(\mathbb{S}^\ell)$ such that for any $\alpha \in X_s$ the infimum $\#_s\alpha$ is attained.*

In this work we are interested in the stability of such results as s changes.

Our first main result is that one can choose the generating set X_s from Theorem 1 *locally stable* as s varies. More precisely we have the following theorem.

Theorem 2. *Fix $n, \ell \geq 1$ with either $(\ell,n) = (1,1)$ or $\ell \geq 2$. Let $\Lambda > 0$ and set for $s \in (0,1)$,*

$$X_s := \Big\{\alpha \in \pi_n(\mathbb{S}^\ell) : \text{there exists a } W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^\ell)\text{-minimizer } u \text{ in } \alpha \\ \text{and } [u]_{W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^\ell)} \leq \Lambda\Big\}.$$

Then for any $t \in (0,1)$ there exists $\delta > 0$ such that

$$Y := \bigcap_{s\in(t-\delta,t+\delta)} X_s$$

spans the same set as X_s, i.e., $X_s \subset \operatorname{span} Y$, for each $s \in (t-\delta, t+\delta)$.

See Theorem 25 for a more precise statement with respect to the dependencies of δ.

Let us stress that Theorem 2 does *not* imply that

$$X_s := \big\{\alpha \in \pi_n(\mathbb{S}^\ell) \setminus \{0\} : \#_s\alpha \text{ is attained}\big\}$$

is unchanged as s varies. Rather, it says that we can choose the set of attained generators of $\pi_n(\mathbb{S}^\ell)$ locally stable. In particular, for $n = \ell$ (when the homotopy class is identified

[1]The case $s \leq \frac{1}{2}$, $n = 1$ was not treated in [15] but is covered in [12].

with the degree), in principle, it could be possible that

$$X_s = \begin{cases} \{1\}, & s < 1/2, \\ \mathbb{Z} \setminus \{0\}, & s = 1/2, \\ \{1, 2, -3\}, & s \in (1/2, 3/4), \\ \{2, -3\}, & s \in (3/4, 1). \end{cases}$$

An important ingredient in the aforementioned Theorem 2, and our second main result, is the following continuity result for the map $s \mapsto \#_s \alpha$. By smooth approximation it is elementary that for $\alpha \in \pi_n(\mathbb{S}^\ell)$ we have for any $t \in (0, 1)$,

$$\#_t \alpha \geq \limsup_{s \to t} \#_s \alpha.$$

But actually, we have full continuity.

Theorem 3. *Assume $\ell, n \geq 1$, with either $(\ell, n) = (1, 1)$ or $\ell \geq 2$. Let $\alpha \in \pi_n(\mathbb{S}^\ell)$. Then the map*

$$s \mapsto \#_s \alpha, \quad s \in (0, 1)$$

is continuous.

Besides being a crucial ingredient for the proof of Theorem 2, Theorem 3 also has several interesting corollaries that we discuss now.

Firstly, it is natural to expect that minimizers exist in the class of degree-one maps in $W^{s,\frac{n}{s}}(\mathbb{S}^n, \mathbb{S}^n)$. Towards this, Berlyand–Mironescu–Rybalko–Sandier obtain in [1, Lemma 3.1] that for maps in $W^{\frac{1}{2},2}(\mathbb{S}^1, \mathbb{S}^1)$, minimizers are attained for any degree; see also [3, Theorem 12.9]. Moreover, Mironescu proved a stability result [17, Theorem 2], which asserts that for $s \in [\frac{1}{2}, \frac{1}{2} + \delta)$, degree-one minimizers exist in $W^{s,\frac{1}{s}}(\mathbb{S}^1, \mathbb{S}^1)$. As a corollary of Theorem 3 we can extend the latter in the other direction.

Corollary 4. *For maps from $\mathbb{S}^1$ to $\mathbb{S}^1$, there exists $\delta > 0$ such that*

$$\#_s 1 = \inf\left\{[u]_{W^{s,\frac{1}{s}}}^{\frac{1}{s}} : u \in W^{s,\frac{1}{s}}(\mathbb{S}^1, \mathbb{S}^1),\ \deg u = 1\right\}$$

is attained for all $s \in (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$.

This provides progress towards [16, Open Problem 1] and [3, Open Problem 24.].

More generally, if $\#_s \alpha$ is attained for an $\alpha \in \pi_n(\mathbb{S}^\ell)$, it is unclear whether $\#_t \alpha$ is attained for $t \approx s$. Corollary 4 works for degree-one maps, because they have the lowest energy level among nontrivial homotopy classes. This observation is true in any dimension and we have the following corollary.

Corollary 5. *Fix $n, \ell \geq 1$ with either $(\ell, n) = (1, 1)$ or $\ell \geq 2$. Assume that for some $t \in (0, 1)$ and $\alpha \in \pi_n(\mathbb{S}^\ell) \setminus \{0\}$ we have*

$$\#_t \alpha \leq \#_t \beta \quad \forall \beta \in \pi_n(\mathbb{S}^\ell) \setminus \{0\}.$$

Then not only is $\#_t\alpha$ attained by [15]*, but also there exists $\delta > 0$ such that $\#_s\alpha$ is attained for all $s \in (t-\delta, t+\delta)$.*

For our next corollary of Theorem 3, we consider the Bourgain–Brezis–Mironescu degree inequality, [2, Theorem 0.6], which says that for maps $u \in W^{s,\frac{n}{s}}(\mathbb{S}^n, \mathbb{S}^n)$,

$$\deg u \leq C_{n,s} [u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^n)}.$$

Let $\overline{C}_{n,s}$ be the minimal constant, i.e.,

$$\overline{C}_{n,s} := \sup_{u \in W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^n)} \frac{\deg u}{[u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^n)}} < \infty.$$

It is natural to discuss the continuity of the map $s \mapsto \overline{C}_{n,s}$.

Corollary 6. *The map $s \mapsto \overline{C}_{n,s}$ is lower semicontinuous.*

More precisely, for any $\Lambda > 0$ the map $s \mapsto \overline{C}_{n,s;\Lambda}$ defined by

$$\overline{C}_{n,s;\Lambda} := \sup_{\substack{u \in W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^n),\\ 0<[u]_{W^{s,\frac{n}{s}}} \leq \Lambda}} \frac{\deg u}{[u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^n)}}$$

is continuous.

Corresponding results hold for the Hopf degree (see [26]):

$$\left(1-\frac{1}{4n}, 1\right) \ni s \mapsto \widetilde{C}_{n,s} := \sup_{\substack{u \in W^{s,\frac{n}{s}}(\mathbb{S}^{4n-1},\mathbb{S}^{2n}),\\ 0<[u]_{W^{s,\frac{4n-1}{s}}}}} \frac{[u]_{\pi_{4n-1}(\mathbb{S}^{2n})}}{[u]^{\frac{4n}{s}}_{W^{s,\frac{4n-1}{s}}(\mathbb{S}^n,\mathbb{S}^n)}},$$

and more generally for maps representing rational homotopy groups of spheres; see [19].

We turn to the main ideas for Theorem 3, and thus Theorem 2. We use the following new ingredient:

$$\text{whenever } \#_s\alpha \text{ is attained, then } \#_t\alpha \leq \#_s\alpha + \varepsilon \text{ for } t \approx s; \tag{1.2}$$

for the precise formulation see Corollary 24. To obtain (1.2) we show that $W^{s,n/s}(\mathbb{S}^n, \mathbb{S}^\ell)$ minimizers actually belong *globally* to $W^{s_1,\frac{n}{s_1}}(\mathbb{S}^n, \mathbb{S}^\ell)$ for an $s_1 > s$; see Theorem 16. Hölder regularity in this situation has been established in [14, 23], after pioneering work for $n = 1$ and $s = \frac{1}{2}$ in [7], but this is a *local* result on domains where the BMO-norm is small. Smallness of the BMO-norm is of course not a scaling invariant property, indeed it depends heavily on the specific minimizer u and one cannot deduce from it a uniform global property. Our higher regularity result in a conformally invariant Sobolev space, Theorem 16, is, on the other hand, uniform and independent of the specific minimizer u. Hence, using stability of the Sobolev norm $W^{s,n/s}$, Proposition 12, we obtain (1.2). Once we have (1.2), the main results follow from combinatorial observations coupled with the Sacks–Uhlenbeck theory developed in [15] and the energy identity from Theorem 15.

Remark 7. We conclude this introduction with a few remarks about possible generalizations of these results.

(1) The modulus of continuity in Theorem 3 and the δ in Theorem 2 are relatively easily to compute. They depend on the regularity theory gain (which can be calculated explicitly) and they get worse with large $\#_s\alpha$.

(2) In this paper, to ensure clarity in our presentation, we focus on the case when the target manifold is a sphere. Nonetheless, it should be easy to extend the results to the case of a compact Lie group in the target.

(3) It seems that an extension of our results to a general target manifold is more challenging – the only obstacle is regularity, but it is unclear even for minimizing maps how to get scaling-invariant higher regularity of Theorem 16.

(4) It would be interesting to study the limiting cases as $s \to 1^-$ and $s \to 0^+$.

Notation

We write α, β, etc. to denote a homotopy group, $\mathbb{S}^n$ for the sphere in the domain, and $\mathbb{S}^\ell$ for the target sphere. For brevity, we write $\precsim$ whenever there is a constant C (not depending on any crucial quantity) such that $A \leq C$B. Similarly, $A \approx B$ means $A \precsim B$ and $B \precsim A$.

2. Preliminary results

Let us emphasize that some of the results of this section can be easily extended to general target manifolds. For brevity we restrict everything to sphere targets.

The first result, which is well known and follows from the embedding of the critical Sobolev space into BMO, is the following; see, e.g., [15, Lemma 2.10].

Proposition 8. *For any $\ell, n \in \mathbb{N}$ there exists $\lambda = \lambda(\ell, n) > 0$ such that whenever $s \in (0, 1]$ and $\alpha \in \pi_n(\mathbb{S}^\ell) \setminus \{0\}$,*

$$\#_s\alpha \geq \lambda.$$

In [28, Theorem 1.2] or [3, Lemma 12.6.] the following is proven.

Proposition 9. *Whenever $s \in (0, 1]$ and α has a free homotopy group decomposition into $(\alpha_i)_{i=1}^N$ then*

$$\#_s\alpha \leq \sum_{i=1}^N \#_s\alpha_i.$$

In the case of $W^{\frac{1}{2},2}(\mathbb{S}^1, \mathbb{S}^1)$ maps, minimizers exist for each degree and their exact energy is known. Precisely, by [1, Lemma 3.1] (see also [3, Theorems 12.9 & 12.10]), we have the following theorem.

Theorem 10. *For maps from $\mathbb{S}^1$ to $\mathbb{S}^1$ we have*

$$\#_{\frac{1}{2}} d = 4\pi^2 |d|.$$

Moreover, $\#_{\frac{1}{2}} d$ is attained for all $d \in \mathbb{Z}$.

By results in [28] and [2] we obtain that if we know that the energy of a map is bounded then the map can belong only to a finite subgroup of $\pi_n(\mathbb{S}^l)$.

Theorem 11. *Fix $\Lambda > 0$ and let $0 < s_0 < s_1 < 1$, $n, \ell \geq 1$ with either $(\ell, n) = (1, 1)$ or $\ell \geq 2$. Then there exists a* finite *subgroup $\mathcal{Q} \subset \pi_n(\mathbb{S}^\ell)$ such that the following holds:*

Whenever for some $s \in (s_0, s_1)$ the map $u \in W^{s,\frac{n}{s}}(\mathbb{S}^n, \mathbb{S}^\ell)$ satisfies

$$[u]_{W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^\ell)} \leq \Lambda$$

then $u \in \mathcal{Q}$.

Proof. In the case when $n = \ell$ the assertion follows from the degree estimate in [2, Theorem 0.6], since for any $s > 0$ we have

$$|\deg u| \precsim [u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^n)}.$$

If $\pi_1(\mathbb{S}^\ell) = \{0\}$, i.e., $\ell \geq 2$, we have for any $\varepsilon > 0$,

$$\begin{aligned}
&\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \chi_{\{|f(x)-f(y)|>\varepsilon\}} \frac{1}{|x-y|^{2n}}\, \mathrm{d}x\, \mathrm{d}y \\
&\quad \leq \varepsilon^{-\frac{n}{s_0}} \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{|f(x)-f(y)|^{\frac{n}{s_0}}}{|x-y|^{2n}}\, \mathrm{d}x\, \mathrm{d}y \\
&\quad \leq \varepsilon^{-\frac{n}{s_0}} 2^{\frac{n}{s_0}-\frac{n}{s_1}} \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{|f(x)-f(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\, \mathrm{d}x\, \mathrm{d}y \\
&\quad \leq \varepsilon^{-\frac{n}{s_0}} 2^{\frac{n}{s_0}-\frac{n}{s_1}} \Lambda^n.
\end{aligned}$$

Hence the assumption in [28, Theorem 1.4] is satisfied and we may conclude. ∎

2.1. Continuity for the $s \mapsto W^{s,\frac{n}{s}}$-norm

We need the following continuity result for the fractional Sobolev norm.

Proposition 12. *Fix $n, \ell \in \mathbb{N}$. Let $\Lambda > 0$, $t_0 > 0$, $t \in (t_0, 1)$, and let $s_1 > t$. For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, \Lambda, |s_1 - t|, n) > 0$ such that the following holds.*

Assume that $u \colon \mathbb{S}^n \to \mathbb{S}^\ell$ satisfies

$$[u]_{W^{s_1,\frac{n}{s_1}}(\mathbb{S}^n,\mathbb{S}^\ell)} \leq \Lambda.$$

Then

$$\sup_{r_1,r_2 \in (t-\delta,t+\delta)} \left| [u]^{\frac{n}{r_1}}_{W^{r_1,\frac{n}{r_1}}(\mathbb{S}^n,\mathbb{S}^\ell)} - [u]^{\frac{n}{r_2}}_{W^{r_2,\frac{n}{r_2}}(\mathbb{S}^n,\mathbb{S}^\ell)} \right| \leq \varepsilon.$$

The proof is based on the following elementary lemma.

Lemma 13. *For any $\varepsilon > 0$, $\Gamma > 0$, $0 < p_0 < p_1$, there exists $\tilde{\delta} = \tilde{\delta}(p_0, p_1, \Gamma, \varepsilon) > 0$ such that if $|p - q| < \tilde{\delta}$, $p, q \in [p_0, p_1]$ then*

$$|p - q| < \tilde{\delta} \quad \Rightarrow \quad |a^p - a^q| < \varepsilon \quad \forall a \in [0, \Gamma]. \tag{2.1}$$

Proof. We may assume that $\varepsilon \in (0, 1)$ and $\Gamma \geq 1$, $p_0 \leq p < q \leq p_1$.

Set $\sigma := (\frac{1}{2}\varepsilon)^{\frac{1}{p_0}} \in (0, 1)$. Then

$$|a^p - a^q| < 2\sigma^{p_0} \leq \varepsilon \quad \forall a \in [0, \sigma].$$

Moreover, with the inequality $|1 - e^t| \leq |t|e^{|t|}$ we find for $\Lambda_\varepsilon := \max_{a \in [\sigma, \Gamma]} |\ln a|$,

$$|a^p - a^q| = a^p|1 - a^{q-p}| \leq \Gamma^{p_1}|1 - a^{q-p}| \leq |q - p|\Gamma^{p_1}\Lambda_\varepsilon e^{\Lambda_\varepsilon 2p_1} \quad \forall a \in [\sigma, \Gamma].$$

So if we set

$$\tilde{\delta} := \frac{1}{\Gamma^{p_1}\Lambda_\varepsilon e^{\Lambda_\varepsilon 2p_1}},$$

we have shown that

$$|a^p - a^q| < \varepsilon \quad \forall |p - q| < \tilde{\delta}, \ a \in [0, \Gamma]. \qquad \blacksquare$$

Proof of Proposition 12. Pick some $\bar{s} \in (t, s_1)$ and take $\frac{t-t_0}{2} < \delta < \frac{\bar{s}-t}{2}$ to be specified later. The relation between the numbers is now

$$0 < t_0 < t - \delta < t < t + \delta < \bar{s} < s_1 < 1.$$

Fix $r_1, r_2 \in (t - \delta, t + \delta)$ such that $r_2 > r_1$. We have

$$\begin{aligned}
&\left| [u]^{\frac{n}{r_1}}_{W^{r_1,\frac{n}{r_1}}(\mathbb{S}^n,\mathbb{S}^\ell)} - [u]^{\frac{n}{r_2}}_{W^{r_2,\frac{n}{r_2}}(\mathbb{S}^n,\mathbb{S}^\ell)} \right| \\
&\quad \leq \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{\left||u(x) - u(y)|^{\frac{n}{r_1}} - |u(x) - u(y)|^{\frac{n}{r_2}}\right|}{|x - y|^{2n}} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{|u(x) - u(y)|^{\frac{n}{s_1}}\left||u(x) - u(y)|^{\frac{n}{r_1} - \frac{n}{s_1}} - |u(x) - u(y)|^{\frac{n}{r_2} - \frac{n}{s_1}}\right|}{|x - y|^{2n}} \,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{2.2}$$

Now set

$$a := |u(x) - u(y)|, \quad p := \frac{n}{r_1} - \frac{n}{s_1}, \quad q := \frac{n}{r_2} - \frac{n}{s_1}.$$

Since $|u| \equiv 1$ we have $a \in [0, 2]$. Also $p, q \in [p_0, p_1]$ for

$$p_0 = \frac{n}{t + \delta} - \frac{n}{s_1} \geq \frac{n}{\bar{s}} - \frac{n}{s_1} > 0 \quad \text{and} \quad p_1 = \frac{n}{t - \delta} - \frac{n}{s_1} \leq \frac{n}{t_0} - \frac{n}{s_1} < \infty.$$

We observe

$$|p - q| = \frac{n}{r_1 r_2}|r_2 - r_1| \leq \frac{2n}{(t_0)^2}\delta.$$

Hence, choosing $\delta = \tilde{\delta}\frac{(t_0)^2}{2n}$, where $\tilde{\delta}$ is from Lemma 13, and combining (2.2) with (2.1) we get

$$\left|[u]^{\frac{n}{r_1}}_{W^{r_1,\frac{n}{r_1}}(\mathbb{S}^n,\mathbb{S}^\ell)} - [u]^{\frac{n}{r_2}}_{W^{r_2,\frac{n}{r_2}}(\mathbb{S}^n,\mathbb{S}^\ell)}\right| \le \sup_{a\in[0,2]} |a^p - a^q| \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{|u(x)-u(y)|^{\frac{n}{s_1}}}{|x-y|^{2n}}\,dx\,dy$$
$$\le \varepsilon \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{|u(x)-u(y)|^{\frac{n}{s_1}}}{|x-y|^{2n}}\,dx\,dy,$$

as desired. ■

3. Existence of minimizers and energy identity

In this section we show how to deduce an energy identity using [15] and [28]. We begin by recalling the following lemma, which we will combine later with Theorem 11.

Lemma 14 ([15, Lemma 7.7]). *Fix $n, \ell \in \mathbb{N}$ with either $(\ell, n) = (1, 1)$ or $\ell \ge 2$, and $s \in (0, 1)$. There is a number $\theta = \theta(s, n, \ell)$ such that the following holds.*[2] *Let $\alpha \in \pi_n(\mathbb{S}^\ell) \setminus \{0\}$. Then either $\#_s\alpha$ is attained or for any $\delta > 0$ there exist $\alpha_1, \alpha_2 \in \pi_n(\mathbb{S}^\ell) \setminus \{0\}$ (possibly depending on δ) such that $\alpha = \alpha_1 + \alpha_2$,*

$$\#_s\alpha_1 + \#_s\alpha_2 \le \#_s\alpha + \delta,$$

and

$$\theta < \#_s\alpha_i < \#_s\alpha - \frac{\theta}{2}, \quad \text{for } i = 1, 2. \tag{3.1}$$

From Lemma 14 we can conclude the following existence and energy identity.

Theorem 15 (Energy identity). *Fix $n, \ell \in \mathbb{N}$, with either $(\ell, n) = (1, 1)$ or $\ell \ge 2$, and $s \in (0, 1)$. For each $\alpha \in \pi_n(\mathbb{S}^\ell) \setminus \{0\}$ there exists a finite sequence $(\alpha_i)_{i=1}^N \subset \pi_n(\mathbb{S}^\ell) \setminus \{0\}$ such that*

(1) $\alpha = \sum_{i=1}^N \alpha_i$,
(2) $\#_s\alpha = \sum_{i=1}^N \#_s\alpha_i$,
(3) *$\#_s\alpha_i$ are attained for each $i \in \{1, \dots, N\}$.*

Proof. Fix $\Lambda := \#_s\alpha + 1$.

Fix $\tilde{\delta} \in (0, 1)$. If $\#_s\alpha$ is attained then we are done. If $\#_s\alpha$ is not attained, we apply Lemma 14 and decompose $\alpha = \alpha_1 + \alpha_2$ with

$$\#_s\alpha_1 + \#_s\alpha_2 \le \#_s\alpha + 2^{-1}\tilde{\delta} \quad \text{and} \quad \theta < \#_s\alpha_i < \#_s\alpha - \frac{\theta}{2} \ \text{ for } i = 1, 2.$$

[2] As mentioned before, the case $s \le \frac{1}{2}$, $n = 1$ was not treated in [15] but is covered in [12].

If both $i \in 1, 2$, $\#_s\alpha_i$ are not attained we finish; if at least one is not attained, say $\#_s\alpha_1$, we apply Lemma 14 again, with $\delta = 2^{-2}\tilde{\delta}$. We decompose $\alpha_1 = \alpha_{1,1} + \alpha_{1,2}$ and obtain

$$\#_s\alpha_{1,1} + \#_s\alpha_{1,2} + \#_s\alpha_2 \leq \#_s\alpha_1 + \#_s\alpha_2 + 2^{-2}\tilde{\delta} + 2^{-1}\tilde{\delta} \leq \#_s\alpha + \sum_{i=1}^{2} 2^{-i}\tilde{\delta},$$

with

$$\theta < \#_s\alpha_{1,i} < \#_s\alpha_1 - \frac{\theta}{2} < \#_s\alpha - \theta,$$

where $\alpha = \alpha_{1,1} + \alpha_{1,2} + \alpha_2$. With an abuse of notation we relabel $\alpha = \alpha_1 + \alpha_2 + \alpha_3$.

We apply Lemma 14 iteratively, whenever the minimizer of one of the decomposed terms is not attained – at the ℓth step we take $\delta = 2^{-\ell}\tilde{\delta}$ and obtain a decomposition into $\ell + 1$ terms with

$$\sum_{i=1}^{\ell+1} \#_s\alpha_i \leq \#_s\alpha + \sum_{i=1}^{\ell+1} 2^{-i}\tilde{\delta}.$$

By (3.1), after $2^{\ell-1}$ iterations there is an $i \in \{1, \ldots, 2^{\ell-1} + 1\}$ for which

$$\theta < \#_s\alpha_i < \#_s\alpha - \ell\frac{\theta}{2} = \Lambda - 1 - \ell\frac{\theta}{2}. \tag{3.2}$$

If however $\ell > \frac{2}{\theta}(\Lambda - 1 - \theta)$, we get a contradiction in (3.2). Hence, we may iterate at most 2^{L-1} times for an $L = L(\theta, \Lambda)$, obtaining a decomposition into $N_{\tilde{\delta}}$ terms, where $N_{\tilde{\delta}} \leq 2^{L-1} + 1$ for all $\tilde{\delta} \in (0, 1)$ and

$$\sum_{i=1}^{N_{\tilde{\delta}}} \alpha_i = \alpha$$

such that $\#_s\alpha_i$ must be attained (otherwise we would have continued the iteration of Lemma 14). Moreover,

$$\sum_{i=1}^{N_{\tilde{\delta}}} \#_s\alpha_i \leq \#_s\alpha + \tilde{\delta}.$$

We want to let $\tilde{\delta} \to 0$. Let us stress that as of now the decomposition of α_i depends on $\tilde{\delta}$.

By Theorem 11,

$$\left\{\beta \in \pi_n(\mathbb{S}^\ell) : \#_s\beta \leq \#_s\alpha + 1\right\}$$

is a finite set. Hence, there is only a finite number of possibilities of $\alpha_i = \alpha_i(\tilde{\delta})$. Thus there exists a sequence $\tilde{\delta}_k \to 0$ such that $\alpha_i(\tilde{\delta}_k) = \alpha_i(\tilde{\delta}_j)$ for all $j, k \in \mathbb{N}$. Thus we have

$$\sum_{i=1}^{N} \#_s\alpha_i \leq \#_s\alpha + \tilde{\delta}_k,$$

where the left-hand side does not depend on $\tilde{\delta}_k$ anymore. Letting $k \to \infty$ we obtain

$$\sum_{i=1}^{N} \#_s \alpha_i \leq \#_s \alpha.$$

On the other hand, since $\sum_{i=1}^{N} \alpha_i = \alpha$, we have by Proposition 9,

$$\#_s \alpha \leq \sum_{i=1}^{N} \#_s \alpha_i.$$

Combining the two last statements we obtain

$$\#_s \alpha = \sum_{i=1}^{N} \#_s \alpha_i. \qquad \blacksquare$$

Let us remark in passing that another strategy to obtain Theorem 15 would be to extend the methods in [8], which mostly rely on the conformal invariance of the norms involved, and thus should be applicable to our case.

4. Globally improved regularity in the conformal scaling

In this section we show that $W^{s,\frac{n}{s}}$-minimizers actually belong *globally* to $W^{\Theta s,\frac{n}{\Theta s}}(\mathbb{S}^n, \mathbb{S}^\ell)$ for a $\Theta > 1$. To do so we adapt the strategy of [23]. It seems to us that we use a somehow unique feature of the Gagliardo seminorm and the fractional p-Laplacian (i.e., we do not see how the argument would work for the classical p-Laplacian): we use intrinsically a feature of differential stability that was observed for the fractional p-Laplacian but is not known for the classical p-Laplacian; see [13, 24].

Theorem 16. *Fix $n, \ell \in \mathbb{N}$. For every $0 < s_0 < s_1 < 1$ there exists $\Theta > 1$ and for any $\Lambda > 0$ there is a constant $C(\Lambda) = C(n, \ell, s_0, s_1, \Lambda) > 0$ such that the following holds.*

If $s_0 < s < s_1$ and $u \in W^{s,\frac{n}{s}}(\mathbb{S}^n, \mathbb{S}^\ell)$ is a $W^{s,\frac{n}{s}}(\mathbb{S}^n, \mathbb{S}^\ell)$-minimizer in its own homotopy group with

$$[u]_{W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^\ell)} \leq \Lambda,$$

then

$$[u]_{W^{\Theta s,\frac{n}{\Theta s}}(\mathbb{S}^n,\mathbb{S}^\ell)} \leq C(\Lambda).$$

Theorem 16 is a consequence of Propositions 18 and 21 below.

Remark 17. A few observations regarding the previous regularity theorem are in order.

(1) The study of the regularity of fractional harmonic maps was initiated by Da Lio–Rivière in their celebrated papers [6, 7].

(2) The argument below applies generally to critical points, not only minimizers. We only state the a priori estimate versions, since (local) higher regularity was discussed in [15].

(3) In [23] it was proven that minimizers as in Theorem 16 are Hölder continuous; see also [14]. Observe that, however, there is no hope to prove

$$[u]_{C^{0,\alpha}(\mathbb{S}^n,\mathbb{S}^\ell)} \precsim [u]_{W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^\ell)}.$$

Indeed, this can be simply disproved by conformal rescaling, concentrating the map u into one point. The right-hand side is conformally invariant, and thus does not change. However, the continuity on the left-hand side becomes worse and worse.

(4) Similarly, there is no hope to obtain

$$[u]_{W^{s_1,p_1}(\mathbb{S}^n,\mathbb{S}^\ell)} \precsim [u]_{W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^\ell)}$$

whenever $s_1 p_1 > n$.

(5) While Theorem 16, does not imply continuity, and therefore seems to be a weaker result than [14, 23], it has the crucial advantage of being a global result on all of $\mathbb{S}^n$.

By the conformal invariance of the energy we may replace the domain $\mathbb{S}^n$ by $\mathbb{R}^n$ and assume $u \in \dot{W}^{s,\frac{n}{s}}(\mathbb{R}^n, \mathbb{S}^\ell)$.

4.1. Improved global estimates for harmonic maps

We begin with the Euler–Lagrange equations for $W^{s,\frac{n}{s}}$-harmonic maps; see, e.g., [23]. For any $\varphi \in L^\infty \cap \dot{W}^{s,\frac{n}{s}}(\mathbb{R}^n, \mathbb{R}^{\ell+1})$, a critical point $u \in W^{s,\frac{n}{s}}(\mathbb{R}^n, \mathbb{S}^\ell)$ of the energy $\mathcal{E}_{s,\frac{n}{s}}$ satisfies the equation

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y)) \wedge (u(x)\varphi(x)-u(y)\varphi(y))}{|x-y|^{2n}} \,\mathrm{d}x\,\mathrm{d}y = 0. \quad (4.1)$$

Here, $\wedge\colon \mathbb{R}^\ell \times \mathbb{R}^\ell$ denotes the wedge product in $\mathbb{R}^{\ell+1}$,

$$(u \wedge v)_{ij} = u^i v^j - u^j v^i, \quad i, j \in \{1, \dots, \ell+1\}.$$

The crucial result is that the equation for fractional $W^{s,\frac{n}{s}}$-harmonic maps improves globally.

Proposition 18. *Let* $0 < s_0 < s_1 < 1$. *There exists a* $\theta \in (0,1)$ *such that the following holds.*

For any $s \in (s_0, s_1)$ *and any* $u \in \dot{W}^{s,\frac{n}{s}}(\mathbb{R}^n, \mathbb{S}^\ell)$, *which is a minimizing* $W^{s,\frac{n}{s}}$*-harmonic map in its own homotopy group, we have*

$$[(-\Delta)^{s}_{\frac{n}{s}} u]_{W^{-\theta s,\frac{n}{n-\theta s}}(\mathbb{R}^n)} \le C(s_0,s_1,n)[u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)};$$

that is, for any $\psi \in \dot{W}^{\theta s, \frac{n}{\theta s}}(\mathbb{R}^n, \mathbb{R}^{\ell+1})$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{2n}}\, dx\, dy \leq C(s_0, s_1, n)[u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}[\psi]_{W^{\theta s,\frac{n}{\theta s}}(\mathbb{R}^n)}. \tag{4.2}$$

In the proof below we will frequently work with the fractional Laplacian $(-\Delta)^{\frac{s}{2}}$ which can be defined as

$$(-\Delta)^{\frac{s}{2}} f = c_s \mathcal{F}^{-1}(|\xi|^s \mathcal{F} f),$$

where $\mathcal{F}$ is the Fourier transform. When $s \in (0,1)$, with a different constant, we also have the representation

$$(-\Delta)^{\frac{s}{2}} f(x) = c_s \int_{\mathbb{R}^n} \frac{f(y)-f(x)}{|x-y|^{n+s}}\, dy.$$

The inverse of the fractional Laplacian is the Riesz potential I^s, defined as

$$I^s f = c_s \mathcal{F}^{-1}(|\xi|^{-s} \mathcal{F} f),$$

or, for $s \in (0,n)$,

$$I^s f(x) = c_s \int_{\mathbb{R}^n} |x-y|^{s-n} f(y)\, dy. \tag{4.3}$$

For mapping properties of the Riesz potential we refer the reader to standard literature, e.g., [9, 10, 21, 27].

In order to prove Proposition 18 we consider the following potential introduced in [23]:

$$T_t u(z) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))(|x-z|^{t-n}-|y-z|^{t-n})}{|x-y|^{2n}}\, dx\, dy. \tag{4.4}$$

Observe that for $t < s$ we have by the representation of the Riesz potential I^t, (4.3),

$$\int_{\mathbb{R}^n} T_t u(z)\varphi(z)\, dz = c \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))(I^t\varphi(x)-I^t\varphi(y))}{|x-y|^{2n}}\, dx\, dy.$$

From the definition of T_t and Hölder's inequality we have

$$\left|\int_{\mathbb{R}^n} T_t u(z)\varphi(z)\, dz\right| \precsim [u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}[I^t\varphi]_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}. \tag{4.5}$$

Thus, if $t < s$, $T_t u$ is a tempered distribution for $u \in W^{s,\frac{n}{s}}(\mathbb{R}^n)$.

Observe that even as a distribution we have

$$\|T_t u\|_{L^{\frac{n}{n-t}}(\mathbb{R}^n)} \precsim \|u \cdot T_t u\|_{L^{\frac{n}{n-t}}(\mathbb{R}^n)} + \|u \wedge T_t u\|_{L^{\frac{n}{n-t}}(\mathbb{R}^n)}, \tag{4.6}$$

whenever the right-hand side is finite. In the next two lemmas we will separately estimate both terms on the right-hand side of (4.6): the orthogonal projection $u \cdot T_t u$ and the tangential projection $u \wedge T_t u$ (orthogonal and tangential are meant with respect to the tangent space $T_u \mathbb{S}^\ell$).

Lemma 19. *Let $0 < s_0 < s_1 < 1$. There exist a $\theta \in (0,1)$ and a constant $C = C(s_0, s_1, n, \ell) > 0$ such that the following holds.*

For any $s \in (s_0, s_1)$, there exists $t < \theta s$ such that if u is a $W^{s,\frac{n}{s}}(\mathbb{R}^n, \mathbb{S}^\ell)$-minimizing harmonic map in its own homotopy group, then for T_t as in (4.4),

$$\|u \cdot T_t u\|_{L^{\frac{n}{n-t}}(\mathbb{R}^n)} \le C [u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}. \tag{4.7}$$

Proof. We argue similarly to the proof of [23, Lemma 6.5]. We note that since $|u| = 1$ we have

$$(u(x) - u(y)) \cdot u(z) = -\frac{1}{2}(u(x) - u(y)) \cdot (u(x) + u(y) - 2u(z))$$

and hence

$$|u \cdot T_t u(z)| \precsim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-1} |u(x)+u(y)-2u(z)| \big| |x-z|^{t-n} - |y-z|^{t-n} \big|}{|x-y|^{2n}} \, dx \, dy.$$

We observe that for $r \in (0,1)$, we have by [25, Proposition 6.6],

$$|u(x) - u(y)| \precsim |x-y|^r (\mathcal{M}(-\Delta)^{\frac{r}{2}} u(x) + \mathcal{M}(-\Delta)^{\frac{r}{2}} u(y)), \tag{4.8}$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal function.

We follow the proof in [23, Proposition 6.3] but replace the use of [23, Proposition 6.2] by (4.8) and consider the three regimes (similarly to [23, Proposition 6.1])

$$\begin{aligned} &\big\{|x-y| \precsim \min\{|x-z|, |y-z|\}\big\}: \quad \text{in this case } |x-z| \approx |y-z|, \\ &\big\{|x-z| \precsim \min\{|y-z|, |x-y|\}\big\}: \quad \text{in this case } |y-z| \approx |x-y|, \\ &\big\{|y-z| \precsim \min\{|x-z|, |x-y|\}\big\}: \quad \text{in this case } |x-z| \approx |x-y|. \end{aligned}$$

We obtain for $\tilde{t} \in (0,t)$ with $r + \tilde{t} \in (0,1)$,

$$\begin{aligned} &|u(x) + u(y) - 2u(z)| \big| |x-z|^{t-n} - |y-z|^{t-n} \big| \\ &\quad \precsim (\mathcal{M}(-\Delta)^{\frac{r}{2}} u(x) + \mathcal{M}(-\Delta)^{\frac{r}{2}} u(y) + \mathcal{M}(-\Delta)^{\frac{r}{2}} u(z)) \\ &\qquad \times |x-y|^{r+\tilde{t}} k_{t-\tilde{t},t}(x,y,z), \end{aligned} \tag{4.9}$$

where $\kappa_{\alpha,\gamma}(x,y,z)$ is given by

$$\begin{aligned} \kappa_{\alpha,\gamma}(x,y,z) &= \min\{|x-z|^{\alpha-n}, |y-z|^{\alpha-n}\} \\ &\quad + \left(\frac{|y-z|}{|x-y|}\right)^{\gamma-\alpha} |y-z|^{\alpha-n} \chi_{\{|x-z| \precsim \min\{|y-z|,|x-y|\}\}} \\ &\quad + \left(\frac{|x-z|}{|x-y|}\right)^{\gamma-\alpha} |x-z|^{\alpha-n} \chi_{\{|y-z| \precsim \min\{|x-z|,|x-y|\}\}}. \end{aligned}$$

Moreover, we have by (4.8),

$$|u(x)-u(y)|^{\frac{n}{s}-1} \precsim |x-y|^{r(\frac{n}{s}-1)}\big((\mathcal{M}(-\Delta)^{\frac{r}{2}}u(x))^{\frac{n}{s}-1}+(\mathcal{M}(-\Delta)^{\frac{r}{2}}u(y))^{\frac{n}{s}-1}\big). \tag{4.10}$$

Combining (4.10) with (4.9) we obtain for a $\varphi \in C_c^\infty(\mathbb{R}^n)$, $\|\varphi\|_{L^{\frac{n}{t}}(\mathbb{R}^n)} \leq 1$,

$$\begin{aligned}
&\|u \cdot T_t u\|_{L^{\frac{n}{n-t}}(\mathbb{R}^n)} \\
&\precsim \int_{\mathbb{R}^n} u(z)\cdot T_t u(z)\varphi(z)\,\mathrm{d}z \\
&\precsim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-1}|u(x)+u(y)-2u(z)| \times \big||x-z|^{t-n}-|y-z|^{t-n}\big|}{|x-y|^{2n}}|\varphi(z)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\precsim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{r\frac{n}{s}+\tilde{t}-n-n}k_{t-\tilde{t},t}(x,y,z)U_r(x,y,z)|\varphi(z)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,
\end{aligned} \tag{4.11}$$

where

$$U_r(x,y,z) := \big((\mathcal{M}(-\Delta)^{\frac{r}{2}}u(x))^{\frac{n}{s}-1}+(\mathcal{M}(-\Delta)^{\frac{r}{2}}u(y))^{\frac{n}{s}-1}\big) \times \big(\mathcal{M}(-\Delta)^{\frac{r}{2}}u(x)+\mathcal{M}(-\Delta)^{\frac{r}{2}}u(y)+\mathcal{M}(-\Delta)^{\frac{r}{2}}u(z)\big).$$

Assuming $r\frac{n}{s}+\tilde{t}-n>0$ we further estimate (4.11) with the help of [23, Proposition 6.4] and Hölder's inequality:

$$\begin{aligned}
&\|u\cdot T_t u\|_{L^{\frac{n}{n-t}}(\mathbb{R}^n)} \\
&\precsim \max_{\substack{t_1+t_2+t_3\\ =r\frac{n}{s}-n+t}} \int_{\mathbb{R}^n} I^{t_1}(\mathcal{M}(-\Delta)^{\frac{r}{2}}u(z))^{\frac{n}{s}-1} I^{t_2}(\mathcal{M}(-\Delta)^{\frac{r}{2}}u(z)) I^{t_3}|\varphi(z)|\,\mathrm{d}z \\
&\precsim \max_{\substack{t_1+t_2+t_3\\ =r\frac{n}{s}-n+t}} \|I^{t_1}(\mathcal{M}(-\Delta)^{\frac{r}{2}}u)^{\frac{n}{s}-1}\|_{L^{\frac{n}{r\frac{n}{s}-r-t_1}}(\mathbb{R}^n)} \\
&\quad\times \|I^{t_2}(\mathcal{M}(-\Delta)^{\frac{r}{2}}u)\|_{L^{\frac{n}{r-t_2}}(\mathbb{R}^n)}\|I^{t_3}|\varphi|\|_{L^{\frac{n}{t-t_3}}(\mathbb{R}^n)}.
\end{aligned} \tag{4.12}$$

Assuming $r\frac{n}{s}-r-t_1 \geq -r-t+n>0$, $t_2 \leq r\frac{n}{s}-n+t<r$, $t_3 \leq r\frac{n}{s}-n+t<t$, we apply Sobolev's inequality and the maximal theorem twice to get

$$\begin{aligned}
\|u\cdot T_t u\|_{L^{\frac{n}{n-t}}(\mathbb{R}^n)} &\precsim \|(\mathcal{M}(-\Delta)^{\frac{r}{2}}u)^{\frac{n}{s}-1}\|_{L^{\frac{n}{r\frac{n}{s}-r}}(\mathbb{R}^n)}\|\mathcal{M}(-\Delta)^{\frac{r}{2}}u\|_{L^{\frac{n}{r}}(\mathbb{R}^n)}\|\varphi\|_{L^{\frac{n}{t}}(\mathbb{R}^n)} \\
&\precsim \|(-\Delta)^{\frac{r}{2}}u\|^{\frac{n}{s}-1}_{L^{(\frac{n}{s}-1)\frac{n}{r\frac{n}{s}-r}}(\mathbb{R}^n)}\|(-\Delta)^{\frac{r}{2}}u\|_{L^{\frac{n}{r}}(\mathbb{R}^n)}\|\varphi\|_{L^{\frac{n}{t}}(\mathbb{R}^n)} \\
&= \|(-\Delta)^{\frac{r}{2}}u\|^{\frac{n}{s}}_{L^{\frac{n}{r}}(\mathbb{R}^n)}\|\varphi\|_{L^{\frac{n}{t}}(\mathbb{R}^n)} \\
&\precsim [u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}.
\end{aligned} \tag{4.13}$$

In the last line we used $r<s$.

The conditions on r and t used in estimates (4.12) and (4.13) are

$$r < s, \quad n < \frac{n}{s} r + t < n + r, \quad r + t < n.$$

If $n = 1$ we choose $r = t$, and then any $t \in (\frac{s}{1+s}, \min\{s, \frac{1}{2}\})$ is admissible, so we can pick

$$t := \frac{\frac{s}{1+s} + \min\{s, \frac{1}{2}\}}{2} \le s \frac{\frac{1}{1+s} + 1}{2} \le s \frac{\frac{1}{1+s_0} + 1}{2} := s\theta.$$

Observe that we do not need to make a distinction between the cases $\frac{n}{s} < 2$ and $\frac{n}{s} \ge 2$.

If $n \ge 2$ it is even easier, since $r + t < n$ becomes a trivial condition. This finishes the proof. ■

Now we estimate the second part of the right-hand side of (4.6).

Lemma 20. *Let $0 < s_0 < s_1 < 1$. There exists a $\theta \in (0, 1)$ such that the following holds.*

For any $s \in (s_0, s_1)$, there exists $t < \theta s$ such that if u is a $W^{s,\frac{n}{s}}(\mathbb{R}^n, \mathbb{S}^\ell)$-minimizing harmonic map in its own homotopy group, then for T_t as in (4.4),

$$\|u \wedge T_t u\|_{L^{\frac{n}{n-t}}(\mathbb{R}^n)} \precsim [u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}. \tag{4.14}$$

Proof. We argue as in [23, Proof of Lemma 3.5].

By duality, there is some $\psi \in C_c^\infty(\mathbb{R}^n)$, $\|\psi\|_{L^{\frac{n}{t}}(\mathbb{R}^n)} \le 1$, for which

$$\|u \wedge T_t u\|_{L^{\frac{n}{n-t}}(\mathbb{R}^n)} \precsim \int_{\mathbb{R}^n} u \wedge T_t u \psi.$$

Take $R \gg 1$ so that $\operatorname{supp} \psi \subset B(0, R)$. Let $\eta_R \in C_c^\infty(B(0, 2R))$ be a cut-off function such that $\eta_R \equiv 1$ in $B(0, R)$, and set

$$\varphi_{1,R} := \eta_R I^t \psi,$$
$$\varphi_{2,R} := (1 - \eta_R) I^t \psi.$$

Then

$$\int_{\mathbb{R}^n} u \wedge T_t u \psi = \int_{\mathbb{R}^n} u \wedge T_t u (-\Delta)^{\frac{t}{2}} \varphi_{1,R} + \int_{\mathbb{R}^n} u \wedge T_t u (-\Delta)^{\frac{t}{2}} \varphi_{2,R}.$$

We observe that with a constant independent of $R \gg 1$,

$$\|(-\Delta)^{\frac{t}{2}} \varphi_{1,R}\|_{L^{\frac{n}{t}}(\mathbb{R}^n)} \precsim \|\psi\|_{L^{\frac{n}{t}}(\mathbb{R}^n)} \le 1,$$

hence

$$\int_{\mathbb{R}^n} u \wedge T_t u \psi \le \sup_{\substack{\varphi \in C_c^\infty(\mathbb{R}^n), \\ \|(-\Delta)^{\frac{t}{2}} \varphi\|_{L^{\frac{n}{t}}(\mathbb{R}^n)} \precsim 1}} \int_{\mathbb{R}^n} u \wedge T_t u (-\Delta)^{\frac{t}{2}} \varphi$$
$$+ \limsup_{R \to \infty} \left| \int_{\mathbb{R}^n} u \wedge T_t u (-\Delta)^{\frac{t}{2}} \varphi_{2,R} \right|.$$

For the second term on the right-hand side we observe that similarly to (4.5), for suitably small $\varepsilon > 0$,

$$\left|\int_{\mathbb{R}^n} u \wedge T_t u(-\Delta)^{\frac{t}{2}} \varphi_{2,R}\right| \precsim [u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} [I^t(u \wedge (-\Delta)^{\frac{t}{2}}\varphi_{2,R})]_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}$$
$$\precsim [u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} \|(-\Delta)^{\frac{s-t+\varepsilon}{2}}(u \wedge (-\Delta)^{\frac{t}{2}}\varphi_{2,R})\|_{L^{\frac{n}{s+\varepsilon}}(\mathbb{R}^n)}$$
$$\precsim [u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} (1+[u]_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}) \|(-\Delta)^{\frac{s+\varepsilon}{2}}\varphi_{2,R}\|_{L^{\frac{n}{s+\varepsilon}}(\mathbb{R}^n)},$$

where in the second estimate we used an embedding in Triebel–Lizorkin spaces; see [21, Theorem 2.2.3].

Now we observe that, due to the support of $1-\eta_R$ and ψ, we have for some $\sigma > 0$

$$\|(-\Delta)^{\frac{s+\varepsilon}{2}}\varphi_{2,R}\|_{L^{\frac{n}{s+\varepsilon}}(\mathbb{R}^n)} \precsim R^{-s-\varepsilon}\|I^t\psi\|_{L^{\frac{n}{s+\varepsilon}}} + \|(1-\eta_R)(-\Delta)^{\frac{s+\varepsilon-t}{2}}\psi\|_{L^{\frac{n}{s+\varepsilon}}}$$
$$\precsim R^{-\sigma} C(\operatorname{supp}\psi)\|\psi\|_{L^{\frac{n}{t}}} \xrightarrow{R\to\infty} 0.$$

Thus, for some $\varphi \in C_c^\infty(\mathbb{R}^n)$, $\|(-\Delta)^{\frac{t}{2}}\varphi\|_{L^{\frac{n}{t}}(\mathbb{R}^n)} \le 1$,

$$\|u \wedge T_t u\|_{L^{\frac{n}{n-t}}(\mathbb{R}^n)} \precsim \int_{\mathbb{R}^n} u \wedge T_t u(-\Delta)^{\frac{t}{2}}\varphi$$
$$= -\int_{\mathbb{R}^n} \varphi(-\Delta)^{\frac{t}{2}} u \wedge T_t u + \underbrace{\int_{\mathbb{R}^n} (-\Delta)^{\frac{t}{2}}(\varphi u) \wedge T_t u}_{\overset{(4.1)}{=}0}$$
$$-\int_{\mathbb{R}^n} H_{(-\Delta)^{\frac{t}{2}}}(\varphi, u) \wedge T_t u. \tag{4.15}$$

Here we use the Leibniz term notation

$$H_{(-\Delta)^{\frac{t}{2}}}(f,g) := (-\Delta)^{\frac{t}{2}}(fg) - f(-\Delta)^{\frac{t}{2}}g - (-\Delta)^{\frac{t}{2}}f\; g.$$

For the last term of (4.15) we observe that similarly to (4.5) for a suitably small $\varepsilon > 0$,

$$\int_{\mathbb{R}^n} H_{(-\Delta)^{\frac{t}{2}}}(\varphi,u) \wedge T_t u \precsim [u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} [I^t H_{(-\Delta)^{\frac{t}{2}}}(\varphi,u)]_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}$$
$$\precsim [u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} \|(-\Delta)^{\frac{s-t+\varepsilon}{2}} H_{(-\Delta)^{\frac{t}{2}}}(\varphi,u)\|_{L^{\frac{n}{s+\varepsilon}}(\mathbb{R}^n)}$$
$$\precsim [u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} \|(-\Delta)^{\frac{t}{2}}\varphi\|_{L^{\frac{n}{t}}(\mathbb{R}^n)}.$$

The last line works as long $s - t + \varepsilon < t$.

For the remaining estimates we abbreviate

$$\operatorname{diff}_{\frac{n}{s}} u(x,y) = |u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y)).$$

By the representation of the Riesz potential, (4.3), and (4.1) we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varphi(-\Delta)^{\frac{t}{2}} u \wedge T_t u \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\operatorname{diff}_{\frac{n}{s}} u(x,y) \wedge \left(I^t(\varphi(-\Delta)^{\frac{t}{2}}u)(x) - I^t(\varphi(-\Delta)^{\frac{t}{2}}u)(y)\right)}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\operatorname{diff}_{\frac{n}{s}} u(x,y) \wedge \left(I^t(\varphi(-\Delta)^{\frac{t}{2}}u)(x) - I^t(\varphi(-\Delta)^{\frac{t}{2}}u)(y) - \frac{1}{2}(u(x)-u(y))(\varphi(x)+\varphi(y))\right)}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Exactly as in the first lines of the proof of [23, Lemma 6.6] we have

$$\begin{aligned}
&\left|I^t(\varphi(-\Delta)^{\frac{t}{2}}u)(x) - I^t(\varphi(-\Delta)^{\frac{t}{2}}u)(y) - \frac{1}{2}(u(x)-u(y))(\varphi(x)+\varphi(y))\right| \\
&\quad\precsim \int_{\mathbb{R}^n} \left||x-z|^{t-n} - |y-z|^{t-n}\right| |(-\Delta)^{\frac{t}{2}}u(z)|\, |\varphi(x)+\varphi(y)-2\varphi(z)|\,\mathrm{d}z.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \varphi(-\Delta)^{\frac{t}{2}} u \wedge T_t u\right| \\
&\quad\precsim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-1}\left||x-z|^{t-n} - |y-z|^{t-n}\right| \times |(-\Delta)^{\frac{t}{2}}u(z)|\,|\varphi(x)+\varphi(y)-2\varphi(z)|}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.
\end{aligned}$$

This is the same situation as in (4.11): the role of $\varphi \in L^{\frac{n}{t}}$ in (4.11) is taken here by $(-\Delta)^{\frac{t}{2}}u \in L^{\frac{n}{t}}$ and the role of $|u(x)+u(y)-2u(z)|$ by $|\varphi(x)+\varphi(y)-2\varphi(z)|$, and observe that $(-\Delta)^{\frac{t}{2}}\varphi \in L^{\frac{n}{t}}$. As was discussedthere we can pick $r \le t$, and thus we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} \varphi(-\Delta)^{\frac{t}{2}} u \wedge T_t u\right| &\precsim \|(-\Delta)^{\frac{r}{2}}u\|_{L^{\frac{n}{r}}(\mathbb{R}^n)}^{\frac{n}{s}-1} \|(-\Delta)^{\frac{r}{2}}\varphi\|_{L^{\frac{n}{r}}(\mathbb{R}^n)}^{\frac{n}{s}-1} \|(-\Delta)^{\frac{t}{2}}u\|_{L^{\frac{n}{t}}(\mathbb{R}^n)} \\
&\precsim [u]_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}^{\frac{n}{s}} \|(-\Delta)^{\frac{t}{2}}\varphi\|_{L^{\frac{n}{t}}(\mathbb{R}^n)}.
\end{aligned}$$

We can conclude. (Again it is worth noting that the proof above does not need to distinguish between the cases $\frac{n}{s} \ge 2$ and $\frac{n}{s} \le 2$.) ■

We are now ready to proceed with the proof of the main result of this section.

Proof of Proposition 18. Combining (4.6) with (4.14) and (4.7) we get for a $t < \theta s$, where t and θ are as in Lemmas 19 and 20,

$$\|T_t u\|_{L^{\frac{n}{n-t}}(\mathbb{R}^n)} \precsim [u]_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}^{\frac{n}{s}}. \tag{4.16}$$

By duality, (4.16) implies, for any $\varphi \in C^\infty \cap L^{\frac{n}{t}}(\mathbb{R}^n, \mathbb{R}^{\ell+1})$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))(I^t\varphi(x)-I^t\varphi(y))}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y$$
$$\precsim [u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} \|\varphi\|_{L^{\frac{n}{t}}(\mathbb{R}^n)}.$$

Thus for any $\psi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y$$
$$\precsim [u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} \|(-\Delta)^{\frac{t}{2}}\psi\|_{L^{\frac{n}{t}}(\mathbb{R}^n)}.$$

Using Sobolev embedding this implies for $t < t_2 < s$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y$$
$$\precsim [u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} [\psi]_{W^{t_2,\frac{n}{t_2}}(\mathbb{R}^n)}.$$

The constant depends on $|t-t_2|$, so by taking $\tilde{\theta}$ slightly larger than θ and $t_2 = \tilde{\theta}s$ we have $\tilde{\theta}s - t > (\tilde{\theta}-\theta)s_0$, so the constant can be chosen uniform and, by density, [21, Section 2.6.2, Proposition 1], we obtain (4.2) for $\tilde{\theta}$. ∎

4.2. A fractional version of Iwaniec's stability result

A fractional version of Iwaniec's stability result was proposed in [24]. However, the result of [24] does not apply in our situation since it only considers stability in the differential direction, without adjusting the integrability. We need the latter, since we need to stay in the scaling-invariant case. Hence, we employ a different version of Iwaniec's stability result [11, Theorem 13.2.1] to obtain the following regularity result.

Proposition 21. *For any $0 < s_0 < s_1 < 1$ there exists an $\varepsilon_0 = \varepsilon_0(s_0, s_1, n) > 0$ such that the following holds.*

For any $s \in (s_0, s_1)$ and any $\Lambda > 0$ there exists a constant $C(\Lambda)$ such that if $u \in L^\infty \cap W^{s,\frac{n}{s}}(\mathbb{R}^n, \mathbb{R}^N)$ satisfies for a $t \in (s-\varepsilon_0, s]$ and for any $\psi \in \dot{W}^{t,\frac{n}{t}}(\mathbb{R}^n, \mathbb{R}^N)$,

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y\right| \leq \Lambda [\psi]_{W^{t,\frac{n}{t}}(\mathbb{R}^n)}, \quad (4.17)$$

then for $r := s\frac{n-t}{n-s} \geq s$,

$$[u]_{W^{r,\frac{n}{r}}(\mathbb{R}^n)} \leq C(\Lambda, \varepsilon_0, s_0, s_1).$$

We first observe that for $p := \frac{n}{s}$ and $\varepsilon := p - \frac{n}{r} = \frac{n}{s} - \frac{n}{r} > 0$ we have $\frac{t}{1-\varepsilon} = r$ and

$$[u]^{\frac{n}{r}}_{W^{r,\frac{n}{r}}} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\begin{array}{c}|u(x)-u(y)|^{p-2}(u(x)-u(y))\\ \times |u(x)-u(y)|^{-\varepsilon}(u(x)-u(y))\end{array}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y. \quad (4.18)$$

We perform a de facto Hodge decomposition:

$$|u(x)-u(y)|^{-\varepsilon}(u(x)-u(y)) = A(x)-A(y)+G(x,y), \tag{4.19}$$

where, in the terminology of [14], we choose A such that

$$(-\Delta)^t A(x) := \operatorname{div}_t\Big(\frac{|u(x)-u(y)|^{-\varepsilon}(u(x)-u(y))}{|x-y|^t}\Big), \tag{4.20}$$

that is, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$(-\Delta)^t A[\varphi] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{-\varepsilon}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2t}}\,dx\,dy.$$

From the linear theory of partial differential equation we have the following lemma.

Lemma 22. *For any $0<t_0<t_1<1$ there exists an $\varepsilon_0>0$ such that whenever $t\in(t_0,t_1)$, an $A\in \dot W^{t,\frac{n}{t}}(\mathbb{R}^n,\mathbb{R}^N)$ as in (4.20) exists and satisfies the estimate*

$$[A]_{W^{t,\frac{n}{t}}(\mathbb{R}^n)} \precsim [u]^{1-\varepsilon}_{W^{\frac{t}{1-\varepsilon},(1-\varepsilon)\frac{n}{t}}(\mathbb{R}^n)} = [u]^{1-\varepsilon}_{W^{r,\frac{n}{r}}(\mathbb{R}^n)}$$

with a constant independent of ε as long as $\varepsilon\in[0,\varepsilon_0]$. The quantity A is unique up to constants.

Proof. Proceeding exactly as in [5, Lemma A.1] we have the a priori estimate

$$\begin{aligned}(-\Delta)^t A[\varphi] &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{-\varepsilon}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2t}}\,dx\,dy\\ &\le [u]^{1-\varepsilon}_{W^{\frac{t}{1-\varepsilon},(1-\varepsilon)\frac{n}{t}}(\mathbb{R}^n)}[\varphi]_{W^{t,\frac{n}{n-t}}(\mathbb{R}^n)}.\end{aligned}$$

Using the identification via Triebel–Lizorkin spaces, [21, Section 2], we have

$$[A]_{W^{t,\frac{n}{t}}(\mathbb{R}^n)} \approx [A]_{\dot F^{t,\frac{n}{t}}_{\frac{n}{t}}} \approx [(-\Delta)^t A]_{\dot F^{-t,\frac{n}{t}}_{\frac{n}{t}}} \approx [(-\Delta)^t A]_{\left(\dot F^{t,\frac{n}{n-t}}_{\frac{n}{n-t}}\right)^*} \precsim [u]^{1-\varepsilon}_{W^{\frac{t}{1-\varepsilon},(1-\varepsilon)\frac{n}{t}}(\mathbb{R}^n)}.$$

The constants depend only on t_0 and t_1 since $t\in(t_0,t_1)$. In particular, A exists since $(-\Delta)^t A\in\big(\dot F^{t,\frac{n}{n-t}}_{\frac{n}{n-t}}\big)^*$. Further, A is unique up to constants, since $[A]_{W^{t,\frac{n}{t}}(\mathbb{R}^n)}=0$ implies that A is a constant. ■

From Lemma 22 and (4.19) we have in particular,

$$\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|G(x,y)|^{\frac{n}{t}}}{|x-y|^{2n}}\,dx\,dy\right)^{\frac{t}{n}} \precsim [u]^{1-\varepsilon}_{W^{r,\frac{n}{r}}(\mathbb{R}^n)}.$$

The latter estimate can, however, be improved.

Proposition 23. *For any $0<t_0<t_1<1$ there exist an $\varepsilon_0>0$ and a constant $C=C(t_0,t_1,\varepsilon_0)$ such that for any $\varepsilon\in(0,\varepsilon_0)$, and $t\in(t_0,t_1)$ and G as in (4.19),*

$$\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|G(x,y)|^{\frac{n}{t}}}{|x-y|^{2n}}\,dx\,dy\right)^{\frac{t}{n}} \le C|\varepsilon|[u]^{1-\varepsilon}_{W^{r,\frac{n}{r}}(\mathbb{R}^n)}.$$

Proof. We follow the approach in [11, Theorem 13.2.1]. Fix ε_0 to be specified later. By density, cf. [21, Section 2.6.2, Proposition 1], we may assume that $u \in C_c^\infty(\mathbb{R}^n)$ (observe that we can change u by a constant without changing the definitions of G and A).

For $z \in \mathbb{C}$, $|z| \le \varepsilon_0$ we set

$$G_z(x,y) = (A_z(x) - A_z(y)) - |u(x)-u(y)|^z (u(x)-u(y)), \tag{4.21}$$

where A_z is defined as the solution to

$$(-\Delta)^t A_z[\varphi] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^z (u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2t}}\,\mathrm{d}x\,\mathrm{d}y, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n). \tag{4.22}$$

This is well defined since $u \in C_c^\infty(\mathbb{R}^n)$ and the right-hand side is a linear functional on a Triebel–Lizorkin space. Assume that for a $\lambda \in (0, \frac{1}{2})$ we have a $q \in (1,\infty)$ such that

$$2t - \frac{n}{q}(1+\Re(z)) \in (\lambda, 1-\lambda). \tag{4.23}$$

Let us remark here that later we will apply this to $q = \frac{n}{t}(1+\Re(z))$ for $|z| \le \varepsilon_0$, so that λ and ε_0 can be chosen depending only on t_0 and t_1.

We get by (4.21),

$$\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|G_z(x,y)|^{\frac{q}{1+\Re(z)}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1+\Re(z)}{q}} \precsim [u]^{1+\Re(z)}_{W^{\frac{n}{q},q}(\mathbb{R}^n)} + [A_z]_{W^{\frac{n}{q}(1+\Re(z)),\frac{q}{1+\Re(z)}}(\mathbb{R}^n)}. \tag{4.24}$$

Arguing with the identification of Triebel–Lizorkin spaces as in Lemma 22 we have

$$[A_z]_{W^{\frac{n}{q}(1+\Re(z)),\frac{q}{1+\Re(z)}}(\mathbb{R}^n)} \approx [A_z]_{\dot{F}^{\frac{n}{q}(1+\Re(z))}_{\frac{q}{1+\Re(z)},\frac{q}{1+\Re(z)}}(\mathbb{R}^n)} \approx [(-\Delta)^t A]_{\dot{F}^{\frac{n}{q}(1+\Re(z))-2t}_{\frac{q}{1+\Re(z)},\frac{q}{1+\Re(z)}}(\mathbb{R}^n)}.$$

Moreover,

$$\dot{F}^{\frac{n}{q}(1+\Re(z))-2t}_{\frac{q}{1+\Re(z)},\frac{q}{1+\Re(z)}}(\mathbb{R}^n) = \left(\dot{F}^{2t-\frac{n}{q}(1+\Re(z))}_{(\frac{q}{1+\Re(z)})',(\frac{q}{1+\Re(z)})'}(\mathbb{R}^n)\right)^* = \left(W^{2t-\frac{n}{q}(1+\Re(z)),(\frac{q}{1+\Re(z)})'}(\mathbb{R}^n)\right)^* =: X^*.$$

Hence, by the equivalence of the norms (the constant depends on λ),

$$\begin{aligned}
&[A_z]_{W^{\frac{n}{q}(1+\Re(z)),\frac{q}{1+\Re(z)}}(\mathbb{R}^n)} \\
&\approx \sup_{[\varphi]_X \le 1} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^z (u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2t}}\,\mathrm{d}x\,\mathrm{d}y \\
&\le \sup_{[\varphi]_X \le 1} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{1+\Re(z)}}{|x-y|^{\frac{n}{q}(1+\Re(z))}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{2t-\frac{n}{q}(1+\Re(z))}} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^n} \\
&\precsim [u]^{1+\Re(z)}_{W^{\frac{n}{q},q}(\mathbb{R}^n)}.
\end{aligned} \tag{4.25}$$

We stress that for $|z| \leq \varepsilon_0$, all of the constants above are independent of z but depend on λ. Hence, combining the estimates (4.24) with (4.25) we obtain

$$\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|G_z(x,y)|^{\frac{q}{1+\Re(z)}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1+\Re(z)}{q}} \precsim [u]^{1+\Re(z)}_{W^{\frac{n}{q},q}(\mathbb{R}^n)}.$$

Fix now $\psi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^N$ such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\psi(x,y)|^q}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y \leq 1 \tag{4.26}$$

and set

$$F_{\psi,u}(z) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\langle G_z(x,y), |\psi(x,y)|^{q-2-\bar{z}}\psi(x,y)\rangle_{\mathbb{C}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y.$$

Then

$$|F_{\psi,u}(z)| \leq \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|G_z(x,y)|^{\frac{q}{1+\Re(z)}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1+\Re(z)}{q}} \precsim [u]^{1+\Re(z)}_{W^{\frac{n}{q},q}(\mathbb{R}^n)},$$

where the constants are independent of q as long as the assumption (4.23) is satisfied.

We observe that $F_{\psi,u}(0) = 0$. Indeed, by the definition (4.22) we have$(-\Delta)^t A_0 = (-\Delta)^t u$, and hence $A = u + c$ for a constant c. This, by definition (4.21), implies $G_0(x,y) \equiv 0$.

Moreover, just as in [11, Theorem 13.2.1], $z \mapsto F_{\psi,u}(z)$ is holomorphic: since $u \in C_c^\infty(\mathbb{R}^n)$ the map $\partial_{\bar{z}} F_{\psi,u}$ is well defined, and we can compute explicitly that the Cauchy–Riemann equations are satisfied.

From the Schwarz lemma for holomorphic functions we have for all $|z| \leq \varepsilon_0$,

$$|F_{\psi,u}(z)| \precsim |z|[u]^{1+\Re(z)}_{W^{\frac{n}{p},p}(\mathbb{R}^n)}, \tag{4.27}$$

with constant independent of ψ as long as (4.26) is satisfied. The constant depends only on λ and thus on t_0, t_1, and ε_0.

Now take $q = \frac{n}{t}(1-\varepsilon)$, $z = -\varepsilon$, $\varepsilon = \frac{n}{s} - \frac{n}{r}$, $r = s\frac{n-t}{n-s}$, so that $\frac{t}{1-\varepsilon} = r$. By (4.27) we get

$$\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|G_{-\varepsilon}(x,y)|^{\frac{n}{t}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{t}{n}} = F_{\psi,u}(-\varepsilon) \leq \sup_{\psi \text{ as in } (4.26)} F_{\psi,u}(-\varepsilon) \precsim |\varepsilon|[u]^{1+\Re(z)}_{W^{\frac{n}{q},q}(\mathbb{R}^n)}.$$

■

We are ready to proceed with the proof of the main result of this section.

Proof of Proposition 21. We have by (4.18) and (4.19),

$$[u]^{\frac{n}{r}}_{W^{r,\frac{n}{r}}(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))(A(x)-A(y))}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))(G(x,y))}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y. \tag{4.28}$$

By (4.17) and Lemma 22 we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))(A(x)-A(y))}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y \leq \Lambda[A]_{W^{t,\frac{n}{t}}(\mathbb{R}^n)} \precsim \Lambda[u]^{1-\varepsilon}_{W^{r,\frac{n}{r}}(\mathbb{R}^n)}. \tag{4.29}$$

As for the second term on the right-hand side of (4.28), we note that $\frac{r}{n}(\frac{n}{s}-1)+\frac{t}{n}=1$. Hence, by Hölder's inequality, Proposition 23, and the observation $\frac{n}{s}-\varepsilon=\frac{n}{r}$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))(G(x,y))}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y \precsim [u]^{\frac{n}{s}-1}_{W^{r,\frac{n}{r}}(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|G(x,y)|^{\frac{n}{t}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{t}{n}} \precsim |\varepsilon|[u]^{\frac{n}{r}}_{W^{r,\frac{n}{r}}(\mathbb{R}^n)}. \tag{4.30}$$

Combining (4.28) with (4.29) and (4.30) we obtain

$$[u]^{\frac{n}{r}}_{W^{r,\frac{n}{r}}(\mathbb{R}^n)} \precsim \Lambda[u]^{1-\varepsilon}_{W^{r,\frac{n}{r}}(\mathbb{R}^n)} + |\varepsilon|[u]^{\frac{n}{r}}_{W^{r,\frac{n}{r}}(\mathbb{R}^n)}.$$

Now $s-t\leq\varepsilon_0$ implies $\varepsilon=\frac{n}{s}-\frac{n}{r}=\frac{n}{s}(\frac{s-t}{n-t})\leq C(s_0,s_1)\varepsilon_0$, so for ε_0 suitably small we can absorb and conclude

$$[u]^{\frac{n}{r}}_{W^{r,\frac{n}{r}}(\mathbb{R}^n)} \precsim C(\varepsilon_0,\Lambda,s_1,s_0). \qquad \blacksquare$$

5. Continuous dependence

The main observation is that the regularity theory of Theorem 16 combined with the stability Proposition 12 imply the following corollary.

Corollary 24. *Fix $n,\ell\in\mathbb{N}$ with either $(\ell,n)=(1,1)$ or $\ell\geq 2$. Fix $0<s_0<s_1<1$ and $\Lambda>0$. Then for any $\varepsilon>0$ there exists $\delta>0$ such that the following holds.*

If $s\in(s_0,s_1)$ and $\#_s\alpha$ is attained with $\#_s\alpha\leq\Lambda$ for a homotopy class $\alpha\in\pi_n(\mathbb{S}^\ell)$ then, for any $\tilde{s}\in(s-\delta,s+\delta)$,

$$\#_s\alpha \geq \#_{\tilde{s}}\alpha - \varepsilon.$$

Proof. Let $u \in \alpha$ be the minimizer of $\mathcal{E}_{s,\frac{n}{s}}$. By Theorem 16 there is a $\Theta > 0$ such that

$$[u]_{W^{\Theta s,\frac{n}{\Theta s}}} \leq C(s_0, s_1, \Lambda).$$

Now we pick δ from Proposition 12 and conclude for any $\tilde{s} \in (s-\delta, s+\delta)$,

$$\#_s\alpha = [u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{S}^n)} \geq [u]^{\frac{n}{\tilde{s}}}_{W^{\tilde{s},\frac{n}{\tilde{s}}}(\mathbb{S}^n)} - \varepsilon \geq \#_{\tilde{s}}\alpha - \varepsilon. \qquad \blacksquare$$

5.1. Continuous dependence of minimal energy: Proof of Theorem 3

Theorem 3 is a pretty straightforward consequence of Corollary 24.

Proof of Theorem 3. Fix $\varepsilon > 0$ and $\alpha \in \pi_n(\mathbb{S}^\ell)$. Take any smooth map $\bar{u} \in C^\infty(\mathbb{S}^n, \mathbb{S}^\ell)$ that represents α. Then we have, for any $0 < s_0 < s_1 < 1$, the estimate

$$\sup_{t\in(s_0,s_1)} \#_t\alpha \leq C(\bar{u}).$$

That is, we have a uniform energy bound that is needed later in the application of Corollary 24.

In view of Theorem 15 there is an $N \in \mathbb{N}$ for which

$$\#_s\alpha = \sum_{i=1}^N \#_s\alpha_i \quad \text{for some} \sum_{i=1}^N \alpha_i = \alpha \text{ such that } \#_s\alpha_i \text{ is attained.}$$

By Corollary 24 there is a $\delta > 0$ such that for all $t \in (s-\delta, s+\delta)$ we have

$$\#_s\alpha = \sum_{i=1}^N \#_s\alpha_i \geq \sum_{i=1}^N \Big(\#_t\alpha_i - \frac{\varepsilon}{N}\Big) \geq \#_t\alpha - \varepsilon,$$

where the last inequality is a consequence of Proposition 9. The converse inequality follows by reversing the roles of t and s. $\blacksquare$

5.2. Proofs of corollaries

Proof of Corollary 4. Fix $0 < s_0 < 1/2 < s_1 < 1$. Assume that $\#_s 1$ is not attained for some $s \in (s_0, s_1)$. Then by Theorem 15,

$$\#_s 1 \geq \#_s d_{1,s} + \#_s d_{2,s}$$

for degrees $d_{1,s}, d_{2,s} \in \mathcal{Q} \setminus \{-1, 0, 1\}$ such that $\#_s d_{i,s}$ is attained for $i = 1, 2$. Theorem 11 implies that $\mathcal{Q}$ can only be a finite set of integers.

By Theorem 3, the family of maps

$$(s_0, s_1) \ni s \mapsto \#_s d, \quad d \in \mathcal{Q}$$

are equicontinuous. So, for every $\varepsilon > 0$, there is a δ such that for every $s \in (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$ we have

$$\#_{\frac{1}{2}} 1 \geq \#_s 1 - \varepsilon \geq \#_s d_{1,s} + \#_s d_{2,s} - \varepsilon$$
$$\geq \#_{\frac{1}{2}} d_{1,s} + \#_{\frac{1}{2}} d_{2,s} - 3\varepsilon.$$

Combining this with Theorem 10 we obtain

$$4\pi^2 = \#_{\frac{1}{2}} 1 \geq \#_{\frac{1}{2}} d_{1,s} + \#_{\frac{1}{2}} d_{2,s} - 3\varepsilon \geq 8\pi^2 - 3\varepsilon.$$

For $\varepsilon < \frac{4}{3}\pi^2$ this is a contradiction. ■

In a very similar way to Corollary 4 we obtain the following proof.

Proof of Corollary 5. The fact that $\#_t \alpha$ is attained is an immediate consequence of [15, Lemma 7.7]. Arguing exactly as in the proof of Corollary 4 – assuming that $\#_s \alpha$ is not attained – we obtain for $\beta_{1,s}, \beta_{2,s} \in \pi_n(\mathbb{S}^\ell) \setminus \{0\}$,

$$\#_t \alpha \geq \#_s \beta_{1,s} + \#_s \beta_{2,s} - \varepsilon \geq \#_t \beta_{1,s} + \#_t \beta_{2,s} - 3\varepsilon.$$

However, by assumption $\#_t \alpha \leq \#_t \beta_{i,s}$ for $i = 1, 2$. Hence we would get $\#_t \alpha \geq 2\#_t \alpha - 3\varepsilon$. This gives a contradiction with Proposition 8 for sufficiently small $\varepsilon > 0$. ■

Proof of Corollary 6. In view of Theorem 11, for each $\Lambda > 0$ there exists $D \in \mathbb{N}$ such that

$$\overline{C}_{n,s;\Lambda} \equiv \sup_{\substack{u \in W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^n),\\ 0<[u]_{W^{s,\frac{n}{s}}} \leq \Lambda}} \frac{\deg u}{[u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^n)}}$$
$$= \max_{\substack{d \in \mathbb{Z}\setminus\{0\},\\ |d| \leq D}} \sup_{\substack{u \in W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^n),\\ [u]_{W^{s,\frac{n}{s}}} \leq \Lambda,\\ \deg u = d}} \frac{d}{[u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^n)}}$$
$$= \max\left\{\frac{d}{\#_s d} : |d| \leq D,\ d \neq 0, \exists u \in W^{s,\frac{n}{s}}(\mathbb{S}^n, \mathbb{S}^n) \text{ with } \deg u = d \text{ and } [u]_{W^{s,\frac{n}{s}}} \leq \Lambda\right\}.$$

For each $d \in \mathbb{Z} \setminus \{0\}$ the map $s \mapsto \frac{d}{\#_s d}$ is continuous, by Theorem 3.

Since

$$\overline{C}_{n,s} = \sup_{\Lambda > 0} \overline{C}_{n,s;\Lambda} = \lim_{\Lambda \to \infty} \overline{C}_{n,s;\Lambda}$$

we have that $s \mapsto \overline{C}_{n,s}$ is lower semicontinuous. ■

5.3. Stability of generators of minimizing $W^{s,\frac{n}{s}}$-harmonic maps: Proof of Theorem 2

Theorem 2 is a consequence of the following more precise theorem.

Theorem 25. *Fix $n, \ell \geq 1$ with either $(\ell, n) = (1,1)$ or $\ell \geq 2$. Let $\Lambda > 0$ and $0 < t_0 < t_1 < 1$. There exists a $\delta = \delta(\ell, n, t_0, t_1, \Lambda) \in (0,1)$ such that the following holds for $t \in (t_0, t_1)$.*

Set

$$X_s := \left\{\alpha \in \pi_n(\mathbb{S}^\ell) : \text{there exists a } W^{s,\frac{n}{s}}(\mathbb{S}^n, \mathbb{S}^\ell)\text{-minimizer } u \text{ in } \alpha \text{ and } [u]_{W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^\ell)} \leq \Lambda\right\}$$

and

$$Y := \bigcap_{s\in(t-\delta,t+\delta)} X_s.$$

Then X_s is generated by Y, i.e., $X_s \subset \operatorname{span} Y$, for each $s \in (t-\delta, t+\delta)$.

Proof. Let $\varepsilon > 0$ be fixed and chosen below. For this ε we take $\delta > 0$ from Corollary 24.

By Theorem 11 there exists a finite number $M \in \mathbb{N}$ such that if for some $s \in (t-\delta, t+\delta)[u]_{W^{s,\frac{n}{s}}(\mathbb{S}^n)} < \Lambda$ then $u \in \mathcal{Q} := \{\alpha_1, \ldots, \alpha_M\} \subset \pi_n(\mathbb{S}^\ell)$. In particular, $X_s \subset \mathcal{Q}$ for all $s \in (t-\delta, t+\delta)$. Observe that for each $s \in (t-\delta, t+\delta)$ we have from Theorem 1,

$$X_s \text{ generates } \mathcal{Q}.$$

Let us enumerate the elements of $Y = \bigcap_{s\in(t-\delta,t+\delta)} X_s$:

$$Y = \{\gamma_1, \ldots, \gamma_K\} \quad \text{for } K \leq M.$$

Of course, for now $K = 0$ is a possibility.

We define $Z \subset \mathcal{Q} \setminus Y$ as the collection of homotopy groups β, where $\#_s\beta$ is not attained for at least one s. More precisely,

$$Z := \left\{\alpha \in \pi_n(\mathbb{S}^\ell) : \#_s\alpha \leq \Lambda \text{ for an } s \in (t-\delta, t+\delta)\right\} \setminus Y = \{\beta_1, \ldots, \beta_L\}.$$

We note that $L + K \leq M$. Now assume that for some $s \in (t-\delta, t+\delta)$ there is α such that $\alpha \in X_s \setminus Y$. Then there must be some $\tilde{s} \in (t-\delta, t+\delta)$ with $\alpha \not\in X_{\tilde{s}}$. By Theorem 15 we find $\beta_k \in Z$ and $\gamma_k \in Y \setminus \{0\}$ such that $\#_{\tilde{s}}\beta_k$ and $\#_{\tilde{s}}\gamma_k$ are attained, $2 \leq A + B \leq M$, and

$$\alpha = \sum_{k=1}^{A} \beta_k + \sum_{k=1}^{B} \gamma_k, \quad \#_{\tilde{s}}\alpha = \sum_{k=1}^{A} \#_{\tilde{s}}\beta_k + \sum_{k=1}^{B} \#_{\tilde{s}}\gamma_k. \tag{5.1}$$

Using (5.1) and applying Corollary 24 twice we obtain

$$\begin{aligned} \#_s\alpha \geq \#_{\tilde{s}}\alpha - \varepsilon &= \sum_{k=1}^{A} \#_{\tilde{s}}\beta_k + \sum_{k=1}^{B} \#_{\tilde{s}}\gamma_k - \varepsilon \\ &\geq \sup_{r\in(t-\delta,t+\delta)} \left(\sum_{k=1}^{A} \#_r\beta_k + \sum_{k=1}^{B} \#_r\gamma_k\right) - (M+1)\varepsilon. \end{aligned} \tag{5.2}$$

In particular, we have

$$\#_s\alpha \geq \#_{\tilde{s}}\alpha - \varepsilon \geq \sum_{k=1}^{A} \#_s\beta_k + \sum_{k=1}^{B} \#_s\gamma_k - (M+1)\varepsilon. \tag{5.3}$$

Observe that this is a contradiction if any of the β_k on the right-hand side are equal to α (assuming ε is small enough).

If $A = 0$ then we are done because then $\alpha = \sum \gamma_k$ and each $\gamma_k \in Y$.

If $A > 0$, then for each β_k in (5.2) we have two possibilities: either $\#_s\beta_k$ is attained (but still $\beta_k \notin Y$) or $\#_s\beta_k$ is not attained. Rearranging the terms we can assume that

- for the terms $\beta_1, \dots, \beta_{A_1}$, $\#_s\beta_k$ is attained, i.e., $\beta_k \in X_s \setminus Y$,
- for the terms $\beta_{A_1+1}, \dots, \beta_A$ the infimum $\#_s\beta_k$ is not attained, i.e., $\beta_k \notin X_s$.

It is possible that $A_1 = 0$ or $A_1 = A$.

In the case $\beta_k \in X_s \setminus Y$ we find an $s_k \in (t-\delta, t+\delta)$ such that $\beta_k \notin X_{s_k}$ and we apply (5.3).

In the case $\beta_k \notin X_s$ we apply Theorem 15.

We obtain, first using Corollary 24 and then using (5.3) and Theorem 15,

$$\begin{aligned}
\sum_{k=1}^{A} \#_s\beta_k &\geq \sum_{k=1}^{A_1} \#_{s_k}\beta_k + \sum_{k=A_1+1}^{A} \#_s\beta_k - M\varepsilon \\
&\geq \sum_{k=1}^{A_1} \left(\sum_{j=1}^{A_{1,k}} \#_s\beta_{k_j} + \sum_{j=1}^{B_{1,k}} \#_s\gamma_{k_j} \right) + \sum_{k=A_1+1}^{A} \left(\sum_{j=1}^{A_{\tilde{1},k}} \#_s\beta_{\tilde{k}_j} + \sum_{j=1}^{B_{\tilde{1},k}} \#_s\gamma_{\tilde{k}_j} \right) \\
&\quad - (2M^2 + M)\varepsilon \\
&= \sum_{i=1}^{\tilde{A}_1} \#_s\beta_{\sigma(i)} + \sum_{i=1}^{\tilde{B}_2} \#_s\gamma_{\zeta(i)} - (2M^2 + M)\varepsilon,
\end{aligned} \tag{5.4}$$

where $\tilde{A}_1, \tilde{B}_1 \leq 2M^2$, $\sigma(i) \in \{1, \dots, L\}$, $\zeta(i) \in \{1, \dots, K\}$ for each i, and at each step $2 \leq A_{1,k} + B_{1,k} \leq M$. Now again, we can repeat the procedure. However, since there are at most M elements in Z we obtain that after a finite number of iterations of this procedure, the same element β_i would appear on the right-hand side and on the left-hand side of (5.4). This would give a contradiction for a sufficiently small ε (depending on M and λ from Proposition 8). Hence, we must obtain at some moment a decomposition of α into terms belonging only to Y. ∎

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References

[1] L. Berlyand, P. Mironescu, V. Rybalko, and E. Sandier, Minimax critical points in Ginzburg–Landau problems with semi-stiff boundary conditions: Existence and bubbling. *Comm. Partial Differential Equations* **39** (2014), no. 5, 946–1005 Zbl 1296.35036 MR 3196191

[2] J. Bourgain, H. Brezis, and P. Mironescu, Lifting, degree, and distributional Jacobian revisited. *Comm. Pure Appl. Math.* **58** (2005), no. 4, 529–551 Zbl 1077.46023 MR 2119868

[3] H. Brezis and P. Mironescu, *Sobolev maps to the circle – From the perspective of analysis, geometry, and topology*. Progr. Nonlinear Differential Equations Appl. 96, Birkhäuser/Springer, New York, 2021 Zbl 1501.46001 MR 4390036

[4] H. Brezis and L. Nirenberg, Degree theory and BMO. I. Compact manifolds without boundaries. *Selecta Math. (N.S.)* **1** (1995), no. 2, 197–263 Zbl 0852.58010 MR 1354598

[5] F. Da Lio, K. Mazowiecka, and A. Schikorra, A fractional version of Rivière’s GL(n)-gauge. *Ann. Mat. Pura Appl. (4)* **201** (2022), no. 4, 1817–1853 Zbl 1494.42031 MR 4454383

[6] F. Da Lio and T. Rivière, Sub-criticality of non-local Schrödinger systems with antisymmetric potentials and applications to half-harmonic maps. *Adv. Math.* **227** (2011), no. 3, 1300–1348 Zbl 1219.58004 MR 2799607

[7] F. Da Lio and T. Rivière, Three-term commutator estimates and the regularity of $\frac{1}{2}$-harmonic maps into spheres. *Anal. PDE* **4** (2011), no. 1, 149–190 Zbl 1241.35035 MR 2783309

[8] F. Duzaar and E. Kuwert, Minimization of conformally invariant energies in homotopy classes. *Calc. Var. Partial Differential Equations* **6** (1998), no. 4, 285–313 Zbl 0909.49008 MR 1624288

[9] L. Grafakos, *Classical Fourier analysis*. 3rd edn., Grad. Texts in Math. 249, Springer, New York, 2014 Zbl 1304.42001 MR 3243734

[10] L. Grafakos, *Modern Fourier analysis*. 3rd edn., Grad. Texts in Math. 250, Springer, New York, 2014 Zbl 1304.42002 MR 3243741

[11] T. Iwaniec and G. Martin, *Geometric function theory and non-linear analysis*. Oxford Math. Monogr., The Clarendon Press, Oxford University Press, New York, 2001 Zbl 1045.30011 MR 1859913

[12] T. Kostrzewa and K. Mazowiecka, in preparation

[13] T. Kuusi, G. Mingione, and Y. Sire, Nonlocal self-improving properties. *Anal. PDE* **8** (2015), no. 1, 57–114 Zbl 1317.35284 MR 3336922

[14] K. Mazowiecka and A. Schikorra, Fractional div-curl quantities and applications to nonlocal geometric equations. *J. Funct. Anal.* **275** (2018), no. 1, 1–44 Zbl 1440.42114 MR 3799622

[15] K. Mazowiecka and A. Schikorra, Minimal $W^{s,\frac{n}{s}}$-harmonic maps in homotopy classes. *J. Lond. Math. Soc. (2)* **108** (2023), no. 2, 742–836 Zbl 1535.58009 MR 4626724

[16] P. Mironescu, Sobolev maps on manifolds: Degree, approximation, lifting. In *Perspectives in nonlinear partial differential equations*, pp. 413–436, Contemp. Math. 446, American Mathematical Society, Providence, RI, 2007 Zbl 1201.46032 MR 2376670

[17] P. Mironescu, Profile decomposition and phase control for circle-valued maps in one dimension. *C. R. Math. Acad. Sci. Paris* **353** (2015), no. 12, 1087–1092 Zbl 1350.46028 MR 3427913

[18] P. Mironescu and A. Pisante, A variational problem with lack of compactness for $H^{1/2}(S^1; S^1)$ maps of prescribed degree. *J. Funct. Anal.* **217** (2004), no. 2, 249–279 Zbl 1067.58009 MR 2102568

[19] W. Park and A. Schikorra, Quantitative estimates for fractional Sobolev mappings in rational homotopy groups. *Nonlinear Anal.* **235** (2023), article no. 113349 Zbl 1530.55011 MR 4623984

[20] T. Rivière, Minimizing fibrations and p-harmonic maps in homotopy classes from S^3 into S^2. *Comm. Anal. Geom.* **6** (1998), no. 3, 427–483 Zbl 0914.58010 MR 1638862

[21] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*. De Gruyter Ser. Nonlinear Anal. Appl. 3, Walter de Gruyter, Berlin, 1996 Zbl 0873.35001 MR 1419319

[22] J. Sacks and K. Uhlenbeck, The existence of minimal immersions of 2-spheres. *Ann. of Math. (2)* **113** (1981), no. 1, 1–24 Zbl 0462.58014 MR 0604040

[23] A. Schikorra, Integro-differential harmonic maps into spheres. *Comm. Partial Differential Equations* **40** (2015), no. 3, 506–539 Zbl 1329.58013 MR 3285243

[24] A. Schikorra, Nonlinear commutators for the fractional p-Laplacian and applications. *Math. Ann.* **366** (2016), no. 1-2, 695–720 Zbl 1351.35255 MR 3552254

[25] A. Schikorra, Boundary equations and regularity theory for geometric variational systems with Neumann data. *Arch. Ration. Mech. Anal.* **229** (2018), no. 2, 709–788 Zbl 1411.35059 MR 3803775

[26] A. Schikorra and J. Van Schaftingen, An estimate of the Hopf degree of fractional Sobolev mappings. *Proc. Amer. Math. Soc.* **148** (2020), no. 7, 2877–2891 Zbl 1487.55020 MR 4099776

[27] E. M. Stein, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*. Princeton Math. Ser. 43, Princeton University Press, Princeton, NJ, 1993 Zbl 0821.42001 MR 1232192

[28] J. Van Schaftingen, Estimates by gap potentials of free homotopy decompositions of critical Sobolev maps. *Adv. Nonlinear Anal.* **9** (2020), no. 1, 1214–1250 Zbl 1437.58008 MR 4042308

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