Ann. Inst. H. Poincaré C
Anal. Non Linéaire 42 (2025), 1247–1312
DOI 10.4171/AIHPC/125
© 2024 Association Publications de l'Institut Henri Poincaré
Published by EMS Press
This work is licensed under a CC BY 4.0 license

Time-periodic solutions of completely resonant Klein–Gordon equations on $\mathbb{S}^3$

Massimiliano Berti, Beatrice Langella, and Diego Silimbani

Abstract. We prove existence and multiplicity of Cantor families of small-amplitude time-periodic solutions of completely resonant Klein–Gordon equations on the sphere $\mathbb{S}^3$ with quadratic, cubic, and quintic nonlinearity, regarded as toy models in general relativity. The solutions are obtained by a variational Lyapunov–Schmidt decomposition, which reduces the problem to the search of mountain pass critical points of a restricted Euler–Lagrange action functional. Compactness properties of its gradient are obtained by Strichartz-type estimates for the solutions of the linear Klein–Gordon equation on $\mathbb{S}^3$.

1. Introduction

Motivated by the stability problem of anti–de Sitter space-time (AdS), the goal of this paper is to prove existence and multiplicity of Cantor families of time-periodic solutions of nonlinear Klein–Gordon equations of the form

$$(-\partial_{tt} + \Delta_{\mathbb{S}^3} - \mathbb{1})\phi = \begin{cases} |\phi|^{p-1}\phi & \text{if } p \text{ is odd, } p \geq 3, \\ \phi^p & \text{if } p \text{ is even,} \end{cases} \tag{1.1}$$

where $\phi\colon \mathbb{R} \times \mathbb{S}^3 \to \mathbb{C}$ and $\Delta_{\mathbb{S}^3}$ is the Laplace–Beltrami operator on the three-dimensional sphere $\mathbb{S}^3$. For $p = 3$, time-periodic solutions of (1.1) have very recently been constructed by Chatzikaleas and Smulevici [22]. A mathematical point of interest of Theorems 1.2 and 1.4 below is that, jointly with [21, 22], they are the only existence results of time-periodic solutions for *completely resonant* Hamiltonian PDEs on a manifold of dimension higher than 1. Their proof is based on a novel combination of *variational* methods and *Strichartz*-type estimates for free solutions of the Klein–Gordon equation on $\mathbb{S}^3$, which we find of theoretical interest in itself and not available anywhere else in the literature. The Strichartz estimates, which are sharper than the Sobolev embeddings, allow one to gain compactness of the gradient of the action functional and smoothness of the critical points.

Let us first briefly outline the physical framework connecting (1.1) with the stability problem of AdS space-time. AdS is the maximally symmetric solution to the vacuum

Mathematics Subject Classification 2020: 37K58 (primary); 58E07, 35L05, 58J45, 83C10 (secondary).
Keywords: time-periodic solutions, Klein–Gordon equation, completely resonant equations, mountain pass theorem, Strichartz estimates, general relativity.

Einstein equations $\mathrm{Ric}(g) = -\Lambda g$ with negative cosmological constant Λ. Unlike de Sitter or Minkowski space-time, its stability properties are nowadays still poorly understood. In particular, the stability of AdS depends on the conformal boundary conditions. While, for instance, it is expected that under dissipative boundary conditions AdS is stable, see [31], it has been conjectured, by Dafermos–Holzegel [25] and by Anderson [2], that AdS is unstable under fully reflective boundary conditions. The latter instability conjecture is supported by the numerical investigations of Bizoń–Rostworowski [16] for the spherically symmetric Einstein-massless-scalar field equations, suggesting that AdS is unstable, against the formation of black holes under arbitrarily small perturbations. Such a phenomenon has been rigorously proven in the works [35, 36] for the Einstein-massless-Vlasov system with spherical symmetry: they prove the existence of a one-parameter family of initial data lying arbitrarily close to AdS whose time evolution gives rise to a black-hole region. Notwithstanding, the work [16] also suggests the existence of small initial data leading to stable solutions in the Einstein-massless-scalar field equation, confirmed later by Maliborski–Rostworowski [34], who constructed formal time-periodic solutions, supported by numerical evidence. The same existence conjecture of time-periodic solutions – called geons – has also been extended to the vacuum Einstein equations in [26, 27].

The nonlinear wave equation (1.1) with $p = 3$ has been introduced in [15, 16, 34] as a toy model of spherically symmetric Einstein-massless-scalar field equations close to the AdS solution. Chatzikaleas [20] constructed formal power series expansions of small-amplitude time-periodic solutions of (1.1) in the spherically symmetric case, which reduces to the one-dimensional wave equation with singular nonlinearity

$$-\partial_{tt}u + \partial_{xx}u = \frac{u^3}{\sin^2(x)}, \quad u(t,0) = u(t,\pi) = 0, \quad x \in (0,\pi).$$

The absence of secular terms in the power series expansions is obtained using the method of Maliborski–Rostworowski [34], developed for the Einstein–Klein–Gordon equation. However, the presence of *small divisors* prevents the convergence of such power series. This difficulty looks analogous to the convergence problem of "Linstedt series" of quasi-periodic solutions in celestial mechanics, devised since Poincaré [37], and successfully overcome during the last century by the celebrated KAM theory. The first rigorous existence result of time-periodic solutions of (1.1) for $p = 3$ with strongly Diophantine frequencies ω is given in the very recent paper [22]. Such work constructs solutions of the following form:

- *spherically symmetric* functions, namely $\phi(t,x) = u(t,\cos(x))$, $x \in (0,\pi)$; see Definition 1.1 below;
- *plane waves in Hopf coordinates*, namely $\phi(t,\eta,\xi_1,\xi_2) = u(t,\eta)e^{\mathrm{i}\mu_1\xi_1}e^{\mathrm{i}\mu_2\xi_2}$, see Definition 1.3, up to restricting to values of the momenta $\mu_1 = \mu_2 \in \{1,\dots,5\}$, or $\mu_1 = \mu_2$ large enough.

The results in [22] rely on an abstract theorem by Bambusi–Paleari [3], which uses a Lyapunov–Schmidt approach and whose main assumption is the existence of a nondegenerate zero of the "resonant system". For a physics-oriented perspective of the status of resonant system approximations of nonlinear dynamics of AdS space-times we refer to [28].

The goal of this paper is to prove existence and multiplicity of periodic solutions of (1.1), for more general values of the nonlinearity degree p and of the momenta μ_1, μ_2. More precisely, for

- $p = 2$ and $p = 5$, we find real-valued spherically symmetric solutions; see Theorem 1.2;
- $p = 3$, we find complex plane waves in Hopf coordinates for *any* value of the momenta $\mu_1, \mu_2 \in \mathbb{Z}$; see Theorem 1.4.

These generalizations require new methods, since the verification of the existence of a nondegenerate zero of the associated resonant system seems unapproachable, if even true. In this work we combine variational methods of mountain pass type, inspired by the works of Berti and Bolle [5–7] for one-dimensional semilinear wave equations, with Strichartz-type estimates for the linear Klein–Gordon equation on $\mathbb{S}^3$.

We now rigorously present our results.

1.1. Main results

Small amplitude time-periodic solutions of (1.1) bifurcate from suitable solutions of the linear Klein–Gordon equation

$$-\partial_{tt} v + \Delta_{\mathbb{S}^3} v - v = 0. \tag{1.2}$$

Since the eigenvalues of $-\Delta_{\mathbb{S}^3} + \mathbb{1}$ are $\{j^2, j \in \mathbb{N}_*\}$, where $\mathbb{N}_* := \{1, 2, 3, \ldots\}$, *all* the solutions of (1.2) are 2π-periodic in time, i.e. have frequency $\omega = 1$. For this reason (1.1) is called a *completely resonant* equation and a major difficulty of the problem is to determine from which free solutions v of (1.2) periodic solutions of the nonlinear Klein–Gordon equation (1.1) branch off.

We look for time-periodic solutions of (1.1) with strongly Diophantine frequency $\omega \sim 1$ belonging to the Cantor set

$$\Omega_\gamma := \left\{\omega \in [\tfrac{1}{2}, 2] : |\omega\ell - j| \geq \tfrac{\gamma}{\ell}\ \forall \ell \in \mathbb{N}_*,\ j \in \mathbb{N},\ \ell \neq j\right\}. \tag{1.3}$$

For $\gamma \in (0, \gamma_0)$ and γ_0 small enough, the set Ω_γ is uncountable, with zero measure, and accumulates to $\omega = 1$, as proved in [3].

We look for time-periodic solutions of (1.1) taking values in Sobolev spaces $H^s(\mathbb{S}^3, d\sigma)$ of scalar functions $\phi\colon \mathbb{S}^3 \to \mathbb{C}$ with $s > \frac{3}{2}$, where

$$H^s(\mathbb{S}^3, d\sigma) := \left\{\phi \in L^2(\mathbb{S}^3, d\sigma) : (-\Delta_{\mathbb{S}^3} + \mathbb{1})^{\frac{s}{2}} \phi \in L^2(\mathbb{S}^3, d\sigma)\right\}, \quad s \in \mathbb{R}, \tag{1.4}$$

and $d\sigma$ denotes the standard Lebesgue measure on the sphere. Each $H^s(\mathbb{S}^3, d\sigma)$ is a Hilbert space endowed with the complex scalar product $\langle \phi_1, \phi_2 \rangle_{H^s(\mathbb{S}^3, d\sigma)} := \langle (-\Delta_{\mathbb{S}^3} + \mathbb{1})^s \phi_1, \phi_2 \rangle_{L^2(\mathbb{S}^3, d\sigma)}$.

For any $s > \frac{3}{2}$ the spaces $H^s(\mathbb{S}^3, d\sigma)$ continuously embed into $L^\infty(\mathbb{S}^3)$ and form an algebra with respect to the product of functions. We also recall the continuous embedding of the Sobolev spaces

$$H^s(\mathbb{S}^3, d\sigma) \hookrightarrow L^p(\mathbb{S}^3, d\sigma), \quad p \le p^*(s,3) := \frac{6}{3-2s}, \tag{1.5}$$

with compact embedding for $p < p^*(s,3)$. In particular, $H^s(\mathbb{S}^3, d\sigma) \hookrightarrow L^6(\mathbb{S}^3, d\sigma)$ for any $s \ge 1$, with compact embedding if $s > 1$, and $H^s(\mathbb{S}^3, d\sigma) \hookrightarrow L^4(\mathbb{S}^3, d\sigma)$ for any $s \ge \frac{3}{4}$, with compact embedding if $s > \frac{3}{4}$.

Our first existence result concerns spherically symmetric solutions, according to the following definition.

Definition 1.1 (Spherically symmetric functions). Consider on $\mathbb{S}^3$ coordinates

$$\begin{aligned} &(x,\theta,\varphi) \in (0,\pi)\times(0,\pi)\times(0,2\pi), \\ &(x,\theta,\varphi) \mapsto (\cos(x), \sin(x)\cos(\theta), \sin(x)\sin(\theta)\cos(\varphi), \sin(x)\sin(\theta)\sin(\varphi)). \end{aligned} \tag{1.6}$$

We say that $\phi\colon \mathbb{S}^3 \to \mathbb{C}$ is *spherically symmetric* if

$$\phi(x,\theta,\varphi) = u(x) \otimes 1_{\theta,\varphi} \quad \forall (x,\theta,\varphi) \in (0,\pi)\times(0,\pi)\times(0,2\pi),\ u\colon (0,\pi) \to \mathbb{C},$$

where $1_{\theta,\varphi}$ is the function identically equal to 1 for any (θ,φ). By (1.6), u has to be of the form $u(x) = U(\cos(x))$, for some $U\colon (-1,1) \to \mathbb{C}$. We say that $\phi\colon \mathbb{R}\times\mathbb{S}^3 \to \mathbb{C}$ is *spherically symmetric* if $\phi(t,\cdot)$ is spherically symmetric for any $t \in \mathbb{R}$.

Our first result is the following:

Theorem 1.2 (Spherically symmetric solutions). *Let $p = 2$ or $p = 5$. Fix $\gamma \in (0,\gamma_0)$ and $\mathtt{d} \in (0, \frac{1}{4})$. For any $n \in \mathbb{N}$, $r > \frac{1}{2}$, and $s > \frac{3}{2}$, there exist $\varepsilon_0 := \varepsilon_0(n,r,s,\mathtt{d},\gamma) > 0$ and $C := C(n,r,s,\mathtt{d}) > 0$ such that for any ε belonging to*

$$\mathcal{E} := \{\varepsilon \in (0,\varepsilon_0) : \omega_\varepsilon \in \Omega_\gamma\}, \quad \omega_\varepsilon^2 := 1 + \varsigma\varepsilon, \quad \varsigma := \begin{cases} -1 & \text{if } p = 2, \\ 1 & \text{if } p = 5, \end{cases} \tag{1.7}$$

with Ω_γ defined in (1.3), there exist n different real-valued, nonzero, T_ε-periodic solutions

$$\{\phi_\varepsilon^{(1)}(t,x), \dots, \phi_\varepsilon^{(n)}(t,x)\}$$

of equation (1.1) with frequency $\omega_\varepsilon := \frac{2\pi}{T_\varepsilon}$, even in time and with spherical symmetry. They are of the following form:

(i) *There exist n different 2π-periodic, even in time, nonzero solutions $\{v_\varepsilon^{(1)}(t,x), \dots, v_\varepsilon^{(n)}(t,x)\}$ of the linear equation (1.2), spherically symmetric, with size*

$$C^{-1}\varepsilon^{\frac{1}{q}} \le \|v_\varepsilon^{(k)}\|_{H^r([0,2\pi], H^s(\mathbb{S}^3, d\sigma))} \le C\varepsilon^{\frac{1}{q}-\mathtt{d}}, \quad q := \begin{cases} 2 & \text{if } p = 2, \\ 4 & \text{if } p = 5, \end{cases} \tag{1.8}$$

such that, as $\varepsilon \to 0$,

$$\|\phi_\varepsilon^{(k)} - v_\varepsilon^{(k)}(\omega_\varepsilon \cdot, \cdot)\|_{H^r([0,T_\varepsilon],H^s(\mathbb{S}^3,d\sigma))} = o(\|v_\varepsilon^{(k)}(\omega_\varepsilon \cdot, \cdot)\|_{H^r([0,T_\varepsilon],H^s(\mathbb{S}^3,d\sigma))}). \tag{1.9}$$

(ii) *Each* $\phi_\varepsilon^{(k)}$ *has minimal period* $T_{k,\varepsilon} := \frac{T_\varepsilon}{m_k}$, *where* $\{m_k\}_{k=1}^n$ *is an increasing sequence of positive integers. Correspondingly, the functions* $\{v_\varepsilon^{(1)}, \dots, v_\varepsilon^{(n)}\}$ *have minimal periods* $T_k := \frac{2\pi}{m_k}$.

We point out that Theorem 1.2 also holds in the case $p = 3$. This is actually the result in [22]. In this case the $v_\varepsilon^{(j)}$ are close to the "one-mode" functions

$$\varepsilon^{\frac{1}{2}} \underline{v}^{(j)}, \quad \underline{v}^{(j)} := \kappa_j \cos((j+1)t) e_j(x), \quad e_j(x) := \frac{\sin((j+1)x)}{\sin(x)}, \tag{1.10}$$

for suitable $\kappa_j \in \mathbb{R} \setminus \{0\}$. Note that the e_j are the spherically symmetric functions $e_j(x) = U_j(\cos(x))$, where $U_j : \mathbb{R} \to \mathbb{R}$ are Chebyshev polynomials of the second kind. The functions $\underline{v}^{(j)}$ in (1.10) are actually solutions of the "zeroth-order bifurcation equation", also called the "resonant system" in [22],

$$(-\Delta_{\mathbb{S}^3} + \mathbb{1})v - \Pi_V(v^3) = 0, \tag{1.11}$$

where Π_V is the L^2-projector on the infinite-dimensional linear space V formed by the solutions of (1.2) (see definitions (3.7), (3.11) below). The name "resonant system" refers to the fact that (1.11) is restricted on the space V of solutions of equation (1.2), which are the superposition of harmonics with the same frequency in space and in time.

On the other hand, for $p = 5$ the functions $v_\varepsilon^{(j)}$ in (1.8) are not close to "one modes" as in (1.10). Actually, the $v_\varepsilon^{(j)}$ are close to functions of the form $\varepsilon^{\frac{1}{4}} \underline{v}^{(j)}$, where $\underline{v}^{(j)}$ are nonzero solutions of the equation

$$(-\Delta_{\mathbb{S}^3} + \mathbb{1})v - \Pi_V(v^5) = 0, \tag{1.12}$$

which *does not* possess one-mode solutions. We actually prove the existence of nontrivial solutions of (1.12), exploiting that it is the Euler–Lagrange equation of the action functional

$$\frac{1}{2}\|v\|^2_{L^2(\mathbb{T}_t,H^1(\mathbb{S}^3,d\sigma))} - \frac{1}{6}\int_{\mathbb{T}_t \times \mathbb{S}^3} v^6(t,z)\, dt\, d\sigma(z), \quad \mathbb{T} := \mathbb{T}_t := \mathbb{R}/(2\pi\mathbb{Z}), \tag{1.13}$$

which, thanks to the time-space Strichartz-type estimates proved in Section 4, admits mountain pass critical points of class C^∞. Strichartz estimates are required to imply compactness properties of the action functional, which are not a consequence of Sobolev embeddings (1.5) on $\mathbb{S}^3$; see Remark 4.6.

The case $p = 2$ is degenerate, since $\Pi_V(v^2) = 0$ (see Lemma 4.4), and the $v_\varepsilon^{(j)}$ are close to functions of the form $\varepsilon^{\frac{1}{2}} \underline{v}^{(j)}$, where $\underline{v}^{(j)}$ are nonzero solutions of the equation

$$(-\Delta_{\mathbb{S}^3} + \mathbb{1})v + \Pi_V(v\mathscr{L}_1^{-1}(v^2)) = 0, \quad \mathscr{L}_1 := -\partial_{tt} + \Delta_{\mathbb{S}^3} - \mathbb{1}. \tag{1.14}$$

It turns out that equation (1.14) admits mountain pass critical points as well. Further comments are postponed until after Theorem 1.4.

In the case $p = 3$ we have new existence results of periodic Hopf plane wave solutions of (1.1) for any value of the momenta (μ_1, μ_2), which we now define (the use of these coordinates was first suggested in [28]):

Definition 1.3 (Hopf plane waves). Consider on $\mathbb{S}^3$ Hopf coordinates

$$(\eta, \xi_1, \xi_2) \in \Big(0, \frac{\pi}{2}\Big) \times \mathbb{T} \times \mathbb{T},$$
$$(\eta, \xi_1, \xi_2) \mapsto (\sin(\eta)\cos(\xi_1), \sin(\eta)\sin(\xi_1), \cos(\eta)\cos(\xi_2), \cos(\eta)\sin(\xi_2)).$$

Given $(\mu_1, \mu_2) \in \mathbb{Z}^2$, we say that $\phi\colon \mathbb{S}^3 \to \mathbb{C}$ is a *Hopf plane wave with momentum* (μ_1, μ_2) if

$$\phi(\eta, \xi_1, \xi_2) = u(\eta) e^{\mathrm{i}\mu_1 \xi_1} e^{\mathrm{i}\mu_2 \xi_2} \quad \forall (\eta, \xi_1, \xi_2) \in \Big(0, \frac{\pi}{2}\Big) \times \mathbb{T} \times \mathbb{T},\ u\colon \Big(0, \frac{\pi}{2}\Big) \to \mathbb{R}. \tag{1.15}$$

We say that $\phi\colon \mathbb{R} \times \mathbb{S}^3 \to \mathbb{C}$ is a *Hopf plane wave with momentum* (μ_1, μ_2) if $\phi(t, \cdot)$ is a Hopf plane wave with momentum (μ_1, μ_2) for any $t \in \mathbb{R}$.

The following result extends [22], which holds for $\mu_1 = \mu_2$ either equal to $\{1, 2, 3, 4, 5\}$ or large enough:

Theorem 1.4 (Hopf plane waves). *Let $p = 3$. Fix $\gamma \in (0, \gamma_0)$ and $\mathtt{d} \in (0, \frac{1}{2})$. For any $n \in \mathbb{N}$, $r > \frac{1}{2}$, $s > \frac{3}{2}$, and any $(\mu_1, \mu_2) \in \mathbb{Z}^2$, there exist $\varepsilon_0 := \varepsilon_0(n, r, s, \mathtt{d}, \gamma, \mu_1, \mu_2) > 0$ and $C := C(n, r, s, \mathtt{d}, \mu_1, \mu_2) > 0$ such that for any ε belonging to the set*

$$\mathcal{E} := \big\{\varepsilon \in (0, \varepsilon_0) : \omega_\varepsilon \in \Omega_\gamma\big\}, \quad \omega_\varepsilon^2 := 1 + \varepsilon, \tag{1.16}$$

there exist n different nonzero, T_ε-periodic Hopf wave solutions (see (1.15)),

$$\big\{\phi_\varepsilon^{(1)}, \dots, \phi_\varepsilon^{(n)}\big\} := \big\{\phi_{\varepsilon,\mu_1,\mu_2}^{(1)}, \dots, \phi_{\varepsilon,\mu_1,\mu_2}^{(n)}\big\}$$

of equation (1.1), with frequency $\omega_\varepsilon := \frac{2\pi}{T_\varepsilon}$, even in time. They are of the following form:

(i) *There exist n different 2π-periodic nonzero Hopf wave solutions $\{v_{\varepsilon,\mu_1,\mu_2}^{(1)}, \dots, v_{\varepsilon,\mu_1,\mu_2}^{(n)}\}$ of the linear equation (1.2), with size*

$$C^{-1}\varepsilon^{\frac{1}{2}} \le \|v_{\varepsilon,\mu_1,\mu_2}^{(k)}\|_{H^r([0,2\pi], H^s(\mathbb{S}^3, d\sigma))} \le C\varepsilon^{\frac{1}{2}-\mathtt{d}}, \tag{1.17}$$

such that $\phi_{\varepsilon,\mu_1,\mu_2}^{(k)} = v_{\varepsilon,\mu_1,\mu_2}^{(j)} + o(v_{\varepsilon,\mu_1,\mu_2}^{(k)})$ as in (1.9).

(ii) *Each $\phi_{\varepsilon,\mu_1,\mu_2}^{(k)}$ has minimal period $T_{k,\varepsilon} := \frac{T_\varepsilon}{m_k}$, where $\{m_k\}_{k=1}^n$ is an increasing sequence of positive integers, and each $v_{\varepsilon,\mu_1,\mu_2}^{(k)}$ has minimal period $T_k := \frac{2\pi}{m_k}$.*

It is proved in [22] that for any μ_1, μ_2 there exist one-mode Hopf plane wave solutions of the resonant system (1.11), of the form

$$\varepsilon^{\frac{1}{2}} \underline{v}^{(j)}, \quad \underline{v}^{(j)} := \kappa_j \cos(\omega_j^{(\mu_1,\mu_2)} t) \mathrm{e}_j^{(\mu_1,\mu_2)}(\eta) e^{\mathrm{i}\mu_1\xi_1} e^{\mathrm{i}\mu_2\xi_2}, \quad \kappa_j \in \mathbb{R} \setminus \{0\}, \tag{1.18}$$

where $\mathrm{e}_j^{(\mu_1,\mu_2)}$ are eigenfunctions of $-\Delta_{\mathbb{S}^3}+\mathbb{1}$ with eigenvalues $\omega_j^{(\mu_1,\mu_2)} := 2j+|\mu_1|+|\mu_2|+1$. However, the proof that they are nondegenerate is obtained only for $\mu_1=\mu_2$ and either $\mu_1 \in \{0,\dots,5\}$ or μ_1 large enough, with explicit computations performed with Mathematica code. The proof of the more general Theorem 1.4 is obtained exploiting variational methods.

We make the following comments, common to both Theorems 1.2 and 1.4:

- (*Regularity and multiplicity*) If $r>\frac{5}{2}$ and $s>\frac{7}{2}$ the solutions $\{\phi_\varepsilon^{(j)}\}$ of (1.1) proved in Theorems 1.2 and 1.4 are *classical*. Actually, the smoother we require the solutions to be in time and in space (i.e. the larger r, s are), the smaller $\varepsilon_0(r,s,n)$ has to be. Analogously, the larger the number of solutions n, the smaller $\varepsilon_0(r,s,n)$ has to be.
- (*Minimal periods*) The solutions $\phi_\varepsilon^{(k)}$ of (1.1) whose existence is stated in Theorems 1.2, 1.4 are *geometrically distinct*, having different minimal time periods $\frac{T_\varepsilon}{m_k}$, as stated in each item (ii).
- (*Increasing norms*) The functions $v_\varepsilon^{(k)}$ turn out to have increasing norms in $k=1,\dots,n$, although, for simplicity, we have stated estimates (1.8) and (1.17) uniformly in k. We point out that the factor $\varepsilon^{-\mathtt{d}}$ in the upper bounds in (1.8), (1.17) (where $\mathtt{d}$ is arbitrarily small) is actually present only for $r>\frac{1}{2}$, $s>\frac{3}{2}$, and not in the low norm case $r+s=1$. In such a case we have $\|v_\varepsilon^{(k)}\|_{H^r([0,T_\varepsilon],H^s(\mathbb{S}^3,d\sigma))} \asymp \varepsilon^{\frac{1}{q}}$. See more comments in Section 1.2.
- (*Critical exponent*) The restrictions on the exponents $p\le 5$ and $p\neq 4$ are *not* technical. In the critical case $p=5$, the functional

$$\mathcal{G}_{p+1}(v) := \frac{1}{p+1}\int_{\mathbb{T}\times\mathbb{S}^3} v^{p+1}(t,z)\,dt\,d\sigma(z)$$

associated to the nonlinearity is finite for any v in the space $L^2(\mathbb{T}_t, H^1(\mathbb{S}^3,d\sigma))$ (which appears in (1.13)) only by Sobolev embedding $H^1(\mathbb{S}^3,d\sigma)\hookrightarrow L^6(\mathbb{S}^3,d\sigma)$. However, it follows to have *compact* gradient by the Strichartz estimates in Proposition 4.5 (see Remark 4.6). For the supercritical exponents $p\ge 7$, the functional $\mathcal{G}_{p+1}(v)$ is not expected to be well defined for any v in $L^2(\mathbb{T}_t, H^1(\mathbb{S}^3,d\sigma))$. If $p=4$, then $\mathcal{G}_5(v)\equiv 0$, as well as for all even values of p. Then the leading term in the action functional of the corresponding resonant system turns out to have degree 8, which is supercritical.

As already mentioned, Theorems 1.2 and 1.4 are inspired by the variational approach of [4–7], developed for one-dimensional semilinear completely resonant wave equations $-\partial_{tt}u+\partial_{xx}u=u^p+\cdots$, with Dirichlet boundary conditions. Major difficulties with respect to these works arise because of the three-dimensional manifold $\mathbb{S}^3$. This becomes evident for instance in the search of spherically symmetric solutions of (1.1), which reduces to solving the wave equation

$$-\partial_{tt}u+\partial_{xx}u=\frac{u^p}{\sin^{p-1}(x)},\quad x\in(0,\pi),\ u(t,0)=u(t,\pi)=0, \tag{1.19}$$

which has a singular nonlinearity at $x = 0, \pi$. Before explaining the main difficulties and ideas of our proof, we present a few related results.

Related literature. The first existence results of 2π-periodic solutions for completely resonant wave equations $\partial_{tt}u - \partial_{xx}u = |u|^{p-2}u$, $p > 2$, have been proved by Rabinowitz starting with [40], via global variational methods. These techniques, as well as those in [18, 19], enable one to find periodic orbits with rational frequency, the reason being that other periods give rise to a small denominator problem.

Independently of these global results, the local bifurcation theory of periodic and quasi-periodic solutions was initiated for *nonresonant* one-dimensional Klein–Gordon equations by Wayne [41], Kuksin [32], Craig–Wayne [24], Pöschel [38], Chierchia–You [23], with KAM methods. For semilinear Klein–Gordon equations on $\mathbb{T}^d$ with convolution potentials, the first result is due to Bourgain [17], later extended by [9, 10] for multiplicative potentials. Bifurcation for periodic and quasi-periodic solutions of nonresonant Klein–Gordon equations was obtained in [12, 14] for Lie groups and homogeneous manifolds, in [11] for Zoll manifolds, and in [30] for the sphere $\mathbb{S}^d$. These results do not cover the completely resonant case (1.1), where all the linear frequencies of oscillations are integers.

The first existence results of Cantor families of small-amplitude time-periodic solutions of one-dimensional *completely resonant* wave equations $-\partial_{tt}u + \partial_{xx}u = u^p$, $p = 3$, was proved in [33] under periodic boundary conditions and in [3] for Dirichlet boundary conditions, for frequencies belonging to the zero measure set (1.3). The latter result was then generalized in [5, 6] to arbitrary exponents p, using variational methods. Existence of periodic solutions for a set of frequencies $\omega \sim 1$ of density 1 was proved in [7, 8] via Nash–Moser implicit function techniques, and in [29] via trees resummation arguments. Existence of time-quasi-periodic solutions with two frequencies of completely resonant nonlinear wave equations on the circle were obtained in [39] and [13].

For completely resonant wave equations, or even more general Hamiltonian PDEs in dimension higher than 1, not much is known about time-periodic solutions besides the aforementioned paper [22] and the present work.

1.2. Ideas of proof

In order to look for bifurcation of small-amplitude time-periodic solutions of (1.1) with frequency $\omega \sim 1$, a natural approach is to implement a Lyapunov–Schmidt decomposition in the spirit of [5–7] for one-dimensional semilinear wave equations. Major difficulties arise due to the higher dimension of the space domain, here the sphere $\mathbb{S}^3$, as we now explain. After a time rescaling, we look for 2π-periodic in time real solutions $u(t, z)$ of $-\omega^2\partial_{tt}u + \Delta_{\mathbb{S}^3}u - u = u^p$. By splitting

$$u = v + w, \quad v := \Pi_V u, \quad w := \Pi_W u, \quad \Pi_W := \mathbb{1} - \Pi_V,$$

where V is the kernel of the operator $-\partial_{tt} + \Delta_{\mathbb{S}^3} - \mathbb{1}$ (namely the space of solutions of the free Klein–Gordon equation (1.2)) and Π_V the corresponding orthogonal projector, it

amounts to the system

$$(\omega^2-1)(-\Delta_{\mathbb{S}^3}+\mathbb{1})v=\Pi_V((v+w)^p), \tag{Bif. eq}$$
$$(-\omega^2\partial_{tt}+\Delta_{\mathbb{S}^3}-\mathbb{1})w=\Pi_W((v+w)^p). \tag{Range eq}$$

For any $\omega\in\Omega_\gamma$ the operator $\mathcal{L}_\omega:=-\omega^2\partial_{tt}+\Delta_{\mathbb{S}^3}-\mathbb{1}$ is invertible on the range $W:=V^\perp$ and, for any fixed $v\in V$ small enough (in some suitable norm), one may first solve the range equation, obtaining $w=w(v)=o(v)$, by a contraction argument. Here, in order to control the nonlinearity $(v+w)^p$, it is natural to close the contraction in Sobolev spaces

$$H^r_t H^s_z:=H^r(\mathbb{T}_t,H^s(\mathbb{S}^3,d\sigma)),\quad r>\frac{1}{2},\ s>\frac{3}{2},$$

which are an algebra with respect to the product of functions, and where $\mathcal{L}_\omega^{-1}|_W$ is bounded. This requires taking v small enough in $H^r_t H^s_z$ as well, which amounts, for functions in the kernel V, to requiring that

$$\|v\|_{\mathcal{V}^{r+s}_{t,z}}\ll 1,\quad \mathcal{V}^{r+s}_{t,z}:=L^2(\mathbb{T}_t,H^{r+s}(\mathbb{S}^3,d\sigma))\cap V,\quad r+s>2. \tag{1.20}$$

On the other hand, one needs then to solve the bifurcation equation (Bif. eq) with $w=w(v)$. As observed in [5,6], this turns out to be the Euler–Lagrange equation of the reduced action functional

$$\Phi(v):=\frac{(\omega^2-1)}{2}\|v\|^2_{\mathcal{V}^1_{t,z}}-\frac{1}{p+1}\int_{\mathbb{T}\times\mathbb{S}^3}(v+w(v))^{p+1}\,dt\,d\sigma. \tag{1.21}$$

A serious problem which arises is thus the following:

> Problem: *The natural space to find mountain pass critical points for the functional Φ in (1.21) is (a small ball in) the space $\mathcal{V}^1_{t,z}$ (modeled with an H^1-norm), associated to its quadratic part. This is clearly in contradiction with solving* (Range eq) *on the much smaller domain $\{\|v\|_{\mathcal{V}^{2+}_{t,z}}\ll 1\}$[1] in (1.20). How do we fill this regularity gap?*

We remark that the previous difficulty *does not* disappear when restricting to search solutions which depend on only one space variable, as spherically symmetric functions or Hopf waves. This is evident for instance in the spherically symmetric case, where the reduced equation (1.19) has a *singular* nonlinearity. If $p=3$, this issue is overcome (cf. [22]) noting that the functional Φ in (1.21) possesses nondegenerate critical points of the explicit form $v=\varepsilon^{\frac{1}{2}}(\underline{v}+\cdots)$, where $\underline{v}$ is a one-mode function as in (1.10), which belong to $\{\|v\|_{\mathcal{V}^{r+s}_{t,z}}\ll 1\}$ for any $r+s>2$.

[1] Here, if $a\in\mathbb{R}$, by $a+$ we mean a number greater than a.

We now describe our strategy. For simplicity, we focus on the case $p = 5$ and we restrict on spherically symmetric functions. The seminal idea is to note that, neglecting $w(v)$, the functional $\Phi(v)$ in (1.21) is a perturbation of the "resonant system" functional

$$\begin{aligned} \Phi_0(v) &:= \frac{\varepsilon}{2}\|v\|^2_{\mathcal{V}^1_{t,z}} - \mathcal{G}_{p+1}(v), \\ \mathcal{G}_{p+1}(v) &:= \frac{1}{p+1}\int_{\mathbb{T}\times\mathbb{S}^3} v^{p+1}\, dt\, d\sigma, \quad \varepsilon := \omega^2 - 1 > 0. \end{aligned} \tag{1.22}$$

The Strichartz estimate (4.10) implies that $\mathcal{G}_6$ is well defined on $\mathcal{V}^1_{t,z}$ and its gradient $\nabla_{\mathcal{V}^1_{t,z}}\mathcal{G}_6$ is a bounded map from $\mathcal{V}^s_{t,z}$ to $\mathcal{V}^1_{t,z}$ for any $s > \frac{5}{6}$, thus compact on $\mathcal{V}^1_{t,z}$. Thus Φ_0 possesses a mountain pass critical point $v \in \mathcal{V}^1_{t,z}$ (see [1]), which by homogeneity has the form $v = \varepsilon^{\frac{1}{4}}\underline{v}$, where $\underline{v}$ solves the rescaled equation $v = (-\Delta_{\mathbb{S}^3} + \mathbb{1})^{-1}\Pi_V v^5$. Such a $\underline{v}$ is not a one-mode function, but it is C^∞ by the following bootstrap argument. By the Strichartz estimate (4.10), one has

$$\|\underline{v}\|_{\mathcal{V}^{\frac{7}{6}-\delta}_{t,z}} = \|\Pi_V \underline{v}^5\|_{\mathcal{V}^{-(\frac{5}{6}+\delta)}_{t,z}} = \sup_{\substack{h\in\mathcal{V}^{\frac{5}{6}+\delta}_{t,z} \\ \|h\|_{\mathcal{V}^{\frac{5}{6}+\delta}_{t,z}}\le 1}} \left|\int_{\mathbb{T}\times\mathbb{S}^3} \underline{v}^5 h\, dt\, d\sigma\right| \le C_\delta \|\underline{v}\|^5_{\mathcal{V}^{\frac{5}{6}+\delta}_{t,z}}.$$

Then, to further increase the regularity of $\underline{v}$, we observe that the Strichartz estimate (4.11) implies

$$\|\underline{v}\|_{\mathcal{V}^{2+\delta'}_{t,z}} = \|\Pi_V \underline{v}^5\|_{\mathcal{V}^{\delta'}_{t,z}} = \sup_{\substack{h\in\mathcal{V}^{\infty}_{t,z} \\ \|h\|_{\mathcal{V}^{-\delta'}_{t,z}}\le 1}} \left|\int_{\mathbb{T}\times\mathbb{S}^3} \underline{v}^5 h\, dt\, d\sigma\right| \le C_\delta \|\underline{v}\|^5_{\mathcal{V}^{1+\delta'}_{t,z}}.$$

Iterating this procedure with increasing values of δ', one deduces that $\underline{v}$ is in C^∞.

In order to adapt the previous arguments to deal with the whole functional Φ in (1.21), we split the bifurcation equation into low and high frequencies. For any $N \in \mathbb{N}$ (to be determined later large enough) the bifurcation equation is equivalent to the system

$$\varepsilon(-\Delta_{\mathbb{S}^3} + \mathbb{1})v_1 = \Pi_{V_{\le N}}((v_1 + v_2 + w)^p), \tag{1.23}$$

$$\varepsilon(-\Delta_{\mathbb{S}^3} + \mathbb{1})v_2 = \Pi_{V_{>N}}((v_1 + v_2 + w)^p), \tag{1.24}$$

where

$$v(t,z) = \sum_{j\in\mathbb{N}} v_j \cos(\omega_j t)e_j(z), \quad v = v_1 + v_2,\ v_1 := \Pi_{V_{\le N}} v,\ v_2 := \Pi_{>N} v,$$

$\omega_j := j + 1$ are the frequencies associated to the eigenfunctions e_j defined in (1.10), and $\Pi_{\le N}$, resp. $\Pi_{>N}$, is the projector on the time-space Fourier frequencies smaller than N, resp. $> N$.

Then we solve both the high-frequency bifurcation equation (1.24) and the range equation (Range eq) arguing by contraction:

- In Section 5 we solve first the high-frequency bifurcation equation (1.24) for v_2 in a small ball of $\mathcal{V}^{2+}_{t,z}$, for any $\|v_1\|_{\mathcal{V}^1_{t,z}} \leq R\varepsilon^{\frac{1}{4}}$ and $\|w\|_{H_t^{\frac{1}{2}+}H_z^{\frac{3}{2}+}} \lesssim \varepsilon^{\frac{5}{4}}N^{5+}$. Here we use the Strichartz-type estimates (4.10)–(4.11).
- In Section 6 we solve the range equation (Range eq) for w in a small ball of $H_t^{\frac{1}{2}+}H_z^{\frac{3}{2}+}$. We exploit algebra properties since $v_2 \in \mathcal{V}^{2+}_{t,z} \subseteq H_t^{\frac{1}{2}+}H_z^{\frac{3}{2}+}$ and v_1 belongs to a finite-dimensional space.
- In Section 7 we solve the finite-dimensional bifurcation equation (1.23), which has a variational structure, applying mountain pass arguments and finding a critical point v_1 satisfying $\|v_1\|_{\mathcal{V}^1_{t,z}} \leq R\varepsilon^{\frac{1}{4}}$. Finally, in Section 8 we prove multiplicity of critical points, distinguished by their minimal period. In Section 9 we prove their higher regularity, where $\|v_1\|_{\mathcal{V}^s_{t,z}}$ is estimated by $\|v_1\|_{\mathcal{V}^1_{t,z}}N^{s-1}$ for any $s \geq 1$.

In all these points, we use smallness conditions of the form $\gamma^{-1}N^{\zeta}\varepsilon \ll 1$, for $\zeta > 0$. Then for technical simplicity we take $N = \varepsilon^{-\frac{1}{\beta}}$, with β arbitrarily large.

In the case $p = 3$ we follow an analogous variational procedure. Remark that in this case the "resonant system" functional (1.22) possesses one-mode Hopf plane wave solutions for any value of the momenta $(\mu_1, \mu_2) \in \mathbb{Z} \times \mathbb{Z}$, but in general their nondegeneracy is not known, except for the particular values considered in [22]. This is because for $\mu_1 \neq \mu_2$ an explicit formula for the product between the eigenfunctions $\{e_j^{(\mu_1,\mu_2)}\}_j$ in (1.18) is not available. Then we split our equation (1.1) into the range equation (Range eq) and the high and low bifurcation equations (1.24), (1.23). We solve the low-frequency bifurcation equation (1.23) using duality arguments, the Hölder inequality, and the Sobolev embedding (1.5), without Strichartz-type estimates.

In the degenerate case $p = 2$, one has $\Pi_V(v^2) = 0$ and the leading nonlinear term in the bifurcation equations (1.23)–(1.24) turns out to be the cubic term $\Pi_V(v\mathcal{L}_\omega^{-1}v^2)$. The Strichartz-type estimates (4.22)–(4.23) are used to solve the high-frequency equation (1.24), avoiding the need to prove whether $\mathcal{L}_\omega^{-1}|_W$ is bounded on $L^q(\mathbb{T}_t, L^q(\mathbb{S}^3, d\sigma))$ spaces.

Notation. We denote by $\mathbb{N} := \{0, 1, 2, \ldots\}$ the set of integer numbers and $\mathbb{N}_* := \{1, 2, \ldots\}$. Given $a \in \mathbb{R}$, we denote $\langle a\rangle := \max\{1, |a|\}$. Given a, b real-valued functions, $a \lesssim b$ means that there exists $C > 0$ such that $a \leq Cb$. If C depends on parameters $\alpha_1, \ldots, \alpha_r$, we write $a \lesssim_{\alpha_1,\ldots,\alpha_r} b$. If $a \lesssim b$ and $b \lesssim a$, we write $a \asymp b$.

2. Functional setting

We describe the Laplace–Beltrami operator in spherical and Hopf coordinates, we describe its spectrum and eigenfunctions, and we define the Sobolev space of spherically symmetric functions and Hopf plane waves.

2.1. Functions with spherical symmetry

According to Definition 1.1, in spherical coordinates the metric tensor is represented with respect to the basis of the tangent space $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}\}$ as

$$g(x,\theta,\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2(x) & 0 \\ 0 & 0 & \sin^2(x)\sin^2(\theta) \end{pmatrix}.$$

Hence the volume form is $d\sigma = \sin^2(x)\sin(\theta)\,dx\,d\theta\,d\varphi$, and the Laplace–Beltrami operator reads

$$\Delta_{\mathbb{S}^3} = \partial_x^2 + 2\frac{\cos(x)}{\sin(x)}\partial_x + \frac{1}{\sin^2(x)}\partial_\theta^2 + \frac{\cos(\theta)}{\sin^2(x)\sin(\theta)}\partial_\theta + \frac{1}{\sin^2(x)\sin^2(\theta)}\partial_\varphi^2. \quad (2.1)$$

For convenience, we introduce the normalized measures

$$đ\sigma := \frac{1}{2\pi^2}\,d\sigma, \quad đx := \frac{2}{\pi}\,dx, \quad đ\theta := \frac{1}{2}\,d\theta, \quad đ\varphi := \frac{1}{2\pi}\,d\varphi,$$

chosen in such a way that the measure of the sphere $\mathbb{S}^3$ is 1. We denote $L^p(\mathbb{S}^3, đ\sigma) := L^p(\mathbb{S}^3)$.

The Laplace–Beltrami operator (2.1) leaves invariant the subspace of spherically symmetric functions (cf. Definition 1.1), acting as

$$\Delta_{\mathbb{S}^3}(u \otimes 1_{\theta,\varphi}) = (\Delta^{\mathrm{ss}}_{\mathbb{S}^3} u) \otimes 1_{\theta,\varphi}, \quad \Delta^{\mathrm{ss}}_{\mathbb{S}^3} := \partial_x^2 + 2\frac{\cos(x)}{\sin(x)}\partial_x.$$

As a consequence, the subspaces of spherically symmetric functions in $H^s(\mathbb{S}^3, đ\sigma)$ coincide with

$$\mathcal{H}^s_x := \left\{u \in L^2([0,\pi], \sin^2(x)\,đx) : (-\Delta^{\mathrm{ss}}_{\mathbb{S}^3} + \mathbb{1})^{\frac{s}{2}} u \in L^2([0,\pi], \sin^2(x)\,đx)\right\}, \quad (2.2)$$

equipped with inner product

$$\langle u_1, u_2\rangle_{\mathcal{H}^s_x} := \langle(-\Delta^{\mathrm{ss}}_{\mathbb{S}^3} + \mathbb{1})^s u_1, u_2\rangle_{L^2([0,\pi],\sin^2(x)\,đx)},$$

in the sense that $u \in \mathcal{H}^s_x$ if and only if $u \otimes 1_{\theta,\varphi} \in H^s(\mathbb{S}^3, đ\sigma)$, with

$$\|u\|_{\mathcal{H}^s_x} = \|u \otimes 1_{\theta,\varphi}\|_{H^s(\mathbb{S}^3, đ\sigma)} \quad \forall u \in \mathcal{H}^s_x. \quad (2.3)$$

We now exhibit a basis of eigenfunctions and eigenvalues for the operator $\Delta^{\mathrm{ss}}_{\mathbb{S}^3}$; see [22]:

Lemma 2.1 (Spectral decomposition of $\Delta^{\mathrm{ss}}_{\mathbb{S}^3}$). *The set of functions $\{e_n\}_{n\in\mathbb{N}}$ defined by*

$$e_n(x) := \frac{\sin((n+1)x)}{\sin(x)} \quad \forall n \in \mathbb{N}, \quad (2.4)$$

is an orthonormal basis for $\mathcal{H}^0_x$ of eigenfunctions of $-\Delta^{\mathrm{ss}}_{\mathbb{S}^3} + \mathbb{1}$, with eigenvalues

$$\omega_n^2, \ \omega_n := n+1 \quad \forall n \in \mathbb{N}.$$

As a consequence, the Sobolev spaces $\mathcal{H}^s_x$ in (2.2) are spectrally characterized as

$$\mathcal{H}^s_x = \{u(x) = \textstyle\sum_{j\in\mathbb{N}} u_j e_j(x) : \|u\|^2_{\mathcal{H}^s_x} := \sum_{j\in\mathbb{N}} |u_j|^2 \omega_j^{2s} < \infty\}, \tag{2.5}$$

where $u_j := \langle u, e_j\rangle_{\mathcal{H}^0_x}$ are the Fourier coefficients with respect to the basis $\{e_j\}$, with scalar product $\langle u, v\rangle_{\mathcal{H}^s_x} = \sum_{j\in\mathbb{N}} u_j \overline{v_j}\omega_j^{2s}$.

The eigenfunctions $\{e_n\}_{n\in\mathbb{N}}$ satisfy the following product rule: for any integer $n \geq m$,

$$e_n(x)e_m(x) = \sum_{k=0}^{m} e_{n-m+2k}(x). \tag{2.6}$$

We shall use property (2.6) to prove the Strichartz-type Propositions 4.5 and 4.12. It can also be used to prove that the spaces $\mathcal{H}^s_x$ with $s > \frac{3}{2}$ enjoy the algebra property.

2.2. Hopf symmetry

According to Definition 1.3, in Hopf coordinates the metric tensor is represented with respect to the basis of the tangent space $\{\frac{\partial}{\partial\eta}, \frac{\partial}{\partial\xi_1}, \frac{\partial}{\partial\xi_2}\}$ as

$$g(\eta,\xi_1,\xi_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2(\eta) & 0 \\ 0 & 0 & \cos^2(\eta) \end{pmatrix}.$$

Hence the volume form is $d\sigma = \frac{1}{2}\sin(2\eta)\,d\eta\,d\xi_1\,d\xi_2$ and the Laplace–Beltrami operator reads

$$\Delta_{\mathbb{S}^3} = \partial^2_\eta + 2\frac{\cos(2\eta)}{\sin(2\eta)}\partial_\eta + \frac{1}{\sin^2(\eta)}\partial^2_{\xi_1} + \frac{1}{\cos^2(\eta)}\partial^2_{\xi_2}. \tag{2.7}$$

We introduce the normalized measure

$$đ\sigma := \frac{1}{2\pi^2}\,d\sigma, \quad đ\eta := d\eta, \quad đ\xi_1 := \frac{1}{2\pi}\,d\xi_1, \quad đ\xi_2 := \frac{1}{2\pi}\,d\xi_2, \tag{2.8}$$

so that the measure of the sphere $\mathbb{S}^3$ is 1.

Representing a function ϕ in Hopf coordinates (see Definition 1.3) and expanding in Fourier series with respect to the variables ξ_1, ξ_2, we have

$$\phi(\eta,\xi_1,\xi_2) = \sum_{\mu_1,\mu_2\in\mathbb{Z}} \hat{\phi}_{\mu_1,\mu_2}(\eta) e^{\mathrm{i}\mu_1\xi_1} e^{\mathrm{i}\mu_2\xi_2}.$$

In these coordinates, the Laplace–Beltrami operator (2.7) reads

$$\Delta_{\mathbb{S}^3}\phi(\eta,\xi_1,\xi_2) = \sum_{\mu_1,\mu_2\in\mathbb{Z}} e^{\mathrm{i}\mu_1\xi_1} e^{\mathrm{i}\mu_2\xi_2} \Delta_{\mu_1,\mu_2}\hat{\phi}_{\mu_1,\mu_2}(\eta)$$

with

$$\Delta_{\mu_1,\mu_2} := \partial^2_\eta + 2\frac{\cos(2\eta)}{\sin(2\eta)}\partial_\eta - \frac{\mu_1^2}{\sin^2(\eta)} - \frac{\mu_2^2}{\cos^2(\eta)}. \tag{2.9}$$

As a consequence, the space of Hopf plane waves is left invariant by $\Delta_{\mathbb{S}^3}$ and recalling (2.7), (2.8), the subspaces of Hopf plane waves in $H^s(\mathbb{S}^3, \mathrm{d}\sigma)$ coincide with

$$\mathcal{H}^s_\eta := \left\{ u \in L^2((0,\tfrac{\pi}{2}), \sin(2\eta)\,d\eta) : (-\Delta_{\mu_1,\mu_2} + \mathbb{1})^{\frac{s}{2}} u \in L^2((0,\tfrac{\pi}{2}), \sin(2\eta)\,d\eta) \right\} \quad (2.10)$$

for any $s \in \mathbb{R}$, equipped with inner product

$$\langle u_1, u_2 \rangle_{\mathcal{H}^s_\eta} = \langle (-\Delta_{\mu_1,\mu_2} + \mathbb{1})^s u_1, u_2 \rangle_{L^2((0,\frac{\pi}{2}),\sin(2\eta)\,d\eta)},$$

in the sense that $u(\eta) \in \mathcal{H}^s_\eta$ if and only if $u(\eta) e^{\mathrm{i}\mu_1\xi_1} e^{\mathrm{i}\mu_2\xi_2} \in H^s(\mathbb{S}^3, \mathrm{d}\sigma)$ (cf. (1.4)), with

$$\|u(\cdot)\|_{\mathcal{H}^s_\eta} = \|u(\cdot) e^{\mathrm{i}\mu_1\xi_1} e^{\mathrm{i}\mu_2\xi_2}\|_{H^s(\mathbb{S}^3,\mathrm{d}\sigma)}. \quad (2.11)$$

We now exhibit a basis of eigenfunctions and eigenvalues of $-\Delta_{\mu_1,\mu_2}$; see [22, Section 3.2].

Lemma 2.2 (Spectral decomposition of Δ_{μ_1,μ_2}). *There exists an L^2-orthonormal basis of eigenfunctions $\{e_j^{(\mu_1,\mu_2)}\}_{j\in\mathbb{N}}$ of $-\Delta_{\mu_1,\mu_2} + \mathbb{1}$, with eigenvalues $(\omega_j^{(\mu_1,\mu_2)})^2$, where*

$$\omega_j^{(\mu_1,\mu_2)} := 2j + 1 + |\mu_1| + |\mu_2|, \quad j \in \mathbb{N}.$$

The $e_j^{(\mu_1,\mu_2)}$ are the real functions

$$e_j^{(\mu_1,\mu_2)}(\eta) = N_j^{(|\mu_1|,|\mu_2|)} (1-\cos(2\eta))^{\frac{|\mu_1|}{2}} (1+\cos(2\eta))^{\frac{|\mu_2|}{2}} P_j^{(|\mu_1|,|\mu_2|)}(\cos(2\eta)), \quad (2.12)$$

where $\{P_j^{(|\mu_1|,|\mu_2|)}\}_{j\in\mathbb{N}}$ are the Jacobi polynomials and $N_j^{(|\mu_1|,|\mu_2|)}$ are suitable normalization constants.

By Lemma 2.2, the Sobolev spaces $\mathcal{H}^s_\eta$ in (2.10) are spectrally characterized as

$$\mathcal{H}^s_\eta := \left\{ u(\eta) := \textstyle\sum_{j\in\mathbb{N}} u_j e_j^{(\mu_1,\mu_2)}(\eta) : \|u\|^2_{\mathcal{H}^s_\eta} := \sum_{j\in\mathbb{N}} |u_j|^2 (\omega_j^{(\mu_1,\mu_2)})^{2s} < \infty \right\}, \quad (2.13)$$

where $u_j := \langle u, e_j^{(\mu_1,\mu_2)} \rangle_{\mathcal{H}^0_\eta}$ are the Fourier coefficients of u with respect to the basis $\{e_j^{(\mu_1,\mu_2)}\}_{j\in\mathbb{N}}$ equipped with scalar product $\langle u, v \rangle_{\mathcal{H}^s_\eta} = \sum_{j\in\mathbb{N}} u_j \overline{v_j} (\omega_j^{(\mu_1,\mu_2)})^{2s}$.

2.3. Sobolev spaces in time-space

Since equation (1.1) is time reversible, we look for functions which are even in time. For this reason, we consider the Sobolev spaces of time-periodic even real functions

$$H^r_t \mathcal{H}^s_z := \left\{ u(t,z) = \textstyle\sum_{\ell,j\in\mathbb{N}} u_{\ell,j} \cos(\ell t) \mathrm{e}_j(z) : \right.$$
$$\left. \|u\|^2_{H^r_t \mathcal{H}^s_z} := \sum_{\ell\in\mathbb{N}} \langle \ell \rangle^{2r} \sum_{j\in\mathbb{N}} \omega_j^{2s} |u_{\ell,j}|^2 < \infty \right\}, \quad (2.14)$$

taking values in

$$\mathcal{H}^s_z := \begin{cases} \mathcal{H}^s_x \text{ defined in (2.5)} & \text{for spherically symmetric functions,} \\ \mathcal{H}^s_\eta \text{ defined in (2.13)} & \text{for Hopf plane waves.} \end{cases} \quad (2.15)$$

In (2.14) the $\{\mathrm{e}_j\}_{j\in\mathbb{N}}$ are respectively the eigenfunctions of $-\Delta^{\mathrm{ss}}_{\mathbb{S}^3} - \mathbb{1}$ and $-\Delta_{\mu_1,\mu_2} + \mathbb{1}$, namely

$$\mathrm{e}_j := \begin{cases} e_j \text{ as in (2.4)} & \text{for spherically symmetric functions,} \\ e_j^{(\mu_1,\mu_2)} \text{ as in Lemma 2.2} & \text{for Hopf plane waves,} \end{cases}$$

corresponding to the eigenvalues $\{\omega_j^2\}_{j\in\mathbb{N}}$, where

$$\omega_j := \begin{cases} j+1 & \text{for spherically symmetric functions,} \\ 2j+1+|\mu_1|+|\mu_2| & \text{for Hopf plane waves,} \end{cases} \tag{2.16}$$

and $u_{\ell,j}$ are the time-space Fourier coefficients of u,

$$u_{\ell,j} := \begin{cases} \displaystyle\int_{\mathbb{T}} \cos(\ell t)\langle u(t,\cdot), e_j\rangle_{\mathcal{H}^0_x}\, đt & \text{for spherically symmetric functions,} \\ \displaystyle\int_{\mathbb{T}} \cos(\ell t)\langle u(t,\cdot), e_j^{(\mu_1,\mu_2)}(\eta)\rangle_{\mathcal{H}^0_\eta}\, đt & \text{for Hopf plane waves,} \end{cases}$$

where

$$đt := \frac{1}{\pi}\, dt. \tag{2.17}$$

For any $r > \frac{1}{2}$ and for any $s \in \mathbb{R}$ the space $H^r_t \mathcal{H}^s_z$ is embedded continuously into $L^\infty(\mathbb{T}_t, \mathcal{H}^s_z)$, with

$$\|u\|_{L^\infty(\mathbb{T}_t,\mathcal{H}^s_z)} \le C_r \|u\|_{H^r_t \mathcal{H}^s_z}, \tag{2.18}$$

for some $C_r > 0$. Moreover, since the spaces $\mathcal{H}^s_z$ for $s > \frac{3}{2}$ are an algebra, also the spaces $H^r_t \mathcal{H}^s_z$ are an algebra for any $r > \frac{1}{2}$ and $s > \frac{3}{2}$: there exists a constant $C_{s,r} > 0$ such that

$$\|u_1 u_2\|_{H^r_t \mathcal{H}^s_z} \le C_{s,r} \|u_1\|_{H^r_t \mathcal{H}^s_z} \|u_2\|_{H^r_t \mathcal{H}^s_z}. \tag{2.19}$$

Lemma 2.3. *There exists $C > 0$ such that for any $u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)} \in L^\infty(\mathbb{T}_t, \mathcal{H}^{\frac{3}{4}}_\eta)$,*

$$\left| \int_{\mathbb{T}} \int_0^{\frac{\pi}{2}} u^{(1)} u^{(2)} u^{(3)} u^{(4)} \sin(2\eta)\, d\eta\, đt \right| \le C \prod_{l=1}^4 \|u^{(l)}\|_{L^\infty(\mathbb{T}_t,\mathcal{H}^{\frac{3}{4}}_\eta)}. \tag{2.20}$$

For any $u^{(1)}, u^{(2)}, u^{(3)} \in L^\infty(\mathbb{T}_t, \mathcal{H}^1_\eta)$, and any $u^{(4)} \in L^\infty(\mathbb{T}_t, \mathcal{H}^0_\eta)$,

$$\begin{aligned} &\left| \int_{\mathbb{T}} \int_0^{\frac{\pi}{2}} u^{(1)} u^{(2)} u^{(3)} u^{(4)} \sin(2\eta)\, d\eta\, đt \right| \\ &\qquad \le C \left(\prod_{l=1}^3 \|u^{(l)}\|_{L^\infty(\mathbb{T}_t,\mathcal{H}^1_\eta)} \right) \|u^{(4)}\|_{L^\infty(\mathbb{T}_t,\mathcal{H}^0_\eta)}. \end{aligned} \tag{2.21}$$

Proof. Let $U^{(1)} := u^{(1)} e^{\mathrm{i}\mu_1\xi_1+\mathrm{i}\mu_2\xi_2}$, $U^{(2)} := u^{(2)} e^{-\mathrm{i}\mu_1\xi_1-\mathrm{i}\mu_2\xi_2}$, $U^{(3)} := u^{(3)} e^{\mathrm{i}\mu_1\xi_1+\mathrm{i}\mu_2\xi_2}$, $U^{(4)} := u^{(4)} e^{-\mathrm{i}\mu_1\xi_1-\mathrm{i}\mu_2\xi_2}$. Recalling the definition of $đ\sigma$, $đ\xi_1$, $đ\xi_2$ in (2.8), one has

$$\int_{\mathbb{T}} \int_0^{\frac{\pi}{2}} \prod_{l=1}^4 u^{(l)}(t,\eta) \sin(2\eta)\, d\eta\, đt = \int_{\mathbb{T}} \int_{\mathbb{S}^3} \prod_{l=1}^4 U^{(l)}(t,z)\, đ\sigma(z)\, đt.$$

Then applying at any time t the generalized Hölder inequality with $p_1 = p_2 = p_3 = p_4 = 4$ for functions on $\mathbb{S}^3$, and the Sobolev embedding $H^{\frac{3}{4}}(\mathbb{S}^3, đ\sigma) \hookrightarrow L^4(\mathbb{S}^3, đ\sigma)$ we get

$$\begin{aligned}\left| \int_{\mathbb{T}} \int_0^{\frac{\pi}{2}} \prod_{l=1}^4 u^{(l)}(t,\eta) \sin(2\eta)\, d\eta\, đt \right| &\le \int_{\mathbb{T}} \prod_{l=1}^4 \|U^{(l)}(t,\cdot)\|_{L^4(\mathbb{S}^3, đ\sigma)}\, đt \\ &\lesssim \int_{\mathbb{T}} \prod_{l=1}^4 \|U^{(l)}(t,\cdot)\|_{H^{\frac{3}{4}}(\mathbb{S}^3, đ\sigma)}\, đt.\end{aligned}$$

Then (2.20) follows because

$$\|U^{(l)}(t,\cdot)\|_{H^{\frac{3}{4}}(\mathbb{S}^3)} = \|u^{(l)}(t)\|_{\mathcal{H}^{\frac{3}{4}}_\eta}$$

by (2.11). The bound (2.21) follows by similarly applying the generalized Hölder inequality with $p_1 = p_2 = p_3 = 6$, and $p_4 = 2$ for functions on $\mathbb{S}^3$, using the embedding $H^1(\mathbb{S}^3, đ\sigma) \hookrightarrow L^6(\mathbb{S}^3, đ\sigma)$ and (2.11). ∎

3. Variational Lyapunov–Schmidt decomposition

We look for time-periodic solutions of (1.1) with time frequency ω close to 1, via a Lyapunov–Schmidt decomposition. More specifically, we look for a $\frac{2\pi}{\omega}$-time-periodic real-valued spherically symmetric solution $u(t,x)$ of (1.1) which solves

$$\begin{aligned} &-\omega^2 \partial_{tt} u(t,x) + (\Delta^{\mathrm{ss}}_{\mathbb{S}^3} - \mathbb{1}) u(t,x) = u^p(t,x), \\ &(t,x) \in \mathbb{T} \times (0,\pi), \quad \partial_x u(t,0) = \partial_x u(t,2\pi) = 0. \end{aligned} \tag{3.1}$$

We consider the cases $p = 2, 5$ only, because the case $p = 3$ is covered in [22].

If $p = 3$ we look for a $\frac{2\pi}{\omega}$-time-periodic Hopf plane wave solution $\phi(t,\eta,\xi_1,\xi_2) = u(t,\eta) e^{i\mu_1\xi_1} e^{i\mu_2\xi_2}$ of (1.1), with $u(t,\eta)$ real. The function $u(t,\eta)$ solves

$$\begin{aligned} &-\omega^2 \partial_{tt} u(t,\eta) + (\Delta_{\mu_1,\mu_2} - \mathbb{1}) u(t,\eta) = u^3(t,\eta), \\ &(t,\eta) \in \mathbb{T} \times \Big(0, \frac{\pi}{2}\Big), \quad \partial_\eta u(t,0) = \partial_\eta u\Big(t, \frac{\pi}{2}\Big) = 0, \end{aligned} \tag{3.2}$$

with Δ_{μ_1,μ_2} defined in (2.9). Both equations in (3.1) and (3.2) are of the form

$$\mathcal{L}_\omega u = u^p, \quad \mathcal{L}_\omega := -\omega^2 \partial_{tt} - A, \tag{3.3}$$

where A denotes the unbounded, self-adjoint, positive operator

$$A := \begin{cases} -\Delta^{\mathrm{ss}}_{\mathbb{S}^3} + \mathbb{1} & \text{for spherically symmetric functions,} \\ -\Delta_{\mu_1,\mu_2} + \mathbb{1} & \text{for Hopf waves.} \end{cases} \tag{3.4}$$

Equation (3.3) admits a variational formulation. It is the formal Euler–Lagrange equation of the action functional

$$\Psi(u) := \frac{1}{2}\int_{\mathbb{T}} \langle \mathcal{L}_\omega u(t), u(t)\rangle_{\mathcal{H}^0_z}\, đt - \mathcal{G}_{p+1}(u), \tag{3.5}$$

with

$$\mathcal{G}_{p+1}(u) := \begin{cases} \dfrac{1}{p+1}\displaystyle\int_{\mathbb{T}}\int_0^{\pi} u^{p+1}(t,x)\sin^2(x)\, đx\, đt & \text{in spherically sym. case,} \\ \dfrac{1}{p+1}\displaystyle\int_{\mathbb{T}}\int_0^{\frac{\pi}{2}} u^{p+1}(t,\eta)\sin(2\eta)\, d\eta\, đt & \text{for Hopf plane waves.} \end{cases} \tag{3.6}$$

We shall exploit the variational structure of (3.3) in Section 7, after a suitable finite-dimensional reduction. We perform a Lyapunov–Schmidt decomposition of equation (3.3). We define

$$\begin{aligned} V &:= \ker(-\partial_{tt} - A) \\ &= \big\{u(t,z) = \textstyle\sum_{j,\ell\in\mathbb{N}} u_{\ell,j}\cos(\ell t)\mathrm{e}_j(z) : u_{\ell,j} = 0\ \forall \ell \neq \omega_j\big\} \\ &= \big\{v(t,z) = \textstyle\sum_{j\in\mathbb{N}} v_j\cos(\omega_j t)\mathrm{e}_j(z)\big\}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} W &:= \mathrm{Rg}(-\partial_{tt} - A) \\ &= \big\{u(t,z) = \textstyle\sum_{j,\ell\in\mathbb{N}} u_{\ell,j}\cos(\ell t)\mathrm{e}_j(z) : u_{\ell,j} = 0\ \forall \ell = \omega_j\big\}. \end{aligned} \tag{3.8}$$

Note that $W = V^\perp$ in any $H^r_t\mathcal{H}^s_z$.

We decompose the space V into low and high frequencies: given $N \in \mathbb{N}$, we define

$$V_1 := V_{\le N} := \big\{v(t,z) = \textstyle\sum_{0\le\omega_j\le N} v_j\cos(\omega_j t)\mathrm{e}_j(z)\big\}, \tag{3.9}$$

$$V_2 := V_{>N} := \big\{v(t,z) = \textstyle\sum_{\omega_j > N} v_j\cos(\omega_j t)\mathrm{e}_j(z)\big\}. \tag{3.10}$$

We denote by Π_V, Π_W, $\Pi_{V_{\le N}} \equiv \Pi_{V_1}$, $\Pi_{V>N} \equiv \Pi_{V_2}$, the orthogonal projectors on V, W, V_1, V_2 respectively, so that any u can be decomposed as

$$\begin{gathered} u = v + w,\ \ v := \Pi_V u = \sum_{j\in\mathbb{N}} u_{\omega_j,j}\cos(\omega_j t)\mathrm{e}_j \in V,\ \ w := \Pi_W u \in W, \\ u = v_1 + v_2 + w,\ \ v_1 := \Pi_{V_1} u \in V_1,\ \ v_2 := \Pi_{V_2} u \in V_2,\ \ w := \Pi_W u \in W. \end{gathered} \tag{3.11}$$

We then observe that a function u satisfies (3.3) if and only if it is a solution of the system

$$(\omega^2 - 1)Av_1 - \Pi_{V_1}(v_1 + v_2 + w)^p = 0, \tag{3.12}$$

$$(\omega^2 - 1)Av_2 - \Pi_{V_2}(v_1 + v_2 + w)^p = 0, \tag{3.13}$$

$$\mathcal{L}_\omega w - \Pi_W(v_1 + v_2 + w)^p = 0. \tag{3.14}$$

We shall solve equation (3.13) for v_2 by a contraction argument in Section 5. Then in Section 6 we shall solve the range equation (3.14), arguing again by a contraction argument and using the following lemma.

Lemma 3.1. *Assume $\omega \in \Omega_\gamma$ with Ω_γ defined in (1.3). Then the linear operator $\mathcal{L}_\omega$ defined in (3.3) is invertible on W, with*

$$\|\mathcal{L}_\omega^{-1}\|_{\mathcal{B}(W\cap H_t^r\mathcal{H}_z^s;W\cap H_t^r\mathcal{H}_z^s)} \leq \frac{2}{\gamma} \quad \forall r,s\in\mathbb{R}.$$

Furthermore, if $\omega = 1$, one has $\|\mathcal{L}_1^{-1}\|_{\mathcal{B}(W\cap H_t^r\mathcal{H}_z^s;W\cap H_t^r\mathcal{H}_z^s)} \leq 1$.

Proof. Let $w(t,z) = \sum_{\ell,j,\ell\neq\omega_j} w_{\ell,j}\cos(\ell t)\mathrm{e}_j(z)$. Then

$$\mathcal{L}_\omega^{-1} w(t,z) = \sum_{\ell\neq\omega_j} \frac{w_{\ell,j}}{\omega^2\ell^2-\omega_j^2}\cos(\ell t)\mathrm{e}_j(z). \tag{3.15}$$

Then it is sufficient to observe that, if $\omega\in\Omega_\gamma$, then

$$\forall \ell\neq\omega_j,\quad |\omega^2\ell^2-\omega_j^2| \geq \frac{\gamma}{2}, \tag{3.16}$$

because for any ℓ and j such that $\ell\neq\omega_j$ and $\ell\neq 0$, one has

$$|(\omega\ell+\omega_j)(\omega\ell-\omega_j)| \geq |\omega\ell|\frac{\gamma}{|\ell|} \geq \frac{\gamma}{2}.$$

Finally, if $\omega = 1$, the estimate immediately follows observing that $|\ell^2-\omega_j^2|\geq 1$, for any $\ell\neq\omega_j$. ∎

4. Properties of functions in V

In this section we prove some properties of functions in the kernel V which will be used to solve system (3.12)–(3.14). Given $s\in\mathbb{R}$, we shall denote $\mathcal{V}_{t,z}^s := V\cap H_t^0\mathcal{H}_z^s$, equipped with norm

$$\|v\|_{\mathcal{V}_{t,z}^s}^2 := \sum_{j\in\mathbb{N}} |v_j|^2\omega_j^{2s}. \tag{4.1}$$

Furthermore, we denote $\mathcal{V}_{t,z}^\infty := \bigcap_{s\geq 0}\mathcal{V}_{t,z}^s$ and $\|\cdot\|_{L^p(\mathbb{T}_t,E)} := (\int_{\mathbb{T}}\|\cdot\|_E^p\,dt)^{\frac{1}{p}}$.

Lemma 4.1. *Let $r,r',s,s'\in\mathbb{R}$ such that $r+s = r'+s'$. Then for any $v\in V$,*

$$\|v\|_{H_t^r\mathcal{H}_z^s} = \|v\|_{H_t^{r'}\mathcal{H}_z^{s'}} = \|v\|_{L^2(\mathbb{T}_t,\mathcal{H}_z^{r+s})}, \tag{4.2}$$

$$\|v\|_{H_t^0\mathcal{H}_z^s}^2 = \|v(0,\cdot)\|_{\mathcal{H}_z^s}^2 = \|v\|_{L^\infty(\mathbb{T}_t,\mathcal{H}_z^s)}^2 = \|v\|_{\mathcal{V}_{t,z}^s}^2. \tag{4.3}$$

Proof. In order to prove (4.2) it is sufficient to observe that

$$\|v\|^2_{H^r_t \mathcal{H}^s_z} = \Big\| \sum_{j\in\mathbb{N}} v_j \cos(\omega_j t) \mathrm{e}_j(z) \Big\|^2_{H^r_t \mathcal{H}^s_z} = \sum_{j\in\mathbb{N}} |v_j|^2 \omega_j^{2r} \omega_j^{2s} = \sum_{j\in\mathbb{N}} |v_j|^2 \omega_j^{2(r+s)}.$$

The identities (4.3) follow because, for any $t \in \mathbb{T}$,

$$\|v(t,\cdot)\|^2_{\mathcal{H}^s_z} = \Big\| \sum_{j\in\mathbb{N}} v_j \cos(\omega_j t) \mathrm{e}_j(z) \Big\|^2_{\mathcal{H}^s_z} = \sum_{j\in\mathbb{N}} |v_j|^2 |\cos(\omega_j t)|^2 \omega_j^{2s},$$

with $\|\cdot\|_{\mathcal{H}^s_z}$ defined according to (2.15) and since $|\cos(\cdot)| \le 1$. ∎

By (4.3) and the algebra property of the spaces $\mathcal{H}^s_z$, for any $v^{(1)}, v^{(2)} \in \mathcal{V}^s_{t,z}$ and $s > \frac{3}{2}$,

$$\|v^{(1)} v^{(2)}\|_{\mathcal{V}^s_{t,z}} \lesssim_s \|v^{(1)}\|_{\mathcal{V}^s_{t,z}} \|v^{(2)}\|_{\mathcal{V}^s_{t,z}}.$$

For any $s < s'$ the following smoothing properties hold (cf. (3.9), (3.10)): for any $v \in V$,

$$\|\Pi_{V_1} v\|_{\mathcal{V}^{s'}_{t,z}} \le N^{s'-s} \|v\|_{\mathcal{V}^s_{t,z}}, \quad \|\Pi_{V_2} v\|_{\mathcal{V}^s_{t,z}} \le N^{-(s'-s)} \|v\|_{\mathcal{V}^{s'}_{t,z}}. \tag{4.4}$$

Since $A\mathrm{e}_j(z) = \omega_j^2 \mathrm{e}_j(z)$ for any j (see Lemmas 2.1 and 2.2) and recalling (4.1) it holds that

$$\|A^{-1} v\|_{\mathcal{V}^{s-2}_{t,z}} \le \|v\|_{\mathcal{V}^s_{t,z}} \quad \forall v \in \mathcal{V}^s_{t,z}. \tag{4.5}$$

We shall also use that by Lemma 4.1 and the Sobolev embedding (1.5), for any $v \in \mathcal{V}^s_{t,x}$ one has

$$\|v\|_{L^p_{t,x}} \lesssim_p \|v\|_{\mathcal{H}^s_{t,x}}, \quad p \le \frac{6}{3-2s}, \tag{4.6}$$

with $L^p_{t,x} := L^p(\mathbb{T}_t, L^p((0,\pi), \sin^2(x)\,đx))$.

Lemma 4.2. *For any $u \in L^2(\mathbb{T}_t, \mathcal{H}^s_z)$ it holds that $\|\Pi_V u\|_{\mathcal{V}^s_{t,z}} \le \|u\|_{H^0_t \mathcal{H}^s_z}$. The same holds if Π_V is replaced by Π_{V_1} or Π_{V_2}.*

Proof. By Lemma 4.1 we have $\|\Pi_V u\|^2_{\mathcal{V}^s_{t,z}} = \|\Pi_V u\|^2_{H^0_t \mathcal{H}^s_z} \le \|u\|^2_{H^0_t \mathcal{H}^s_z}$. ∎

Lemma 4.3. *Let $s > \frac{3}{2}$, $r > \frac{1}{2}$, and $q \in \mathbb{N}$. Then there exists a positive constant $C = C(s,r,q)$ such that for any $j < q$ and any $v^{(1)}, \dots, v^{(j)} \in \mathcal{V}^s_{t,z}$, $u^{(j+1)}, \dots, u^{(q)} \in H^r_t \mathcal{H}^s_z$,*

$$\begin{aligned} &\|\Pi_V (v^{(1)} \cdots v^{(j)} u^{(j+1)} \cdots u^{(q)})\|_{\mathcal{V}^s_{t,z}} \\ &\quad \le C \Big(\prod_{l=1}^{j} \|v^{(l)}\|_{\mathcal{V}^s_{t,z}} \Big) \Big(\prod_{l=j+1}^{q} \|u^{(l)}\|_{H^r_t \mathcal{H}^s_z} \Big). \end{aligned} \tag{4.7}$$

The same statement holds if Π_V is replaced by Π_{V_1} or Π_{V_2}.

Proof. Estimate (4.7) follows from Lemma 4.2, since $\|u\|_{H^0_t \mathcal{H}^s_z} \lesssim \|u\|_{L^\infty_t \mathcal{H}^s_z}$, (2.19), and (2.18). ∎

Lemma 4.4. *For any $q \in \mathbb{N}$ odd and any $v^{(1)}, \dots, v^{(q)} \in V$ one has*

$$\int_{\mathbb{T}} \int_0^\pi v^{(1)}(t,x) \cdots v^{(q)}(t,x) \sin^2(x)\, đx\, đt = 0. \tag{4.8}$$

In particular, if n is even then $\Pi_V(v^{(1)} \cdots v^{(n)}) = 0$ for any $v^{(1)}, \dots, v^{(n)} \in V$.

Proof. In the integral in (4.8), performing the change of variables $(t,x) \mapsto (t',x')$, with $t := t' + \pi$, $x := \pi - x'$, one has $v^{(l)}(t+\pi, \pi - x) = -v^{(l)}(t,x)$ and, thus, since q is odd,

$$\begin{aligned}
\mathcal{I} &:= \int_{\mathbb{T}} \int_0^\pi v^{(1)}(t,x) \cdots v^{(q)}(t,x) \sin^2(x)\, đx\, đt \\
&= \int_{\mathbb{T}} \int_0^\pi v^{(1)}(t'+\pi, \pi - x') \cdots v^{(q)}(t'+\pi, \pi - x') \sin^2(\pi - x')\, đx'\, đt' \\
&= \int_{\mathbb{T}} \int_0^\pi (-1)^q v^{(1)}(t',x') \cdots v^{(q)}(t',x') \sin^2(x')\, đx'\, đt' = -\mathcal{I},
\end{aligned}$$

namely $\mathcal{I} = 0$. ∎

4.1. Strichartz-type estimates for $p = 5$

The aim of this section is to prove a set of Strichartz-type estimates for solutions of (1.2) in the case of spherical symmetry. We shall use the following duality property: for any $s \in \mathbb{R}$,

$$\|v\|_{\mathcal{V}^s_{t,x}} = \sup_{\substack{h \in \mathcal{V}^\infty_{t,x} \\ \|h\|_{\mathcal{V}^{-s}_{t,x}} \le 1}} \int_{\mathbb{T}} \int_0^\pi v(t,x) h(t,x) \sin^2(x)\, đx\, đt. \tag{4.9}$$

Proposition 4.5 (Generalized Strichartz-type estimates). *The following estimates hold:*

(1) *For any $\delta > 0$ there exists a constant $C_\delta > 0$ such that, for any $v^{(1)}, \dots, v^{(6)} \in \mathcal{V}^{\frac{5}{6}+\delta}_{t,x}$,*

$$\left| \int_{\mathbb{T}} \int_0^\pi v^{(1)}(t,x) \cdots v^{(6)}(t,x) \sin^2(x)\, đx\, đt \right| \le C_\delta \prod_{n=1}^6 \|v^{(n)}\|_{\mathcal{V}^{\frac{5}{6}+\delta}_{t,x}}. \tag{4.10}$$

(2) *For any $\delta > 0$ there exists a constant $C_\delta > 0$ such that, for any $v^{(1)}, \dots, v^{(5)} \in \mathcal{V}^{1+\delta}_{t,x}$ and $v^{(6)} \in \mathcal{V}^1_{t,x}$,*

$$\begin{aligned}
&\left| \int_{\mathbb{T}} \int_0^\pi v^{(1)}(t,x) \cdots v^{(6)}(t,x) \sin^2(x)\, đx\, đt \right| \\
&\qquad \le C_\delta \Big(\prod_{n=1}^5 \|v^{(n)}\|_{\mathcal{V}^{1+\delta}_{t,x}} \Big) \|v^{(6)}\|_{\mathcal{V}^{-\delta}_{t,x}}.
\end{aligned} \tag{4.11}$$

Remark 4.6. From (4.10) the functional $\mathcal{G}_6(v) := \frac{1}{6}\int_{\mathbb{T}}\int_0^\pi v^6 \sin^2(x)\, đx\, đt$ is well defined on $\mathcal{V}^1_{t,x}$ with compact gradient.

Remark 4.7. By (4.10) with $v^{(1)} = \cdots = v^{(6)} =: v$, using (4.3) and (2.3), any solution v of the Cauchy problem $\partial_{tt} v + (-\Delta_{\mathbb{S}^3} + \mathbb{1})v = 0$, $\partial_t v(0,\cdot) = 0$, $v(0,\cdot) = v_0 \in \mathcal{H}^{\frac{5}{6}+\delta}_x$, satisfies the Strichartz estimate $\|v\|_{L^6(\mathbb{T}_t\times\mathbb{S}^3, đt\, đσ)} \lesssim_\delta \|v\|_{\mathcal{V}^{\frac{5}{6}+\delta}_{t,x}} = \|v_0\|_{\mathcal{H}^{\frac{5}{6}+\delta}_x}$.

Proposition 4.5 enables us to deduce the following:

Lemma 4.8. *For any $\delta > 0$ there exists $C_\delta > 0$ such that for any $v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}, v^{(5)} \in \mathcal{V}^{1+2\delta}_{t,x}$,*

$$\|\Pi_{V_2}(v^{(1)}v^{(2)}v^{(3)}v^{(4)}v^{(5)})\|_{\mathcal{V}^{2\delta}_{t,x}} \le C_\delta \prod_{n=1}^5 \|v^{(n)}\|_{\mathcal{V}^{1+2\delta}_{t,x}}.$$

Proof. By (4.9) we have

$$\begin{aligned}
\|\Pi_{V_2}(v^{(1)}v^{(2)}v^{(3)}v^{(4)}v^{(5)})\|_{\mathcal{V}^{2\delta}_{t,x}} &= \sup_{\substack{h\in V_2\\ \|h\|_{\mathcal{V}^{-2\delta}_{t,x}}\le 1}} \int_{\mathbb{T}}\int_0^\pi \Pi_{V_2}\Big(\prod_{n=1}^5 v^{(n)}\Big) h \sin^2(x)\, đx\, đt \\
&\lesssim_\delta \sup_{\substack{h\in V_2\\ \|h\|_{\mathcal{V}^{-2\delta}_{t,x}}\le 1}} \prod_{n=1}^5 \|v^{(n)}\|_{\mathcal{V}^{1+2\delta}_{t,x}} \|h\|_{\mathcal{V}^{-2\delta}_{t,x}} \\
&\lesssim_\delta \prod_{n=1}^5 \|v^{(n)}\|_{\mathcal{V}^{1+2\delta}_{t,x}},
\end{aligned}$$

where in the second passage we have used Proposition 4.5 (2). ∎

The rest of this section is devoted to the proof of Proposition 4.5. We use the following definition.

Definition 4.9. Given $j_1, j_2, j_3, j_4, j_5, j_6 \in \mathbb{N}$, we define $j_{\min_1}, \dots, j_{\min_6} \in \mathbb{N}$ by the property that $\{j_1, \dots, j_6\} = \{j_{\min_1}, \dots, j_{\min_6}\}$, and

$$\min\{j_1, \dots, j_6\} =: j_{\min} := j_{\min_1} \le j_{\min_2} \le j_{\min_3} \le \cdots \le j_{\min_6} := \max\{j_1, \dots, j_6\}.$$

Furthermore, we denote by $\delta(a = b) := \delta_{a,b}$ for any $a, b \in \mathbb{N}$, the Kronecker delta.

The following lemma is a direct computation, recalling (2.17):

Lemma 4.10 (Integral in time). *Given $\vec{\omega} := (\omega_{j_1}, \omega_{j_2}, \omega_{j_3}, \omega_{j_4}, \omega_{j_5}, \omega_{j_6}) \in \mathbb{N}^6_*$ and $\vec{\sigma} \in \{\pm 1\}^6$,*

$$\int_{\mathbb{T}} \prod_{k=1}^6 \cos(\omega_{j_k} t)\, đt = 2^{-5} \sum_{\vec{\sigma}\in\{\pm1\}^6} \delta(\vec{\sigma}\cdot\vec{\omega} = 0).$$

The next lemma exploits properties of the eigenfunctions $\{e_n\}_{n\in\mathbb{N}}$ of $-\Delta^{ss}_{\mathbb{S}^3} + \mathbb{1}$, defined in (2.4).

Lemma 4.11 (Integral in space). *For any $j_1, j_2, j_3, j_4, j_5, j_6 \in \mathbb{N}$, the integral*

$$\mathcal{I}_{j_1,\dots,j_6} := \int_0^\pi e_{j_1}(x)\cdots e_{j_6}(x)\sin^2(x)\,đx \tag{4.12}$$

satisfies

$$0 \le \mathcal{I}_{j_1,\dots,j_6} \le \omega_{j_{\min}}\omega_{j_{\min_2}}\omega_{j_{\min_3}}. \tag{4.13}$$

Proof. With no loss of generality, we suppose that $j_1 \le j_2 \le \cdots \le j_6$. By the product rule (2.6),

$$e_{j_1}e_{j_4} = \sum_{k=0}^{j_1} e_{j_4-j_1+2k}, \quad e_{j_3}e_{j_5} = \sum_{l=0}^{j_3} e_{j_5-j_3+2l},$$

and

$$e_{j_1}e_{j_4}e_{j_2} = \sum_{k=0}^{j_1}\sum_{h=0}^{\min\{j_2,j_4-j_1+2k\}} e_{|j_2-(j_4-j_1+2k)|+2h},$$
$$e_{j_3}e_{j_5}e_{j_6} = \sum_{l=0}^{j_3}\sum_{m=0}^{\min\{j_6,j_5-j_3+2l\}} e_{|j_6-(j_5-j_3+2l)|+2m}.$$

Since by Lemma 2.1 the $\{e_j\}_{j\in\mathbb{N}}$ are orthonormal on $L^2([0,\pi],\sin^2(x)\,đx)$, one has

$$\begin{aligned}\mathcal{I}_{j_1,\dots,j_6} &= \int_0^\pi e_{j_1}(x)\cdots e_{j_6}(x)\sin^2(x)\,đx\\ &= \sum_{k=0}^{j_1}\sum_{h=0}^{\min\{j_2,j_4-j_1+2k\}}\sum_{l=0}^{j_3}\sum_{m=0}^{\min\{j_6,j_5-j_3+2l\}} \delta(|j_2-(j_4-j_1+2k)|+2h\\ &= |j_6-(j_5-j_3+2l)|+2m).\end{aligned} \tag{4.14}$$

Now, for each fixed value of $j_1,\dots,j_6$, k, h, and l, there is at most one value of m such that $\delta(|j_2-(j_4-j_1+2k)|+2h = |j_6-(j_5-j_3+2l)|+2m) \neq 0$. Moreover, the sum over k runs over $j_1+1 = \omega_{j_1}$ elements, the sum over h runs over $\le j_2+1 = \omega_{j_2}$ elements, and the sum over l runs over $\le j_3+1 = \omega_{j_3}$ elements. This proves that the integral (4.14) satisfies $\mathcal{I}_{j_1,\dots,j_6} \le \omega_{j_1}\omega_{j_2}\omega_{j_3}$. Also, the lower bound $\mathcal{I}_{j_1,\dots,j_6} \ge 0$ directly follows because (4.14) is the sum of nonnegative integers. ∎

We now prove Proposition 4.5.

Proof of Proposition 4.5 (1). We show that for any $v \in V$ and $N \in \mathbb{N}$, the function $v_N := \Pi_{V_{\le N}} v$, satisfies

$$\int_{\mathbb{T}}\int_0^\pi v_N^6(t,x)\sin^2(x)\,đx\,đt \lesssim_\delta \|v_N\|^6_{\mathcal{V}^{\frac{5}{6}+\delta}_{t,x}}. \tag{4.15}$$

Then (4.10) follows since, using the generalized Hölder inequality with $p_1 = \cdots = p_6 = \frac{1}{6}$, Fatou's lemma, and the fact that $v_N^{(n)} \to v^{(n)}$ in $\mathcal{V}_{t,x}^{\frac{5}{6}+\delta}$ and, up to subsequence, $v_N^{(n)}(t,x) \to v^{(n)}(t,x)$ a.e.,

$$\left| \int_{\mathbb{T}} \int_0^\pi \prod_{n=1}^6 v^{(n)}(t,x) \sin^2(x) \, đx \, đt \right| \leq \prod_{n=1}^6 \|v^{(n)}\|_{L^6_{t,x}}$$
$$\leq \liminf_{N\to\infty} \prod_{n=1}^6 \|v_N^{(n)}\|_{L^6_{t,x}} \overset{(4.15)}{\lesssim_\delta} \lim_{N\to\infty} \prod_{n=1}^6 \|v_N^{(n)}\|_{\mathcal{V}_{t,x}^{\frac{5}{6}+\delta}} \lesssim_\delta \prod_{n=1}^6 \|v^{(n)}\|_{\mathcal{V}_{t,x}^{\frac{5}{6}+\delta}}. \quad (4.16)$$

We now prove (4.15). By Lemma 4.10 and recalling (4.12), one has

$$\mathcal{I} := \int_{\mathbb{T}} \int_0^\pi v_N^6(t,x) \sin^2(x) \, đx \, đt$$
$$= \int_{\mathbb{T}} \int_0^\pi \sum_{\substack{j_1,\dots,j_6 \in \mathbb{N} \\ j_k \leq N \, \forall k}} \prod_{k=1}^6 v_{j_k} \cos(\omega_{j_k} t) e_{j_k}(x) \sin^2(x) \, đx \, đt$$
$$= 2^{-5} \sum_{\sigma_1,\dots,\sigma_6 \in \{\pm 1\}} \sum_{\substack{j_1,\dots,j_6 \in \mathbb{N} \\ j_k \leq N \, \forall k}} \delta(\sigma_1 \omega_{j_1} + \cdots + \sigma_6 \omega_{j_6} = 0) v_{j_1} \cdots v_{j_6} \mathcal{I}_{j_1,\dots,j_6} \quad (4.17)$$
$$\overset{(4.13)}{\lesssim} \sum_{\sigma_1,\dots,\sigma_6 \in \{\pm 1\}} \sum_{\substack{j_1 \leq \cdots \leq j_6 \leq N \\ \sigma_1 \omega_{j_1} + \cdots + \sigma_6 \omega_{j_6} = 0}} |v_{j_1} \cdots v_{j_6}| \omega_{j_1} \omega_{j_2} \omega_{j_3}, \quad (4.18)$$

by the symmetry of (4.17) with respect to $j_1, \dots, j_6$. Since $\omega_{j_1} \leq \cdots \leq \omega_{j_6}$, one has $\omega_{j_1} \leq \omega_{j_1}^{\frac{5}{6}} \omega_{j_2}^{\frac{1}{6}}$, $\omega_{j_2} \leq \omega_{j_2}^{\frac{4}{6}} \omega_{j_3}^{\frac{2}{6}}$, and $\omega_{j_3} \leq \omega_{j_4}^{\frac{1}{3}} \omega_{j_5}^{\frac{1}{3}} \omega_{j_6}^{\frac{1}{3}}$, thus

$$\omega_{j_1} \omega_{j_2} \omega_{j_3} \leq \omega_{j_1}^{\frac{1}{3}} \omega_{j_2}^{\frac{1}{3}} \omega_{j_3}^{\frac{1}{3}} \omega_{j_4}^{\frac{1}{3}} \omega_{j_5}^{\frac{5}{6}} \omega_{j_6}^{\frac{5}{6}}. \quad (4.19)$$

Moreover, recalling $\omega_j = j + 1$, the sum in (4.18) is restricted to

$$j_6 = \sigma_6^{-1}(\sigma_1 \omega_{j_1} + \cdots + \sigma_5 \omega_{j_5}) - 1 =: f(\sigma_1, \dots, \sigma_6, j_1, \dots, j_5) =: f(\vec{\sigma}, \hat{\jmath}),$$

where we set $\vec{\sigma} := (\sigma_1, \dots, \sigma_6)$ and $\hat{\jmath} := (j_1, \dots, j_5)$. By (4.18), (4.19), and the Cauchy–Schwarz inequality, one has

$$\mathcal{I} \lesssim \sum_{\substack{j_1 \leq N, \dots, j_4 \leq N \\ \sigma_1,\dots,\sigma_6 \in \{\pm 1\}}} |v_{j_1} \cdots v_{j_4}| \omega_{j_1}^{\frac{1}{3}} \cdots \omega_{j_4}^{\frac{1}{3}} \sum_{\substack{j_5 \leq N \text{ s.t.} \\ f(\vec{\sigma},\hat{\jmath}) \leq N}} |v_{j_5}| \omega_{j_5}^{\frac{5}{6}} |v_{f(\vec{\sigma},\hat{\jmath})}| \omega_{f(\vec{\sigma},\hat{\jmath})}^{\frac{5}{6}}$$
$$\lesssim \sum_{\substack{j_1 \leq N, \dots, j_4 \leq N \\ \sigma_1,\dots,\sigma_6 \in \{\pm 1\}}} |v_{j_1}| \omega_{j_1}^{\frac{1}{3}} \cdots |v_{j_4}| \omega_{j_4}^{\frac{1}{3}} \Bigg(\sum_{j_5 \leq N} |v_{j_5}|^2 \omega_{j_5}^{\frac{5}{6}2} \Bigg)^{\frac{1}{2}} \Bigg(\sum_{j_5 : f(\vec{\sigma},\hat{\jmath}) \leq N} |v_{f(\vec{\sigma},\hat{\jmath})}|^2 \omega_{f(\vec{\sigma},\hat{\jmath})}^{\frac{5}{6}2} \Bigg)^{\frac{1}{2}}$$
$$\lesssim \|v_N\|_{\mathcal{V}_{t,x}^{\frac{5}{6}}}^2 \Bigg(\sum_{j \leq N} |v_j| \omega_j^{\frac{1}{3}} \Bigg)^4 \lesssim_\delta \|v_N\|_{\mathcal{V}_{t,x}^{\frac{5}{6}+\delta}}^6,$$

where in the last passage we have used the Cauchy–Schwarz inequality to bound

$$\sum_{j\le N}|v_j|\omega_j^{\frac{1}{3}}\le\Big(\sum_{j\le N}|v_j|^2\omega_j^{2(\frac{5}{6}+\delta)}\Big)^{\frac{1}{2}}\Big(\sum_{j\le N}\omega_j^{-(1+2\delta)}\Big)^{\frac{1}{2}}\lesssim_\delta \|v_N\|_{\mathcal{V}^{\frac{5}{6}+\delta}_{t,x}}.$$

This proves (4.15). ∎

Proof of Proposition 4.5 (2). We are going to show that, for any $n=1,\dots,6$ and $N\in\mathbb{N}$, the functions $v_N^{(n)}:=\Pi_{\le N}v^{(n)}$ satisfy

$$\Big|\int_{\mathbb{T}}\int_0^\pi v_N^{(1)}(t,x)\cdots v_N^{(6)}(t,x)\sin^2(x)\,đx\,đt\Big|\lesssim_\delta\prod_{n=1}^5\|v_N^{(n)}\|_{\mathcal{V}^{1+\delta}_{t,x}}\|v_N^{(6)}\|_{\mathcal{V}^{-\delta}_{t,x}}. \tag{4.20}$$

This implies (4.11). Indeed, $v_N^{(n)}\to v^{(n)}$ in $\mathcal{V}^1_{t,x}$ and, by (4.6), in $L^6_{t,x}$ and a.e. for any $n=1,\dots,6$. Then passing to the limit one obtains (4.11).

We now prove (4.20). By Lemma 4.10 and recalling (4.12), one has

$$\begin{aligned}\mathcal{I}_N&:=\Big|\int_{\mathbb{T}}\int_0^\pi v_N^{(1)}(t,x)\dots v_N^{(6)}(t,x)\sin^2(x)\,đx\,đt\Big|\\&\le\sum_{\sigma_1,\dots,\sigma_6\in\{\pm1\}}\sum_{\substack{j_1\le N,\dots,j_6\le N\\ \sigma_1\omega_{j_1}+\cdots+\sigma_j\omega_{j_6}=0}}|v_{j_1}^{(1)}\cdots v_{j_6}^{(6)}|\mathcal{I}_{j_1,\dots,j_6}\\&\overset{(4.13)}{\lesssim}\sum_{\sigma_1,\dots,\sigma_6\in\{\pm1\}}\sum_{\substack{j_1\le\cdots\le j_5\le N\\ j_6=f(\vec\sigma,\hat\jmath)\le N}}|v_{j_1}^{(1)}|\cdots|v_{j_5}^{(5)}||v_{j_6}^{(6)}|\omega_{j_1}\omega_{j_2}\omega_{j_6},\end{aligned} \tag{4.21}$$

by the symmetry of (4.21) in $j_1,\dots,j_5$. Since $\omega_{j_6}=\sigma_6^{-1}(\sigma_1\omega_{j_1}+\cdots+\sigma_5\omega_{j_5})\le5\omega_{j_5}$ and

$$\omega_{j_6}\le5^{1+\delta}\omega_{j_6}^{-\delta}\omega_{j_5}^{1+\delta},\quad \omega_{j_1}\omega_{j_2}\le\omega_{j_1}^{\frac12}\omega_{j_2}^{\frac12}\omega_{j_3}^{\frac12}\omega_{j_4}^{\frac12},$$

by (4.21) and using Cauchy–Schwarz, we have

$$\begin{aligned}\mathcal{I}_N&\lesssim_\delta\sum_{\sigma_1,\dots,\sigma_6\in\{\pm1\}}\sum_{\substack{j_1\le\cdots\le j_5\le N\\ j_6=f(\vec\sigma,\hat\jmath)\le N}}|v_{j_1}^{(1)}|\cdots|v_{j_5}^{(5)}||v_{j_6}^{(6)}|\omega_{j_1}^{\frac12}\cdots\omega_{j_4}^{\frac12}\omega_{j_5}^{1+\delta}\omega_{f(\vec\sigma,\hat\jmath)}^{-\delta}\\&\lesssim_\delta\sum_{\substack{j_1\le N,\dots,j_4\le N\\ \sigma_1,\dots,\sigma_6\in\{\pm1\}}}|v_{j_1}^{(1)}|\omega_{j_1}^{\frac12}\cdots|v_{j_4}^{(4)}|\omega_{j_4}^{\frac12}\sum_{\substack{j_5\le N\text{ s.t.}\\ f(\vec\sigma,\hat\jmath)\le N}}|v_{j_5}^{(5)}|\omega_{j_5}^{1+\delta}|v_{f(\vec\sigma,\hat\jmath)}^{(6)}|\omega_{f(\vec\sigma,\hat\jmath)}^{-\delta}\\&\lesssim_\delta\sum_{j_1\le N,\dots,j_4\le N}|v_{j_1}^{(1)}|\omega_{j_1}^{1+\delta}\omega_{j_1}^{-(\frac12+\delta)}\cdots|v_{j_4}^{(4)}|\omega_{j_4}^{1+\delta}\omega_{j_4}^{-(\frac12+\delta)}\|v_N^{(5)}\|_{\mathcal{V}^{1+\delta}_{t,x}}\|v_N^{(6)}\|_{\mathcal{V}^{-\delta}_{t,x}}\\&\lesssim_\delta\|v_N^{(1)}\|_{\mathcal{V}^{1+\delta}_{t,x}}\cdots\|v_N^{(4)}\|_{\mathcal{V}^{1+\delta}_{t,x}}\|v_N^{(5)}\|_{\mathcal{V}^{1+\delta}_{t,x}}\|v_N^{(6)}\|_{\mathcal{V}^{-\delta}_{t,x}}\end{aligned}$$

proving (4.20). ∎

4.2. Strichartz-type estimates for $p = 2$

The following result will play a central role in studying (1.1) in the case $p = 2$:

Proposition 4.12 (Generalized Strichartz-type estimates). *For any $\delta > 0$ there exists $C_\delta > 0$ such that the following hold:*

(1) *For any $v^{(1)}, \dots, v^{(4)} \in \mathcal{V}_{t,x}^{\frac{1}{2}+\delta}$,*

$$\left| \int_{\mathbb{T}} \int_0^\pi v^{(1)} v^{(2)} \mathcal{L}_\omega^{-1}(v^{(3)} v^{(4)}) \sin^2(x)\, đx\, đt \right| \le C_\delta \gamma^{-1} \prod_{n=1}^{4} \|v^{(n)}\|_{\mathcal{V}_{t,x}^{\frac{1}{2}+\delta}}. \quad (4.22)$$

(2) *For any $v^{(1)}, \dots, v^{(4)} \in V \cap \mathcal{V}_{t,x}^{\frac{2}{3}+\delta}$, for any $l = 1, \dots, 4$,*

$$\left| \int_{\mathbb{T}} \int_0^\pi v^{(1)} v^{(2)} \mathcal{L}_\omega^{-1}(v^{(3)} v^{(4)}) \sin^2(x)\, đx\, đt \right| \le C_\delta \gamma^{-1} \prod_{\substack{n=1 \\ n \ne l}}^{4} \|v^{(n)}\|_{\mathcal{V}_{t,x}^{\frac{2}{3}+\delta}} \|v^{(l)}\|_{\mathcal{V}_{t,x}^{-\delta}}. \quad (4.23)$$

Furthermore, if $\omega = 1$, estimates (4.22) and (4.23) hold with the factor γ^{-1} on the right-hand side replaced by 1.

Note that, due to Lemma 4.4, one has $v^{(3)} v^{(4)} \in W$, thus $\mathcal{L}_\omega^{-1}(v^{(3)} v^{(4)})$ is well defined. The rest of this section is devoted to the proof of Proposition 4.12.

Lemma 4.13 (Integral in space). *For any $j_1, j_2, j_3, j_4 \in \mathbb{N}$, the integral*

$$\mathcal{I}_{j_1,\dots,j_4} := \int_0^\pi e_{j_1}(x) \cdots e_{j_4}(x) \sin^2(x)\, đx$$

satisfies $0 \le \mathcal{I}_{j_1,\dots,j_4} \le \omega_{j_{\min}}$.

Proof. With no loss of generality, we suppose that $j_1 \le j_2 \le j_3 \le j_4$. By the product rule (2.6) one has

$$e_{j_1} e_{j_2} e_{j_3} e_{j_4} = \sum_{k=0}^{j_1} \sum_{h=0}^{j_3} e_{j_2 - j_1 + 2k} e_{j_4 - j_3 + 2h}. \quad (4.24)$$

Thus, since by Lemma 2.1 the $\{e_j\}_{j \in \mathbb{N}}$ are orthonormal on $L^2([0, \pi], \sin^2(x)\, đx)$, one has

$$\mathcal{I}_{j_1,\dots,j_4} = \sum_{k=0}^{j_1} \sum_{h=0}^{j_3} \delta(j_2 - j_1 + 2k = j_4 - j_3 + 2h). \quad (4.25)$$

Now for each fixed j_1, j_2, j_3, j_4, k there is at most one value of h such that $j_2 - j_1 + 2k = j_4 - j_3 = 2h$. Moreover, the sum over k runs over $j_1 + 1 = \omega_1$ elements. This proves $\mathcal{I}_{j_1 j_2 j_3 j_4} \le \omega_{j_1}$. The lower bound $\mathcal{I}_{j_1 j_2 j_3 j_4} \ge 0$ directly follows because (4.25) is the sum of nonnegative integers. ∎

Proof of Proposition 4.12 (1). For any $v \in V$ and $N \in \mathbb{N}_*$ we have $v_N := \Pi_{V_{\leq N}} v$. We are going to show that

$$\int_{\mathbb{T}} \int_0^\pi v_N^4(t,x) \sin^2(x) \, đx \, đt \lesssim_\delta \|v_N\|^4_{\mathcal{V}^{\frac{1}{2}+\delta}_{t,x}}. \tag{4.26}$$

Then estimate (4.22) follows. Indeed, by (4.26), one has, arguing as for (4.16),

$$\|v\|_{L^4_{t,x}} \lesssim_\delta \|v\|_{\mathcal{V}^{\frac{1}{2}+\delta}_{t,x}}. \tag{4.27}$$

Furthermore, for any $\omega \in \Omega_\gamma$, using the Cauchy–Schwarz inequality and Lemma 3.1, we get

$$\begin{aligned}
\left| \int_{\mathbb{T}} \int_0^\pi v^{(1)} v^{(2)} \mathcal{L}_\omega^{-1}(v^{(3)} v^{(4)}) \sin^2(x) \, đx \, đt \right| &\leq \|v^{(1)} v^{(2)}\|_{L^2_{t,x}} \|\mathcal{L}_\omega^{-1}(v^{(3)} v^{(4)})\|_{L^2_{t,x}} \\
&\leq 2\gamma^{-1} \prod_{n=1}^4 \|v^{(n)}\|_{L^4_{t,x}} \\
&\overset{(4.27)}{\leq} C_\delta \gamma^{-1} \prod_{n=1}^4 \|v^{(n)}\|_{\mathcal{V}^{\frac{1}{2}+\delta}_{t,x}},
\end{aligned}$$

proving (4.22). If $\omega = 1$ the thesis follows since, by Lemma 3.1, $\|\mathcal{L}_1^{-1}\|_{\mathcal{B}(\mathcal{H}^0_{t,x}, \mathcal{H}^0_{t,x})} \leq 1$.

We now prove (4.26). Arguing as for (4.18), we have

$$\begin{aligned}
\mathcal{I}_N &:= \int_{\mathbb{T}} \int_0^\pi v_N^4(t,x) \sin^2(x) \, đx \, đt \\
&\lesssim \sum_{\sigma_1,\dots,\sigma_4 \in \{\pm 1\}} \sum_{\substack{j_1 \leq \cdots \leq j_4 \leq N \\ \sigma_1 \omega_{j_1} + \cdots + \sigma_4 \omega_{j_4} = 0}} |v_{j_1} \cdots v_{j_4}| \mathcal{I}_{j_1,\dots,j_4}.
\end{aligned} \tag{4.28}$$

By Lemma 4.13 and estimating $\mathcal{I}_{j_1,\dots,j_4} \leq \omega_{j_1} \leq \omega_{j_3}^{\frac{1}{2}} \omega_{j_4}^{\frac{1}{2}}$, and defining

$$j_4 = \sigma_4^{-1}(\sigma_1 \omega_{j_1} + \cdots + \sigma_3 \omega_{j_3}) - 1 =: g(\sigma_1, \dots, \sigma_4, j_1, \dots, j_3) =: g(\vec{\sigma}, \hat{\jmath}),$$

we have, using the Cauchy–Schwarz inequality and (4.28),

$$\begin{aligned}
\mathcal{I}_N &\lesssim \sum_{\substack{j_1, j_2 \leq N \\ \sigma_1,\dots,\sigma_4 \in \{\pm 1\}}} |v_{j_1}| |v_{j_2}| \sum_{\substack{j_3 \leq N \\ g(\vec{\sigma},\hat{\jmath}) \leq N}} |v_{j_3}| |v_{g(\vec{\sigma},\hat{\jmath})}| \omega_{j_3}^{\frac{1}{2}} \omega_{g(\vec{\sigma},\hat{\jmath})}^{\frac{1}{2}} \\
&\lesssim \sum_{j_1 \leq N} |v_{j_1}| \omega_{j_1}^{\frac{1}{2}+\delta} \omega_{j_1}^{-(\frac{1}{2}+\delta)} \sum_{j_2 \leq N} |v_{j_2}| \omega_{j_2}^{\frac{1}{2}+\delta} \omega_{j_2}^{-(\frac{1}{2}+\delta)} \Big(\sum_{j_3 \leq N} |v_{j_3}|^2 \omega_{j_3} \Big)^{\frac{1}{2}} \\
&\quad \cdot \Big(\sum_{\substack{j_3 \leq N \text{ s.t.} \\ g(\vec{\sigma},\hat{\jmath}) \leq N}} |v_{g(\vec{\sigma},\hat{\jmath})}|^2 \omega_{g(\vec{\sigma},\hat{\jmath})} \Big)^{\frac{1}{2}} \\
&\lesssim_\delta \|v_N\|^2_{\mathcal{V}^{\frac{1}{2}+\delta}_{t,x}} \|v_N\|^2_{\mathcal{V}^{\frac{1}{2}}_{t,x}},
\end{aligned}$$

which gives (4.26). ∎

Proof of Proposition 4.12 (2). Defining $v_N^{(k)} := \Pi_{\le N} v^{(k)}$, $N \in \mathbb{N}$, first we prove that one has

$$\begin{aligned}&\left| \int_{\mathbb{T}} \int_0^\pi v_N^{(1)} v_N^{(2)} \mathcal{L}_\omega^{-1}(v_N^{(3)} v_N^{(4)}) \sin^2(x) \, đx \, đt \right| \\ &\quad \lesssim_\delta \frac{1}{\gamma} \|v^{(1)}\|_{\mathcal{V}_{t,x}^{\frac{2}{3}+\delta}} \|v^{(2)}\|_{\mathcal{V}_{t,x}^{\frac{2}{3}+\delta}} \|v^{(3)}\|_{\mathcal{V}_{t,x}^{\frac{2}{3}+\delta}} \|v^{(4)}\|_{\mathcal{V}_{t,x}^{-\delta}}, \end{aligned} \tag{4.29}$$

with the factor γ^{-1} in (4.29) replaced by 1 if $\omega = 1$. Once that (4.29) has been proved, (2) follows by the following claim: for any $v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)} \in \mathcal{V}_{t,x}^{\frac{1}{2}+\delta}$ it holds that

$$v_N^{(1)} v_N^{(2)} \xrightarrow{L^2_{t,x}} v^{(1)} v^{(2)}, \quad \mathcal{L}_\omega^{-1}(v_N^{(3)} v_N^{(4)}) \xrightarrow{L^2_{t,x}} \mathcal{L}_\omega^{-1}(v^{(3)} v^{(4)}), \quad \text{as } N \to \infty, \tag{4.30}$$

so that $v_N^{(1)} v_N^{(2)} \mathcal{L}_\omega^{-1}(v_N^{(3)} v_N^{(4)})$ converges to $v^{(1)} v^{(2)} \mathcal{L}_\omega^{-1}(v^{(3)} v^{(4)})$ in L^1. Claim (4.30) follows because

$$\begin{aligned}&\|v_N^{(1)} v_N^{(2)} - v^{(1)} v^{(2)}\|_{L^2_{t,x}} \\ &\quad \le \|v_N^{(1)} - v^{(1)}\|_{L^4_{t,x}} \|v_N^{(2)}\|_{L^4_{t,x}} + \|v^{(2)} - v_N^{(2)}\|_{L^4_{t,x}} \|v^{(1)}\|_{L^4_{t,x}} \\ &\quad \overset{(4.27)}{\lesssim_\delta} \|v_N^{(1)} - v^{(1)}\|_{\mathcal{V}_{t,x}^{\frac{1}{2}+\delta}} \|v_N^{(2)}\|_{\mathcal{V}_{t,x}^{\frac{1}{2}+\delta}} + \|v^{(2)} - v_N^{(2)}\|_{\mathcal{V}_{t,x}^{\frac{1}{2}+\delta}} \|v^{(1)}\|_{\mathcal{V}_{t,x}^{\frac{1}{2}+\delta}} \to 0 \end{aligned}$$

as $N \to \infty$. Similarly, $\mathcal{L}_\omega^{-1} v_N^{(3)} v_N^{(4)} \to \mathcal{L}_\omega^{-1}(v^{(3)} v^{(4)})$ in $L^2_{t,x}$ using Lemma 3.1 also. Moreover, (4.23) for a general index l follows by self-adjointness of $\mathcal{L}_\omega^{-1}$.

The first step in the proof of (4.29) is the following:

Lemma 4.14. *We have*

$$\begin{aligned}&\left| \int_{\mathbb{T}} \int_0^\pi v_N^{(1)} v_N^{(2)} \mathcal{L}_\omega^{-1}(v_N^{(3)} v_N^{(4)}) \sin^2(x) \, đx \, đt \right| \lesssim \begin{cases} \gamma^{-1} \mathcal{J} & \text{if } \omega \ne 1, \\ \mathcal{J} & \text{if } \omega = 1, \end{cases} \\ &\mathcal{J} := \sum_{\sigma,\sigma',\sigma_1,\sigma_2 \in \{\pm 1\}} \sum_{j_1,j_2,j_3 \le N} |v_{j_1}^{(1)} v_{j_2}^{(2)} v_{j_3}^{(3)} v_{h(\vec\sigma,\hat\jmath)}^{(4)}| \omega_{\min\{j_1,j_2,j_3,h(\vec\sigma,\hat\jmath)\}}, \end{aligned} \tag{4.31}$$

with

$$\begin{aligned} h(\vec\sigma, \hat\jmath) &:= -\sigma'\sigma_2\sigma_1\omega_{j_1} - \sigma'\sigma_2\sigma_1\sigma\omega_{j_2} - \sigma'\omega_{j_3} - 1, \\ \vec\sigma &:= (\sigma, \sigma', \sigma_1, \sigma_2), \quad \hat\jmath := (j_1, j_2, j_3). \end{aligned} \tag{4.32}$$

Proof. By (2.6) and Lemma 3.1, we compute

$$\begin{aligned}&\int_{\mathbb{T}} \int_0^\pi v_N^{(1)} v_N^{(2)} \mathcal{L}_\omega^{-1}(v_N^{(3)} v_N^{(4)}) \\ &\quad = \sum_{\sigma,\sigma' \in \{\pm 1\}} \sum_{j_1 \le N, \dots, j_4 \le N} v_{j_1}^{(1)} v_{j_2}^{(2)} v_{j_3}^{(3)} v_{j_4}^{(4)} A_{j_1,j_2,j_3,j_4}^{(\sigma,\sigma')}, \end{aligned} \tag{4.33}$$

where $A^{(\sigma,\sigma')}_{j_1,j_2,j_3,j_4}$, $\sigma,\sigma' \in \{-1,1\}$, are equal to

$$\sum_{h=0}^{\min\{j_1,j_2\}} \sum_{k=0}^{\min\{j_3,j_4\}} \int_{\mathbb{T}} \frac{\cos((\omega_{j_1}+\sigma\omega_{j_2})t)\cos((\omega_{j_3}+\sigma'\omega_{j_4})t)}{4(\omega^2(\omega_{j_3}+\sigma'\omega_{j_4})^2-\omega^2_{|j_4-j_3|+2k})}\,đt \cdot \int_0^\pi e_{|j_4-j_3|+2k} e_{|j_2-j_1|+2h} \sin^2(x)\,đx.$$

Using $\omega \in \Omega_\gamma$ and (3.16), we have the lower bound

$$|\omega^2(\omega_{j_3}+\sigma'\omega_{j_4})^2 - (\omega_{|j_4-j_3|+2k})^2| \geq \frac{\gamma}{2}. \tag{4.34}$$

Moreover,

$$\int_{\mathbb{T}} \cos(\alpha_1 t)\cos(\alpha_2 t)\,đt = \frac{1}{2}\sum_{\substack{\sigma_1=\pm1\\ \sigma_2=\pm1}} \delta(\sigma_1\alpha_1+\sigma_2\alpha_2),$$

thus $A^{(\sigma,\sigma')}_{j_1,j_2,j_3,j_4} \neq 0$ only if $\sigma_1(\omega_{j_1}+\sigma\omega_{j_2})+\sigma_2(\omega_{j_3}+\sigma'\omega_{j_4})=0$, which gives $j_4 = h(\vec\sigma,\hat\jmath)$ with $\vec\sigma$, $\hat\jmath$ and $h(\vec\sigma,\hat\jmath)$ as in (4.32). Furthermore, by orthogonality of $\{e_j\}_j$ as in Lemma 2.1, by (4.24) and by Lemma 4.13, one has

$$\sum_{h=0}^{\min\{j_1,j_2\}} \sum_{k=0}^{\min\{j_3,j_4\}} \left| \int_0^\pi e_{|j_4-j_3|+2k} e_{|j_2-j_1|+2h} \sin^2(x)\,đx \right| = \mathcal{I}_{j_1,j_2,j_3,j_4} \leq \omega_{\min\{j_1,j_2,j_3,j_4\}}. \tag{4.35}$$

Note that, if $\omega = 1$, the factor γ^{-1} in (4.34) can be replaced by 1. Thus, combining (4.33), (4.34), and (4.35), one gets (4.31). ∎

The sum $\mathcal{J}$ in (4.31), using its symmetry in the indexes j_1, j_2, j_3, is bounded by

$$\mathcal{J} \lesssim \sum_{\vec\sigma\in\{\pm1\}^4} \sum_{\substack{j_1\leq j_2\leq j_3\leq N\\ h(\vec\sigma,\hat\jmath)\leq N}} |v^{(1)}_{j_1} v^{(2)}_{j_2} v^{(3)}_{j_3}|\,|v^{(4)}_{h(\vec\sigma,\hat\jmath)}|\omega_{j_{\min}}. \tag{4.36}$$

Since $\omega_{h(\vec\sigma,\hat\jmath)} \leq 4\omega_{j_3}$,

$$\omega_{j_{\min}} \leq \omega_{j_1}^{\frac16}\omega_{j_2}^{\frac16}\omega_{j_3}^{\frac23} \lesssim_\delta \omega_{j_1}^{\frac16}\omega_{j_2}^{\frac16}\omega_{j_3}^{\frac23+\delta}\omega^{-\delta}_{h(\vec\sigma,\hat\jmath)}. \tag{4.37}$$

Then, using the Cauchy–Schwarz inequality, (4.36), and (4.37), one has

$$\begin{aligned}\mathcal{J} &\lesssim_\delta \sum_{\vec\sigma\in\{\pm1\}^4} \sum_{j_1\leq N, j_2\leq N} |v^{(1)}_{j_1} v^{(2)}_{j_2}|\omega_{j_1}^{\frac16}\omega_{j_2}^{\frac16} \sum_{\substack{j_3\leq N \text{ s.t.}\\ h(\vec\sigma,\hat\jmath)\leq N}} |v^{(3)}_{j_3}|\omega_{j_3}^{\frac23+\delta}|v^{(4)}_{h(\vec\sigma,\hat\jmath)}|\omega^{-\delta}_{h(\vec\sigma,\hat\jmath)}\\ &\lesssim_\delta \|v^{(1)}_N\|_{\mathcal{V}^{\frac23+\delta}_{t,x}}\|v^{(2)}_N\|_{\mathcal{V}^{\frac23+\delta}_{t,x}}\|v^{(3)}_N\|_{\mathcal{V}^{\frac23+\delta}_{t,x}}\|v^{(4)}_N\|_{\mathcal{V}^{-\delta}_{t,x}},\end{aligned}$$

from which (4.29) follows. ∎

5. Solution of the v_2 equation

In this section we solve equation (3.13) for the high-frequency components v_2 in the kernel. We argue separately for the cases $p = 5$, $p = 3$, and for the degenerate case $p = 2$. Given $\rho_1 \in (0,1)$, $\rho_2 \in (0,1)$, $\rho_3 \in (0,1)$, we define

$$\mathcal{D}_{\rho_1} := \{v_1 \in V_1 : \|v_1\|_{\mathcal{V}^1_{t,z}} \le \rho_1\}, \tag{5.1}$$

and for some $\delta > 0$,

$$\mathcal{D}^{V_2}_{\rho_2} := \{v_2 \in V_2 \cap \mathcal{V}^{2+2\delta}_{t,z} : \|v_2\|_{\mathcal{V}^{2+2\delta}_{t,z}} \le \rho_2\}, \tag{5.2}$$

$$\mathcal{D}^{W}_{\rho_3} := \{w \in H_t^{\frac12+\delta}\mathcal{H}_z^{\frac32+\delta} \cap W : \|w\|_{H_t^{\frac12+\delta}\mathcal{H}_z^{\frac32+\delta}} \le \rho_3\}. \tag{5.3}$$

In the sequel, δ will always denote a positive small constant.

5.1. Case $p = 5$

For any $R > 0$, $\delta > 0$, and $\gamma \in (0, \gamma_0)$, let

$$\rho_1 := \varepsilon^{\frac14} R, \quad \rho_2 := \mathtt{c}_2 N^{10\delta} R^5 \varepsilon^{\frac14}, \quad \rho_3 := \mathtt{c}_3 \gamma^{-1} N^{5+10\delta} R^5 \varepsilon^{\frac54}, \quad N := \varepsilon^{-\frac{1}{\beta}}, \tag{5.4}$$

where $\varepsilon = \omega^2 - 1 > 0$ according to (1.7), $\mathtt{c}_2, \mathtt{c}_3 > 0$, and $\beta > 1$.

Proposition 5.1 (Solution of the v_2 equation for $p = 5$). *For any $\delta \in (0, \frac18)$, $R > 0$, and $\gamma \in (0, \gamma_0)$, let ρ_1, ρ_2, ρ_3, N be as in (5.20). There exist $\mathtt{c}_2 := \mathtt{c}_2(\delta) > 0$, $\beta_\delta > 1$, $\zeta := \zeta(\delta) > 0$, $\epsilon_{\delta,R} > 0$, and $C_{1,\delta}, C_{2,\delta} > 0$ such that, for any $\beta > \beta_\delta$ and any $\varepsilon > 0$ such that*

$$\varepsilon N^{\zeta} \gamma^{-1} \le \epsilon_{\delta,R}, \tag{5.5}$$

there exists a C^1 function $v_2 \colon \mathcal{D}_{\rho_1} \times \mathcal{D}^W_{\rho_3} \to V_2 \cap \mathcal{V}^{2+2\delta}_{t,x}$, $(v_1, w) \mapsto v_2(v_1, w)$, where $\mathcal{D}_{\rho_1}$ and $\mathcal{D}^W_{\rho_3}$ are defined as in (5.1) and (5.3), satisfying $v_2(0,0) = 0$, and

$$\|v_2(v_1,w)\|_{\mathcal{V}^{2+2\delta}_{t,x}} \le \rho_2, \tag{5.6}$$

$$\|\partial_{v_1} v_2(v_1,w)\|_{\mathcal{B}(V_1 \cap \mathcal{V}^1_{t,x}, \mathcal{V}^{2+2\delta}_{t,x})} \le C_{1,\delta} N^{10\delta} R^4, \tag{5.7}$$

$$\|\partial_w v_2(v_1,w)\|_{\mathcal{B}(W \cap H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}, \mathcal{V}^{2+2\delta}_{t,x})} \le C_{2,\delta} N^{\frac12+5\delta} R^4, \tag{5.8}$$

such that $v_2(v_1, w)$ solves

$$\varepsilon(-\Delta^{\mathrm{ss}}_{\mathbb{S}^3} + \mathbb{1}) v_2(v_1,w) - \Pi_{V_2}\big((v_1 + v_2(v_1,w) + w)^5\big) = 0. \tag{5.9}$$

In the rest of this section we prove Proposition 5.1. For any $(v_1, w) \in \mathcal{D}_{\rho_1} \times \mathcal{D}^W_{\rho_3}$, we look for a solution of (5.9) as a fixed point of the map

$$v_2 \mapsto \mathcal{T}_{v_1,w}(v_2) := \varepsilon^{-1} A^{-1} \Pi_{V_2}(v_1 + v_2 + w)^5, \tag{5.10}$$

where, according to (3.4), we set $A := -\Delta^{\mathrm{ss}}_{\mathbb{S}^3} + \mathbb{1}$. The next lemma is based on the Strichartz estimates of Section 4.

Lemma 5.2 (Contraction). *Let $\zeta \geq \frac{11}{2} + 5\delta$. There exist $C_\delta > 0$ and $\epsilon_{\delta,R} > 0$ such that, if (5.5) holds, the map $\mathcal{T}_{v_1,w}$ maps $\mathcal{D}^{V_2}_{\rho_2}$ into itself, with*

$$\|\mathcal{T}_{v_1,w}(v_2) - \mathcal{T}_{v_1,w}(v_2')\|_{\mathcal{Y}^{2+2\delta}_{t,x}} \leq C_\delta R^4 N^{-1+8\delta} \|v_2 - v_2'\|_{\mathcal{Y}^{2+2\delta}_{t,x}} \quad \forall v_2, v_2' \in \mathcal{D}^{V_2}_{\rho_2}. \tag{5.11}$$

As a consequence, for any $(v_1, w) \in \mathcal{D}_{\rho_1} \times \mathcal{D}^W_{\rho_3}$ there exists a unique $v_2(v_1, w) \in \mathcal{D}^{V_2}_{\rho_2}$ solving (5.9) and such that $v_2(0,0) = 0$.

Proof. We write $\mathcal{T}_{v_1,w}(v_2) = \varepsilon^{-1} A^{-1} \Pi_{V_2} \sum_{j_1+j_2+j_3=5} c_{j_1 j_2 j_3} v_1^{j_1} v_2^{j_2} w^{j_3}$. We estimate the terms where w does not appear using the Strichartz-type estimate in Lemma 4.8, and the terms with w using algebra property (2.19). If $j_3 = 0$, by (4.5), (4.4), and Lemma 4.8, for any j_1, j_2 one has

$$\begin{aligned} \|A^{-1}\Pi_{V_2}(v_1^{j_1} v_2^{j_2})\|_{\mathcal{Y}^{2+2\delta}_{t,x}} &\lesssim_\delta (N^{2\delta}\|v_1\|_{\mathcal{Y}^1_{t,x}})^{j_1} (N^{-1}\|v_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}})^{j_2} \\ &\lesssim_\delta (N^{2\delta}\rho_1)^{j_1} (N^{-1}\rho_2)^{j_2}, \end{aligned} \tag{5.12}$$

for any $\|v_1\|_{\mathcal{Y}^1_{t,x}} \leq \rho_1$ and $\|v_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}} \leq \rho_2$. Recalling the definitions of ρ_1, ρ_2, N in (5.4), one then gets $N^{-1}\rho_2 \leq N^{2\delta}\rho_1$ and thus for any $j_1 + j_2 = 5$,

$$\|A^{-1}\Pi_{V_2}(v_1^{j_1} v_2^{j_2})\|_{\mathcal{Y}^{2+2\delta}_{t,x}} \lesssim_\delta N^{10\delta}\rho_1^5. \tag{5.13}$$

On the other hand, if $j_3 \neq 0$, by estimates (4.5), (4.4), and Lemma 4.3 one has

$$\begin{aligned} &\|A^{-1}\Pi_{V_2}(v_1^{j_1} v_2^{j_2} w^{j_3})\|_{\mathcal{Y}^{2+2\delta}_{t,x}} \\ &\lesssim_\delta N^{-\frac{3}{2}+\delta}(N^{\frac{1}{2}+\delta}\|v_1\|_{\mathcal{Y}^1_{t,x}})^{j_1}(N^{-\frac{1}{2}-\delta}\|v_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}})^{j_2}\|w\|^{j_3}_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}} \\ &\lesssim_\delta N^{-\frac{3}{2}+\delta}(N^{\frac{1}{2}+\delta}\rho_1)^{j_1}(\rho_2 N^{-\frac{1}{2}-\delta})^{j_2}\rho_3^{j_3}, \end{aligned} \tag{5.14}$$

for any $\|v_1\|_{\mathcal{Y}^1_{t,x}} \leq \rho_1$, $\|v_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}} \leq \rho_2$, and $\|w\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}} \leq \rho_3$. Assuming (5.5) with $\zeta = \frac{11}{2} + 5\delta$ and $\epsilon_{\delta,R}$ small enough, one has $\rho_3 \leq N^{\frac{1}{2}+\delta}\rho_1$ and $N^{-\frac{1}{2}-\delta}\rho_2 \leq N^{\frac{1}{2}+\delta}\rho_1$, and recalling $j_3 \geq 1$, one gets

$$\|A^{-1}\Pi_{V_2}(v_1^{j_1} v_2^{j_2} w^{j_3})\|_{\mathcal{Y}^{2+2\delta}_{t,x}} \lesssim_\delta N^{\frac{1}{2}+5\delta}\rho_1^4\rho_3 \lesssim_\delta N^{10\delta}\rho_1^5. \tag{5.15}$$

Thus, combining (5.13) and (5.15), there exists a constant $C_\delta > 0$ such that we have

$$\|\mathcal{T}_{v_1,w}(v_2)\|_{\mathcal{Y}^{2+2\delta}_{t,x}} \leq C_\delta \varepsilon^{-1} N^{10\delta}\rho_1^5 \overset{(5.4)}{=} C_\delta N^{10\delta} R^5 \varepsilon^{\frac{1}{4}} < \mathtt{c}_2 N^{10\delta} R^5 \varepsilon^{\frac{1}{4}} = \rho_2,$$

provided $\mathtt{c}_2 > C_\delta$. Thus $\mathcal{T}_{v_1,w}$ maps $\mathcal{D}^{V_2}_{\rho_2}$ into itself. We now prove that it is a contraction. For any $h_2 \in \mathcal{V}^{2+2\delta}_{t,x} \cap V_2$, we have

$$\begin{aligned} \partial_{v_2}\mathcal{T}_{v_1,w}(v_2)[h_2] &= 5\varepsilon^{-1}A^{-1}\Pi_{V_2}((v_1+v_2+w)^4 h_2) \\ &= 5\varepsilon^{-1}\sum_{j_1+j_2+j_3=4} c_{j_1 j_2 j_3} A^{-1}\Pi_{V_2}(v_1^{j_1} v_2^{j_2} w^{j_3} h_2). \end{aligned} \tag{5.16}$$

If $j_3 = 0$, we argue as in (5.12) to get

$$\begin{aligned}\|A^{-1}\Pi_{V_2}(v_1^{j_1}v_2^{j_2}h_2)\|_{\mathcal{Y}^{2+2\delta}_{t,x}}&\\ \lesssim_\delta (N^{2\delta}\rho_1)^{j_1}(N^{-1}\rho_2)^{j_2}N^{-1}\|h_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}} &\lesssim_\delta N^{-1+8\delta}\rho_1^4\|h_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}},\end{aligned} \tag{5.17}$$

whereas if $j_3 \neq 0$ we argue as in (5.14) to get

$$\begin{aligned}\|A^{-1}\Pi_{V_2}(v_1^{j_1}v_2^{j_2}w^{j_3}h_2)\|_{\mathcal{Y}^{2+2\delta}_{t,x}} &\lesssim_\delta N^{-2}(\rho_1 N^{\frac12+\delta})^{j_1}(\rho_2 N^{-\frac12-\delta})^{j_2}\rho_3^{j_3}\|h_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}}\\ &\lesssim_\delta N^{-\frac12+3\delta}\rho_1^3\rho_3\|h_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}}\\ &\lesssim_\delta N^{-1+8\delta}\rho_1^4\|h_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}}.\end{aligned} \tag{5.18}$$

Thus, by (5.16), (5.17), and (5.18), and since $\rho_1 = \varepsilon^{\frac14}R$, we deduce

$$\begin{aligned}\|\partial_{v_2}\mathcal{T}_{v_1,w}(v_2)[h_2]\|_{\mathcal{Y}^{2+2\delta}_{t,x}} &\le C_\delta\varepsilon^{-1}N^{-1+8\delta}\varepsilon R^4\|h_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}}\\ &= C_\delta R^4 N^{-1+8\delta}\|h_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}},\end{aligned} \tag{5.19}$$

from which (5.11) follows. Thus $\mathcal{T}_{v_1,w}$ is a contraction on $\mathcal{D}^{V_2}_{\rho_2}$. ∎

Lemma 5.3 (Differentiability of v_2). *The function $v_2(v_1, w)$ is differentiable in v_1, w, and it satisfies estimates* (5.7), (5.8).

Proof. By (5.19), the operator $\mathbb{1} - \partial_{v_2}\mathcal{T}_{v_1,w}(v_2)$ is invertible on $\mathcal{Y}^{2+2\delta}_{t,x}$, with operator norm bounded by 2. Furthermore, arguing as for (5.12) and (5.14), one obtains

$$\begin{aligned}\|\partial_{v_1}\mathcal{T}_{v_1,w}(v_2)[h_1]\|_{\mathcal{Y}^{2+2\delta}_{t,x}} &\lesssim_\delta N^{10\delta}R^4\|h_1\|_{\mathcal{Y}^{1}_{t,x}},\\ \|\partial_{w}\mathcal{T}_{v_1,w}(v_2)[h_3]\|_{\mathcal{Y}^{2+2\delta}_{t,x}} &\lesssim_\delta \varepsilon^{-1}N^{\frac12+5\delta}\rho_1^4\|h_3\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}}\\ &\lesssim_\delta N^{\frac12+5\delta}R^4\|h_3\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}}.\end{aligned}$$

Then $\partial_{v_1}v_2(v_1, w) = (\mathbb{1} - \partial_{v_2}\mathcal{T}_{v_1,w}(v_2))^{-1}\partial_{v_1}\mathcal{T}_{v_1,w}(v_2)$ satisfies (5.7) and, in addition, $\partial_w v_2(v_1, w) = (\mathbb{1} - \partial_{v_2}\mathcal{T}_{v_1,w}(v_2))^{-1}\partial_w\mathcal{T}_{v_1,w}(v_2)$ satisfies (5.8). ∎

5.2. Case $p = 3$

For any $R > 0$, $\delta > 0$, and $\gamma \in (0, \gamma_0)$, let

$$\rho_1 := \varepsilon^{\frac12}R, \quad \rho_2 := \mathtt{c}_2R^3N^{4\delta}\varepsilon^{\frac12}, \quad \rho_3 := \mathtt{c}_3\gamma^{-1}N^{3+6\delta}R^3\varepsilon^{\frac32}, \quad N := \varepsilon^{-\frac{1}{\beta}}, \tag{5.20}$$

where we recall that $\varepsilon = \omega^2 - 1 > 0$, according to (1.16), $\beta > 1$, and $\mathtt{c}_2, \mathtt{c}_3 > 0$.

Proposition 5.4 (Solution of the v_2 equation for $p = 3$). *For any $\delta \in (0, \frac{1}{8})$, $R > 0$, and $\gamma \in (0, \gamma_0)$, let ρ_1, ρ_2, ρ_3, N as in (5.20). There exist $\mathtt{c}_2 := \mathtt{c}_2(\delta) > 0$, $\beta_\delta > 1$, $\zeta := \zeta(\delta) > 1$, $\epsilon_{\delta,R} > 0$, and $C_{1,\delta}, C_{2,\delta} > 0$ such that, for any $\beta > \beta_\delta$ and any ε such that*

$$\varepsilon N^{\zeta} \gamma^{-1} \leq \epsilon_{\delta,R}, \tag{5.21}$$

there exists a C^1 function $v_2 \colon \mathcal{D}_{\rho_1} \times \mathcal{D}^W_{\rho_3} \to V_2 \cap \mathcal{V}^{2+2\delta}_{t,\eta}$, $(v_1, w) \mapsto v_2(v_1, w)$, where $\mathcal{D}_{\rho_1}$ and $\mathcal{D}^W_{\rho_3}$ are defined as in (5.1) and (5.3), satisfying $v_2(0,0) = 0$ and

$$\|v_2(v_1, w)\|_{\mathcal{V}^{2+2\delta}_{t,\eta}} \leq \rho_2, \tag{5.22}$$

$$\|\partial_{v_1} v_2(v_1, w)\|_{\mathcal{B}(V_1 \cap \mathcal{V}^1_{t,\eta}, \mathcal{V}^{2+2\delta}_{t,\eta})} \leq C_{1,\delta} R^2 N^{4\delta}, \tag{5.23}$$

$$\|\partial_w v_2(v_1, w)\|_{\mathcal{B}(W \cap H^{\frac{1}{2}+\delta}_t \mathcal{H}^{\frac{3}{2}+\delta}_\eta, \mathcal{V}^{2+2\delta}_{t,\eta})} \leq C_{2,\delta} R^2 N^{-\frac{1}{2}+3\delta}, \tag{5.24}$$

such that $v_2(v_1, w)$ solves

$$\varepsilon A v_2(v_1, w) - \Pi_{V_2}((v_1 + v_2(v_1, w) + w)^3) = 0, \tag{5.25}$$

where $A := -\Delta_{\mu_1,\mu_2} + \mathbb{1}$ according to (3.4).

We now prove Proposition 5.4. We define the map

$$v_2 \mapsto \mathcal{T}_{v_1,w}(v_2) := \varepsilon^{-1} A^{-1} \Pi_{V_2}((v_1 + v_2 + w)^3) \tag{5.26}$$

and show that it is a contraction.

Lemma 5.5 (Contraction for $p = 3$). *Let $\zeta \geq \frac{5}{2} + 5\delta$. There exist $\epsilon_{\delta,R} > 0$ and $C_\delta > 0$ such that, if (5.21) holds, then $\mathcal{T}_{v_1,w}$ defined as in (5.26) maps $\mathcal{D}^{V_2}_{\rho_2}$ into itself, with*

$$\|\mathcal{T}_{v_1,w}(v_2) - \mathcal{T}_{v_1,w}(v_2')\|_{\mathcal{V}^{2+2\delta}_{t,\eta}} \leq C_\delta R^2 N^{-1} \|v_2 - v_2'\|_{\mathcal{V}^{2+2\delta}_{t,\eta}} \quad \forall v_2, v_2' \in \mathcal{D}^{V_2}_{\rho_2}. \tag{5.27}$$

As a consequence, for any $(v_1, w) \in \mathcal{D}_{\rho_1} \times \mathcal{D}^W_{\rho_3}$ there exists a unique solution $v_2(v_1, w) \in \mathcal{D}^{V_2}_{\rho_2}$ satisfying (5.25) and $v_2(0,0) = 0$.

Proof. We start expanding $\mathcal{T}_{v_1,w}(v_2) = \varepsilon^{-1} \sum_{j_1+j_2+j_3=3} c_{j_1,j_2,j_3} A^{-1} \Pi_{V_2}(v_1^{j_1} v_2^{j_2} w^{j_3})$. By (4.4) and Lemma 4.3, one has

$$\begin{aligned} \|A^{-1} \Pi_{V_2}(v_1^{j_1} v_2^{j_2} w^{j_3})\|_{\mathcal{V}^{2+2\delta}_{t,\eta}} &\leq N^{-\frac{3}{2}+\delta} \|\Pi_{V_2}(v_1^{j_1} v_2^{j_2} w^{j_3})\|_{\mathcal{V}^{\frac{3}{2}+\delta}_{t,\eta}} \\ &\lesssim_\delta N^{-\frac{3}{2}+\delta} (N^{\frac{1}{2}+\delta} \rho_1)^{j_1} (N^{-\frac{1}{2}-\delta} \rho_2)^{j_2} \rho_3^{j_3} \end{aligned}$$

for any $v_1 \in \mathcal{D}_{\rho_1}$, $v_2 \in \mathcal{D}^{V_2}_{\rho_2}$, $w \in \mathcal{D}^W_{\rho_3}$. By (5.20) and the smallness assumption (5.21), if $\zeta \geq \frac{5}{2} + 5\delta$ one has $N^{-\frac{1}{2}-\delta} \rho_2 \leq N^{\frac{1}{2}+\delta} \rho_1$ and $\rho_3 \leq N^{\frac{1}{2}+\delta} \rho_1$. Then recalling $j_1 + j_2 + j_3 = 3$, there exists $C_\delta > 0$ such that

$$\|\mathcal{T}_{v_1,w}(v_2)\|_{\mathcal{V}^{2+2\delta}_{t,\eta}} \leq C_\delta \varepsilon^{-1} N^{4\delta} \rho_1^3 = C_\delta \varepsilon^{\frac{1}{2}} N^{4\delta} R^3 \leq \mathtt{c}_2 \varepsilon^{\frac{1}{2}} N^{4\delta} R^3 = \rho_2,$$

provided $\mathtt{c}_2 \geq C_\delta$. Thus $\mathcal{T}_{v_1,w}$ maps $\mathcal{D}_{\rho_2}^{V_2}$ into itself. We now prove that it is a contraction. One has

$$\partial_{v_2}\mathcal{T}_{v_1,w}(v_2)[h_2] = 3A^{-1}\Pi_{V_2}((v_1+v_2+w)^2 h_2) \quad \forall h_2 \in \mathcal{V}_{t,\eta}^{2+2\delta} \cap V_2.$$

Applying Lemma 4.3, (4.4), using (5.20) and the smallness condition (5.21), one obtains

$$\begin{aligned}
&\|A^{-1}\Pi_{V_2}(v_1^{j_1}v_2^{j_2}w^{j_3}h_2)\|_{\mathcal{V}_{t,\eta}^{2+2\delta}} \\
&\quad\leq\ N^{-\frac{3}{2}+\delta}(N^{\frac{1}{2}+\delta}\rho_1)^{j_1}(N^{-\frac{1}{2}-\delta}\rho_2)^{j_2}\rho_3^{j_3}N^{-\frac{1}{2}-\delta}\|h_2\|_{\mathcal{V}_{t,\eta}^{2+2\delta}} \\
&\quad\lesssim_\delta\ N^{-1+2\delta}\rho_1^2\|h_2\|_{\mathcal{V}_{t,\eta}^{2+2\delta}} \lesssim_\delta\ N^{-1+2\delta}\varepsilon R^2,
\end{aligned}$$

using that $j_1+j_2+j_3 = 2$. Thus

$$\|\partial_{v_2}\mathcal{T}_{v_1,w}(v_2)[h_2]\|_{\mathcal{V}_{t,x}^{2+\delta}} \lesssim_\delta\ N^{-1+2\delta}R^2\|h_2\|_{\mathcal{V}_{t,\eta}^{2+2\delta}},$$

which is (5.27). ∎

Differentiability of the function $v_2(v_1,w)$ with estimates (5.23) and (5.24) follows similarly.

5.3. Case $p = 2$

For any $\delta > 0$, $R > 0$, and $\gamma \in (0,\gamma_0)$, let

$$\begin{gathered}
\rho_1 := R\varepsilon^{\frac{1}{2}}, \quad \rho_2 := \mathtt{c}_2\gamma^{-1}R^3\varepsilon^{\frac{1}{2}}, \quad \rho_3 := \mathtt{c}_3\gamma^{-2}\varepsilon\sqrt{\varepsilon}R^3N^b, \\
b := 3+6\delta, \quad N := \varepsilon^{-\frac{1}{\beta}},
\end{gathered} \tag{5.28}$$

where $\varepsilon > 0$ is defined by (1.7), namely $\omega^2 = 1-\varepsilon$, $\mathtt{c}_2, \mathtt{c}_3 > 0$, and $\beta > 1$. Since equations (3.12)–(3.14) for $p = 2$ are degenerate, in the sense that $\Pi_{V_2}(v_1+v_2)^2 = 0$, we perform the translation

$$w = \mathcal{L}_\omega^{-1}(v_1+v_2)^2 + \widetilde{w}. \tag{5.29}$$

We then rewrite (3.13), (3.14) in terms of $v_1, v_2, \widetilde{w}$, and since $\Pi_V(v_1+v_2)^2 = 0$ by Lemma 4.4, we obtain

$$-\varepsilon A v_2 = \Pi_{V_2}\big(2(v_1+v_2)(\mathcal{L}_\omega^{-1}(v_1+v_2)^2+\widetilde{w}) + (\mathcal{L}_\omega^{-1}(v_1+v_2)^2+\widetilde{w})^2\big), \tag{5.30}$$

$$\mathcal{L}_\omega\widetilde{w} = \Pi_W\big(2(v_1+v_2)(\mathcal{L}_\omega^{-1}(v_1+v_2)^2+\widetilde{w}) + (\mathcal{L}_\omega^{-1}(v_1+v_2)^2+\widetilde{w})^2\big). \tag{5.31}$$

Proposition 5.6 (Solution of the v_2 equation for $p = 2$). *For any $\delta \in (0,\frac{1}{10})$, $R > 0$, and $\gamma \in (0,\gamma_0)$, let ρ_1, ρ_2, ρ_3, N be as in (5.28). There exist $\mathtt{c}_2 := \mathtt{c}_2(\delta) > 0$, $\beta_\delta > 1$, $\mathtt{b} := \mathtt{b}(\delta) > 0$, $\epsilon_{\delta,R} > 0$, and $C_{1,\delta}, C_{2,\delta} > 0$ such that, for any $\beta > \beta_\delta$ and any $\varepsilon > 0$ such that $\omega \in \Omega_\gamma$ and*

$$N^{-\mathtt{b}}\gamma^{-1} \leq \epsilon_{\delta,R}, \quad N := \varepsilon^{-\frac{1}{\beta}}, \tag{5.32}$$

there exists a C^1 function $v_2\colon \mathcal{D}_{\rho_1}\times\mathcal{D}^W_{\rho_3}\to V_2\cap\mathcal{V}^{2+2\delta}_{t,x}$, $(v_1,\widetilde{w})\mapsto v_2(v_1,\widetilde{w})$, where $\mathcal{D}_{\rho_1}$ and $\mathcal{D}^W_{\rho_3}$ are defined as in (5.1) and (5.3), satisfying $v_2(0,0)=0$ and

$$\|v_2(v_1,\widetilde{w})\|_{\mathcal{V}^{2+2\delta}_{t,x}}\le\rho_2, \tag{5.33}$$

$$\|\partial_{v_1}v_2(v_1,\widetilde{w})\|_{\mathcal{B}(V_1\cap\mathcal{V}^1_{t,x},\mathcal{V}^{2+2\delta}_{t,x})}\le C_{1,\delta}\gamma^{-1}R^2, \tag{5.34}$$

$$\|\partial_{\widetilde{w}}v_2(v_1,\widetilde{w})\|_{\mathcal{B}(W\cap H^{\frac12+\delta}_t\mathcal{H}^{\frac32+\delta}_x,\mathcal{V}^{2+2\delta}_{t,x})}\le C_{2,\delta}\varepsilon^{-\frac12}N^{-1+2\delta}R, \tag{5.35}$$

such that $v_2(v_1,\widetilde{w})$ solves equation (5.30).

We now prove Proposition 5.6. For any $(v_1,\widetilde{w})\in\mathcal{D}_{\rho_1}\times\mathcal{D}^W_{\rho_3}$ we look for a solution of (5.30) as a fixed point of the map which to v_2 associates

$$\begin{aligned}\mathcal{T}_{v_1,\widetilde{w}}(v_2):=-\varepsilon^{-1}A^{-1}\Pi_{V_2}\big(&2(v_1+v_2)(\mathcal{L}_\omega^{-1}(v_1+v_2)^2+\widetilde{w})\\&+(\mathcal{L}_\omega^{-1}(v_1+v_2)^2+\widetilde{w})^2\big)\end{aligned} \tag{5.36}$$

with $A=-\Delta^{\mathrm{ss}}_{\mathbb{S}^3}+\mathbb{1}$ as in (3.4). We shall use the following technical lemma:

Lemma 5.7. *Let ρ_1 and ρ_2 be as in (5.28). There exists $\epsilon_{R,\mathtt{c}_2}>0$ such that, if $N^{-1-2\delta}\gamma^{-1}<\epsilon_{R,\mathtt{c}_2}$, then for any $s\in[0,2+2\delta]$, any $v_1\in\mathcal{D}_{\rho_1}$ and $v_2\in\mathcal{D}^{V_2}_{\rho_2}$, one has*

$$\|v_1\|_{\mathcal{V}^s_{t,x}}\le N^{\max\{0,s-1\}}\rho_1,\quad \|v_2\|_{\mathcal{V}^s_{t,x}}\le N^{\max\{0,s-1\}}\rho_1.$$

Proof. The estimate on $\|v_1\|_{\mathcal{V}^s_{t,x}}$ follows from (4.4). For any $s\in[0,2+2\delta]$ and $v_2\in\mathcal{D}^{V_2}_{\rho_2}$, by (4.4) and (5.28) one has

$$\begin{aligned}\|v_2\|_{\mathcal{V}^s_{t,x}}&\le N^{-2-2\delta+s}\|v_2\|_{\mathcal{V}^{2+2\delta}_{t,x}}\\&\le N^{-2-2\delta+s}\rho_2=N^{-2-2\delta+s}\mathtt{c}_2\gamma^{-1}R^2\rho_1\le N^{\max\{0,s-1\}}\rho_1,\end{aligned}$$

since $-2-2\delta+s\le\max\{0,s-1\}-1-2\delta$ and $N^{-1-2\delta}\gamma^{-1}<(\mathtt{c}_2R^2)^{-1}=:\epsilon_{R,\mathtt{c}_2}$. ∎

The next lemma is based on the Strichartz-type estimates of Proposition 4.12:

Lemma 5.8 (Contraction). *There exist $\mathtt{b}_\delta\in(0,1)$, $\epsilon_{\delta,R}>0$, and $C_\delta>0$ such that, if (5.32) holds with $\mathtt{b}\ge\mathtt{b}_\delta$, then for any $v_1\in\mathcal{D}_{\rho_1}$ and $\widetilde{w}\in\mathcal{D}^W_{\rho_3}$, $\mathcal{T}_{v_1,\widetilde{w}}$ defined in (5.36) maps $\mathcal{D}^{V_2}_{\rho_2}$ into itself, with*

$$\begin{aligned}&\|\mathcal{T}_{v_1,\widetilde{w}}(v_2)-\mathcal{T}_{v_1,\widetilde{w}}(v_2')\|_{\mathcal{V}^{2+2\delta}_{t,x}}\\&\quad\le C_\delta\gamma^{-1}R^2N^{-\frac43}\|v_2-v_2'\|_{\mathcal{V}^{2+2\delta}_{t,x}}\quad\forall v_2,v_2'\in\mathcal{D}^{V_2}_{\rho_2}.\end{aligned} \tag{5.37}$$

As a consequence, for any $(v_1,\widetilde{w})\in\mathcal{D}_{\rho_1}\times\mathcal{D}^W_{\rho_3}$ there exists a unique $v_2(v_1,\widetilde{w})$ solving (5.30), satisfying $v_2(0,0)=0$.

Proof. We expand $\mathcal{T}_{v_1,\widetilde{w}}(v_2)$ in (5.36) as $\mathcal{T}_{v_1,\widetilde{w}}(v_2) = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5$, with

$$\begin{aligned}
\mathcal{T}_1 &:= -2\varepsilon^{-1} A^{-1}\Pi_{V_2}((v_1+v_2)\mathcal{L}_\omega^{-1}(v_1+v_2)^2),\\
\mathcal{T}_2 &:= -2\varepsilon^{-1} A^{-1}\Pi_{V_2}((v_1+v_2)\widetilde{w}),\\
\mathcal{T}_3 &:= -\varepsilon^{-1} A^{-1}\Pi_{V_2}\big((\mathcal{L}_\omega^{-1}(v_1+v_2)^2)^2\big),\\
\mathcal{T}_4 &:= -\varepsilon^{-1} A^{-1}\Pi_{V_2}\widetilde{w}^2,\\
\mathcal{T}_5 &:= -2\varepsilon^{-1} A^{-1}\Pi_{V_2}(\widetilde{w}\mathcal{L}_\omega^{-1}(v_1+v_2)^2),
\end{aligned}$$

and we estimate each term separately. First, $\mathcal{T}_1$ is estimated using Proposition 4.12 (2), which gives

$$\begin{aligned}
\|\mathcal{T}_1\|_{\mathcal{Y}^{2+2\delta}_{t,x}} &\le 2\varepsilon^{-1}\|\Pi_{V_2}((v_1+v_2)\mathcal{L}_\omega^{-1}(v_1+v_2)^2)\|_{\mathcal{Y}^{2\delta}_{t,x}}\\
&= 2\varepsilon^{-1}\sup_{\substack{h_2\in V_2\cap\mathcal{Y}^{-2\delta}_{t,x}\\ \|h_2\|_{\mathcal{Y}^{-2\delta}_{t,x}}\le 1}}\left|\int_{\mathbb{T}}\int_0^\pi (v_1+v_2)h_2\mathcal{L}_\omega^{-1}(v_1+v_2)^2\sin^2(x)\,đx\,đt\right|\\
&\lesssim_\delta \varepsilon^{-1}\gamma^{-1}\|v_1+v_2\|^3_{\mathcal{Y}^{\frac{2}{3}+2\delta}_{t,x}} \lesssim_\delta \varepsilon^{-1}\gamma^{-1}(\|v_1\|_{\mathcal{Y}^{\frac{2}{3}+2\delta}_{t,x}} + \|v_2\|_{\mathcal{Y}^{\frac{2}{3}+2\delta}_{t,x}})^3. \qquad (5.38)
\end{aligned}$$

Then $\mathcal{T}_2$ is estimated using (4.4), and Lemma 4.3: one has

$$\begin{aligned}
\|\mathcal{T}_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}} &\le \varepsilon^{-1}\|\Pi_{V_2}(v_1+v_2)\widetilde{w}\|_{\mathcal{Y}^{2\delta}_{t,x}} \lesssim_\delta \varepsilon^{-1}N^{-\frac{3}{2}+\delta}\|\Pi_{V_2}(v_1+v_2)\widetilde{w}\|_{\mathcal{Y}^{\frac{3}{2}+\delta}_{t,x}}\\
&\lesssim_\delta \varepsilon^{-1}N^{-\frac{3}{2}+\delta}(\|v_1\|_{\mathcal{Y}^{\frac{3}{2}+\delta}_{t,x}} + \|v_2\|_{\mathcal{Y}^{\frac{3}{2}+\delta}_{t,x}})\|\widetilde{w}\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}}. \qquad (5.39)
\end{aligned}$$

Next, $\mathcal{T}_3$ and $\mathcal{T}_5$ are estimated using (2.19), (4.4), (4.3), (2.18), and Lemma 3.1, which gives

$$\begin{aligned}
\|\mathcal{T}_3\|_{\mathcal{Y}^{2+2\delta}_{t,x}} &\lesssim_\delta \varepsilon^{-1}\big\|\Pi_{V_2}\big((\mathcal{L}_\omega^{-1}(v_1+v_2)^2)^2\big)\big\|_{\mathcal{Y}^{2\delta}_{t,x}}\\
&\lesssim_\delta \varepsilon^{-1}N^{-\frac{3}{2}+\delta}\|(\mathcal{L}_\omega^{-1}(v_1+v_2)^2)^2\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}}\\
&\lesssim_\delta \gamma^{-2}\varepsilon^{-1}N^{-\frac{3}{2}+\delta}(\|v_1\|_{\mathcal{Y}^{2+2\delta}_{t,x}} + \|v_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}})^4, \qquad (5.40)
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{T}_5\|_{\mathcal{Y}^{2+2\delta}_{t,x}} &\lesssim_\delta \varepsilon^{-1}N^{-\frac{3}{2}+\delta}\|\Pi_{V_2}(\widetilde{w}\mathcal{L}_\omega^{-1}(v_1+v_2)^2)\|_{\mathcal{Y}^{\frac{3}{2}+\delta}_{t,x}}\\
&\lesssim_\delta \gamma^{-1}\varepsilon^{-1}N^{-\frac{3}{2}+\delta}\|\widetilde{w}\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}}(\|v_1\|_{\mathcal{Y}^{2+2\delta}_{t,x}} + \|v_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}})^2. \qquad (5.41)
\end{aligned}$$

Finally, $\mathcal{T}_4$ is estimated using algebra property (2.19):

$$\|\mathcal{T}_4\|_{\mathcal{Y}^{2+2\delta}_{t,x}} \lesssim_\delta \varepsilon^{-1}\|\Pi_{V_2}\widetilde{w}^2\|_{\mathcal{Y}^{2\delta}_{t,x}} \lesssim_\delta \varepsilon^{-1}N^{-\frac{3}{2}+\delta}\|\widetilde{w}\|^2_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}}. \qquad (5.42)$$

Then, by Lemma 5.7, recalling the definitions of ρ_1, ρ_2, ρ_3 as in (5.28) and combining (5.38), (5.39), (5.40), (5.42), (5.41), there exists a positive constant C_δ such that

$$\begin{aligned}\|\mathcal{T}_{v_1,\widetilde{w}}(v_2)\|_{\mathcal{Y}^{2+2\delta}_{t,x}} &\leq C_\delta(\gamma^{-1}\sqrt{\varepsilon}R^3 + \mathtt{c}_3\gamma^{-2}\varepsilon N^{-1+2\delta+b}R^4 + \gamma^{-2}\varepsilon N^{\frac{5}{2}+9\delta}R^4 \\ &\quad + \mathtt{c}_3^2\gamma^{-4}R^6N^{-\frac{3}{2}+\delta+2b}\varepsilon^2 + \mathtt{c}_3\gamma^{-3}\varepsilon\sqrt{\varepsilon}R^5N^{\frac{1}{2}+5\delta+b}) \\ &\leq 2C_\delta\gamma^{-1}\sqrt{\varepsilon}R^3 \leq \gamma^{-1}\mathtt{c}_2\sqrt{\varepsilon}R^3 = \rho_2,\end{aligned}$$

provided (5.32) holds for some $\mathtt{b} \in (0,1)$ and $\epsilon_{R,\delta}$ small enough and $\mathtt{c}_2 \geq 2C_\delta$. We now prove that $\mathcal{T}_{v_1,\widetilde{w}}$ is a contraction. We actually prove that $\partial_{v_2}\mathcal{T}_{v_1,\widetilde{w}} \in \mathcal{B}(V_2 \cap \mathcal{Y}^{2+2\delta}_{t,x}, \mathcal{Y}^{2+2\delta}_{t,x})$. Indeed, one has $\partial_{v_2}\mathcal{T}_{v_1,\widetilde{w}}[h_2] = D_1[h_2] + D_2[h_2] + D_3[h_2] + D_4[h_2]$, with

$$\begin{aligned}D_1[h_2] &:= -2\varepsilon^{-1}A^{-1}\Pi_{V_2}(h_2\mathcal{L}_\omega^{-1}(v_1+v_2)^2), \\ D_2[h_2] &:= -2\varepsilon^{-1}A^{-1}\Pi_{V_2}(h_2\widetilde{w}), \\ D_3[h_2] &:= -4\varepsilon^{-1}A^{-1}\Pi_{V_2}\big((v_1+v_2)\mathcal{L}_\omega^{-1}((v_1+v_2)h_2)\big), \\ D_4[h_2] &:= -4\varepsilon^{-1}A^{-1}\Pi_{V_2}\big((\mathcal{L}_\omega^{-1}(v_1+v_2)^2 + \widetilde{w})\mathcal{L}_\omega^{-1}((v_1+v_2)h_2)\big).\end{aligned}$$

We proceed by estimating all terms separately. By Proposition 4.12 (2), by Lemma 5.7, and using the definitions of the parameters ρ_1, ρ_2, ρ_3, one has

$$\begin{aligned}\|D_1[h_2]\|_{\mathcal{Y}^{2+2\delta}_{t,x}} &\lesssim_\delta \varepsilon^{-1}\|\Pi_{V_2}(h_2\mathcal{L}_\omega^{-1}(v_1+v_2)^2)\|_{\mathcal{Y}^{2\delta}_{t,x}} \\ &\lesssim_\delta \gamma^{-1}\varepsilon^{-1}\|h_2\|_{\mathcal{Y}^{\frac{2}{3}+2\delta}_{t,x}}\|v_1+v_2\|^2_{\mathcal{Y}^{\frac{2}{3}+2\delta}_{t,x}} \\ &\lesssim_\delta \gamma^{-1}\varepsilon^{-1}N^{-\frac{4}{3}}\|h_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}}\|v_1+v_2\|^2_{\mathcal{Y}^{\frac{2}{3}+2\delta}_{t,x}} \\ &\lesssim_\delta \gamma^{-1}\varepsilon^{-1}\rho_1^2N^{-\frac{4}{3}}\|h_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}} \lesssim_\delta \gamma^{-1}R^2N^{-\frac{4}{3}}\|h_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}}.\end{aligned} \tag{5.43}$$

The estimate of $D_3[h_2]$ is the same, and gives

$$\|D_3[h_2]\|_{\mathcal{Y}^{2+2\delta}_{t,x}} \lesssim_\delta \gamma^{-1}R^2N^{-\frac{4}{3}}\|h_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}}. \tag{5.44}$$

The estimate of $D_2[h_2]$ is analogous to the estimate of $\mathcal{T}_2$, and yields

$$\begin{aligned}\|D_2[h_2]\|_{\mathcal{Y}^{2+2\delta}_{t,x}} &\lesssim_\delta \varepsilon^{-1}N^{-\frac{3}{2}+\delta}\|h_2\|_{\mathcal{Y}^{\frac{3}{2}+\delta}_{t,x}}\|\widetilde{w}\|_{H^{\frac{1}{2}+\delta}_t\mathcal{H}^{\frac{3}{2}+\delta}_x} \\ &\lesssim_\delta \varepsilon^{-1}N^{-2}\|h_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}}\rho_3 \lesssim_\delta \mathtt{c}_3\gamma^{-2}R^3\sqrt{\varepsilon}N^{-2+b}\|h_2\|_{\mathcal{Y}^{2+2\delta}_{t,x}}.\end{aligned} \tag{5.45}$$

We finally estimate $D_4[h_2]$ using algebra property (2.19) and Lemma 3.1. One gets

$$\begin{aligned}&\|D_4[h_2]\|_{\mathcal{Y}^{2+2\delta}_{t,x}} \\ &\lesssim_\delta N^{-\frac{3}{2}+\delta}\varepsilon^{-1}\big\|\Pi_{V_2}\big((\mathcal{L}_\omega^{-1}(v_1+v_2)^2 + \widetilde{w})\mathcal{L}_\omega^{-1}((v_1+v_2)h_2)\big)\big\|_{\mathcal{Y}^{\frac{3}{2}+\delta}_{t,x}}\end{aligned}$$

$$\begin{aligned}
&\lesssim_\delta N^{-\frac{3}{2}+\delta}\varepsilon^{-1}\|\mathcal{L}_\omega^{-1}(v_1+v_2)^2\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}}\|\mathcal{L}_\omega^{-1}((v_1+v_2)h_2)\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}}\\
&\quad + N^{-\frac{3}{2}+\delta}\varepsilon^{-1}\|\widetilde{w}\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}}\|\mathcal{L}_\omega^{-1}((v_1+v_2)h_2)\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}}\\
&\lesssim_\delta N^{-\frac{3}{2}+\delta}\gamma^{-2}\varepsilon^{-1}(\|v_1+v_2\|^2_{\mathcal{V}^{2+2\delta}_{t,x}}+\|\widetilde{w}\|_{H^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}})\|v_1+v_2\|_{\mathcal{V}^{2+2\delta}_{t,x}}\|h_2\|_{\mathcal{V}^{2+2\delta}_{t,x}}.
\end{aligned}$$

Then using Lemma 5.7 and the definitions of parameters ρ_1, ρ_2, ρ_3 as in (5.28) one has

$$\begin{aligned}
&\|D_4[h_2]\|_{\mathcal{V}^{2+2\delta}_{t,x}}\\
&\quad\lesssim_\delta N^{-\frac{3}{2}+\delta}\gamma^{-2}\varepsilon^{-1}(\varepsilon R^2N^{2(1+2\delta)}+\mathtt{c}_3\gamma^{-2}R^3\varepsilon^{\frac{3}{2}}N^b)R\sqrt{\varepsilon}N^{1+2\delta}\|h_2\|_{\mathcal{V}^{2+2\delta}_{t,x}}\\
&\quad\lesssim_\delta \gamma^{-2}R^3N^{3(1+2\delta)-\frac{3}{2}+\delta}\sqrt{\varepsilon}\|h_2\|_{\mathcal{V}^{2+2\delta}_{t,x}}. \qquad (5.46)
\end{aligned}$$

Thus, combining (5.43), (5.45), (5.44), (5.46) and assuming (5.32), one gets

$$\|\partial_{v_2}\mathcal{T}_{v_1,\widetilde{w}}(v_2)[h_2]\|_{\mathcal{V}^{2+2\delta}_{t,x}}\lesssim_\delta \gamma^{-1}R^2N^{-\frac{4}{3}}\|h_2\|_{\mathcal{V}^{2+2\delta}_{t,x}},$$

which implies (5.37) and that $\mathcal{T}_{v_1,\widetilde{w}}$ is a contraction. Finally, since

$$\mathcal{T}_{0,0}=\varepsilon^{-1}A^{-1}\Pi_{V_2}\big(2v_2\mathcal{L}_\omega^{-1}(v_2^2)+(\mathcal{L}_\omega^{-1}(v_2^2))^2\big)$$

vanishes at $v_2=0$, we also have $v_2(0,0)=0$. ∎

Finally, with analogous arguments to those in the proof of Lemma 5.8, one obtains differentiability of $v_2(v_1,\widetilde{w})$ with respect to v_1 and $\widetilde{w}$ with estimates (5.34), (5.35).

6. Solution of the range equation

In this section we solve the range equation (3.14) in the algebra spaces $H_t^{\frac{1}{2}+\delta}\mathcal{H}_z^{\frac{3}{2}+\delta}$, where $v_2=v_2(v_1,w)$ is the solution of (3.13); namely we find w such that

$$\mathcal{L}_\omega w-\Pi_W(v_1+v_2(v_1,w)+w)^p=0. \qquad (6.1)$$

6.1. Cases $p=5$ and $p=3$

Proposition 6.1 (Solution of the range equation for $p=5$). *For any $\delta\in(0,\frac{1}{8})$, $\gamma\in(0,\gamma_0)$, and $R>0$, let ρ_1, ρ_2, ρ_3, N be as in (5.4). There exist $\beta_\delta>0$, $\mathtt{c}_3:=\mathtt{c}_3(\delta)>0$, $\zeta:=\zeta(\delta)>0$, $\epsilon_{\delta,R}>0$, and $C_\delta>0$ such that, for any $\beta>\beta_\delta$ and any $\varepsilon>0$ such that $\omega\in\Omega_\gamma$ and (5.5) holds, there exists a C^1 function $w\colon\mathcal{D}_{\rho_1}\to W\cap H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}$, $v_1\mapsto w(v_1)$, satisfying $w(0)=0$,*

$$\|w\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}}\le\rho_3, \qquad (6.2)$$

$$\|d_{v_1}w(v_1)\|_{\mathcal{B}(V_1\cap\mathcal{V}^1_{t,x},H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta})}\le C_\delta\gamma^{-1}N^{5+10\delta}\varepsilon R^4, \qquad (6.3)$$

which solves equation (6.1).

We are going to prove that the map

$$\mathcal{T}_{v_1}\colon w \mapsto \mathcal{L}_\omega^{-1}\Pi_W\big((v_1 + v_2(v_1, w) + w)^5\big) \tag{6.4}$$

is a contraction, with $v_2(v_1, w)$ as in Proposition 5.1.

Lemma 6.2 (Contraction). *Assume the smallness condition (5.5) holds with $\zeta \geq \frac{9}{2} + 9\delta$. The map $\mathcal{T}_{v_1}$ in (6.4) maps $\mathcal{D}^W_{\rho_3}$ into itself, and there exists $C_\delta > 0$ such that*

$$\begin{aligned}
&\|\mathcal{T}_{v_1}(w) - \mathcal{T}_{v_1}(w')\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}} \\
&\quad \leq C_\delta \gamma^{-1} N^{\frac{9}{2}+13\delta} R^8 \|w - w'\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}} \quad \forall w, w' \in \mathcal{D}^W_{\rho_3}.
\end{aligned} \tag{6.5}$$

As a consequence, for any $v_1 \in \mathcal{D}_{\rho_1}$ there exists a unique $w(v_1) \in \mathcal{D}^W_{\rho_3}$ solving (6.1), and such that $w(0) = 0$.

Proof. For brevity, we denote $v_2(v_1, w) = v_2$. By (6.4), Lemma 3.1, algebra property (2.19), (4.4), and Lemma 4.1, we have

$$\begin{aligned}
\|\mathcal{T}_{v_1}(w)\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}} &\lesssim \gamma^{-1} \sum_{j_1+j_2+j_3=5} \|\Pi_W(v_1^{j_1} v_2^{j_2} w^{j_3})\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}} \\
&\lesssim_\delta \gamma^{-1} \sum_{j_1+j_2+j_3=5} (N^{1+2\delta}\|v_1\|_{\mathcal{V}^1_{t,x}})^{j_1} \|v_2\|^{j_2}_{\mathcal{V}^{2+2\delta}_{t,x}} \|w\|^{j_3}_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}} \\
&\lesssim_\delta \gamma^{-1} \sum_{j_1+j_2+j_3=5} (N^{1+2\delta}\rho_1)^{j_1} \rho_2^{j_2} \rho_3^{j_3}.
\end{aligned} \tag{6.6}$$

Then by the smallness assumption (5.5), we have $\rho_2 \leq N^{1+2\delta}\rho_1$ and $\rho_3 \leq N^{1+2\delta}\rho_1$, thus by (6.6) there exists a constant $C_\delta > 0$ such that

$$\|\mathcal{T}_{v_1}(w)\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}} \leq C_\delta \gamma^{-1} N^{5+10\delta}\varepsilon^{\frac{5}{4}} R^5 < \mathtt{c}_3 \gamma^{-1} N^{5+10\delta}\varepsilon^{\frac{5}{4}} R^5 = \rho_3,$$

provided $\mathtt{c}_3 > C_\delta$. This proves that $\mathcal{T}_{v_1}$ maps $\mathcal{D}^W_{\rho_3}$ into itself. We now prove (6.5). Let $h_3 \in H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}$; then arguing as to obtain (6.6), and using estimate (5.8), we obtain

$$\begin{aligned}
&\|\partial_w \mathcal{T}_{v_1}(w)[h_3]\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}} \\
&\quad = 5\|\mathcal{L}_\omega^{-1}\Pi_W((v_1 + v_2 + w)^4(h_3 + \partial_w v_2[h_3]))\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}} \\
&\quad \lesssim \gamma^{-1} \sum_{j_1+j_2+j_3=4} \|\Pi_W(v_1^{j_1} v_2^{j_2} w^{j_3}(h_3 + \partial_w v_2[h_3]))\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}} \\
&\quad \lesssim_\delta \gamma^{-1} \sum_{j_1+j_2+j_3=4} (N^{1+2\delta}\rho_1)^{j_1} \rho_2^{j_2} \rho_3^{j_3} (1 + N^{\frac{1}{2}+5\delta} R^4)\|h_3\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}} \\
&\quad \lesssim_\delta \gamma^{-1} \sum_{j_1+j_2+j_3=4} N^{\frac{9}{2}+13\delta} R^8 \varepsilon \|h_3\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}},
\end{aligned} \tag{6.7}$$

where we have used $\rho_2 \le N^{1+2\delta}\rho_1$ and $\rho_3 \le N^{1+2\delta}\rho_1$. Estimate (6.7) then gives (6.5). Then the map $\mathcal{T}_{v_1}$ is a contraction, and since $v_2(0,0) = 0$ by Proposition 5.1, we have $w(0) = 0$. ∎

Lemma 6.3 (Differentiability of $w(v_1)$). *The function $w(v_1)$ is differentiable in v_1 and* (6.3) *holds.*

Proof. Due to (6.7) and the smallness condition (5.5), one has that $[\mathbb{1} - \partial_w \mathcal{T}_{v_1}(w)]^{-1}$ is a bounded operator onto $H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}$, with norm bounded by 2. For brevity let us denote $v_2(v_1, w(v_1)) = v_2$ and $w(v_1) = w$. Then for any $h_1 \in V_1$, by Lemma 3.1, (2.19), (5.4), smallness condition (5.5), and estimate (5.8), one has

$$\begin{aligned}
&\|\partial_{v_1}\mathcal{T}_{v_1}(w)[h_1]\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}} \\
&\quad= 5\|\mathcal{L}_\omega^{-1}\Pi_W((v_1+v_2+w)^4(h_1+\partial_{v_1}v_2[h_1]))\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}} \\
&\quad\lesssim \gamma^{-1}\sum_{j_1+j_2+j_3=4}\|\Pi_W(v_1^{j_1}v_2^{j_2}w^{j_3}(h_1+\partial_{v_1}v_2[h_1]))\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}} \\
&\quad\lesssim_\delta \gamma^{-1}\sum_{j_1+j_2+j_3=4}(N^{1+2\delta}\rho_1)^{j_1}\rho_2^{j_2}\rho_3^{j_3}(N^{1+2\delta}+N^{10\delta}R^4)\|h_1\|_{\mathcal{V}^1_{t,x}} \\
&\quad\lesssim_\delta \gamma^{-1}N^{5+10\delta}\varepsilon R^4\|h_1\|_{\mathcal{V}^1_{t,x}}.
\end{aligned}$$

This gives (6.3). ∎

If $p = 3$, the proof of the existence of a solution w of (6.1) follows arguing as in the case $p = 5$.

Proposition 6.4 (Solution of the range equation for $p = 3$). *For any $\delta \in (0, \frac{1}{8})$, $\gamma \in (0, \gamma_0)$, and $R > 0$, let ρ_1, ρ_2, ρ_3, N be as in* (5.20). *There exist $\beta_\delta > 0$, $\mathsf{c}_3 := \mathsf{c}_3(\delta) > 0$, $\zeta := \zeta(\delta) > 0$, $\epsilon_{R,\delta} > 0$, and $C_\delta > 0$ such that, for any $\beta > \beta_\delta$ and any $\varepsilon > 0$ such that $\omega \in \Omega_\gamma$ and* (5.21) *holds, there exists a C^1 function $w\colon \mathcal{D}_{\rho_1} \to W \cap H_t^{\frac{1}{2}+\delta}\mathcal{H}_\eta^{\frac{3}{2}+\delta}$, $v_1 \mapsto w(v_1)$, satisfying $w(0) = 0$,*

$$\|w\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_\eta^{\frac{3}{2}+\delta}} \le \rho_3, \quad \|d_{v_1}w(v_1)\|_{\mathcal{B}(V_1\cap\mathcal{V}^1_{t,\eta}, H_t^{\frac{1}{2}+\delta}\mathcal{H}_\eta^{\frac{3}{2}+\delta})} \le C_\delta\varepsilon\gamma^{-1}N^{3+6\delta}R^2,$$

which solves (6.1).

6.2. Case $p = 2$

Here we solve equation (5.31) with $v_2 = v_2(v_1, \widetilde{w})$, which reads

$$\begin{aligned}
\mathcal{L}_\omega\widetilde{w} &= \Pi_W\big(2(v_1+v_2(v_1,\widetilde{w}))\big(\mathcal{L}_\omega^{-1}(v_1+v_2(v_1,\widetilde{w}))^2+\widetilde{w}\big)\big) \\
&\quad+\Pi_W\big(\mathcal{L}_\omega^{-1}(v_1+v_2(v_1,\widetilde{w}))^2+\widetilde{w}\big)^2,
\end{aligned} \tag{6.8}$$

where $v_2(v_1, \widetilde{w})$ is the function in Proposition 5.6.

Proposition 6.5 (Solution of the range equation for $p = 2$). *For any $\delta \in (0, \frac{1}{8})$, $\gamma \in (0, \gamma_0)$, and $R > 0$, assume that ρ_1, ρ_2, ρ_3, N are as in (5.28) and satisfy the assumptions of Proposition 5.6. There exist $\mathtt{c}_3 := \mathtt{c}_3(\delta) > 0$, $\beta_\delta > 0$, $\mathtt{b} := \mathtt{b}(\delta) > 0$, $\epsilon_{\delta,R} > 0$, and $C_\delta > 0$ such that, for any $\beta > \beta_\delta$ and any $\varepsilon > 0$ such that $\omega \in \Omega_\gamma$ and (5.32) holds, there exists a C^1 function $\widetilde{w} \colon \mathcal{D}_{\rho_1} \to W \cap H_t^{\frac{1}{2}+\delta} \mathcal{H}_x^{\frac{3}{2}+\delta}$, $v_1 \mapsto \widetilde{w}(v_1)$, where $\mathcal{D}_{\rho_1}$ is defined as in (5.1), satisfying $\widetilde{w}(0) = 0$, and*

$$\begin{aligned} &\|\widetilde{w}(v_1)\|_{H_t^{\frac{1}{2}+\delta} \mathcal{H}_x^{\frac{3}{2}+\delta}} \leq \rho_3, \\ &\|d_{v_1}\widetilde{w}(v_1)\|_{\mathcal{B}(V_1 \cap \mathcal{V}^1_{t,x}, H_t^{\frac{1}{2}+\delta} \mathcal{H}_x^{\frac{3}{2}+\delta})} \leq C_\delta \gamma^{-2} N^{3(1+2\delta)} \varepsilon R^2, \end{aligned} \tag{6.9}$$

which solves equation (6.8).

We show that the map

$$\begin{aligned} \widetilde{w} \mapsto \mathcal{T}_{v_1}(\widetilde{w}) := {} & \mathcal{L}_\omega^{-1} \Pi_W \big(2(v_1 + v_2(v_1, \widetilde{w}))\big(\mathcal{L}_\omega^{-1}(v_1 + v_2(v_1, \widetilde{w}))^2 + \widetilde{w}\big)\big) \\ & + \mathcal{L}_\omega^{-1} \Pi_W \Big(\big(\mathcal{L}_\omega^{-1}(v_1 + v_2(v_1, \widetilde{w}))^2 + \widetilde{w}\big)^2\Big) \end{aligned} \tag{6.10}$$

is a contraction.

Lemma 6.6 (Contraction). *The function $\mathcal{T}_{v_1}$ defined in (6.10) maps $\mathcal{D}^W_{\rho_3}$ into itself, with*

$$\begin{aligned} &\|\mathcal{T}_{v_1}(\widetilde{w}) - \mathcal{T}_{v_1}(\widetilde{w}')\|_{H_t^{\frac{1}{2}+\delta} \mathcal{H}_x^{\frac{3}{2}+\delta}} \\ &\quad \leq C_\delta \gamma^{-2} \sqrt{\varepsilon} N^{1+6\delta} R^3 \|\widetilde{w} - \widetilde{w}'\|_{H_t^{\frac{1}{2}+\delta} \mathcal{H}_x^{\frac{3}{2}+\delta}} \quad \forall \widetilde{w}, \widetilde{w}' \in \mathcal{D}^W_{\rho_3}, \end{aligned} \tag{6.11}$$

for some $C_\delta > 0$. As a consequence, for any $v_1 \in \mathcal{D}_{\rho_1}$ there exists a unique $\widetilde{w}(v_1)$ solving (6.8) and such that $\widetilde{w}(0) = 0$.

Proof. We denote $v_2 := v_2(v_1, \widetilde{w})$. By (2.19), Lemmas 3.1, 4.1, and 5.7, (5.28), and the smallness condition (5.32), one has

$$\begin{aligned} \big\|\mathcal{L}_\omega^{-1}\big(\Pi_W(2(v_1 + v_2)\mathcal{L}_\omega^{-1}(v_1 + v_2)^2)\big)\big\|_{H_t^{\frac{1}{2}+\delta} \mathcal{H}_x^{\frac{3}{2}+\delta}} &\lesssim_\delta \gamma^{-2} \|v_1 + v_2\|^3_{\mathcal{V}^{2+2\delta}_{t,x}} \\ &\lesssim_\delta \gamma^{-2} \varepsilon \sqrt{\varepsilon} N^{3(1+2\delta)} R^3, \\ \|\mathcal{L}_\omega^{-1} \Pi_W (2(v_1 + v_2)\widetilde{w})\|_{H_t^{\frac{1}{2}+\delta} \mathcal{H}_x^{\frac{3}{2}+\delta}} &\lesssim_\delta \gamma^{-1} N^{1+2\delta} \rho_1 \rho_3 \\ &\lesssim_\delta \gamma^{-3} N^{4+8\delta} \varepsilon^2 R^4, \\ \big\|\mathcal{L}_\omega^{-1} \Pi_W \big((\mathcal{L}_\omega^{-1}(v_1 + v_2)^2 + \widetilde{w})^2\big)\big\|_{H_t^{\frac{1}{2}+\delta} \mathcal{H}_x^{\frac{3}{2}+\delta}} &\lesssim_\delta \gamma^{-1} (\gamma^{-1}(N^{1+2\delta}\rho_1)^2 + \rho_3)^2 \\ &\lesssim_\delta \gamma^{-3} \varepsilon^2 R^4 N^{4(1+2\delta)}. \end{aligned}$$

Therefore, there exists a constant $C_\delta > 0$ such that

$$\|\mathcal{T}_{v_1}(\widetilde{w})\|_{H_t^{\frac{1}{2}+\delta} \mathcal{H}_x^{\frac{3}{2}+\delta}} \leq C_\delta \gamma^{-2} \varepsilon \sqrt{\varepsilon} N^{3(1+2\delta)} R^3 < \rho_3,$$

where the last inequality holds taking $\mathtt{c}_3 = \mathtt{c}_3(\delta) > C_\delta$ in the definition of ρ_3 (see (5.28)). We now show that $\mathcal{T}_{v_1}$ is a contraction. We have that $\partial_{\widetilde{w}}\mathcal{T}_{v_1}(\widetilde{w})[h_3] = F_1[h_3] + F_2[h_3] + F_3[h_3]$, with

$$
\begin{aligned}
F_1[h_3] &:= \mathcal{L}_\omega^{-1}\Pi_W[2\partial_{\widetilde{w}} v_2[h_3](\mathcal{L}_\omega^{-1}(v_1+v_2)^2 + \widetilde{w})],\\
F_2[h_3] &:= \mathcal{L}_\omega^{-1}\Pi_W\big[2(v_1+v_2)\big(2\mathcal{L}_\omega^{-1}((v_1+v_2)\partial_{\widetilde{w}} v_2[h_3]) + h_3\big)\big],\\
F_3[h_3] &:= 2\mathcal{L}_\omega^{-1}\Pi_W\big[(\mathcal{L}_\omega^{-1}(v_1+v_2)^2 + \widetilde{w})\big(2\mathcal{L}_\omega^{-1}((v_1+v_2)\partial_{\widetilde{w}} v_2[h_3]) + h_3\big)\big].
\end{aligned}
$$

By Lemma 3.1, algebra property (2.19), Lemmas 5.7 and 4.1, estimate (5.35), (5.28), and the smallness condition (5.32), one gets

$$
\begin{aligned}
\|F_1[h_3]\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}} &\lesssim_\delta \gamma^{-1}\|\partial_{\widetilde{w}} v_2[h_3]\|_{\mathcal{V}^{2+2\delta}_{t,x}}(\gamma^{-1}\|v_1+v_2\|^2_{\mathcal{V}^{2+2\delta}_{t,x}} + \|\widetilde{w}\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}})\\
&\lesssim_\delta \gamma^{-2}\varepsilon^{\frac12}N^{1+6\delta}R^3\|h_3\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}}.
\end{aligned}
$$

Similarly F_2 and F_3 satisfy the estimates

$$
\begin{aligned}
\|F_2[h_3]\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}} &\lesssim_\delta \gamma^{-2}N^{1+6\delta}R^3\varepsilon^{\frac12}\|h_3\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}},\\
\|F_3[h_3]\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}} &\lesssim_\delta \gamma^{-3}N^{2+8\delta}R^4\varepsilon\|h_3\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}}.
\end{aligned}
$$

Combining the estimates on F_1, F_2, F_3 and assuming that condition (5.32) holds for suitable parameters $\mathtt{b}$ and $\epsilon_{R,\delta}$, one gets

$$
\|\partial_{\widetilde{w}}\mathcal{T}_{v_1}(\widetilde{w})[h_3]\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}} \lesssim_\delta \gamma^{-2}\sqrt{\varepsilon}N^{1+6\delta}R^3\|h_3\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}},
$$

which gives (6.11). Thus $\mathcal{T}_{v_1}$ is a contraction. ∎

The proof of (6.9) follows by similar arguments, using Lemma 3.1, algebra property (2.19), Lemma 5.7, estimate (5.34), (5.28), and the smallness condition (5.32).

7. Solution of the bifurcation equation

In this section we solve

$$
\begin{aligned}
&(\omega^2-1)Av_1 - \Pi_{V_1}(v_1 + \mathtt{v}_2(v_1) + w(v_1))^p = 0,\\
&\mathtt{v}_2(v_1) := \begin{cases} v_2(v_1, w(v_1)) & \text{if } p = 3,\ p = 5,\\ v_2(v_1, \widetilde{w}(v_1)) & \text{if } p = 2, \end{cases}
\end{aligned}
\tag{7.1}
$$

where

- if $p = 5$, $v_2(v_1, w(v_1))$ is the solution of (3.13), whose existence has been proved in Proposition 5.1 (resp. in Proposition 5.4 if $p = 3$), and $w(v_1)$ is the solution of (6.1), whose existence has been proved in Proposition 6.1 (resp. in Proposition 6.4 if $p = 3$);

- if $p = 2$, $v_2(v_1, \widetilde{w}(v_1))$ is the solution of (5.30) as in Proposition 5.6, $\widetilde{w}(v_1)$ is the solution of (5.31), as in Proposition 6.5, and (cf. (5.29))

$$w(v_1) := \mathcal{L}_\omega^{-1}(v_1 + \mathrm{v}_2(v_1))^2 + \widetilde{w}(v_1). \tag{7.2}$$

Throughout this section, we suppose that ρ_1, ρ_2, ρ_3, N, and ε satisfy the hypotheses of Sections 5 and 6.

7.1. Restricted Euler–Lagrange functional

We start by observing that (7.1) has a variational structure.

Lemma 7.1. *Equation (7.1) is the Euler–Lagrange equation of the restricted action functional*

$$\breve{\Psi}\colon \mathcal{D}_{\rho_1} \to \mathbb{R}, \quad v_1 \mapsto \breve{\Psi}(v_1) := \Psi(v_1 + \mathrm{v}_2(v_1) + w(v_1)), \tag{7.3}$$

where Ψ is the action functional defined in (3.5). In particular, $u = v_1 + \mathrm{v}_2(v_1) + w(v_1)$ solves equation (7.1) if and only if v_1 is a critical point of $\breve{\Psi}$.

Proof. We prove the result for $p = 5$ for definiteness. In view of (3.5)–(3.6), the fact that $-\partial_{tt} v_1 = A v_1$, and $\mathrm{v}_2 := \mathrm{v}_2(v_1) \in \mathcal{V}^{2+2\delta}_{t,x}$, resp. $w := w(v_1) \in H^{\frac{1}{2}+\delta}_t \mathcal{H}^{\frac{3}{2}+\delta}_x$, solves equation (5.9), resp. (6.1), one has

$$\begin{aligned}
\breve{\Psi}(v_1) = {} & \frac{\omega^2 - 1}{2} \|v_1\|^2_{\mathcal{V}^1_{t,z}} + \frac{1}{2}\int_{\mathbb{T}}\int_0^\pi \mathrm{v}_2(v_1)\Pi_{V_2}(v_1 + \mathrm{v}_2 + w)^5 \sin^2(x)\, đx\, đt \\
& + \frac{1}{2}\int_{\mathbb{T}}\int_0^\pi w\Pi_W(v_1 + \mathrm{v}_2 + w)^5 \sin^2(x)\, đx\, đt \\
& - \frac{1}{6}\int_{\mathbb{T}}\int_0^\pi (v_1 + \mathrm{v}_2 + w)^6 \sin^2(x)\, đx\, đt,
\end{aligned}$$

and by (2.19) the functions $\mathrm{v}_2\Pi_{V_2}(v_1 + \mathrm{v}_2 + w)^5$, $w\Pi_W(v_1 + \mathrm{v}_2 + w)^5$, and $(v_1 + \mathrm{v}_2 + w)^6$ are in $H^{\frac{1}{2}+\delta}_t \mathcal{H}^{\frac{3}{2}+\delta}_x \subset L^2_{t,x} \subset L^1_{t,x}$. Thus $\breve{\Psi}$ is well posed and differentiable since by Propositions 5.1 and 6.1 also $d_{v_1}\mathrm{v}_2[h_1]$ and $d_{v_1} w[h_1]$ are in $H^{\frac{1}{2}+\delta}_t \mathcal{H}^{\frac{3}{2}+\delta}_x$ for any $h_1 \in V_1$. Differentiating equations (5.9) and (6.1), we get

$$\begin{aligned}
d\breve{\Psi}(v_1)[h_1] = {} & \langle \mathcal{L}_\omega v_1, h_1\rangle_{L^2_{t,x}} + \frac{1}{2}\int_{\mathbb{T}}\int_0^\pi d\mathrm{v}_2[h_1]\mathcal{L}_\omega \mathrm{v}_2 \sin^2(x)\, đx\, đt \\
& + \frac{1}{2}\int_{\mathbb{T}}\int_0^\pi \mathrm{v}_2\mathcal{L}_\omega(d\mathrm{v}_2[h_1]) \sin^2(x)\, đx\, đt \\
& + \frac{1}{2}\int_{\mathbb{T}}\int_0^\pi dw[h_1]\mathcal{L}_\omega w \sin^2(x)\, đx\, đt \\
& + \frac{1}{2}\int_{\mathbb{T}}\int_0^\pi w\mathcal{L}_\omega(dw[h_1]) \sin^2(x)\, đx\, đt
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{T}}\int_0^\pi (v_1+\mathtt{v}_2+w)^5(h_1+d\mathtt{v}_2[h]+dw[h])\sin^2(x)\,đx\,đt\\
&=\langle \mathscr{L}_\omega v_1,h_1\rangle_{L^2_{t,x}}-\int_{\mathbb{T}}\int_0^\pi \Pi_{V_1}(v_1+\mathtt{v}_2+w)^5 h_1\sin^2(x)\,đx\,đt,
\end{aligned}
$$

so that $d\breve{\Psi}(v_1)[h_1]=0$ if and only if v_1 solves (7.1). ∎

By Lemma 7.1, the following result ensures the existence of one solution of equations (3.1)–(3.2).

Theorem 7.2 (Existence of one critical point). *Let $\delta\in(0,\frac{1}{100})$. Then the following hold:*

Case $p=5$, $p=3$: *There exist $R_0>1$, $\epsilon_{\delta,R}>0$, $\zeta:=\zeta(\delta)>1$, and $\beta_\delta>1$ such that, if $R>R_0$, (5.5) holds, $\beta>\beta_\delta$, ρ_1, ρ_2, ρ_3, N are as in (5.4) if $p=5$ and in (5.20) if $p=3$, and $\omega>1$, $\omega\in\Omega_\gamma$, then the functional $\breve{\Psi}$ defined in (7.3) admits a critical point $v_1^{(1)}\in\mathcal{D}_{\rho_1}$ with norm*

$$
\|v_1^{(1)}\|_{\mathcal{Y}^1_{t,z}}\asymp\begin{cases}\varepsilon^{\frac14} & \text{if } p=5,\\ \varepsilon^{\frac12} & \text{if } p=3.\end{cases}
\tag{7.4}
$$

Case $p=2$: *There exist $R_0>1$, $\epsilon_{\delta,R}>0$, $\mathtt{b}:=\mathtt{b}(\delta)>1$, and $\beta_\delta>1$ such that, if $R>R_0$, (5.32) holds, $\beta>\beta_\delta$, ρ_1, ρ_2, ρ_3, N are as in (5.28), and $\omega<1$, $\omega\in\Omega_\gamma$, then the functional $\breve{\Psi}$ defined in (7.3) admits a critical point $v_1^{(1)}\in\mathcal{D}_{\rho_1}$ with norm $\|v_1^{(1)}\|_{\mathcal{Y}^1_{t,x}}\asymp\varepsilon^{\frac12}$ as $\varepsilon\to0$.*

In order to prove Theorem 7.2 we first provide a suitable decomposition of the functional $\breve{\Psi}$, using Lemma 7.1. We argue separately for the cases $p=5$, $p=3$ and for the degenerate case $p=2$.

Lemma 7.3 ($\breve{\Psi}$ for $p=5$, $p=3$). *The functional $\breve{\Psi}$ defined in (7.3) has the form*

$$
\breve{\Psi}(v_1)=\frac{\varepsilon}{2}\|v_1\|^2_{\mathcal{Y}^1_{t,z}}-\mathscr{G}_{p+1}(v_1)+\mathscr{R}_{p+1}(v_1)\quad\forall v_1\in\mathcal{D}_{\rho_1},
\tag{7.5}
$$

where $\mathscr{G}_{p+1}$ is defined in (3.6), $\mathscr{R}_{p+1}(0)=0$, and

$$
\begin{aligned}
&d\mathscr{R}_{p+1}(v_1)[v_1]\\
&=\begin{cases}-\displaystyle\int_{\mathbb{T}}\int_0^\pi\big((v_1+\mathtt{v}_2(v_1)+w(v_1))^5-v_1^5\big)\sin^2(x)\,đx\,đt & \text{if } p=5,\\ -\displaystyle\int_{\mathbb{T}}\int_0^{\frac\pi2}\big((v_1+\mathtt{v}_2(v_1)+w(v_1))^3-v_1^3\big)\sin(2\eta)\,d\eta\,đt & \text{if } p=3.\end{cases}
\end{aligned}
\tag{7.6}
$$

If $p=5$ the functions $\mathtt{v}_2(v_1)$ and $w(v_1)$ satisfy

$$
\begin{aligned}
\|\mathtt{v}_2(v_1)\|_{\mathcal{Y}^{2+2\delta}_{t,x}}&\lesssim_\delta N^{10\delta}R^4\|v_1\|_{\mathcal{Y}^1_{t,x}},\\
\|w(v_1)\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}}&\lesssim_\delta \varepsilon\gamma^{-1}N^{5+10\delta}R^4\|v_1\|_{\mathcal{Y}^1_{t,z}},
\end{aligned}
\tag{7.7}
$$

whereas if $p = 3$,

$$\begin{aligned} \|\mathrm{v}_2(v_1)\|_{\mathcal{V}^{2+2\delta}_{t,\eta}} &\lesssim_\delta N^{4\delta} R^2 \|v_1\|_{\mathcal{V}^1_{t,\eta}}, \\ \|w(v_1)\|_{H^{\frac{1}{2}+\delta}_t \mathcal{H}^{\frac{3}{2}+\delta}_\eta} &\lesssim_\delta \varepsilon\gamma^{-1} N^{3+6\delta} R^2 \|v_1\|_{\mathcal{V}^1_{t,\eta}}. \end{aligned} \tag{7.8}$$

Here, estimates (7.7), (7.8) follow by Propositions 5.1, 6.1, 5.4, 6.4.

The case $p = 2$ is different since Lemma 4.4 implies that $\int_{\mathbb{T}} \int_0^\pi v_1^3 \sin^2(x)\, đx\, đt = 0$ identically vanishes. We perform a different decomposition.

Lemma 7.4 ($\breve{\Psi}$ in the case $p = 2$). *For any* $v_1 \in \mathcal{D}_{\rho_1}$ *one has*

$$\breve{\Psi}(v_1) = -\frac{\varepsilon}{2}\|v_1\|^2_{\mathcal{V}^1_{t,x}} - \breve{\mathcal{G}}_4(v_1) + \breve{\mathcal{R}}_4(v_1), \tag{7.9}$$

$$\breve{\mathcal{G}}_4(v_1) := \frac{1}{2}\int_{\mathbb{T}}\int_0^\pi v_1^2 \mathcal{L}_1^{-1} v_1^2 \sin^2(x)\, đx\, đt, \tag{7.10}$$

with $\breve{\mathcal{R}}_4(0) = 0$ *and*

$$\begin{aligned} d\breve{\mathcal{R}}_4(v_1)[v_1] = {} & 2\int_{\mathbb{T}}\int_0^\pi v_1^2(\mathcal{L}_1^{-1} - \mathcal{L}_\omega^{-1}) v_1^2 \sin^2(x)\, đx\, đt \\ & - \int_{\mathbb{T}}\int_0^\pi (2v_1 \mathrm{v}_2(v_1) w(v_1) + w(v_1)^2 v_1) \sin^2(x)\, đx\, đt \\ & - 2\int_{\mathbb{T}}\int_0^\pi v_1^2 (w(v_1) - \mathcal{L}_\omega^{-1} v_1^2) \sin^2(x)\, đx\, đt. \end{aligned} \tag{7.11}$$

We prove Theorem 7.2 as an application of the following abstract result, which is a particular case of [6, Theorem 2.3]:

Theorem 7.5 (Abstract mountain pass theorem). *Let* E *be a finite-dimensional Hilbert space equipped with scalar product* $\langle\cdot,\cdot\rangle$ *and norm* $\|\cdot\|^2 = \langle\cdot,\cdot\rangle$. *Let* $\mathcal{I}\colon B_{\rho_1} \subset E \to \mathbb{R}$ *be a* C^1 *functional defined on the ball* $B_{\rho_1} := \{v \in E : \|v\| < \rho_1\}$ *for some* $\rho_1 > 0$, *of the form*

$$\mathcal{I}(v) = \frac{\varepsilon}{2}\|v\|^2 - \mathcal{G}(v) + \mathcal{R}(v),$$

where $\mathcal{G} \in C^1(E, \mathbb{R})$ *is a nonzero homogeneous functional of degree* $p+1$, $p > 1$, *and* $\mathcal{R} \in C^1(B_{\rho_1}, \mathbb{R})$ *satisfies* $\mathcal{R}(0) = 0$. *Define*

$$m^+(\mathcal{G}) := \sup_{v\neq 0} \frac{\mathcal{G}(v)}{\|v\|^{p+1}}, \quad m^-(\mathcal{G}) := \inf_{v\neq 0} \frac{\mathcal{G}(v)}{\|v\|^{p+1}},$$

$$m(\mathcal{G}) := \begin{cases} m^+(\mathcal{G}) & \text{if } m^+(\mathcal{G}) > 0, \\ -m^-(\mathcal{G}) & \text{if } m^-(\mathcal{G}) < 0, \end{cases}$$

and suppose $\varepsilon > 0$ *(resp.* $\varepsilon < 0$*) if* $m^+(\mathcal{G}) > 0$ *(resp.* $m^-(\mathcal{G}) < 0$*).*

Then there exists a positive constant $\underline{C}$, depending on p only, such that, if

$$\alpha(\mathcal{R}) := \sup_{v\in B_{\rho_1}\setminus\{0\}} \frac{|d\mathcal{R}(v)[v]|}{\|v\|^{p+1}} \le \underline{C} m(\mathcal{G}), \quad \Big(\frac{|\varepsilon|}{m(\mathcal{G})}\Big)^{\frac{1}{p-1}} \le \underline{C}\rho_1, \tag{7.12}$$

the functional $\mathcal{I}$ has a critical point $v \in B_{\rho_1}$ on a critical level

$$c := \frac{p-1}{2} m(\mathcal{G}) \Big(\frac{|\varepsilon|}{(p+1)m(\mathcal{G})}\Big)^{(p+1)/(p-1)} \Big(1 + \mathcal{O}\Big(\frac{\alpha(\mathcal{R})}{m(\mathcal{G})}\Big)\Big).$$

Moreover,

$$v = \Big(1 + \mathcal{O}\Big(\frac{\alpha(\mathcal{R})}{m(\mathcal{G})}\Big)\Big)\Big(\frac{|\varepsilon|}{(p+1)m(\mathcal{G})}\Big)^{1/(p-1)} y, \tag{7.13}$$

for some $y \in E$ with $\|y\| = 1$ and $\mathcal{G}(y) = m^+(\mathcal{G}) + \mathcal{O}(\alpha(\mathcal{R}))$ (resp., $\mathcal{G}(y) = m^-(\mathcal{G}) + \mathcal{O}(\alpha(\mathcal{R}))$).

We shall apply Theorem 7.5 to $\breve{\Psi}$ in (7.3) with $E = (V_1, \langle\cdot,\cdot\rangle_{\mathcal{V}^1_{t,z}})$.

7.2. Cases $p = 5$ and $p = 3$

We first consider the case $p = 5$. Since the functional $\mathcal{G}_6$ in (3.6) is positive, we have $m(\mathcal{G}_6) = m^+(\mathcal{G}_6)$.

Lemma 7.6 (Estimate of $m(\mathcal{G}_6)$). *There exists $C > 0$, independent of N, such that*

$$\frac{5}{48} \le m(\mathcal{G}_6) \le C. \tag{7.14}$$

Proof. By Proposition 4.5 (1) with $\delta = \frac{1}{6}$, one has $\mathcal{G}(v_1) \lesssim \|v_1\|^6_{\mathcal{V}^1_{t,x}}$, which gives $m(\mathcal{G}_6) \le C$ for some $C > 0$. We now estimate $m(\mathcal{G}_6)$ from below. We observe that, letting $\check{v}_1(t,x) := \cos(t)e_0(x) = \cos(t)$, then $\check{v}_1 \in V_1$, and by (4.1) we have $\|\check{v}_1\|^2_{\mathcal{V}^1_{t,x}} = 1$. One has

$$\mathcal{G}_6(\check{v}_1) = \frac{1}{6}\int_{\mathbb{T}} \cos^6(t)\, đt \int_0^\pi \sin^2(x)\, đx = \frac{5}{48},$$

since $\int_{\mathbb{T}} \cos^6(t)\, đt = \frac{5}{8}$, and estimate (7.14) is proved. ∎

We now prove properties of the functional $\mathcal{R}_6$ defined in Lemma 7.3.

Lemma 7.7 (Estimate of $\alpha(\mathcal{R}_6)$). *There exists $C_\delta > 0$ such that $\alpha(\mathcal{R}_6) \le C_\delta N^{-\frac{7}{6}+9\delta} R^4$.*

Proof. We set $\mathrm{v}_2 := \mathrm{v}_2(v_1)$ and $w := w(v_1)$. By (7.6), one has

$$\begin{aligned} |d\mathcal{R}_6(v_1)[v_1]| \lesssim{}& \sum_{\substack{j_1+j_2=5\\ j_2\ge 1}} \Big|\int_{\mathbb{T}}\int_0^\pi v_1^{j_1+1}\mathrm{v}_2^{j_2}\sin^2(x)\, đx\, đt\Big| \\ &+ \sum_{\substack{j_1+j_2+j_3=5\\ j_3\ge 1}} \Big|\int_{\mathbb{T}}\int_0^\pi v_1^{j_1+1}\mathrm{v}_2^{j_2} w^{j_3}\sin^2(x)\, đx\, đt\Big|. \end{aligned}$$

Using (4.10) for the first term one has, for any j_1, j_2 with $j_1 + j_2 = 5$ and $j_2 \geq 1$,

$$\begin{aligned}\left|\int_{\mathbb{T}}\int_0^{\pi} v_1^{j_1+1}\mathrm{v}_2^{j_2}\sin^2(x)\,đx\,đt\right| &\lesssim_\delta \|v_1\|^{j_1+1}_{\mathcal{V}^{\frac{5}{6}+\delta}_{t,x}}\|\mathrm{v}_2\|^{j_2}_{\mathcal{V}^{\frac{5}{6}+\delta}_{t,x}}\\ &\overset{(4.4)}{\lesssim_\delta} N^{-(\frac{7}{6}+\delta)j_2}\|v_1\|^{j_1+1}_{\mathcal{V}^1_{t,x}}\|\mathrm{v}_2\|^{j_2}_{\mathcal{V}^{2+2\delta}_{t,x}}\\ &\overset{(7.7)}{\lesssim_\delta} N^{-(\frac{7}{6}+\delta)j_2}(N^{10\delta}R^4)^{j_2}\|v_1\|^6_{\mathcal{V}^1_{t,x}}\\ &\lesssim_\delta N^{-\frac{7}{6}+9\delta}R^4\|v_1\|^6_{\mathcal{V}^1_{t,x}},\end{aligned}\tag{7.15}$$

since $N^{-\frac{7}{6}+9\delta}R^4 < 1$, due to the smallness condition in (5.5). The second term is estimated using Lemma 4.3 and, recalling $j_1 + j_2 + j_3 = 5$, $j_3 \geq 1$, one obtains

$$\begin{aligned}&\left|\int_{\mathbb{T}}\int_0^{\pi}\Pi_{V_1}(v_1^{j_1}\mathrm{v}_2^{j_2}w^{j_3})v_1\sin^2(x)\,đx\,đt\right|\\ &\lesssim_\delta \|\Pi_{V_1}(v_1^{j_1}\mathrm{v}_2^{j_2}w^{j_3})\|_{\mathcal{V}^0_{t,x}}\|v_1\|_{\mathcal{V}^0_{t,x}}\\ &\lesssim_\delta \|v_1\|^{j_1}_{\mathcal{V}^{\frac{3}{2}+\delta}_{t,x}}\|\mathrm{v}_2\|^{j_2}_{\mathcal{V}^{\frac{3}{2}+\delta}_{t,x}}\|w\|^{j_3}_{H^{\frac{1}{2}+\delta}_t\mathcal{H}^{\frac{3}{2}+\delta}_x}\|v_1\|_{\mathcal{V}^1_{t,x}}\\ &\overset{(4.4)}{\lesssim_\delta} N^{j_1(\frac{1}{2}+\delta)}\|v_1\|^{j_1+1}_{\mathcal{V}^1_{t,x}}\|\mathrm{v}_2\|^{j_2}_{\mathcal{V}^{2+2\delta}_{t,x}}\|w\|^{j_3}_{H^{\frac{1}{2}+\delta}_t\mathcal{H}^{\frac{3}{2}+\delta}_x}\\ &\overset{(7.7),(5.5)}{\lesssim_\delta} N^{4(\frac{1}{2}+\delta)+5+10\delta}R^4\gamma^{-1}\varepsilon\|v_1\|^6_{\mathcal{V}^1_{t,x}},\end{aligned}\tag{7.16}$$

which is $\lesssim_\delta N^{-\frac{7}{6}+9\delta}R^4\|v_1\|^6_{\mathcal{V}^1_{t,x}}$, using the smallness condition (5.5) again. Estimates (7.15) and (7.16) imply $|d\mathcal{R}_6(v_1)[v_1]| \lesssim_\delta N^{-\frac{7}{6}+9\delta}R^4\|v_1\|^6_{\mathcal{V}^1_{t,x}}$, which gives the thesis. ∎

Proof of Theorem 7.2 for $p = 5$. We now verify that the two conditions in (7.12) are satisfied. Let $\underline{C}$ be the positive constant defined in Theorem 7.5: by (7.14) and Lemma 7.7,

$$\frac{\alpha(\mathcal{R}_6)}{m(\mathcal{G}_6)} \leq \frac{C_\delta N^{-\frac{7}{6}+9\delta}R^4}{\frac{5}{48}} \leq \underline{C},\tag{7.17}$$

provided $N^{\frac{7}{6}-9\delta} \geq \frac{48C_\delta R^4}{5\underline{C}}$. Furthermore, by (7.14), one has

$$\left(\frac{\varepsilon}{m(\mathcal{G}_6)}\right)^{\frac{1}{4}} \leq \varepsilon^{\frac{1}{4}}\left(\frac{48}{5}\right)^{\frac{1}{4}} \leq \underline{C}\rho_1 \overset{(5.4)}{=} \underline{C}R\varepsilon^{\frac{1}{4}},$$

provided $R \geq (\frac{48}{5})^{\frac{1}{4}}\underline{C}^{-1}$. Then Theorem 7.5 ensures the existence of a critical point $v_1^{(1)} \in V_1$. Finally, (7.14), (7.17), and $N := \varepsilon^{-\frac{1}{\beta}}$ imply $\frac{\alpha(\mathcal{R}_6)}{m(\mathcal{G}_6)} = o(1)$ as $\varepsilon \to 0$. Then equation (7.13) of Theorem 7.5 implies $\|v_1^{(1)}\|_{\mathcal{V}^1_{t,x}} \asymp \varepsilon^{\frac{1}{4}}$ as $\varepsilon \to 0$. ∎

We now consider the case $p = 3$. Since $\mathcal{G}_4$ defined in (3.6) is positive, we have $m(\mathcal{G}_4) = m^+(\mathcal{G}_4)$.

Lemma 7.8 (Estimate of $m(\mathcal{G}_4)$). *There exist $C^+ > 0$ and $C^-_{\mu_1,\mu_2} > 0$, independent of N, such that*

$$C^-_{\mu_1,\mu_2} \le m(\mathcal{G}_4) \le C^+. \tag{7.18}$$

Proof. By (2.20) and by Lemma 4.1, for any $v_1 \in V_1$ one has

$$\mathcal{G}_4(v_1) = \frac{1}{4}\int_{\mathbb{T}}\int_0^{\frac{\pi}{2}} v_1^4 \sin(2\eta)\, d\eta\, đt \lesssim \|v_1\|^4_{\mathcal{V}^1_{t,\eta}},$$

which gives the upper bound in (7.18). The lower bound follows since there exists $\check{v}_1$ such that $\mathcal{G}_4(\check{v}_1) > 0$. For example, $\check{v}_1(t,\eta) := \cos(\omega_0 t)\mathrm{e}_0(\eta)$, with $\mathrm{e}_0 = e_0^{(\mu_1,\mu_2)}$ as in (2.12). ∎

We now estimate the functional $\mathcal{R}_4$ defined in Lemma 7.3:

Lemma 7.9 (Estimate of $\alpha(\mathcal{R}_4)$). *There exists $C_\delta > 0$ such that $\alpha(\mathcal{R}_4) \le C_\delta N^{-\frac{5}{4}+2\delta} R^2$.*

Proof. We set $\mathrm{v}_2 := \mathrm{v}_2(v_1)$ and $w := w(v_1)$. By (7.6), one has

$$\begin{aligned} |d\mathcal{R}_4(v_1)[v_1]| \lesssim & \sum_{\substack{j_1+j_2=3\\ j_2\ge 1}} \left| \int_{\mathbb{T}}\int_0^{\frac{\pi}{2}} v_1^{j_1+1}\mathrm{v}_2^{j_2} \sin(2\eta)\, d\eta\, đt \right| \\ & + \sum_{\substack{j_1+j_2+j_3=3\\ j_3\ge 1}} \left| \int_{\mathbb{T}}\int_0^{\frac{\pi}{2}} v_1^{j_1+1}\mathrm{v}_2^{j_2} w^{j_3} \sin(2\eta)\, d\eta\, đt \right|, \end{aligned}$$

and we estimate the two terms separately. Using Lemmas 2.3 and 4.1, (4.4), (7.8), and (5.21), and recalling $j_1 + j_2 = 3$, $j_2 \ge 1$, for the first term one has

$$\begin{aligned} \left| \int_{\mathbb{T}}\int_0^{\frac{\pi}{2}} v_1^{j_1+1}\mathrm{v}_2^{j_2} \sin(2\eta)\, d\eta\, đt \right| &\lesssim_\delta N^{(-\frac{5}{4}+2\delta)j_2} R^{2j_2} \|v_1\|^4_{\mathcal{V}^1_{t,\eta}} \\ &\lesssim_\delta N^{-\frac{5}{4}+2\delta} R^2 \|v_1\|^4_{\mathcal{V}^1_{t,\eta}}. \end{aligned} \tag{7.19}$$

For the second term, using Lemma 4.3, (4.4), (7.8), (5.21) and recalling $j_1 + j_2 + j_3 = 3$ and $j_3 \ge 1$, one obtains

$$\begin{aligned} \left| \int_{\mathbb{T}}\int_0^{\frac{\pi}{2}} \Pi_{V_1}(v_1^{j_1}\mathrm{v}_2^{j_2} w^{j_3}) v_1 \sin(2\eta)\, d\eta\, đt \right| &\lesssim_\delta \varepsilon\gamma^{-1} R^2 N^{4+8\delta} \|v_1\|^4_{\mathcal{V}^1_{t,\eta}} \\ &\lesssim_\delta N^{-\frac{5}{4}+2\delta} R^2 \|v_1\|^4_{\mathcal{V}^1_{t,\eta}}. \end{aligned} \tag{7.20}$$

Combining estimates (7.19) and (7.20), one then gets Lemma 7.9. ∎

Proof of Theorem 7.2 for $p = 3$. By (7.18) and Lemma 7.9, one has

$$\frac{\alpha(\mathcal{R}_4)}{m(\mathcal{G}_4)} \leq \frac{C_\delta N^{-\frac{5}{4}+2\delta} R^2}{C^-_{\mu_1,\mu_2}} \leq \underline{C},$$

with $\underline{C}$ the constant whose existence is stated in Theorem 7.2, provided (5.21) holds with $\epsilon_{R,\delta}$ small enough. By (7.18) one observes that

$$\Big(\frac{\varepsilon}{m(\mathcal{G}_4)}\Big)^{\frac{1}{2}} \leq \varepsilon^{\frac{1}{2}} (C^-_{\mu_1,\mu_2})^{-\frac{1}{2}} \leq \underline{C}\rho_1 \overset{(5.20)}{=} \underline{C} R\varepsilon^{\frac{1}{2}},$$

provided $R \geq (C^-_{\mu_1,\mu_2})^{-\frac{1}{2}} \underline{C}^{-1}$. Then the existence of a critical point $v_1^{(1)} \in \mathcal{V}^1_{t,\eta}$ such that $\|v_1^{(1)}\|_{\mathcal{V}^1_{t,\eta}} \asymp \varepsilon^{\frac{1}{2}}$ as $\varepsilon \to 0$ follows. ∎

7.3. Case $p = 2$

In the next lemma we show that $\check{\mathcal{G}}_4$ in (7.10) also assumes negative values. Thus $m(\check{\mathcal{G}}_4) = -m^-(\check{\mathcal{G}}_4)$.

Lemma 7.10 (Estimate of $m(\check{\mathcal{G}}_4)$). *There exists $C > 0$, independent of N, such that*

$$\frac{5}{24} \leq m(\check{\mathcal{G}}_4) \leq C. \tag{7.21}$$

Proof. By Proposition 4.12 (1) with $\delta = \frac{1}{2}$, for any $v_1 \in V_1$ one has

$$-\check{\mathcal{G}}_4(v_1) \leq \frac{1}{2}\Big|\int_{\mathbb{T}}\int_0^\pi v_1^2 \mathcal{L}_1^{-1} v_1^2 \sin^2(x)\, đx\, đt\Big| \leq C\|v_1\|^4_{\mathcal{V}^1_{t,x}},$$

for some $C > 0$, which gives the second inequality in (7.21). We now prove the first inequality in (7.21). Let $\bar{v}_1(t,x) := \cos(t) e_0(x) = \cos(t)$, with $e_0 = 1$ by (2.4). Then $\bar{v}_1 \in V_1$ and $\|\bar{v}_1\|_{\mathcal{V}^1_{t,x}} = 1$. We now compute $\check{\mathcal{G}}_4(\bar{v})$. Recalling (3.15), one has

$$\begin{aligned}
&\int_{\mathbb{T}}\int_0^\pi \bar{v}_1^2 \mathcal{L}_1^{-1} \bar{v}_1^2 \sin^2(x)\, đx\, đt \\
&\quad = \int_{\mathbb{T}}\int_0^\pi \Big(\frac{1}{2}\cos(2t) + \frac{1}{2}\Big)\Big(\frac{1}{6}\cos(2t) - \frac{1}{2}\Big)\sin^2(x)\, đx\, đt = -\frac{5}{12},
\end{aligned}$$

thus

$$m(\check{\mathcal{G}}_4) = -\inf_{v_1 \in V_1 \setminus \{0\}} \frac{\check{\mathcal{G}}_4(v_1)}{\|v_1\|^4_{\mathcal{V}^1_{t,x}}} \geq -\check{\mathcal{G}}_4(\bar{v}) > \frac{5}{24}.$$ ∎

We now estimate $\alpha(\check{\mathcal{R}}_4)$ where $\check{\mathcal{R}}_4$ is defined in (7.11). We use the following auxiliary lemmas.

Lemma 7.11. *There exists $C_\delta > 0$ such that*

$$\|\mathrm{v}_2(v_1)\|_{\mathcal{Y}^{2+2\delta}_{t,x}} \lesssim_\delta \gamma^{-1} R^2 \|v_1\|_{\mathcal{Y}^1_{t,x}}, \tag{7.22}$$

$$\|w(v_1)\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}} \le C_\delta \gamma^{-1} N^{2(1+2\delta)} \sqrt{\varepsilon} R \|v_1\|_{\mathcal{Y}^1_{t,x}}, \tag{7.23}$$

$$\|\widetilde{w}(v_1)\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}} \le C_\delta \gamma^{-2} R N^{3(1+2\delta)} \sqrt{\varepsilon} \|v_1\|^2_{\mathcal{Y}^1_{t,x}}. \tag{7.24}$$

Proof. By the chain rule we have

$$d_{v_1}\mathrm{v}_2(v_1)[h_1] = (\partial_{v_1} v_2)(v_1, \widetilde{w}(v_1))[h_1] + (\partial_w v_2)(v_1, \widetilde{w}(v_1))[\partial_{v_1}\widetilde{w}(v_1)[h_1]],$$

hence (7.22) follows by (5.34), (5.35), (6.9), and (5.32) and recalling that $\mathrm{v}_2(0) = 0$.

Concerning $w(v_1)$, we have $w(0) = 0$ since $v_2(0,0) = 0$ and $\widetilde{w}(0) = 0$. For any $h_1 \in V_1$, by algebra property (2.19), by Lemmas 3.1 and 5.7, (7.22), (6.9), (5.28), and (5.32) one has

$$\begin{aligned}
&\|d_{v_1} w(v_1)[h_1]\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}} \\
&\quad \le \|2\mathcal{L}_\omega^{-1}((v_1+\mathrm{v}_2)(h_1 + d_{v_1}\mathrm{v}_2[h_1]))\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}} + \|d_{v_1}\widetilde{w}[h_1]\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}} \\
&\quad \lesssim_\delta \gamma^{-1} N^{1+2\delta}\rho_1(N^{1+2\delta} + \gamma^{-1}R^2)\|h_1\|_{\mathcal{Y}^1_{t,x}} + \gamma^{-2}N^{3(1+2\delta)}\varepsilon R^2\|h_1\|_{\mathcal{Y}^1_{t,x}} \\
&\quad \lesssim_\delta \gamma^{-1} N^{2(1+2\delta)} R\sqrt{\varepsilon}\|h_1\|_{\mathcal{Y}^1_{t,x}},
\end{aligned}$$

which implies (7.23), since $w(0) = 0$.

We now prove (7.24). First we observe that, by (6.8), $\widetilde{w}(v_1)$ satisfies

$$\widetilde{w}(v_1) = \mathcal{L}_\omega^{-1}\Pi_W(2(v_1+\mathrm{v}_2)w + w^2) = \mathcal{L}_\omega^{-1}\Pi_W((v_1+\mathrm{v}_2+w)^2 - (v_1+\mathrm{v}_2)^2).$$

Then by algebra property (2.19) and Lemma 3.1 we have

$$\begin{aligned}
&\|\widetilde{w}(v_1)\|_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}} \\
&\quad \lesssim_\delta \gamma^{-1} \sum_{\substack{j_1+j_2+j_3=2 \\ j_3\ge 1}} (N^{(1+2\delta)}\|v_1\|_{\mathcal{Y}^1_{t,x}})^{j_1}\|\mathrm{v}_2(v_1)\|^{j_2}_{\mathcal{Y}^{2+2\delta}_{t,x}}\|w(v_1)\|^{j_3}_{H_t^{\frac12+\delta}\mathcal{H}_x^{\frac32+\delta}}.
\end{aligned}$$

Now (7.24) follows by (7.22) and (7.23). ∎

Lemma 7.12. *Assume $\omega \in \Omega_\gamma$. For any $w \in W \cap H_t^{r+1}\mathcal{H}_x^s$, $r, s \in \mathbb{R}$, one has*

$$\|(\mathcal{L}_\omega^{-1} - \mathcal{L}_1^{-1})w\|_{H_t^r\mathcal{H}_x^s} \le 2\gamma^{-1}\varepsilon\|w\|_{H_t^{r+1}\mathcal{H}_x^s}. \tag{7.25}$$

Proof. By (3.15) we have

$$(\mathcal{L}_\omega^{-1} - \mathcal{L}_1^{-1})w(t,x) = \sum_{\ell\ne\omega_j,\ell\ne 0} w_{\ell,j}\Big(\frac{1}{\omega^2\ell^2 - \omega_j^2} - \frac{1}{\ell^2-\omega_j^2}\Big)\cos(\ell t)e_j(x), \tag{7.26}$$

and

$$\Big|\frac{1}{\omega^2\ell^2-\omega_j^2}-\frac{1}{\ell^2-\omega_j^2}\Big|=\Big|\frac{\ell^2(1-\omega^2)}{(\omega^2\ell^2-\omega_j^2)(\ell^2-\omega_j^2)}\Big|\leq\frac{|\ell|^2\varepsilon}{|\ell|\frac{\gamma}{2}}=\frac{2|\ell|\varepsilon}{\gamma},\tag{7.27}$$

using (3.16) and $|\ell^2-\omega_j^2|\geq|\ell+\omega_j|\geq|\ell|$. Combining (7.26) and (7.27), one deduces (7.25). ∎

We now exhibit an upper bound for $\alpha(\breve{\mathcal{R}}_4)$.

Lemma 7.13 (Estimate of $\alpha(\breve{\mathcal{R}}_4)$). *There exists $C_\delta>0$ such that*

$$\alpha(\breve{\mathcal{R}}_4)\leq C_\delta\gamma^{-2}N^{-\frac{3}{2}-\delta}R^2.$$

Proof. For brevity, we denote $\mathrm{v}_2:=\mathrm{v}_2(v_1)$, $w:=w(v_1)$, $\widetilde{w}:=\widetilde{w}(v_1)$. By (7.11) and (7.2), we have $d\breve{\mathcal{R}}_4(v_1)[v_1]=A(v_1)+B(v_1)+C(v_1)+D(v_1)$, where

$$\begin{aligned}
A(v_1)&:=-2\int_{\mathbb{T}}\int_0^\pi\Pi_{V_1}((v_1+\mathrm{v}_2)\widetilde{w})v_1\sin^2(x)\,đx\,đt,\\
B(v_1)&:=\underbrace{-2\int_{\mathbb{T}}\int_0^\pi v_1^2\mathcal{L}_\omega^{-1}(2v_1\mathrm{v}_2+\mathrm{v}_2^2)\sin^2(x)\,đx\,đt}_{:=B_1(v_1)}\\
&\quad\underbrace{-2\int_{\mathbb{T}}\int_0^\pi v_1\mathrm{v}_2\mathcal{L}_\omega^{-1}(v_1+\mathrm{v}_2)^2\sin^2(x)\,đx\,đt}_{:=B_2(v_1)},\\
C(v_1)&:=-\int_{\mathbb{T}}\int_0^\pi w^2v_1\sin^2(x)\,đx\,đt,\\
D(v_1)&:=2\int_{\mathbb{T}}\int_0^\pi v_1^2(\mathcal{L}_1^{-1}-\mathcal{L}_\omega^{-1})v_1^2\sin^2(x)\,đx\,đt.
\end{aligned}$$

Estimate of $A(v_1)$. By the Cauchy–Schwarz inequality, Lemmas 4.3 and 7.11, (4.4), and (5.32), we estimate

$$\begin{aligned}
|A(v_1)|&\leq\ 2\|\Pi_{V_1}((v_1+\mathrm{v}_2)\widetilde{w})\|_{\mathcal{V}^0_{t,x}}\|v_1\|_{\mathcal{V}^0_{t,x}}\\
&\lesssim_\delta\ (N^{\frac{1}{2}+\delta}\|v_1\|_{\mathcal{V}^1_{t,x}}+N^{-\frac{1}{2}-\delta}\|\mathrm{v}_2\|_{\mathcal{V}^{2+2\delta}_{t,x}})\|\widetilde{w}\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_x^{\frac{3}{2}+\delta}}\|v_1\|_{\mathcal{V}^1_{t,x}}\\
&\lesssim_\delta\ \gamma^{-2}RN^{\frac{7}{2}+7\delta}\sqrt{\varepsilon}\|v_1\|^4_{\mathcal{V}^1_{t,x}}.
\end{aligned}\tag{7.28}$$

Estimate of $B(v_1)$. We claim that

$$|B(v_1)|\lesssim_\delta\gamma^{-2}N^{-\frac{3}{2}-\delta}R^2\|v_1\|^4_{\mathcal{V}^1_{t,x}}.\tag{7.29}$$

By Proposition 4.12, (4.4), (7.22), and (5.32),

$$\begin{aligned}|B_1(v_1)| &\lesssim_\delta \gamma^{-1}\|v_1\|^2_{\mathcal{V}^1_{t,x}} N^{-\frac{3}{2}-\delta}\|v_2\|_{\mathcal{V}^{2+2\delta}_{t,x}}(\|v_1\|_{\mathcal{V}^1_{t,x}} + N^{-\frac{3}{2}-\delta}\|v_2\|_{\mathcal{V}^{2+2\delta}_{t,x}})\\ &\lesssim_\delta \gamma^{-2}N^{-\frac{3}{2}-\delta}R^2\|v_1\|^4_{\mathcal{V}^1_{t,x}}.\end{aligned}$$

Similarly, one gets $|B_2(v_1)| \lesssim_\delta \gamma^{-2}N^{-\frac{3}{2}-\delta}R^2\|v_1\|^4_{\mathcal{V}^1_{t,x}}$.

Estimate of $C(v_1)$. Recalling (7.2) and using (2.19), Lemmas 3.1 and 7.11, (5.28), and (5.32) results in

$$|C(v_1)| \lesssim_\delta \gamma^{-2}N^{4(1+2\delta)}\|v_1\|^5_{\mathcal{V}^1_{t,x}} \lesssim_\delta \gamma^{-2}\sqrt{\varepsilon}RN^{4(1+2\delta)}\|v_1\|^4_{\mathcal{V}^1_{t,x}}. \tag{7.30}$$

Estimate of $D(v_1)$. Using (4.27), Lemma 7.12, and (2.19), one has

$$\begin{aligned}|D(v_1)| &\lesssim \|v_1^2\|_{\mathcal{V}^0_{t,x}}\|(\mathcal{L}_\omega^{-1}-\mathcal{L}_1^{-1})v_1^2\|_{\mathcal{V}^0_{t,x}} \lesssim_\delta \|v_1\|^2_{\mathcal{V}^{\frac{3}{2}+\delta}_{t,x}}\varepsilon\gamma^{-1}\|v_1^2\|_{H^1_t\mathcal{H}^0_x}\\ &\lesssim_\delta \varepsilon\gamma^{-1}\|v_1\|^2_{\mathcal{V}^{\frac{3}{2}+\delta}_{t,x}}\|v_1^2\|_{H^1_t\mathcal{H}^{\frac{3}{2}+\delta}_x} \lesssim_\delta \varepsilon\gamma^{-1}\|v_1\|^2_{\mathcal{V}^{\frac{3}{2}+\delta}_{t,x}}\|v_1\|^2_{\mathcal{V}^{\frac{5}{2}+\delta}_{t,x}}\\ &\lesssim_\delta \varepsilon\gamma^{-1}N^{2(\frac{3}{2}+\delta)+2(\frac{1}{2}+\delta)}\|v_1\|^4_{\mathcal{V}^1_{t,x}} \lesssim_\delta \gamma^{-1}\varepsilon N^{4+4\delta}\|v_1\|^4_{\mathcal{V}^1_{t,x}}.\end{aligned} \tag{7.31}$$

Combining estimates (7.28), (7.29), (7.30), and (7.31), one gets $\alpha(\breve{\mathcal{R}}_4) \lesssim_\delta \gamma^{-3}N^{-\frac{3}{2}-\delta}R^2$. ∎

Proof of Theorem 7.2 for $p = 2$. We verify conditions (7.12). Let $\underline{C}$ be the positive constant defined in Theorem 7.5: by Lemmas 7.10 and 7.13, one has

$$\frac{\alpha(\breve{\mathcal{R}}_4)}{m(\breve{\mathcal{G}}_4)} \le \frac{C_\delta\gamma^{-2}N^{-\frac{3}{2}-\delta}R^2}{\frac{5}{24}} \le \underline{C},$$

provided $C_\delta\gamma^{-2}N^{-\frac{3}{2}-\delta}R^2 \le \frac{5\underline{C}}{24}$, which is satisfied due to (5.32). Furthermore,

$$\Big(\frac{|\varepsilon|}{m(\breve{\mathcal{G}}_4)}\Big)^{\frac{1}{2}} \le \Big(\frac{24\varepsilon}{5}\Big)^{\frac{1}{2}} \le \underline{C}\rho_1 \overset{(5.28)}{=} \underline{C}R\varepsilon^{\frac{1}{2}},$$

which is satisfied provided $R \ge R_0 := \underline{C}^{-1}(\frac{24}{5})^{\frac{1}{2}}$. Finally, $\|v_1^{(1)}\|_{\mathcal{V}^1_{t,x}} \asymp \varepsilon^{\frac{1}{2}}$ by (7.13) of Theorem 7.5 and from the fact that $m(\breve{\mathcal{G}}_4)$ is uniformly bounded in N, as stated in Lemma 7.10. ∎

8. Multiplicity of solutions

In this section we prove multiplicity of solutions.

Theorem 8.1 (Multiplicity of solutions with different minimal periods). *For any $p = 2, 3, 5$, there exists a sequence of integers $\{\mathtt{n}_k\}_{k\in\mathbb{N}}$ with $\mathtt{n}_0 := 1$ and $\mathtt{n}_{k+1} > \mathtt{n}_k$ for any k such that the following holds. For any $\delta \in (0, \frac{1}{100})$ and $k_* \in \mathbb{N}_*$, there exist $R_{k_*} > 0$ and $\epsilon_{k_*,\delta,R} > 0$ such that, if R, ε, N are as in the assumptions of Theorem 7.2 and $R \geq R_{k_*}$, and if (5.5) holds with $\epsilon_{\delta,R} := \epsilon_{k_*,\delta,R}$ in the case $p = 5$ (respectively (5.21) and (5.32) in the cases $p = 3$ and $p = 2$), there exist 2π-periodic distinct solutions $u^{(1)}, \dots, u^{(k_*)}$ of the form*

$$u^{(k)} := v_1^{(k)} + \mathtt{v}_2(v_1^{(k)}) + w(v_1^{(k)}) \quad \textit{of} \quad \begin{cases} (3.1) & \textit{if } p = 5,\ p = 2, \\ (3.2) & \textit{if } p = 3, \end{cases} \tag{8.1}$$

with minimal period

$$T_k \in \Big\{ \frac{2\pi}{\mathtt{n}_k - 1}, \dots, \frac{2\pi}{\mathtt{n}_{k-1}} \Big\}, \quad k = 1, \dots, k_*.$$

The following estimates hold:

$$\|v_1^{(k)}\|_{\mathcal{V}^1_{t,z}} \asymp \varepsilon^{\frac{1}{\mathtt{q}}}, \quad \mathtt{q} := \begin{cases} 4 & \textit{if } p = 5, \\ 2 & \textit{if } p = 2,\ p = 3, \end{cases}$$

and

$$\|v_1^{(k)}\|_{\mathcal{V}^1_{t,z}} \leq \rho_1, \quad \|\mathtt{v}_2(v_1^{(k)})\|_{\mathcal{V}^{2+2\delta}_{t,z}} \leq \rho_2,$$

$$\|w(v_1^{(k)})\|_{H_t^{\frac{1}{2}+\delta}\mathcal{H}_z^{\frac{3}{2}+\delta}} \leq \begin{cases} \rho_3 & \textit{if } p = 3,\ p = 5, \\ C_\delta \gamma^{-1} N^{2(1+2\delta)} \varepsilon R^2 & \textit{if } p = 2, \end{cases}$$

with ρ_1, ρ_2, ρ_3 defined in (5.4) if $p = 5$, respectively in (5.20) if $p = 3$ and in (5.28) if $p = 2$.

The remaining part of this section is devoted to proving Theorem 8.1. Since the dependence of the spaces V_1, V_2 on the parameter N plays a significant role, in this section we denote them respectively by $V_{\leq N}$, $V_{>N}$. We regard equations (3.1), (3.2) on the space of $\frac{2\pi}{\mathtt{n}}$-time-periodic functions

$$X_{\mathtt{n}} := \big\{ u(t,z) = \textstyle\sum_{\ell\in\mathbb{N}} \sum_{j\in\mathbb{N}} u_{\ell,j} \cos(\mathtt{n}\ell t)\mathrm{e}_j(z) \big\}.$$

We define the restrictions to $X_{\mathtt{n}}$ of the kernel and range subspaces V, W, $V_{\leq N}$, $V_{>N}$ defined in (3.7), (3.8), (3.9), (3.10):

$$V_{\mathtt{n}} := V \cap X_{\mathtt{n}}, \quad W_{\mathtt{n}} := W \cap X_{\mathtt{n}}, \quad V_{\leq N,\mathtt{n}} := V_{\leq N} \cap X_{\mathtt{n}}, \quad V_{>N,\mathtt{n}} := V_{>N} \cap X_{\mathtt{n}}.$$

We note that for any $\mathtt{n} \in \mathbb{N}_*$ the space $X_{\mathtt{n}}$ is left invariant by both the spatial operator A defined in (3.4) and $\mathcal{L}_\omega$ defined in (3.3).

Lemma 8.2 (Kernel on $\frac{2\pi}{\mathtt{n}}$-periodic functions). *A function $v \in V_{\mathtt{n}}$ if and only if*

$$v = \begin{cases} \displaystyle\sum_{\ell \in \mathbb{N}_*} v_\ell \cos(\mathtt{n}\ell t) e_{\mathtt{n}\ell-1}(x) & \text{if } p = 5,\ p = 2, \\ \displaystyle\sum_{\substack{\ell \in \mathbb{N}_*, \ell \geq \underline{\mu} \\ \mathtt{n}\ell - \underline{\mu} \text{ is even}}} v_\ell \cos(\mathtt{n}\ell t) e^{(\mu_1,\mu_2)}_{\frac{\mathtt{n}\ell-\underline{\mu}}{2}}(\eta) & \text{if } p = 3, \end{cases} \tag{8.2}$$

where $\underline{\mu} := |\mu_1| + |\mu_2| + 1$.

Proof. If $p = 2$ or $p = 5$, the thesis follows by (2.16) and by (3.7). If $p = 3$, it is sufficient to observe that, by (2.16), $\omega_j = 2j + \underline{\mu}$, thus $\ell = 2j + \underline{\mu}$ for some $j \in \mathbb{N}_*$ if and only if $\ell - \underline{\mu}$ is an even positive number. Thus one has $v \in V$ if and only if

$$v(t, \eta) = \sum_{\substack{\ell \in \mathbb{N}, \ell \geq \underline{\mu} \\ \ell - \underline{\mu} \text{ is even}}} v_\ell \cos(\ell t) \mathrm{e}_{\frac{\ell-\underline{\mu}}{2}}(\eta),$$

and (8.2) follows restricting to the indexes ℓ such that $\ell = \mathtt{n}\ell'$ for some $\ell' \in \mathbb{N}_*$. ∎

Remark 8.3. In the case $p = 3$, if $\underline{\mu}$ is odd, then $V_{\mathtt{n}} = \{0\}$ for any $\mathtt{n}$ even, and $V_{\mathtt{n}} \neq \{0\}$ if and only if $\mathtt{n}$ belongs to

$$\mathcal{Z}^{(\mu_1,\mu_2)} := \begin{cases} \mathbb{N}_* & \text{if } \underline{\mu} \text{ is even} \\ \text{odd integers} & \text{if } \underline{\mu} \text{ is odd.} \end{cases}$$

Lemma 8.4. *For any $s < s'$ and any $v \in V_{\mathtt{n}}$, one has $\|v\|_{\mathcal{V}^s_{t,z}} \leq \|v\|_{\mathcal{V}^{s'}_{t,z}} \mathtt{n}^{s-s'}$.*

Proof. One has

$$\|v\|^2_{\mathcal{V}^s_{t,z}} = \sum_{\ell \in \mathbb{N}} |v_\ell|^2 (\mathtt{n}\ell)^{2s} = \mathtt{n}^{2s} \sum_{\ell \in \mathbb{N}} |v_\ell|^2 \ell^{2s} \leq \mathtt{n}^{2s} \sum_{\ell \in \mathbb{N}} |v_\ell|^2 \ell^{2s'} = \mathtt{n}^{2(s-s')} \|v\|^2_{\mathcal{V}^{s'}_{t,z}}. \quad ∎$$

We look for $\frac{2\pi}{\mathtt{n}}$-periodic solutions of (3.12)–(3.14). The Lyapunov–Schmidt decomposition defined in Sections 5 and 6 preserves the spaces of $\frac{2\pi}{\mathtt{n}}$-periodic functions:

Lemma 8.5. *Given $\mathtt{n} \in \mathbb{N}_*$, let δ, ρ_1, ρ_2, ρ_3, ε, $N \geq \mathtt{n}$ and R be as in the assumptions of Propositions 5.1, 6.1 if $p = 5$, resp. Propositions 5.4, 6.4 if $p = 3$, and Propositions 5.6 and 6.5 if $p = 2$. For any $v_1 \in V_{\leq N,\mathtt{n}} \cap \mathcal{D}_{\rho_1}$ let $\mathtt{v}_2(v_1)$ and $w(v_1)$ be the solutions to (3.13) and (3.14); then $\mathtt{v}_2(v_1) \in V_{>N,\mathtt{n}}$ and $w(v_1) \in W_{\mathtt{n}}$.*

Proof. The functions v_2 and w are respectively obtained as the fixed point of the contractions $\mathcal{T}_{v_1,w(v_1)}$ and $\mathcal{T}_{v_1}$, defined in (5.10) and (6.4) in the case $p = 5$ (the cases $p = 3$, $p = 2$ are analogous). Then the lemma follows by observing that, for any $v_1 \in V_{\leq N,\mathtt{n}} \cap \mathcal{D}_{\rho_1}$ and $w \in W_{\mathtt{n}} \cap \mathcal{D}^W_{\rho_3}$, the operator $\mathcal{T}_{v_1,w}$ maps $V_{>N,\mathtt{n}}$ into itself, and that for any $v_1 \in V_{\leq N,\mathtt{n}} \cap \mathcal{D}_{\rho_1}$, the operator $\mathcal{T}_{v_1}$ maps $W_{\mathtt{n}}$ into itself. ∎

In order to find $\frac{2\pi}{\mathtt{n}}$-periodic solutions of (7.1), we look for critical points of

$$\breve{\Psi}_{\mathtt{n}} := \breve{\Psi}|_{V_{\leq N,\mathtt{n}}\cap\mathcal{D}_{\rho_1}}.$$

We recall that $\breve{\Psi}$ has expansion (7.5) in the cases $p=5$, $p=3$ and (7.9) in the case $p=2$.

Proposition 8.6 (Critical point with minimal period). *For any* $\mathtt{n}\in\mathbb{N}_*$ *if* $p=2$ *or* $p=5$, *resp.* $\mathtt{n}\in\mathcal{Z}^{(\mu_1,\mu_2)}$ *if* $p=3$, *define*

$$\alpha_{\mathtt{n}}(\mathcal{R}) := \alpha(\mathcal{R}|_{V_{\leq N,\mathtt{n}}\cap\mathcal{D}_{\rho_1}}),\quad m_{\mathtt{n}}(\mathcal{G}) := m(\mathcal{G}|_{V_{\leq N,\mathtt{n}}\cap\mathcal{D}_{\rho_1}}), \tag{8.3}$$

where

$$\mathcal{G} := \begin{cases} \mathcal{G}_{p+1} \text{ defined in (3.6)} & \text{if } p=5,\ p=3,\\ \breve{\mathcal{G}}_4 \text{ defined in (7.10)} & \text{if } p=2,\end{cases}$$

$$\mathcal{R} := \begin{cases} \mathcal{R}_{p+1} \text{ as in (7.5)} & \text{if } p=5,\ p=3,\\ \breve{\mathcal{R}}_4 \text{ as in (7.9)} & \text{if } p=2,\end{cases}$$

and $m(\cdot)$ *and* $\alpha(\cdot)$ *are defined in Theorem 7.5. Suppose that there exist* $\beta\in(0,1)$ *and* $\mathtt{m}_0>0$ *such that for any* $\mathtt{m}>\mathtt{m}_0$,

$$\sup_{v_1\in V_{\leq N,\mathtt{nm}}\setminus\{0\}}\frac{\mathcal{G}(v_1)}{\|v_1\|^q_{\mathcal{V}^1_{t,z}}}\leq\beta\sup_{v_1\in V_{\leq N,\mathtt{n}}\setminus\{0\}}\frac{\mathcal{G}(v_1)}{\|v_1\|^q_{\mathcal{V}^1_{t,z}}},\quad q:=\begin{cases}p+1 & \text{if } p=3,\ p=5,\\ 4 & \text{if } p=2.\end{cases} \tag{8.4}$$

Then there exist positive constants $\underline{C}$ *and* $C_1:=C_1(\beta)$ *such that, if*

$$\alpha_{\mathtt{n}}(\mathcal{R})\leq C_1 m_{\mathtt{n}}(\mathcal{G}),\quad \Big(\frac{|\varepsilon|}{m_{\mathtt{n}}(\mathcal{G})}\Big)^{\frac{1}{q-2}}\leq\underline{C}\rho_1, \tag{8.5}$$

the functional $\breve{\Psi}$ *has a critical point* $v_1^{(\mathtt{n})}$ *with minimal period* $T_{\mathtt{n}}\in\{\frac{2\pi}{\mathtt{m}_0\mathtt{n}},\dots,\frac{2\pi}{\mathtt{n}}\}$, *satisfying*

$$\|v_1^{(\mathtt{n})}\|_{\mathcal{V}^1_{t,z}}\asymp\varepsilon^{\frac{1}{q-2}}.$$

Furthermore, $v_1^{(\mathtt{n})}$ *is also a critical point of* $\breve{\Psi}$.

Proof. By Theorem 7.5 and (8.5), $\breve{\Psi}_{\mathtt{n}}$ admits a critical point $v_1^{(\mathtt{n})}\in V_{\leq N,\mathtt{n}}$ which is proportional to a point $y^{(\mathtt{n})}$ satisfying

$$\|y^{(\mathtt{n})}\|_{\mathcal{V}^1_{t,z}}=1,\quad \mathcal{G}(y^{(\mathtt{n})})=m_{\mathtt{n}}(\mathcal{G})+r^{(\mathtt{n})},\quad r^{(\mathtt{n})}=\mathcal{O}(\alpha_{\mathtt{n}}(\mathcal{R})). \tag{8.6}$$

We note that, since $r^{(\mathtt{n})}=\mathcal{O}(\alpha_{\mathtt{n}}(\mathcal{R}))$, provided $\frac{\alpha_{\mathtt{n}}(\mathcal{R})}{m_{\mathtt{n}}(\mathcal{G})}<C$ with $C=C(\beta)$ small enough, one has $(1-\beta)m_{\mathtt{n}}(\mathcal{G})>|r^{(\mathtt{n})}|$, which by (8.6) gives

$$\mathcal{G}(y^{(\mathtt{n})})=m_{\mathtt{n}}(\mathcal{G})+r^{(\mathtt{n})}\geq m_{\mathtt{n}}(\mathcal{G})-|r^{(\mathtt{n})}|>\beta m_{\mathtt{n}}(\mathcal{G}).$$

Combining the latter inequality with hypothesis (8.4), one gets

$$\frac{\mathcal{G}(y^{(\mathtt{n})})}{\|y^{(\mathtt{n})}\|^q_{\mathcal{V}^1_{t,z}}} > \beta m_{\mathtt{n}}(\mathcal{G}) \geq \sup_{v_1 \in V_{\leq N,\mathtt{nm}}\setminus\{0\}} \frac{\mathcal{G}(v_1)}{\|v_1\|^q_{\mathcal{V}^1_{t,z}}},$$

thus for any $\mathtt{m} > \mathtt{m}_0$ one has that $y^{(\mathtt{n})}$ belongs to $V_{\leq N,\mathtt{n}} \subset V$ but $y^{(\mathtt{n})}$ does not belong to $V_{\leq N,\mathtt{mn}}$, namely $y^{(\mathtt{n})}$ has minimal period $\geq \frac{2\pi}{\mathtt{m}_0\mathtt{n}}$. Since $v_1^{(\mathtt{n})}$ and $y^{(\mathtt{n})}$ are proportional, the same holds for $v_1^{(\mathtt{n})}$. It remains to prove that $v_1^{(\mathtt{n})}$ is also a critical point for the functional $\breve{\Psi}$. To fix ideas, we prove the result for $p = 5$. The cases $p = 3$, $p = 2$ follow analogously. By Lemma 7.1, a point $v_1 \in \mathcal{D}_{\rho_1}$ is critical for $\breve{\Psi}$ if and only if

$$\int_{\mathbb{T}} \int_0^\pi \big(A v_1 + \Pi_{V_{\leq N}}(v_1 + \mathtt{v}_2(v_1) + w(v_1))^5\big) h \sin^2(x)\, đx\, đt = 0 \quad \forall h \in V_{\leq N}. \tag{8.7}$$

Since $v_1^{(\mathtt{n})}$ is critical for $\breve{\Psi}_{\mathtt{n}}$, one already has that (8.7) holds for $h \in V_{\leq N,\mathtt{n}}$, thus it remains to prove it for $h \in V_{\leq N} \cap V^\perp_{\leq N,\mathtt{n}}$. Then we observe that, by Lemma 8.5, $\mathtt{v}_2(v_1^{(\mathtt{n})}) \in V_{>N,\mathtt{n}}$ and $w(v_1^{(\mathtt{n})}) \in W_{\mathtt{n}}$, since $v_1^{(\mathtt{n})} \in V_{\leq N,\mathtt{n}}$. Thus $A v_1^{(\mathtt{n})} + \Pi_{V_{\leq N}}(v_1^{(\mathtt{n})} + \mathtt{v}_2(v_1^{(\mathtt{n})}) + w(v_1^{(\mathtt{n})}))^5$ belongs to $V_{\leq N,\mathtt{n}}$, namely it is orthogonal to any $h \in V_{\leq N} \cap V^\perp_{\leq N,\mathtt{n}}$, which gives the thesis. ∎

Theorem 8.1 follows from an iterative application of Lemma 8.5 and Proposition 8.6. In the next sections we verify the assumptions (8.4), arguing separately for the cases $p = 5$, $p = 3$, $p = 2$.

8.1. Cases $p = 5$ and $p = 3$

We start with $p = 5$ and we prove lower and upper bounds for $m_{\mathtt{n}}(\mathcal{G}_6)$ defined in (8.3).

Lemma 8.7 (Estimate of $m_{\mathtt{n}}(\mathcal{G}_6)$). *For any $\delta > 0$ there exists $C_\delta > 0$ and for any $\mathtt{n} \in \mathbb{N}_*$ there exists $\kappa_{\mathtt{n}} > 0$ such that*

$$\kappa_{\mathtt{n}} \leq m_{\mathtt{n}}(\mathcal{G}_6) \leq C_\delta \mathtt{n}^{-1+6\delta}. \tag{8.8}$$

Proof. We take $v_{\mathtt{n}} = \cos(\mathtt{n}t) e_{\mathtt{n}-1}(x)$, then $\|v_{\mathtt{n}}\|_{\mathcal{V}^1_{t,x}} = \mathtt{n}$. One has

$$\mathcal{G}_6(v_{\mathtt{n}}) = \frac{1}{6} \int_{\mathbb{T}} \cos^6(\mathtt{n}t)\, đt \int_0^\pi e^6_{\mathtt{n}-1}(x) \sin^2(x)\, đx =: \alpha_{\mathtt{n}} > 0,$$

and the lower bound in (8.8) follows setting $\kappa_{\mathtt{n}} := \frac{\alpha_{\mathtt{n}}}{\mathtt{n}^6}$. The upper bound follows because for any $v \in V_{\mathtt{n}}$, by Proposition 4.5 and Lemma 8.4 we have

$$\mathcal{G}_6(v) \lesssim_\delta \|v\|^6_{\mathcal{V}^{\frac{5}{6}+\delta}_{t,x}} \lesssim_\delta \mathtt{n}^{-1+6\delta} \|v\|^6_{\mathcal{V}^1_{t,x}}.$$

∎

Proof of Theorem 8.1 *for* $p = 5$. By Lemma 8.7 with $\delta = \frac{1}{12}$, there exist $C > 0$, $\kappa_{\mathtt{n}} > 0$ such that

$$\sup_{v \in V_{\mathtt{nm}} \setminus \{0\}} \frac{\mathcal{G}_6(v)}{\|v\|^6_{\mathcal{V}^1_{t,\eta}}} \leq \frac{C}{(\mathtt{nm})^{\frac{1}{2}}} \leq \frac{1}{2}\kappa_{\mathtt{n}} \leq \frac{1}{2} m_{\mathtt{n}}(\mathcal{G}_6),$$

provided $\mathtt{m} \geq \mathtt{m}_0(\mathtt{n}) := \frac{1}{\mathtt{n}}(\frac{C}{2\kappa_{\mathtt{n}}})^2$. This proves that for any $\mathtt{n}, \mathtt{m} \in \mathbb{N}$ there exists $\mathtt{m}_0 = \mathtt{m}_0(\mathtt{n}) \in \mathbb{N}$ such that, if $\mathtt{m} > \mathtt{m}_0(\mathtt{n})$ and $N \geq \mathtt{n}$, one has

$$\sup_{v \in V_{\mathtt{mn}} \setminus \{0\}} \frac{\mathcal{G}_6(v)}{\|v\|^6_{\mathcal{V}^1_{t,\eta}}} \leq \frac{1}{2} \sup_{v_1 \in V_{\leq N, \mathtt{n}} \setminus \{0\}} \frac{\mathcal{G}_6(v_1)}{\|v_1\|^6_{\mathcal{V}^1_{t,\eta}}},$$

namely (8.4) is satisfied. We then define $\mathtt{n}_0 := 1$, $\mathtt{n}_{k+1} := \mathtt{m}_0(\mathtt{n}_k)\mathtt{n}_k + 1$ and we apply Proposition 8.6 with $\mathtt{n} = \mathtt{n}_k$ for any $k = 1, \dots, k_*$. In particular, assumptions (8.5) hold for any $\mathtt{n}_k$, observing that $\alpha_{\mathtt{n}_k}(\mathcal{R}_6) \leq \alpha(\mathcal{R}_6) \lesssim_\delta N^{-\frac{7}{6}+9\delta} R^4$ by Lemma 7.7 and $m_{\mathtt{n}_k}(\mathcal{G}_6) \geq \kappa_{\mathtt{n}}$ by Lemma 8.7, and assuming that $R \geq R_{k_*} = \max_k(\kappa_{\mathtt{n}_k} \underline{C})^{-1}$ and $N \lesssim_\delta \inf_k (R^4 \kappa_{\mathtt{n}_k})^{1/(\frac{7}{6}-9\delta)}$, which is ensured by (5.5). Thus by Proposition 8.6 the functional $\breve{\Psi}$ admits a critical point $v_1^{(k)} := v_1^{(\mathtt{n}_k)}$ with minimal period $T_{\mathtt{n}_k} \in \{\frac{2\pi}{\mathtt{m}_0(\mathtt{n}_k)\mathtt{n}_k}, \dots, \frac{2\pi}{\mathtt{n}_k}\}$. Finally, by Lemma 8.5, $u^{(k)} := v_1^{(k)} + \mathtt{v}_2(v_1^{(k)}) + w(v_1^{(k)})$ has the same minimal period $T_{\mathtt{n}_k}$. ∎

Remark 8.8. With careful estimates on $m(\mathcal{G}_6)$ one can obtain $\mathtt{n}_{k+1} = 3\mathtt{n}_k + 1$ for any k.

For $p = 3$, defining $m_{\mathtt{n}}(\mathcal{G}_4)$ as in (8.3), we prove the following:

Lemma 8.9 (Estimate of $m_{\mathtt{n}}(\mathcal{G}_4)$). *There exists $C > 0$ and for any $\mathtt{n} \in \mathcal{Z}^{(\mu_1,\mu_2)}$ there exists $\kappa_{\mathtt{n}} := \kappa_{\mathtt{n}}(\mu_1, \mu_2) > 0$ such that*

$$\kappa_{\mathtt{n}} \leq m_{\mathtt{n}}(\mathcal{G}_4) \leq C\mathtt{n}^{-1}. \tag{8.9}$$

Proof. We take $v_{\mathtt{n}}(t, \eta) = \cos(\mathtt{n}\underline{\mu} t) e_{\frac{(\mathtt{n}-1)\underline{\mu}}{2}}(\eta)$. Then $\|v_{\mathtt{n}}\|^4_{\mathcal{V}^1_{t,\eta}} = \mathtt{n}^4 \underline{\mu}^4$ and

$$\mathcal{G}_4(v_{\mathtt{n}}) = \frac{1}{4} \int_{\mathbb{T}} \cos^4(\mathtt{n}\underline{\mu} t)\, đt \int_0^{\frac{\pi}{2}} \big(e^{(\mu_1,\mu_2)}_{\frac{\mathtt{n}\ell - \underline{\mu}}{2}}(\eta)\big)^4 \sin(2\eta)\, d\eta := \alpha_{\mathtt{n}} > 0.$$

Then the lower bound in (8.9) follows with $\kappa_{\mathtt{n}}(\mu_1, \mu_2) := \frac{\alpha_{\mathtt{n}}}{\mathtt{n}^4 \underline{\mu}^4}$. For the upper bound we observe that, by Lemmas 2.3 and 4.1, for any $\mathtt{n} \in \mathcal{Z}^{(\mu_1,\mu_2)}$ and $v \in V_{\mathtt{n}}$ there exists $C > 0$ such that

$$\mathcal{G}_4(v) = \frac{1}{4} \int_{\mathbb{T}} \int_0^{\frac{\pi}{2}} v^4(t, \eta) \sin(2\eta)\, d\eta\, đt \leq \frac{C}{4} \|v\|^4_{\mathcal{V}^{\frac{3}{4}}_{t,\eta}} \leq \frac{C}{4} \mathtt{n}^{-1} \|v\|^4_{\mathcal{V}^1_{t,\eta}},$$

by Lemma 8.4. ∎

Proof of Theorem 8.1 *for* $p = 3$. By Lemma 8.9 there exist $C > 0$ and $\kappa_{\mathtt{n}} := \kappa_{\mathtt{n}}(\mu_1, \mu_2)$ such that

$$\sup_{v \in V_{\mathtt{nm}} \setminus \{0\}} \frac{\mathcal{G}_4(v)}{\|v\|^4_{\mathcal{V}^1_{t,\eta}}} \leq \frac{C}{4\mathtt{nm}} \leq \frac{1}{2}\kappa_{\mathtt{n}} \leq \frac{1}{2} m_{\mathtt{n}}(\mathcal{G}_4),$$

provided $\mathtt{m} \geq \mathtt{m}_0(\mathtt{n}) := \lfloor \frac{C}{2\mathtt{n}\kappa_\mathtt{n}} \rfloor + 1$. Then for any $\mathtt{n}, \mathtt{m} \in \mathbb{N}$ there exists $\mathtt{m}_0 = \mathtt{m}_0(\mathtt{n}) \in \mathbb{N}$ such that, if $\mathtt{m} > \mathtt{m}_0$ and $N \geq \mathtt{n}$, one has

$$\sup_{v \in V_{\mathtt{mn}} \setminus \{0\}} \frac{\mathcal{G}_4(v)}{\|v\|^4_{\mathcal{V}^1_{t,\eta}}} \leq \frac{1}{2} \sup_{v_1 \in V_{\leq N,\mathtt{n}} \setminus \{0\}} \frac{\mathcal{G}_4(v_1)}{\|v_1\|^4_{\mathcal{V}^1_{t,\eta}}};$$

namely for any $\mathtt{n}$ there exists $\mathtt{m}_0(\mathtt{n}) \in \mathbb{N}_*$ such that (8.4) is satisfied for any $\mathtt{m} > \mathtt{m}_0(\mathtt{n})$. We then define $\mathtt{n}_0 := 1$, $\mathtt{n}_{k+1} := \mathtt{m}_0(\mathtt{n}_{k-1})\mathtt{n}_k + 1$ and Theorem 8.1 follows by Proposition 8.6, with $\mathtt{n} = \mathtt{n}_k$ for any $k = 1, \dots, k_*$. ∎

8.2. Case $p = 2$

Lemma 8.10 (Estimate of $m_\mathtt{n}(\breve{\mathcal{G}}_4)$). *For any $\delta > 0$ there exist $C_\delta, \kappa > 0$ and $\underline{\mathtt{n}} > 0$ such that for any $\mathtt{n} \geq \underline{\mathtt{n}}$ and $N \geq \mathtt{n}$,*

$$\frac{\kappa}{\mathtt{n}^4} \leq m_\mathtt{n}(\breve{\mathcal{G}}_4) := -\inf_{v_1 \in V_{\leq N,\mathtt{n}} \setminus \{0\}} \frac{\breve{\mathcal{G}}_4(v_1)}{\|v_1\|^4_{\mathcal{V}^1_{t,x}}} \leq \frac{C_\delta}{\mathtt{n}^{2-4\delta}}. \tag{8.10}$$

Proof. Let $\bar{v}_\mathtt{n} := \cos(\mathtt{n}t)e_{\mathtt{n}-1}(x)$. We compute $\breve{\mathcal{G}}_4(\bar{v}_\mathtt{n})$. By (7.10), (2.6), using (3.15) and Lemma 2.1, one has

$$\begin{aligned}\breve{\mathcal{G}}_4(\bar{v}_\mathtt{n}) &= \frac{1}{8}\Big(\frac{1}{4\mathtt{n}}\sum_{k=0}^{\mathtt{n}-1}\frac{1}{2\mathtt{n}+2k+1} + \frac{1}{8\mathtt{n}}\sum_{k=0}^{\mathtt{n}-1}\frac{1}{2\mathtt{n}-2k-1} - \sum_{k=0}^{\mathtt{n}-1}\frac{2}{(2k+1)^2}\Big)\\ &=: \frac{1}{8}(S_1(\mathtt{n}) + S_2(\mathtt{n}) + S_3(\mathtt{n})).\end{aligned}$$

One has

$$S_1(\mathtt{n}) + S_2(\mathtt{n}) + S_3(\mathtt{n}) \leq \frac{1}{8\mathtt{n}} + \frac{\ln(\mathtt{n})}{\mathtt{n}} - 2 \leq -1$$

if $\mathtt{n} \geq \underline{\mathtt{n}}$ large enough. Thus we conclude that

$$m_\mathtt{n}(\breve{\mathcal{G}}_4) = -\inf_{v_1 \in V_{\leq N,\mathtt{n}} \setminus \{0\}} \frac{\breve{\mathcal{G}}_4(v_1)}{\|v_1\|^4_{\mathcal{V}^1_{t,x}}} \geq -\frac{\breve{\mathcal{G}}_4(\bar{v}_\mathtt{n})}{\|\bar{v}_\mathtt{n}\|^4_{\mathcal{V}^1_{t,x}}} \gtrsim \frac{1}{\mathtt{n}^4}.$$

This proves the lower bound in (8.10). To prove the upper bound we observe that by Proposition 4.12 and Lemma 8.4, for any $v \in V_\mathtt{n} \setminus \{0\}$,

$$-\breve{\mathcal{G}}_4(v) \leq |\breve{\mathcal{G}}_4(v)| = \frac{1}{2}\Big|\int_\mathbb{T}\int_0^\pi v^2 \mathcal{L}_1^{-1} v^2 \sin^2(x)\, đx\, đt\Big| \lesssim_\delta \|v\|^4_{\mathcal{V}^{\frac{1}{2}+\delta}_{t,x}} \lesssim_\delta \mathtt{n}^{-2-4\delta}\|v\|^4_{\mathcal{V}^1_{t,x}}.$$

Then the second estimate in (8.10) follows since

$$-\inf_{v \in V_\mathtt{n} \setminus \{0\}} \frac{\breve{\mathcal{G}}_4(v)}{\|v\|^4_{\mathcal{V}^1_{t,x}}} = \sup_{v \in V_\mathtt{n} \setminus \{0\}} \Big(-\frac{\breve{\mathcal{G}}_4(v)}{\|v\|^4_{\mathcal{V}^1_{t,x}}}\Big).$$

This completes the proof. ∎

Proof of Theorem 8.1 for $p = 2$. By Lemma 8.10 with $\delta = \frac{1}{8}$ there exists $\underline{\mathtt{n}} \in \mathbb{N}$ such that for any $\mathtt{n} \geq \underline{\mathtt{n}}$ there exist $C > 0$ and $\kappa_{\mathtt{n}} > 0$ such that

$$\sup_{v \in V_{\mathtt{nm}} \setminus \{0\}} \frac{\breve{\mathscr{G}}_4(v)}{\|v\|^4_{\mathcal{V}^1_{t,\eta}}} \leq \frac{C}{(\mathtt{nm})^{\frac{3}{2}}} \leq \frac{1}{2}\kappa_{\mathtt{n}} \leq \frac{1}{2} m_{\mathtt{n}}(\breve{\mathscr{G}}_4),$$

provided $\mathtt{m} \geq \mathtt{m}_0(\mathtt{n})$, with $\mathtt{m}_0(\mathtt{n})$ such that

$$\frac{C}{\mathtt{n}^{\frac{3}{2}} \mathtt{m}_0(\mathtt{n})^{\frac{3}{2}}} \leq \frac{1}{2}\kappa_{\mathtt{n}}.$$

Then for any $\mathtt{n} \geq \underline{\mathtt{n}}$ and $\mathtt{m} \in \mathbb{N}$ there exists $\mathtt{m}_0 = \mathtt{m}_0(\mathtt{n}) \in \mathbb{N}$ such that, if $\mathtt{m} > \mathtt{m}_0$ and $N \geq \mathtt{n}$, one has

$$\sup_{v \in V_{\mathtt{mn}} \setminus \{0\}} \frac{\mathscr{G}_4(v)}{\|v\|^4_{\mathcal{V}^1_{t,\eta}}} \leq \frac{1}{2} \sup_{v \in V_{\leq N, \mathtt{n}} \setminus \{0\}} \frac{\mathscr{G}_4(v)}{\|v\|^4_{\mathcal{V}^1_{t,\eta}}};$$

namely for any $\mathtt{n} \geq \underline{\mathtt{n}}$ there exists $\mathtt{m}_0(\mathtt{n}) \in \mathbb{N}_*$ such that (8.4) is satisfied with $\beta = \frac{1}{2}$ for any $\mathtt{m} > \mathtt{m}_0(\mathtt{n})$. We then define $\mathtt{n}_0 := 1$, $\mathtt{n}_1 := \mathtt{m}_0(\underline{\mathtt{n}}) + \underline{\mathtt{n}} + 1$, $\mathtt{n}_{k+1} := \mathtt{m}_0(\mathtt{n}_{k-1})\mathtt{n}_k + 1$, and Theorem 8.1 follows by Proposition 8.6 with $\mathtt{n} = \mathtt{n}_k$ for any $k = 1, \dots, k_*$. ∎

9. Strong solutions

In this section we prove higher regularity of the solutions found in Theorem 8.1.

Theorem 9.1 (Regularity). *Let R, ε, N as in the assumptions of Theorem 8.1 and for any $k_* \in \mathbb{N}_*$ let $\{u^{(k)}\}_{k=1}^{k_*}$ be the functions in (8.1). Then for any $r > \frac{1}{2}$, $s > \frac{3}{2}$, there exist $\epsilon_{r,s,k_*,R} > 0$, $\zeta_{r,s} > 0$, $\mathtt{A}_{r,s} > 0$ and $\mathtt{B}_{r,s} > 0$ such that we have the following:*

Cases $p = 3, 5$: *If $\gamma^{-1}\varepsilon N^{\zeta_{r,s}} \leq \epsilon_{r,s,k_*,R}$, each solution $u^{(k)}$ in (8.1) belongs to $H^r_t \mathcal{H}^s_z$ and*

$$\begin{gathered} \|v_1^{(k)}\|_{\mathcal{V}^{r+s}_{t,z}} \leq C_{1,R,r,s,k_*} \varepsilon^{\frac{1}{p-1}} N^{r+s-1}, \quad \|\mathtt{v}_2(v_1^{(k)})\|_{\mathcal{V}^{r+s}_{t,z}} \leq C_{2,R,r,s,k_*} \varepsilon^{\frac{1}{p-1}} N^{\mathtt{A}_{r,s}}, \\ \|w(v_1^{(k)})\|_{H^r_t \mathcal{H}^s_z} \leq C_{3,R,r,s,k_*} \gamma^{-1} \varepsilon^{\frac{p}{p-1}} N^{\mathtt{B}_{r,s}}, \end{gathered} \tag{9.1}$$

for some positive constants C_{l,R,r,s,k_*}, $l = 1, 2, 3$.

Case $p = 2$: *If $\gamma^{-2}\varepsilon N^{\zeta_{r,s}} \leq \epsilon_{r,s,k_*,R}$, each solution $u^{(k)}$ in (8.1) belongs to $H^r_t \mathcal{H}^s_x$ and*

$$\begin{gathered} \|v_1^{(k)}\|_{\mathcal{V}^{r+s}_{t,x}} \leq C_{1,R,r,s,k_*} \varepsilon^{\frac{1}{2}} N^{r+s-1}, \quad \|\mathtt{v}_2(v_1^{(k)})\|_{\mathcal{V}^{r+s}_{t,x}} \leq C_{2,R,r,s,k_*} \varepsilon^{\frac{1}{2}} N^{\mathtt{A}_{r,s}}, \\ \|w(v_1^{(k)})\|_{H^r_t \mathcal{H}^s_x} \leq C_{3,R,r,s,k_*} \varepsilon N^{\mathtt{B}_{r,s}} \end{gathered}$$

for some positive constants C_{l,R,r,s,k_*}, $l = 1, 2, 3$.

Theorem 9.1 immediately implies Theorems 1.2 and 1.4. Let us prove Theorem 1.2. Theorem 1.4 follows in analogous way.

Proof of Theorem 1.2. We prove it for $p = 5$; the case $p = 2$ follows similarly. Let n, r, s, and $\mathtt{d}$ be as in the assumptions of Theorem 1.2. For any $k = 1, \dots, k_* = n$ define $\phi_\varepsilon^{(k)}(t,x) := u^{(k)}(\omega_\varepsilon t, x)$, $v_\varepsilon^{(k)} := v_1^{(k)} + \mathtt{v}_2(v_1^{(k)})$, for any $k = 1, \dots, n$. Then, recalling that $N := \varepsilon^{-\frac{1}{\beta}}$ (see (5.4)), it is sufficient to choose $\beta := \beta(r,s,\mathtt{d}) > 1$ such that

$$\varepsilon^{\frac14} N^{\max\{r+s-1, \mathtt{A}_{r,s}\}} := \varepsilon^{\frac14 - \frac{\max\{r+s-1,\mathtt{A}_{r,s}\}}{\beta_{r,s,\mathtt{d}}}} \le \varepsilon^{\frac14 - \mathtt{d}}, \tag{9.2}$$

and the upper bound in (1.8), (1.9) follows from (9.1). As a consequence of (9.2), (9.1), and (7.4), one has $\varepsilon^{\frac14} \lesssim \|v_\varepsilon^{(k)}\|_{H_t^r \mathcal{H}_z^s} \lesssim \varepsilon^{\frac14} N^{\max\{r+s-1,\mathtt{A}_{r,s}\}} \lesssim \varepsilon^{\frac14 - \mathtt{d}}$, proving (1.8) and (1.9). ∎

Theorem 9.1 is a consequence of the iterative application of the following two lemmas.

Lemma 9.2 (Regularity bootstrap for $\mathtt{v}_2$). *Let $r > \frac12$ and $s > \frac32$. Assume that $v_1 \in \mathcal{D}_{\rho_1}$, $\mathtt{v}_2 := \mathtt{v}_2(v_1, w(v_1))$, and $w := w(v_1)$ satisfy*

$$\|v_1\|_{\mathcal{V}^s_{t,z}} \le \tilde\rho_1, \quad \|\mathtt{v}_2\|_{\mathcal{V}^s_{t,z}} \le \tilde\rho_2, \tag{9.3}$$

$$\|w\|_{H_t^r \mathcal{H}_z^s} \le \max\{\tilde\rho_1, \tilde\rho_2\}. \tag{9.4}$$

Then $\mathtt{v}_2$ belongs to $\mathcal{V}^{s+2}_{t,z}$ and

$$\|\mathtt{v}_2\|_{\mathcal{V}^{s+2}_{t,z}} \lesssim_{r,s} \begin{cases} \varepsilon^{-1} \max\{\tilde\rho_1, \tilde\rho_2\}^p & \text{if } p = 3,\ p = 5, \\ \varepsilon^{-1} \max\{\tilde\rho_1, \tilde\rho_2\} \|w\|_{H_t^r \mathcal{H}_x^s} & \text{if } p = 2. \end{cases} \tag{9.5}$$

Proof. For the cases $p = 5, 3$, since $\mathtt{v}_2$ solves (5.9), resp. (5.25), (4.5), Lemma 4.3, (9.3), we get

$$\|\mathtt{v}_2\|_{\mathcal{V}^{s+2}_{t,z}} \lesssim_{s,r} \varepsilon^{-1} \sum_{j_1+j_2+j_3=p} \tilde\rho_1^{j_1} \tilde\rho_2^{j_2} \|w\|^{j_3}_{H_t^r \mathcal{H}_x^s} \overset{(9.4)}{\lesssim_{s,r}} \varepsilon^{-1} \max\{\tilde\rho_1, \tilde\rho_2\}^p,$$

which gives (9.5). If $p = 2$ then $\mathtt{v}_2 = \varepsilon^{-1} A^{-1} \Pi_{V_2}((2(v_1 + \mathtt{v}_2) + w)w)$ (cf. (5.30)) and estimate (9.5) follows similarly. ∎

Lemma 9.3 (Regularity bootstrap for w). *Let $\lambda > \frac12$, $\mu > \frac32$. There exists $K_{\lambda,\mu,p} > 0$, depending only on p and on the algebra constant $C_{\lambda,\mu}$ in (2.19), such that, if $v_1 \in \mathcal{D}_{\rho_1}$ and $\mathtt{v}_2(v_1)$ defined in (7.1) satisfy*

$$\gamma^{-1} \max\{\|v_1\|_{\mathcal{V}^{\lambda+\mu}_{t,z}}, \|\mathtt{v}_2(v_1)\|_{\mathcal{V}^{\lambda+\mu}_{t,z}}\}^{p-1} < K_{\lambda,\mu,p}, \tag{9.6}$$

then

(1) *if $p = 5$, resp. $p = 3$, then the solution $w(v_1)$ of (6.1) found in Proposition 6.1 (resp. Proposition 6.4) belongs to $H_t^\lambda \mathcal{H}_z^\mu$ and*

$$\|w(v_1)\|_{H_t^\lambda \mathcal{H}_z^\mu} \le K^{-1}_{\lambda,\mu,p} \gamma^{-1} \max\{\|v_1\|_{\mathcal{V}^{\lambda+\mu}_{t,z}}, \|\mathtt{v}_2(v_1)\|_{\mathcal{V}^{\lambda+\mu}_{t,z}}\}^p; \tag{9.7}$$

(2) *if $p = 2$ then the function $w(v_1) = \widetilde{w}(v_1) + \mathcal{L}_\omega^{-1}(v_1 + \mathrm{v}_2(v_1))^2$, where $\widetilde{w}(v_1)$ is the solution of (6.8), found in Proposition 6.5 belongs to $H_t^\lambda \mathcal{H}_x^\mu$ and*

$$\|w(v_1)\|_{H_t^\lambda \mathcal{H}_x^\mu} \leq \frac{1}{2K_{\lambda,\mu,2}} \gamma^{-1} \max\{\|v_1\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}, \|\mathrm{v}_2(v_1)\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}\}^2. \tag{9.8}$$

Proof. For brevity we denote $\mathrm{v}_2 := \mathrm{v}_2(v_1)$ and $w := w(v_1)$. Let $p = 3, 5$.

Step 1: The sequence $(\bar{w}_k)_{k\in\mathbb{N}}$ defined by $\bar{w}_0 := 0$ and $\bar{w}_{k+1} := \mathcal{L}_\omega^{-1}\Pi_W((v_1 + \mathrm{v}_2 + \bar{w}_k)^p)$ has limit $\lim_{k\to\infty} \bar{w}_k = w$ in $H_t^{\frac{1}{2}+\delta} \mathcal{H}_z^{\frac{3}{2}+\delta}$. In fact, arguing as in Lemma 6.2, the map $\bar{w} \mapsto \mathcal{L}_\omega^{-1}\Pi_W((v_1 + \mathrm{v}_2 + \bar{w})^p)$ is a contraction on $\mathcal{D}_{\rho_3}^W$. Hence it admits a unique fixed point $\hat{w} = \lim_{k\to\infty} \bar{w}_k$ in $H_t^{\frac{1}{2}+\delta} \mathcal{H}_z^{\frac{3}{2}+\delta}$ satisfying the equation $\hat{w} = \mathcal{L}_\omega^{-1}\Pi_W((v_1 + \mathrm{v}_2 + \hat{w})^p)$, which is (6.1). It implies that $w = \hat{w} = \lim_{k\to\infty} \bar{w}_k$.

Step 2: For any $k \in \mathbb{N}$ each $\bar{w}_k$ satisfies (9.7). We proceed by induction. Clearly $\bar{w}_0 = 0$ satisfies (9.7). Now assume that $\bar{w}_{k-1}$ satisfies (9.7). By Lemma 3.1 and (2.19), there exists $C_{\lambda,\mu,p} > 0$ such that

$$\begin{aligned} &\|\mathcal{L}_\omega^{-1}\Pi_W(v_1 + \mathrm{v}_2 + \bar{w})^p\|_{H_t^\lambda \mathcal{H}_z^\mu} \\ &\quad \leq \gamma^{-1} C_{\lambda,\mu,p} \max\{\|v_1\|_{\mathcal{V}_{t,z}^{\lambda+\mu}}, \|\mathrm{v}_2\|_{\mathcal{V}_{t,z}^{\lambda+\mu}}, \|w\|_{H_t^\lambda \mathcal{H}_z^\mu}\}^p. \end{aligned} \tag{9.9}$$

Then take $K_{\lambda,\mu,p} := {C_{\lambda,\mu,p}}^{-1}$ in (9.6). By (9.9), the fact that $\bar{w}_{k-1}$ satisfies (9.7),

$$\begin{aligned} &\|\bar{w}_k\|_{H_t^\lambda \mathcal{H}_z^\mu} \\ &\leq \gamma^{-1} C_{\lambda,\mu,p} \max\{\|v_1\|_{\mathcal{V}_{t,z}^{\lambda+\mu}}, \|\mathrm{v}_2\|_{\mathcal{V}_{t,z}^{\lambda+\mu}}, \|\bar{w}_{k-1}\|_{H_t^\lambda \mathcal{H}_z^\mu}\}^p \\ &\leq \gamma^{-1} C_{\lambda,\mu,p} \max\big\{\|v_1\|_{\mathcal{V}_{t,z}^{\lambda+\mu}}, \|\mathrm{v}_2\|_{\mathcal{V}_{t,z}^{\lambda+\mu}}, K_{\lambda,\mu,p}^{-1}\gamma^{-1} \max\{\|v_1\|_{\mathcal{V}_{t,z}^{\lambda+\mu}}, \|\mathrm{v}_2\|_{\mathcal{V}_{t,z}^{\lambda+\mu}}\}^p\big\}^p \\ &\overset{(9.6)}{\leq} \gamma^{-1} K_{\lambda,\mu,p}^{-1} \max\{\|v_1\|_{\mathcal{V}_{t,z}^{\lambda+\mu}}, \|\mathrm{v}_2\|_{\mathcal{V}_{t,z}^{\lambda+\mu}}\}^p. \end{aligned}$$

Step 3: Proof of (9.7). By Step 2, the bounded sequence $\{\bar{w}_k\}_{k\in\mathbb{N}}$ converges up to subsequences to a weak limit $\bar{w} \in H_t^\lambda \mathcal{H}_z^\mu$ satisfying

$$\|\bar{w}\|_{H_t^\lambda \mathcal{H}_z^\mu} \leq K_{\lambda,\mu,p}^{-1} \gamma^{-1} \max\{\|v_1\|_{\mathcal{V}_{t,z}^{\lambda+\mu}}, \|\mathrm{v}_2\|_{\mathcal{V}_{t,z}^{\lambda+\mu}}\}^p.$$

Since $H_t^\lambda \mathcal{H}_z^\mu$ is compactly embedded into $H_t^{\frac{1}{2}+\delta} \mathcal{H}_z^{\frac{3}{2}+\delta}$ for $\delta > 0$ small enough, and using Step 1, we deduce that $\bar{w} = w$. This proves that w satisfies (9.7).

We now consider the case $p = 2$.

Step 1: The sequence $(\breve{w}_k)_{k\in\mathbb{N}}$ defined by $\breve{w}_0 := 0$ and

$$\begin{aligned} \breve{w}_{k+1} := \mathcal{T}(v_1, \mathrm{v}_2, \breve{w}_k) := {} & \mathcal{L}_\omega^{-1}\Pi_W\big(2(v_1 + \mathrm{v}_2)(\mathcal{L}_\omega^{-1}(v_1 + \mathrm{v}_2)^2 + \breve{w}_k)\big) \\ & + \mathcal{L}_\omega^{-1}\Pi_W\big((\mathcal{L}_\omega^{-1}(v_1 + \mathrm{v}_2)^2 + \breve{w}_k)^2\big), \end{aligned} \tag{9.10}$$

has limit $\lim_{k\to\infty} \breve{w}_k = \widetilde{w}$ in $H_t^{\frac{1}{2}+\delta} \mathcal{H}_x^{\frac{3}{2}+\delta}$. Arguing as in Lemma 6.6, the map $\breve{w} \mapsto \mathcal{T}(v_1, \mathrm{v}_2, \breve{w})$ is a contraction on $\mathcal{D}_{\rho_3}^W$, thus it admits a unique fixed point in $\mathcal{D}_{\rho_3}^W$ which solves (6.8) and therefore it coincides with $\widetilde{w}$.

Step 2: For any $k \in \mathbb{N}$ the function $\breve{w}_k$ satisfies

$$\|\breve{w}_k\|_{H_t^\lambda \mathcal{H}_x^\mu} \le \frac{1}{4} K_{\lambda,\mu,2}^{-2} \gamma^{-2} \max\{\|v_1\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}, \|\mathrm{v}_2(v_1)\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}\}^3. \tag{9.11}$$

We proceed by induction. Clearly $\breve{w}_0$ satisfies (9.11). Then suppose $\breve{w}_k$ satisfies (9.11). Let $C_{\lambda,\mu,2}$ be the algebra constant in (2.19) and take $K_{\lambda,\mu,2} := \frac{1}{32 C_{\lambda,\mu,2}}$. By Lemma 3.1, (2.19), and (9.3),

$$\|\mathcal{L}_\omega^{-1}(v_1 + \mathrm{v}_2)^2\|_{H_t^\lambda \mathcal{H}_x^\mu} \le 8\gamma^{-1} C_{\lambda,\mu,2} \max\{\|v_1\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}, \|\mathrm{v}_2\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}\}^2. \tag{9.12}$$

For any $k \in \mathbb{N}$ we define $\underline{w}_k := \mathcal{L}_\omega^{-1}(v_1 + \mathrm{v}_2)^2 + \breve{w}_k$. By (9.10), Lemma 3.1, (2.19), and (4.4) one has

$$\begin{aligned} &\|\breve{w}_{k+1}\|_{H_t^\lambda \mathcal{H}_x^\mu} \\ &\quad \le 2\gamma^{-1} C_{\lambda,\mu,2} \|\underline{w}_k\|_{H_t^\lambda \mathcal{H}_x^\mu} (4 \max\{\|v_1\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}, \|\mathrm{v}_2\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}\} + \|\underline{w}_k\|_{H_t^\lambda \mathcal{H}_x^\mu}). \end{aligned} \tag{9.13}$$

By (9.12), the inductive assumption, assumption (9.6), and $K_{\lambda,\mu,2} = \frac{1}{32 C_{\lambda,\mu,2}}$ we have

$$\begin{aligned} \|\underline{w}_k\|_{H_t^\lambda \mathcal{H}_x^\mu} &\le 8\gamma^{-1} C_{\lambda,\mu,2} \max\{\|v_1\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}, \|\mathrm{v}_2\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}\}^2 \\ &\quad + \frac{1}{4} K_{\lambda,\mu,2}^{-2} \gamma^{-2} \max\{\|v_1\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}, \|\mathrm{v}_2\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}\}^3 \\ &\le 16\gamma^{-1} C_{\lambda,\mu,2} \max\{\|v_1\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}, \|\mathrm{v}_2\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}\}^2 \qquad (9.14) \\ &= \frac{1}{2} \max\{\|v_1\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}, \|\mathrm{v}_2\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}\}. \qquad (9.15) \end{aligned}$$

Then, by (9.13), (9.14), (9.15), one gets

$$\|\breve{w}_{k+1}\|_{H_t^\lambda \mathcal{H}_x^\mu} \le \frac{1}{4} K_{\lambda,\mu,2}^{-2} \gamma^{-2} \max\{\|v_1\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}, \|v_2\|_{\mathcal{V}_{t,x}^{\lambda+\mu}}\}^3,$$

proving the claim.

By Steps 1 and 2 we conclude, as in the cases $p = 3, 5$, that $\widetilde{w}$ satisfies (9.11). Finally, (9.12), (9.11) for $\widetilde{w}$ and assumption (9.6) and $K_{\lambda,\mu,2} = \frac{1}{32 C_{\lambda,\mu,2}}$ imply (9.8). ∎

We start by proving Theorem 9.1 in the cases $p = 5$, $p = 3$. Given $\delta \in (0, \frac{1}{100})$, $r_0 := \frac{1}{2} + \delta$, and $s_0 := \frac{3}{2} + \delta$, define for all $l \ge 0$ the quantities

$$\sigma_0 := s_0 := \frac{3}{2} + \delta, \qquad \sigma_{l+1} := \sigma_l + \frac{3}{2} - \delta, \tag{9.16}$$

$$\alpha_0 := \begin{cases} 10\delta & \text{if } p = 5, \\ 4\delta & \text{if } p = 3, \end{cases} \qquad a_0 := \alpha_0, \tag{9.17}$$

$$\alpha_{l+1} := p \max\{\sigma_l - 1, a_l\}, \qquad a_{l+1} := \alpha_{l+1} - \frac{1}{2} - \delta, \tag{9.18}$$

$$b_0 := \begin{cases} 5+11\delta & \text{if } p=5, \\ 3+7\delta & \text{if } p=3, \end{cases} \quad b_{l+1} := p\max\{\sigma_l+1, \alpha_{l+1}\}, \tag{9.19}$$

$$\zeta_l := \max\{b_l - \max\{\sigma_l - 1, a_l\}, (p-1)\max\{\sigma_l+1, \alpha_{l+1}\}\}. \tag{9.20}$$

Lemma 9.4 (Iterative regularity bootstrap). *Let $p = 3, 5$ and $v_1^{(1)} \in \mathcal{D}_{\rho_1}$ as in Theorem 7.2. For any $l \geq 0$ and $\delta \in (0, \frac{1}{100})$ there exists $\epsilon_{R,\delta,l} > 0$ such that, if ε, R, N are as in Theorem 7.2 and $0 < \gamma^{-1}\varepsilon N^{\zeta_l} \leq \epsilon_{R,\delta,l}$, then for any $\lambda_l \geq r_0$ and $\mu_l \geq s_0$ such that $\lambda_l + \mu_l = \sigma_l + 2$, the function $u = v_1^{(1)} + \mathrm{v}_2(v_1^{(1)}) + w(v_1^{(1)})$ belongs to $H_t^{\lambda_l} \mathcal{H}_z^{\mu_l}$ and*

$$\begin{gathered} \|v_1^{(1)}\|_{\mathcal{V}_{t,z}^{\sigma_l+2}} \leq R\varepsilon^{\frac{1}{p-1}} N^{\sigma_l+1}, \quad \|\mathrm{v}_2(v_1^{(1)})\|_{\mathcal{V}_{t,z}^{\sigma_l+2}} \lesssim_{R,\delta,l} \varepsilon^{\frac{1}{p-1}} N^{\alpha_{l+1}}, \\ \|w(v_1^{(1)})\|_{H_t^{\lambda_l}\mathcal{H}_z^{\mu_l}} \lesssim_{R,\delta,l} \gamma^{-1}\varepsilon^{\frac{p}{p-1}} N^{b_{l+1}}. \end{gathered} \tag{9.21}$$

Proof. By (4.4), $v_1 \in \mathcal{D}_{\rho_1}$ and (5.4), (5.20), we have the first estimate in (9.21) for any l. We denote $v_1 := v_1^{(1)}$, $\mathrm{v}_2 := \mathrm{v}_2(v_1^{(1)})$, $w := w(v_1^{(1)})$. The proof of the second and third inequalities in (9.21) proceeds by induction.

Initialization. If $l = 0$, by (9.16), Theorem 7.2, (4.4), the definitions of ρ_1, ρ_2, ρ_3 in (5.4), (5.20), (9.17), (9.19), for N large enough we have

$$\begin{gathered} \|v_1\|_{\mathcal{V}_{t,z}^{\sigma_0}} \leq N^{\sigma_0-1}\rho_1 = N^{\sigma_0-1}R\varepsilon^{\frac{1}{p-1}}, \quad \|\mathrm{v}_2\|_{\mathcal{V}_{t,z}^{\sigma_0}} \leq \rho_2 = \mathtt{c}_2(\delta)R^p\varepsilon^{\frac{1}{p-1}}N^{a_0}, \\ \|w\|_{H_t^{r_0}\mathcal{H}_z^{\sigma_0}} \leq \rho_3 \leq \gamma^{-1}R^p\varepsilon^{\frac{p}{p-1}}N^{b_0}. \end{gathered} \tag{9.22}$$

Claim 1: $\|\mathrm{v}_2\|_{\mathcal{V}_{t,z}^{\sigma_0+2}} \lesssim_{R,\delta} \varepsilon^{\frac{1}{p-1}} N^{\alpha_1}$. We apply Lemma 9.2 with $s \rightsquigarrow \sigma_0$, $r \rightsquigarrow r_0$, $\tilde{\rho}_1 \rightsquigarrow N^{\sigma_0-1}\rho_1$, $\tilde{\rho}_2 \rightsquigarrow \rho_2$. By (9.22), taking $\gamma^{-1}\varepsilon N^{\zeta_0} \lesssim_{R,\delta} 1$ with ζ_0 defined in (9.20), one deduces (9.3), (9.4). Thus the claim follows by (9.5), recalling that $\alpha_1 \geq p(\sigma_0 - 1)$ by (9.18).

Claim 2: *For any $\lambda_0 + \mu_0 = \sigma_0 + 2$, with $\lambda_0 \geq r_0$ and $\mu_0 \geq s_0$, the function w satisfies (9.21) for $l = 0$.* We apply Lemma 9.3. Assumption (9.6) is satisfied since, by (4.4), (5.4), (5.20), and Claim 1, one has

$$\gamma^{-1}\max\{\|v_1\|_{\mathcal{V}_{t,z}^{\sigma_0+2}}, \|\mathrm{v}_2\|_{\mathcal{V}_{t,z}^{\sigma_0+2}}\}^{p-1} \lesssim_R \gamma^{-1}\varepsilon N^{(p-1)\max\{\sigma_0+1,\alpha_1\}} \overset{(9.20)}{\lesssim_R} \gamma^{-1}\varepsilon N^{\zeta_0},$$

hence

$$\gamma^{-1}\max\{\|v_1\|_{\mathcal{V}_{t,z}^{\sigma_0+2}}, \|\mathrm{v}_2\|_{\mathcal{V}_{t,z}^{\sigma_0+2}}\}^{p-1} \leq \inf_{\substack{\lambda_0\in[r_0,\sigma_0+2] \\ \mu\in[s_0,\sigma_0+2]}} K_{\lambda_0,\mu_0,p},$$

provided $\gamma^{-1}\varepsilon N^{\zeta_0}$ is small enough. Thus by (9.7), (4.4), $v_1 \in \mathcal{D}_{\rho_1}$, and Claim 1 one gets

$$\|w\|_{H_t^{\lambda_0}\mathcal{H}_z^{\mu_0}} \lesssim_{R,\delta} \gamma^{-1}\max\{\varepsilon^{\frac{1}{p-1}}N^{\sigma_0+1}, \varepsilon^{\frac{1}{p-1}}N^{\alpha_1}\}^p \overset{(9.19)}{\lesssim_{R,\delta}} \gamma^{-1}\varepsilon^{\frac{p}{p-1}}N^{b_1},$$

which is the second estimate in (9.21) for $l = 0$.

Induction. We now assume that (9.21) holds for $l-1$ and we prove it for l.

*Claim l*1: $\|v_2\|_{\mathcal{Y}^{\sigma_l+2}_{t,z}} \lesssim_{R,\delta,l} \varepsilon^{\frac{1}{p-1}} N^{\alpha_{l+1}}$. Assuming $\gamma^{-1}\varepsilon N^{\zeta_l} \lesssim_{R,\delta,l} 1$, choosing $\lambda_{l-1} = r_0$ and $\mu_{l-1} = \sigma_{l-1} + 2 - \lambda_{l-1} = \sigma_l$, by the induction hypothesis and using (4.4), (9.16), (9.18) we have

$$\begin{gathered} \|v_1\|_{\mathcal{Y}^{\sigma_l}_{t,z}} \le R\varepsilon^{\frac{1}{p-1}} N^{\sigma_l - 1}, \quad \|v_2\|_{\mathcal{Y}^{\sigma_l}_{t,z}} \lesssim_{R,\delta,l} \varepsilon^{\frac{1}{p-1}} N^{a_l}, \\ \|w\|_{H^{r_0}_t \mathcal{H}^{\sigma_l}_z} \lesssim_{R,\delta,l} \gamma^{-1}\varepsilon^{\frac{p}{p-1}} N^{b_l}. \end{gathered} \tag{9.23}$$

We apply Lemma 9.2 with $s = \sigma_l$, $\tilde{\rho}_1 = R\varepsilon^{\frac{1}{p-1}} N^{\sigma_l - 1}$, and $\tilde{\rho}_2 \gtrsim_{R,\delta,l} \varepsilon^{\frac{1}{p-1}} N^{a_l}$. Then by (9.23) assumptions (9.3) and (9.4) are satisfied, taking $\gamma^{-1}\varepsilon N^{\zeta_l} \lesssim_{R,\delta,l} 1$, with ζ_l given by (9.20). Thus (9.5) and (9.18) imply the claim.

*Claim l*2: *For any $\lambda_l + \mu_l = \sigma_l + 2$ with $\lambda_l \ge r_0$ and $\mu_l \ge s_0$, the function w satisfies* (9.21). We apply Lemma 9.3 with $\lambda = \lambda_l$ and $\mu = \mu_l$. Indeed, since $\gamma^{-1}\varepsilon N^{\zeta_l} \lesssim_{R,\delta,l} 1$ by (9.20), and using (5.4), (5.20), and Claim *l*1, assumption (9.6) of Lemma 9.3 is satisfied, and thus by (9.7), $\|w\|_{H^{\lambda_l}_t \mathcal{H}^{\mu_l}_z} \lesssim_{R,\delta,l} \gamma^{-1}\varepsilon^{\frac{p}{p-1}} N^{b_{l+1}}$. This concludes the inductive step. ∎

The proof of Theorem 9.1 for $p = 2$ follows with similar arguments: given $\delta \in (0, \frac{1}{100})$, $r_0 \ge \frac{1}{2} + \delta$, define for any $l \ge 0$ the quantities

$$\begin{gathered} \sigma_0 := s_0 := \frac{3}{2} + \delta, \quad \sigma_{l+1} := \sigma_l + \frac{3}{2} - \delta, \\ \alpha_0 := 0, \quad a_0 := 0, \quad b_0 := 2 + 5\delta, \\ \alpha_{l+1} := b_l + \max\{\sigma_l - 1, a_l\} + 1, \quad a_{l+1} := \alpha_{l+1} - \frac{1}{2} - \delta, \\ b_{l+1} := 2\max\{\sigma_l + 1, \alpha_{l+1}\}, \\ \zeta_l := 2\max\{b_l - \max\{\sigma_l - 1, a_l\}, \max\{\sigma_l + 1, \alpha_{l+1}\}\}. \end{gathered}$$

Then one proves the following:

Lemma 9.5 (Iterative regularity bootstrap). *Let $p = 2$ and $v_1^{(1)} \in \mathcal{D}_{\rho_1}$ as in Theorem 7.2. For any $l \ge 0$ and $\delta \in (0, \frac{1}{100})$ there exists $\epsilon_{R,\delta,l} > 0$ such that, if ε, R, N are as in Theorem 7.2 and $0 < \gamma^{-2}\varepsilon N^{\zeta_l} \le \epsilon_{R,\delta,l}$, then for any $\lambda_l \ge r_0$ and $\mu_l \ge s_0$ such that $\lambda_l + \mu_l = \sigma_l + 2$, the function $u = v_1^{(1)} + v_2(v_1^{(1)}) + w(v_1^{(1)})$ belongs to $H^{\lambda_l}_t \mathcal{H}^{\mu_l}_x$ and*

$$\begin{gathered} \|v_1^{(1)}\|_{\mathcal{Y}^{\sigma_l+2}_{t,x}} \le R\varepsilon^{\frac{1}{2}} N^{\sigma_l+1}, \quad \|v_2(v_1^{(1)})\|_{\mathcal{Y}^{\sigma_l+2}_{t,x}} \lesssim_{\delta,l} \varepsilon^{\frac{1}{2}} N^{\alpha_{l+1}}, \\ \|w(v_1^{(1)})\|_{H^{\lambda_l}_t \mathcal{H}^{\mu_l}_x} \lesssim_{R,\delta,l} \gamma^{-1}\varepsilon N^{b_{l+1}}. \end{gathered}$$

Acknowledgments. We thank Philippe Bolle for useful comments.

Funding. Research supported by PRIN 2020 (2020XB3EFL001) "Hamiltonian and dispersive PDEs" and by PRIN 2022 (2022HSSYPN) "TESEO – Turbulent effects vs stability in equations from oceanography".

References

[1] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications. *J. Functional Analysis* **14** (1973), 349–381 Zbl 0273.49063 MR 0370183

[2] M. T. Anderson, On the uniqueness and global dynamics of AdS spacetimes. *Classical Quantum Gravity* **23** (2006), no. 23, 6935–6953 Zbl 1107.83015 MR 2271154

[3] D. Bambusi and S. Paleari, Families of periodic solutions of resonant PDEs. *J. Nonlinear Sci.* **11** (2001), no. 1, 69–87 Zbl 0994.37040 MR 1819863

[4] M. Berti, *Nonlinear oscillations of Hamiltonian PDEs*. Progr. Nonlinear Differential Equations Appl. 74, Birkhäuser, Boston, MA, 2007 Zbl 1146.35002 MR 2345400

[5] M. Berti and P. Bolle, Periodic solutions of nonlinear wave equations with general nonlinearities. *Comm. Math. Phys.* **243** (2003), no. 2, 315–328 Zbl 1072.35015 MR 2021909

[6] M. Berti and P. Bolle, Multiplicity of periodic solutions of nonlinear wave equations. *Nonlinear Anal.* **56** (2004), no. 7, 1011–1046 Zbl 1064.35119 MR 2038735

[7] M. Berti and P. Bolle, Cantor families of periodic solutions for completely resonant nonlinear wave equations. *Duke Math. J.* **134** (2006), no. 2, 359–419 Zbl 1103.35077 MR 2248834

[8] M. Berti and P. Bolle, Cantor families of periodic solutions of wave equations with C^k nonlinearities. *NoDEA Nonlinear Differential Equations Appl.* **15** (2008), no. 1-2, 247–276 Zbl 1155.35004 MR 2408354

[9] M. Berti and P. Bolle, Sobolev quasi-periodic solutions of multidimensional wave equations with a multiplicative potential. *Nonlinearity* **25** (2012), no. 9, 2579–2613 Zbl 1262.35015 MR 2967117

[10] M. Berti and P. Bolle, *Quasi-periodic solutions of nonlinear wave equations on the d-dimensional torus*. EMS Monogr. Math., EMS Publishing House, Berlin, 2020 Zbl 1454.37003 MR 4321994

[11] M. Berti, P. Bolle, and M. Procesi, An abstract Nash–Moser theorem with parameters and applications to PDEs. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **27** (2010), no. 1, 377–399 Zbl 1203.47038 MR 2580515

[12] M. Berti, L. Corsi, and M. Procesi, An abstract Nash–Moser theorem and quasi-periodic solutions for NLW and NLS on compact Lie groups and homogeneous manifolds. *Comm. Math. Phys.* **334** (2015), no. 3, 1413–1454 Zbl 1312.35157 MR 3312439

[13] M. Berti and M. Procesi, Quasi-periodic solutions of completely resonant forced wave equations. *Comm. Partial Differential Equations* **31** (2006), no. 4-6, 959–985 Zbl 1100.35011 MR 2233048

[14] M. Berti and M. Procesi, Nonlinear wave and Schrödinger equations on compact Lie groups and homogeneous spaces. *Duke Math. J.* **159** (2011), no. 3, 479–538 Zbl 1260.37045 MR 2831876

[15] P. Bizoń, B. Craps, O. Evnin, D. Hunik, V. Luyten, and M. Maliborski, Conformal flow on S^3 and weak field integrability in AdS_4. *Comm. Math. Phys.* **353** (2017), no. 3, 1179–1199 Zbl 1378.35300 MR 3652487

[16] P. Bizoń and A. Rostworowski, Weakly turbulent instability of anti–de Sitter spacetime. *Phys. Rev. Lett.* **107** (2011), no. 3, article no. 031102

[17] J. Bourgain, *Green's function estimates for lattice Schrödinger operators and applications*. Ann. of Math. Stud. 158, Princeton University Press, Princeton, NJ, 2005 Zbl 1137.35001 MR 2100420

[18] H. Brézis, J.-M. Coron, and L. Nirenberg, Free vibrations for a nonlinear wave equation and a theorem of P. Rabinowitz. *Comm. Pure Appl. Math.* **33** (1980), no. 5, 667–684 Zbl 0484.35057 MR 0586417

[19] H. Brézis and L. Nirenberg, Forced vibrations for a nonlinear wave equation. *Comm. Pure Appl. Math.* **31** (1978), no. 1, 1–30 Zbl 0378.35040 MR 0470377

[20] A. Chatzikaleas, On time periodic solutions to the conformal cubic wave equation on the Einstein cylinder. *J. Math. Phys.* **61** (2020), no. 11, article no. 111505 Zbl 1454.81094 MR 4173810

[21] A. Chatzikaleas and J. Smulevici, Time periodic solutions and Nekhoroshev stability to nonlinear massive Klein–Gordon equations in anti-de Sitter. 2023, arXiv:2304.12784v1

[22] A. Chatzikaleas and J. Smulevici, Non-linear periodic waves on the Einstein cylinder. arXiv:2201.05447, to appear in *Anal. PDE*

[23] L. Chierchia and J. You, KAM tori for 1D nonlinear wave equations with periodic boundary conditions. *Comm. Math. Phys.* **211** (2000), no. 2, 497–525 Zbl 0956.37054 MR 1754527

[24] W. Craig and C. E. Wayne, Newton's method and periodic solutions of nonlinear wave equations. *Comm. Pure Appl. Math.* **46** (1993), no. 11, 1409–1498 Zbl 0794.35104 MR 1239318

[25] M. Dafermos and G. Holzegel, Dynamic instability of solitons in $4 + 1$ dimensional gravity with negative cosmological constant. 2006, https://www.dpmms.cam.ac.uk/~md384/ADSinstability.pdf visited on 24 April 2024

[26] Ó. J. C. Dias, G. T. Horowitz, D. Marolf, and J. E. Santos, On the nonlinear stability of asymptotically anti-de Sitter solutions. *Classical Quantum Gravity* **29** (2012), no. 23, article no. 235019 Zbl 1258.83022 MR 3002881

[27] Ó. J. C. Dias, G. T. Horowitz, and J. E. Santos, Gravitational turbulent instability of anti-de Sitter space. *Classical Quantum Gravity* **29** (2012), no. 19, article no. 194002 Zbl 1254.83030 MR 2978943

[28] O. Evnin, Resonant Hamiltonian systems and weakly nonlinear dynamics in AdS spacetimes. *Classical Quantum Gravity* **38** (2021), no. 20, article no. 203001 Zbl 1479.83019 MR 4318547

[29] G. Gentile, V. Mastropietro, and M. Procesi, Periodic solutions for completely resonant nonlinear wave equations with Dirichlet boundary conditions. *Comm. Math. Phys.* **256** (2005), no. 2, 437–490 Zbl 1094.35021 MR 2160800

[30] B. Grébert and E. Paturel, KAM for the Klein Gordon equation on $\mathbb{S}^d$. *Boll. Unione Mat. Ital.* **9** (2016), no. 2, 237–288 Zbl 1343.37075 MR 3502160

[31] G. Holzegel, J. Luk, J. Smulevici, and C. Warnick, Asymptotic properties of linear field equations in anti–de Sitter space. *Comm. Math. Phys.* **374** (2020), no. 2, 1125–1178 Zbl 1448.83003 MR 4072237

[32] S. B. Kuksin, *Analysis of Hamiltonian PDEs*. Oxford Lecture Ser. Math. Appl. 19, Oxford University Press, Oxford, 2000 Zbl 0960.35001 MR 1857574

[33] B. V. Lidskiĭ and E. I. Shul'man, Periodic solutions of the equation $u_{tt} - u_{xx} + u^3 = 0$. *Funktsional. Anal. i Prilozhen.* **22** (1988), no. 4, 88–89 Zbl 0837.35012 MR 0977006

[34] M. Maliborski and A. Rostworowski, Time-periodic solutions in an Einstein AdS–massless-scalar-field system. *Phys. Rev. Lett.* **111** (2013), no. 5, article no. 051102

[35] G. Moschidis, A proof of the instability of AdS for the Einstein-null dust system with an inner mirror. *Anal. PDE* **13** (2020), no. 6, 1671–1754 Zbl 1452.83009 MR 4150259

[36] G. Moschidis, A proof of the instability of AdS for the Einstein-massless Vlasov system. *Invent. Math.* **231** (2023), no. 2, 467–672 Zbl 07661328 MR 4542703

[37] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste. Tome I*. Les Grands Classiques Gauthier-Villars, Librairie Scientifique et Technique Albert Blanchard, Paris, 1987 Zbl 0651.70002 MR 0926906

[38] J. Pöschel, A KAM-theorem for some nonlinear partial differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **23** (1996), no. 1, 119–148 Zbl 0870.34060 MR 1401420

[39] M. Procesi, Quasi-periodic solutions for completely resonant non-linear wave equations in 1D and 2D. *Discrete Contin. Dyn. Syst.* **13** (2005), no. 3, 541–552 Zbl 1086.35007 MR 2152330

[40] P. H. Rabinowitz, Free vibrations for a semilinear wave equation. *Comm. Pure Appl. Math.* **31** (1978), no. 1, 31–68 Zbl 0341.35051 MR 0470378

[41] C. E. Wayne, Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory. *Comm. Math. Phys.* **127** (1990), no. 3, 479–528 Zbl 0708.35087 MR 1040892

Received 5 September 2023; revised 16 February 2024; accepted 26 February 2024.

Massimiliano Berti
Mathematics Area, International School for Advanced Studies (SISSA), Via Bonomea 265, 34136 Trieste, Italy; berti@sissa.it

Beatrice Langella
Mathematics Area, International School for Advanced Studies (SISSA), Via Bonomea 265, 34136 Trieste, Italy; beatrice.langella@sissa.it

Diego Silimbani
Mathematics Area, International School for Advanced Studies (SISSA), Via Bonomea 265, 34136 Trieste, Italy; dsilimba@sissa.it