Generalized theta functions, projectively flat vector bundles and noncommutative tori

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Abstract. In this paper, the well-known relationship between theta functions and Heisenberg group actions thereon is resumed by combining complex algebraic and noncommutative geometric techniques in that we describe Hermitian–Einstein vector bundles on 2-tori via representations of noncommutative tori, thereby reconstructing Matsushima's (1976) setup and elucidating the ensuing Fourier–Mukai–Nahm (FMN) aspects. We prove the existence of noncommutative torus actions on the space of smooth sections of Hermitian–Einstein vector bundles on 2-tori preserving the eigenspaces of a natural Laplace operator. Motivated by the Coherent State Transform approach to theta functions (Florentino, Mourão, Nunes (2002), Tyurin (2003)), we extend the latter to vector valued thetas and develop an additional algebraic reinterpretation of Matsushima's theory making FMN-duality manifest again.

1. Introduction

In this paper, we address, in the simplest case, the well-known intriguing and multifaceted relationship between theta functions and representations of Heisenberg groups (both infinite and finite [12, 14, 17]), from a blended complex differential geometric viewpoint – focused on holomorphic vector bundles on 2-tori – and a noncommutative geometric one – involving (rational) noncommutative 2-tori – possibly bringing in some novel insights and, in particular, improving the treatment given in [23]. Noncommutative geometry arose with the aim of studying singular objects, such as orbit spaces, generally intractable via traditional topological, analytical and geometrical tools (see, in particular, the comprehensive [4]), and it is ultimately based on the transition from points in a topological space to functions thereon and thence to general algebras. Noncommutative tori provide a simple yet highly non-trivial testing ground for carrying out such a programme. They appear naturally in condensed matter physics issues, see, e.g., [2, 5] and they also implicitly crop up in the theory of

projectively flat vector bundles over tori, see, e.g., [11, 12]; it is precisely this aspect that is dealt with in the present work.

Specifically, in Section 3, via a series of propositions, we prove the existence of a representation of the noncommutative torus $\mathcal{A}_{1/\theta}$ ($\theta = q/r$, q and r being coprime positive integers) on the space of sections $\Gamma(\mathcal{E}_{r,q})$ of a projectively flat *Hermitian–Einstein* holomorphic vector bundle (or *HE*-vector bundle for short) $\mathcal{E}_{r,q}$ of rank r and degree q on a two-dimensional torus. This representation will actually preserve the eigenspaces ("Landau levels") of a natural Laplace operator (essentially, a quantum harmonic oscillator), hence, in particular, its holomorphic sections, thereby recovering the classical algebraic-geometric portrait. The vector bundle $\mathcal{E}_{r,q}$ itself, in turn, can be manufactured from a representation of $\mathcal{A}_\theta$ on its typical fibre. Another representation of $\mathcal{A}_\theta$ on $\Gamma(\mathcal{E}_{r,q})$, commuting with the representation of $\mathcal{A}_{1/\theta}$, is produced out of the parallel transport pertaining to the Chern connection on $\mathcal{E}_{r,q}$. The above developments bring in a vivid portrait of the Fourier–Mukai–Nahm (FMN, [13, 15]) duality between $\mathcal{E}_{r,q}$ and $\mathcal{E}_{q,r}$ together with their respective Chern connections. Actually, all objects, representations and bundles, will come in (torus-) families (moduli). In Section 4, upon resorting to the well-known heat equation interpretation of theta functions (described via the so-called *Coherent State Transform (CST)* of [8]) and further insisting on a noncommutative torus perspective, we present a "matrix" description of Matsushima's theory making again the above duality manifest. Finally, we prove that, as pre-C^*-algebras (and for the unique C^*-tensor product involved), $\mathcal{A}_{q/r} \otimes \mathcal{A}_{r/q}$ and $\mathcal{A}_{1/rq}$ are isomorphic. This will be a byproduct of a "categorical" reinterpretation of Gauss sums identities also shedding light on Fourier–Mukai–Nahm transform issues. Moreover, a vector analogue of the *CST* will be set up. The layout of the paper is completed by Section 2 – gathering together background material from different areas in order to fix notation and to pave the ground for the successive developments in Sections 3 and 4 – and by Section 5, pointing out possible applications and further research directions.

2. Preliminary tools

In this section, we establish our notation and collect several miscellaneous technical tools for the benefit of a wider readership.

2.1. *k*-level theta functions and the Coherent State Transform

We begin by providing minimal background on k-level theta functions and on their relationship with the *heat equation* closely following the exposition of [8] (see also [25]) – up to slight notational changes – and referring to it for a complete treat-

ment. We restrict to the genus one case, namely to an Abelian torus (M, τ), $\tau \in \mathbb{C}$, Im $\tau > 0$.

Let us start from the following (tempered) distributions on S^1

$$\theta_\ell^0(x) = \sum_{n \in \mathbb{Z}} e^{2\pi i(\ell + kn)x}$$

with $\ell = 0, 1, \ldots, k - 1$. They are mapped, via the so-called *Coherent State Transform (CST)*,

$$\mathrm{CST}(\theta_\ell^0)(z) = \vartheta_\ell(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i(\ell + kn)(\tau/k)(\ell + kn)} e^{2\pi i(\ell + kn)z}$$

to the *k-level theta functions*. These, in turn, are interpreted as holomorphic sections of the k-th power of the so-called *Theta line bundle*, and yield a basis thereof, as a consequence of the Riemann–Roch theorem. A far reaching generalization for HE-vector bundles has been developed by Matsushima [12] and his theory will be retrieved and elaborated on in what follows.

2.2. Review of Matsushima's theory

In this subsection, we outline Matsushima's theory [12], tailoring the exposition to our purposes and referring to [11], especially Chapter IV-7 and to [23, Section 3.2], for background material. Here we just recall that an irreducible holomorphic vector bundle (i.e., without proper holomorphic direct summands) admits a HE-metric if and only if it is stable in the algebraic-geometric sense: this is the celebrated *Kobayashi–Hitchin* correspondence, fully established for compact Kähler manifolds in [26]. HE-vector bundles are poly-stable, i.e., direct sums of stable bundles. In the present work we shall consider the special class of HE-bundles consisting of projectively flat holomorphic vector bundles on complex tori, which are equipped with a Hermitian metric whose corresponding canonical (Chern) connection has constant curvature.

Let r and q be coprime positive integers, i.e., $\gcd(r, q) = 1$. Let V be a one-dimensional complex vector space and let us consider a complex torus V/L where $L \cong \mathbb{Z}^2$ is a lattice. Let $L' \subset L$ be a complete sublattice of L. Specifically, if $L = \langle \omega_1, \omega_2 \rangle$ is the lattice generated by a (real) basis $\{\omega_j\}_{j=1,2}$ of V, let $L' = \langle r\omega_1, \omega_2 \rangle$ and

$$K := L/L' \cong \mathbb{Z}_r,$$

thus we have an r-covering of complex tori

$$\varphi : V/L' \to V/L.$$

Let A be the $\mathbb{Q}$-valued form defined by $A(\omega_1, \omega_2) = q/r$, and $A' = rA$ the $\mathbb{Z}$-valued form fulfilling $A'(\omega_1, \omega_2) = q$ (the pull-back of A via φ). The form A gives rise to a HE-vector bundle $\mathcal{E}_{r,q} \to V/L$ – of rank r and degree q – i.e., such that its canonical (Chern) connection has constant curvature

$\Omega = -2\pi i A \otimes \mathrm{Id}_{\mathcal{E}_{r,q}}$. Correspondingly, one has a HE-line bundle $\mathcal{E}_{1,q} \to V/L'$, the q-level theta line bundle over V/L', related to the form A'. Let us denote, as usual, by $H^0(X, E)$ the space of holomorphic sections of a holomorphic vector bundle $E \to X$, X being the base manifold, with dimension $h^0(X, E)$. It is known (by Riemann–Roch and the vanishing of $H^1(X, E)$ for tori) that

$$h^0(V/L, \mathcal{E}_{r,q}) = h^0(V/L', \mathcal{E}_{1,q}) = q,$$

thus the corresponding section spaces are (non-canonically) isomorphic. Now, given an orthonormal basis of $H^0(V/L', \mathcal{E}_{1,q})$ made up by q-level theta functions $\{\vartheta_m\}_{m=0}^{q-1}$, one has, according to Matsushima, a splitting of $\varphi^* \mathcal{E}_{r,q} \to V/L'$ as

$$\varphi^* \mathcal{E}_{r,q} \cong \bigoplus_{\sigma \in K} (\mathcal{E}_{1,q})_\sigma$$

where $(\mathcal{E}_{1,q})_\sigma$ is a translate of $\mathcal{E}_{1,q}$ and any two different translates being non-isomorphic as holomorphic line bundles.

Therefore, one has an injective correspondence

$$\mathcal{M} : H^0(V/L, \mathcal{E}_{r,q}) \to \bigoplus_{\sigma \in \mathbb{Z}_r} H^0(V/L', (\mathcal{E}_{1,q})_\sigma)$$

given by (picking an orthonormal basis $\{s_m\}$, $m = 0, 1, \ldots, q-1$)

$$\mathcal{M} : s_m \mapsto \mathrm{vec}(\vartheta_m) := [(\sigma \cdot \vartheta_m)_{\sigma \in \mathbb{Z}_r}]$$

where the map vec arranges the translates of $\vartheta_m \in H^0(V/L', \mathcal{E}_{1,q})$ into a column vector.

We do not spell out the action of σ in the original Matsushima picture in detail, since we shall essentially recover it anew in what follows, see Section 4.2.

2.3. The noncommutative torus

General references for the present subsection are, among others, [3, 4, 16]. Recall that a C^*-algebra $\mathcal{A}$ is a Banach $*$-algebra whose norm $\|\cdot\|$ satisfies $\|a^* a\| = \|a\|^2$ for all $a \in \mathcal{A}$. The algebra $C(X)$ of continuous functions on a locally compact topological space X – vanishing at infinity if X is not compact – is a commutative C^*-algebra, with the product given by the pointwise product of functions, the involution $*$ being complex conjugation and $\|\cdot\|$ the supremum norm. If X is not compact, $C(X)$ will

not possess an identity, which can nevertheless be adjoined, this corresponding to the one-point compactification of X. Conversely, according to the Gel'fand–Naimark theorem, any commutative C^*-algebra $\mathcal{A}$ can be realized as $C(\sigma(\mathcal{A}))$, with $\sigma(\mathcal{A})$ being the spectrum of $\mathcal{A}$, i.e., the set of maximal ideals $\mathcal{I} \subset \mathcal{A}$, equipped with the weak $*$-topology. In $C(X)$, a maximal ideal is of the form $\mathcal{I}_x = \{f \in C(X) \mid f(x) = 0\}$, where $x \in X$. Therefore, the category of locally compact spaces and proper maps is dual to the category of commutative C^*-algebras and $*$-homomorphisms. In particular, a standard 2-torus $\mathbb{T}^2 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_j| = 1, j = 1, 2\}$ can be traded for $C(\mathbb{T}^2)$, which is in turn generated, in view of Fourier theory, by the (unitary) multiplication operators $u_j := z_j\cdot$ acting on the Hilbert space $L^2(\mathbb{T}^2, \mathfrak{m})$, with $\mathfrak{m}$ the Lebesgue measure. "Deformation" of the above algebra produces what is called a noncommutative torus. It is then possible to select the "smooth" part of it, akin to the smooth functions on a manifold. This is done immediately below.

Let $\theta \in \mathbb{R}$. The noncommutative torus is the pre-C^*-algebra $\mathcal{A}_\theta$ consisting of rapidly decaying series

$$a = \sum_{n,m=-\infty}^{\infty} a_{nm} u^n v^m, \qquad a_{n,m} \in \mathbb{C}$$

where u, v are unitary operators in a Hilbert space $\mathcal{H}$ satisfying the relation

$$vu = e^{2\pi i \theta} uv. \tag{1}$$

We have a natural smooth structure on $\mathcal{A}_\theta$ given by the noncommutative integral

$$\tau(a) = a_{00}, \qquad a \in \mathcal{A}_\theta,$$

and noncommutative derivatives

$$\partial_1(u^n v^m) = in u^n v^m, \qquad \partial_2(u^n v^m) = im u^n v^m.$$

In the sequel, we shall take $\theta \in \mathbb{Q}$, $\theta > 0$ and, ultimately, we shall deal with $\theta = q/r$, with q and r positive and coprime. Also, we notationally distinguish $\mathcal{A}_\theta$ from its C^*-completion A_θ.

We remark from the outset that finite-dimensional irreducible unitary representations of $\mathcal{A}_\theta$ exist, see, e.g., [4, 23]: indeed set $v := q/r$, with q and r relatively prime positive integers, and take $u = \mathrm{diag}(1, e(v), e(2 \cdot v), \ldots, e((r-1) \cdot v))$ and $v = $ matrix of the shift map $e_i \to e_{i-1}$, $i = 1, 2, \ldots, r$, $e_0 = e_r$, with $(e_1, \ldots, e_r)$ being the canonical basis of $\mathbb{C}^r$ and where we defined, for real x, $e(x) := e^{2\pi i x}$. Then (1) is satisfied; also notice that $u^r = v^r = 1$ (unit matrix), which entails irreducibility.

Proposition 3.3 below will show that (up to phase factors and unitary equivalence) this is indeed the typical example. In Section 4 we shall present a distributional realization of the Matsushima spaces carrying explicit noncommutative tori representations akin to the one just discussed.

2.4. The Canonical Commutation Relations and the quantum harmonic oscillator

In this subsection, we assemble basic information on the quantum harmonic oscillator and its relationship to the Canonical Commutation Relations and the associated Stone–von Neumann theorem [27], referring to the comprehensive survey [21] for elucidation of their modern ramifications.

Let us consider a representation of the *Canonical* (or Weyl–Heisenberg) *Commutation Relations* (CCR) on a (necessarily infinite-dimensional) separable Hilbert space $\mathcal{H}$,

$$[Q, P] = i\mathbf{1}$$

(one degree of freedom), with Q and P ("position" and "momentum" operators, respectively) unbounded self-adjoint operators on a suitable domain. In order to avoid problems arising from the latter issue (see however [18, Section X.6], for amplification and further use, together with [7]), the CCR are reformulated (Weyl) in integral form,

$$U(a)V(b) = e^{iab}V(b)U(a), \qquad a, b \in \mathbb{R}$$

with P and Q becoming the infinitesimal generators of the one parameter unitary groups $U(\cdot)$ and $V(\cdot)$, respectively.

The quantum harmonic oscillator Hamiltonian reads

$$H = \frac{1}{2}(P^2 + Q^2) = A^\dagger A + \frac{1}{2}\mathbf{1} = \frac{1}{2}(A^\dagger A + AA^\dagger)$$

in terms of *annihilation* and *creation* operators

$$A = \frac{1}{\sqrt{2}}(Q + iP), \qquad A^\dagger = \frac{1}{\sqrt{2}}(Q - iP)$$

subject to the commutation relation

$$AA^\dagger - A^\dagger A = \mathbf{1}.$$

In the irreducible case, the spectrum of H only consists of simple eigenvalues $\{n + \frac{1}{2}\}_{n=0}^{\infty}$ and the n-th eigenspace $\mathcal{H}_n$ is generated by $\phi_n = (1/\sqrt{n!})(A^\dagger)^n \phi_0$, with the ground state ϕ_0 fulfilling $A\phi_0 = 0$. The operator $A^\dagger A$, namely, the Hamiltonian without constant term ("zero-point energy") is called the *number operator*.

In general the multiplicity of a representation of the CCR (phrased into Weyl's integral form) is given by $k = \dim \mathcal{H}_0$: this is a version of the Stone–von Neumann uniqueness theorem (see, e.g., [27]).

2.5. Gauss sums

First of all, let us recall the celebrated *Gauss sums* (see [9, Section 5.6]):

$$S(\mu, r) := \sum_{0 \leq \ell \leq r-1} e^{2\pi i \ell^2 \frac{\mu}{r}},$$

for integers μ and r, the latter different from zero, together with the well-known multiplicative formula

$$S(\mu q, r) S(\mu r, q) = S(\mu, rq)$$

valid for coprime integers r and q and any integer μ (cf. [9, Theorem 64]).

Here is a quick outline of the proof. The right-hand side reads

$$\sum_{k=0}^{rq-1} e^{2\pi i \frac{\mu}{rq} k^2}.$$

Now, upon exploiting the group isomorphism

$$\mathbb{Z}_r \times \mathbb{Z}_q \cong \mathbb{Z}_{rq}$$

stemming from the equation

$$q \cdot [\ell]_r + r \cdot [m]_q = [k]_{rq}$$

which, given a residue class $[k]$ modulo rq, yields unique residue classes $[\ell]$ modulo r and $[m]$ modulo q (the converse being clear), we see that, setting $k = \ell q + mr$ (no dependence on representatives), the right-hand side splits into the product appearing in the left-hand side. Explicitly,

$$\frac{k^2}{rq} = \frac{(\ell q + mr)^2}{rq} = \frac{\ell^2 q}{r} + \frac{m^2 r}{q} + 2\ell m$$

and the last term in the right-hand side exponentiates to 1. Notice that the problem of finding $[k]_{rq}$ such that $[k]_r = [\ell]_r$ and $[k]_q = [m]_q$, with given classes $[\ell]_r$ and $[m]_q$ is solved via the Chinese Remainder Theorem: if a and b are integers such that $ar + bq = 1$, then $k = qb\ell + ram$, see again [9, Theorem 121].

3. Representations of noncommutative tori and HE-vector bundles

In this Section, we reinterpret the Matsushima construction of holomorphic HE-vector bundles over a two-dimensional torus $\mathbb{C}/\Lambda$ – with lattice $\Lambda = \langle 1, \tau \rangle$ and Im $\tau > 0$ – via representations of the two-dimensional noncommutative torus, see also [23]. This will be unfolded through the following series of propositions.

Proposition 3.1. *Given $\theta \in \mathbb{Q}$, $\theta > 0$, an irreducible representation of $\mathcal{A}_\theta$ on a finite-dimensional Hilbert space $\mathcal{H}$ produces a HE-vector bundle $E_\theta \to \mathbb{C}/\Lambda$ over a two-dimensional torus $\mathbb{C}/\Lambda$, where $\Lambda = \langle 1, \tau \rangle$, $\mathrm{Im}(\tau) > 0$ with degree $\theta \dim(\mathcal{H})$ and rank $\dim(\mathcal{H})$.*

Proof. Let u, v be unitary operators on a finite-dimensional Hilbert space $\mathcal{H}$ satisfying (1) (abuse of notation); then if $\gamma = n + \tau m \in \Lambda$ the function (theta multiplier)

$$J_\gamma^{-1}(z) = e^{\frac{\theta\pi}{\mathrm{Im}(\tau)}(z\bar\gamma + \frac{1}{2}|\gamma|^2)} e^{\pi i \theta n m} u^{-n} v^{-m}, \qquad z = x + \tau y \in \mathbb{C}, \ \gamma \in \Lambda,$$

satisfying

$$J_{\gamma+\delta}^{-1}(z) = J_\gamma^{-1}(z + \delta) J_\delta^{-1}(z),$$

defines a holomorphic vector bundle E with typical fibre $\mathcal{H}$ over the torus $\mathbb{C}/\Lambda$ given as the quotient

$$(\mathbb{C} \times \mathcal{H})/\sim$$

where

$$(z + \gamma, \mathbf{v}) \sim (z, J_\gamma^{-1}(z)\mathbf{v}),$$

$\mathbf{v} \in \mathcal{H}$. Its (smooth) sections $s : \mathbb{C} \to \mathcal{H}$ (collectively denoted by $\Gamma(E)$) are then characterized by the following periodicity conditions:

$$s(z + 1) = e^{\frac{\theta\pi}{\mathrm{Im}(\tau)}(z+\frac{1}{2})} u^*(s(z)), \qquad s(z + \tau) = e^{\frac{\theta\pi}{\mathrm{Im}(\tau)}(z\bar\tau + \frac{1}{2}|\tau|^2)} v^*(s(z)). \quad (2)$$

This can be ascertained via the following computation. Write

$$\widetilde{s}(z) = s^j(z)\mathbf{e}_j(z)$$

with $\mathbf{e}_j(z)$, $j = 1, 2, \ldots, \dim\mathcal{H}$ a smooth frame (Einstein's convention employed). Then

$$\widetilde{s}(z) = \widetilde{s}(z + \gamma) = s^j(z + \gamma)\mathbf{e}_j(z + \gamma) = s^j(z + \gamma)[J_\gamma^{-1}(z)]_j^i \mathbf{e}_i(z)$$

whence

$$s^i(z + \gamma) = [J_\gamma(z)]_j^i s^j(z),$$

yielding (2).

On $\Gamma(E)$ we have a Hermitian structure $(\cdot|\cdot)$ given by

$$(s|s')(z) = \langle s(z), s'(z) \rangle_{\mathcal{H}} h(z), \qquad h(z) = e^{-\frac{\theta\pi}{\mathrm{Im}(\tau)}|z|^2}$$

with Chern connection (the unique connection compatible with the Hermitian and the holomorphic structure)

$$\nabla = \left(d - \frac{\theta\pi}{\mathrm{Im}(\tau)} \bar z \, dz \right) \otimes \mathbf{1}_{\mathcal{H}}$$

having constant curvature and Chern class

$$\frac{i}{2\pi}\nabla^2 = \theta\omega \otimes \mathbf{1}_{\mathcal{H}}, \qquad c_1(E) = \theta \dim(\mathcal{H})\omega$$

with

$$\omega = \frac{i}{2\operatorname{Im}(\tau)}dz \wedge d\bar{z}.$$

Indeed, a short computation shows that, if $Q := i\nabla_{\frac{\partial}{\partial x}}$, $P := i\nabla_{\frac{\partial}{\partial x}}$, then

$$\frac{1}{2\pi i}[Q, P] = \theta\mathbf{1}_{\Gamma(E)}.$$

Moreover, it is clear that the rank of E is $c_0(E) = \dim(\mathcal{H})$.

This vector bundle will be our E_θ. ∎

Proposition 3.2. *The correspondence that assigns to each representation π of $\mathcal{A}_\theta$ the holomorphic vector bundle $\mathcal{E}_\pi$ over the torus $\mathbb{C}/\Lambda$ is functorial.*

Proof. Let $\pi : \mathcal{A}_\theta \to \mathcal{B}(\mathcal{H}_\pi)$ and $\sigma : \mathcal{A}_\theta \to \mathcal{B}(\mathcal{H}_\sigma)$ be two such representations and let $T : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ be an intertwining unitary map. Then the map on sections

$$\psi_T : \Gamma(\mathcal{E}_\pi) \to \Gamma(\mathcal{E}_\sigma)$$

given by

$$(\psi_T(s))(z) = T(s(z))$$

is an isomorphism of $C^\infty(\mathbb{T}^2)$-modules. The above map is indeed well defined, i.e., it maps sections to sections,

$$\begin{aligned}
(\psi_T(s))(z + n + \tau n) &= e^{\frac{\theta\pi}{\operatorname{Im}(\tau)}(z\bar{\gamma} + \frac{1}{2}|\gamma|^2)}e^{\pi i\theta nm}T\pi(u)^{-n}\pi(v)^{-m}s(z) \\
&= e^{\frac{\theta\pi}{\operatorname{Im}(\tau)}(z\bar{\gamma} + \frac{1}{2}|\gamma|^2)}e^{\pi i\theta nm}\sigma(u)^{-n}\sigma(v)^{-m}Ts(z) \\
&= e^{\frac{\theta\pi}{\operatorname{Im}(\tau)}(z\bar{\gamma} + \frac{1}{2}|\gamma|^2)}e^{\pi i\theta nm}\sigma(u)^{-n}\sigma(v)^{-m}(\psi_T s)(z). \qquad ∎
\end{aligned}$$

Proposition 3.3. *Let $\pi : \mathcal{A}_{q/r} \to \mathcal{B}(\mathfrak{S})$ be an irreducible finite-dimensional representation of the noncommutative torus $\mathcal{A}_{q/r}$ where r and q are positive and coprime. Then, the dimension of $\mathfrak{S}$ is r and $\pi(u) =: \mathfrak{u}$, $\pi(v) =: \mathfrak{v}$ fulfill*

$$\mathfrak{u}^r = \mu\mathbf{1} \qquad \text{and} \qquad \mathfrak{v}^r = \nu\mathbf{1},$$

for some $\mu, \nu \in S^1$.

Proof. Let $\mathfrak{u}, \mathfrak{v}$ define a finite-dimensional irreducible representation of $\mathcal{A}_{q/r}$ on $\mathfrak{S}$ of dimension d. Let us write $\theta := q/r$ with $\gcd(r, q) = 1$. Taking the determinant of $\mathfrak{v}\mathfrak{u} = e^{2\pi i\theta}\mathfrak{u}\mathfrak{v}$ we see that $\theta d \in \mathbb{Z}$. Also observe that, since $\mathfrak{u}^r$ and $\mathfrak{v}^r$ com-

mute with the representation by Schur's lemma, there is a constant $\mu \in S^1$ such that $\mathfrak{u}^r = \mu\mathbf{1}$; therefore, the minimal polynomial of u, call it $\mathcal{P}$, has to divide $x^r - \mu$ and has degree at most r. Moreover, $\mathcal{P}$ satisfies $\mathfrak{v}\mathcal{P}(\mathfrak{u})\mathfrak{v}^* = \mathcal{P}(\mathfrak{u}e^{2\pi i\theta}) = 0$ and the polynomial $\mathcal{Q}(x) = \mathcal{P}(xe^{2\pi i\theta})$ must also satisfy $\mathcal{Q}(\mathfrak{u}) = 0$; thus, given a root λ of $\mathcal{P}$ we see that $e^{2\pi i\theta}\lambda$ is a different root of it, whence the polynomial $\widetilde{\mathcal{P}}(x) := (x - \lambda)(x - e^{2\pi i/r}\lambda)\cdots(x - e^{2\pi i(r-1)/r}\lambda)$ divides $\mathcal{P}$ and, having the same degree as $\mathcal{P}$, coincides with it. In particular, $\mathcal{P}(x) = x^r - \lambda^r$. Let $\varphi \neq 0$ be an eigenvector for $\mathfrak{u}$, then $\{\mathfrak{u}^n\mathfrak{v}^m\varphi\}_{n,m=0}^{r-1}$ generate the whole Hilbert space, and one checks that $\mathfrak{u}^n\mathfrak{v}^m\varphi = \lambda^n e^{-2\pi inm\theta}\mathfrak{v}^m\varphi$ so the dimension d of the Hilbert space $\mathfrak{H}$ is at most r, and therefore equal to r. ∎

Note that if $s, t \in S^1$ and $s^r \neq 1$, $t^r \neq 1$ then

$$\mathfrak{u}' \equiv \pi'(u) := s\mathfrak{u} = s\pi(u), \qquad \mathfrak{v}' \equiv \pi'(uv) := t\mathfrak{v} = t\pi(v),$$

defines a second irreducible representation $\pi' \ncong \pi$, since the minimal polynomial of $\mathfrak{u}'$ is now given by $\mathfrak{u}'^r - s^r\mu\mathbf{1}$ (notational abuse) and it is intrinsic to a representation.

Recall that two noncommutative tori $A := A_\vartheta$ and $B := A_{\vartheta'}$ are called dual, or strongly Morita equivalent if there exists an $A - B$-bimodule E such that they are each other's endomorphism algebra, see, e.g., [20]. This is tantamount to require that ϑ and ϑ' are on the same SL$(2, \mathbb{Z})$-orbit. It turns out that all *rational* noncommutative tori are strongly Morita equivalent to $C(\mathbb{T}^2)$, the C^*-algebra of continuous functions on the torus $\mathbb{T}^2$ ([19]). This entails the following.

Corollary 3.1. *Since $\mathcal{A}_\theta$ is strongly Morita equivalent to the classical torus, its representations are indexed by points in $\mathbb{T}^2$, so they all produce,* topologically, *the same vector bundle.*

Proof. Let π denote a finite-dimensional irreducible representation of the noncommutative torus $\mathcal{A}_\theta$ via operators u, v as above, then any other representation on the same space is given by $u' := \mu u$, $v' := \nu v$, $\mu, \nu \in S^1$. Accordingly, we get another theta character J' and a corresponding holomorphic vector bundle $\mathcal{E}'$, together with a naturally induced isomorphism with $\mathcal{E}$. ∎

Proposition 3.4. *Let $\theta = \frac{q}{r}$, with q and r positive and coprime. Let π be an irreducible representation of $\mathcal{A}_\theta$ on a finite-dimensional Hilbert space $\mathcal{H}$ via operators u, v, inducing as above a HE-vector bundle $E_\theta \to \mathbb{C}/\Lambda$. Then, on the space $\widehat{\mathcal{H}}$ consisting of the L^2-sections of E_θ and in particular on the holomorphic ones (i.e., the Matsushima generalised (vector) theta functions) we have:*

(i) *a natural C^*-representation of $\mathcal{A}_\theta$ given by operators $\check{u} := e^{iQ}$, $\check{v} := e^{iP}$;*

(ii) *a natural C^*-representation of $\mathcal{A}_{1/\theta}$ given by operators $\hat{u}$, $\hat{v}$,*

$$(\hat{u}s)(z) = e^{\frac{\theta\pi}{\mathrm{Im}(\tau)}(\frac{\bar{\tau}}{\theta}z - \frac{1}{2}|\frac{\tau}{\theta}|^2)}s\left(z - \frac{1}{\theta}\tau\right),$$

$$(\hat{v}s)(z) = e^{\frac{\theta\pi}{\mathrm{Im}(\tau)}(\frac{1}{\theta}z - \frac{1}{2}|\frac{1}{\theta}|^2)}s\left(z - \frac{1}{\theta}\right),$$

that is, $\hat{u}$, $\hat{v}$ are unitary operators satisfying $\hat{v}\hat{u} = e^{2\pi i \frac{1}{\theta}}\hat{u}\hat{v}$. The two representations mutually commute.

Proof. Assertion (ii) being clear via a straightforward computation, let us elaborate on (i). By virtue of Proposition 3.3, dim $\mathcal{H} = r$, so we shall denote E_θ also as $\mathcal{E}_{r,q}$. Then notice that $Q = i\nabla_{\frac{\partial}{\partial x}}$, $P = i\nabla_{\frac{\partial}{\partial y}}$ are symmetric and essentially self-adjoint on $\Gamma(\mathcal{E}_{r,q})$ since for all $s, s' \in \Gamma(\mathcal{E}_{r,q})$

$$\int_{\mathbb{C}/\Lambda}\left[(\nabla_{\frac{\partial}{\partial x}}s|s') + (s|\nabla_{\frac{\partial}{\partial x}}s')\right] = \int_{\mathbb{C}/\Lambda}\frac{\partial}{\partial x}(s|s') = 0$$

and similarly for $\nabla_{\frac{\partial}{\partial y}}$. Essential self-adjointness ultimately follows from Nelson's analytic vector theorem, see, e.g., [18, X.6, together with Example 2]. Then, on the same domain, we have

$$\frac{1}{2\pi i}[Q, P] = \theta \cdot \mathbf{1}_{\widehat{\mathcal{H}}}$$

and we shall check below that P and Q and hence $\nabla_{\frac{\partial}{\partial \bar{z}}} = -\frac{\tau}{2\,\mathrm{Im}(\tau)i}Q + \frac{1}{2\,\mathrm{Im}(\tau)i}P$ commute with $\hat{u}$, $\hat{v}$. So in the space $\widehat{\mathcal{H}}$ we have operators satisfying

$$\hat{v}\hat{u} = e^{2\pi i \frac{1}{\theta}}\hat{u}\hat{v},$$

$$U(a)V(b) = e^{2\pi i \theta ab}V(b)U(a), \qquad a, b \in \mathbb{R}$$

where $U(a) = e^{iaP}$, $V(b) = e^{ibQ}$ (parallel transport operators along the fundamental directions). In particular, setting $\check{u} := V(1)$ and $\check{v} := U(1)$ (caveat) we get

$$\check{v}\check{u} = e^{2\pi i\theta}\check{u}\check{v}.$$

It is then enough to show that, on $\Gamma(\mathcal{E}_{r,q})$,

$$[Q, \hat{u}] = 0 = [Q, \hat{v}]$$

and that the same relation holds for P. Let us start with the proof for Q. Let $\alpha = \frac{\theta\pi}{\mathrm{Im}(\tau)}$. We have

$$-iQ\hat{v}s(z) = \frac{\partial}{\partial x}\hat{v}s(z) - \alpha z\hat{v}s(z),$$

$$-i\hat{v}Qs(z) = \hat{v}\frac{\partial s}{\partial x}(z) - \alpha\left(z - \frac{1}{\theta}\right)\hat{v}s(z),$$

so if $(\widehat{v}s)(z) = e^{\beta(z)}s(z - \frac{1}{\theta})$,

$$[-iQ, \widehat{v}]s(z) = \left[\frac{\partial}{\partial x}, \widehat{v}\right]s(z) - \frac{\alpha}{\theta}\widehat{v}s(z) = \frac{\partial\beta}{\partial x}e^{\beta}s\left(z - \frac{1}{\theta}\right) - \frac{\alpha}{\theta}e^{\beta}s\left(z - \frac{1}{\theta}\right) = 0.$$

Now, repeating the computation for P we find

$$-iP\widehat{v}s(z) = \frac{\partial}{\partial y}\widehat{v}s(z) - \alpha\tau z\widehat{v}s(z),$$

$$-i\widehat{v}Ps(z) = \widehat{v}\frac{\partial s}{\partial y}(z) - \alpha\tau\left(z - \frac{1}{\theta}\right)\widehat{v}s(z).$$

Thus

$$[-iP, \widehat{v}]s(z) = \left[\frac{\partial}{\partial y}, \widehat{v}\right]s(z) - \tau\frac{\alpha}{\theta}\widehat{v}s(z) = \frac{\partial\beta}{\partial y}e^{\beta}s\left(z - \frac{1}{\theta}\right) - \tau\frac{\alpha}{\theta}e^{\beta}s\left(z - \frac{1}{\theta}\right) = 0.$$

In the same vein, if $\widehat{u}s(z) = e^{\gamma}s(z - \frac{\tau}{\theta})$,

$$[-iQ, \widehat{u}]s(z) = \left[\frac{\partial}{\partial x}, \widehat{u}\right]s(z) - \alpha\overline{\tau}\frac{1}{\theta}e^{\gamma}s\left(z - \frac{\tau}{\theta}\right)$$

$$= \left(\frac{\partial\gamma}{\partial q} - \alpha\frac{\overline{\tau}}{\theta}\right)e^{\gamma}s\left(z - \frac{\tau}{\theta}\right) = 0,$$

$$[-iP, \widehat{u}]s(z) = \left[\frac{\partial}{\partial y}, \widehat{u}\right]s(z) - \alpha|\tau|^2\frac{1}{\theta}e^{\gamma}s\left(z - \frac{\tau}{\theta}\right)$$

$$= \left(\frac{\partial\gamma}{\partial y} - \alpha\frac{|\tau|^2}{\theta}\right)e^{\gamma}s\left(z - \frac{\tau}{\theta}\right) = 0$$

yielding the conclusion. ∎

We may rephrase the previous result in the following manner.

Proposition 3.5. *Within the above setting, we have a representation of the Heisenberg group* $\mathbb{H} = \mathbb{C} \times S^1$ *with parameter* θ *and a representation of the discrete Heisenberg group* $\mathbb{H}_r = \Lambda \times S^1$ *with parameter* r

$$W(z, t) = e^{2\pi it\theta}e^{i\pi\theta xy}V(x)U(y), \qquad z = x + \tau y \in \mathbb{C},$$

$$\widehat{w}(n, t) = t^r e^{i\pi\theta n_1 n_2}\widehat{v}^{n_1}\widehat{u}^{n_2}, \qquad\qquad n = n_1 + \tau n_2 \in \Lambda$$

where the group structures are given, respectively, by

$$(z, t)(z', t') = (z + z', tt'e^{i\pi(x'y - y'x)}), \qquad z, z' \in \mathbb{C}, t, t' \in S^1$$

and

$$(n, t)(n', t') = (n + n', tt'e^{i\pi\frac{1}{r}(n'_1 n_2 - n_1 n'_2)}), \qquad n, n' \in \Lambda.$$

These representations commute,

$$W(z,t)\hat{w}(n,s) = \hat{w}(n,s)W(z,t)$$

for all z, t, n, s. ∎

We also notice the following consequence.

Proposition 3.6. *We have $\hat{v}^q = \mu \mathbf{1}_{\hat{\mathscr{H}}}$, $\hat{u}^q = \nu \mathbf{1}_{\hat{\mathscr{H}}}$, where $\mu, \nu \in S^1$ are constants given by $u^r = \mu \mathbf{1}_{\mathscr{H}}$, $v^r = \nu \mathbf{1}_{\mathscr{H}}$.*

Proof. We find, successively

$$(\hat{v}^q s)(z) = e^{\frac{\theta\pi}{\mathrm{Im}(\tau)}(\frac{1}{\theta}(z+(z-1/\theta)-\cdots-(z-(q-1)/\theta))-\frac{q}{2}|\frac{1}{\theta}|^2)} s\left(z - \frac{q}{\theta}\right)$$

$$= e^{\frac{\theta\pi}{\mathrm{Im}(\tau)}(\frac{1}{\theta}(qz-\frac{q(q-1)}{2\theta})-\frac{q}{2}|\frac{1}{\theta}|^2)} s(z - r)$$

$$= e^{\frac{\theta\pi}{\mathrm{Im}(\tau)}(rz-\frac{r^2(q-1)}{2q}-\frac{r^2}{2q})} s(z - r)$$

$$= e^{\frac{\theta\pi}{\mathrm{Im}(\tau)}(rz-\frac{r^2}{2})} s(z - r)$$

$$= e^{\frac{\theta\pi}{\mathrm{Im}(\tau)}(rz-\frac{r^2}{2})} e^{\frac{\theta\pi}{\mathrm{Im}(\tau)}(-rz+\frac{r^2}{2})}(u^r)(s(z))$$

$$= \mu s(z),$$

where we used (2) in the second to last equality. Similarly,

$$(\hat{u}^q s)(z) = e^{\frac{\theta\pi}{\mathrm{Im}(\tau)}(\frac{\bar{\tau}}{\theta}(z+(z-\tau/\theta)-\cdots-(z-(q-1)\tau/\theta))-\frac{q}{2}|\frac{\tau}{\theta}|^2)} s\left(z - \frac{\tau q}{\theta}\right)$$

$$= e^{\frac{\theta\pi}{\mathrm{Im}(\tau)}(\frac{\bar{\tau}}{\theta}(qz-\frac{\tau q(q-1)}{2\theta})-\frac{q}{2}|\frac{\tau}{\theta}|^2)} s(z - \tau r)$$

$$= e^{\frac{\theta\pi}{\mathrm{Im}(\tau)}(r\bar{\tau}z-\frac{r^2(q-1)|\tau|^2}{2q}-\frac{r^2|\tau|^2}{2q})} s(z - \tau r)$$

$$= e^{\frac{\theta\pi}{\mathrm{Im}(\tau)}(r\bar{\tau}z-\frac{r^2|\tau|^2}{2})} s(z - \tau r)$$

$$= e^{\frac{\theta\pi}{\mathrm{Im}(\tau)}(r\bar{\tau}z-\frac{r^2|\tau|^2}{2})} e^{\frac{\theta\pi}{\mathrm{Im}(\tau)}(-r\bar{\tau}z+\frac{r^2|\tau|^2}{2})}(v^r)(s(z))$$

$$= \nu s(z).$$ ∎

Note that, in accordance with the Stone–von Neumann theorem, the representation W is not irreducible: indeed, by Riemann–Roch, its multiplicity is precisely $\theta \dim(\mathscr{H}) = (q/r) \cdot r = q$, also cf. [22, 23].

Corollary 3.2. *Each k-level theta line bundle over the two-dimensional torus produces an irreducible finite-dimensional representation of $\mathcal{A}_\theta$ with $\theta = 1/k$.*

In view of the preceding discussion (Section 2.4) on the harmonic oscillator we have (with obvious and inessential notational changes), the following result.

Proposition 3.7. *Let $\Delta = (\nabla_{\partial_{\bar{z}}})^*\nabla_{\partial_{\bar{z}}} \equiv A^*A$ be the Laplacian on E_θ (the "number operator"). Then its spectrum only consists of eigenvalues, whose eigenspaces are finite-dimensional with the same dimension q and each carrying a representation of $\mathcal{A}_{1/\theta} = \mathcal{A}_{r/q}$ (with generators $\hat{u}$, $\hat{v}$).*

Remark 3.1. The above proposition, together with the preceding developments, reformulate and possibly improve (in the classical case) the celebrated results establishing the action of finite Heisenberg groups on spaces of theta functions (viewed as holomorphic sections of line bundles on complex tori), see, e.g., [14] (especially volume III) and [17] (cf. in particular, the remark in Section 3.1).

Proposition 3.8 (Bimodule structure). *The HE-vector bundle $\mathcal{E}_{r,q} \to \mathbb{C}/\Lambda$ (actually, its space of smooth sections $\Gamma(\mathcal{E}_{r,q})$) comes equipped with a $\mathcal{A}_\theta - \mathcal{A}_{-1/\theta}$ bimodule structure, where $\mathcal{A}_\theta$ acts on the left by $\check{u}$ and $\check{v}$, and $\mathcal{A}_{1/\theta}$ acts on the right by $\hat{u}$ and $\hat{v}$.*

Proof. This is clear in view of the preceding discussion. The minus sign comes from regarding $\hat{u}$ and $\hat{v}$ as acting on the right. ∎

Remark 3.2. Regarding the above mentioned strong Morita equivalence between $\mathcal{A}_\theta$ and $\mathcal{A}_{-1/\theta}$, it would be interesting to explicitly compare the two algebra-valued Hermitian structures involved, see, e.g., [20].

4. Gauss sums, vector theta functions and the FMN-transform

The above construction can be interpreted in terms of the so-called Fourier–Mukai–Nahm (FMN*) transform (plus dualization) as in [23] (see, in particular, Section 4.3). For background on the FMN-transform see, e.g., – in addition to the original sources [13, 15] – the article [24] and the textbook [1].

Specifically, in view of Proposition 3.3, an irreducible representation π' of $\mathcal{A}_{1/\theta}$ on a finite-dimensional Hilbert space $\mathcal{H}'$ yields in turn, *á la* Matsushima, a HE-vector bundle $E_{1/\theta} \to \mathbb{C}/\Lambda$ with rank $q = \dim H^0(E_\theta) = \dim \mathcal{H}'$ and degree r (FMN*-dual to $E_\theta \to \mathbb{C}/\Lambda$) equipped with a Chern connection ∇' having constant curvature and Chern class given respectively by

$$\frac{i}{2\pi}\nabla'^2 = (1/\theta)\omega\mathbf{1}_{\mathcal{H}'}, \qquad c_1(E_{\frac{1}{\theta}}) = (1/\theta)\dim(\mathcal{H}')\omega.$$

The above connection can be also readily computed via noncommutative geometric tools as in [23]. In general, the moduli dependence is governed by Proposition 3.6.

In the following sections, we shall reformulate the Matsushima approach by a further enhancement of a noncommutative torus perspective and by enforcing FMN* from the outset via a matrix portrait and by building upon the Coherent State Transform of [8].

4.1. δ-description of vector theta functions and Gauss sums

Let us consider coprime positive integers r and q, set, for $x \in S^1$,

$$\delta_\ell^{(q)}(x) := \delta[(x - \ell/r)q], \qquad \ell = 0, 1, \ldots, r-1.$$

If $q = 1$ we simply write δ_ℓ instead of $\delta_\ell^{(q)}$. From the (distributional) Fourier expansion (involving a q-covering $S^1 \to S^1$ and, dually, the subgroup $q\mathbb{Z} \subset \mathbb{Z}$)

$$\delta_0^{(q)}(x) = \delta(qx) = \sum_{n \in \mathbb{Z}} e^{2\pi i n q x} \equiv \mathcal{F}\delta^{(q)} \equiv \theta_0^{(q)}$$

one gets

$$\delta_\ell^{(q)}(x) = \sum_{0 \le \ell' \le r-1} e^{-2\pi i \ell' \ell \frac{q}{r}} \sum_{n \in \mathbb{Z}} e^{2\pi i (\ell' + rn)qx} = \sum_{0 \le \ell' \le r-1} e^{-2\pi i \ell' \ell \frac{q}{r}} \theta_{\ell'}^{(q)} \equiv a_\ell^{\ell'} \theta_{\ell'}^{(q)}$$

via the introduction of the (invertible) $r \times r$ matrix (cf. [8])

$$A := (a_\ell^{\ell'}) = (e^{-2\pi i \ell' \ell \frac{q}{r}})$$

(Einstein's convention is employed) relating the δ and (boundary) theta descriptions; thus

$$\text{Tr}\, A = \sum_{0 \le \ell \le r-1} e^{-2\pi i \ell^2 \frac{q}{r}} = \overline{S(q, r)},$$

i.e., a Gauss sum. Similarly (obvious notation, with $y \in S^1$), one has

$$\delta_m^{(r)}(y) := \delta[(y - m/q)r] = \sum_{0 \le m' \le q-1} e^{-2\pi i m' m \frac{r}{q}} \sum_{n \in \mathbb{Z}} e^{2\pi i (m' + qn)ry} \equiv b_m^{m'} \theta_{m'}^{(r)}$$

with a corresponding matrix

$$B := (b_m^{m'}) = (e^{-2\pi i m' m \frac{r}{q}})$$

with

$$\text{Tr}\, B = \sum_{0 \le m \le q-1} e^{-2\pi i m^2 \frac{r}{q}} = \overline{S(r, q)}.$$

Then consider the tensor product distributions

$$\delta_\ell^{(q)}(x)\delta_m^{(r)}(y), \qquad x, y \in S^1.$$

Upon fixing $m \in \{0, 1, \ldots, q-1\}$, one has an obvious r-component *column* vector, representing a model for the m-th Matsushima holomorphic section for the vector bundle $\mathcal{E}_{r,q} \to V/L$. More precisely, we have (with a natural abridged notation), upon

suitably reinterpreting Matsushima's construction ([12, Section 8], and our previous discussion on the CST in Section 2.1),

$$\vec{\delta}_{\cdot,m} := (\delta_{0,m}, \delta_{1,m}, \dots, \delta_{r-1,m})^T \leftrightarrow \delta_{0,m}.$$

The q columns thus obtained yield a basis for a q-dimensional Hilbert space $\mathcal{H}^q \cong H^0(V/L, \mathcal{E}_{r,q})$.

Similarly, fixing $\ell \in \{0, 1, \dots, r-1\}$, we get a *row* vector, giving rise to a model for the ℓ-th holomorphic section of the (FMN*) dual vector bundle $\mathcal{E}_{q,r} \to V/L$, namely,

$$\vec{\delta}_{\ell,\cdot} := (\delta_{\ell,0}, \delta_{\ell,1}, \dots, \delta_{\ell,q-1}) \leftrightarrow \delta_{\ell,0}.$$

and the ensuing r rows yield a basis for an r-dimensional Hilbert space $\mathcal{H}^r \cong H^0(V/L, \mathcal{E}_{q,r})$.

Also, in view of the previous considerations (Section 2.5, we can naturally establish a bijective correspondence

$$\delta_\ell^{(q)}(x)\delta_m^{(r)}(y) \leftrightarrow \delta_k(z) = \delta(z - k/rq), \qquad z \in S^1$$

($k \in \{0, 1, \dots, rq - 1\}$), with the rq-level thetas, viewed as holomorphic sections of the complex line bundle $\mathcal{E}_{1,rq} \to V/L'$.

Therefore, one finds a third matrix

$$C := (c_k^{k'} = e^{-2\pi i k'k\frac{1}{rq}})$$

with

$$\text{Tr}\, C = \sum_{0 \le k \le rq-1} e^{-2\pi i k^2 \frac{1}{rq}} = \overline{S(1, rq)}.$$

The above matrix is related to the former ones in the following way. Let us consider the following rq-dimensional Hilbert spaces: $\mathcal{H}^{rq}$, generated by the orthonormal basis $\delta_\ell^{(q)}(x)\delta_m^{(r)}(y)$, and $\mathbb{H}^{rq}$, generated by the orthonormal basis $\delta_k(z)$; we have then a natural unitary transformation $\mathcal{T} : \mathcal{H}^{rq} \to \mathbb{H}^{rq}$

$$\mathcal{T}(\delta_\ell^{(q)}\delta_m^{(r)}) := \delta_k$$

whereby

$$\mathcal{T}(\theta_\ell^{(q)}\theta_m^{(r)}) = \theta_k$$

as well (shorthand notation), this easily leading to

$$C = \mathcal{T}(A \otimes B)\mathcal{T}^{-1}.$$

Therefore, from

$$\text{Tr}\, C = \text{Tr}[\mathcal{T}(A \otimes B)\mathcal{T}^{-1}] = \text{Tr}(A \otimes B) = \text{Tr}\, A \cdot \text{Tr}\, B$$

we get a special case of the above multiplicative formula for Gauss sums with $\mu = 1$.

Actually, the general formula is also obtained via the same technique, after introducing from the outset another μ-covering $S^1 \to S^1$, resulting in an extra factor μ in the numerators of all arguments of the exponentials.

This may be viewed as a sort of *categorification* of Gauss sums in the sense that, as numerical objects, they come from the multiplicativity of tensor product traces.

A variant of the above procedure consists in exploiting the (algebra) isomorphism $M_{rq}(\mathbb{C}) \cong M_r(\mathbb{C}) \otimes M_q(\mathbb{C})$ via the elementary matrix bases E_{ij} (i.e., the matrices whose (i, j)-entry is 1 and all others are zero),

$$E_{kk'} \leftrightarrow E_{\ell\ell'} \otimes E_{mm'}$$

(abridged notation: k, k' and so on are taken modulo the size of the respective matrices).

Remark 4.1. A few words about the *heuristics* behind the above discussion are maybe in order: upon *formally* multiplying the deltas labelled by ℓ and m after taking the same argument $x = y$ (this is an ill-defined object!), one has, for the product of their Fourier series, after an obvious index relabelling, the (meaningless) expression

$$\sum_{n,N\in\mathbb{Z}} e^{2\pi i[\ell q + mr + rqN]x} \tag{3}$$

which, upon *discarding* the sum in n, yields the distributional Fourier series expressing $\delta(z - k/rq)$ – after changing x to z – with $[k]_{rq}$ obeying the above equation.

4.2. Noncommutative torus aspects of the δ-formulation

Set (abridged notation) $\delta_{\ell m} := \delta_\ell \delta_m$ and define, in $\mathcal{H}^{rq}$, for $\mu, \nu, \tilde{\mu}, \tilde{\nu} \in S^1$

$$\mathfrak{U}\delta_{\ell m} := \mu\delta_{\ell,m-1}, \qquad \mathfrak{V}\delta_{\ell m} := \nu e^{-2\pi i \frac{m}{q}}\delta_{\ell m}$$

and

$$\tilde{\mathfrak{U}}\delta_{\ell m} := \tilde{\mu}\delta_{\ell-1,m}, \qquad \tilde{\mathfrak{V}}\delta_{\ell m} := \tilde{\nu} e^{-2\pi i \frac{\ell}{r}}\delta_{\ell m}$$

(cyclic ordering understood: for instance, $\mathfrak{U}\delta_{\ell,0} := \mu\delta_{\ell,q}$ et cetera). One has, upon restriction to the spaces indicated, $\mathfrak{U}^q = \mu^q 1_{\mathcal{H}^q}$, $\mathfrak{V}^q = \nu^q 1_{\mathcal{H}^q}$, $\tilde{\mathfrak{U}}^r = \tilde{\mu}^r 1_{\mathcal{H}^r}$, $\tilde{\mathfrak{V}}^r = \tilde{\nu}^r 1_{\mathcal{H}^r}$ and subsequently

$$\mathfrak{U}\mathfrak{V} = e^{-2\pi i \frac{1}{q}}\mathfrak{V}\mathfrak{U}, \qquad \tilde{\mathfrak{U}}\tilde{\mathfrak{V}} = e^{-2\pi i \frac{1}{r}}\tilde{\mathfrak{V}}\tilde{\mathfrak{U}}.$$

Then define

$$\mathbb{U}\delta_{\ell m} := \mu\delta_{\ell,m-1} = \mathfrak{U}\delta_{\ell m}, \qquad \mathbb{V}\delta_{\ell m} := \nu^r e^{-2\pi i \frac{r}{q}m}\delta_{\ell m} = \mathfrak{V}^r\delta_{\ell m},$$

yielding

$$\mathbb{U}\mathbb{V} = e^{-2\pi i \frac{r}{q}} \mathbb{V}\mathbb{U},$$

that is, a representation of $\mathcal{A}_{1/\theta}$, $\theta = q/r$ (cyclic ordering again understood), with $\mathbb{U}^q = \mu^q 1_{\mathcal{H}^q}$, $\mathbb{V}^q = \nu^{rq} 1_{\mathcal{H}^q}$. Similarly, upon setting

$$\widetilde{\mathbb{U}}\delta_{\ell m} := \widetilde{\mu}\delta_{\ell-1,m} = \widetilde{\mathfrak{U}}\delta_{\ell m}, \qquad \widetilde{\mathbb{V}}\delta_{\ell m} := \widetilde{\nu}^q e^{-2\pi i \frac{q}{r}\ell}\delta_{\ell m} = \widetilde{\mathfrak{V}}^q \delta_{\ell m},$$

we get

$$\widetilde{\mathbb{U}}\widetilde{\mathbb{V}} = e^{-2\pi i \frac{q}{r}}\widetilde{\mathbb{V}}\widetilde{\mathbb{U}}$$

(a representation of $\mathcal{A}_\theta$), together with $\widetilde{\mathbb{U}}^r = \widetilde{\mu}^r 1_{\mathcal{H}^r}$, $\widetilde{\mathbb{V}}^r = \widetilde{\nu}^{rq} 1_{\mathcal{H}^r}$. These two representations mutually commute (since they do not mix first and second subscripts) and they are exchanged upon application of the FMN*-transform.

The action of the various operators involved can be cast in a more compact way as follows: in $\mathcal{H}^q$ one has

$$\mathbb{U}\vec{\delta}_{\cdot,m} = \mu\vec{\delta}_{\cdot,m-1}, \qquad \mathbb{V}\vec{\delta}_{\cdot,m} = \nu^r e^{-2\pi i \frac{r}{q}m}\vec{\delta}_{\cdot,m}, \qquad m = 0, 1, \ldots, q-1$$

with the "tilded" operators acting as the identity:

$$\widetilde{\mathbb{U}}\vec{\delta}_{\cdot,m} = \vec{\delta}_{\cdot,m}, \qquad \widetilde{\mathbb{V}}\vec{\delta}_{\cdot,m} = \vec{\delta}_{\cdot,m}, \qquad m = 0, 1, \ldots, q-1.$$

A similar portrait, *mutatis mutandis*, holds in $\mathcal{H}^r$. Summarizing, we have the following proposition.

Proposition 4.1. *The above operators $\mathbb{U}$, $\mathbb{V}$ realize a representation of $\mathcal{A}_{r/q}$ unitarily equivalent to a representation induced by $\widehat{u}$, $\widehat{v}$ in Proposition 3.4 above, after restriction of the latter to the space of holomorphic sections of $\mathcal{E}_{r,q}$. An analogous statement is true for the tilded operators and the FMN*-transformed bundle $\mathcal{E}_{q,r}$.*

Remark 4.2. Geometrically, the above "toric" families of representations correspond to tensoring the initial holomorphic bundle $E_\theta \to \mathbb{C}/\Lambda$ with the flat line bundle $\mathcal{P}_\xi \to \mathbb{C}/\Lambda$, the restriction of the Poincaré bundle to $\mathbb{C}/\Lambda \times \{\xi\} \cong \mathbb{C}/\Lambda$, where $\xi = (\mu, \nu)$ and similarly for $E_{1/\theta} \to \mathbb{C}/\Lambda$. Also notice that the torus also classifies holonomies of the different Chern connections, see also [23].

We recover the standard Matsushima correspondence involving the holomorphic vector bundle $\mathcal{E}_{r,q} \to V/L$ and the q-level theta line bundle $\mathcal{E}_{1,q} \to V/L'$ via the Coherent State Transform CST through the following steps (obvious abridged notation), also setting $\mu = \widetilde{\mu} = \nu = \widetilde{\nu} = 1$ for simplicity. Indeed,

$$\vartheta_{j,m} = \mathrm{CST}(\delta_{j,m}) = \mathrm{CST}(\widetilde{U}^j \delta_{0,m}) = \mathrm{CST}(\widetilde{\mathbb{U}}^j \delta_{0,m}), \qquad j = 0, 1, \ldots, r-1$$

whence

$$\vec{\vartheta}_{\cdot,m} := (\vartheta_{j,m})_{j=0,\ldots,r-1}^T =: \mathrm{CST}(\vec{\delta}_{\cdot,m}) \leftrightarrow \vartheta_{0,m}.$$

Actually, in this manner we have defined a *vector* version of the Coherent State Transform:

$$\mathcal{CST} = \mathcal{M}^{-1} \circ \text{vec} \circ \text{CST} \circ \mathcal{F}$$

mapping $\delta_{0,m} \mapsto s_m$ and spelled out as follows

$$\delta_{0,m} \overset{\mathcal{F}}{\longmapsto} \theta_{0,m}(x) \overset{\text{CST}}{\longmapsto} \vartheta_{0,m}(z) \overset{\text{vec}}{\longmapsto} \vec{\vartheta}_{\cdot,m}(z) \overset{\mathcal{M}^{-1}}{\longmapsto} s_m.$$

The notation $\mathcal{M}^{-1}$ is justified since $\mathcal{M}$ is injective and $\text{Im}(\text{vec}) \subset \text{Im } \mathcal{M}$. We may recap the previous discussion via the isomorphism

$$H^0(V/L, \mathcal{E}_{r,q}) \otimes H^0(V/L, \mathcal{E}_{q,r}) \cong H^0(V/L', \mathcal{E}_{1,rq})$$

induced by the correspondence

$$s_m \otimes s_\ell \leftrightarrow s_k$$

where again

$$[k]_{rq} = q[l]_r + r[m]_q$$

and by further noticing the following "categorical" result.

Proposition 4.2. *Under the above assumptions, we have*

$$\mathcal{A}_{1/rq} \cong \mathcal{A}_{q/r} \otimes \mathcal{A}_{r/q}.$$

Proof. First observe that noncommutative tori are nuclear C^*-algebras, so their C^*-tensor product appearing in the right-hand side is uniquely determined ([3]). Then, starting from

$$uv = e^{-2\pi i \frac{r}{q}} vu$$

and

$$\widetilde{u}\widetilde{v} = e^{-2\pi i \frac{q}{r}} \widetilde{v}\widetilde{u}$$

(all tilded operators commuting with untilded ones), define (same notation as before: $k = \ell q + mr$ and so on)

$$\mathcal{U}^k := u^m \widetilde{u}^\ell, \qquad \mathcal{V}^{k'} := v^{m'} \widetilde{v}^{\ell'}.$$

A straightforward computation then yields

$$\mathcal{U}^k \mathcal{V}^{k'} = e^{-2\pi i \frac{kk'}{rq}} \mathcal{V}^{k'} \mathcal{U}^k.$$

The above reasoning is clearly invertible, achieving the sought for conclusion. ∎

Remark 4.3. Upon further requiring that $u^q = v^q = 1$ and $\widetilde{u}^r = \widetilde{v}^r = 1$, we get $\mathcal{U}^{rq} = \mathcal{V}^{rq} = 1$ (with 1 the identity in the respective algebras).

Also notice that, at the vector bundle level, this reflects an operation (denoted by $\star$)

$$\mathcal{E}_{r,q} \star \mathcal{E}_{q,r} = \mathcal{E}_{1,rq}.$$

casting light on FMN-duality via a "Gauss" perspective.

5. Conclusions and outlook

In this paper, we used complex algebraic-geometric and noncommutative geometric techniques in order to understand and possibly enhance, at least in the classical case (i.e., on the complex field) the relationship between theta functions and Heisenberg groups. Our research is strongly motivated by the Quantum Hall Effect as well: the symmetry $\theta \leftrightarrow 1/\theta$ discussed in the paper may ultimately lead to an explanation of the duality occurring between Hofstadter's and Harper's regimes, see, e.g., [5]. Also, our results may help in providing a clear-cut mathematical formulation of the important *Laughlin gauge principle* for a toral configuration, see, e.g., the comprehensive review [10]. Finally, instances of the vector bundles dealt with in the present paper also appear in the works [5, 6], devoted to a far reaching generalization of the TKNN equations. These questions will be possibly tackled elsewhere.

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