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Topological semi-conjugacy between dynamical systems induced on the space of probability measures and subshifts of finite type, and chaos

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Abstract. This paper establishes topological semi-conjugacy between a symbolic dynamical system (Σ_A, σ_A) and the system $(M(X), T_M)$ induced on the space of probability measures by a compact topological dynamical system (X, T). Some conditions on an original system (X, T) are obtained for the induced system $(M(X), T_M)$ to have a subsystem topologically semi-conjugate to (Σ_A, σ_A), and some relations between an original system (X, T) and the induced system $(M(X), T_M)$ concerning semi-conjugacy to (Σ_A, σ_A) are given. By using these results, several criteria of Li–Yorke chaos and Devaney chaos for the induced system $(M(X), T_M)$ are established. Also, a characterization of the $\mathcal{F}$-mixing dynamical system (X, T) with Furstenberg family $\mathcal{F}$ is obtained by means of the system $(M(X), T_M)$, which gives an answer to the question posed by Fu and Xing [Chaos Solitons Fractals (2012), 439–443].

1. Introduction

Symbolic dynamical systems have rich dynamical behaviors including chaos and play an important role in the study of chaos theory and dynamical systems. A topological conjugacy (or a semi-conjugacy) between two dynamical systems implies equivalency between many of dynamical properties of the two dynamical systems. So we often consider whether or not a given topological dynamical system is topologically conjugate (or semi-conjugate) to a symbolic dynamical system when determining whether or not it has certain dynamical complexity such as chaos.

A number of papers have been dedicated to the study of the topological conjugacy or semi-conjugacy between symbolic dynamical systems and other dynamical systems (see [17] for more details).

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The notion of the A-coupled-expanding map for a transitive matrix A which is a generalization of the famous Smale's horseshoe, was used as one of some criteria for a map to be topologically conjugate or semi-conjugate to a subshift of finite type (Σ_A, σ_A) (see [8] and [17] for detail). In many papers it has been shown that a (strictly A-)coupled-expanding map has a subsystem topologically conjugate or topologically semi-conjugate to the subshift of finite type (Σ_A, σ_A) under certain conditions (for example, see [2–4, 7–11, 15–17, 19, 21–23]).

Ju, H., Shao, H., Choe, Y., and Shi, Y. [8] obtained more relaxed necessary and sufficient conditions for a dynamical system to have a subsystem topologically conjugate or semi-conjugate to the subshift of finite type (Σ_A, σ_A) in the context of Hausdorff space. They proved that two of the necessary and sufficient conditions obtained in [23] and [17] for it, are not independent of each other and either condition can be deleted, that is, they obtained a simplified criterion consisted of only one for it, which makes it easier for us to check theoretically the criterion of the topological conjugacy (or semi-conjugacy) with a subshift of finite type (Σ_A, σ_A) for given a dynamical system.

The present paper is concerned with the conditions on an original topological dynamical system for its induced system on the space of all Borel probability measures to have a subsystem topologically semi-conjugate to a subshift of finite type (Σ_A, σ_A).

Given a topological dynamical system (t.d.s. for short) (X, T) where X is a compact metric space and T is a continuous map on X, it induces three systems in natural manner; the first one is a hyperspace dynamical system $(K(X), T_K)$ on the hyperspace $K(X)$ consisting of all nonempty closed subsets of X endowed with the Hausdorff metric, the second one is a dynamical system $(M(X), T_M)$ on the space $M(X)$ consisting of all Borel probability measures equipped with the weak*-topology and the third one is a fuzzy dynamical system (see [7, 12, 20] for more detail).

Wang, Y. and Wei, G. [19] obtained conditions on an original compact t.d.s. ensuring that the induced hyperspace dynamical system contains at least one subsystem topologically (semi-)conjugate to a symbolic dynamical system, that is, they proved that, if a t.d.s. (X, T) is strictly coupled-expanding, then its induced system $(K(X), T_K)$ has at least one subsystem which is topologically semi-conjugate to a full shift (Σ_m, σ).

Ju, H., Kim, C., Choe, Y., and Chen, M. [7] obtained a sufficient condition on an original compact t.d.s. for the induced system $(K(X), T_K)$ to have a subsystem topologically (semi-)conjugate to a subshift of finite type (Σ_A, σ_A), which generalizes the result of [19]. And in [7] they also obtained a sufficient condition on an original compact t.d.s. for its induced fuzzy dynamical system to have a subsystem topologically (semi-)conjugate to a subshift of finite type (Σ_A, σ_A). In [14], Li, Z., Wang, M. and Wei, G. investigated the expansivity and topological entropy of the hyperspace dynamical system $(K(\Sigma), \sigma_K)$ induced by full shift (Σ, σ).

Shao, H., Chen, G., and Shi, Y. [15] studied topological conjugacy between induced non-autonomous hyperspace systems and (Σ_A, σ_A).

However, for the induced system $(M(X), T_M)$ on the space $M(X)$ of Borel probability measures, there is no result concerned with this problem yet to the best of our knowledge, while it has been solved as for two induced systems, that is, the hyperspace dynamical system and the fuzzy dynamical system.

So first, we devote this paper to fill this gap.

In fact, given a state space X of some system, the elements of the space $M(X)$ of probability measures on X, can be viewed as statistical states, representing imperfect knowledge of the system. When T is a continuous map on X and T_M the corresponding induced map of $M(X)$, $(M(X), T_M)$ can be viewed as a system where the dynamics is deterministic and the configurations are stochastic [1]. So the system $(M(X), T_M)$ has many applications not only in classical statistical mechanics but also in several fields. For example, [6] showed applications of the induced system $(M(X), T_M)$ to dynamic programming and stochastic game theory. So many previous works have dealt with the dynamical properties including the chaos for the induced system on the space of probability measures (see [1, 6, 12, 13, 20], etc.).

Generally it is known that while there are similarities in the dynamics of these three t.d.s.s, (X, T), $(K(X), T_K)$ and $(M(X), T_M)$, they are not straightforward and the relations between an original system (X, T) and the induced systems are not always the same for $(K(X), T_K)$ and $(M(X), T_M)$ (for example, see [12] and the references given therein). In [12] Li, J., Oprocha, P., and Wu, X. investigated some relations of sensitivity between the original system (X, T) and its induced system $(M(X), T_M)$. These imply that it needs to study about the above mentioned gap.

Next, this paper is also concerned with the question posed in [5].

In [5], Fu, H. and Xing, Z. studied mixing properties of the induced system $(K(X), T_K)$ via Furstenberg families, and posed a question: what is the analogous equivalent characterization in $(M(X), T_M)$ corresponding to their one result (Theorem 5.1) (which characterized the $\mathcal{F}$-mixing system (X, T) by means of the induced system $(K(X), T_K)$)?

Using a result obtained in this paper, we show a characterization of the mixing original system (X, T) by means of the induced system $(M(X), T_M)$ to give an answer to the question.

The rest of the present paper is organized as follows. In Section 2, some basic concepts are introduced. In Section 3, the topological semi-conjugacy between a subshift of finite type (Σ_A, σ_A) and its induced system $(M(X), T_M)$ is studied. It is shown a condition on an original system (X, T) for the induced system $(M(X), T_M)$ to have a subsystem topologically semi-conjugate to (Σ_A, σ_A). Here the simplified criterion obtained in [8] is a key to obtain the condition without big difficulty. And some relations between an original system (X, T) and the induced system $(M(X), T_M)$

concerning topological semi-conjugacy to (Σ_A, σ_A) are given. In Section 4, by applying the above obtained results, several criteria of Li–Yorke chaos and Devaney chaos for the induced system $(M(X), T_M)$ are established. In Section 5, a characterization of the $\mathcal{F}$-mixing system (X, T) by means of the system $(M(X), T_M)$ is obtained, and an answer to the question posed in [5] is given.

2. Preliminary

2.1. Topological dynamical system

By a *topological dynamical system* (t.d.s. for short) we mean a pair (X, T), where X is a compact metric space with a metric d and $T : X \to X$ is a continuous map.

Let $m \geq 2$ and $\mathcal{A} = \{1, \ldots, m\}$. $\mathbb{N}_0$ denotes the set of all non-negative integers. The set $\Sigma_m = \{\alpha = (a_0, a_1, \ldots) : a_i \in \mathcal{A},\ i \in \mathbb{N}_0\}$ is a compact metric space with the metric

$$\rho(\alpha, \beta) = \begin{cases} 0 & \text{if } \alpha = \beta, \\ 2^{-(k+1)} & \text{if } \alpha \neq \beta \text{ and } k = \min\{i : a_i \neq b_i\}, \end{cases}$$

where $\alpha = (a_0, a_1, \ldots)$, $\beta = (b_0, b_1, \ldots) \in \Sigma_m$. The *full shift* $\sigma : \Sigma_m \to \Sigma_m$ is a continuous map defined by $\sigma(\alpha) = (a_1, a_2, \ldots)$, where $\alpha = (a_0, a_1, a_2, \ldots) \in \Sigma_m$. The t.d.s. (Σ_m, σ) is called a *symbolic dynamical system* by full shift (shortly, *full shift*). Let $A = (a_{ij})$ be an $m \times m$ transitive matrix, that is, $a_{ij} = 0$ or 1 for all i, j. The set

$$\Sigma_A = \{(b_0, b_1, \ldots) \in \Sigma_m : a_{b_i b_{i+1}} = 1,\ i \in \mathbb{N}_0\}$$

is a compact σ-invariant set. The map $\sigma_A = \sigma|_{\Sigma_A}$ is called a *subshift map of finite type* for the matrix A and the subsystem (Σ_A, σ_A) is called a *symbolic dynamical system* by *subshift of finite type* for the matrix A (shortly, *subshift of finite type*).

The definitions of chaos in the sense of Li–Yorke and Devaney are well known in the field.

Definition ([17]). Let (X, d) be a metric space and $T : V \subset X \to X$. Suppose that $A = (a_{ij})$ is an $m \times m$ transitive matrix for some $m \geq 2$. If there exist m nonempty subsets V_i $(1 \leq i \leq m)$ of V with pairwise disjoint interiors such that

$$T(V_i) \supset \bigcup_{\substack{j, \\ a_{ij}=1}} V_j$$

for all $1 \leq i \leq m$, then the map T is said to be *A-coupled-expanding* in V_i $(1 \leq i \leq m)$. Moreover, the map T is said to be *strictly A-coupled-expanding* in V_i $(1 \leq i \leq m)$ if

$d(V_i, V_j) > 0$ for all $1 \leq i \neq j \leq m$, where $d(V_i, V_j) = \inf\{d(x, y) : x \in V_i,\ y \in V_j\}$. If all entries of the matrix A are equal to 1, then a (strictly) A-coupled-expanding map T is shortly called (strictly) *coupled-expanding* map.

2.2. The induced system on the hyperspace

Let (X, T) be a t.d.s.. We consider the following hyperspace:

$$K(X) = \{A \subseteq X : A \text{ is nonempty and closed}\}.$$

Let $d(x, A) = \inf\{d(x, y) : y \in A)\}$ and $A^\varepsilon = \{x \in X : d(x, A) < \varepsilon\}$ for $x \in X$ and $A \subset X$. We can endow this space with the Hausdorff metric D_H defined as follows:

$$D_H(A, B) = \inf\{\varepsilon > 0 : A \subset B^\varepsilon,\ B \subset A^\varepsilon\},$$

where $A, B \in K(X)$.

It is known that whenever a crisp space X is compact, complete and separable, then the metric space $(K(X), D_H)$ has those three properties. When (X, d) is a compact metric space, then the topology induced by the Hausdorff metric D_H and the Vietoris topology are consistent on $K(X)$. A base for the Vietoris topology is given by the collection of sets of the form

$$\langle U_1, U_2, \ldots, U_m\rangle = \left\{A \in K(X) : A \subset \bigcup_{i=1}^{m} U_i,\ A \cap U_i \neq \emptyset,\ ; i = 1, 2, \ldots, m\right\},$$

where $U_1, U_2, \ldots, U_m$ are nonempty open subsets of X. The continuous map $T : X \to X$ induces a *set-valued extension* $T_K : K(X) \to K(X)$ in a natural manner as follows:

$$T_K(A) = T(A), \quad A \in K(X).$$

Consequently, we obtain new *induced system* $(K(X), T_K)$ by (X, T).

2.3. The induced system on the space of probability measures

Let (X, T) be a t.d.s. and $M(X)$ be the space of all Borel probability measures on X equipped with the *Prohorov metric* D_P defined by

$$D_P(\mu, \nu) = \inf\{\varepsilon : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \text{ for any Borel subset } A \subset X\}$$

for $\mu, \nu \in M(X)$. The topology induced by D_P coincides with the weak*- topology for measures.

It is well known that a map $T_M : M(X) \to M(X)$ in a natural manner is induced by T as follows:

$$T_M(\mu)(A) = \mu(T^{-1}(A)),$$

for any $\mu \in M(X)$ and Borel subset $A \subset X$, and then $(M(X), T_M)$ is also a (compact) t.d.s.. More details can be found in [1].

3. Topological semi-conjugacy of the induced system on the space of probability measures to the subshift of finite type

In this section, we study some conditions on an original system (X, T) for the induced system $(M(X), T_M)$ on the space of probability measures to have a subsystem topologically semi-conjugate to a subshift of finite type (Σ_A, σ_A) and some relations between the original system (X, T) and the induced system $(M(X), T_M)$ concerning topological semi-conjugacy to (Σ_A, σ_A).

Let (X, T) be a t.d.s.. For $A \subset X$, we define

$$M(A) = \{\mu \in M(X) : \mathrm{supp}\mu \subset A\}.$$

First, we prove some propositions and lemmas.

Proposition 3.1. *If $A \in K(X)$, then $M(A)$ is a compact subset of $M(X)$.*

Proof. We take any sequence $(\mu_n)_{n=1}^{\infty}$ in $M(A)$ such that $\mu_n \to \mu$ $(n \to \infty)$. Then, for any open set $U \subset X \setminus A$ we have that $\mu(U) = 0$ by using $\mu_n(U) = 0$ for any $n \in \mathbb{N}$ and

$$\liminf_{n\to\infty} \mu_n(U) \geq \mu(U),$$

which means $\mu \in M(A)$. ■

Proposition 3.2. *Given a measure $\mu \in M(X)$, it holds that*

$$\mathrm{supp}\, T_M^n(\mu) = T^n(\mathrm{supp}\, \mu),$$

for any $n \in \mathbb{N}$.

Proof. We now prove by induction on n. If $n = 1$, we have that $T(\mathrm{supp}\, \mu) = \mathrm{supp}\, T_M(\mu)$ for $\mu \in M(X)$. In fact, for any open subset $U \subset X \setminus T(\mathrm{supp}\, \mu)$ we have that

$$\begin{aligned} T_M(\mu)(U) &= \mu(T^{-1}(U)) \\ &\leq \mu(T^{-1}(X \setminus T(\mathrm{supp}\, \mu))) \end{aligned}$$

$$
\begin{aligned}
&= \mu(T^{-1}(X) \backslash T^{-1}(T(\operatorname{supp}\mu))) \\
&\le \mu(T^{-1}(X) \backslash \operatorname{supp}\mu) = 0,
\end{aligned}
$$

which implies that $\operatorname{supp} T_M(\mu) \subset T(\operatorname{supp}\mu)$.

Next, to show that $\operatorname{supp} T_M(\mu) \supset T(\operatorname{supp}\mu)$, it is sufficient to show that if $B \subset X$ is closed and $T_M(\mu)(B) = 1$ then $T(\operatorname{supp}\mu) \subset B$, which follows from the fact that $T_M(\mu)(B) = 1$ implies $\mu(T^{-1}(B)) = 1$.

Now we suppose that $\operatorname{supp} T_M^k(\mu) = T^k(\operatorname{supp}\mu)$ for any $k > 1$. Then, we have that

$$
\begin{aligned}
\operatorname{supp} T_M^{k+1}(\mu) &= \operatorname{supp} T_M^k(T_M(\mu)) = T^k(\operatorname{supp} T_M(\mu)) \\
&= T^k(T(\operatorname{supp}\mu)) = T^{k+1}(\operatorname{supp}\mu),
\end{aligned}
$$

which completes the proof of the proposition. ∎

We have the following corollary from Proposition 3.2.

Corollary 3.1. *If $A \in K(X)$, then $M(T^{-n}(A)) = T_M^{-n}(M(A))$ for all $n \in \mathbb{N}$.*

Lemma 3.1. *Suppose that $K_n \in K(X)$ for any $n \in \mathbb{N}$. If $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$, then*

$$
M\left(\bigcap_{n=1}^{\infty} K_n\right) = \bigcap_{n=1}^{\infty} M(K_n).
$$

Proof. It is obvious from the definition on measure. ∎

Lemma 3.2. *Let (X, T) be a t.d.s. and let A be an $m \times m$ transitive matrix. Then, the following statements are equivalent:*

(1) *There exist m pairwise disjoint nonempty subsets $V_1, V_2, \dots, V_m \in K(X)$ such that for any $\alpha = (a_0, a_1, a_2, \dots) \in \Sigma_A$,*

$$
V_\alpha = \bigcap_{s=0}^{\infty} T^{-s}(V_{a_s}) \neq \emptyset.
$$

(2) *There exist m pairwise disjoint nonempty compact subsets $M_1, M_2, \dots, M_m \subset M(X)$ such that m sets*

$$
Q_i = \overline{\bigcup_{\mu \in M_i} \operatorname{supp}\mu} \quad (1 \le i \le m)
$$

are pairwise disjoint in X, and for any $\alpha = (a_0, a_1, a_2, \dots) \in \Sigma_A$,

$$
M_\alpha = \bigcap_{s=0}^{\infty} T_M^{-s}(M_{a_s}) \neq \emptyset.
$$

Proof. First, we prove that (1) $\Rightarrow$ (2). For any $i \in \{1, 2, \ldots, m\}$, put $M_i = M(V_i)$, then $M_1, M_2, \ldots, M_m$ are pairwise disjoint nonempty compact subsets. By Corollary 3.1 and Lemma 3.1, we have

$$\bigcap_{s=0}^{\infty} T_M^{-s}(M_{a_s}) = \bigcap_{s=0}^{\infty} T_M^{-s}(M(V_{a_s})) = \bigcap_{s=0}^{\infty} M(T^{-s}(V_{a_s}))$$
$$= M\left(\bigcap_{s=0}^{\infty} T^{-s}(V_{a_s})\right) \neq \emptyset.$$

By assumption, it follows that $Q_i = V_i$ for any $i \in \{1, 2, \ldots, m\}$, hence, $Q_1, Q_2, \ldots, Q_m$ are pairwise disjoint.

Next, we claim (2) $\Rightarrow$ (1). Let $V_i = Q_i$ for any $i \in \{1, 2, \ldots, m\}$. Then we have that

$$V_\alpha = \bigcap_{s=0}^{\infty} T^{-s}(V_{a_s}) = \bigcap_{s=0}^{\infty} T^{-s}\left(\overline{\bigcup_{\mu \in M_{a_s}} \operatorname{supp} \mu}\right)$$

for any $\alpha = (a_0, a_1, \ldots) \in \Sigma_A$. Take $\mu_0 \in \bigcap_{s=0}^{\infty} T_M^{-s}(M_{a_s}) \neq \emptyset$. Then, for any $s \in \mathbb{N}_0$, $T_M^s(\mu_0) \in M_{a_s}$, which means that $\operatorname{supp} T_M^s(\mu_0) \subset \overline{\bigcup_{\mu \in M_{a_s}} \operatorname{supp} \mu}$. Hence, we have that

$$\operatorname{supp} T_M^s(\mu_0) = T^s(\operatorname{supp} \mu_0) \subset \overline{\bigcup_{\mu \in M_{a_s}} \operatorname{supp} \mu},$$

by Proposition 3.1. Since

$$\operatorname{supp} \mu_0 \subset T^{-s}(T^s(\operatorname{supp} \mu_0)) \subset T^{-s}\left(\overline{\bigcup_{\mu \in M_{a_s}} \operatorname{supp} \mu}\right)$$

and $\operatorname{supp} \mu_0 \neq \emptyset$, we can see that

$$\operatorname{supp} \mu_0 \subset \bigcap_{s=0}^{\infty} T^{-s}\left(\overline{\bigcup_{\mu \in M_{a_s}} \operatorname{supp} \mu}\right) \neq \emptyset,$$

which implies that $\bigcap_{s=0}^{\infty} T^{-s}(V_{a_s}) \neq \emptyset$. ■

Now we are ready to obtain a condition for the induced systems $(M(X), T_M)$ to have a subsystem topologically semi-conjugate to (Σ_A, σ_A).

Theorem 3.1. *Let (X, T) be a t.d.s. and let A be an $m \times m$ ($m \geq 2$) transitive matrix. If there exist m pairwise disjoint nonempty closed subsets $V_1, V_2, \ldots, V_m \subset X$ such that for any $\alpha = (a_0, a_1, a_2, \ldots) \in \Sigma_A$,*

$$V_\alpha = \bigcap_{s=0}^{\infty} T^{-s}(V_{a_s}) \neq \emptyset, \qquad (*)$$

then there exists a subsystem $(M_0, T_M|_{M_0})$ *of* $(M(X), T_M)$ *such that* $(M_0, T_M|_{M_0})$ *is topologically semi-conjugate to* (Σ_A, σ_A).

Proof. Let $M_i = M(V_i)$ for $i \in \{1, 2, \ldots, m\}$. By Proposition 3.1, $M_1, M_2, \ldots, M_m$ are pairwise disjoint nonempty compact subsets of $M(X)$. Also, by Lemma 3.2, for any $\alpha = (a_0, a_1, a_2, \ldots) \in \Sigma_A$,

$$M_\alpha = \bigcap_{s=0}^{\infty} T_M^{-s}(M_{a_s}) = \bigcap_{s=0}^{\infty} T_M^{-s}(M(V_{a_s})) = \bigcap_{s=0}^{\infty} M(T^{-s}(V_{a_s})) = M(V_\alpha) \neq \emptyset.$$

Now put $M_0 = \bigcup_{\alpha \in \Sigma_A} M_\alpha$, then M_0 is T_M-invariant. In fact,

$$\begin{aligned} T_M(M_0) &= \bigcup_{\alpha \in \Sigma_A} T_M(M_\alpha) = \bigcup_{\alpha \in \Sigma_A} T_M\left(\bigcap_{s=0}^{\infty} T_M^{-s}(M_{a_s})\right) \\ &= \bigcup_{\alpha \in \Sigma_A} \bigcap_{s=0}^{\infty} T_M^{-s+1}(M_{a_s}) = \bigcup_{\alpha \in \Sigma_A} M(\sigma(\alpha)) \subseteq M_0. \end{aligned}$$

Therefore, by [8, Theorem 4.1], it follows that the subsystem $(M_0, T_M|_{M_0})$ of $(M(X), T_M)$ is topologically semi-conjugate to (Σ_A, σ_A). ■

Here it is worth noting that the simplified criterion obtained in [8, Theorem 4.1] for a system to have a subsystem topologically semi-conjugate to (Σ_A, σ_A) is a key to prove Theorem 3.1 without big difficulties as you can see from the proof.

Remark 3.1. Given a point $x \in X$, let $\delta_x \in M(X)$ be the *Dirac point measure* of x defined by

$$\delta_x(A) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases}$$

where A is a Borel subset of X. Obviously, the map $x \mapsto \delta_x$ is an embedding map from X to $M(X)$ and a homeomorphic map (see [1]). If the assumptions of Theorem 3.1 are satisfied, then by [8, Theorem 4.1] there exists a subsystem $(\Lambda, T|_\Lambda)$ of (X, T) such that $(\Lambda, T|_\Lambda)$ is topologically semi-conjugate to (Σ_A, σ_A). Here Λ is defined by $\Lambda = \bigcup_{\alpha \in \Sigma_A} V_\alpha$. Put $\Lambda_M = \{\delta_x \in M(X) : x \in \Lambda\}$. Then it is clear that the subsystem $(\Lambda_M, T_M|_{\Lambda_M})$ of $(M(X), T_M)$ is also topologically semi-conjugate to (Σ_A, σ_A). Therefore, if the subsystem $(\Lambda, T|_\Lambda)$ of the original system (X, T) is topologically semi-conjugate to (Σ_A, σ_A), so is its corresponding induced subsystem $(\Lambda_M, T_M|_{\Lambda_M})$ of $(M(X), T_M)$. In this sense, we call this subsystem $(\Lambda_M, T_M|_{\Lambda_M})$ of $(M(X), T_M)$ *trivial* subsystem topologically semi-conjugate to (Σ_A, σ_A).

If for every $\alpha \in \Sigma_A$, V_α consists of only one point, the above subsystem $(\Lambda, T|_\Lambda)$ of (X, T) is topologically conjugate to (Σ_A, σ_A) by [8, Theorem 3.1]. Then we can see that in this case the trivial subsystem $(\Lambda_M, T_M|_{\Lambda_M})$ of $(M(X), T_M)$ is also topologically conjugate with (Σ_A, σ_A). However, if for some $\alpha \in \Sigma_A$, V_α consists of more than two points, then since $M_\alpha = \bigcap_{s=0}^{\infty} T_M^{-s}(M_{a_s}) = M(V_\alpha)$ as shown in the proof of Theorem 3.1, it is clear that $\sharp(V_\alpha) < \sharp(M_\alpha)$ where $\sharp(\cdot)$ denotes the cardinality of a set. Then

$$\sharp(M_0) = \sharp\Big(\bigcup_{\alpha\in\Sigma_A} M_\alpha\Big) > \sharp\Big(\bigcup_{\alpha\in\Sigma_A} V_\alpha\Big) = \sharp(\Lambda) = \sharp(\Lambda_M),$$

and thus $\Lambda_M \subset M_0$ and $\Lambda_M \neq M_0$. So the subsystem $(M_0, T_M|_{M_0})$ of $(M(X), T_M)$ in Theorem 3.1 obviously is *nontrivial* subsystem topologically semi-conjugate to (Σ_A, σ_A).

Thus we can restate Theorem 3.1 as follows.

Theorem 3.2. *If a t.d.s. (X, T) satisfies the assumptions of Theorem 3.1, then its induced system $(M(X), T_M)$ has a nontrivial subsystem topologically semi-conjugate to (Σ_A, σ_A).*

Remark 3.2. Given a compact metric space (X, d), the embedding mapping $i : X \to K(X)$ defined by $i(x) = \{x\} \in K(X)$, is an isometric and homeomorphic embedding. Hence, (X, T) can be viewed as (i.e., homeomorphic to) a subsystem of its induced hyperspace system $(K(X), T_K)$. As a result, if (X, T) has a subsystem topologically conjugate (respectively, semi-conjugate) to (Σ_A, σ_A), then its induced hyperspace system $(K(X), T_K)$ will also have a subsystem topologically conjugate (respectively, semi-conjugate) to (Σ_A, σ_A) (see [19]). Here we also call such subsystem of $(K(X), T_K)$ *trivial* subsystem topologically semi-conjugate to (Σ_A, σ_A).

The following corollaries show some relations between the original system (X, T) and the induced systems $(K(X), T_K)$ and $(M(X), T_M)$ concerning topological semi-conjugacy to (Σ_A, σ_A).

Corollary 3.2. *Let (X, T) be a t.d.s. and let A be an $m \times m$ ($m \geq 2$) transitive matrix. If the system (X, T) has a subsystem topologically semi-conjugate to (Σ_A, σ_A), then the induced systems $(K(X), T_K)$ and $(M(X), T_M)$ have some nontrivial subsystems topologically semi-conjugate to (Σ_A, σ_A), respectively.*

Proof. From [8, Theorem 4.1], condition $(*)$ in Theorem 3.1 is a necessary and sufficient condition for (X, T) to have a subsystem topologically semi-conjugate to (Σ_A, σ_A). Therefore desired results hold by Theorem 3.1 and [7, Theorem 3.1.5]. ■

Corollary 3.3. *Let (X, T) be a t.d.s. and let A be an $m \times m$ ($m \geq 2$) transitive matrix. If there exist m pairwise disjoint nonempty compact subsets $M_1, M_2, \ldots, M_m$ in $M(X)$ such that*

$$Q_i = \overline{\bigcup_{\mu\in M_i} \operatorname{supp}\mu} \quad (1 \leq i \leq m)$$

are pairwise disjoint, and for any $\alpha \in \Sigma_A$,

$$M_\alpha = \bigcap_{s=0}^{\infty} T_M^{-s}(M_{a_s}) \neq \emptyset,$$

then there exists a subsystem of (X, T) *which is topologically semi-conjugate to* (Σ_A, σ_A).

Proof. By Lemma 3.2 and [8, Theorem 4.1] it is obvious. ■

In practice, it is not easy to check the condition $(*)$ of Theorem 3.1 though it makes it easy to discuss theoretically. But using the concept of A-coupled expanding, we can obtain more convenient conditions to check as follows.

Theorem 3.3. *Let* (X, T) *be a t.d.s. and let* A *be an* $m \times m$ *(*$m \geq 2$*) transitive matrix. Let* $V_1, V_2, \ldots, V_m$ *be pairwise disjoint closed subsets in* X. *If* T *is an* A*-coupled-expanding on* $V_1, V_2, \ldots, V_m$, *then the induced system* $(M(X), T_M)$ *has a nontrivial subsystem which is topologically semi-conjugate to* (Σ_A, σ_A).

Proof. It follows from Theorem 3.1 in this paper and [8, Theorem 4.2]. ■

Theorem 3.4. *Let* (X, f) *be a t.d.s. and let* A *be an* $m \times m$ *transition matrix such that* (Σ_A, σ_A) *is topologically transitive. Let* $\alpha_0 \in \Sigma_A$ *be a transitive point of* (Σ_A, σ_A). *If there exist* m *pairwise disjoint nonempty compact subsets* $V_1, V_2, \ldots, V_m \subset X$ *such that* $V_{\alpha_0} \neq \emptyset$, *then there exists a nontrivial subsystem in* $(M(X), T_M)$ *which is topologically semi-conjugate to* (Σ_A, σ_A).

Proof. By the assumption in this theorem and [7, Proposition 3.1.8] we see that $V_\alpha \neq \emptyset$ for any $\alpha \in \Sigma_A$. Thus, by Theorem 3.1, the proof is completed. ■

4. The chaos of the induced system on the space of probability measures

Now we study some chaotic properties of the system $(M(X), T_M)$ induced by a t.d.s. (X, T).

Theorem 4.1. *Let* (X, T) *be a* t.d.s.. *Suppose that* $A = (a_{ij})$ *is an* $m \times m$ *transitive matrix satisfying that* $\Sigma_{j=1}^m a_{i_0 j} \geq 2$ *for some* $i_0 \in \{1, 2, \ldots, m\}$. *If there exist* m *nonempty compact subsets* $V_1, V_2, \ldots, V_m \subset X$ *such that* V_α *is a singleton for any* $\alpha \in \Sigma_A$, *then the induced system* $(M(X), T_M)$ *has a subsystem which is chaotic in the sense of Devaney.*

Proof. Consider a natural embedding map from X to $M(X)$, $X \ni x \mapsto \delta_x \in M(X)$, which is homeomorphic (see [6]). We can assume $\mathrm{diam}(X) < 1$. Then, by the definition of the Prohorov metric in $M(X)$, this map is isometric (see [6]), where $\mathrm{diam}(X)$ denotes the diameter of the set X. Now we denote the Prohorov metric of $M(X)$ as d_M. Then, for any $n \in \mathbb{N}$, we have

$$d(T^n(x), T^n(y)) = d_M(\delta_{T^n(x)}, \delta_{T^n(y)}) = d_M(T_M^n(\delta_x), T_M^n(\delta_y)).$$

Since $V_\alpha = \{x\}$ is a singleton for any $\alpha \in \Sigma_A$, M_α is so, that is, $M_\alpha = \{\delta_x\}$. Therefore, by Theorem 3.1 and [8, Theorem 5.1], the proof is completed. ∎

Theorem 4.2. *Let (X, T) be a* t.d.s.. *Suppose that $A = (a_{ij})$ is an $m \times m$ transitive matrix satisfying that $\Sigma_{j=1}^m a_{i_0 j} \geq 2$ for some $i_0 \in \{1, 2, \dots, m\}$. If there exist m pairwise disjoint and nonempty compact subsets $V_1, V_2, \dots, V_m \subset X$ such that V_α is nonempty for any $\alpha \in \Sigma_A$, then the system $(M(X), T_M)$ is chaotic in the sense of Li–Yorke.*

Proof. By Theorem 3.1, there exists a subsystem $(M_0, T_M|_{M_0})$ of $(M(X), T_M)$ which is topologically semi-conjugate to (Σ_A, σ_A). Therefore,

$$h(T_M) \geq h(T_M|_{M_0}) \geq h(\sigma_A) > 0.$$

Hence, $(M(X), T_M)$ is chaotic in the sense of Li–Yorke. ∎

Corollary 4.1. *Let (X, T) be a* t.d.s.. *Suppose that $A = (a_{ij})$ is an $m \times m$ transitive matrix satisfying that $\Sigma_{j=1}^m a_{i_0 j} \geq 2$ for some $i_0 \in \{1, 2, \dots, m\}$. If there exist m pairwise disjoint and nonempty compact subsets $V_1, V_2, \dots, V_m \subset X$ such that T is an A-coupled expanding on $V_1, V_2, \dots, V_m$, then the system $(M(X), T_M)$ is chaotic in the sense of Li–Yorke.*

5. A characterization of the $\mathcal{F}$-mixing system (X, T) by means of $(M(X), T_M)$

In this section, we consider some properties of the induced system $(M(X), T_M)$ relevant to the question posed in [5] using a result of the present paper. In [5], Fu, H. and Xing, Z. obtained a characterization of the $\mathcal{F}$-mixing (X, T) via a Furstenberg family $\mathcal{F}$ by means of $(K(X), T_K)$ as follows.

Theorem 5.1 ([5]). *Let (X, T) be a t.d.s. and let $\mathcal{F} \subset \mathcal{F}_{\mathrm{inf}}$ be a Furstenberg Family. Then the following statements are equivalent:*

(1) *(X, T) is $\mathcal{F}$-mixing.*

(2) *For any* $A \in K(X)$ *with* $\operatorname{int}(A) \neq \emptyset$,

$$\tau\mathcal{F} - \lim T_K^n(A) = X,$$

where $\operatorname{int}(A)$ *denotes the interior of* A.

And they posed the following question:

Bauer and Sigmund [1] have studied the interplay of (X, T) with $(M(X), T_M)$ as well as with $(K(X), T_K)$, and pointed out that dynamical properties of $(M(X), T_M)$ are in parallel with those of $(K(X), T_K)$. As have been found later, there still exist a few discrepancies between them. Then, what is the analogous equivalent characterization in $(M(X), T_M)$ corresponding to the above Theorem 5.1?

Here we give an answer to this question.

Before stating our result, we briefly introduce some more concepts concerning Furstenberg families.

Let $2^{\mathbb{N}_0}$ be the set of all subsets of $\mathbb{N}_0$ and $\mathcal{F} \subset 2^{\mathbb{N}_0}$. We say that $\mathcal{F}$ is a *Furstenberg family* if it is hereditary upwards, that is, for $F_1 \in \mathcal{F}$, $F_1 \subset F_2$ implies that $F_2 \in \mathcal{F}$. For an $\mathcal{F}$, we define its *dual family* as

$$k\mathcal{F} = \{F \in 2^{\mathbb{N}_0} : \mathbb{N}_0 \backslash F \notin \mathcal{F}\}.$$

For $i \in \mathbb{N}_0$, we define a map $g^i : \mathbb{N}_0 \to \mathbb{N}_0$ by $g^i(j) = i + j$. For a Furstenberg family $\mathcal{F}$, let

$$\tau\mathcal{F} = \left\{F \in 2^{\mathbb{N}_0} : \bigcap_{j=1}^{n} g^{-i_j}(F) \in \mathcal{F}, \text{ for any } n \in \mathbb{N} \right.$$
$$\left. \text{and any } \{i_1, i_2, \ldots, i_n\} \subset \mathbb{N}_0 \right\}.$$

$\mathcal{F}_{\text{inf}}$ denotes the set of all infinite subsets of $\mathbb{N}_0$.

For $A, B \subset X$, we define a set $N(A, B) = \{n \in \mathbb{N}_0 : A \cap T^n(B) \neq \emptyset\}$. Let $\mathcal{F}$ be a Furstenberg family. We say that the t.d.s. (X, T) is $\mathcal{F}$*-transitive* if $N(U, V) \in \mathcal{F}$ for any nonempty open subsets U, V of X, and (X, T) is $\mathcal{F}$*-mixing* if $(X \times X, T \times T)$ is $\mathcal{F}$-transitive.

Let $\{x_n\}$ be a sequence of points in X and $\mathcal{F} \subset \mathcal{F}_{\text{inf}}$ be a Furstenberg family. We say that the sequence $\{x_n\}$ $\mathcal{F}$*-converges* to a point $x \in X$ (denoted by $\mathcal{F}\text{-}\lim x_n = x$) if $\{n : x_n \in V\} \in \mathcal{F}$ for every neighborhood V of the point x.

The following theorem is a characterization of the $\mathcal{F}$-mixing system (X, T) corresponding to the above Theorem 5.1, by means of $(M(X), T_M)$.

Theorem 5.2. *Let* (X, T) *be a t.d.s. and let* $\mathcal{F} \subset \mathcal{F}_{\text{inf}}$ *be a Furstenberg Family. Then the following statements are equivalent:*

(1) (X, T) *is* $\mathcal{F}$*-mixing.*

(2) *For any* $\mu \in M(X)$ *with* $\operatorname{int}(\operatorname{supp}\mu) \neq \emptyset$, $\tau\mathcal{F} - \lim\,(\operatorname{supp} T_M^n(\mu)) = X$ *in the sense of the topology in* $K(X)$.

Proof. (1) $\Rightarrow$ (2). Let $\mu \in M(X)$ such that $\operatorname{int}(\operatorname{supp}\mu) \neq \emptyset$. Put $A = \operatorname{supp}\mu$. Then we have that $A \in K(X)$ and by Theorem 5.1,

$$\tau\mathcal{F} - \lim T_K^n(A) = X.$$

Using Proposition 3.2, it follows that

$$T_K^n(A) = T^n(A) = \operatorname{supp} T_M^n(\mu),$$

which implies that

$$\tau\mathcal{F} - \lim(\operatorname{supp} T_M^n(\mu)) = X.$$

(2) $\Rightarrow$ (1). For any $A \in K(X)$, we can always choose some $\mu \in M(X)$ such that $\operatorname{supp}\mu = A$. Since

$$T_K^n(A) = \operatorname{supp} T_M^n(A)$$

from Proposition 3.2, if $\operatorname{int}(A) \neq \emptyset$, then we have

$$\tau\mathcal{F} - \lim T_K^n(A) = \tau\mathcal{F} - \lim\,(\operatorname{supp} T_M^n(\mu)) = X.$$

Therefore, T is $\mathcal{F}$-mixing by Theorem 5.1. ■

Remark 5.1. The question posed in [5] is what an analogous equivalent characterization in $(M(X), T_M)$ corresponding to Theorem 5.1 (2) is. Regarding Theorem 5.1, the analogue seems to be as follows.

(2)′ *For any* $\mu \in M(X)$ *with* $\operatorname{int}(\operatorname{supp}\mu) \neq \emptyset$, *there exists a* $\nu \in M(X)$ *such that* $\operatorname{supp}\nu = X$ *and* $\tau\mathcal{F} - \lim T_M^n(\mu) = \nu$ *in the sense of the topology in* $M(X)$.

But (1) $\Rightarrow$ (2)′ does not hold generally. More precisely, it does not hold in the case when $\mathcal{F} = \mathcal{F}_{\text{cf}}$. In fact, if we set $\mathcal{F} = \mathcal{F}_{\text{cf}}$, then $\tau\mathcal{F} = \mathcal{F}_{\text{cf}}$ and $\mathcal{F}$-mixing coincides with strongly mixing and $\tau\mathcal{F}$-convergence is equal to the normal convergence. Assume that (1) $\Rightarrow$ (2)′ holds for $\mathcal{F}_{\text{cf}}$. If T is strongly mixing, then for any $\mu \in M(X)$ with $\operatorname{int}(\operatorname{supp}\mu) \neq \emptyset$ there exists a $\nu \in M(X)$ such that $\operatorname{supp}\nu = X$ and $T_M^n(\mu) \to \nu \in M(X)$ $(n \to \infty)$ in the weak*-topology. Then, ν is an invariant measure and thus (X, T) is an E-system.

However, there is an example which is a strongly mixing system but not an E-system. The subshift (X, σ) in [18, Section 5] is strongly mixing by [18, Proposition 2]. But, if $a \in X$ is not a minimal point ($a \neq 00\cdots$), then there exists a neighborhood U of a such that the upper Banach density of $N(a, U)$ is zero (see the proof of [18, Lemma 2]). It is well known that a transitive system (X, T) is an E-system if and only if $N(x, U)$ has a positive upper Banach density for any transitive point

$x \in \mathrm{Trans}(X, T)$ and any neighborhood U of x. Therefore this implies that the subshift (X, σ) is not an E-system.

So we can conclude that $(2)'$ is not an analogous equivalent characterization in $(M(X), T_M)$ corresponding to Theorem 5.1 (2).

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