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Classification of trusses whose retracts are cyclic groups

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Abstract. This article is devoted to study some relationships of trusses with rings. We classify all trusses whose retracts are finite cyclic groups.

1. Introduction

In the 1920s, H. Prüfer and R. Baer defined a heap as an algebraic system consisting of a set with a ternary operation which fulfils conditions that allow one to associate an isomorphic group to every element. Moreover, with every group one can connect a heap by taking special operation (see [3, 8]). In 2007, W. Rump introduced braces as algebraic systems corresponding to solutions of set-theoretic Yang–Baxter equations [9]. A brace is a triple $(G, +, \cdot)$ where $(G, +)$ is an abelian group, $(G, \cdot)$ is a group and the following distributive law holds, for all $a, b, c \in G$, $a \cdot (b + c) = a \cdot b - a + a \cdot c$; see [6]. Through their connection with set-theoretic Yang–Baxter equations, braces have become an intensive field of studies. In particular, it has been shown that a brace allows one to construct a non-degenerate involutive set-theoretic solution of the Yang–Baxter equation.

A truss [1, 4] is an algebraic system consisting of a set with a ternary operation making it into an abelian heap [3,8] and an associative binary operation that distributes over the ternary one.

Later, it turned out that there are interesting relations of trusses with various algebraic structures: groups, rings, brace-like systems, etc. (cf. [9]). These relationships allow these structures to be studied using new tools. Research in this area is very promising and currently being intensively developed; a whole series of articles by T. Brzeziński and his associates are devoted to these topics: [1, 4, 5]. We will now quickly recall main truss concepts that we will be using, and we refer the reader to the cited literature for detailed information and discussion.

It is worth mentioning that trying to describe various algebraic structures on fixed classes of groups (abelian) is a common practice. The aim of our article is to present

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a complete classification of trusses whose retracts are cyclic groups. This fits into the current trend of research on the previously mentioned structures. For example, in paper [5] (see Theorem 3.51) Brzeziński described all trusses whose retracts are isomorphic to the infinite cyclic group. The results presented in this work are a natural continuation of his research. Namely, we classify all trusses whose retracts are finite cyclic groups. Additionally, it is worth noting that Prof. Rump conducted research in a similar vein. Namely, in article [10] he classified cyclic braces with primary additive group.

2. Trusses

We will now quickly recall main truss concepts that we will be using, and we refer the reader to the cited literature for detailed information and discussion. The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{P}$ stand for the set of natural numbers, the set of integers and the set of all prime numbers, respectively. The prime field of characteristic p is denoted by $\mathbb{Z}_p$. The additive group of a ring R is denoted by R^+.

An abelian heap is a set H with a ternary operation $[-,-,-]$ such that, for all $h_1, h_2, h_3, h_4, h_5 \in H$,

$$[h_1, h_2, [h_3, h_4, h_5]] = [[h_1, h_2, h_3], h_4, h_5], \tag{2.1a}$$

$$[h_1, h_1, h_2] = h_2 \quad \text{and} \quad [h_1, h_2, h_2] = h_1, \tag{2.1b}$$

$$[h_1, h_2, h_3] = [h_3, h_2, h_1]. \tag{2.1c}$$

For any $e \in H$ we define the binary operation $+_e$ on the set H by

$$a +_e b = [a, e, b] \quad \text{for all } a, b \in H. \tag{2.2}$$

Then $(H, +_e, e)$ is an abelian group, known as *a retract* of H and $-_e h := [e, h, e]$ for any $h \in H$. Moreover, for all $e, f \in H$ the groups $(H, +_e, e)$ and $(H, +_f, f)$ are isomorphic. Conversely, if $(A, +)$ is an abelian group and $[a, b, c] = a - b + c$ for any $a, b, c \in A$, then $(A, [-,-,-])$ is an abelian heap.

A truss is a set T together with a ternary operation $[-,-,-]$ and an associative binary operation $\cdot$ called multiplication, such that $(T, [-,-,-])$ is an abelian heap and, for all $a, b, c, d \in T$,

$$a \cdot [b, c, d] = [a \cdot b, a \cdot c, a \cdot d] \quad \text{and} \quad [a, b, c] \cdot d = [a \cdot d, b \cdot d, c \cdot d].$$

We say that a truss $(T, [-,-,-], \cdot)$ is *commutative*, if $a \cdot b = b \cdot a$ for all $a, b \in T$. Moreover, by *a retract of a truss* $(T, [-,-,-], \cdot)$ we call any retract of an abelian heap $(T, [-,-,-])$. An element $e \in T$ is called *an idempotent*, if $e \cdot e = e$. If $u \cdot a = a \cdot u = a$ for any $a \in T$, then we say that u is *an identity of a truss* T.

Let $(T_1, [-,-,-]_1, \cdot_1)$ and $(T_2, [-,-,-]_2, \cdot_2)$ be trusses. A mapping $f : T_1 \to T_2$ is called a *homomorphism of trusses*, if for all $a, b, c \in T_1$,

$$f([a,b,c]_1) = [f(a), f(b), f(c)]_2 \quad \text{and} \quad f(a \cdot_1 b) = f(a) \cdot_2 f(b).$$

If, additionally, f is a bijection, then we say that f is an *isomorphism of trusses*.

It is easy to check that if $(T_i, [-,-,-]_i, \cdot_i)$ is a truss for every $i \in I$ $(I \neq \emptyset)$, then $T = \prod_{i \in I} T_i$ with operations $[-,-,-]$ and $\cdot$ defined by the formulas

$$[(a_i)_{i\in I}, (b_i)_{i\in I}, (c_i)_{i\in I}] := ([a_i, b_i, c_i]_i)_{i\in I} \quad \text{and} \quad (a_i)_{i\in I} \cdot (b_i)_{i\in I} := (a_i \cdot_i b_i)_{i\in I}$$

is also a truss. This truss is called *a direct product of trusses* $(T_i, [-,-,-]_i, \cdot_i)$ for $i \in I$ and denoted by $\prod_{i\in I} T_i$. Note that, if $(T_i', [-,-,-]_i', \cdot_i')$ is a truss and $f_i : T_i \to T_i'$ is a truss isomorphism for every $i \in I$, then a function f given by a formula $f((x_i)_{i\in I}) := (f_i(x_i))_{i\in I}$ is an isomorphism of a truss $\prod_{i\in I} T_i$ onto a truss $\prod_{i\in I} T_i'$.

Remark 2.1. The problem of classification of trusses whose retract is isomorphic to a given abelian group $(A, +, 0)$ comes down to the classification of trusses of the form $(A, [-,-,-], \cdot)$ such that $[a,b,c] = a - b + c$ for any $a, b, c \in A$. Indeed, if $(T, [-,-,-]', \circ)$ is a truss, $e \in T$ and $f : (T, +_e, e) \to A$ is an isomorphism of groups and $a \cdot b = f(f^{-1}(a) \circ f^{-1}(b))$ for $a, b \in A$, then $(A, [-,-,-], \cdot)$ is a truss and f is an isomorphism of a truss $(T, [-,-,-]', \circ)$ onto a truss $(A, [-,-,-], \cdot)$. In this way, we free ourselves from the too general form of ternary action and reduce it to an action given by a formula $[a,b,c] = a - b + c$ for $a, b, c \in A$.

This simple observation also allows us to choose different forms of groups A, e.g., instead $A = \mathbb{Z}_n^+$ one can consider a finite product of simple groups of the form $\mathbb{Z}_{p^k}^+$, where $p \in \mathbb{P}$ and $k \in \mathbb{N}$.

Example 2.2. Let $(A, +, 0)$ be an abelian group and let $[a,b,c] = a - b + c$ for $a, b, c \in A$. Moreover, let for any $a, b \in A$

$$a \cdot_l b := a, \quad a \cdot_r b := b, \tag{2.3a}$$

$$a \cdot_0 b := 0, \quad a \circ_0 b := a + b. \tag{2.3b}$$

A standard check shows that $(A, [-,-,-], \cdot_l)$, $(A, [-,-,-], \cdot_r)$, $(A, [-,-,-], \cdot_0)$, $(A, [-,-,-], \circ_0)$ are trusses where the last two of them are commutative and the first two are not commutative, whenever $A \neq \{0\}$. Note that 0 is the identity of a truss $(A, [-,-,-], \circ_0)$, and a truss $(A, [-,-,-], \cdot_0)$ does not have identity, if only $A \neq \{0\}$. Moreover, trusses $(A, [-,-,-], \cdot)$ and $(A, [-,-,-], \cdot_l)$ are isomorphic if and only if they are equal and $(A, [-,-,-], \cdot)$ and $(A, [-,-,-], \cdot_r)$ are isomorphic if and only if they are equal. If $A \neq \{0\}$, then no two of these four trusses are isomorphic.

A pair (λ, ρ) of additive mutually commuting endomorphisms of the additive group of a ring R is called *double homothetism* on an associative ring $(R, +, \cdot, 0)$, if for any $a, b \in R$ the conditions

$$a \cdot \lambda(b) = (a)\rho \cdot b, \quad \lambda(a \cdot b) = \lambda(a) \cdot b, \quad (a \cdot b)\rho = a \cdot (b)\rho$$

hold. Examples of double homotheties on any associative ring $(R, +, \cdot, 0)$ are pairs $(k \cdot \mathrm{id}_R, k \cdot \mathrm{id}_R)$ for $k \in \mathbb{Z}$ and (l_a, r_a), where for fixed $a \in R$ and any $x \in R$: $l_a(x) := a \cdot x$ and $(x)r_a := x \cdot a$.

Remark 2.3. Let $e \in T$, where $(T, [-,-,-], \cdot)$ is a truss. Let, for any $a, b \in T$,

$$a \cdot_e b := a \cdot b -_e a \cdot e -_e e \cdot b +_e e \cdot e. \tag{2.4}$$

In article [1], the authors proved that $(T, +_e, \cdot_e, e)$ is a ring and a pair (λ, ρ), where

$$\lambda(x) = e \cdot x -_e e \cdot e \quad \text{and} \quad (x)\rho = x \cdot e -_e e \cdot e \quad \text{and} \quad x \in T \tag{2.5}$$

is a double homothetism on that ring, $\lambda(\alpha) = (\alpha)\rho$ for $\alpha = e \cdot e$, and for every $x \in T$,

$$\lambda(\lambda(x)) = \lambda(x) +_e \alpha \cdot_e x \quad \text{and} \quad ((x)\rho)\rho = (x)\rho +_e x \cdot_e \alpha. \tag{2.6}$$

Moreover, for any $a, b \in T$ we have

$$a \cdot b = a \cdot_e b +_e (a)\rho +_e \lambda(b) +_e \alpha. \tag{2.7}$$

In [1] it is also proved that for any elements $e, f \in T$ the rings $(T, +_e, \cdot_e, e)$ and $(T, +_f, \cdot_f, f)$ are isomorphic. Therefore, we say that a truss $(T, [-,-,-], \cdot)$ determinates the ring $(T, +_e, \cdot_e, e)$.

Remark 2.4. Let φ be a homomorphism of a truss $(T, [-,-,-], \cdot)$ into a truss $(T_1, [-,-,-]_1, \cdot_1)$ and let $f = \varphi(e)$. Then, by formula (2.2) we get that φ is a homomorphism of an abelian group $(T, +_e, e)$ into a group $(T_1, +_f, f)$. Hence, and by (2.4) we get that $\varphi(a \cdot_e b) = \varphi(a) \cdot_f \varphi(b)$ for all $a, b \in T$. Therefore, φ is a ring homomorphism. Moreover, for every element $x \in T$ by (2.5) we have that $\varphi(\lambda(x)) = \lambda_1(\varphi(x))$ and $\varphi((x)\rho) = (\varphi(x))\rho_1$, where $\lambda_1(y) = f \cdot_1 y -_f f \cdot_1 f$ and $(y)\rho_1 = y \cdot_1 f -_f f \cdot_1 f$ for $y \in T_1$. In particular, isomorphic trusses determine isomorphic rings.

A standard computation based on formula (2.4) shows that any truss from Example 2.2 determines (for $e = 0$) the same ring with a zero multiplication with an additive group $(A, +, 0)$. Thus, non-isomorphic trusses can determinate isomorphic rings. Notice that the truss $(T, [-,-,-], \cdot)$ determines a double homothetism (λ, ρ) and an element $\alpha \in T$ such that $\lambda(\alpha) = (\alpha)\rho$ and formulas (2.6) and (2.7) are fulfilled.

Therefore, some of the information about this truss is hidden not only in the ring determinated by it, but also in the double homothetism (λ, ρ) and an element α.

Conversely, if $(R, +, \cdot, 0)$ is a ring and (λ, ρ) is a double homothetism on that ring and for $\alpha \in R$ we have $\lambda(\alpha) = (\alpha)\rho$ and for any $x \in R$, $\lambda(\lambda(x)) = \lambda(x) + \alpha \cdot x$ and $((x)\rho)\rho = (x)\rho + x \cdot \alpha$, then $(R, [-,-,-], \circ)$ is a truss, if for any $a, b, c \in R$ we have $[a, b, c] = a - b + c$ and

$$a \circ b = a \cdot b + (a)\rho + \lambda(b) + \alpha. \tag{2.8}$$

Moreover, we have that $(R, +_0, \circ_0, 0) = (R, +, \cdot, 0)$ and for non-isomorphic rings correspond non-isomorphic trusses. A truss $(R, [-,-,-], \circ)$ will be denoted by $T(R, \alpha, \lambda, \rho)$.

We will now present the construction of a quotient structure on a truss of the form $T(R, \alpha, \lambda, \rho)$.

Remark 2.5. Let (λ, ρ) be double homothetism on a ring $(R, +, \cdot, 0)$ and let $\alpha \in R$ be such that $T = T(R, \alpha, \lambda, \rho)$ is a truss. Let I be an ideal of the ring R, such that $\lambda(I) \subseteq I$ and $(I)\rho \subseteq I$. Set $\overline{R} = R/I$ and $\overline{\alpha} = \alpha + I$. Moreover, let $\overline{\lambda}(x + I) = \lambda(x) + I$ and $(x + I)\overline{\rho} = (x)\rho + I$ for $x \in R$. A standard computation shows that $(\overline{\lambda}, \overline{\rho})$ is a double homothetism on the quotient ring $\overline{R}$. Moreover, $\overline{\lambda}(\overline{\alpha}) = \lambda(\alpha) + I = (\alpha)\rho + I = (\overline{\alpha})\overline{\rho}$ and for every $\overline{x} = x + I \in R/I$ we have $\overline{\lambda}^2(\overline{x}) = \overline{\lambda}(\overline{x}) + \overline{\alpha} \cdot \overline{x}$ and $(\overline{x})\overline{\rho}^2 = (\overline{x})\overline{\rho} + \overline{x} \cdot \overline{\alpha}$. Therefore, by virtue of Remark 2.4 we have a truss $\overline{R}, \overline{\alpha}, \overline{\lambda}, \overline{\rho}$ with the multiplication $\overline{\circ}$ given by the formula

$$\overline{a} \,\overline{\circ}\, \overline{b} = \overline{a} \cdot \overline{b} + \overline{(a)\rho} + \overline{\lambda(b)} + \overline{\alpha},$$

for any $a, b \in R$.

Since a mapping $a \mapsto a + I$ is a ring homomorphism of R onto a ring $\overline{R}$, the truss $\overline{T} = T(\overline{R}, \overline{\alpha}, \overline{\lambda}, \overline{\rho})$ is a homomorphic image of a truss T.

The next theorem can be considered as an equivalent of the well-known First Isomorphism Theorem, for trusses.

Theorem 2.6. *Let $T = T(R, \alpha, \lambda, \rho)$ be a truss determined by the ring $(R, +, \cdot, 0)$. Then the truss S is a homomorphic image of the truss T if and only if there exists an ideal I of the ring R such that $\lambda(I) \subseteq I$, $(I)\rho \subseteq I$ and $S \cong T(\overline{R}, \overline{\alpha}, \overline{\lambda}, \overline{\rho})$.*

Proof. The implication of $\Leftarrow$ follows immediately from Remark 2.5.

$\Rightarrow$. Let ψ be a homomorphism of a truss T onto a truss S. By Remark 2.4 we have that ψ is a homomorphism of a ring $(R, +, \cdot, 0)$ onto a ring $(S, +_s, \cdot_s, s)$ where $s = \psi(0)$. Let $I = \operatorname{Ker} \psi$. Then, from the general ring theory, a function $\overline{\psi}\colon R/I \to S$ given by $\overline{\psi}(x + I) = \psi(x)$ for $x \in R$ is an isomorphism of the ring $\overline{R} = R/I$ onto the ring S. It remains to prove that the truss S is of the form $T(\overline{R}, \overline{\alpha}, \overline{\lambda}, \overline{\rho})$.

Let $i \in I$. Then

$$\begin{aligned}\psi(\lambda(i)) &= \psi(0 \cdot i - 0 \cdot 0) = \psi([0 \cdot i, 0, 0 \cdot 0]) = [\psi(0 \cdot i), \psi(0), \psi(0 \cdot 0)]_S \\ &= s \cdot \psi(i) -_s s \cdot_s s = s \cdot s -_s s \cdot_s s = s,\end{aligned}$$

which shows that $\lambda(i) \in I$ and in a consequence $\lambda(I) \subseteq I$. Similarly, it can be shown that $(I)\rho \subseteq I$.

Denote $\overline{\alpha} = \psi(\alpha)$ and let for $y \in R/I$: $\overline{\lambda}(y) = \psi(\lambda(x))$ and $(y)\overline{\rho} = \psi((x)\rho)$, where $x \in R$ and $\psi(x) = y$. By assumptions we have that $\overline{\lambda}$ and $\overline{\rho}$ are well-defined functions of a ring R/I in R/I. Standard check shows that $(\overline{\lambda}, \overline{\rho})$ is a double homothetism on a ring R/I and $\overline{\lambda}(\overline{\alpha}) = (\overline{\alpha})\overline{\lambda} = \psi(\lambda(\alpha))$ and for every $y \in R/I$ we have $\overline{\lambda}^2(y) = \overline{\lambda}(y) + \overline{\alpha} \cdot y$ and $(y)\overline{\rho}^2 = (y)\overline{\rho} + y \cdot \overline{\alpha}$. Therefore, by virtue of Remark 2.4 we have a truss $T(\overline{R}, \overline{\alpha}, \overline{\lambda}, \overline{\rho})$ built on a ring $\overline{R}$ with a multiplication given by the formula

$$\psi(a) \circ \psi(b) = \psi(a) \cdot \psi(b) + \psi((a)\rho) + \psi(\lambda(b)) + \overline{\alpha},$$

for all $a, b \in R$. In particular, it follows that ψ is an isomorphism of a truss T onto a truss $T(\overline{R}, \overline{\alpha}, \overline{\lambda}, \overline{\rho})$. ■

The next theorems proved in [1] provide necessary and sufficient conditions for trusses of the form $T(R, \alpha, \lambda, \rho)$ to be isomorphic.

Theorem 2.7 ([1, Lemma 3.7 & Lemma 3.8]). *Let $T = T(R, \alpha, \lambda, \rho)$ be a truss that determines the ring $R = (R, +, \cdot, 0)$ and let Φ be an automorphism of R and let $\beta \in R$. Moreover, let*

$$\alpha_1 = \Phi(\alpha + \beta + \beta \cdot \beta - (\beta)\rho - \lambda(\beta)), \tag{2.9}$$

$$\rho_1 = \Phi(\rho - r_\beta)\Phi^{-1}, \tag{2.10}$$

$$\lambda_1 = \Phi(\lambda - l_\beta)\Phi^{-1}. \tag{2.11}$$

Then $T_1 = T(R, \alpha_1, \lambda_1, \rho_1)$ is a truss that determines the ring R and $T_1 \cong T$.

Theorem 2.8. *Let $T = T(R, \alpha, \lambda, \rho)$ and $T_1 = T(R, \alpha_1, \lambda_1, \rho_1)$ be the trusses determined by the ring $R = (R, +, 0, \cdot)$. These trusses are isomorphic if and only if there exists $\beta_0 \in R$ and there exists an automorphism Φ of the ring R such that relations (2.9)–(2.11) hold.*

The next result is a generalization of the results presented above, better suited to our purposes.

Theorem 2.9. *Let $T_i = T(R_i, \alpha_i, \lambda_i, \rho_i)$ be a truss determined by a ring $(R_i, +_i, \cdot_i, 0_i)$ for $i = 1, 2$. A function $\varphi\colon R_1 \to R_2$ is a truss homomorphism of T_1 into a truss T_2 if* *and only if there exist a ring homomorphism* ψ *of a ring* R_1 *into a ring* R_2 *and there exists* $\beta \in R_2$ *such that*

$$\psi(\alpha_1) +_2 \beta = \beta \cdot_2 \beta +_2 (\beta)\rho_2 +_2 \lambda_2(\beta) +_2 \alpha_2$$

and $\varphi(x) = \psi(x) +_2 \beta$ *for every* $x \in R_1$ *and for any* $a, b \in R_1$*,*

$$\psi(\lambda_1(b)) = \beta \cdot_2 \psi(b) +_2 \lambda_2(\psi(b)) \quad \text{and} \quad \psi((a)\rho_1) = \psi(a) \cdot_2 \beta +_2 (\psi(a))\rho_2.$$

In particular, $\lambda_1(x) \in \operatorname{Ker}\psi$ *and* $(x)\rho_1 \in \operatorname{Ker}\psi$ *for every* $x \in \operatorname{Ker}\psi$ *and* $\varphi(T_1) \cong \overline{T_1}$*, where* $\overline{R_1} = R_1/\operatorname{Ker}\psi$*.*

Proof. $\Rightarrow$. Let $\beta := \varphi(0_1)$ and $\psi(x) := \varphi(x) -_2 \varphi(0_1)$ for $x \in R_1$. Then for $a, b \in R_1$ we have

$$\begin{aligned}\psi(a +_1 b) &= \varphi(a +_1 b) -_2 \varphi(0) = \varphi([a, 0_1, b]_1) -_2 \varphi(0_1)\\ &= [\varphi(a), \varphi(0_1), \varphi(b)]_2 -_2 \varphi(0_1) = \varphi(a) -_2 \varphi(0_1) +_2 \varphi(b) -_2 \varphi(0_1)\\ &= \psi(a) +_2 \psi(b).\end{aligned}$$

Hence ψ is a group homomorphism of a group $(R_1, +_1, 0_1)$ into a group $(R_2, +_2, 0_2)$. Next, by formula (2.8), $\alpha_1 = 0_1 \circ_1 0_1$, so

$$\varphi(\alpha_1) = \varphi(0_1) \circ_2 \varphi(0_1) = \beta \cdot_2 \beta +_2 (\beta)\rho_2 +_2 \lambda_2(\beta) +_2 \alpha_2.$$

Thus

$$\psi(\alpha_1) +_2 \beta = \beta \cdot_2 \beta +_2 (\beta)\rho_2 +_2 \lambda_2(\beta) +_2 \alpha_2.$$

Similarly, for any $a, b \in R_1$ we have $\varphi(a \circ_1 b) = \varphi(a) \circ_2 \varphi(b)$, so

$$\begin{aligned}&\psi(a \cdot_1 b +_1 (a)\rho_1 +_1 \lambda_1(b) +_1 \alpha_1) +_2 \beta\\ &\quad = (\psi(a) +_2 \beta) \cdot_2 (\psi(b) +_2 \beta) +_2 (\psi(a) +_2 \beta)\rho_2 +_2 \lambda_2(\psi(b) +_2 \beta) +_2 \alpha_2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\psi(a \cdot_1 b) +_2 \psi((a)\rho_1) +_2 \psi(\lambda_1(b))\\ &\quad = \psi(a) \cdot_2 \psi(b) +_2 \psi(a) \cdot_2 \beta +_2 \beta \cdot_2 \psi(b) +_2 (\psi(a))\rho_2 +_2 \lambda_2(\psi(b)).\end{aligned}$$

Substituting $a = 0$ and $b = 0$ we get that $\psi(\lambda_1(b)) = \beta \cdot_2 \psi(b) +_2 \lambda_2(\psi(b))$ and $\psi((a)\rho_1) = \psi(a) \cdot_2 \beta +_2 (\psi(a))\rho_2$, therefore consequently $\psi(a \cdot_1 b) = \psi(a) \cdot_2 \psi(b)$. Hence ψ is a ring homomorphism of R_1 into a ring R_2.

$\Leftarrow$. For any $a, b \in R_1$ we have

$$\begin{aligned}\varphi([a, b, c]_1) &= \varphi(a -_1 b +_1 c) = \psi(a -_1 b +_1 c) +_2 \beta\\ &= \psi(a) -_2 \psi(b) +_2 \psi(c) +_2 \beta\end{aligned}$$

$$= (\psi(a) +_2 \beta) -_2 (\psi(b) +_2 \beta) +_2 (\psi(c) +_2 \beta)$$
$$= [\varphi(a), \varphi(b), \varphi(c)]_2$$

and

$$\begin{aligned}\varphi(a \circ_1 b) &= \psi(a \cdot_1 b +_1 (a)\rho_1 +_1 \lambda_1(b) +_1 \alpha_1) +_2 \beta \\ &= \psi(a) \cdot_2 \psi(b) +_2 \psi((a)\rho_1) +_2 \psi(\lambda_1(b)) +_2 \psi(\alpha_1) +_2 \beta.\end{aligned}$$

Moreover,

$$\begin{aligned}&\varphi(a) \circ_2 \varphi(b) \\ &\quad= (\psi(a) +_2 \beta) \cdot_2 (\psi(b) +_2 \beta) +_2 (\psi(a) +_2 \beta)\rho_2 +_2 \lambda_2(\psi(b) +_2 \beta) +_2 \alpha_2 \\ &\quad= \psi(a) \cdot_2 \psi(b) +_2 \psi(a) \cdot_2 \beta +_2 \beta \cdot_2 \psi(b) +_2 \beta \cdot_2 \beta +_2 (\psi(a))\rho_2 \\ &\qquad +_2 (\beta)\rho_2 +_2 \lambda_2(\psi(b)) +_2 \lambda_2(\beta) +_2 \alpha_2,\end{aligned}$$

so after taking into account the assumptions $\varphi(a \circ_1 b) = \varphi(a) \circ_2 \varphi(b)$. Hence φ is a truss homomorphism of T_1 into a truss T_2.

A standard check shows that the function $y \mapsto y +_2 \beta$ for $y \in \psi(R)$ is an isomorphism of a truss $\psi(T_1)$ to a truss $\varphi(T_1)$. Hence, and by Theorem 2.6, we have that $\varphi(T_1) \cong \overline{T_1}$, where $\overline{R_1} = R_1/\operatorname{Ker}\psi$. ■

Remark 2.10. Remarks 2.1 and 2.3 and Theorems 2.7 and 2.8 justify the correctness of the following determination procedure, up to an isomorphism, of all trusses with retracts isomorphic to a fixed abelian group $(A, +, 0)$:

(1) It is enough to consider the trusses of the form $(A, [-,-,-], \cdot)$, where $[a, b, c] = a - b + c$ for all $a, b, c \in A$.

(2) It is necessary to determine all, up to an isomorphism, rings of the form $R_s = (A, +, \cdot_s, 0)$ for $s \in S$ with an additive group $(A, +, 0)$.

(3) For every $s \in S$ determine, up to an isomorphism, all trusses T_i for $i \in I_s$ of the form $T_i = T(R_s, \alpha, \lambda, \rho)$.

(4) Up to isomorphism, all trusses with retract isomorphic to $(A, +, 0)$ are trusses T_i for $i \in I_s$ and $s \in T$.

Example 2.11. Let $(R, +, \cdot, 0)$, where $R \neq \{0\}$, be a ring with zero multiplication (i.e., $a \cdot b = 0$ for any $a, b \in R$) and let the group $(R, +, 0)$ be indecomposable. Formula (2.6) shows that if (λ, ρ) is a double homothetism on this ring, then $\lambda^2 = \lambda$ and $\rho^2 = \rho$. Hence $R^+ = \operatorname{Ker}\lambda \oplus \operatorname{Im}\lambda$ and $R^+ = \operatorname{Ker}\rho \oplus \operatorname{Im}\rho$, so from the indecomposability of the group R^+ it follows that $\lambda, \rho \in \{0_R, \mathrm{id}_R\}$.

If $\lambda = \rho = 0_R$, then by (2.8) we have that $a \circ b = \alpha$ for any $a, b \in R$. Hence the function $\varphi\colon R \to R$ given by $\varphi(x) = x - \alpha$ is an isomorphism of the truss

$T(R, \alpha, 0_R, 0_R)$ onto the truss $(R, [-,-,-], *)$ such that $[a, b, c] = a - b + c$ and $a * b = 0$ for any $a, b, c \in R$.

If $\lambda = \rho = \mathrm{id}_R$, then by (2.8) we have that $a \circ b = a + b + \alpha$ for any $a, b \in R$. Hence the function $\varphi\colon R \to R$ given by $\varphi(x) = x + \alpha$ is an isomorphism of the truss $T(R, \alpha, \mathrm{id}_R, \mathrm{id}_R)$ onto the truss $(R, [-,-,-], *)$ such that $[a, b, c] = a - b + c$ and $a * b = a + b$ for any $a, b, c \in R$.

If $\lambda = 0_R$ and $\rho = \mathrm{id}_R$, then from the condition $\lambda(\alpha) = (\alpha)\rho$ it follows $\alpha = 0$ and by (2.8) we have that $a \circ b = a$ for any $a, b \in R$. So in this case we have a truss $(R, [-,-,-], \cdot_l)$.

If $\lambda = 0_R$ and $\rho = \mathrm{id}_R$, then from the condition $\lambda(\alpha) = (\alpha)\rho$ it follows $\alpha = 0$ and by (2.8) we have that $a \circ b = b$ for any $a, b \in R$. So in this case we have a truss $(R, [-,-,-], \cdot_r)$.

The four obtained trusses are pairwise non-isomorphic by Example 2.2.

Remark 2.12. Formulas (2.4) and (2.5) show that if a truss $(T, [-,-,-], \cdot)$ is commutative, then the ring determined by this truss is also commutative and $\lambda(x) = (x)\rho$ for each $x \in T$. Conversely, from formula (2.8) we get that if the ring $(R, +, \cdot, 0)$ is commutative and $\lambda(x) = (x)\rho$ for each $x \in R$, then the truss $T(R, \alpha, \lambda, \rho)$ is commutative. In [2, Theorem 4.7], it was proved that if the ring $(R, +, \cdot, 0)$ is commutative, $R^2 \neq \{0\}$ and the group R^+ is indecomposable, then every truss of the form $T(R, \alpha, \lambda, \rho)$ is commutative, so in particular, we have that $\lambda(x) = (x)\rho$ for every $x \in R$.

Example 2.13. Let $(R, +, \cdot, 0)$ be any ring. Then $(0_R, 0_R)$ and $(\mathrm{id}_R, \mathrm{id}_R)$ are double homotheties on this ring, so by Remark 2.3 we have two trusses: $T(R, 0, 0_R, 0_R)$ and $T(R, 0, \mathrm{id}_R, \mathrm{id}_R)$ with multiplications given by the formulas $a * b = a \cdot b$ and $a \circ b = a \cdot b + a + b$ for $a, b \in R$. We will call the first of these trusses *ring-type*, and the second one *circle-type*. Note that 0 is the identity of the circle-type truss. Hence it follows that if these trusses are isomorphic, then the ring R has an identity. Conversely, let 1 be the identity of the ring R. It is easy to check that the function φ given by $\varphi(x) = x + 1$ is an isomorphism of a circle-type truss into a ring-type truss. We have thus shown that a circle-type truss is isomorphic to a ring-type truss if and only if the ring has identity.

Let us also note that if 1 is an identity of the ring R, then every double homothetism on this ring is of the form (l_c, r_c) for some $c \in R$. Hence, for the truss $T(R, \alpha, l_c, r_c)$ we have that $\alpha = c^2 - c$ and the truss multiplication is given by

$$a * b = a \cdot b + a \cdot c + c \cdot b + c^2 - c = (a + c) \cdot (b + c) - c$$

for $a, b \in R$. A simple check shows that the function f given by the formula $f(x) = x + c$ is an isomorphism of the truss $T(R, \alpha, l_c, r_c)$ onto the ring-type truss

$T(R, 0, 0_R, 0_R)$. Therefore, the ring-type truss is the only truss, up to an isomorphism, determined by the ring R with identity.

3. Trusses that have an idempotent

We will start with a simple conclusion from Remark 2.3.

Corollary 3.1. *Let e be an idempotent of a truss $T = (T, [-,-,-], \cdot)$. It is easy to check that for all $a, b \in T$,*

$$a \cdot_e e = a \cdot e -_e a \cdot e -_e e \cdot e +_e e \cdot e = e$$

and

$$e \cdot_e b = e \cdot b -_e e \cdot e -_e e \cdot b +_e e \cdot e = e,$$

so e is the zero of the ring $(T, +_e, \cdot_e, e)$ and

$$\lambda(x) = e \cdot x -_e e \cdot e = e \cdot x \quad \text{and} \quad (x)\rho = x \cdot e -_e e \cdot e = x \cdot e \quad \text{for } x \in T,$$

$\lambda(e) = (e)\rho$ and for every $x \in T$,

$$\lambda(\lambda(x)) = \lambda(x) \quad \text{and} \quad ((x)\rho)\rho = (x)\rho.$$

Moreover, for any $a, b \in T$ we have

$$a \cdot b = a \cdot_e b +_e (a)\rho +_e \lambda(b) = a \cdot_e b +_e a \cdot e +_e e \cdot b.$$

Substituting in Theorem 2.7 $\Phi = \mathrm{id}_R$ and $\beta = -e$, we immediately obtain the following theorem.

Theorem 3.2. *Let $(R, +, \cdot, 0)$ be a ring with a double homothetism (λ, ρ) and an element $\alpha \in R$ such that the truss $T = T(R, \alpha, \lambda, \rho)$ has an idempotent element e. Let $\rho_1 = \rho + r_e$ and $\lambda_1 = \lambda_1 + l_e$. Then (λ_1, ρ_1) is a double homothetism on the ring R and $T_1 = T(R, 0, \lambda_1, \rho_1)$ is a truss such that $(T, e) \cong (T_1, 0)$ and the function φ given by $\varphi(x) = x - e$ is an isomorphism of the truss T onto the truss T_1 and $\varphi(e) = 0$.*

We will now investigate when the truss $T(R, \alpha, \lambda, \rho)$ has an idempotent element. We start with the following theorem.

Theorem 3.3. *Let $(R, +, \cdot, 0)$ be a ring and let $T = T(R, \alpha, \lambda, \rho)$ be a truss. Then there exists a truss $T' = T(R, 4\alpha^3 - 3\alpha^2, \lambda', \rho')$ isomorphic to the truss T. In particular, if the element α is nilpotent, then there exists a truss $T_1 = T(R, 0, \lambda_1, \rho_1)$ that is isomorphic to the truss T, with $\lambda_1^2 = \lambda_1$ and $\rho_1^2 = \rho_1$ and then the truss T has an idempotent element.*

Proof. The assumptions of the theorem imply that $(\alpha)\rho = \lambda(\alpha)$, $\lambda^2 = \lambda + l_\alpha$ and $\rho^2 = \rho + r_\alpha$. Moreover, $\alpha \cdot \lambda(\alpha) = (\alpha)\rho \cdot \alpha = \lambda(\alpha) \cdot \alpha$. Thus, for $\beta = 2\lambda(\alpha) - \alpha$ we have $\beta^2 = 4[\lambda(\alpha)]^2 - 4\lambda(\alpha) \cdot \alpha + \alpha^2$ and

$$\begin{aligned}(\beta)\rho &= (2(\alpha)\rho - \alpha)\rho = 2(\alpha)\rho^2 - (\alpha)\rho = 2((\alpha)\rho + \alpha^2) - (\alpha)\rho \\ &= (\alpha)\rho + 2\alpha^2 = \lambda(\alpha) + 2\alpha^2.\end{aligned}$$

Similarly, $\lambda(\beta) = 2\lambda^2(\alpha) - \lambda(\alpha) = 2\lambda(\alpha) + 2\alpha^2 - \lambda(\alpha) = \lambda(\alpha) + 2\alpha^2$. Additionally,

$$\begin{aligned}[\lambda(\alpha)]^2 &= \lambda(\alpha) \cdot \lambda(\alpha) = \lambda(\alpha \cdot \lambda(\alpha)) = \lambda(\lambda(\alpha) \cdot \alpha)) = \lambda^2(\alpha^2) \\ &= \lambda(\alpha^2) + \alpha \cdot \alpha^2 = \lambda(\alpha) \cdot \alpha + \alpha^3,\end{aligned}$$

so

$$\begin{aligned}\alpha' &= \alpha + \beta + \beta^2 - (\beta)\rho - \lambda(\beta) \\ &= \alpha + 2\lambda(\alpha) - \alpha + 4[\lambda(\alpha) \cdot \alpha + \alpha^3] - 4\lambda(\alpha) \cdot \alpha + \alpha^2 - 2[\lambda(\alpha) + 2\alpha^2] \\ &= 4\alpha^3 - 3\alpha^2.\end{aligned}$$

Hence, the existence of a truss T' follows from Theorem 2.7.

If, additionally, the element $\alpha \neq 0$ is nilpotent of nilpotency index $n > 1$, then $n = 2k$ or $n = 2k + 1$ for some $k \in \mathbb{N}$, so for the element $\alpha_1 = 4\alpha^3 - 3\alpha^2$ we have $\alpha_1^k = 0$ in the first case. In the second, $\alpha_1^{k+1} = 0$, which shows that the element α_1 has less nilpotency index than the element α. Therefore, repeating our procedure at most n times, we will obtain the element $\alpha_n = 0$ and by Theorem 2.7 the proof is complete. ■

The situation becomes much simpler in the case of finite trusses.

Theorem 3.4. *Every finite truss has an idempotent.*

Proof. If $(T, [-, -, -], \cdot)$ is a finite truss, then $(T, \cdot)$ is a finite semigroup and the assertion follows from [7, Theorem 1.9]. ■

4. Endomorphisms of certain direct sums of abelian groups

The following theorem is well known and its proof is standard, so we will omit it.

Theorem 4.1. *Let I be a nonempty set and let $(A_i, +_i, 0_i)$ for $i \in I$ be abelian groups such that* $\mathrm{Hom}(A_i, A_j) = 0$ *for all distinct $i, j \in I$. Then f is an endomorphism of the group $A = \bigoplus_{i \in I} A_i$ if and only if for each $i \in I$ there is an endomorphism f_i of the group A_i, with $f(x) = (f_i(x_i))_{i \in I}$ for each $x = (x_i)_{i \in I} \in A$. Moreover, $(A, +, \cdot)$ is an associative ring if and only if for each $i \in I$ there exists an associative ring $(A_i, +_i, \cdot_i)$, with $a \cdot b = (a_i \cdot_i b_i)_{i \in I}$ for any $a = (a_i)_{i \in I}$, $b = (b_i)_{i \in I} \in A$.*

We will now describe trusses of the form $(A, [-,-,-], *)$, where $A = \bigoplus_{i\in I} A_i$, I is a finite set and $\mathrm{Hom}(A_i, A_j) = 0$ for all distinct $i, j \in I$, with $[a,b,c] = a - b + c$ for any $a, b, c \in A$. By Remark 2.3 there is an associative ring $(A, +, \cdot)$ and there is an element $\alpha = (\alpha_i)_{i\in I} \in A$ and there is a double homothetism (λ, ρ) on this ring such that $\lambda(\alpha) = (\alpha)\rho$, $\lambda^2 = \lambda + l_\alpha$ and $\rho^2 = \rho + r_\alpha$ and $a * b = a \cdot b + (a)\rho + \lambda(b) + \alpha$ for any $a, b \in A$.

Therefore, by Remark 2.3 for each $i \in I$ there exist $\lambda_i, \rho_i \in \mathrm{End}(A_i)$ and there exist associative rings $(A_i, +_i, \cdot_i)$ such that if $a = (a_i)_{i\in I}$ and $b = (b_i)_{i\in I}$, then $a \cdot b = (a_i \cdot_i b_i)_{i\in I}$, $\lambda(a) = (\lambda_i(a_i))_{i\in i}$ and $(b)\rho = ((b_i)\rho_i)_{i\in I}$ and $\lambda_i(\alpha_i) = (\alpha_i)\rho_i$ for every $i \in I$. Moreover, the standard check shows that for every $i \in I$, (λ_i, ρ_i) is a double homothetism on the ring $(A_i, +_i, \cdot_i)$, $\lambda_i^2 = \lambda_i +_i l_{\alpha_i}$ and $\rho_i^2 = \rho_i +_i r_{\alpha_i}$. Therefore, by Remark 2.3, $(A_i, [-,-,-]_i, *_i)$, is a truss, if for $u, v, w \in A_i$ we set $[u, v, w]_i = u -_i v +_i w$ and $u *_i v = u \cdot_i v +_i (u)\rho_i +_i \lambda_i(v) +_i \alpha_i$.

Now let $T'_i = (A_i, [-,-,-]_i, \circ_i)$ for $i \in I$ be a truss. Then $T = (A, [-,-,-], \circ)$, where for $a = (a_i)_{i\in I}, b = (b_i)_{i\in I} \in A$: $a \circ b = (a_i \circ_i b_i)_{i\in I}$, is a truss. Suppose that the trusses T' and $T = (A, [-,-,-], *)$ are isomorphic. Let F be an isomorphism of the truss T onto the truss T'. Then F is a bijection of the set A onto the set A, hence the function $f : A \to A$ given by $f(x) = F(x) - F(0)$ is also a bijection. Moreover, for $x, y \in A$ we have that $F(x + y) = F(x - 0 + y) = F([x, 0, y]) = [F(x), F(0), F(y)] = F(x) - F(0) + F(y)$, hence $f(x + y) = f(x) + f(y)$. Therefore, $f \in \mathrm{End}(A)$ and by Remark 2.3 for each $i \in I$ there exists $f_i \in \mathrm{End}(A_i)$ such that for each $x = (x_i)_{i\in I} \in A$ we have $f(x) = (f_i(x_i))_{i\in I}$. Moreover, $F(0) = (a_i)_{i\in I}$ where $a_i \in A_i$ for $i \in I$, so if $g_i(u) = f_i(u) +_i a_i$ for $u \in A_i$ and for $i \in I$, then $F(x) = (g_i(x_i))_{i\in I}$. The standard check shows that g_i is an isomorphism of the truss $(A_i, [-,-,-]_i, *_i)$ onto the truss $(A_i, [-,-,-]_i, \circ_i)$ for every $i \in I$. Hence, by Remark 2.3 we have that g_i is an isomorphism of the ring $(A_i, +_i, *_{0_i})$ determined by the truss $(A_i, [-,-,-]_i, *_i)$ into the ring $(A_i, +_i, \circ_{0_i})$ determined by the truss $(A_i, [-,-,-]_i, \circ_i)$.

Thus we have proved the following theorem.

Theorem 4.2. *Let I be a nonempty finite set and let $(A_i, +_i, 0_i)$ for $i \in I$ be abelian groups such that* $\mathrm{Hom}(A_i, A_j) = 0$ *for all distinct $i, j \in I$. Then every truss with a retract $A = \bigoplus_{i\in I} A_i$ is isomorphic to the truss T of the form $T = \prod_{i\in J} T_i$, where $T_i = (A_i, [-,-,-]_i, *_i)$. Moreover, the truss T is determined, up to isomorphism, by the trusses T_i for $i \in I$, and these isomorphisms are isomorphisms of the rings determined by the appropriate trusses.*

Remark 4.3. An abelian group $(A, +, 0)$ is *torsion* if each of its elements has a finite order. For every prime number p, $A_p := \{a \in A : p^n a = 0 \text{ for some } n \in \mathbb{N}\}$ is a subgroup of the group A and of course A_p is a p-group. Note that for any distinct primes p and q we have that $\mathrm{Hom}(A_p, A_q) = 0$. Moreover, it is known that $A \cong$

$\bigoplus_{p\in\mathbb{P}} A_p$. If a non-trivial abelian group $(A,+,0)$ has a finite exponent, i.e., there is a smallest natural number $n>1$ such that $nx=0$ for every $x\in A$, then the set Π of all prime divisors of n is nonempty and finite and then $A\cong\prod_{p\in\Pi} A_p$, so in this case, by Theorem 4.2 every truss with a retract isomorphic to A is isomorphic to the truss T of the form $T=\prod_{p\in\Pi} T_p$, where the truss $T_p=(A_p,[-,-,-]_p,*_p)$. Moreover, the truss T is determined, up to an isomorphism, by the trusses T_p for $p\in\Pi$.

In particular, when A is a finite cyclic group of order $n>1$, then there exist natural numbers $s,\alpha_1,\ldots,\alpha_s$ and there exist distinct primes $p_1,\ldots,p_s$ such that $n=p_1^{\alpha_1}\cdots p_s^{\alpha_s}$ and $A\cong\mathbb{Z}^+_{p_1^{\alpha_1}}\times\ldots\times\mathbb{Z}^+_{p_s^{\alpha_s}}$. Therefore, every truss with a retract isomorphic to A is isomorphic to the direct product of s trusses $T_1,\ldots,T_s$ such that the retract of truss T_i is isomorphic to the group $\mathbb{Z}^+_{p_i^{\alpha_i}}$, where the truss T_i is determined by a certain ring with the additive group $\mathbb{Z}^+_{p_i^{\alpha_i}}$ for each $i=1,\ldots,s$. Therefore, the classification of trusses whose retracts are finite cyclic groups comes down to the classification of trusses of the form $T(R,\alpha,\lambda,\rho)$, where R is a ring with an additive group $\mathbb{Z}^+_{p^k}$ and $p\in\mathbb{P}$ and $k\in\mathbb{N}$. Theorems 3.4 and 3.2 reduce the problem to the case where $\alpha=0$ and $\lambda^2=\lambda$ and $\rho^2=\rho$. Moreover, the group $\mathbb{Z}^+_{p^k}$ is indecomposable, so by Example 2.11, if $a\cdot b=0$ for $a,b\in R$, then we have exactly four non-isomorphic trusses: $T(R,0,0_R,0_R)$, $T(R,0,0_R,\mathrm{id}_R)$, $T(R,0,\mathrm{id}_R,0_R)$ and $T(R,0,\mathrm{id}_R,\mathrm{id}_R)$. Otherwise, by Examples 2.11 and 2.13, when the ring R has identity, we have only one truss (namely the ring-type truss), and if the ring R does not have identity, then we have exactly two non-isomorphic trusses: ring-type truss and circle-type truss. Therefore, it remains to classify all, up to an isomorphism, rings with the additive group $\mathbb{Z}^+_{p^k}$.

5. Rings with the additive group $\mathbb{Z}^+_{p^k}$

Theorem 5.1. *For any prime number p and for a positive integer k we have*

(i) $\mathrm{End}(\mathbb{Z}^+_{p^k})=\{l_a : a\in\mathbb{Z}_{p^k}\}$, *where $l_a(x)=a\cdot x$ for all $x\in\mathbb{Z}_{p^k}$,*

(ii) *every ring (not necessarily associative) with the additive group $\mathbb{Z}^+_{p^k}$ is commutative and associative and has the form $(\mathbb{Z}_{p^k},+,\cdot_c,0)$ for some $c\in\mathbb{Z}_{p^k}$, where $a\cdot_c b=a\cdot c\cdot b$ for $a,b\in\mathbb{Z}_{p^k}$,*

(iii) *a group $\mathbb{Z}^+_{p^k}$ is indecomposable,*

(iv) *for $c,d\in\mathbb{Z}_{p^k}$, the rings $(\mathbb{Z}_{p^k},+,\cdot_c,0)$ and $(\mathbb{Z}_{p^k},+,\cdot_d,0)$ are isomorphic if and only if $d=c\cdot u$ for some $u\in\mathbb{Z}^*_{p^k}$,*

(v) *all, up to an isomorphism, trusses with retract $(\mathbb{Z}_{p^k},+,0)$ that determines a ring with zero multiplication, are trusses with multiplications given by formulas (2.3a) and (2.3b).*

Proof. (i)–(iv). Well known.

(v). The assertion follows directly from Example 2.11. ∎

Theorem 5.2. *For every prime p and for every positive integer n up to an isomorphism there exist exactly $2n+3$ trusses with retract being a cyclic group of order p^n. Each of these trusses is a heap $(\mathbb{Z}_{p^n}, [-,-,-])$ such that $[a,b,c] = a-b+c$ for $a,b,c \in \mathbb{Z}_{p^n}$ and determines a certain commutative ring with the additive group $(\mathbb{Z}_{p^n}, +, 0)$. There are exactly $n+1$ such rings. Trusses that determine the zero ring have multiplication given by one of the formulas $a \circ b = 0$, $a \circ b = a+b$, $a \circ b = a$ and $a \circ b = b$ for all $a,b \in \mathbb{Z}_{p^n}$. The only truss determining the ring $\mathbb{Z}_{p^n}$ is a ring-type truss. Whereas for $k = 1,\dots,n-1$ we have two trusses determining the ring $(\mathbb{Z}_{p^n}, +, \cdot_{p^k}, 0)$ and they are: ring-type truss and circle-type truss.*

Proof. From Theorem 5.1 each ring with an additive group $\mathbb{Z}^+_{p^n}$ has a multiplication $\cdot_c$ for some $c \in P$ given by the formula $a \cdot cb = a \cdot c \cdot b$. Since each nonzero element c of the ring $\mathbb{Z}_{p^n}$ can be written in the form $c = p^k \cdot u$, where $k \in \{0,1,\dots,n-1\}$ and $u \in (\mathbb{Z}_{p^n})^*$, the rings $(\mathbb{Z}_{p^n}, +, \cdot_c, 0)$ and $(\mathbb{Z}_{p^n}, +, \cdot_{p^k}, 0)$ are isomorphic by Theorem 5.1. Note that $o(p^k \cdot u) = p^{n-k}$ in the group $\mathbb{Z}^+_{p^n}$, so again by Theorem 5.1 the rings $(\mathbb{Z}_{p^n}, +, \cdot_{p^k}, 0)$ for $k = 0,1,\dots,n-1$ are pairwise non-isomorphic and obviously none of them is with zero multiplication. Hence none of them is isomorphic to the ring $(\mathbb{Z}_{p^n}, +, \cdot_0, 0)$. This means that up to an isomorphism there exist exactly $n+1$ rings with the additive groups $\mathbb{Z}^+_{p^n}$ and they are rings $(\mathbb{Z}_{p^n}, +, \cdot_{p^k}, 0)$ for $k = 0,1,\dots,n-1$, and the ring $(\mathbb{Z}_{p^n}, +, \cdot_0, 0)$. As we know, the group $\mathbb{Z}^+_{p^n}$ is indecomposable. So the rest of our theorem follows from Remark 4.3. ∎

From Remark 4.3 and Theorem 5.2 we immediately obtain the following theorem classifying trusses with finite cyclic retracts.

Theorem 5.3. *Let $s, n_1, \dots, n_s \in \mathbb{N}$ and let $p_1, \dots, p_s$ be different prime numbers. Then, up to isomorphism for $m = p_1^{n_1} \cdots p_s^{n_s}$ there exist exactly $(2n_1+3)\cdots(2n_s+3)$ trusses with a retract that is a cyclic group of order m. Moreover, each of these trusses is a direct product of trusses with retracts $\mathbb{Z}^+_{p_i^{n_i}}$ for $i = 1,\dots,s$ described in Theorem 5.2.*

Let us note that for m from the above theorem, the number of trusses, with a retract that is a cyclic group of order m can be neatly expressed by the formula $\tau((m \cdot \mathrm{rad}(m))^2)$, where $\tau(x)$ denotes the number of divisors of an integer x and $\mathrm{rad}(y)$ the radical of a positive integer y.

The following example shows a surprising result that not all obvious facts about rings have equivalents for trusses.

Example 5.4. Every ring whose additive group is a cyclic group of order n is a homomorphic image of a certain ring whose additive group is $\mathbb{Z}^+$, so it is natural to ask: Is every truss whose retract is a finite cyclic group a homomorphic image of some truss whose retract is isomorphic with the group $\mathbb{Z}^+$? We will show that this is not the case. For this purpose, let us take any distinct prime numbers p and q and consider a truss $(\mathbb{Z}_p \times \mathbb{Z}_q, [-, -, -], \circ)$ where $(a_1, b_1) \circ (a_2, b_2) = (a_1, b_2)$ for any $(a_1, b_1), (a_2, b_2) \in \mathbb{Z}_p \times \mathbb{Z}_q$. This truss is not commutative, because, e.g., $(1,1) = (1,0) \circ (0,1) \neq (0,1) \circ (1,0) = (0,0)$. Let us assume that our truss is a homomorphic image of a certain truss whose retract is an infinite cyclic group. Then by [5, Theorem 3.51] our truss is a homomorphic image of a certain non-commutative truss built on an abelian heap $(\mathbb{Z}, [-, -, -])$, where $[a,b,c] = a - b + c$ for $a, b, c \in \mathbb{Z}^+$. Therefore, by [5, Theorem 3.51] we have $a * b = a$ for all $a, b \in \mathbb{Z}$ or $a * b = b$ for all $a, b \in \mathbb{Z}$, where $*$ is a truss multiplication. Therefore, there is homomorphism f of a truss $(\mathbb{Z}, [-, -, -], *)$ onto a truss $(\mathbb{Z}_p \times \mathbb{Z}_q, [-, -, -], \circ)$. Hence $(1,0) = f(a)$ and $(0,1) = f(b)$ for some $a, b \in \mathbb{Z}$ and $(1,1) = (1,0) \circ (0,1) = f(a) \circ f(b) = f(a * b)$. But $f(a * b) = f(a) = (1,0)$ or $f(a * b) = f(b) = (0,1)$, so we get a contradiction.

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