Port. Math. 82 (2025), 227–257
DOI 10.4171/PM/2139

Weighted $\frac{N}{2}$-biharmonic equations with exponential growth nonlinearity without the Ambrosetti–Rabinowitz condition

Imed Abid, Sami Baraket, and Rached Jaidane

Abstract. In this paper, we establish the existence of nontrivial solutions for a logarithmic weighted $\frac{N}{2}$-biharmonic problem within the unit ball B of $\mathbb{R}^N$, without imposing the Ambrosetti–Rabinowitz condition. The nonlinearity exhibits critical or subcritical growth in view of weighted Adams inequalities. Our proof relies on minimax techniques and the pass mountain theorem applied to Cerami sequences. In the critical case, the associated energy does not adhere to the compactness constraint. We introduce a novel growth condition and emphasise the importance of avoiding the compactness level.

1. Introduction and main results

This paper investigates innovative solutions for a weighted problem, without considering the Ambrosetti–Rabinowitz condition. Specifically, we focus on addressing the following problem:

$$\begin{cases} \Delta\big(\sigma_\beta(x)|\Delta u|^{\frac{N}{2}-2}\Delta u\big) = f(x,u) & \text{in } B, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

where B is the unit ball of $\mathbb{R}^N$ and the function $f(|x|,t)$ has a maximal growth in t with respect to the weighted Laplacian norm. The weight σ_β is given by

$$\sigma_\beta(x) = \Big(\log \frac{e}{|x|}\Big)^{\beta(\frac{N}{2}-1)}, \quad \beta \in (0,1). \tag{1.2}$$

Inspired by the pioneering work of Lazer and McKenna [19] on wave movement in suspension bridges, researchers have since delved into investigating the existence of multiple solutions for nonlinear biharmonic equations and p-biharmonic equations

Mathematics Subject Classification 2020: 35J20 (primary); 35J30, 35J35, 35J60 (secondary).
Keywords: Adams' inequality, Moser–Trudinger's inequality, nonlinearity of exponential growth, mountain pass method, compactness level.

using methods from nonlinear functional analysis. Bhakta [7] specifically investigated the presence, variety, and characteristics of complete solutions of the p-biharmonic equations with Hardy term. Bueno et al. [9] discovered multiple solutions for p-biharmonic problems involving concave-convex nonlinearities. Wang and Zhao [23] explored the existence of multiple solutions for equations of p-biharmonic type with critical growth. For additional insights into this subject, one can refer to [6, 21] and related literature.

Now, we will look at the historical origins of Adams inequalities. The notion of critical exponential growth was then extended to higher order Sobolev spaces by Adams [1]. More precisely, Adams proved the following result: for $m \in \mathbb{N}$ and Ω an open bounded set of $\mathbb{R}^N$ such that $m < N$, there exists a positive constant $C_{m,N}$ such that

$$\sup_{u\in W_0^{m,\frac{N}{m}}(\Omega),|\nabla^m u|_{\frac{N}{m}}\leq 1} \int_\Omega \exp(\beta_0|u|^{\frac{N}{N-m}})\,dx \leq C_{m,N}|\Omega|, \tag{1.3}$$

where $W_0^{m,\frac{N}{m}}(\Omega)$ denotes the mth-order Sobolev space, $\nabla^m u$ denotes the mth-order gradient of u, namely

$$\nabla^m u := \begin{cases} \Delta^{\frac{m}{2}} u, & \text{if } m \text{ is even,} \\ \nabla\Delta^{\frac{m-1}{2}} u, & \text{if } m \text{ odd,} \end{cases}$$

and

$$\beta_0 = \beta_0(m,N) := \frac{N}{\omega_{N-1}} \begin{cases} \left[\dfrac{\pi^{\frac{N}{2}} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{N-m}{2})}\right]^{\frac{N}{N-m}}, & \text{if } m \text{ is even,} \\ \left[\dfrac{\pi^{\frac{N}{2}} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{N-m+1}{2})}\right]^{\frac{N}{N-m}}, & \text{if } m \text{ odd.} \end{cases}$$

In the particular case where $N = 4$ and $m = 2$, inequality (1.3) takes the form

$$\sup_{u\in W_0^{2,2}(\Omega),|\Delta u|_2\leq 1} \int_\Omega \exp(32\pi^2|u|^2)\,dx \leq C|\Omega|.$$

Recently, an extension of Adams inequalities to Sobolev spaces involving logarithmic weights has been achieved. Wang and Zhao [23] have recently established the following result.

Theorem 1.1 ([23]). *Let $\beta \in (0,1)$ and let $\omega_\beta = (\log(\frac{e}{|x|}))^\beta$, then*

$$\sup_{u\in W_{0,\mathrm{rad}}^{2,2}(B,\omega_\beta),\|u\|\leq 1} \int_B \exp(\alpha|u|^{\frac{2}{1-\beta}})\,dx < \infty \quad \Leftrightarrow \quad \alpha \leq \alpha_\beta = 4[8\pi^2(1-\beta)]^{\frac{1}{1-\beta}},$$

where $W^{2,2}_{0,\mathrm{rad}}(B,\omega_\beta)$ *denotes the weighted Sobolev space of radial functions given by*

$$W^{2,2}_{0,\mathrm{rad}}(B,\omega_\beta) = \text{closure}\left\{u \in C^\infty_{0,\mathrm{rad}}(B) \mid \int_B \omega_\beta(x)|\Delta u|^2\,dx < \infty\right\},$$

endowed with the norm $\|u\|_{W^{2,2}_{0,\mathrm{rad}}(B,\omega_\beta)} = (\int_B \omega_\beta(x)|\Delta u|^2\,dx)^{\frac{1}{2}}$*,* B *is the unit open ball in* $\mathbb{R}^4$*.*

As an application of Theorem 1.1, Dridi and Jaidane [16] considered the following problem:

$$\begin{cases} \Delta(\omega_\beta(x)\Delta u) - \Delta u + V(x)u = f(x,u) & \text{in } B, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial B, \end{cases}$$

where B is the unit open ball in $\mathbb{R}^4$, $f(x,t)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp(\alpha t^{\frac{2}{1-\beta}})$ as $t \to +\infty$ for some $\alpha > 0$, and the potential V is positive and continuous on $\overline{B}$ and bounded away from zero in B. The authors proved that there is a nontrivial weak solution to the above problem by using the mountain pass theorem combined with the Trudinger–Moser inequality. Additionally, Jaidane [18] employed the same techniques to study a Kirchhoff-type biharmonic problem involving non-linearities with exponential growth in the sense of the Theorem 1.1.

Recent research has focused on works that employ operators characterized by logarithmic weights and exponential growth nonlinearities. These studies utilize Trudinger–Moser inequalities with logarithmic weights, as highlighted in recent investigations (refer to [5, 10, 12, 14, 17]).

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\sigma_\beta \in L^1(\Omega)$ be a non-negative function. The weighted Sobolev space is defined as

$$W^{2,\frac{N}{2}}_0(\Omega,\sigma_\beta) = \text{closure}\left\{u \in C^\infty_0(\Omega) \mid \int_B |\Delta u|^{\frac{N}{2}}\sigma_\beta(x)\,dx < \infty\right\}.$$

To obtain meaningful results, it is necessary to restrict attention to radial functions. Therefore, we consider the subspace of radial functions and focus on the following class of functions:

$$E = W^{2,\frac{N}{2}}_{0,\mathrm{rad}}(B,\sigma_\beta) = \text{closure}\left\{u \in C^\infty_{0,\mathrm{rad}}(B) \mid \int_B |\Delta u|^{\frac{N}{2}}\sigma_\beta(x)\,dx < \infty\right\}$$

which is endowed with the norm

$$\|u\| = \left(\int_B |\Delta u|^{\frac{N}{2}}\sigma_\beta(x)\,dx\right)^{\frac{2}{N}}.$$

The choice of the weight in (1.2) and the space $W^{2,\frac{N}{2}}_{0,\mathrm{rad}}(B,\sigma_\beta)$ is motivated by the particular significance of logarithmic weights and their consideration as limiting situations in the embedding of the spaces $W^{2,\frac{N}{2}}_0(\Omega,\sigma_\beta)$. Additionally, this choice is influenced by the following exponential inequalities.

Theorem 1.2 ([25]). *Let $\beta \in (0,1)$ and let σ_β given by (1.2), then*

$$\sup_{\substack{u \in W^{2,\frac{N}{2}}_{0,\mathrm{rad}}(B,\sigma_\beta), \\ \int_B \sigma_\beta(x)|\Delta u|^{\frac{N}{2}} dx \le 1}} \int_B \exp(\alpha |u|^{\frac{N}{(N-2)(1-\beta)}})\, dx < +\infty \quad \Leftrightarrow \quad \alpha \le \alpha_\beta, \tag{1.4}$$

where $\alpha_\beta = N[(N-2)N V_N]^{\frac{2}{(N-2)(1-\beta)}}(1-\beta)^{\frac{1}{(1-\beta)}}$ and V_N is the volume of the unit ball B in $\mathbb{R}^N$.

Let $\gamma = \gamma(N,\beta) := \frac{N}{(N-2)(1-\beta)}$. In view of inequality (1.4), the function f is said to have subcritical growth at $+\infty$ if

$$\forall \alpha > 0 \qquad \lim_{|s|\to+\infty} \frac{|f(x,s)|}{\exp(\alpha s^\gamma)} = 0$$

and f has a critical growth at $+\infty$ if there exists some $\alpha_0 > 0$, such that

$$\begin{aligned} &\lim_{|s|\to+\infty} \frac{|f(x,s)|}{\exp(\alpha s^\gamma)} = 0, && \forall \alpha > \alpha_0, \\ &\lim_{|s|\to+\infty} \frac{|f(x,s)|}{\exp(\alpha s^\gamma)} = +\infty, && \forall \alpha < \alpha_0 \text{ uniformly in } x \in \overline{B}. \end{aligned} \tag{1.5}$$

In the context of this paper, we consider problem (1.1) with subcritical or critical growth nonlinearities $f(x,t)$. Additionally, it becomes imperative to impose specific assumptions on the behaviour of f. Precisely, we shall consider the following conditions:

(A1) The function $f : B \times \mathbb{R} \to \mathbb{R}$ is continuous, positive, radial in x, and $f(x,t) = 0$ for $t \le 0$.

(A2) We have

$$\lim_{t\to+\infty} \frac{F(x,t)}{t^{\frac{N}{2}}} = +\infty \quad \text{uniformly in } x \in B,$$

where

$$F(x,t) = \int_0^t f(x,s)\, ds.$$

(A3) There are $\overline{C} \ge 0$ and $\theta \ge 1$ such that

$$H(x,t) \le \theta H(x,s) + \overline{C}, \quad \text{for all } 0 < t < s,\ \forall x \in B,$$

where

$$H(x,t) = t f(x,t) - \frac{N}{2} F(x,t).$$

(A4) We have

$$\limsup_{t\to 0} \frac{NF(x,t)}{2t^{\frac{N}{2}}} < \lambda_1 \quad \text{uniformly in } x \in B.$$

The first eigenvalue of problem (1.1) is defined as

$$\lambda_1 = \inf_{\substack{u\in W^{2,\frac{N}{2}}_{0,\mathrm{rad}}(B,\sigma_\beta),\\ u\neq 0}} \frac{\int_B |\Delta u|^{\frac{N}{2}}\sigma_\beta(x)\,dx}{\int_B |u|^{\frac{N}{2}}\,dx}.$$

This eigenvalue exists, and the corresponding eigenfunction is positive and belongs to $L^\infty(B)$, as shown in [15].

(A5) In the critical case,

$$\lim_{t\to\infty} \frac{f(x,t)t}{\exp(\alpha_0 t^\gamma)} \geq \gamma_0 \quad \text{uniformly in } x \in B,$$

with

$$\gamma_0 > \frac{(\frac{\alpha_\beta}{\alpha_0})^{\frac{N}{2\gamma}}}{V_N \exp(N(1-\log(2e)))}.$$

(A6) For any $\{u_n\} \in E$, if $u_n \rightharpoonup 0$ weakly in E and $f(x,u_n) \to 0$ in $L^1(B)$, then,

$$F(x,u_n) \to 0 \quad \text{in } L^1(B).$$

We recall the Ambrosetti–Rabinowitz condition (AR): there exist constants $\theta > N$ and $t_0 > 0$ such that for all $x \in B$ and $|t| > t_0$,

$$0 < \theta F(x,t) \leq f(x,t)t. \tag{AR}$$

The significance of the (AR) condition lies in its influence on the mountain pass geometry associated with the Euler–Lagrange functional linked to problem (1.1). Additionally, it guarantees the boundedness of the Palais–Smale sequence for this functional. Despite its pivotal role, the (AR) condition imposes considerable restrictions, leading to the exclusion of various intriguing and crucial nonlinearities. Notably, the (AR) condition implies the existence of positive constants θ, a_1, and a_2 such that

$$F(x,t) \geq a_1|t|^\theta - a_2, \quad \forall (x,t) \in B \times \mathbb{R},$$

where $\theta > N$. As a consequence, functions such as

$$f(x,t) = |t|^{\frac{N}{2}-2} t \log(1+|t|)$$

do not satisfy (AR) for any $\theta > N$. However, these functions do satisfy conditions (A2), (A3) and (A4).

Motivated by the works cited above, we aim to establish the existence of a nontrivial solution for problem (1.1) without relying on the (AR) condition. In the context of subcritical exponential growth, we present the following result.

Theorem 1.3. *Let $f(x,t)$ be a function with subcritical growth at $+\infty$ and satisfying (A1), (A2), (A3), and (A4). Then problem (1.1) has a nontrivial radial solution.*

In the critical exponential growth, the following result holds.

Theorem 1.4. *Assume that $f(x,t)$ has a critical growth at $+\infty$ for some α_0 and satisfies the conditions (A1), (A2), (A3) with $\theta = 1$ and $\overline{C} = 0$, and additionally satisfies (A4), (A5), and (A6), then problem (1.1) admits a nontrivial solution.*

Our methodology relies on a suitable version of the mountain pass theorem introduced by G. Cerami [11]. Problem (1.1) exhibits variational structure, and finding weak solutions in the Banach space $E = W^{2,\frac{N}{2}}_{0,\mathrm{rad}}(B, \sigma_\beta)$ is equivalent to identifying critical points of the $\mathcal{C}^1$ Euler–Lagrange functional $\mathcal{J} : E \to \mathbb{R}$ defined as follows:

$$\mathcal{J}(u) = \frac{2}{N}\int_B \sigma_\beta(x)|\Delta u|^{\frac{N}{2}}\,dx - \int_B F(x,u)\,dx. \tag{1.6}$$

The geometric prerequisites of the mountain pass theorem arise from assumptions about the nonlinear reaction term f, with the challenge lying in validating the compactness condition. We establish that when f exhibits subcritical growth, the functional $\mathcal{J}$ satisfies the necessary compactness requirement as outlined in the Ambrosetti–Rabinowitz theorem [4]. However, in the case of critical growth, this compactness dissipates. To overcome this, we verify the compactness of the Euler–Lagrange functional at a suitable level by choosing testing functions that are extremal to the weighted Adams inequality.

In summary, the study of fourth-order partial differential equations remains a captivating area with wide-ranging applications.

This paper is structured as follows: Section 2 provides relevant knowledge and helpful lemmas. In Section 3, we establish that the energy $\mathcal{J}$ fulfils the two geometric properties, and we estimate the minimax level of the Euler–Lagrange functional associated with problem (1.1). Section 4 focuses on the compactness analysis and the proof of the main results.

Throughout this work, the constant C exhibits variability across different contexts, and to emphasise this variability and its evolution, we sometimes employ indexing for these constants.

2. Preliminaries

In the subsequent discussion, we introduce key definitions, notations, and fundamental results that will play a pivotal role in this paper.

Let $\|u\|_{L^p(B)}$ denote the standard norm in the Lebesgue space $L^p(B)$ for $1 \leq p < \infty$. It is defined as

$$\|u\|_{L^p(B)} = \left(\int_B |u|^p dx\right)^{\frac{1}{p}}.$$

Furthermore, we use $\|u\|$ the norm defined in the weighted Sobolev space $E = W^{2,\frac{N}{2}}_{0,\text{rad}}(B, \sigma_\beta)$,

$$\|u\| = \left(\int_B |\Delta u|^{\frac{N}{2}} \sigma_\beta(x)\, dx\right)^{\frac{2}{N}}.$$

This space is shown to be a Banach and reflexive space. The classical Sobolev embedding theorem asserts the continuous embedding $W^{2,\frac{N}{2}}_{0,\text{rad}}(B, \sigma_\beta) \hookrightarrow L^q$ for all $q \geq 1$. Moreover, the Rellich–Kondrachov theorem affirms the compactness of the embedding

$$W^{2,\frac{N}{2}}_{0,\text{rad}}(B, \sigma_\beta) \hookrightarrow\hookrightarrow L^q \quad \text{for all } q > \frac{N}{2}.$$

For more detailed information on the compactness of embeddings in weighted Sobolev spaces, particularly in the radial case, see [2]. This compact embedding will be crucial in the proof of our multiplicity result.

Now, we introduce a crucial definition.

Definition 2.1. Let $u \in E$ be a solution of problem (1.1) if $f(x, u) \in L^1(B)$, and for all $\varphi \in E$, it satisfies

$$\int_B \sigma_\beta(x)|\Delta u|^{\frac{N}{2}-2}\Delta u.\Delta\varphi\, dx = \int_B f(x, u)\varphi\, dx.$$

Considering that the nonlinearity f exhibits critical or subcritical growth, we establish the existence of positive constants c_1 and c_2 such that

$$|f(x,t)| \leq c_1 \exp(c_2|t|^\gamma), \quad \forall x \in B,\ \forall t \in \mathbb{R}. \tag{2.1}$$

Using (A1) and inequality (2.1), we ensure the well-defined nature of the functional $\mathcal{J}$ given by (1.6) and that it is of class $\mathcal{C}^1$.

Now, we proceed to another definition.

Definition 2.2. Consider a sequence (u_n) in a Banach space E and let $\mathcal{J} \in \mathcal{C}^1(E, \mathbb{R})$ with $c \in \mathbb{R}$. We define (u_n) as a Palais–Smale sequence at level c (or $(\text{PS})_c$ sequence) for the functional $\mathcal{J}$ if

$$\mathcal{J}(u_n) \to c \quad \text{and} \quad \mathcal{J}'(u_n) \to 0 \quad \text{in } E'.$$

Moreover, the functional $\mathcal{J}$ satisfies the Palais–Smale condition $(\mathrm{PS})_c$ at the level c if every $(\mathrm{PS})_c$ sequence (u_n) is relatively compact in E.

Additionally, a functional $\mathcal{J}$ satisfies $(\mathrm{C})_c$, the Cerami condition, at a level $c \in \mathbb{R}$ if any sequence $(u_n) \subset E$ such that

$$\mathcal{J}(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)\mathcal{J}'(u_n) \to 0,$$

has a convergent subsequence.

Within critical point theory, there are situations where a Palais–Smale sequence may not result in a critical point, whereas a Cerami sequence can yield one. This is grounded in the concept of linking (refer to [22] for more details and examples). Importantly, the Cerami condition is relatively weaker than the Palais–Smale condition, as the former encompasses the latter.

In the sequel, we need the following radial lemma introduced and proved in [25] which is of crucial importance.

Lemma 1 ([25]). *Let B be the unit ball in $\mathbb{R}^N$, and $u \in W^{2,\frac{N}{2}}_{0,\mathrm{rad}}(B, w)$. If $w(x) = (\log(\frac{e}{|x|}))^{\beta(\frac{N}{2}-1)}$ and $0 < \beta < 1$, then we have*

$$|u(x)| \le \left(\frac{N}{\alpha_\beta}\left(\left|\log \frac{e}{|x|}\right| - 1\right)\right)^{\frac{1}{\gamma}} \|u\|.$$

Remark 2.1. Following from Lemma 1 and by density, we obtain $\int_B \exp(|u|^\gamma)\,dx < +\infty$, $\forall u \in E$.

Next, we present a useful lemma.

Lemma 2 ([13]). *Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be a bounded domain and $f : \overline{\Omega} \times \mathbb{R}$ a continuous function. Let $\{u_n\}_n$ be a sequence in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $f(x, u_n)$ and $f(x, u)$ are also in $L^1(\Omega)$. If*

$$\int_\Omega |f(x, u_n)u_n|\,dx \le C,$$

where C is a positive constant, then

$$f(x, u_n) \to f(x, u) \quad \text{in } L^1(\Omega).$$

3. The geometrical properties and the minimax level

3.1. The mountain pass structure

We establish the mountain pass geometry of the functional $\mathcal{J}$. Specifically, we prove the following result.

Proposition 3.1. *Suppose that (A1)–(A4) hold. Then there exist $a > 0$ and $\rho > 0$ such that $\mathcal{J}(u) \geq a$ for all $u \in E$ with $\|u\| = \rho$.*

Proof. By hypothesis (A4), there exist a small constant $\varepsilon_0 \in (0, 1)$ and $\delta > 0$ such that for all $|t| \leq \delta$,

$$F(x,t) \leq \frac{2}{N}\lambda_1(1-\varepsilon_0)|t|^{\frac{N}{2}}. \tag{3.1}$$

From (A4), we have

$$\limsup_{t\to 0} \frac{NF(x,t)}{2t^{\frac{N}{2}}} < \lambda_1 \quad \text{uniformly in } x \in B,$$

which implies

$$\inf_{\delta>0} \sup\left\{\frac{NF(x,t)}{2t^{\frac{N}{2}}},\ 0 < t < \delta\right\} < \lambda_1.$$

The strict inequality allows us to find $\varepsilon_0 > 0$ such that

$$\inf_{\delta>0} \sup\left\{\frac{NF(x,t)}{2t^{\frac{N}{2}}},\ 0 < t < \delta\right\} < \lambda_1 - \varepsilon_0.$$

Thus, there exists $\delta > 0$, such that

$$\sup\left\{\frac{NF(x,t)}{2t^{\frac{N}{2}}},\ 0 < t < \delta\right\} < \lambda_1 - \varepsilon_0,$$

leading to

$$\forall |t| < \delta, \quad F(x,t) \leq \frac{2}{N}\lambda_1(1-\varepsilon_0)t^{\frac{N}{2}}.$$

By inequality (2.1) and choosing $q > \frac{N}{2}$, there exists a constant c_3 such that

$$F(x,t) \leq c_3|t|^q \exp(c_2|t|^\gamma), \quad \text{for } |t| \geq \delta. \tag{3.2}$$

Therefore, from (3.1) and (3.2), we conclude that

$$F(x,t) \leq \frac{2}{N}\lambda_1(1-\varepsilon_0)|t|^{\frac{N}{2}} + c_3|t|^q \exp(c_2|t|^\gamma), \quad \text{for } t \in \mathbb{R}.$$

Using this inequality, we obtain

$$\begin{aligned}\mathcal{J}(u) &\geq \frac{2}{N}\|u\|^{\frac{N}{2}} - \frac{2}{N}\lambda_1(1-\varepsilon_0)\int_B |u|^{\frac{N}{2}}\,dx - c_3\int_B |u|^q \exp(c_2 u^\gamma)\,dx \\ &\geq \frac{2\varepsilon_0}{N}\|u\|^{\frac{N}{2}} - c_3\int_B |u|^q \exp(c_2 u^\gamma)\,dx.\end{aligned}$$

By applying the Hölder inequality, we get

$$\mathcal{J}(u) \geq \frac{2\varepsilon_0}{N}\|u\|^{\frac{N}{2}} - c_3\left(\int_B \exp\left(\frac{N}{2}c_2|u|^\gamma\right)dx\right)^{\frac{2}{N}}\left(\int_B |u|^{\frac{Nq}{N-2}}\,dx\right)^{\frac{N-2}{N}}.$$

From Theorem 1.2, if we choose $u \in E$ such that

$$\frac{N}{2}c_2\|u\|^{\gamma} \leq \alpha_{\beta}, \tag{3.3}$$

we get

$$\int_B \exp\Big(\frac{N}{2}c_2|u|^{\gamma}\Big)\,dx = \int_B \exp\Big(\frac{N}{2}c_2\|u\|^{\gamma}\Big(\frac{|u|}{\|u\|}\Big)^{\gamma}\Big)\,dx < +\infty.$$

By Lemma 1, there exists a constant $c_4 > 0$, such that

$$\|u\|_{L^{\frac{Nq}{N-2}}(B)} \leq c_4\|u\|.$$

Thus,

$$\mathcal{J}(u) \geq \frac{2\varepsilon_0}{N}\|u\|^{\frac{N}{2}} - c_5\|u\|^q,$$

for all $u \in E$ satisfying (3.3). Since $q > \frac{N}{2}$, we can choose $\rho = \|u\|$ as the maximum point of the function $g(\rho) = \frac{2\varepsilon_0}{N}\rho^{\gamma} - c_5\rho^q$ on the interval

$$\Bigg[0, \frac{\alpha_{\beta}^{\frac{1}{\gamma}}}{(2c_2)^{\frac{1}{\gamma}}}\Bigg],$$

and let $a = \mathcal{J}(\rho)$. Then, Proposition 3.1 follows. ∎

As a second geometric property of the energy $\mathcal{J}$, we establish the following result.

Proposition 3.2. *Suppose (A1) and (A2) hold. Then, there exists $\overline{u} \in E$ such that $\|\overline{u}\| > \rho$ and $\mathcal{J}(\overline{u}) < 0$.*

Proof. Let $u_0 \in E \setminus \{0\}$ with $u \geq 0$. By (A2), for all $\varepsilon > 0$, there exists $D = D_{\varepsilon}$ such that for all $(x,t) \in B \times \mathbb{R}^+$,

$$F(x,t) \geq \varepsilon t^{\frac{N}{2}} - D.$$

Now, consider $\mathcal{J}(tu_0)$ for $t \geq 1$:

$$\begin{aligned}\mathcal{J}(tu_0) &= \frac{2t^{\frac{N}{2}}}{N}\|u_0\|^{\frac{N}{2}} - \int_B F(x,tu_0)\,dx.\\ &\leq \frac{2|t|^{\frac{N}{2}}}{N}\|u_0\|^N - \varepsilon|t|^{\frac{N}{2}}\|u_0\|^{\frac{N}{2}}_{\frac{N}{2}} + \frac{1}{N}w_{N-1}D\\ &= |t|^{\frac{N}{2}}\Bigg(\frac{\|u_0\|^{\frac{N}{2}}}{\frac{N}{2}} - \varepsilon\|u_0\|^{\frac{N}{2}}_{\frac{N}{2}}\Bigg) + \frac{1}{N}w_{N-1}D.\end{aligned}$$

Choosing $\varepsilon > \frac{2\|u_0\|^{\frac{N}{2}}}{N\|u_0\|^{\frac{N}{2}}_{\frac{N}{2}}}$, we obtain

$$\mathcal{J}(tu_0) \to -\infty, \quad \text{as } t \to +\infty.$$

Then, Proposition 3.2 follows. ∎

3.2. Estimation of the minimax level C_M

According to Propositions 3.1 and 3.2, we define

$$C_M := \inf_{\gamma\in\Lambda}\max_{t\in[0,1]} \mathcal{J}(\gamma(t)) > 0$$

and

$$\Lambda := \{\gamma \in \mathcal{C}([0,1], E) \text{ such that } \gamma(0) = 0 \text{ and } \mathcal{J}(\gamma(1)) < 0\}.$$

Our objective is to estimate the minimax value C_M of the functional $\mathcal{J}$. The idea is to construct a sequence of functions $(v_n) \in E$, and estimate $\max\{\mathcal{J}(tv_n) \mid t \geq 0\}$. For this goal, consider the following Adam's function defined for all $n \geq 3$ as

$$w_n(x) = \begin{cases} \left(\frac{\log(e\sqrt[N]{n})}{\alpha_\beta}\right)^{\frac{1}{\gamma}} - \frac{N|x|^{2(1-\beta)}}{2(\alpha_\beta)^{\frac{1}{\gamma}}(\frac{1}{n})^{\frac{2(1-\beta)}{N}}(\log(e\sqrt[N]{n}))^{\frac{\gamma-1}{\gamma}}} \\ \quad + \frac{N}{2(\alpha_\beta)^{\frac{1}{\gamma}}(\log(e\sqrt[N]{n}))^{\frac{\gamma-1}{\gamma}}} & \text{if } 0 \leq |x| \leq \frac{1}{\sqrt[N]{n}}, \\ \frac{N^{(1-\beta)}}{\alpha_\beta^{\frac{1}{\gamma}}(\log e\sqrt[N]{n})^{\frac{2(1-\beta)}{N}}}\left(\log\left(\frac{e}{|x|}\right)\right)^{1-\beta} & \text{if } \frac{1}{\sqrt[N]{n}} \leq |x| \leq \frac{1}{2}, \\ \zeta_n & \text{if } \frac{1}{2} \leq |x| \leq 1, \end{cases}$$

where $\zeta_n \in C^\infty_{0,\mathrm{rad}}(B)$ is such that

$$\zeta_n\left(\frac{1}{2}\right) = \frac{N^{(1-\beta)}}{\alpha_\beta^{\frac{1}{\gamma}}(\log e\sqrt[N]{n})^{\frac{2(1-\beta)}{N}}}(\log 2e)^{1-\beta},$$

$$\frac{\partial\zeta_n}{\partial r}\left(\frac{1}{2}\right) = \frac{-2(1-\beta)N^{(1-\beta)}}{\alpha_\beta^{\frac{1}{\gamma}}(\log e\sqrt[N]{n})^{\frac{2(1-\beta)}{N}}}(\log(2e))^{-\beta},$$

$$\zeta_n(1) = \frac{\partial\zeta_n}{\partial r}(1) = 0,$$

and ξ_n, $\nabla\xi_n$, $\Delta\xi_n$ are all $O\left(\frac{1}{[\log(e\sqrt[N]{n})]^{\frac{2(1-\beta)}{N}}}\right)$. Here, $\frac{\partial\zeta_n}{\partial r}$ denotes the first derivative of ζ_n in the radial variable $r = |x|$.

Let $v_n(x) = \frac{w_n}{\|w_n\|}$. It follows that $v_n \in E$ and $\|v_n\|^{\frac{N}{2}} = 1$. By direct computation, we find

$$\Delta w_n(x) = \begin{cases} -\frac{N(1-\beta)(4-2\beta)|x|^{-2\beta}}{\alpha_\beta^{\frac{1}{\gamma}}(\frac{1}{n})^{\frac{2(1-\beta)}{N}}(\log(e\sqrt[N]{n}))^{\frac{\gamma-1}{\gamma}}} & \text{if } 0 \leq |x| \leq \frac{1}{\sqrt[N]{n}}, \\ \frac{(1-\beta)N^{(1-\beta)}}{\alpha_\beta^{\frac{1}{\gamma}}(\log e\sqrt[N]{n})^{\frac{2(1-\beta)}{N}}}\left(\log\left(\frac{e}{|x|}\right)\right)^{-\beta} \\ \quad \cdot\frac{1}{|x|^2}\left(\frac{-\beta}{\log\frac{e}{|x|}} - (N-2)\right) & \text{if } \frac{1}{\sqrt[N]{n}} \leq |x| \leq \frac{1}{2}, \\ \Delta\zeta_n & \text{if } \frac{1}{2} \leq |x| \leq 1. \end{cases}$$

We compute the weighted Sobolev norm of w_n and obtain

$$\int_B |\Delta w_n|^{\frac{N}{2}} \sigma_\beta(x)\,dx = \underbrace{N V_N \int_0^{\frac{1}{\sqrt[N]{n}}} r^{N-1} |\Delta w_n(x)|^{\frac{N}{2}} \Big(\log\frac{e}{r}\Big)^{\beta(\frac{N}{2}-1)}\,dr}_{I_1}$$
$$+ \underbrace{N V_N \int_{\frac{1}{\sqrt[N]{n}}}^{\frac{1}{2}} r^{N-1} |\Delta w_n(x)|^{\frac{N}{2}} \Big(\log\frac{e}{r}\Big)^{\beta(\frac{N}{2}-1)}\,dr}_{I_2}$$
$$+ \underbrace{N V_N \int_{\frac{1}{2}}^{1} r^{N-1} |\Delta w_n(x)|^{\frac{N}{2}} \Big(\log\frac{e}{r}\Big)^{\beta(\frac{N}{2}-1)}\,dr}_{I_3}.$$

By using integration by parts, we obtain

$$\begin{aligned}
I_1 &= N V_N \int_0^{\frac{1}{\sqrt[N]{n}}} \frac{N^{\frac{N}{2}}(1-\beta)^{\frac{N}{2}}(4-2\beta)^{\frac{N}{2}} r^{N(1-\beta)-1}}{(\alpha_\beta)^{\frac{N}{2\gamma}}(\frac{1}{n})^{1-\beta}(\log(e\sqrt[N]{n}))^{\frac{N(\gamma-1)}{2\gamma}}} \Big(\log\frac{e}{r}\Big)^{\beta(\frac{N}{2}-1)}\,dr \\
&= N V_N \frac{N^{\frac{N}{2}}(1-\beta)^{\frac{N}{2}}(4-2\beta)^{\frac{N}{2}}}{(\alpha_\beta)^{\frac{N}{2\gamma}}(\frac{1}{n})^{1-\beta}(\log(e\sqrt[N]{n}))^{\frac{N(\gamma-1)}{2\gamma}}} \Big[\frac{r^{N(1-\beta)}}{N(1-\beta)}\Big(\log\frac{e}{r}\Big)^{\beta(\frac{N}{2}-1)}\Big]_0^{\frac{1}{\sqrt[N]{n}}} \\
&\quad + N V_N \frac{N^{\frac{N}{2}}\beta(\frac{N}{2}-1)(1-\beta)^{\frac{N}{2}}(4-2\beta)^{\frac{N}{2}}}{2(\alpha_\beta)^{\frac{N}{2\gamma}}(\frac{1}{n})^{1-\beta}(\log(e\sqrt[N]{n}))^{\frac{N(\gamma-1)}{2\gamma}}} \\
&\quad \cdot \int_0^{\frac{1}{\sqrt[N]{n}}} r^{N(1-\beta)-1}\Big(\log\frac{e}{r}\Big)^{\beta(\frac{N}{2}-1)-1}\,dr \\
&= O\Big(\frac{1}{\log e\sqrt[N]{n}}\Big).
\end{aligned}$$

Also,

$$\begin{aligned}
I_2 &= N V_N \frac{(1-\beta)^{\frac{N}{2}} N^{\frac{N(1-\beta)}{2}}}{(\alpha_\beta)^{\frac{N}{2\gamma}}(\log(e\sqrt[N]{n}))^{(1-\beta)}} \int_{\frac{1}{\sqrt[N]{n}}}^{\frac{1}{2}} \frac{1}{r}\Big(\frac{1}{\log\frac{e}{r}}\Big)^{-\beta}\Big(\frac{-\beta}{\log\frac{e}{r}}-(N-2)\Big)^{\frac{N}{2}}\,dr \\
&= -N V_N \frac{(1-\beta)^{\frac{N}{2}} N^{\frac{N(1-\beta)}{2}}}{(\alpha_\beta)^{\frac{N}{2\gamma}}(\log(e\sqrt[N]{n}))^{(1-\beta)}} \int_{\frac{1}{\sqrt[N]{n}}}^{\frac{1}{2}} \Big(\Big|\frac{\beta}{\log\frac{e}{r}}+(N-2)\Big|\Big)^{\frac{N}{2}}\Big(\log\frac{e}{r}\Big)^{-\beta}\frac{dr}{r} \\
&= -N V_N \frac{(1-\beta)^{\frac{N}{2}} N^{\frac{N(1-\beta)}{2}}}{(\alpha_\beta)^{\frac{N}{2\gamma}}(\log(e\sqrt[N]{n}))^{(1-\beta)}} \\
&\quad \cdot \int_{\frac{1}{\sqrt[N]{n}}}^{\frac{1}{2}} \Big(\log\frac{e}{r}\Big)^{-\beta}(N-2)^{\frac{N}{2}}\Big(1+O\Big(\frac{1}{\log\frac{e}{r}}\Big)\Big)\frac{dr}{r}
\end{aligned}$$

$$= NV_N \frac{(1-\beta)^{\frac{N}{2}} N^{\frac{N(1-\beta)}{2}} (N-2)^{\frac{N}{2}}}{(\alpha_\beta)^{\frac{N}{2\gamma}} (\log(e \sqrt[N]{n}))^{(1-\beta)}} \left[\frac{1}{1-\beta} \Big(\log \frac{e}{r}\Big)^{1-\beta} \right]_{\frac{1}{2}}^{\frac{1}{\sqrt[N]{n}}}$$

$$- NV_N \frac{(1-\beta)^{\frac{N}{2}} N^{\frac{N(1-\alpha)}{2}} (N-2)^{\frac{N}{2}}}{(\alpha_\beta)^{\frac{N}{2\gamma}} (\log(e \sqrt[N]{n}))^{(1-\beta)}} \int_{\frac{1}{\sqrt[N]{n}}}^{\frac{1}{2}} \Big(\frac{1}{\log \frac{e}{r}}\Big)^{-\beta} O\Big(\frac{1}{\log \frac{e}{r}}\Big) \frac{dr}{r}$$

$$= 1 + O\Big(\frac{1}{(\log e \sqrt[N]{n})^{1-\beta}}\Big)$$

and $I_3 = O(\frac{1}{(\log e \sqrt[N]{n})^{\frac{2}{\gamma}}})$. Hence, we conclude that

$$\int_B |\Delta w_n|^{\frac{N}{2}} w_\beta(x)\, dx = 1 + O\Big(\frac{1}{(\log e \sqrt[N]{n})^{(1-\beta)}}\Big) \quad \text{as } n \to +\infty.$$

3.3. Key lemmas

Finally, we give the desired estimate.

Lemma 3. *Assume that (A5) holds and that α_0 is the real given by the definition of critical growth, then*

$$C_M < \frac{2}{N} \Big(\frac{\alpha_\beta}{\alpha_0}\Big)^{\frac{N}{2\gamma}}.$$

Proof. Let $v_n \geq 0$ and $\|v_n\| = 1$. From Proposition 3.2 $\mathcal{J}(tv_n) \to -\infty$ as $t \to +\infty$. As a consequence,

$$C_M \leq \max_{t \geq 0} \mathcal{J}(tv_n).$$

By contradiction, suppose that for all $n \geq 1$,

$$\max_{t \geq 0} \mathcal{J}(tv_n) \geq \frac{2}{N} \Big(\frac{\alpha_\beta}{\alpha_0}\Big)^{\frac{N}{2\gamma}}.$$

Therefore, for any $n \geq 1$, there exists $t_n > 0$ such that

$$\max_{t \geq 0} \mathcal{J}(tv_n) = \mathcal{J}(t_n v_n) \geq \frac{2}{N} \Big(\frac{\alpha_\beta}{\alpha_0}\Big)^{\frac{N}{2\gamma}}$$

and so,

$$\frac{2}{N} t_n^{\frac{N}{2}} - \int_B F(x, t_n v_n)\, dx \geq \frac{2}{N} \Big(\frac{\alpha_\beta}{\alpha_0}\Big)^{\frac{N}{2\gamma}}.$$

By using (A1), we have

$$t_n^{\frac{N}{2}} \geq \Big(\frac{\alpha_\beta}{\alpha_0}\Big)^{\frac{N}{2\gamma}}. \tag{3.4}$$

On the other hand,

$$\frac{d}{dt} \mathcal{J}(tv_n)\big|_{t=t_n} = t_n^{\frac{N}{2}-1} - \int_B f(x, t_n v_n) v_n\, dx = 0,$$

then,

$$t_n^{\frac{N}{2}} = \int_B f(x, t_n v_n) t_n v_n \, dx. \tag{3.5}$$

Now, we claim that the sequence (t_n) is bounded in $(0, +\infty)$. Indeed, it follows from (A5) that for all $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$f(x,t)t \geq (\gamma_0 - \varepsilon) e^{\alpha_0 |t|^\gamma} \quad \forall |t| \geq t_\varepsilon, \quad \text{uniformly in } x \in B. \tag{3.6}$$

Using (3.5), we get

$$t_n^{\frac{N}{2}} = \int_B f(x, t_n v_n) t_n v_n \, dx \geq \int_{0 \leq |x| \leq \frac{1}{\sqrt[N]{n}}} f(x, t_n v_n) t_n v_n \, dx.$$

We have for all $0 \leq |x| \leq \frac{1}{\sqrt[N]{n}}$, $w_n^\gamma \geq (\frac{\log(e \sqrt[N]{n})}{\alpha_\beta})$. From (3.4) and the result of Lemma 2,

$$t_n v_n \geq \frac{t_n}{\|w_n\|} \Big(\frac{\log(e \sqrt[N]{n})}{\alpha_\beta} \Big)^{\frac{1}{\gamma}} \to \infty \quad \text{as } n \to +\infty.$$

Hence, it follows from (3.6) that for all $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$

$$t_n^{\frac{N}{2}} \geq (\gamma_0 - \varepsilon) \int_{0 \leq |x| \leq \frac{1}{\sqrt[N]{n}}} \exp(\alpha_0 t_n^\gamma |v|_n^\gamma) \, dx$$

and

$$t_n^{\frac{N}{2}} \geq N V_N (\gamma_0 - \varepsilon) \int_0^{\frac{1}{\sqrt[N]{n}}} r^{N-1} \exp\Big(\alpha_0 t_n^\gamma \Big(\frac{\log(e \sqrt[N]{n})}{\|w_n\|^\gamma \alpha_\beta} \Big) \Big) \, dr. \tag{3.7}$$

Hence,

$$1 \geq N V_N (\gamma_0 - \varepsilon) \exp\Big(\alpha_0 t_n^\gamma \Big(\frac{\log(e \sqrt[N]{n})}{\|w_n\|^\gamma \alpha_\beta} \Big) - \log Nn - \frac{N}{2} \log t_n \Big).$$

Therefore (t_n) is bounded. Also, we have from formula (3.5),

$$\lim_{n \to +\infty} t_n^{\frac{N}{2}} \geq \Big(\frac{\alpha_\beta}{\alpha_0} \Big)^{\frac{N}{2\gamma}}.$$

Now, suppose that

$$\lim_{n \to +\infty} t_n^{\frac{N}{2}} > \Big(\frac{\alpha_\beta}{\alpha_0} \Big)^{\frac{N}{2\gamma}},$$

then for n large enough, there exists some $\delta > 0$ such that $t_n^\gamma \geq \frac{\alpha_\beta}{\alpha_0} + \delta$. Consequently, the right hand side of (3.7) tends to infinity and this contradicts the boundedness of (t_n). Since (t_n) is bounded, we get

$$\lim_{n \to +\infty} t_n^{\frac{N}{2}} = \Big(\frac{\alpha_\beta}{\alpha_0} \Big)^{\frac{N}{2\gamma}}.$$

Let us consider the sets

$$\mathcal{A}_n = \{x \in B \mid t_n v_n \geq t_\varepsilon\} \quad \text{and} \quad \mathcal{C}_n = B \setminus \mathcal{A}_n.$$

We have,

$$\begin{aligned} t_n^{\frac{N}{2}} &= \int_B f(x, t_n v_n) t_n v_n \, dx = \int_{\mathcal{A}_n} f(x, t_n v_n) t_n v_n \, dx + \int_{\mathcal{C}_n} f(x, t_n v_n) t_n v_n \, dx \\ &\geq (\gamma_0 - \varepsilon) \int_{\mathcal{A}_n} \exp(\alpha_0 t_n^\gamma v_n^\gamma) \, dx + \int_{\mathcal{C}_n} f(x, t_n v_n) t_n v_n \, dx \\ &= (\gamma_0 - \varepsilon) \int_B \exp(\alpha_0 t_n^\gamma v_n^\gamma) \, dx - (\gamma_0 - \varepsilon) \int_{\mathcal{C}_n} \exp(\alpha_0 t_n^\gamma v_n^\gamma) \, dx \\ &\quad + \int_{\mathcal{C}_n} f(x, t_n v_n) t_n v_n \, dx. \end{aligned}$$

Since $v_n \to 0$ a.e. in B, $\chi_{\mathcal{C}_n} \to 1$ a.e. in B, therefore, using the dominated convergence theorem, we get

$$\int_{\mathcal{C}_n} f(x, t_n v_n) t_n v_n \, dx \to 0 \quad \text{and} \quad \int_{\mathcal{C}_n} \exp(\alpha_0 t_n^\gamma v_n^\gamma) \, dx \to N V_N.$$

Then,

$$\lim_{n \to +\infty} t_n^{\frac{N}{2}} = \left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{N}{2\gamma}} \geq (\gamma_0 - \varepsilon) \lim_{n \to +\infty} \int_B \exp(\alpha_0 t_n^\gamma v_n^\gamma) \, dx - (\gamma_0 - \varepsilon) N V_N.$$

On the other hand,

$$\int_B \exp(\alpha_0 t_n^\gamma v_n^\gamma) \, dx \geq \int_{\frac{1}{\sqrt[N]{n}} \leq |x| \leq \frac{1}{2}} \exp(\alpha_0 t_n^\gamma v_n^\gamma) \, dx + \int_{\mathcal{C}_n} \exp(\alpha_0 t_n^\gamma v_n^\gamma) \, dx.$$

Then, using (3.4),

$$\begin{aligned} \lim_{n \to +\infty} t_n^{\frac{N}{2}} &\geq \lim_{n \to +\infty} (\gamma_0 - \varepsilon) \int_B \exp(\alpha_0 t_n^\gamma v_n^\gamma) \, dx \\ &\geq \lim_{n \to +\infty} (\gamma_0 - \varepsilon) N V_N \int_{\frac{1}{\sqrt[N]{n}}}^{\frac{1}{2}} r^{N-1} \exp\Big(C^\gamma(N, \beta) \frac{(\log \frac{e}{r})^{\frac{N}{N-2}}}{(\log(e \sqrt[N]{n}))^{\frac{2}{N-2}} \|w_n\|^\gamma}\Big) \, dr. \end{aligned}$$

Therefore, making the change of variable

$$s = \frac{C(N, \beta)^\gamma (\log \frac{e}{r})}{(\log(e \sqrt[N]{n}))^{\frac{2}{N-2}} \|w_n\|^\gamma} = P \frac{(\log \frac{e}{r})}{\|w_n\|^\gamma}, \quad \text{with } P = \frac{C(N, \beta)^\gamma}{(\log(e \sqrt[N]{n}))^{\frac{2}{N-2}}},$$

we get

$$\begin{aligned}
\lim_{n\to+\infty} t_n^{\frac{N}{2}} &\geq \lim_{n\to+\infty} (\gamma_0-\varepsilon)\int_B \exp(\alpha_0 t_n^\gamma v_n^\gamma)\,dx \\
&\geq \lim_{n\to+\infty} N V_N(\gamma_0-\varepsilon)\frac{\|w_n\|^\gamma}{P}\int_{\frac{P\log(2e)}{\|w_n\|^\gamma}}^{\frac{P\log(e\sqrt[N]{n})}{\|w_n\|^\gamma}} \exp\Big(N\Big(1-\frac{s\|w_n\|^\gamma}{P}\Big) \\
&\quad + \frac{\|w_n\|^{\frac{2\gamma}{N-2}}}{P^{\frac{N}{N-2}}} s^{\frac{N}{N-2}}\Big)\,ds \\
&\geq \lim_{n\to+\infty} N V_N(\gamma_0-\varepsilon)\frac{\|w_n\|^\gamma}{P} e^N \int_{\frac{P\log(2e)}{\|w_n\|^\gamma}}^{\frac{P\log(e\sqrt[N]{n})}{\|w_n\|^\gamma}} \exp\Big(-\frac{N}{P}\|w_n\|^\gamma s\Big)\,ds \\
&= \lim_{n\to+\infty} (\gamma_0-\varepsilon) N V_N \frac{\exp(N)}{N}(-\exp(-N\log(e\sqrt[N]{n})) \\
&\quad + \exp(-N\log(2e))) \\
&= (\gamma_0-\varepsilon)V_N\exp(N(1-\log(2e))).
\end{aligned}$$

It follows that

$$\Big(\frac{\alpha_\beta}{\alpha_0}\Big)^{\frac{N}{2\gamma}} \geq (\gamma_0-\varepsilon)V_N\exp(N(1-\log(2e))),$$

for all $\varepsilon > 0$. So,

$$\gamma_0 \leq \frac{(\frac{\alpha_\beta}{\alpha_0})^{\frac{N}{2\gamma}}}{V_N\exp(N(1-\log(2e)))},$$

which is in contradiction with condition (A5). The lemma is proved. ∎

4. The Cerami sequences and proof of the main results

4.1. Lions-type concentration lemma

To prove a compactness condition for the energy $\mathcal{J}$, we require a Lions-type result [20] on an improved Adams inequality.

Lemma 4. *Let $(u_k)_k$ be a sequence in E. Suppose that, $\|u_k\| = 1$, $u_k \rightharpoonup u$ weakly in E, $u_k(x) \to u(x)$ a.e. $x \in B$, $\Delta u_k(x) \to \Delta u(x)$ a.e. $x \in B$ and $u \not\equiv 0$. Then*

$$\sup_k \int_B \exp(p\alpha_\beta |u_k|^\gamma)\,dx < +\infty,$$

where $\alpha_\beta = N[(N-2)NV_N]^{\frac{2}{(N-2)(1-\beta)}}(1-\beta)^{\frac{1}{(1-\beta)}}$, for all $1 < p < U(u)$ where $U(u)$ is given by

$$U(u) := \begin{cases} \dfrac{1}{(1-\|u\|^{\frac{N}{2}})^{\frac{2\gamma}{N}}} & \text{if } \|u\| < 1, \\ +\infty & \text{if } \|u\| = 1. \end{cases}$$

Proof. For $a, b \in \mathbb{R}$, $q > 1$. If q' is its conjugate, i.e., $\frac{1}{q} + \frac{1}{q'} = 1$, we utilize the Young inequality, that

$$e^{a+b} \leq \frac{1}{q} e^{qa} + \frac{1}{q'} e^{q'b}.$$

Additionally, we have

$$(1+a)^q \leq (1+\varepsilon)a^q + \Big(1 - \frac{1}{(1+\varepsilon)^{\frac{1}{q-1}}}\Big)^{1-q}, \quad \forall a \geq 0,\ \forall \varepsilon > 0,\ \forall q > 1.$$

Thus, we obtain

$$\begin{aligned} |u_k|^\gamma &= |u_k - u + u|^\gamma \\ &\leq (|u_k - u| + |u|)^\gamma \\ &\leq (1+\varepsilon)|u_k - u|^\gamma + \Big(1 - \frac{1}{(1+\varepsilon)^{\frac{1}{\gamma-1}}}\Big)^{1-\gamma} |u|^\gamma, \end{aligned}$$

which implies

$$\begin{aligned} \int_B \exp(p\alpha_\beta |u_k|^\gamma)\, dx \leq{}& \frac{1}{q} \int_B \exp(pq\alpha_\beta (1+\varepsilon)|u_k - u|^\gamma)\, dx \\ &+ \frac{1}{q'} \int_B \exp\Big(pq'\alpha_\beta \Big(1 - \frac{1}{(1+\varepsilon)^{\frac{1}{\gamma-1}}}\Big)^{1-\gamma} |u|^\gamma\Big)\, dx, \end{aligned}$$

for any $p > 1$.

From Lemma 1 the last integral is finite. To conclude the proof, we need to verify that for any p such that $1 < p < U(u)$, the following inequality holds:

$$\sup_k \int_B \exp(pq\alpha_\beta (1+\varepsilon)|u_k - u|^\gamma)\, dx < +\infty, \tag{4.1}$$

for some $\varepsilon > 0$ and $q > 1$.

Assuming $|u| < 1$ (the case for $|u| = 1$ is similar), when

$$p < \frac{1}{(1 - \|u\|^{\frac{N}{2}})^{\frac{2\gamma}{N}}},$$

there exists $\nu > 0$ such that

$$p(1 - \|u\|^{\frac{N}{2}})^{\frac{2\gamma}{N}}(1+\nu) < 1.$$

By Brezis–Lieb's lemma [8] we have

$$\|u_k - u\|^{\frac{N}{2}} = \|u_k\|^{\frac{N}{2}} - \|u\|^{\frac{N}{2}} + o(1) \quad \text{where } o(1) \to 0 \text{ as } k \to +\infty.$$

Then,

$$\|u_k - u\|^{\frac{N}{2}} = 1 - \|u\|^{\frac{N}{2}} + o(1),$$

and so

$$\lim_{k\to+\infty} \|u_k - u\|^{\gamma} = (1 - \|u\|^{\frac{N}{2}})^{\frac{2\gamma}{N}}.$$

Therefore, for every $\varepsilon > 0$, there exists $k_\varepsilon \geq 1$ such that

$$\|u_k - u\|^{\gamma} \leq (1+\varepsilon)(1 - \|u\|^{\frac{N}{2}})^{\frac{2\gamma}{N}}, \quad \forall k \geq k_\varepsilon.$$

If we take $q = 1 + \varepsilon$ with $\varepsilon = \sqrt[3]{1+\nu} - 1$, then $\forall k \geq k_\varepsilon$, we have

$$pq(1+\varepsilon)\|u_k - u\|^{\gamma} \leq 1.$$

Therefore,

$$\begin{aligned}
&\int_B \exp(pq\alpha_\beta(1+\varepsilon)|u_k - u|^{\gamma})\, dx \\
&\quad\leq \int_B \exp\Big((1+\varepsilon)pq\alpha_\beta\Big(\frac{|u_k - u|}{\|u_k - u\|}\Big)^{\gamma}\|u_k - u\|^{\gamma}\Big)\, dx \\
&\quad\leq \int_B \exp\Big(\alpha_\beta\Big(\frac{|u_k - u|}{\|u_k - u\|}\Big)^{\gamma}\Big)\, dx \\
&\quad\leq \sup_{\|u\|\leq 1} \int_B \exp(\alpha_\beta |u|^{\gamma})\, dx < +\infty.
\end{aligned}$$

Now, (4.1) follows from (1.4). This completes the proof of Lemma 4. ∎

4.2. Proof of Theorem 1.3

We recall the following version of the mountain pass theorem.

Lemma 5 ([4]). *Let E be a real Banach space and $\mathcal{J} \in C^1(E, \mathbb{R})$. Assume that $\mathcal{J}$ satisfies the* $(\mathrm{C})_c$ *condition for any $c \in \mathbb{R}$ and the following geometric assumptions.*

(1) *$\mathcal{J}(0) = 0$ and there exist positive constants R and α such that*

$$\mathcal{J}(u) \geq \alpha, \quad \textit{for all } u \in E \textit{ with } \|u\| = R.$$

(2) *There exists $u_0 \in E$ such that $\|u_0\| > R$ and $\mathcal{J}(u_0) \leq 0$.*

Then there exists $u \in E$ such that $\mathcal{J}(u) = c$ and $\mathcal{J}'(u) = 0$. Furthermore, the critical value c is characterized by

$$c := \inf_{g\in\Gamma} \max_{u\in g([0,1])} \mathcal{J}(u),$$

where

$$\Gamma := \Big\{g \in C([0,1], E) \mid g(0) = 0,\ g(1) = u_0\Big\}.$$

Now, we prove that the functional $\mathcal{J}$ satisfies the Cerami condition at all levels $c \in \mathbb{R}$ in the subcritical case.

Lemma 6. *Suppose that (A1), (A2), (A3), and (A4) hold. Assume that the function $f(x,t)$ has subcritical growth at $+\infty$. Then the functional $\mathcal{J}$ satisfies the $(\mathrm{C})_c$ condition for any $c \in \mathbb{R}$.*

Proof. Let (u_n) be a $(\mathrm{C})_c$ sequence in E for some $c \in \mathbb{R}$, then

$$\mathcal{J}(u_n) = \frac{2}{N}\|u_n\|^{\frac{N}{2}} - \int_B F(x,u_n)\,dx \to c, \quad n \to +\infty \tag{4.2}$$

and for all $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$,

$$\begin{aligned}
&(1+\|u_n\|)|\mathcal{J}'(u_n)v| \\
&\quad = (1+\|u_n\|)\left|\int_B \sigma_\beta(x)|\Delta u_n|^{\frac{N}{2}-2}\Delta u_n.\Delta v\,dx - \int_B f(x,u_n)v\,dx\right| \\
&\quad \leq \varepsilon\|v\|,
\end{aligned} \tag{4.3}$$

for all $v \in E$ and hence for $\varepsilon_n \to 0$, up to a subsequence,

$$\begin{aligned}
&(1+\|u_n\|)|\mathcal{J}'(u_n)v| \\
&\quad = (1+\|u_n\|)\left|\int_B \sigma_\beta(x)|\Delta u_n|^{\frac{N}{2}-2}\Delta u_n.\Delta v\,dx - \int_B f(x,u_n)v\,dx\right| \\
&\quad \leq \varepsilon_n\|v\|,
\end{aligned} \tag{4.4}$$

for all $v \in E$. We will show that $\{u_n\}$ is bounded.

Assume, by contradiction, that $\|u_n\| \to +\infty$. Define

$$v_n = \frac{u_n}{\|u_n\|},$$

then $\|v_n\| = 1$. Without loss of generality, suppose that $v_n \rightharpoonup v$ in E (up to a subsequence). We want to prove that $v_n \rightharpoonup 0$ in E. By the Sobolev embedding we have $v_n(x) \to v(x)$ a.e. in B and $v_n \to v$ a.e. in $L^p(B)$ for all $p \geq 1$. Let $B^* = \{x \in B \mid v(x) \neq 0\}$ and assume $\mu(B^*) > 0$, where μ is the Lebesgue measure. Then in B^*, we have

$$\lim_{n\to+\infty} |u_n(x)| = \lim_{n\to+\infty} |v_n(x)|\|u_n\| = +\infty \quad \text{a.e. in } B^*.$$

Since $\mathcal{J}(u_n) \to c$, then $\frac{\mathcal{J}(u_n)}{\|u_n\|} \to 0$ as $n \to +\infty$ and it follows that

$$o(1) = \frac{2}{N} - \int_{B*} \frac{F(x,u_n)}{\|u_n\|^{\frac{N}{2}}} - \int_{B\setminus B^*} \frac{F(x,u_n)}{\|u_n\|^{\frac{N}{2}}}. \tag{4.5}$$

Using condition (A2), we obtain

$$\lim_{n\to+\infty}\frac{F(x,u_n(x))}{\|u_n(x)\|^{\frac{N}{2}}} = \lim_{n\to+\infty}\frac{F(x,u_n(x))}{|u_n(x)|^{\frac{N}{2}}}\frac{|u_n(x)|^{\frac{N}{2}}}{\|u_n(x)\|^{\frac{N}{2}}}$$
$$= \lim_{n\to+\infty}\frac{F(x,u_n(x))}{|u_n(x)|^{\frac{N}{2}}}|v_n(x)|^{\frac{N}{2}} = +\infty \quad \text{a.e. in } B^*.$$

This implies that

$$\int_{B^*}\frac{F(x,u_n)}{\|u_n\|^{\frac{N}{2}}}\,dx \to +\infty \quad \text{as } n\to+\infty. \tag{4.6}$$

On the one hand, from (A2), there exists a positive constant $K>0$ such that

$$F(x,t) \geq -K, \quad \forall (x,t)\in \overline{B}\times\mathbb{R}. \tag{4.7}$$

Then, using (4.7), we deduce

$$\int_{B\setminus B^*}\frac{F(x,u_n)}{\|u_n\|^{\frac{N}{2}}} \geq -\frac{K}{\|u_n(x)\|^{\frac{N}{2}}}|B\setminus B^*|.$$

Consequently, from (4.6) and (4.7), we obtain a contradiction with (4.5). Now let $t_n\in[0,1]$ such that

$$\mathcal{J}(t_nu_n) = \max_{t\in[0,1]}\mathcal{J}(tu_n).$$

Given f is subcritical at $+\infty$, for any given $R>0$ there exists $C=C(R)>0$ such that

$$F(x,s)\leq C|s|^{\frac{N}{2}} + |s|\exp\Big(\frac{N}{N-2}\frac{\alpha_\beta}{R^\gamma}|s|^\gamma\Big), \quad \forall (x,s)\in B\times(0,+\infty).$$

Since $\|u_n\|\to+\infty$, we have

$$\mathcal{J}(t_nu_n)\geq \mathcal{J}\Big(\frac{Ru_n}{\|u_n\|}\Big) = \mathcal{J}(Rv_n).$$

Applying (4.8) and using the Hölder inequality, we have

$$\frac{N}{2}\mathcal{J}(Rv_n)\geq R^{\frac{N}{2}} - \frac{N}{2}CR^{\frac{N}{2}}\int_B|v_n(x)|^{\frac{N}{2}}\,dx - \frac{N}{2}\Big(\int_B|v_n|^{\frac{N}{2}}dx\Big)^{\frac{2}{N}}$$
$$\cdot\Big(\int_B\exp\Big(\frac{\alpha_\beta}{R^\gamma}|v_n|^\gamma\Big)\,dx\Big)^{\frac{N-2}{N}}. \tag{4.8}$$

The last integral on the right side is finite in view of Theorem 1.2. Moreover, $v_n\rightharpoonup 0$ in E, then we have $\int_B|v_n(x)|^{\frac{N}{2}}dx\to 0$ as $n\to+\infty$. Letting $n\to+\infty$ in (4.8) and $R\to+\infty$, we obtain

$$\mathcal{J}(t_nu_n)\to+\infty. \tag{4.9}$$

As $\mathcal{J}(0) = 0$ and $\mathcal{J}(u_n) \to c$, we can assume that $t_n \in (0, 1)$. On the one hand, we have that $\mathcal{J}'(t_n u_n) t_n u_n = 0$, then

$$t_n^{\frac{N}{2}} \|u_n\|^{\frac{N}{2}} = \int_B f(x, t_n u_n) t_n u_n \, dx.$$

By (4.2) and (A3), we get

$$\begin{aligned} \frac{N}{2} \mathcal{J}(t_n u_n) &= t_n^{\frac{N}{2}} \|u_n\|^{\frac{N}{2}} - \frac{N}{2} \int_B F(x, t_n u_n) \, dx \\ &= \int_B \Big(f(x, t_n u_n) - \frac{N}{2} F(x, t_n u_n) \Big) \, dx \\ &\leq \theta \int_B \Big(f(x, u_n) - \frac{N}{2} F(x, u_n) \Big) \, dx + \bar{C}. \end{aligned}$$

Applying (4.3),

$$\int_B \Big(f(x, u_n) - \frac{N}{2} F(x, u_n) \Big) \, dx = \frac{N}{2} c + o_n(1).$$

This contradicts (4.9), and hence, u_n is bounded in E. Up to a subsequence, and without loss of generality, we may assume that

$$\begin{cases} \|u_n\| \leq K & \text{in } E, \\ u_n \rightharpoonup u & \text{weakly in } E, \\ u_n \to u & \text{strongly in } L^q(B) \forall q \geq 1, \\ u_n(x) \to u(x) & \text{almost everywhere in } B. \end{cases}$$

Since f is subcritical at $+\infty$, there exists a constant $C_K > 0$ such that

$$f(x, s) \leq C_K e^{\frac{\alpha_\beta}{2K^\gamma} s^\gamma}, \quad \forall (x, s) \in B \times (0, +\infty)\cdot$$

Then, by the Hölder inequality

$$\begin{aligned} \left| \int_B f(x, u_n)(u_n - u) \, dx \right| &\leq \int_B |f(x, u_n)(u_n - u)| \, dx \\ &\leq \Big(\int_B |f(x, u_n)|^2 \, dx \Big)^{\frac{1}{2}} \Big(\int_B |u_n - u|^2 \, dx \Big)^{\frac{1}{2}} \\ &\leq C \Big(\int_B \exp\Big(\frac{\alpha_\beta}{K^\gamma} |u_n|^\gamma \Big) \, dx \Big)^{\frac{1}{2}} \|u_n - u\|_{L^2(B)} \\ &\leq C \Big(\int_B \exp\Big(\alpha_\beta \frac{|u_n|^\gamma}{\|u_n\|^\gamma} \Big) \, dx \Big)^{\frac{1}{2}} \|u_n - u\|_{L^2(B)} \\ &\leq C \|u_n - u\|_{L^2(B)} \to 0 \qquad \text{as } n \to +\infty. \end{aligned}$$

Also, using the fact that $\int_B f(x,u)(u_n - u)\,dx \to 0$ ($u_n \rightharpoonup u$ in E), we get

$$\int_B (f(x,u_n) - f(x,u))(u_n - u)\,dx \to 0.$$

From (4.4) with $v = u_n - u$, we get

$$\int_B \sigma_\beta(x)|\Delta u_n|^{N-2}\Delta u_n.(\Delta u_n - \Delta u)\,dx - \int_B f(x,u_n)(u_n - u)\,dx = o_n(1). \tag{4.10}$$

On the other hand, since $u_n \rightharpoonup u$ weakly in E,

$$\int_B \sigma_\beta(x)|\Delta u|^{\frac{N}{2}-2}\Delta u.(\Delta u_n - \Delta u)dx = o_n(1). \tag{4.11}$$

Combining (4.10) and (4.11), we obtain

$$\begin{aligned}\int_B \sigma_\beta(x)(|\Delta u_n|^{N-2}\Delta u_n - |\Delta u|^{N-2}\Delta u).(\Delta u_n - \Delta u)\,dx \\ = \int_B f(x,u_n)(u_n - u)\,dx + o_n(1).\end{aligned}$$

Using the well-known inequality

$$(|x|^{N-2}x - |y|^{N-2}y).(x-y) \geq 2^{2-N}|x-y|^N, \quad \forall x, y \in \mathbb{R}^N \text{ and } N \geq 2,$$

we obtain

$$0 \leq 2^{2-N}\int_B \sigma_\beta(x)|\Delta u_n - \Delta u|^{\frac{N}{2}}\,dx \leq \int_B f(x,u_n)(u_n - u)\,dx + o_n(1).$$

Using the above results we get,

$$2^{2-N}\int_B \sigma_\beta(x)|\Delta u_n - \Delta u|^{\frac{N}{2}}dx \leq \int_B f(x,u_n)(u_n - u)dx + o_n(1) \to 0.$$

Thus,

$$\|u_n - u\| \to 0 \quad \text{as } n \to \infty.$$

Consequently, $\mathcal{J}$ satisfies the $(C)_c$ condition for all $c \in \mathbb{R}$, and the proof of Lemma 6 is completed. ■

Lemma 6 confirms that the functional $\mathcal{J}$ satisfies condition $(C)_c$ at each level c. Therefore, using Proposition 3.1 and Proposition 3.2, we conclude that the functional $\mathcal{J}$ has a non-zero critical point u in the space E. This concludes the proof of Theorem 1.3. ■

4.3. Proof of Theorem 1.4

We begin by recalling the following result.

Lemma 7 ([11]). *Let E be a real Banach space, $\mathcal{J} \in C^1(E,\mathbb{R})$ and $\mathcal{J}(0) = 0$. Assume that $\mathcal{J}$ satisfies the following geometric assumptions.*

(i) *There exist positive constants R and α such that*

$$\mathcal{J}(u) \geq \alpha, \quad \text{for all } u \in E \text{ with } \|u\| = R.$$

(ii) *There exists $u_0 \in E$ such that $\|u_0\| > R$ and $\mathcal{J}(u_0) \leq 0$.*

Let C_M be characterized by

$$C_M := \inf_{g\in\Gamma} \max_{u\in g([0,1])} \mathcal{J}(u),$$

where

$$\Gamma := \{g \in C([0,1], E) \mid g(0) = 0,\ g(1) = u_0\}.$$

Then $\mathcal{J}$ possesses a $(\mathrm{C})_{C_M}$ sequence.

Now, in the critical case, we will prove that the functional $\mathcal{J}$ satisfies the $(\mathrm{C})_{C_M}$ condition.

Lemma 8. *Suppose that (A1), (A2), (A3), (A4) and (A6) hold. Assume that the function $f(x,t)$ has critical growth at $+\infty$. Then the functional $\mathcal{J}$ satisfies the $(\mathrm{C})_{C_M}$ condition.*

Proof. According to Propositions 3.1 and 3.2, there exists a $(\mathrm{C})_{C_M}$ sequence $\{u_n\}$ in E such that

$$\mathcal{J}(u_n) = \frac{2}{N}\|u_n\|^{\frac{N}{2}} - \int_B F(x,u_n)\,dx \to C_M, \quad n \to +\infty \tag{4.12}$$

and for $\varepsilon_n \to 0$, up to a subsequence

$$\begin{aligned}(1 + \|u_n\|)|\mathcal{J}'(u_n)v| \\ &= (1 + \|u_n\|)\left|\int_B \sigma_\beta(x)|\Delta u_n|^{N-2}\Delta u_n.\Delta v\,dx - \int_B f(x,u_n)v\,dx\right| \\ &\leq \varepsilon_n\|v\|,\end{aligned} \tag{4.13}$$

for all $v \in E$.

We will show that $\{u_n\}$ is bounded. We argue by contradiction and suppose that

$$\|u_n\| \to +\infty.$$

Let $v_n = \frac{u_n}{\|u_n\|}$ then $\|v_n\| = 1$. We may suppose that $v_n \rightharpoonup v$ in E (up to a subsequence). As in the subcritical case, we can similarly show that $v_n \rightharpoonup 0$ in E. Again, let $t_n \in [0,1]$ such that

$$\mathcal{J}(t_n u_n) = \max_{t\in[0,1]} \mathcal{J}(tu_n).$$

Let $R \in (0, (\frac{\alpha_\beta}{\alpha_0})^{\frac{1}{\gamma}})$ and choose $\varepsilon = \frac{N}{N-2}\frac{\alpha_\beta}{R^\gamma} - \alpha_0 > 0$. By the criticality growth condition and (A4), for all $\varepsilon > 0$, there exist two positive constants C and C' such that

$$F(x,s) \leq C|s|^{\frac{N}{2}} + C'|s|\exp((\alpha_0+\varepsilon)|s|^\gamma), \quad \forall (x,s) \in B \times (0,+\infty).$$

Since, $\|u_n\| \to +\infty$, we have

$$\mathcal{J}(t_n u_n) \geq \mathcal{J}\Big(\frac{Ru_n}{\|u_n\|}\Big) = \mathcal{J}(Rv_n). \tag{4.14}$$

From (4.14) and using the Hölder inequality, we get

$$\frac{N}{2}\mathcal{J}(Rv_n) \geq R^{\frac{N}{2}} - \frac{N}{2}CR^{\frac{N}{2}}\int_B |v_n(x)|^{\frac{N}{2}}\,dx - R\frac{N}{2}C'\Big(\int_B |v_n(x)|^{\frac{N}{2}}\Big)^{\frac{2}{N}}$$
$$\cdot\Big(\int_B \exp\Big(\frac{N}{N-2}(\alpha_0+\varepsilon)R^\gamma|v_n|^\gamma\Big)\,dx\Big)^{\frac{N-2}{N}}.$$

The last integral on the right side is finite in view of Theorem 1.2. Moreover, $v_n \rightharpoonup 0$ in E, then we have $\int_B |v_n(x)|^{\frac{N}{2}}\,dx \to 0$ as $n \to +\infty$. Letting $n \to +\infty$ in (4.14) and $R \to (\frac{\alpha_\beta}{\alpha_0})^{\frac{1}{\gamma}}$, we get

$$\liminf_{n\to+\infty} \mathcal{J}(t_n u_n) \geq \frac{2}{N}\Big(\frac{\alpha_\beta}{\alpha_0}\Big)^{\frac{N}{2\gamma}} > C_M. \tag{4.15}$$

We have $\mathcal{J}(0) = 0$ and $\mathcal{J}(u_n) \to C_M$. We can suppose that $t_n \in (0,1)$.

On the one hand, we have that $\mathcal{J}'(t_n u_n)t_n u_n = 0$, then

$$t_n^{\frac{N}{2}}\|u_n\|^{\frac{N}{2}} = \int_B f(x,t_n u_n)t_n u_n\,dx.$$

On the other hand, by hypothesis (A3) with $\theta = 1$ and $\overline{C} = 0$, we get

$$\begin{aligned} N\mathcal{J}(t_n u_n) &= t_n^N\|u_n\|^N - N\int_B F(x,t_n u_n)\,dx \\ &= \int_B (f(x,t_n u_n)t_n u_n - NF(x,t_n u_n))\,dx \\ &\leq \int_B (f(x,u_n)u_n - NF(x,u_n))\,dx. \end{aligned}$$

Since

$$\int_B (f(x, t_n u_n) t_n u_n - N F(x, t_n u_n))\, dx = N C_M + o_n(1),$$

we reach a contradiction with (4.15). Therefore, $\{u_n\}$ is bounded in E. Up to a subsequence, and without loss of generality, we may assume that

$$\begin{cases} \|u_n\| \le M & \text{in } E, \\ u_n \rightharpoonup u & \text{weakly in } E, \\ u_n \to u & \text{strongly in } L^q(B) \forall q \ge 1, \\ u_n(x) \to u(x) & \text{almost everywhere in } B. \end{cases}$$

We follow the schema of [3] to show the convergence almost everywhere of the Laplacian $\Delta u_n(x) \to \Delta u(x)$ a.e. $x \in B$.

From (4.13), we obtain

$$0 < \int_B f(x, u_n) u_n\, dx \le C.$$

Also, from (4.12), we have

$$0 < \int_B F(x, u_n)\, dx \le C.$$

Consequently,

$$(|\Delta u_n|^{\frac{N}{2}-2} \Delta u_n) \quad \text{is bounded in } (L^{\frac{N}{N-2}}(B, \sigma_\beta))^N,$$
$$|\Delta u_n|^{N-2} \Delta u_n \rightharpoonup |\Delta u|^{\frac{N}{2}-2} \Delta u \quad \text{in } (L^{\frac{N}{N-2}}(B, \sigma_\beta))^N \text{ as } n \to +\infty.$$

By [13, Lemma 2.1], we have

$$f(x, u_n) \to f(x, u) \quad \text{in } L^1(B) \text{ as } n \to +\infty.$$

According to hypothesis (A6), we have

$$F(x, u_n) \to F(x, u) \quad \text{in } L^1(B) \text{ as } n \to +\infty. \tag{4.16}$$

By (4.12), we obtain

$$\lim_{n \to +\infty} \|u_n\|^{\frac{N}{2}} = \frac{N}{2}\left(C_M + \int_B F(x, u)\, dx\right).$$

Therefore, passing to the limit in (4.13) and using the same argument as in the subcritical case, we get

$$\int_B \sigma_\beta(x) |\Delta u|^{\frac{N}{2}-2} \Delta u . \Delta \varphi\, dx = \int_B f(x, u) \varphi\, dx, \quad \forall \varphi \in E. \tag{4.17}$$

Hence, u is a weak solution of problem (1.1).

Next, we are going to make some assertions.

Assertion 1. At this stage we affirm that $u \neq 0$. Indeed, we argue by contradiction and suppose that $u \equiv 0$. Therefore, $\int_B F(x, u_n)\, dx \to 0$ and consequently we get

$$\frac{2}{N}\|u_n\|^{\frac{N}{2}} \to C_M < \frac{2}{N}\Big(\frac{\alpha_\beta}{\alpha_0}\Big)^{\frac{N}{2\gamma}}.$$

First we claim that there exists $q > 1$ such that

$$\int_B |f(x, u_n)|^q\, dx \le C. \tag{4.18}$$

Using (4.13), we obtain

$$\left|\|u_n\|^{\frac{N}{2}} - \int_B f(x, u_n)u_n\, dx\right| \le \frac{C\varepsilon_n}{(1 + \|u_n\|)} \le C\varepsilon_n.$$

This yields

$$\|u_n\|^{\frac{N}{2}} \le C\varepsilon_n + \Big(\int_B |f(x, u_n)|^q\, dx\Big)^{\frac{1}{q}}\Big(\int_B |u_n|^{q'}\, dx\Big)^{\frac{1}{q'}},$$

where q' is the conjugate of q. Since (u_n) converges to 0 in $L^{q'}(B)$,

$$\lim_{n\to+\infty} \|u_n\|^{\frac{N}{2}} = 0.$$

By Brezis–Lieb's Lemma [8], $u_n \to 0$ in E. Therefore, $\mathcal{J}(u_n) \to 0$ which is a contradiction with $C_M > 0$.

For the proof of the claim (4.18), given the critical growth of f for every $\varepsilon > 0$ and $q > 1$ there exist $t_\varepsilon > 0$ and $C > 0$ such that for all $|t| \ge t_\varepsilon$, we have

$$|f(x, t)|^q \le C \exp(\alpha_0(1 + \varepsilon)t^\gamma).$$

Consequently,

$$\begin{aligned}\int_B |f(x, u_n)|^q\, dx &= \int_{\{|u_n|\le t_\varepsilon\}} |f(x, u_n)|^q\, dx + \int_{\{|u_n|>t_\varepsilon\}} |f(x, u_n)|^q\, dx\\ &\le \omega_{N-1} \max_{\overline{B}\times[-t_\varepsilon,t_\varepsilon]} |f(x, t)|^q + C\int_B \exp(\alpha_0(1 + \varepsilon)|u_n|^\gamma)\, dx.\end{aligned}$$

Since $(\frac{N}{2}C_M)^{\frac{2\gamma}{N}} < (\frac{\alpha_\beta}{\alpha_0})$, there exists $\eta \in (0, \frac{1}{2})$, such that $(\frac{N}{2}C_M)^{\frac{2\gamma}{N}} = (1-2\eta)(\frac{\alpha_\beta}{\alpha_0})$. From (4.12), $\|u_n\|^\gamma \to (\frac{N}{2}c)^{\frac{2\gamma}{N}}$, so there exists $n_\eta \in \mathbb{N}$ such that $\|u_n\|^\gamma \le (1-\eta)\frac{\alpha_\beta}{\alpha_0}$, for all $n \ge n_\eta$. Therefore,

$$\alpha_0(1 + \varepsilon)\Big(\frac{|u_n|}{\|u_n\|}\Big)^\gamma \|u_n\|^\gamma \le (1 + \varepsilon)(1 - \eta)\alpha_\beta.$$

We choose $\varepsilon > 0$ small enough to get

$$(1+\varepsilon)(1-\eta) < 1.$$

Hence the second integral is uniformly bounded in view of (1.4).

Assertion 2. We affirm that $\mathcal{J}(u) = C_M$. Indeed, since (u_n) is bounded, up to a subsequence, $\|u_n\| \to \rho > 0$. By using (A3) with $\theta = 1$ and $\overline{C} = 0$, we obtain

$$\mathcal{J}(u) \geq \frac{2}{N} \int_B \Big[f(x,u)u - \frac{N}{2} F(x,u) \Big]\, dx \geq 0. \tag{4.19}$$

Now, using the semicontinuity of the norm and (4.16), we have

$$\mathcal{J}(u) \leq \frac{2}{N} \liminf_{n\to\infty} \|u_n\|^{\frac{N}{2}} - \int_B F(x,u)\, dx = C_M.$$

Suppose that

$$\mathcal{J}(u) < C_M.$$

Then, it implies

$$\|u\|^{\frac{N}{2}} < \rho^{\frac{N}{2}}.$$

Furthermore, we have

$$\frac{2}{N} \lim_{n\to+\infty} \|u_n\|^{\frac{N}{2}} = \Big(C_M + \int_B F(x,u)\, dx \Big), \tag{4.20}$$

which means that

$$\rho^{\frac{N}{2}} = \frac{N}{2} \Big(C_M + \int_B F(x,u)\, dx \Big).$$

Set

$$v_n = \frac{u_n}{\|u_n\|} \quad \text{and} \quad v = \frac{u}{\rho}\cdot$$

We have $\|v_n\| = 1$, $v_n \rightharpoonup v$ in E, $v \not\equiv 0$ and $\|v\| < 1$. So, by Lemma 4, we get

$$\sup_n \int_B \exp(p\alpha_\beta |v_n|^\gamma)\, dx < \infty,$$

for $1 < p < U(v) = (1 - \|v\|^{\frac{N}{2}})^{\frac{-2\gamma}{N}}$.

By (4.16), (4.17) and (4.20), we have the following equality:

$$\frac{N}{2} C_M - \frac{N}{2} \mathcal{J}(u) = \rho^{\frac{N}{2}} - \|u\|^{\frac{N}{2}}.$$

From (4.19), (4.20) and the last equality, we obtain

$$\rho^{\frac{N}{2}} \leq \frac{N}{2} C_M + \|u\|^{\frac{N}{2}} < \Big(\frac{\alpha_\beta}{\alpha_0} \Big)^{\frac{N}{2\gamma}} + \|u\|^{\frac{N}{2}}. \tag{4.21}$$

Since

$$\rho^{\frac{N}{2}} < \frac{(\frac{\alpha_\beta}{\alpha_0})^{\frac{N}{2\gamma}}}{1 - \|v\|^{\frac{N}{2}}},$$

we deduce from (4.21) that

$$\rho^{\frac{N}{2}} < \frac{(\frac{\alpha_\beta}{\alpha_0})^{\frac{N}{2\gamma}}}{1 - \|v\|^{\frac{N}{2}}}. \tag{4.22}$$

On one hand, we have $\int_B |f(x,u_n)|^q\,dx < C$. Indeed, For $\varepsilon > 0$,

$$\begin{aligned} \int_B |f(x,u_n)|^q\,dx &= \int_{\{|u_n| \le t_\varepsilon\}} |f(x,u_n)|^q\,dx + \int_{\{|u_n| > t_\varepsilon\}} |f(x,u_n)|^q\,dx \\ &\le N V_N \max_{B\times[-t_\varepsilon,t_\varepsilon]} |f(x,t)|^q + C\int_B \exp(\alpha_0(1+\varepsilon)|u_n|^\gamma)\,dx \\ &\le C_\varepsilon + C\int_B \exp(\alpha_0(1+\varepsilon)\|u_n\|^\gamma |v_n|^\gamma)\,dx \le C, \end{aligned}$$

provided $\alpha_0(1+\varepsilon)\|u_n\|^\gamma \le p\alpha_\beta$, for p such that $1 < p < (1 - \|v\|^{\frac{N}{2}})^{-\frac{2\gamma}{N}}$.

From (4.22), there exists $\delta \in (0, \frac{1}{2})$ such that

$$\rho^\gamma = (1-2\delta)\frac{\frac{\alpha_\beta}{\alpha_0}}{(1 - \|v\|^{\frac{N}{2}})^{\frac{2\gamma}{N}}}.$$

Since $\lim_{n\to+\infty} \|u_n\|^\gamma = \rho^\gamma$, then for n large enough

$$\alpha_0(1+\varepsilon)\|u_n\|^\gamma \le \alpha_0(1+\varepsilon)\rho^\gamma \le (1+\varepsilon)(1-\delta)\frac{\alpha_\beta}{(1 - \|v\|^{\frac{N}{2}})^{\frac{2\gamma}{N}}}.$$

We choose $\varepsilon > 0$ small enough such that $(1+\varepsilon)(1-\delta) < 1$ to have

$$\alpha_0(1+\varepsilon)\|u_n\|^\gamma < \frac{\alpha_\beta}{(1 - \|v\|^{\frac{N}{2}})^{\frac{2\gamma}{N}}}.$$

So, the sequence $(f(x,u_n))$ is bounded in $L^q(B)$, $q > 1$. Using the Hölder inequality and the Sobolev embedding, we deduce that

$$\begin{aligned} \left|\int_B f(x,u_n)(u_n - u)\,dx\right| &\le \left(\int_B |f(x,u_n)|^q\,dx\right)^{\frac{1}{q}} \left(\int_B |u_n - u|^{q'}\,dx\right)^{\frac{1}{q'}} \\ &\le C\left(\int_B |u_n - u|^{q'}\right)^{\frac{1}{q'}} dx \to 0 \qquad \text{as } n \to +\infty \end{aligned}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Since $\langle \mathcal{J}'(u_n), u_n - u\rangle = o_n(1)$, it follows that

$$\int_B \sigma_\beta(x)|\Delta u_n|^{\frac{N}{2}-2}\Delta u_n(\Delta u_n - \Delta u)\,dx \to 0. \tag{4.23}$$

On the other hand, since $u_n \rightharpoonup u$ weakly in E, then

$$\int_B \sigma_\beta(x)|\Delta u|^{\frac{N}{2}-2}\Delta u(\Delta u_n - \Delta u)\,dx = o_n(1). \tag{4.24}$$

Also, using the well-known inequality

$$(|x|^{p-2}x - |y|^{p-2}y).(x-y) \geq 2^{2-p}|x-y|^p, \quad \forall x, y \in \mathbb{R}^m,\ m \in \mathbb{N} \text{ and } p \geq 2,$$

we get that

$$\int_B \sigma_\beta(x)(|\Delta u_n|^{\frac{N}{2}-2}\Delta u_n - |\Delta u|^{\frac{N}{2}-2}\Delta u).(\Delta u_n - \Delta u)\,dx \\ \geq 2^{2-\frac{N}{2}}\int_B \sigma_\beta(x)|\Delta u_n - \Delta u|^{\frac{N}{2}}.$$

Passing to the limit in the last inequality and using (4.23) and (4.24), we get

$$\lim_{n\to+\infty}\|u_n - u\| = 0.$$

Then, by Brezis lemma $\|u_n\| \to \|u\|$, we get

$$\rho^N - \|u\|^N = 0,$$

therefore $\|u\| = \rho$. This is a contradiction with (4.3). Consequently, $\mathcal{J}(u) = C_M$ and Assertion 2 is proved. ∎

As a consequence, again by Brezis–Lieb's Lemma $u_n \to u$ in E. We also have by (4.17), $\mathcal{J}'(u) = 0$. The proof of Theorem 1.4 is established.

References

[1] D. R. Adams, A sharp inequality of J. Moser for higher order derivatives. *Ann. of Math. (2)* **128** (1988), no. 2, 385–398 Zbl 0672.31008 MR 0960950

[2] R. A. Adams and J. J. F. Fournier, *Sobolev spaces*. Second edn., Pure Appl. Math. (Amst.) 140, Elsevier/Academic Press, Amsterdam, 2003 Zbl 1098.46001 MR 2424078

[3] Adimurthi and Y. Yang, An interpolation of Hardy inequality and Trudinger–Moser inequality in $\mathbb{R}^N$ and its applications. *Int. Math. Res. Not. IMRN* (2010), no. 13, 2394–2426 Zbl 1198.35012 MR 2669653

[4] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications. *J. Functional Analysis* **14** (1973), 349–381 Zbl 0273.49063 MR 0370183

[5] S. Baraket and R. Jaidane, Non-autonomous weighted elliptic equations with double exponential growth. *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.* **29** (2021), no. 3, 33–66 Zbl 1524.35190 MR 4346160

[6] F. Bernis, J. García Azorero, and I. Peral, Existence and multiplicity of nontrivial solutions in semilinear critical problems of fourth order. *Adv. Differential Equations* **1** (1996), no. 2, 219–240 Zbl 0841.35036 MR 1364002

[7] M. Bhakta, Entire solutions for a class of elliptic equations involving p-biharmonic operator and Rellich potentials. *J. Math. Anal. Appl.* **423** (2015), no. 2, 1570–1579 Zbl 1307.35021 MR 3278215

[8] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*. Universitext, Springer, New York, 2011 Zbl 1220.46002 MR 2759829

[9] H. Bueno, L. Paes-Leme, and H. Rodrigues, Multiplicity of solutions for p-biharmonic problems with critical growth. *Rocky Mountain J. Math.* **48** (2018), no. 2, 425–442 Zbl 1391.35131 MR 3810206

[10] M. Calanchi, B. Ruf, and F. Sani, Elliptic equations in dimension 2 with double exponential nonlinearities. *NoDEA Nonlinear Differential Equations Appl.* **24** (2017), no. 3, article no. 29 Zbl 1377.35091 MR 3656913

[11] G. Cerami, An existence criterion for the critical points on unbounded manifolds. *Istit. Lombardo Accad. Sci. Lett. Rend. A* **112** (1978), no. 2, 332–336 (1979) Zbl 0436.58006 MR 0581298

[12] R. Chetouane and R. Jaidane, Ground state solutions for weighted N-Laplacian problem with exponential non linear growth. *Bull. Belg. Math. Soc. Simon Stevin* **29** (2022), no. 1, 37–61 Zbl 1522.35185 MR 4546028

[13] D. G. de Figueiredo, O. H. Miyagaki, and B. Ruf, Elliptic equations in $\mathbf{R}^2$ with nonlinearities in the critical growth range. *Calc. Var. Partial Differential Equations* **3** (1995), no. 2, 139–153 Zbl 0820.35060 MR 1386960

[14] S. Deng, T. Hu, and C.-L. Tang, N-Laplacian problems with critical double exponential nonlinearities. *Discrete Contin. Dyn. Syst.* **41** (2021), no. 2, 987–1003 Zbl 1459.35118 MR 4191533

[15] P. Drábek, A. Kufner, and F. Nicolosi, *Quasilinear elliptic equations with degenerations and singularities*. De Gruyter Ser. Nonlinear Anal. Appl. 5, Walter de Gruyter & Co., Berlin, 1997 Zbl 0894.35002 MR 1460729

[16] B. Dridi and R. Jaidane, Existence solutions for a weighted biharmonic equation with critical exponential growth. *Mediterr. J. Math.* **20** (2023), no. 2, article no. 102 Zbl 1512.35253 MR 4547417

[17] R. Jaidane, Ground state solution for a weighted elliptic problem under double exponential nonlinear growth. *Z. Anal. Anwend.* **42** (2023), no. 3-4, 253–281 Zbl 1542.35182 MR 4699881

[18] R. Jaidane, Weighted fourth order equation of Kirchhoff type in dimension 4 with nonlinear exponential growth. *Topol. Methods Nonlinear Anal.* **61** (2023), no. 2, 889–916 Zbl 1533.35104 MR 4645829

[19] A. C. Lazer and P. J. McKenna, Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis. *SIAM Rev.* **32** (1990), no. 4, 537–578 Zbl 0725.73057 MR 1084570

[20] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoamericana* **1** (1985), no. 1, 145–201 Zbl 0704.49005 MR 0834360

[21] M. Makvand Chaharlang and A. Razani, A fourth order singular elliptic problem involving p-biharmonic operator. *Taiwanese J. Math.* **23** (2019), no. 3, 589–599 Zbl 1418.35123 MR 3952242

[22] M. Schechter, The use of Cerami sequences in critical point theory. *Abstr. Appl. Anal.* (2007), article no. 58948 Zbl 1156.58006 MR 2320802

[23] W. Wang and P. Zhao, Nonuniformly nonlinear elliptic equations of p-biharmonic type. *J. Math. Anal. Appl.* **348** (2008), no. 2, 730–738 Zbl 1156.35045 MR 2445772

[24] M. Willem and W. Zou, On a Schrödinger equation with periodic potential and spectrum point zero. *Indiana Univ. Math. J.* **52** (2003), no. 1, 109–132 Zbl 1030.35068 MR 1970023

[25] H. Zhao and M. Zhu, Critical and supercritical Adams' inequalities with logarithmic weights. *Mediterr. J. Math.* **20** (2023), no. 6, article no. 313 Zbl 1536.46031 MR 4651957

Received 5 February 2024; revised 14 November 2024.

Imed Abid
Department of Mathematics, Higher Institute of medicals technologies of Tunis, University of Tunis El Manar, 9 street Dr. Zouhair Essafi, 1006 Tunis, Tunisia; imed.abid@istmt.utm.tn

Sami Baraket
Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), 11623 Riyadh, Saudi Arabia; smbaraket@imamu.edu.sa

Rached Jaidane
Department of Mathematics, Faculty of Science, University of Tunis El Manar, 2092 Tunis, Tunisia; rachedjaidane@gmail.com