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Non-dense orbit sets carry full metric mean dimension for maps with gluing orbit property

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Abstract. In this paper, we analyze the non-dense orbit sets for maps with gluing orbit property. More precisely, let $f : X \to X$ be a continuous map with the gluing orbit property on a compact metric space X, in the setting, we show that for any non-transitive point $z_0 \in X$, the set $E(z_0)$ is empty or carries full metric mean dimension.

1. Introduction

Topological entropy and metric mean dimensions are two measurements of the dynamical complexity, which are particularly important for continuous dynamical systems. While the first is a topological invariant, it is typically infinite for a C^0-Baire generic subset of homeomorphisms on compact topological manifolds with or without boundary and the dimension of M is greater than one [22].

On the other hand, the second one, inspired by Gromov [12] and proposed by Lindenstrauss and Weiss, is a sort of dynamical analogue of the topological dimension, depends on the metric and it is bounded above by the dimension of the ambient space [16]. In this way, the metric mean dimension may be used to distinguish the topological complexity of homeomorphisms on compact manifolds with infinite topological entropy.

Given a continuous map $f : X \to X$ on a compact metric space (X, d), for any $x \in X$ the orbit of x is $O_f(x) := \{x, f(x), \dots, f^n(x), \dots\}$. Also, for any $z \in X$ consider

$$E(z) = \{x \in X; z \notin \overline{O_f(x)}\}.$$

Note that any point of $E(z)$ has a non-dense forward orbit in X. Yang, Chen and Zhou, in [21], show that $E(z)$ is empty or carries full Bowen upper and lower metric

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mean dimension for maps with specification property. In this sense, we show here that this also applies to maps with gluing orbit property, i.e., the set $E(z)$ is empty or carries full metric mean dimension.

Our work was inspired by [21], but we used some techniques from the work of [11] and [14] to demonstrate our result, mainly in the proof of the main theorem.

The gluing orbit property, recently introduced by Bomfim–Varandas in [6], is a topological invariant and is weaker than specification. With a solid understanding of the basic concepts, we prove that under the gluing orbit property, the set of non-dense orbits $E(z_0)$ for any non-transitive point $z_0 \in X$ is empty or carries full Bowen upper and lower metric mean dimension. More precisely, our main theorem is the following.

Theorem A. *Let X a compact metric space and $f : X \to X$ a continuous map satisfying the gluing orbit property on X. For any non-transitive point $z_0 \in X$ either $E(z_0) = \emptyset$, or carries full metric mean dimension. It is,*

$$\overline{\text{mdim}}_B(E(z_0), f, d) = \overline{\text{mdim}}(X, f, d),$$

and

$$\underline{\text{mdim}}_B(E(z_0), f, d) = \underline{\text{mdim}}(X, f, d).$$

The proof of this Theorem A will be exposed in Section 3. Before, some preliminary notions and concepts related to the content of Theorem A and its demonstration will be presented in Section 2 and finally, two examples in which Theorem A is applicable will appear in Section 4.

2. Preliminaries

2.1. Specification and gluing orbit properties

The concept of reconstruction of orbits in topological dynamics gained substantial importance for its wide range of applications in ergodic theory. Among these properties it is worth mentioning the shadowing, specification and the gluing orbit properties. Throughout this subsection $f : X \to X$ denotes a continuous map on a compact metric space X.

Definition 2.1. A continuous map $f : X \to X$ satisfies the *specification property* if for any $\varepsilon > 0$ there exists an integer $m = m(\varepsilon) \geq 1$ such that for any points $x_1, x_2, \ldots, x_k \in X$ and any positive integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ with $a_i - b_{i-1} \geq m$ for all $2 \leq i \leq k$, there is a point $y \in X$ so that $d(f^j(y), f^j(x_1)) \leq \varepsilon$ for every $a_i \leq j \leq b_i$ and $1 \leq i \leq k$. If, in addition, $y \in X$ can be chosen to be periodic with period q for some $q \geq m + b_k - a_1$ then we say that f satisfies the *periodic specification property*.

Now, we present the definition of the gluing orbit property, introduced in [6].

Definition 2.2. A continuous map $f : X \to X$ satisfies the *gluing orbit property* if for any $\varepsilon > 0$ there exists an integer $m = m(\varepsilon) \geq 1$ so that for any points $x_1, x_2, \ldots, x_k \in X$ and any positive integers $n_1, \ldots, n_k$ there are $0 \leq p_1, \ldots, p_{k-1} \leq m(\varepsilon)$ and a point $y \in X$ so that $d(f^j(y), f^j(x_1)) \leq \varepsilon$ for every $0 \leq j \leq n_1$ and

$$d(f^{j+n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(y), f^j(x_i)) \leq \varepsilon$$

for every $2 \leq i \leq k$ and $0 \leq j \leq n_i$. If, in addition, $y \in X$ can be chosen to be periodic with period $\sum_{i=1}^{k}(n_i + p_i)$ for some $0 \leq p_k \leq m(\varepsilon)$ then we say that f satisfies the *periodic gluing orbit property*.

It is clear that the specification property implies the gluing orbit property, which implies transitivity. It is not hard to check that irrational rotations satisfy the gluing orbit property [4], but fail to satisfy the shadowing or specification properties. Partially hyperbolic examples exhibiting the same kind of behavior have been constructed in [5]. In this sense, our Theorem A is a version of the result obtained in [21].

The gluing orbit property was recently applied in some works, related to the study of the form and stability of rotation sets of typical homeomorphisms in $\mathbb{T}^d$ homotopic to identity [7, 14], triviality of centralizer in the context of C^r-generic diffeomorphisms restrict to hyperbolic basic sets [19], the understanding of some topological aspects of incompressible flows [2], among many others.

2.2. Metric mean dimension

In this subsection, we recall two important measurements of topological complexity, namely the concepts of topological entropy and metric mean dimension, and introduce a relative notion of the latter. Our focus on the latter concept arises from the observation that all maps in a generic set of homeomorphisms (and continuous maps) on a compact Riemannian manifold possesses infinite topological entropy [22]. Conversely, metric mean dimension remains constrained by the dimensionality of the compact manifold and can be interpreted as a smoothed gauge of topological complexity, as elucidated in the following discussion.

Classical metric mean dimension. Let (X, d) be a compact metric space. Given $\varepsilon > 0$ and $n \in \mathbb{N}$, a set $E \subset X$ is an (n, ε)-*spanning set* if for every $x \in X$, there exists $y \in E$ with $d_n(x, y) \leq \varepsilon$. A set $E \subset X$ is an (n, ε)-*separated set* if $d_n(x, y) > \varepsilon$ for every $x \neq y \in E$, where $d_n(x, y) = \max\{d(f^j(x), f^j(y)); j = 0, \ldots, n-1\}$ is the Bowen's distance. Moreover, $B_n(x, \varepsilon) = \{y \in X; d_n(x, y) < \varepsilon\}$ denote the Bowen dynamic balls. If $s(n, \varepsilon)$ denotes the maximal cardinality of an (n, ε)-separated subset of X, then the *topological entropy* is defined by

$$h_{\mathrm{top}}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon).$$

The previous notion of entropy does not depend on the metric d and is a topological invariant. Moreover, the topological entropy of C^0-generic homeomorphisms on a closed manifold of dimension at least two is infinite [22] (the same holds for the topological pressure as a consequence of the variational principle), in which case neither the topological entropy nor the topological pressure can distinguish such dynamics. Thus, Gromov [12] proposed an invariant for dynamical systems called *mean dimension*, that was further studied by Lindenstrauss and Weiss [16]. The upper and lower *metric mean dimension*, which may depend on the metric d, are defined in [15, 16] by

$$\begin{aligned} \overline{\mathrm{mdim}}(X, f, d) &= \limsup_{\varepsilon \to 0} \frac{\limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon)}{|\log \varepsilon|}, \\ \text{and} \quad \underline{\mathrm{mdim}}(X, f, d) &= \liminf_{\varepsilon \to 0} \frac{\liminf_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon)}{|\log \varepsilon|}, \end{aligned} \tag{2.1}$$

respectively. Of course, $\overline{\mathrm{mdim}}(X, f, d) = 0 = \underline{\mathrm{mdim}}(X, f, d)$ whenever $h_{\mathrm{top}}(f) < \infty$ and, moreover, when X is a compact smooth boundaryless manifold with dimension strictly greater than one and d is a metric compatible with the smooth structure of X, there exists a C^0-Baire residual subset $\mathcal{R} \subset \mathrm{Homeo}(X, d)$ such that $\overline{\mathrm{mdim}}(X, f, d) = \dim X$ for all $f \in \mathcal{R}$ (according to [9]). A variational principle connecting rate distortion function to metric mean dimension appeared in [15] and more recently, other variational principles for metric mean dimension without any strong assumption were obtained in [3, 8], connecting rate distortion function to metric mean dimension.

Relative metric mean dimension. Since we aim to describe the topological complexity of (not necessarily compact) f-invariant subsets we now introduce a concept of relative metric mean dimension using a Carathéodory structure. Let $Z \subset X$ be an f-invariant Borel set. Given $s \in \mathbb{R}$, define

$$m(Z, \lambda, N, \varepsilon) := \inf_{\Gamma} \Big\{ \sum_{i \in I} \exp(-\lambda n_i) \Big\},$$

where the infimum is taken over all countable collections of dynamic balls $\Gamma = \{B_{n_i}(x_i, \varepsilon)\}_i$ that cover Z and so that $n_i \geq N$. Since the function $m(Z, \lambda, N, \varepsilon)$ is non-decreasing in N the limit $m(Z, \lambda, \varepsilon) = \lim_{N \to \infty} m(Z, \lambda, N, \varepsilon)$ does exist. Then let

$$h_Z(f, \varepsilon) = \inf\{\lambda \in \mathbb{R}; m(Z, \lambda, \varepsilon) = 0\} = \sup\{\lambda \in \mathbb{R}; m(Z, \lambda, \varepsilon) = \infty\}.$$

The existence of $h_Z(f,\varepsilon)$ follows by the Carathéodory structure as described in [18]. Moreover, the *Bowen's topological entropy* of f on Z is defined by

$$h_Z(f) = \lim_{\varepsilon\to 0} h_Z(f,\varepsilon)$$

and finally, the *upper and lower relative metric mean dimension* of Z are

$$\begin{aligned} \overline{\mathrm{mdim}}_B(Z,f,d) &= \limsup_{\varepsilon\to 0} \frac{h_Z(f,\varepsilon)}{|\log\varepsilon|}, \\ \text{and} \quad \underline{\mathrm{mdim}}_B(Z,f,d) &= \liminf_{\varepsilon\to 0} \frac{h_Z(f,\varepsilon)}{|\log\varepsilon|}, \end{aligned} \tag{2.2}$$

respectively.

Definition 2.3. Given a continuous map $f : X \to X$ and an f-invariant subset $Z \subset X$, we say that Z has full metric mean dimension if

$$\underline{\mathrm{mdim}}_B(Z,f,\varepsilon) = \underline{\mathrm{mdim}}(X,f,d)$$

and

$$\overline{\mathrm{mdim}}_B(Z,f,\varepsilon) = \overline{\mathrm{mdim}}(X,f,d).$$

Note that the metric mean dimension vanishes if the topological entropy is finite. The following proposition is immediate. The works [14] and [11] explore the relationship between metric mean dimension and dynamical systems with the gluing orbit property. More specifically, they demonstrate that for continuous maps (or continuous flows) possessing the gluing orbit property, the irregular sets are either empty or form a Baire residual subset.

Proposition 2.4. *If Z_1, Z_2 are subsets of X such that $Z_1 \subset Z_2$, then*

$$\overline{\mathrm{mdim}}_B(Z_1,f,d) \le \overline{\mathrm{mdim}}_B(Z_2,f,d)$$

and

$$\underline{\mathrm{mdim}}_B(Z_1,f,d) \le \underline{\mathrm{mdim}}_B(Z_2,f,d).$$

The following Proposition 2.5 establishes equality between the classical upper (and lower) metric mean dimension (2.1) and the Bowen's upper (and lower) relative metric mean dimension (2.2) of an f-invariant borelian set $Z \subset X$, and its complete proof is shown in [17].

Proposition 2.5. *If $f : X \to X$ is a continuous map and Z is any nonempty compact f-invariant subset of X, then*

$$\overline{\mathrm{mdim}}_B(Z,f,d) = \overline{\mathrm{mdim}}(Z,f,d)$$

and

$$\underline{\mathrm{mdim}}_B(Z,f,d) = \underline{\mathrm{mdim}}(Z,f,d).$$

3. Proof of Theorem A.

3.1. Proof for the case of the upper metric mean dimension

Assuming $E(z_0) \neq \emptyset$ and as $E(z_0) \subset X$, by Propositions 2.4 and 2.5 it follows that $\overline{\text{mdim}}_B(E(z_0), f, d) \leq \overline{\text{mdim}}(X, f, d)$. To conclude the other inequality it is sufficient to prove that

$$\overline{\text{mdim}}_B(E(z_0), f, d) \geq C$$

for any arbitrary fixed constant $C < \overline{\text{mdim}}(X, f, d)$. For this purpose, consider a non-transitive point $z_0 \in X$ and choose $u \in X$ and $\varepsilon_0 > 0$ such that

$$d(u, \overline{O_f(z_0)}) \geq 3\varepsilon_0. \tag{3.1}$$

Fixing $\gamma > 0$, we choose an $\varepsilon < \varepsilon_0$ and a sequence $\{n_k\}_{k\geq 1} \subset \mathbb{N}$ such that there exists a maximal $(n_k, 7\varepsilon)$-separated set S_k of X which is always an $(n_k, 7\varepsilon)$-spanning set such that

$$\#S_k \geq \exp\{(n_k(C-\gamma))|\log 7\varepsilon|\}, \tag{3.2a}$$

$$\frac{h_{E(z_0)}(f,\varepsilon)}{|\log \varepsilon|} \leq \overline{\text{mdim}}_B(E(z_0), f, d) + \gamma, \quad \text{and} \tag{3.2b}$$

$$(\overline{\text{mdim}}_B(E(z_0), f, d) + \gamma) \cdot \frac{|\log \varepsilon|}{|\log 7\varepsilon|} \leq \overline{\text{mdim}}_B(E(z_0), f, d) + 2\gamma. \tag{3.2c}$$

Moreover, $\{n_k\}_{k\geq 1}$ can be chosen so that $n_k \geq 2^m$, where m is as in Definition 2.2.

3.1.1. Part 1: Construction of a Moran-like fractal R. Let $\varepsilon > 0$ be arbitrary and fixed, $m = m(\varepsilon)$ be given by the gluing orbit property (Definition 2.2), and S_k be a maximal $(n_k, 7\varepsilon)$-separated subset X. Now, index the elements of S_k by x_k^j, for $1 \leq j \leq \#S_k$ and choose also a strictly increasing sequence of integers $\{N_k\}_{k\geq 1}$. The gluing orbit property ensures that for every $\underline{x_k} := (x_1^k, \ldots, x_{N_k}^k) \in S_k^{N_k}$ there exists a point $y = y(\underline{x_k}) \in X$ and transition time functions

$$p_{i,j}^k : S_k^{N_k} \times \{y\} \to \mathbb{N}, \quad \text{where } i = 1, 2, \ldots, N_k - 1 \text{ and } j = 1, 2$$

bounded above by m so that

$$d_{n_k}(f^{\alpha_j}(y), x_j^k) < \varepsilon, \quad \text{for every } j = 1, 2, \ldots, N_k - 1 \tag{3.3}$$

and

$$d(f^{\beta_j}(y), u) < \varepsilon, \quad \text{for every } j = 1, 2, \ldots, N_k - 1, \tag{3.4}$$

where

$$\alpha_j = \begin{cases} 0 & \text{if } j = 1, \\ (j-1)n_k + \sum_{r=1}^{j-1} p_{r,1}^k + \sum_{r=1}^{j-1} p_{r,2}^k & \text{if } j = 2, \dots, N_k, \end{cases}$$

and

$$\beta_j = \begin{cases} n_k + p_{1,1}^k & \text{if } j = 1, \\ jn_k + \sum_{r=1}^{j} p_{r,1}^k + \sum_{r=1}^{j-1} p_{r,2}^k & \text{if } j = 2, \dots, N_k. \end{cases}$$

Remark 3.1. Note that the functions $p_{i,1}^k$ above describe the time lag that the orbit of $y = y(\underline{x_k})$ takes to jump from an ε-neighborhood of $f^{n_k}(x_j^k)$ to an ε-neighborhood of u and $p_{i,2}^k$ an ε-neighborhood of u to an ε-neighborhood of x_{j+1}^k (see Figure 1).

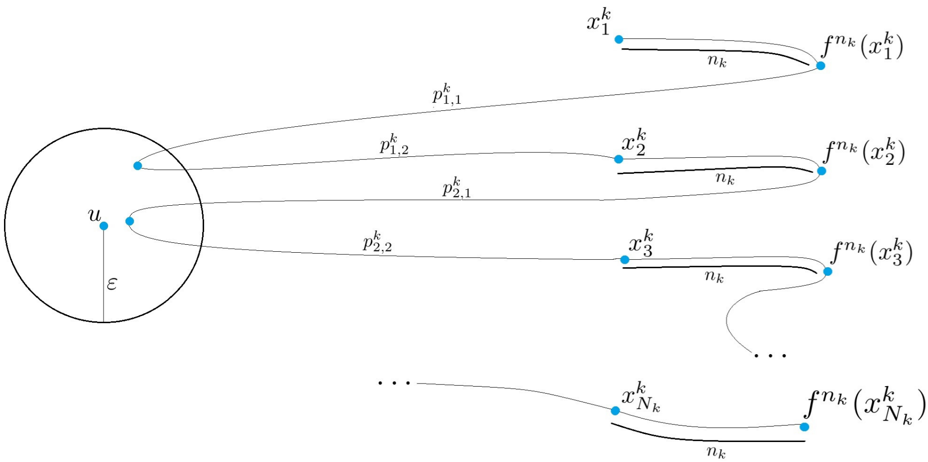

Figure 1. Shadowing of the orbit in $S_k^{N_k}$

We proceed to make a recursive construction of a family $\{L_k\}_{k\geq 1}$ of sets (guaranteed by the gluing orbit property) contained in a neighborhood of x_1^1:

(1) $L_1 = \{z = z(\underline{x_1}) \in X; \underline{x_1} \in S_1^{N_1,}\}$, where

  (a) $t_1 := N_1 n_1 + \sum_{r=1}^{N_1-1} p_{r,1}^1 + \sum_{r=1}^{N_1-1} p_{r,2}^1$,

  (b) $l_1 := t_1$ and $z = z(\underline{x_1})$ satisfies $d_{l_1}(z, y(\underline{x_1})) < \varepsilon$;

(2) $L_2 = \{z = z(w, y(\underline{x_2})) \in X; w \in L_1 \text{ and } \underline{x_2} \in S_2^{N_2}\}$, where

  (a) $t_2 := N_2 n_2 + \sum_{r=1}^{N_2-1} p_{r,1}^2 + \sum_{r=1}^{N_2-1} p_{r,2}^2$,

  (b) $l_2 := l_1 + p_0^1 + t_2$,

(c) $z = z(w, y(\underline{x_2}))$ satisfies $d_{l_1}(z, w) < \varepsilon$ and $d_{t_2}(f^{l_1+p_0^1}(z), y(\underline{x_2})) < \varepsilon$, with

(d) $0 \le p_0^1 \le m(\varepsilon)$ is given by the gluing orbit property (p_0^1 is the time lag that the orbit of z takes to jump from an ε-neighborhood of $f^{n_1}(x_{N_1}^1)$ to an ε-neighborhood of x_1^2);

and more generally, for all $k \ge 2$ let

(3) $L_k = \{z = z(w, y(\underline{x_k})) \in X; w \in L_{k-1}$ and $\underline{x_k} \in S_k^{N_k}\}$, where

(a) $t_k = N_k n_k + \sum_{r=1}^{N_k-1} p_{r,1}^k + \sum_{r=1}^{N_k-1} p_{r,2}^k$,

(b) $l_k = l_{k-1} + p_0^{k-1} + t_k$,

(c) $z = z(w, y(\underline{x_k}))$ satisfies $d_{l_{k-1}}(z, w) < \varepsilon$ and

$$d_{t_k}(f^{l_{k-1}+p_0^{k-1}}(z), y(\underline{x_k})) < \varepsilon,$$

(d) $0 \le p_0^{k-1} \le m(\varepsilon)$ is given by the gluing orbit property (p_0^{k-1} is the time lag that the orbit of z takes to jump from an ε-neighborhood of $f^{n_{k-1}}(x_{N_{k-1}}^{k-1})$ to an ε-neighborhood of x_1^k).

The previous points $y = y(\underline{x_k})$, $k \ge 1$, are defined as in (3.3) and (3.4). By construction,

$$l_k = \sum_{r=1}^{k} N_r n_r + \sum_{r=1}^{k-1} p_0^r + \sum_{s=1}^{k} \sum_{r=1}^{N_s-1} p_{r,1}^s + \sum_{s=1}^{k} \sum_{r=1}^{N_s-1} p_{r,2}^s$$

for every $k \ge 1$. Also, for every $k \ge 0$ and $\varepsilon > 0$ let

$$R_k = \bigcup_{z \in L_k} \widetilde{B}_{l_k}(z, \varepsilon) \quad \text{and} \quad R = \bigcap_{k \ge 1} R_k,$$

where $\widetilde{B}_{l_k}(z, \delta)$ is the set of points $x \in X$ so that $d(f^\alpha(z), f^\alpha(x)) < \delta$ for all iterates $0 \le \alpha \le l_{k-1} - 1$ and $d(f^\beta(z), f^\beta(x)) \le \delta$ for every $l_{k-1} \le \beta \le l_k - 1$.

The following lemma, similar to [21, Proposition 3.1], ensures that R is a subset of $E(z_0)$.

Lemma 3.2. *If $x \in R$, then $x \in E(z_0)$. This is, $R \subset E(z_0)$.*

Proof. Let $x \in R$, first we will prove that $O_f(x) \cap \widetilde{B} = \emptyset$, where $\widetilde{B} := \widetilde{B}_{l_2+m}(z_0, \varepsilon)$. In fact, if $O_f(x) \cap \widetilde{B} \neq \emptyset$ it is possible to choose $j \in \mathbb{N}$ such that $f^j(x) \in \widetilde{B}$ and, by the construction of R, for any k with $t_k \gg j$ there exists some $z \in L_k$ such that $x \in B_{l_k}(z, \varepsilon)$. Hence, we can choose $q < l_2 + m$ such that $d(f^{j+q}(z), f^{j+q}(x)) \le \varepsilon$, $d(f^{j+q}(z), u) < \varepsilon$ and $d(f^{j+q}(x), f^q(z_0)) < \varepsilon$. Therefore,

$$\begin{aligned} d(u, f^q(z_0)) &\le d(u, f^{j+q}(z)) + d(f^{j+q}(z), f^{j+q}(x)) + d(f^{j+q}(x), f^q(z_0)) \\ &\le \varepsilon + \varepsilon + \varepsilon < 3\varepsilon_0, \end{aligned}$$

which contradicts with (3.1).

Finally, once concluded that $O_f(x) \cap \widetilde{B} = \emptyset$, it follows that $O_f(x) \subset X \setminus \widetilde{B}$ and consequently, $\overline{O_f(x)} \subset X \setminus \widetilde{B}$, implying that $z_0 \notin \overline{O_f(x)}$ or, in others words, $x \in E(z_0)$. ∎

3.1.2. Part 2: Construction of a special sequence of measurements μ_k. For each $\underline{x_k} := (x_1^k, \ldots, x_{N_k}^k) \in S_k^{N_k}$ choose $z(\underline{x_k}) \in L_k$. Observe that

$$\#L_k = \prod_{i=1}^{k} (\#S_i)^{N_i}.$$

Moreover, the following Lemma 3.3 shows that set L_k is $(l_k, 5\varepsilon)$-separated.

Lemma 3.3. *For every $x \in L_k$ and distinct $\underline{x_k}, \underline{y_k} \in S_k^{N_k}$, it holds that*

$$d_{l_{k+1}}(z(x, \underline{x_k}), z(x, \underline{y_k})) > 2\varepsilon.$$

That is, $z(x, \underline{x_k})$ and $z(x, \underline{y_k})$ are $(t_k, 5\varepsilon)$-separated points.

Proof. Let $\underline{x_k}, \underline{y_k} \in S_k^{N_k}$, distinct and called $z_1 := z(x, \underline{x_k})$ and $z_2 := z(x, \underline{y_k})$. Since $\underline{x_k}, \underline{y_k} \in S_k^{N_k}$, there exists $q \in \{1, 2, \ldots, N_{k-1}\}$ such that $x_q^k \neq y_q^k$ and $x_i = y_i$ for each $i < q$. By definition of R, there exists $0 \leq s \leq n_k - 1$ such that

$$d\Big(f^{l_{i-k} + p_0^{k-1} + (q-1)n_k + \sum_{r=1}^{q-1} p_{r,1}^k + \sum_{r=1}^{q-1} p_{r,2}^i + s}(z_1),$$
$$f^{(q-1)n_k + \sum_{r=1}^{q-1} p_{r,1}^k + \sum_{r=1}^{q-1} p_{r,2}^k + s}(x_q^k)\Big) < \varepsilon$$

and

$$d\Big(f^{l_{k-1} + p_0^{k-1} + (q-1)n_k + \sum_{r=1}^{q-1} p_{r,1}^k + \sum_{r=1}^{q-1} p_{r,2}^k + s}(z_2),$$
$$f^{(q-1)n_k + \sum_{r=1}^{q-1} p_{r,1}^k + \sum_{r=1}^{q-1} p_{r,2}^k + s}(y_q^k)\Big) < \varepsilon.$$

Note that as $x_q^k, y_q^k \in S_k$, they are $(n_k, 7\varepsilon)$-separated points and so

$$\begin{aligned}
d_{l_k}(z_1, z_2) &\geq d_{t_k}(z_1, z_2) \\
&\geq d\Big(f^{l_{k-1} + p_0^{k-1} + (q-1)n_k + \sum_{r=1}^{q-1}(p_{r,1}^k + p_{r,2}^k) + s}(z_1), \\
&\qquad f^{l_{k-1} + p_0^{k-1} + (q-1)n_k + \sum_{r=1}^{q-1}(p_{r,1}^k + p_{r,2}^k) + s}(z_2)\Big) \\
&\geq d\Big(f^{(q-1)n_k + \sum_{r=1}^{q-1}(p_{r,1}^k + p_{r,2}^k) + s}(x_q^k), \\
&\qquad f^{(q-1)n_k + \sum_{r=1}^{q-1} p_{r,1}^k + \sum_{r=1}^{q-1} p_{r,2}^k + s}(y_q^k)\Big)
\end{aligned}$$

$$
\begin{aligned}
&- d\Big(f^{l_{k-1}p_0^{k-1}+(q-1)n_i+\sum_{r=1}^{q-1}(p_{r,1}^k+p_{r,2}^k)+s}(z_1),\\
&\qquad f^{(q-1)n_i+\sum_{r=1}^{q-1}(p_{r,1}^k+p_{r,2}^k)+s}(x_q^k)\Big)\\
&- d\Big(f^{l_{k-1}+p_0^{k-1}+(q-1)n_k+\sum_{r=1}^{q-1}(p_{r,1}^k+p_{r,2}^k)+s}(z_2),\\
&\qquad f^{(q-1)n_k+\sum_{r=1}^{q-1}(p_{r,1}^k+p_{r,2}^k)+s}(y_q^k)\Big)\\
&\geq 7\varepsilon-\varepsilon-\varepsilon=5\varepsilon. \qquad \blacksquare
\end{aligned}
$$

Now, consider for any k the probability measure supported on R, defined by

$$\mu_k = \frac{1}{\# L_k}\sum_{z\in L_K}\delta_z.$$

The proof of the following lemma is a simple modification of the arguments of [20, Lemma 3.7] (where it is considered the case where f satisfies the specification property).

Lemma 3.4. *The sequence of measures $\{\mu_k\}_{k\in\mathbb{N}}$ converges to a measure in $\mathcal{M}(X)$ with respect to the weak*-topology μ. Moreover, $\mu(R)=1$.*

Let $\mathcal{B}=B_n(q,\varepsilon)$ be an arbitrary ball which intersects R, where k is the unique number which satisfies $l_k\leq n<l_{k+1}$ and $j\in\{0,1,\dots,N_{k+1}-1\}$ be the unique number such that

$$
\begin{aligned}
&l_k+p_0^{k+1}+jn_{k+1}+\sum_{r=1}^{j}(p_{r,1}^{k+1}+p_{r,2}^{k+1})\\
&\qquad\leq n<l_k+p_0^{k+1}+(j+1)n_{k+1}+\sum_{r=1}^{j+1}(p_{r,1}^{k+1}+p_{r,2}^{k+1}).
\end{aligned}
$$

Lemma 3.5. *Suppose that $\mu_{k+p}(\mathcal{B})>0$ for any $p\geq 1$. In this context,*

$$\mu_{k+p}(\mathcal{B})\leq\frac{1}{\# L_k\cdot(\# S_{k+1})^j}.$$

Proof. First, consider $p=1$. Suppose $\mu_{k+1}(B)>0$, then $L_{k+1}\cap B\neq\emptyset$. Let $z=z(w,y(\underline{x_{k+1}}))\in L_{k+1}\cap\mathcal{B}$ where $w\in L_k$ and $\underline{x_{k+1}}\in(S_{k+1})^{N_{k+1}}$. Define

$$
\begin{aligned}
\mathcal{A}_{k+1,j}:=\big\{z(w,y(\underline{\tilde{x}_{k+1}}))\in L_{k+1};\ &\tilde{x}_1^{k+1}=x_1^{k+1},\tilde{x}_2^{k+1}=x_2^{k+1},\dots,\\
&\tilde{x}_j^{k+1}=x_j^{k+1}\big\},
\end{aligned}
$$

for some $1\leq j\leq N_{k+1}$, $\tilde{w}\in L_k$ and $\underline{\tilde{x}_{k+1}}\in(S_{k+1})^{N_{k+1}}$. If $\tilde{z}=z(\tilde{w},\tilde{z}_{k+1})\in L_{k+1}\cap\mathcal{B}$, since $d_n(z,\tilde{z})<2\varepsilon$, by Lemma 3.3, we have $w=\tilde{w}$ and for $l\in\{1,\dots,j\}$.

Thus,

$$\mu_{k+1}(\mathcal{B}) = \frac{1}{\#L_{k+1}} \sum_{z \in L_{K+1}} \delta_z(\mathcal{B}) \leq \sum_{z \in \mathcal{A}_{k+1,j}} \frac{1}{\#L_{k+1}} \delta_z$$
$$\leq \frac{\#S_{k+1}^{N_{k+1}-j}}{\#L_{k+1}} = \frac{1}{\#L_k \cdot (\#S_{k+1})^j}.$$

In the case $p > 1$,

$$\mu_{k+p}(\mathcal{B}) \leq \frac{\#S_{k+1}^{N_{k+1}-j} \cdot \#S_{k+2}^{N_{k+2}} \cdots \#S_{k+p}^{N_{k+p}}}{\#L_{k+p}} = \frac{1}{\#L_k \cdot (\#S_{k+1})^j}. \qquad \blacksquare$$

The following and last lemma will be used in part 3 (Section 3.1.3) of the demonstration of Theorem A. To do this, we will use the previous Lemma 3.5 to prove it.

Lemma 3.6. *There exists $N \in \mathbb{N}$ such that for any $n \geq N$,*

$$\mu(\mathcal{B}) \leq \exp\{-n(C - 3\gamma)|\log 7\varepsilon|\}.$$

Proof. It follows from the relations in (3.2) that

$$\begin{aligned}
&\#L_k \cdot (\#S_{k+1})^j \\
&\quad = \#S_1^{N_1} \cdots \#S_k^{N_k} \cdot \#S_{k+1}^j \\
&\quad \geq \prod_{i=1}^{k} \exp(N_i n_i (C-\gamma))|\log 7\varepsilon| \cdot \exp\{(j \cdot n_{k+1}(C-\gamma))|\log 7\varepsilon|\} \\
&\quad = \exp\left\{(C-\gamma)|\log 7\varepsilon|\left(\sum_{i=1}^{k} N_i n_i + j n_{k+1}\right)\right\} \\
&\quad \geq \exp\left\{(C-2\gamma)|\log 7\varepsilon|\left(\sum_{i=1}^{k}(N_i n_i + p_0^i) + \sum_{s=1}^{k}\sum_{r=1}^{N_s-1}(p_{r,1}^s + p_{r,2}^s)\right.\right. \\
&\qquad \left.\left. + \; j n_{k+1} + \sum_{r=1}^{j+1}(p_{r,1}^{k+1} + p_{r,2}^{k+1})\right)\right\} \\
&\quad = \exp\left\{(C-2\gamma)|\log 7\varepsilon|\left(l_k + p_0^k + j n_{k+1} + \sum_{r=1}^{j+1}(p_{r,1}^{k+1} + p_{r,2}^{k+1})\right)\right\} \\
&\quad \geq \exp\{n(C-3\gamma)|\log 7\varepsilon|\}.
\end{aligned}$$

In the fourth line, we are able to add in the extra terms with an arbitrarily small change to the constant because n_k is much larger than transition time functions.

Finally, using Lemma 3.5, we conclude that

$$\mu(\mathcal{B}) \leq \liminf_{p \to \infty} \mu_{k+p}(\mathcal{B}) \leq \exp\{-n(C-3\gamma)|\log 7\varepsilon|\}. \qquad \blacksquare$$

3.1.3. Part 3: Now we will finish the proof of the Theorem A and for this purpose we will use the pressure distribution principle type argument. Let $N \in \mathbb{N}$ and $\Gamma = \{B_{n_i}(x_i, \varepsilon)\}_{i \in I}$ be any finite cover of R by dynamic balls with $n_i \geq N$ for all $i \in I$. Without loss of generality, we may assume that $B_{n_i}(x_i, \varepsilon) \neq \emptyset$ for every $i \in I$. An application of Lemma 3.6 on each $B_{n_i}(x_i, \varepsilon)$ implies in the inequality

$$\sum_{i \in I} \exp\{-n(C - 3\gamma)|\log 7\varepsilon|\} \geq \sum_{i \in I} \mu(B_{n_i}(x_i, \varepsilon)) \geq \mu(R) = 1.$$

As Γ is arbitrary, we have $m(R, (C - 2\gamma)|\log 7\varepsilon|, N, \varepsilon) \geq 1 > 0$. So, since $m(R, (C - 3\gamma)|\log 7\varepsilon|, N, \varepsilon)$ does not decrease as N increases, $m(R, (C - 2\gamma)|\log 7\varepsilon|, \varepsilon) \geq 1 > 0$, which implies that

$$h_R(f, \varepsilon) \geq (C - 3\gamma)|\log 7\varepsilon|.$$

So, by Lemma 3.2 and relation (3.2) it follows that

$$\begin{aligned} C - 3\gamma \leq \frac{h_R(f, \varepsilon)}{|\log 7\varepsilon|} &\leq \frac{h_{E(z_0)}(f, \varepsilon)}{|\log 7\varepsilon|} = \frac{h_{E(z_0)}(f, \varepsilon)}{|\log \varepsilon|} \frac{|\log \varepsilon|}{|\log 7\varepsilon|} \\ &\leq (\overline{\mathrm{mdim}}_B(E(z_0), f, d) + \gamma) \frac{|\log \varepsilon|}{|\log 7\varepsilon|} \leq (\overline{\mathrm{mdim}}_B(E(z_0), f, d) + 2\gamma). \end{aligned}$$

So, $\overline{\mathrm{mdim}}_B(E(z_0), f, d) \geq C - 5\gamma$. As $\gamma > 0$ and C are arbitrary, we have

$$\overline{\mathrm{mdim}}_B(E(z_0), f, d) \geq \overline{\mathrm{mdim}}(X, f, d).$$

3.2. The case of the lower metric mean dimension

Now, we will present the proof of our main theorem for the case of the lower metric mean dimension. That is, we show that $\underline{\mathrm{mdim}}_B(E(z_0), f, d) = \underline{\mathrm{mdim}}(X, f, d)$.

Assuming $E(z_0) \neq \emptyset$, as $E(z_0) \subset X$, by Remark 2.4 and Proposition 2.5 it follows that $\underline{\mathrm{mdim}}_B(E(z_0), f, d) \leq \underline{\mathrm{mdim}}(X, f, d)$. To conclude the other inequality it is sufficient to prove that

$$\underline{\mathrm{mdim}}_B(E(z_0), f, d) \geq C'$$

for any arbitrary fixed constant $C' < \underline{\mathrm{mdim}}(X, f, d)$.

Fixing $\gamma > 0$, as in relations (3.2), it is possible to obtain an $\varepsilon' < \varepsilon_0$ and a sequence $\{n_k\}_{k \geq 1} \subset \mathbb{N}$ such that there exists a maximal $(n_k, 7\varepsilon)$-separated set S_k of X so that

$$\#S_k \geq \exp(n_k(C' - \gamma))|\log 7\varepsilon'|, \tag{3.5a}$$

$$\frac{h_{E(z_0)}(f, \varepsilon')}{|\log \varepsilon'|} \leq \underline{\mathrm{mdim}}_B(E(z_0), f, d) + \gamma, \quad \text{and} \tag{3.5b}$$

$$(\underline{\mathrm{mdim}}_B(E(z_0), f, d) + \gamma) \cdot \frac{|\log \varepsilon'|}{|\log 7\varepsilon'|} \leq \underline{\mathrm{mdim}}_B(E(z_0), f, d) + 2\gamma. \tag{3.5c}$$

Moreover, the sequence n_k can be chosen so that $n_k \geq 2^m$, where m is as in Definition 2.2.

We can use the parallel proof in the previous case to show that there exist a Moran-like fractal R' and a measure μ' concentrated on R' satisfying the following lemma.

Lemma 3.7. *There exists $N' \in \mathbb{N}$ such that for any $n \geq N'$, if $\mathcal{B}_n(z, \varepsilon') \cap R' \neq \emptyset$, then $\mu(B_n(z, \varepsilon')) \leq \exp\{-n(C' - 3\gamma)|\log 7\varepsilon'|\}$.*

Let N' be the number defined in Lemma 3.7 Let $\Gamma = \{B_{n_i}(x_i, \varepsilon')\}_{i\in I}$ be any finite cover of R' with $n_i \geq N'$ for all $i \in I$. Without loss of generality, we may assume that $B_{n_i}(x_i, \varepsilon') \cap R' \neq \emptyset$ for every $i \in I$. Applying Lemma 3.7 on each $B_{n_i}(x_i, \varepsilon')$, we obtain

$$\sum_{i\in I} \exp\{-n(C' - 3\gamma)|\log 7\varepsilon'|\} \geq \sum_{i\in I} \mu(B_{n_i}(x_i, \varepsilon')) \geq \mu(R') = 1.$$

As Γ is arbitrary, we have $m(R', (C' - 3\gamma)|\log 7\varepsilon'|, N, \varepsilon') \geq 1 > 0$. So, since $m(R', (C' - 3\gamma)|\log 7\varepsilon'|, N, \varepsilon')$ does not decrease as N increases,

$$m(R', (C' - 2\gamma)|\log 7\varepsilon'|, \varepsilon') \geq 1 > 0,$$

which implies that

$$h_{R'}(f, \varepsilon') \geq (C' - 3\gamma)|\log 7\varepsilon'|.$$

Thus, as in Lemma 3.2 we have that $R' \subset E(z_0)$ and by relation (3.5), it follows that

$$\begin{aligned} C' - 3\gamma &\leq \frac{h_{R'}(f, \varepsilon')}{|\log 7\varepsilon'|} \leq \frac{h_{E(z_0)}(f, \varepsilon')}{|\log 7\varepsilon'|} = \frac{h_{E(z_0)}(f, \varepsilon')}{|\log \varepsilon'|} \frac{|\log \varepsilon'|}{|\log 7\varepsilon'|} \\ &\leq (\underline{\mathrm{mdim}}_B(E(z_0), f, d) + \gamma)\frac{|\log \varepsilon'|}{|\log 7\varepsilon'|} \leq (\underline{\mathrm{mdim}}_B(E(z_0), f, d) + 2\gamma). \end{aligned}$$

So, $\underline{\mathrm{mdim}}_B(E(z_0), f, d) \geq C' - 5\gamma$. As $\gamma > 0$ and C' are arbitrary, we have

$$\underline{\mathrm{mdim}}_B(E(z_0), f, d) \geq \underline{\mathrm{mdim}}(X, f, d).$$

4. Examples

In this Section, we present two examples involving shifts with infinite symbols and generic incompressible homeomorphisms, in which Theorem A is applicable.

Example 4.1. Shift in $[0, 1]$. Let $X = [0, 1]^{\mathbb{N}}$ with the distance $d((a_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}}) = \sup_{i\in\mathbb{N}} \frac{|a_i - b_i|}{2^{i-1}}$. The right shift $\sigma : X \to X$, given to $\sigma(a_1, a_2, a_3, \ldots) = (a_2, a_3, \ldots)$

is a continuous and surjective map in (X, d) such that $h_{\text{top}}(\sigma)$ is infinite. We claim that σ satisfies the gluing orbit property described in Definition 2.2.

In fact, given $\varepsilon > 0$ arbitrary, initially observe that if $m = m(\varepsilon) = \min\{i \in \mathbb{N};\ 1/2^{i-1} < \varepsilon\}$, then $d((a_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}}) < \varepsilon$ iff $\frac{|a_i-b_i|}{2^{i-1}} < \varepsilon$ for all $1 \le i \le m$. So, given any points $x^1 = (x_i^1)_{i\in\mathbb{N}}, \ldots, x^k = (x_i^k)_{i\in\mathbb{N}} \in X$ and any positive integers $n_1, \ldots, n_k$, let $p_1 = \cdots = p_{k-1} = m$.

Now, construct $y = (y_i)_{i\in\mathbb{N}} \in X$ defining it by parts, putting

$$y_i = \begin{cases} x_i^1 & \text{if } 1 \le i \le n_1 + m, \\ x^2_{i-(n_1+m)} & \text{if } n_1 + m + 1 \le i \le n_1 + n_2 + 2m, \\ x^3_{i-(n_1+n_2+2m)} & \text{if } n_1 + n_2 + 2m + 1 \le i \le n_1 + n_2 + n_3 + 3m, \\ & \vdots \\ x^k_{i-\sum_{l=1}^{k-1}(n_l+m)} & \text{if } \sum_{l=1}^{k-1}(n_l + m) + 1 \le i \le \sum_{l=1}^{k}(n_l + m), \\ 0 & \text{if } i \ge \sum_{l=1}^{k}(n_l + m) + 1. \end{cases}$$

Observe that y $d(\sigma^j(y), \sigma^j(x^1)) \le \varepsilon$ for every $0 \le j \le n_1$ and

$$d(\sigma^{j+n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(y), \sigma^j(x^i)) \le \varepsilon$$

for every $2 \le i \le k$ and $0 \le j \le n_i$. In other words, σ satisfies the gluing orbit property. Consequently, our Theorem A is applicable in this Example 4.1. Further, $(3/4, 3/4, 3/4, \ldots) \in X$ is a non-transitive point such that $Y = [0, 1/2]^{\mathbb{N}}$ is a (positively) σ-invariant subset of X and contained in $E(3/4, 3/4, 3/4, \ldots)$.

On the other hand, observe that the homeomorphism $h : [0, 1/2]^{\mathbb{N}} \to [0, 1]^{\mathbb{N}}$ given by $h((x_i)_{i\in\mathbb{N}}) = (2x_i)_{i\in\mathbb{N}}$ is a topological conjugation between σ and the restriction $\sigma_{|[0,1/2]^{\mathbb{N}}}$ (in fact, $\sigma_{|[0,1/2]^{\mathbb{N}}} = h^{-1} \circ \sigma \circ h$) such that $d_{|[0,1/2]^{\mathbb{N}}}((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) = 1/2 d(h((x_i)_{i\in\mathbb{N}}), h((y_i)_{i\in\mathbb{N}}))$. These facts imply that

$$\overline{\text{mdim}}(X, \sigma, d) = \overline{\text{mdim}}([0, 1/2]^{\mathbb{N}}, \sigma, d)$$

and consequently, $\overline{\text{mdim}}_B(E(3/4, 3/4, 3/4, \ldots), \sigma, d) = \overline{\text{mdim}}(X, \sigma, d) = 1$.

Example 4.2. A generic incompressible homeomorphism f in a compact manifold M is chaotic in the sense of Devaney, i.e., transitive and its periodic points are dense in M [10]; satisfies the gluing orbity property [1], and has full metric mean dimension [13]. So, if z_0 is a periodic point of such a generic incompressible homeomorphism f, the set $E(z_0)$ is nonempty because it contains all periodic points of f not in the orbit of z_0 and so, Theorem A implies that $\overline{\text{mdim}}_B(E(z_0), f, d) = \overline{\text{mdim}}(M, f, d) = \dim M$.

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