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L_p-dual Brunn–Minkowski inequality for intersection bodies

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Abstract. In 2003, associated with the radial Minkowski additions of star bodies, Zhao and Leng established the dual Brunn–Minkowski inequality for intersection bodies. In this paper, associated with the L_p-radial Minkowski combinations of star bodies, we firstly prove the L_p-dual Brunn–Minkowski inequality for intersection bodies. Further, associated with the L_p-Minkowski combinations of convex bodies, we give the L_p-Brunn–Minkowski inequality for star dualities of intersection bodies.

1. Introduction and main results

The setting for this paper is the n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in $\mathbb{R}^n$, for the set of convex bodies containing the origin in their interiors in $\mathbb{R}^n$, we write $\mathcal{K}_o^n$. Let $\mathcal{S}_o^n$ denote the set of star bodies (with respect to origin) in $\mathbb{R}^n$. Let B denote the n-dimensional Euclidean unit ball centered at the origin, and the surface of B is written S^{n-1}. We use $V(K)$ to denote the n-dimensional volume of a body K.

The famous Brunn–Minkowski inequality for the volume is an important inequality in the theory of convex bodies. One form of it states the following: *if $K, L \in \mathcal{K}^n$, then*

$$V(K+L)^{\frac{1}{n}} \geq V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}}, \tag{1.1}$$

with equality if and only if K and L are homothetic. Here, $K + L$ denotes the Minkowski addition of K and L.

In 1962, Firey (see [5]) introduced the L_p-Minkowski combinations of convex bodies (also called the Firey p-combinations) and established the following L_p-Brunn–Minkowski inequality. *If $K, L \in \mathcal{K}_o^n$ and $1 \leq p \leq +\infty$ (for $p = 1$, it can be assumed that $K, L \in \mathcal{K}^n$), then*

$$V(K+_p L)^{\frac{p}{n}} \geq V(K)^{\frac{p}{n}} + V(L)^{\frac{p}{n}}. \tag{1.2}$$

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Equality holds in (1.2) *for* $p = 1$ *if and only if* K *and* L *are homothetic, for* $1 < p < +\infty$ *if and only if* K *and* L *are dilates, for* $p = +\infty$ *if and only if* $K \subseteq L$ *or* $L \subseteq K$. Here, $K +_p L$ denotes the L_p-Minkowski addition of K and L.

The dual form of the L_p-Brunn–Minkowski inequality is the following L_p-dual Brunn–Minkowski inequality (see [6]). *If* $K, L \in \mathcal{S}_o^n$ *and real* $p \neq 0$, *then for* $0 < p < n$,

$$V(K \,\tilde{+}_p\, L)^{\frac{p}{n}} \leq V(K)^{\frac{p}{n}} + V(L)^{\frac{p}{n}}; \tag{1.3}$$

for $-\infty \leq p < 0$ *or* $n < p \leq +\infty$, *then*

$$V(K \,\tilde{+}_p\, L)^{\frac{p}{n}} \geq V(K)^{\frac{p}{n}} + V(L)^{\frac{p}{n}}. \tag{1.4}$$

For $p \neq \pm\infty$, *equality hold in* (1.3) *and* (1.4) *if and only if* K *and* L *are dilates; for* $p = \pm\infty$, *equality holds in* (1.4) *if and only if* $K \subseteq L$ *or* $L \subseteq K$. Here, $K \,\tilde{+}_p\, L$ denotes the L_p-radial Minkowski addition of K and L.

In particular, the case $p = 1$ of inequality (1.3) shows the dual Brunn–Minkowski inequality as follows: *If* $K, L \in \mathcal{S}_o^n$, *then*

$$V(K \,\tilde{+}\, L)^{\frac{1}{n}} \leq V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}},$$

with equality if and only if K *and* L *are dilates.* Here, $K \,\tilde{+}\, L = K \,\tilde{+}_1\, L$ denotes the radial Minkowski addition of K and L.

The researches of the Brunn–Minkowski inequality and its dual versions in various forms have made a lot of achievements. For extensive and beautiful surveys on it we refer, e.g., to [1–4, 6–9, 12, 13, 15–17, 19–22, 25].

Associated with the radial Minkowski additions of star bodies, Zhao and Leng (see [23]) established the dual Brunn–Minkowski inequality for intersection bodies as follows.

Theorem 1.A. *If* $K, L \in \mathcal{S}_o^n$, *then*

$$V(I(K \,\tilde{+}\, L))^{\frac{1}{n(n-1)}} \leq V(IK)^{\frac{1}{n(n-1)}} + V(IL)^{\frac{1}{n(n-1)}}, \tag{1.5}$$

with equality if and only if K *and* L *are dilates. Here* IM *denotes the intersection body of* $M \in \mathcal{S}_o^n$.

In this paper, associated with the L_p-radial Minkowski combinations of star bodies, we firstly give the L_p-dual Brunn–Minkowski inequality for intersection bodies as follows.

Theorem 1.1. *If* $K, L \in \mathcal{S}_o^n$, $\lambda \in [0, 1]$ *and* p *is any real, then for* $0 < p < n - 1$,

$$V(I(\lambda \odot K \,\tilde{+}_p\, (1-\lambda) \odot L))^{\frac{p}{n(n-1)}} \leq \lambda V(IK)^{\frac{p}{n(n-1)}} + (1-\lambda) V(IL)^{\frac{p}{n(n-1)}}; \tag{1.6}$$

for $-\infty \le p < 0$ *or* $n(n-1) < p \le +\infty$,

$$V(I(\lambda \odot K \,\tilde{+}_p\, (1-\lambda) \odot L))^{\frac{p}{n(n-1)}} \ge \lambda V(IK)^{\frac{p}{n(n-1)}} + (1-\lambda) V(IL)^{\frac{p}{n(n-1)}}; \tag{1.7}$$

for $p = 0$,

$$V(I(\lambda \odot K \,\tilde{+}_0\, (1-\lambda) \odot L)) \le V(IK)^{\lambda} V(IL)^{1-\lambda}. \tag{1.8}$$

When $\lambda \in (0,1)$, *equality holds in every above inequality for* $p \ne \pm\infty$ *if and only if* K *and* L *are dilates, for* $p = \pm\infty$ *if and only if* $K \subseteq L$ *or* $L \subseteq K$. *When* $\lambda = 0$ *or* $\lambda = 1$, *above inequalities all become equalities.*

Let $p = 1$ and $\lambda = \frac{1}{2}$ in Theorem 1.1, then inequality (1.6) yields inequality (1.5).

Next, respective to the L_p-Minkowski combinations of convex bodies, we prove the following L_p-Brunn–Minkowski inequality for star dualities of intersection bodies.

Theorem 1.2. *If* $K, L \in \mathcal{K}_o^n$, $1 \le p \le +\infty$ *(for* $p = 1$, *it can be assumed that* $K, L \in \mathcal{K}^n$*) and* $\lambda \in [0,1]$, *then*

$$\begin{aligned} &V(I^{\circ}(\lambda \cdot K +_p (1-\lambda) \cdot L))^{-\frac{p}{n(n-1)}} \\ &\qquad \ge \lambda V(I^{\circ}K)^{-\frac{p}{n(n-1)}} + (1-\lambda) V(I^{\circ}L)^{-\frac{p}{n(n-1)}}. \end{aligned} \tag{1.9}$$

When $\lambda \in (0,1)$, *equality holds in* (1.9) *for* $p = 1$ *if and only if* K *and* L *are homothetic, for* $1 < p < +\infty$ *if and only if* K *and* L *are dilates, for* $p = +\infty$ *if and only if* $K \subseteq L$ *or* $L \subseteq K$. *When* $\lambda = 0$ *or* $\lambda = 1$, (1.9) *becomes an equality. Here,* $I^{\circ}M = (IM)^{\circ}$ *denotes the star duality of the intersection body* IM.

Obviously, the case $p = 1$ and $\lambda = \frac{1}{2}$ of inequality (1.9) implies a dual form of inequality (1.5) as follows: *If* $K, L \in \mathcal{K}^n$, *then*

$$V(I^{\circ}(K+L))^{-\frac{1}{n(n-1)}} \ge V(I^{\circ}K)^{-\frac{1}{n(n-1)}} + V(I^{\circ}L)^{-\frac{1}{n(n-1)}},$$

with equality if and only if K *and* L *are homothetic.*

2. Background material

2.1. Support functions and L_p-Minkowski combinations

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K, \cdot) : \mathbb{R}^n \to (-\infty, +\infty)$, is defined by (see [7, 16])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

For $1 \leq p \leq +\infty$, the L_p-Minkowski combinations (or called the Firey L_p-combinations) of convex bodies were introduced by Firey (see [5, 12]). For $K, L \in \mathcal{K}_o^n$, $1 \leq p < +\infty$ (for $p = 1$, it can be assumed that $K, L \in \mathcal{K}^n$) and $\lambda, \mu \geq 0$ (not both zero), the L_p-Minkowski combination, $\lambda \cdot K +_p \mu \cdot L \in \mathcal{K}_o^n$, of K and L is defined by

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot) = [\lambda h(K, \cdot)^p + \mu h(L, \cdot)^p]^{\frac{1}{p}}, \tag{2.1}$$

where $\lambda \cdot K = \lambda^{\frac{1}{p}} K$. If $\lambda = \mu = 1$, then $K +_p L$ is called the L_p-Minkowski addition of K and L. Obviously, if $p = 1$, then $\lambda \cdot K +_1 \mu \cdot L = \lambda K + \mu L$ is the Minkowski combination of K and L. For $p = +\infty$, according to the fact for $a, b \geq 0$,

$$\lim_{p \to +\infty} [\lambda a^p + \mu b^p]^{\frac{1}{p}} = \max\{a, b\}, \tag{2.2}$$

we define for $K, L \in \mathcal{K}_o^n$ (see [16]),

$$\lambda \cdot K +_{+\infty} \mu \cdot L = \mathrm{conv}(K \cup L). \tag{2.3}$$

From (2.1) and the Jensen's inequality, we easily know that *if* $K, L \in \mathcal{K}_o^n$, $1 < p < +\infty$ *and* $\lambda + \mu = 1$ *(*$\lambda, \mu \geq 0$*), then*

$$\lambda K + \mu L \subseteq \lambda \cdot K +_p \mu \cdot L. \tag{2.4}$$

Equality holds in (2.4) *if and only if* $K = L$.

Here, we deal with the equality condition of (2.4). Indeed, if $\lambda K + \mu L = \lambda \cdot K +_p \mu \cdot L$, i.e., for any $u \in S^{n-1}$,

$$[\lambda h(K, u) + \mu h(L, u)]^p = \lambda h(K, u)^p + \mu h(L, u)^p.$$

This implies $h(K, u) = h(L, u)$ for any $u \in S^{n-1}$, i.e., $K = L$. Obviously, if $K = L$, then equality holds in (2.4).

2.2. Radial functions and L_p-radial Minkowski combinations

If K is a compact star-shaped set (about the origin) in $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \backslash \{0\} \to [0, +\infty)$, is defined by (see [7])

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \backslash \{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (about the origin).

From the radial function, we have the following volume formula of a body K:

$$V(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^n du. \tag{2.5}$$

For the radial function, we see that if $K \in \mathcal{S}_o^n$ and ζ is a subspace of $\mathbb{R}^n$, then for any $u \in S^{n-1} \cap \zeta$ (see [7]),

$$\rho(K \cap \zeta, u) = \rho(K, u). \tag{2.6}$$

For $-\infty \leq p \leq +\infty$, the L_p-radial Minkowski combinations of star bodies are defined as follows: For $K, L \in \mathcal{S}_o^n$, $\lambda, \mu \geq 0$ (not both zero), $-\infty < p < +\infty$ and $p \neq 0$, the L_p-radial Minkowski combination, $\lambda \odot K \,\widetilde{+}_p\, \mu \odot L \in \mathcal{S}_o^n$, of K and L is defined by (see [16])

$$\rho(\lambda \odot K \,\widetilde{+}_p\, \mu \odot L, \cdot) = [\lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p]^{\frac{1}{p}}. \tag{2.7}$$

If $\lambda = \mu = 1$, then $K \,\widetilde{+}_p\, L$ is called the L_p-radial Minkowski addition of K and L. In particular, $K \,\widetilde{+}_1\, L = K \,\widetilde{+}\, L$ is the radial Minkowski addition of K and L. For $p = \pm\infty$, from (2.2) and

$$\lim_{p \to -\infty} [\lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p]^{\frac{1}{p}} = \min\{\rho_K, \rho_L\}, \tag{2.8}$$

we define for $K, L \in \mathcal{S}_o^n$,

$$\lambda \odot K \,\widetilde{+}_{+\infty}\, \mu \odot L = K \cup L, \tag{2.9}$$

$$\lambda \odot K \,\widetilde{+}_{-\infty}\, \mu \odot L = K \cap L. \tag{2.10}$$

For $p = 0$, $\lambda \in [0, 1]$, define $\lambda \odot K \,\widetilde{+}_0\, (1 - \lambda) \odot L$ which is called the log-radial Minkowski combination of K and L by (see [18])

$$\begin{aligned} \rho(\lambda \odot K \,\widetilde{+}_0\, (1-\lambda) \odot L, \cdot) &= \lim_{p \to 0} \rho(\lambda \odot K \,\widetilde{+}_p\, (1-\lambda) \odot L, \cdot) \\ &= \lim_{p \to 0} [\lambda \rho(K, \cdot)^p + (1-\lambda) \rho(L, \cdot)^p]^{\frac{1}{p}} \\ &= \rho(K, \cdot)^{\lambda} \rho(L, \cdot)^{1-\lambda}. \end{aligned} \tag{2.11}$$

For the log-radial Minkowski combination, Wang and Liu (see [18]) established the following dual log-Brunn–Minkowski inequality: *If $K, L \in \mathcal{S}_o^n$ and $\lambda \in [0, 1]$, then*

$$V(\lambda \odot K \,\widetilde{+}_0\, (1-\lambda) \odot L) \leq V(K)^{\lambda} V(L)^{1-\lambda}, \tag{2.12}$$

with equality for $\lambda \in (0, 1)$ if and only if K and L are dilates. For $\lambda = 0$ or $\lambda = 1$, (2.12) *becomes an equality.*

2.3. Star dualities

In 1999, Moszyńska (see [14]) introduced the notion of star duality. For $K \in \mathcal{S}_o^n$, the star duality, K°, of K is given by

$$\rho(K^\circ, u) = \frac{1}{\rho(K, u)}, \tag{2.13}$$

for all $u \in \mathcal{S}^{n-1}$. From (2.13), we easily see that for $\lambda > 0$,

$$(\lambda K)^\circ = \frac{1}{\lambda} K^\circ. \tag{2.14}$$

2.4. Intersection bodies

Intersection bodies were first explicitly defined and named by Lutwak (see [11]). For each $K \in \mathcal{S}_o^n$, the intersection body, IK, of K is an origin-symmetric star body whose radial function in the direction $u \in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of K by the $(n-1)$-dimensional subspace $u^\perp$ orthogonal to u. That is, for any $u \in S^{n-1}$,

$$\rho(IK, u) = V_{n-1}(K \cap u^\perp), \tag{2.15}$$

where V_{n-1} denotes $(n-1)$-dimensional volume.

From (2.15), we know that the intersection body has the following property: If $K \in \mathcal{S}_o^n$, then for $\lambda > 0$,

$$I(\lambda K) = \lambda^{n-1} IK. \tag{2.16}$$

The intersection body is a very important object of study in the Brunn–Minkowski theory. A number of important results regarding intersection bodies come together in books [7, 16].

From (2.13) and (2.15), for the star duality $I^\circ K$ of intersection body IK, we have that for any $u \in S^{n-1}$,

$$\rho(I^\circ K, u)^{-1} = \rho(IK, u) = V_{n-1}(K \cap u^\perp). \tag{2.17}$$

Hence, (2.14), (2.16) and (2.17) show that for $\lambda > 0$,

$$I^\circ(\lambda K) = \frac{1}{\lambda^{n-1}} I^\circ K. \tag{2.18}$$

For the works of the star dualities of intersection bodies, also see [10, 24].

3. L_p-dual Brunn–Minkowski inequality for intersection bodies

Theorem 1.1 shows the L_p-dual Brunn–Minkowski inequality for intersection bodies. Here, we will prove Theorem 1.1.

Lemma 3.1. *If $K, L \in \mathcal{S}_o^n$, p is any real and $\lambda \in [0,1]$, then for any $u \in S^{n-1}$,*

$$(\lambda \odot K \tilde{+}_p (1-\lambda) \odot L) \cap u^\perp = \lambda \odot (K \cap u^\perp) \tilde{+}_p (1-\lambda) \odot (L \cap u^\perp). \quad (3.1)$$

Here $u^\perp$ denotes the $(n-1)$-dimensional subspace orthogonal to u.

Proof. For $p \neq 0$, according to (2.6) and (2.7), we have for any $v \in S^{n-1} \cap u^\perp$,

$$\begin{aligned}\rho((\lambda \odot K \tilde{+}_p (1-\lambda) \odot L) \cap u^\perp, v)^p &= \rho(\lambda \odot K \tilde{+}_p (1-\lambda) \odot L, v)^p \\ &= \lambda\rho(K,v)^p + (1-\lambda)\rho(L,v)^p \\ &= \lambda\rho(K \cap u^\perp, v)^p + (1-\lambda)\rho(L \cap u^\perp, v)^p \\ &= \rho(\lambda \odot (K \cap u^\perp) \tilde{+}_p (1-\lambda) \odot (L \cap u^\perp), v)^p.\end{aligned}$$

This gives the case $p \neq 0$ of (3.1).

For $p = 0$, by (2.6) and (2.11) we have that for any $v \in S^{n-1} \cap u^\perp$,

$$\begin{aligned}\rho((\lambda \odot K \tilde{+}_0 (1-\lambda) \odot L) \cap u^\perp, v) &= \rho(\lambda \odot K \tilde{+}_0 (1-\lambda) \odot L, v) \\ &= \rho(K,v)^\lambda \rho(L,v)^{1-\lambda} = \rho(K \cap u^\perp, v)^\lambda \rho(L \cap u^\perp, v)^{1-\lambda} \\ &= \rho(\lambda \odot (K \cap u^\perp) \tilde{+}_0 (1-\lambda) \odot (L \cap u^\perp), v).\end{aligned}$$

This provides the case $p = 0$ of (3.1). ∎

Proof of Theorem 1.1. (1) For $0 < p < n-1$, applying inequality (1.3) to the $(n-1)$-dimensional case and combining with (3.1), we have that for $\lambda \in [0,1]$ and any $u \in S^{n-1}$,

$$\begin{aligned}&V_{n-1}((\lambda \odot K \tilde{+}_p (1-\lambda) \odot L) \cap u^\perp)^{\frac{p}{n-1}} \\ &\quad = V_{n-1}(\lambda \odot (K \cap u^\perp) \tilde{+}_p (1-\lambda) \odot (L \cap u^\perp))^{\frac{p}{n-1}} \\ &\quad \leq \lambda V_{n-1}(K \cap u^\perp)^{\frac{p}{n-1}} + (1-\lambda) V_{n-1}(L \cap u^\perp)^{\frac{p}{n-1}}. \quad (3.2)\end{aligned}$$

According to the equality condition of inequality (1.3), we see that equality holds in (3.2) for $\lambda \in (0,1)$ if and only if $K \cap u^\perp$ and $L \cap u^\perp$ are dilates for any $u \in S^{n-1}$, i.e., K and L are dilates (see [7, Theorem 7.1.1]).

Notice that $0 < p < n-1$ implies $\frac{n(n-1)}{p} > 1$. From this, by (2.5), (2.15), (3.2) and the Minkowski integral inequality we infer that

$$\begin{aligned}&V(I(\lambda \odot K \tilde{+}_p (1-\lambda) \odot L))^{\frac{p}{n(n-1)}} \\ &\quad = \left[\frac{1}{n}\int_{S^{n-1}} \rho(I(\lambda \odot K \tilde{+}_p (1-\lambda) \odot L), u)^n du\right]^{\frac{p}{n(n-1)}}\end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{1}{n}\int_{S^{n-1}} V_{n-1}((\lambda \odot K \,\widetilde{+}_p\, (1-\lambda)\odot L)\cap u^{\perp})^n du\right]^{\frac{p}{n(n-1)}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} [V_{n-1}((\lambda \odot K \,\widetilde{+}_p\, (1-\lambda)\odot L)\cap u^{\perp})^{\frac{p}{n-1}}]^{\frac{n(n-1)}{p}} du\right]^{\frac{p}{n(n-1)}} \\
&\le \left[\frac{1}{n}\int_{S^{n-1}} [\lambda V_{n-1}(K\cap u^{\perp})^{\frac{p}{n-1}}\right. \\
&\quad \left. + (1-\lambda) V_{n-1}(L\cap u^{\perp})^{\frac{p}{n-1}}]^{\frac{n(n-1)}{p}} du\right]^{\frac{p}{n(n-1)}} \\
&\le \lambda\left[\frac{1}{n}\int_{S^{n-1}} V_{n-1}(K\cap u^{\perp})^n du\right]^{\frac{p}{n(n-1)}} \\
&\quad + (1-\lambda)\left[\frac{1}{n}\int_{S^{n-1}} V_{n-1}(L\cap u^{\perp})^n du\right]^{\frac{p}{n(n-1)}} \\
&= \lambda\left[\frac{1}{n}\int_{S^{n-1}} \rho(IK,u)^n du\right]^{\frac{p}{n(n-1)}} \\
&\quad + (1-\lambda)\left[\frac{1}{n}\int_{S^{n-1}} \rho(IL,u)^n du\right]^{\frac{p}{n(n-1)}} \\
&= \lambda V(IK)^{\frac{p}{n(n-1)}} + (1-\lambda)V(IL)^{\frac{p}{n(n-1)}}. \qquad (3.3)
\end{aligned}
$$

From the equality conditions of inequality (3.2) and the Minkowski integral inequality, we know that equality holds in inequality (3.3) if and only if K and L are dilates. This gives inequality (1.6) and its equality condition.

(2) For $-\infty < p < 0$ or $n(n-1) < p < +\infty$, similar to the proof of inequality (1.6), applying inequality (1.4) to the $(n-1)$-dimensional case, we may obtain the reverse form of inequality (3.2). This together with the Minkowski integral inequality and notice $\frac{n(n-1)}{p} < 0$ or $0 < \frac{n(n-1)}{p} < 1$, we get inequality (1.7) and its equality condition.

(3) For $p = +\infty$, inequality (1.7) takes the following form:

$$
\begin{aligned}
&\lim_{p\to+\infty} V(I(\lambda \odot K \,\widetilde{+}_p\, (1-\lambda)\odot L))^{\frac{1}{n(n-1)}} \\
&\qquad \ge \lim_{p\to+\infty} [\lambda V(IK)^{\frac{p}{n(n-1)}} + (1-\lambda)V(IL)^{\frac{p}{n(n-1)}}]^{\frac{1}{p}}.
\end{aligned}
$$

This, together with (2.2) and (2.9), becomes that

$$
V(I(K\cup L))^{\frac{1}{n(n-1)}} \ge \max\{V(IK)^{\frac{1}{n(n-1)}}, V(IL)^{\frac{1}{n(n-1)}}\},
$$

i.e.,

$$
V(I(K\cup L)) \ge \max\{V(IK), V(IL)\}. \qquad (3.4)
$$

Because $K \cup L \supseteq K, L$, thus for $u \in S^{n-1}$,

$$\begin{aligned}\rho(I(K\cup L),u) &= V_{n-1}((K\cup L)\cap u^{\perp})\\ &\geq \max\{V_{n-1}(K\cap u^{\perp}), V_{n-1}(L\cap u^{\perp})\}\\ &= \max\{\rho(IK,u),\rho(IL,u)\}.\end{aligned}$$

This means that (3.4) is true. Hence the case $p = +\infty$ of inequality (1.7) holds.

Equality holds in (3.4) if and only if $K \subseteq L$ or $L \subseteq K$. Indeed, if $K \subseteq L$ or $L \subseteq K$, then equality holds in (3.4). Conversely, for instance, assuming $\max\{V(IK), V(IL)\} = V(IK)$, if $V(I(K\cup L)) = V(IK)$, and notice that $K \cup L \supseteq K$, then $K \cup L = K$. This implies $L \subseteq K$.

For $p = -\infty$, inequality (1.7) becomes as follows:

$$\begin{aligned}&\lim_{p\to-\infty} V(I(\lambda\odot K \,\widetilde{+}_p\, (1-\lambda)\odot L))^{\frac{1}{n(n-1)}}\\ &\quad\leq \lim_{p\to-\infty}[\lambda V(IK)^{\frac{p}{n(n-1)}} + (1-\lambda)V(IL)^{\frac{p}{n(n-1)}}]^{\frac{1}{p}}.\end{aligned}$$

This, together with (2.8) and (2.10), gives that

$$V(I(K\cap L))^{\frac{1}{n(n-1)}} \leq \min\{V(IK)^{\frac{1}{n(n-1)}}, V(IL)^{\frac{1}{n(n-1)}}\},$$

i.e.,

$$V(I(K\cap L)) \leq \min\{V(IK), V(IL)\}. \tag{3.5}$$

Similar to the proof of (3.4), we can obtain inequality (3.5) and its equality condition, i.e., the case $p = -\infty$ of inequality (1.7) and its equality condition are true.

(4) For $p = 0$, applying inequality (2.12) to the $(n-1)$-dimensional case and combining with (3.1), we know that for $\lambda \in [0, 1]$ and any $u \in S^{n-1}$,

$$\begin{aligned}&V_{n-1}((\lambda\odot K \,\widetilde{+}_0\, (1-\lambda)\odot L)\cap u^{\perp})\\ &\quad= V_{n-1}(\lambda\odot(K\cap u^{\perp}) \,\widetilde{+}_0\, (1-\lambda)\odot(L\cap u^{\perp}))\\ &\quad\leq V_{n-1}(K\cap u^{\perp})^{\lambda} V_{n-1}(L\cap u^{\perp})^{1-\lambda}.\end{aligned} \tag{3.6}$$

According to the equality condition of inequality (1.7), we see that equality holds in (3.6) for $\lambda \in (0, 1)$ if and only if $K \cap u^{\perp}$ and $L \cap u^{\perp}$ are dilates for any $u \in S^{n-1}$, i.e., K and L are dilates.

From (2.5), (2.15), (3.6) and the Hölder integral inequality, we deduce that for $\lambda \in (0, 1)$,

$$\begin{aligned}&V(I(\lambda\odot K \,\widetilde{+}_0\, (1-\lambda)\odot L))\\ &\quad= \frac{1}{n}\int_{S^{n-1}} \rho(I(\lambda\odot K \,\widetilde{+}_0\, (1-\lambda)\odot L),u)^n du\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n}\int_{S^{n-1}} V_{n-1}((\lambda \odot K \,\tilde{+}_0\, (1-\lambda)\odot L)\cap u^{\perp})^n du\\
&\le \frac{1}{n}\int_{S^{n-1}} [V_{n-1}(K\cap u^{\perp})^{\lambda} V_{n-1}(L\cap u^{\perp})^{1-\lambda}]^n du\\
&= \frac{1}{n}\int_{S^{n-1}} \rho(IK,u)^{n\lambda}\rho(IL,u)^{n(1-\lambda)} du\\
&\le \left[\frac{1}{n}\int_{S^{n-1}} (\rho(IK,u)^{n\lambda})^{\frac{1}{\lambda}} du\right]^{\lambda}\left[\frac{1}{n}\int_{S^{n-1}} (\rho(IL,u)^{n(1-\lambda)})^{\frac{1}{1-\lambda}} du\right]^{1-\lambda}\\
&= \left[\frac{1}{n}\int_{S^{n-1}} \rho(IK,u)^{n} du\right]^{\lambda}\left[\frac{1}{n}\int_{S^{n-1}} \rho(IL,u)^{n} du\right]^{1-\lambda}\\
&= V(IK)^{\lambda}V(IL)^{1-\lambda}.
\end{aligned}
$$

This yields inequality (1.8). And the equality conditions of (3.6) and the Hölder integral inequality give that equality holds in (1.8) for $\lambda \in (0,1)$ if and only if K and L are dilates.

To sum up, we complete the proof of Theorem 1.1. ■

4. L_p-Brunn–Minkowski inequality for star dualities of intersection bodies

Theorem 1.2 deals with the L_p-Brunn–Minkowski inequality for star dualities of intersection bodies. Now, we give its proof.

Lemma 4.1. *If $K, L \in \mathcal{K}^n$ and $\lambda \in [0,1]$, then for any $u \in S^{n-1}$,*

$$\lambda(K\cap u^{\perp}) + (1-\lambda)(L\cap u^{\perp}) \subseteq (\lambda K + (1-\lambda)L)\cap u^{\perp}. \tag{4.1}$$

When $\lambda \in (0,1)$, equality holds in (4.1) if K and L are homothetic.

Proof. For $\lambda \in (0,1)$ and any $u \in S^{n-1}$,

$$
\begin{aligned}
&\forall x = x_1 + x_2 \in \lambda(K\cap u^{\perp}) + (1-\lambda)(L\cap u^{\perp})\\
&\quad \Leftrightarrow x_1 \in \lambda(K\cap u^{\perp}) \text{ and } x_2 \in (1-\lambda)(L\cap u^{\perp})\\
&\quad \Leftrightarrow \lambda^{-1}x_1 \in K\cap u^{\perp} \text{ and } (1-\lambda)^{-1}x_2 \in L\cap u^{\perp}\\
&\quad \Leftrightarrow \lambda^{-1}x_1 \in K \text{ and } \lambda^{-1}x_1 \in u^{\perp},\ (1-\lambda)^{-1}x_2 \in L \text{ and } (1-\lambda)^{-1}x_2 \in u^{\perp}\\
&\quad \Leftrightarrow x_1 \in \lambda K \text{ and } x_1 \in \lambda u^{\perp},\ x_2 \in (1-\lambda)L \text{ and } x_2 \in (1-\lambda)u^{\perp}\\
&\quad \Leftrightarrow x_1 \in \lambda K \text{ and } x_1 \in u^{\perp},\ x_2 \in (1-\lambda)L \text{ and } x_2 \in u^{\perp}\\
&\quad \Rightarrow x = x_1 + x_2 \in \lambda K + (1-\lambda)L \text{ and } x = x_1 + x_2 \in u^{\perp}\\
&\quad \Leftrightarrow x \in [\lambda K + (1-\lambda)L]\cap u^{\perp}.
\end{aligned}
$$

This gives (4.1). We easily verify that equality holds in (4.1) for $\lambda \in (0,1)$ if K and L are homothetic. ■

Proof of Theorem 1.2. (1) If $p = 1$, from (4.1) and the $(n-1)$-dimensional case of inequality (1.1), we obtain that for $K, L \in \mathcal{K}^n$ and any $u \in S^{n-1}$,

$$
\begin{aligned}
V_{n-1}([\lambda K + (1-\lambda)L] \cap u^{\perp})^{\frac{1}{n-1}} \\
&\geq V_{n-1}(\lambda(K \cap u^{\perp}) + (1-\lambda)(L \cap u^{\perp}))^{\frac{1}{n-1}} \\
&\geq \lambda V_{n-1}(K \cap u^{\perp})^{\frac{1}{n-1}} + (1-\lambda)V_{n-1}(L \cap u^{\perp})^{\frac{1}{n-1}}.
\end{aligned}
\tag{4.2}
$$

Now we give the equality condition of (4.2). If equality holds in (4.2), i.e.,

$$
\begin{aligned}
V_{n-1}([\lambda K + (1-\lambda)L] \cap u^{\perp})^{\frac{1}{n-1}} \\
&= \lambda V_{n-1}(K \cap u^{\perp})^{\frac{1}{n-1}} + (1-\lambda)V_{n-1}(L \cap u^{\perp})^{\frac{1}{n-1}},
\end{aligned}
$$

this, and (4.2) imply that

$$
\begin{aligned}
V_{n-1}(\lambda(K \cap u^{\perp}) + (1-\lambda)(L \cap u^{\perp}))^{\frac{1}{n-1}} \\
&= \lambda V_{n-1}(K \cap u^{\perp})^{\frac{1}{n-1}} + (1-\lambda)V_{n-1}(L \cap u^{\perp})^{\frac{1}{n-1}}.
\end{aligned}
$$

This, together with the equality condition of inequality (1.1), means that when $\lambda \in (0,1)$, $K \cap u^{\perp}$ and $L \cap u^{\perp}$ are homothetic for any $u \in S^{n-1}$, i.e., K and L are homothetic. Conversely, if K and L are homothetic, then equality holds in (4.2). Thus, equality holds in (4.2) for $\lambda \in (0,1)$ if and only if K and L are homothetic.

Hence, by (2.5), (4.2), (2.17) and the Minkowski integral inequality, we deduce that

$$
\begin{aligned}
V(I^{\circ}(\lambda K + (1-\lambda)L))^{-\frac{1}{n(n-1)}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \rho(I^{\circ}(\lambda K + (1-\lambda)L), u)^n du\right]^{-\frac{1}{n(n-1)}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} [\rho(I^{\circ}(\lambda K + (1-\lambda)L), u)^{-1}]^{-n} du\right]^{-\frac{1}{n(n-1)}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} V_{n-1}((\lambda K + (1-\lambda)L) \cap u^{\perp})^{-n} du\right]^{-\frac{1}{n(n-1)}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} [V_{n-1}((\lambda K + (1-\lambda)L) \cap u^{\perp})^{\frac{1}{n-1}}]^{-n(n-1)} du\right]^{-\frac{1}{n(n-1)}} \\
&\geq \left[\frac{1}{n}\int_{S^{n-1}} [\lambda V_{n-1}(K \cap u^{\perp})^{\frac{1}{n-1}} \right. \\
&\qquad \left. + (1-\lambda)V_{n-1}(L \cap u^{\perp})^{\frac{1}{n-1}}]^{-n(n-1)} du\right]^{-\frac{1}{n(n-1)}}
\end{aligned}
$$

$$\geq \lambda\left[\frac{1}{n}\int_{S^{n-1}} V_{n-1}(K\cap u^{\perp})^{-n}du\right]^{-\frac{1}{n(n-1)}}$$
$$+(1-\lambda)\left[\frac{1}{n}\int_{S^{n-1}} V_{n-1}(L\cap u^{\perp})^{-n}du\right]^{-\frac{1}{n(n-1)}}$$
$$=\lambda\left[\frac{1}{n}\int_{S^{n-1}} \rho(I^{\circ}K,u)^{n}du\right]^{-\frac{1}{n(n-1)}}$$
$$+(1-\lambda)\left[\frac{1}{n}\int_{S^{n-1}} \rho(I^{\circ}L,u)^{n}du\right]^{-\frac{1}{n(n-1)}}$$
$$=\lambda V(I^{\circ}K)^{-\frac{1}{n(n-1)}}+(1-\lambda)V(I^{\circ}L)^{-\frac{1}{n(n-1)}}. \tag{4.3}$$

According to the equality conditions of inequality (4.2) and the Minkowski integral inequality, we know that equality holds in (4.3) for $\lambda\in(0,1)$ if and only if K and L are homothetic. From this, we obtain the case $p=1$ of inequality (1.9) and its equality condition.

(2) If $1<p<+\infty$, for $K,L\in\mathcal{K}_o^n$, let $\alpha=V(I^{\circ}K)^{-\frac{1}{n(n-1)}}$, $\beta=V(I^{\circ}L)^{-\frac{1}{n(n-1)}}$, $\overline{K}=\frac{1}{\alpha}K$, $\overline{L}=\frac{1}{\beta}L\in\mathcal{K}_o^n$. Then by (2.18) we get that

$$V(I^{\circ}\overline{K})^{-\frac{1}{n(n-1)}}=V\left(I^{\circ}\left(\frac{1}{\alpha}K\right)\right)^{-\frac{1}{n(n-1)}}=\frac{1}{\alpha}V(I^{\circ}K)^{-\frac{1}{n(n-1)}}=1,$$

i.e., $V(I^{\circ}\overline{K})=1$. Similarly, $V(I^{\circ}\overline{L})=1$. Since for $\lambda\in[0,1]$, $\overline{\lambda}=\frac{\lambda\alpha^p}{\lambda\alpha^p+(1-\lambda)\beta^p}\in[0,1]$, thus for any $u\in S^{n-1}$,

$$\begin{aligned}
&h(\overline{\lambda}\cdot\overline{K}+_p(1-\overline{\lambda})\cdot\overline{L},u)^p\\
&\quad=\overline{\lambda}h(\overline{K},u)^p+(1-\overline{\lambda})h(\overline{L},u)^p\\
&\quad=\frac{\lambda\alpha^p}{\lambda\alpha^p+(1-\lambda)\beta^p}h(\overline{K},u)^p+\frac{(1-\lambda)\beta^p}{\lambda\alpha^p+(1-\lambda)\beta^p}h(\overline{L},u)^p\\
&\quad=\frac{\lambda h(K,u)^p+(1-\lambda)h(L,u)^p}{\lambda\alpha^p+(1-\lambda)\beta^p}=\frac{h(\lambda\cdot K+_p(1-\lambda)\cdot L,u)^p}{\lambda\alpha^p+(1-\lambda)\beta^p}.
\end{aligned}$$

This and (2.4) yield that

$$\begin{aligned}
\lambda\cdot K+_p(1-\lambda)\cdot L&=[\lambda\alpha^p+(1-\lambda)\beta^p]^{\frac{1}{p}}[\overline{\lambda}\cdot\overline{K}+_p(1-\overline{\lambda})\cdot\overline{L}]\\
&\supseteq[\lambda\alpha^p+(1-\lambda)\beta^p]^{\frac{1}{p}}[\overline{\lambda}\overline{K}+(1-\overline{\lambda})\overline{L}].
\end{aligned}\tag{4.4}$$

The equality condition of (2.4) implies that equality holds in (4.4) if and only if $\overline{K}=\overline{L}$, i.e., $K=\frac{\alpha}{\beta}L$ which means that K and L are dilates.

From this, (4.4), (2.17) and (2.18) we deduce that

$$I^\circ(\lambda\cdot K+_p(1-\lambda)\cdot L)\subseteq I^\circ([\lambda\alpha^p+(1-\lambda)\beta^p]^{\frac{1}{p}}[\overline{\lambda}\overline{K}+(1-\overline{\lambda})\overline{L}])$$
$$=[\lambda\alpha^p+(1-\lambda)\beta^p]^{-\frac{n-1}{p}}I^\circ(\overline{\lambda}\overline{K}+(1-\overline{\lambda})\overline{L}). \quad (4.5)$$

Equality holds in (4.5) if and only if K and L are dilates.

Therefore, from (4.5), (2.18) and (4.3), and notice that $V(I^\circ\overline{K})=V(I^\circ\overline{L})=1$, we infer that

$$\begin{aligned}
&V(I^\circ(\lambda\cdot K+_p(1-\lambda)\cdot L))^{-\frac{1}{n(n-1)}}\\
&\quad\geq V([\lambda\alpha^p+(1-\lambda)\beta^p]^{-\frac{n-1}{p}}I^\circ(\overline{\lambda}\overline{K}+(1-\overline{\lambda})\overline{L}))^{-\frac{1}{n(n-1)}}\\
&\quad=[(\lambda\alpha^p+(1-\lambda)\beta^p)^{-\frac{n(n-1)}{p}}]^{-\frac{1}{n(n-1)}}V(I^\circ(\overline{\lambda}\overline{K}+(1-\overline{\lambda})\overline{L}))^{-\frac{1}{n(n-1)}}\\
&\quad\geq[\lambda\alpha^p+(1-\lambda)\beta^p]^{\frac{1}{p}}[\overline{\lambda}V(I^\circ\overline{K})^{-\frac{1}{n(n-1)}}+(1-\overline{\lambda})V(I^\circ\overline{L})^{-\frac{1}{n(n-1)}}]\\
&\quad=[\lambda\alpha^p+(1-\lambda)\beta^p]^{\frac{1}{p}}(\overline{\lambda}+1-\overline{\lambda})=[\lambda\alpha^p+(1-\lambda)\beta^p]^{\frac{1}{p}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
V(I^\circ(\lambda\cdot K+_p(1-\lambda)\cdot L))^{-\frac{p}{n(n-1)}}&\geq\lambda\alpha^p+(1-\lambda)\beta^p\\
&=\lambda V(I^\circ K)^{-\frac{p}{n(n-1)}}+(1-\lambda)V(I^\circ L)^{-\frac{p}{n(n-1)}}.
\end{aligned}$$

This is the case $1<p<+\infty$ of inequality (1.9). The equality condition of (4.5) gives that the equality holds in the case $1<p<+\infty$ of inequality (1.9) if and only if K and L are dilates.

(2) For $p=+\infty$, by (2.2) and (2.3), inequality (1.9) becomes the following form:

$$V(I^\circ(\mathrm{conv}(K\cup L)))^{-\frac{1}{n(n-1)}}\geq\max\{V(I^\circ K)^{-\frac{1}{n(n-1)}},V(I^\circ L)^{-\frac{1}{n(n-1)}}\},$$

or equivalently,

$$V(I^\circ(\mathrm{conv}(K\cup L)))\leq\min\{V(I^\circ K),V(I^\circ L)\}. \quad (4.6)$$

Since $K,L\subseteq\mathrm{conv}(K\cup L)$, thus $IK,IL\subseteq I(\mathrm{conv}(K\cup L))$. This together with (2.13) implies $I^\circ K,I^\circ L\supseteq I^\circ(\mathrm{conv}(K\cup L))$. Therefore, (4.6) is true.

Similar to the derivation of the equality condition of (3.4), we easily see that equality holds in (4.6) if and only if $K\subseteq L$ or $L\subseteq K$. So the case of $p=+\infty$ of inequality (1.9) is proven.

In summary, we complete the proof of Theorem 1.2. ∎

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