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The combinatorial structure of symmetric strongly shifted ideals

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Abstract. Symmetric strongly shifted ideals constitute a class of monomial ideals which are equipped with an action of the symmetric group and are analogous to the well-studied class of strongly stable monomial ideals. In this paper, we focus on algebraic and combinatorial properties of symmetric strongly shifted ideals. On the algebraic side, we elucidate properties that pertain to behavior under ideal operations, primary decomposition, and the structure of their Rees algebra. On the combinatorial side, we develop a notion of partition Borel generators which leads to connections to discrete polymatroids, convex polytopes, and permutohedral toric varieties.

Contents

1. Introduction

Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K and consider the set of partitions

$$\mathrm{P}_n = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : 0 \le \lambda_1 \le \lambda_2 \le \dots \le \lambda_n\}.$$

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Orbits of monomials under the natural action of the symmetric group $\mathfrak{S}_n$ can be identified with elements of P_n. This induces a bijection between $\mathfrak{S}_n$-fixed monomial ideals $I \subset R$ and sets of partitions $P(I) \subset P_n$ given by

$$\mathrm{P}(I) = \{\lambda \in \mathrm{P}_n : x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n} \in I\}.$$

The central objects of study in this note are the following classes of $\mathfrak{S}_n$-fixed monomial ideals, which were introduced in [2].

Definition 1.1. Let $I \subset R$ be an $\mathfrak{S}_n$-fixed monomial ideal. We say that I is a *symmetric shifted ideal*, or *ssi*, if, for every $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathrm{P}(I)$ and $1 \leq i < n$ with $\lambda_i < \lambda_n$, one has $x^\lambda(x_i/x_n) \in I$. We say that I is a *symmetric strongly shifted ideal*, or *sssi*, if, for every $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathrm{P}(I)$ and $1 \leq i < j \leq n$ with $\lambda_i < \lambda_j$, one has $x^\lambda(x_i/x_j) \in I$. Monomials $x^\lambda(x_i/x_n)$ and $x^\lambda(x_i/x_j)$ satisfying the conditions above are referred to as being obtained from x^λ by a *Borel move*.

The definitions of symmetric shifted and strongly shifted ideals are inspired by the definition of stable and strongly stable ideals (which we recall in Definition 3.1). These are the most important classes of monomial ideals in computational algebra since, e.g., in characteristic zero generic initial initials are strongly stable. Moreover, stable and strongly stable ideals have well-understood minimal graded free resolutions. These were constructed by Eliahou and Kervaire, who also gave a formula for their graded Betti numbers in terms of the data of their minimal systems of monomial generators [17]. Analogous results for symmetric shifted ideals were obtained in [2].

In this article, we initiate a comprehensive study of the algebraic properties and combinatorial structure of symmetric strongly shifted ideals. While the only $\mathfrak{S}_n$-fixed ideals which are also strongly stable are the powers of the homogeneous maximal ideal, we discover that the class of symmetric strongly shifted ideals exhibits several similarities to the class of strongly stable ideals. A key tool we develop to unveil these analogies is the notion of *partition Borel generators* of a symmetric strongly shifted ideal (see Definition 3.3), which is inspired by the notion of Borel generators of a strongly stable ideal. (We refer the reader to Section 3 for any unexplained terminology.) In particular, we obtain the following results.

(1) Like strongly stable ideals (see [24]), symmetric strongly shifted ideals are closed under taking sums, intersections, products, and powers (Proposition 2.1 and Proposition 2.2). Moreover, under these ideal operations, one can keep track of partition Borel generators (see Proposition 3.8 and Proposition 3.9) similarly as one does for the Borel generators of a strongly stable ideal (see [20]).

(2) Like strongly stable ideals are sums of *principal Borel ideals* (i.e., strongly stable ideals with exactly one Borel generator), symmetric strongly shifted

ideals are sums of *principal Borel sssi's* (i.e., sssi's with exactly one partition Borel generator).

(3) A principal Borel ideal can be written as a product of prime ideals and as an intersection of powers of prime ideals (see [20]). Similarly, a principal Borel sssi can be factored as a product of *square-free Veronese ideals* (Theorem 3.12) and decomposed as an intersection of symbolic powers of square-free Veronese ideals (Theorem 5.6).

Recall that a square-free Veronese ideal I of degree d is the ideal generated by all square-free monomials of a degree d and can be equivalently written as

$$I = \bigcap_{\sigma \in \mathfrak{S}_n} \sigma(x_1, \dots, x_d),$$

i.e., as the symmetrization of a monomial prime ideal in d variables. Ideals of this kind are the only square-free symmetric strongly shifted ideals (see Remark 2.6). Our first main result says that *every* sssi is the symmetrization of a strongly stable ideal.

Theorem A (cf. Theorem 3.6). *An ideal I is symmetric strongly shifted if and only if I is the symmetrization of a strongly stable ideal J in the following sense:*

$$I = \bigcap_{\sigma \in \mathfrak{S}_n} \sigma(J).$$

Moreover, under this symmetrization process, the partition Borel generators of I correspond to the Borel generators of J.

The symmetrization process described by Theorem A sometimes allows to transfer desirable algebraic properties from the class of strongly stable ideals to the class of symmetric strongly shifted ideals. For instance, in Proposition 4.3, we identify classes of normal strongly shifted ideals which are the symmetrization of normal strongly stable ideals, while in Theorem 5.6 we find a primary decomposition of a principal Borel sssi by symmetrization of a primary decomposition of a principal Borel ideal. Moreover, by refining such primary decomposition to an irredundant one (Theorem 5.8), we determine the associated primes of ordinary powers of principal Borel sssi's (Theorem 5.9), similarly as in [29] for principal Borel ideals.

The analogies between principal Borel ideals and principal Borel sssi's extend beyond their algebraic structure into their combinatorial nature. A key observation in this sense is that both principal Borel ideals and square-free Veronese ideals are *polymatroidal ideals*; namely, the exponent vectors of their monomial generators form *discrete polymatroids*. The latter were introduced by Herzog and Hibi in [26] as a generalization of the notion of matroid (see Definition 3.13 for more details). Surprisingly, the class of principal Borel sssi's coincides with the class of symmetric polymatroidal ideals.

Theorem B (cf. Theorem 3.14). *A monomial ideal is symmetric and polymatroidal if and only if it is a symmetric strongly shifted ideal with exactly one partition Borel generator.*

The proof of Theorem B relies on the factorization property of a principal Borel sssi given by Theorem 3.12, which also allows for a beautiful description of their *toric ideal* (see Section 6). In more detail, our main result is the following.

Theorem C (cf. Theorem 6.5 and Corollary 6.13). *The toric ideal of a sssi with exactly one partition Borel generator is generated by quadratic polynomials, namely, the symmetric exchange relations. Moreover, the toric ring is a Cohen–Macaulay normal domain which has rational singularities in characteristic zero and is strongly F-regular in positive characteristic.*

In particular, our result provides supporting evidence for a longstanding conjecture of White, Herzog, and Hibi [26, 51], which states that, for an arbitrary polymatroidal ideal I, the toric ideal of I is generated by the symmetric exchange relations. Theorem C shows that the conjecture holds for all *symmetric* polymatroidal ideals.

From a geometric perspective, a principal Borel sssi defines a normal *toric permutohedral variety*, i.e., an algebraic variety associated with a well-studied convex polytope dubbed the *permutohedron* (see Proposition 4.6 and Corollary 6.14).

While the combinatorial structure of symmetric strongly shifted ideals with an arbitrary number of Borel generators remains more mysterious, the knowledge of their syzygies from [2] offers a valuable complementary source of information. In particular, it allows us to determine the depths of powers of an equigenerated sssi I (Proposition 5.1) and a structure theorem for its *Rees algebra* $\mathcal{R}(I) \cong \bigoplus_{k\geq 1} I^k$. More specifically, in Theorem 6.2, we prove that $\mathcal{R}(I)$ is the quotient of a polynomial ring modulo relations which are either linear or arise from the toric ideal of I. The latter statement is analogous to a well-known result on the Rees algebra of an equigenerated strongly stable ideal [28, Theorem 5.1].

Structure of the paper. In Section 2, we analyze the behavior of symmetric (strongly) shifted ideals under various algebraic operations. Remarkably, in Proposition 2.10, we show that symbolic powers of symmetric strongly shifted ideals are strongly shifted, generalizing what was known for square-free sssi's by [2, Theorem 4.3]. In Section 3, we introduce partition Borel generators and prove Theorem A. We study principal Borel sssi's in Section 3.1. In Section 3.2, we give combinatorial formulas for several numerical invariants of symmetric monomial ideals.

In Section 4, we discuss the normality property of symmetric strongly shifted ideals; applications to convex polytopes and permutohedra are included in Section 4.1. Next, we exploit the combinatorial structure of sssi's to study their ordinary and symbolic powers. In Section 5, we study the associated primes of powers of symmetric strongly shifted ideals. Finally, we study the Rees algebra of symmetric strongly shifted ideals in Section 6, where we prove the structure theorems 6.2 and 6.5 and discuss the geometry of toric varieties associated with a principal Borel sssi.

Notational conventions. We say that a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of nonnegative integers is a *partition* of d of length n, if $\lambda_1 \leq \cdots \leq \lambda_n$ and $|\boldsymbol{\lambda}| = \lambda_1 + \cdots + \lambda_n = d$. We opt for the less standard convention of ordering the parts nondecreasingly for our conventions to match those in [2], where symmetric shifted ideals were originally introduced. If λ has distinct parts $p_1, \ldots, p_s$ which occur with multiplicities $n_1, \ldots, n_s$, respectively, we sometimes use the alternate notation $\lambda = (p_1^{n_1}, \ldots, p_s^{n_s})$. Throughout the paper, the notation e_i stands for the i-th standard basis vector in $\mathbb{Z}^n$.

For a monomial $u = x_1^{a_1} \cdots x_n^{a_n}$, we write $\operatorname{part}(u) \in P_n$ for the partition obtained from $(a_1, \ldots, a_n)$ by ordering its entries nonincreasingly. If a monomial ideal $I \subset R$ is $\mathfrak{S}_n$-fixed, then a monomial u is in I if and only if $x^{\operatorname{part}(u)}$ is in I. The set $P(I)$ contains a partition $\lambda \neq (0^n)$ if and only if $I \neq R$. Throughout the paper, we assume that $I \neq R$.

2. Symmetric shifted ideals under ideal operations

In [9, Proposition 1], Cimpoeaş observes that the class of ideals of Borel type (which generalizes strongly stable ideals) is closed under sum, intersection, product, and colon operations. The class of strongly stable ideals is also closed under the same operations as demonstrated in [24, Proposition 1.2] and also under taking symbolic powers [24, Theorem 3.8]. (In the latter work, symbolic powers are taken by retaining the primary components associated to *minimal primes* of I.)

In this section, we show that a majority of these statements are also true for symmetric strongly shifted ideals and fewer also hold for symmetric shifted ideals. Towards this end, it will be convenient to simplify Definition 1.1 slightly. Denoting by $G(I)$ the set of minimal monomial generators of I allows to single out the *partition generators* of I, namely,

$$\Lambda(I) = \{\lambda \in P_n : x^\lambda \in G(I)\}.$$

In view of [2, Lemmas 2.2 and 2.3], it suffices to check the conditions of Definition 1.1 for the partition generators $\lambda \in \Lambda(I)$ rather than for arbitrary elements λ in $P(I)$.

We begin by proving that the class of symmetric shifted ideals is closed under sums and intersections.

Proposition 2.1. *Let I, J be symmetric (strongly) shifted ideals. Then, $I + J$ and $I \cap J$ are symmetric (strongly) shifted ideals.*

Proof. We prove the statement assuming that I and J are symmetric strongly shifted. The proof for symmetric shifted ideals is analogous, hence left to the diligent reader.

First, notice that $I + J$ and $I \cap J$ are $\mathfrak{S}_n$-fixed, as I and J are. Now, let $\lambda \in P_n$ be such that $\lambda_i < \lambda_j$. If $x^\lambda \in I + J$, then $x^\lambda \in I$ or $x^\lambda \in J$. In the first case, $x^\lambda x_i/x_j \in I$ by the strongly shifted property of I, while in the second case, $x^\lambda x_i/x_j \in J$ because J is strongly shifted. In either case, $x^\lambda x_i/x_j \in I + J$; thus, $I + J$ is a sssi. Similarly, if $x^\lambda \in I \cap J$, then $x^\lambda x_i/x_j \in I \cap J$ by the strongly shifted property of I and J. Hence, $I \cap J$ is a sssi. ∎

We next show that the class of symmetric strongly shifted ideals is closed under products.

Proposition 2.2. *Products, and hence powers, of symmetric strongly shifted ideals are symmetric strongly shifted.*

Proof. It suffices to prove the statement for the product of two symmetric strongly shifted ideals I, J. It is clear that the ideal

$$IJ = \left(\{\sigma(x^\lambda)\tau(x^\mu) : \sigma, \tau \in \mathfrak{S}_n, \lambda \in P(I), \mu \in P(J)\}\right)$$

is fixed by $\mathfrak{S}_n$. It remains to show that IJ is strongly shifted. Towards this end, let $p \in P_n$ be such that $x^p = \sigma(x^\lambda)\tau(x^\mu) \in IJ$ and denote the relevant monomials by $\sigma(x^\lambda) = x^{\sigma(\lambda)} \in I$ and $\tau(x^\mu) = x^{\tau(\mu)} \in J$, respectively. Now, assume that

$$p_i = (\sigma(\lambda) + \tau(\mu))_i < (\sigma(\lambda) + \tau(\mu))_j = p_j.$$

Then, $\sigma(\lambda)_i < \sigma(\lambda)_j$ or $\tau(\mu)_i < \tau(\mu)_j$. Equivalently, $\lambda_{\sigma^{-1}(i)} < \lambda_{\sigma^{-1}(j)}$ or $\mu_{\tau^{-1}(i)} < \mu_{\tau^{-1}(j)}$.

Suppose the former case holds. Since I is symmetric and $\sigma(x^\lambda) \in I$, we have that $x^\lambda \in I$. Since I is additionally strongly shifted and $\lambda_{\sigma^{-1}(i)} < \lambda_{\sigma^{-1}(j)}$, one deduces that $x^\lambda x_{\sigma^{-1}(i)}/x_{\sigma^{-1}(j)} \in I$. Finally, applying σ yields

$$\sigma(x^\lambda x_{\sigma^{-1}(i)}/x_{\sigma^{-1}(j)}) = \sigma(x^\lambda)x_i/x_j \in I,$$

and thus, $x^p x_i/x_j = \sigma(x^\lambda)\tau(x^\mu) \in IJ$. The latter case is identical, hence omitted. ∎

On the other hand, the class of symmetric shifted ideals is not closed under taking products.

Example 2.3. In $k[x_1, x_2, x_3, x_4]$, consider the symmetric shifted ideal I with

$$\Lambda(I) = \{(1, 1, 2, 2), (0, 2, 2, 2), (0, 1, 2, 3)\}.$$

Notice that I is not a sssi, since $(0, 1, 2, 3) \in P(I)$ but $(1, 1, 1, 3) \notin P(I)$; see [2, Example 2.5]. Moreover, the maximal ideal $\mathfrak{m} = (x_1, x_2, x_3, x_4)$ is symmetric strongly shifted, with $\Lambda(\mathfrak{m}) = \{(0,0,0,1)\}$. Then, the monomial ideal $I\mathfrak{m}$ is $\mathfrak{S}_n$-invariant with

$$\Lambda(I\mathfrak{m}) = \{(1,2,2,2), (1,1,2,3), (0,2,2,3), (0,1,2,4), (0,1,3,3)\}$$

but is not symmetric shifted, since $(0,1,2,4) \in P(I\mathfrak{m})$ but $(1,1,1,4) \notin P(I\mathfrak{m})$.

We do not know of an example that would show that powers of symmetric shifted ideals may not be symmetric shifted. Thus, we are left with the following.

Question 2.4. Is the class of symmetric shifted ideals closed under taking powers?

We are also interested in understanding symbolic powers of symmetric (strongly) shifted ideals. In the literature, two notions of symbolic powers make an appearance. Some authors define the symbolic powers of an ideal I in terms of the *minimal primes* of I as

$$I^{(m)_{\mathrm{Min}}} = \bigcap_{P \in \mathrm{Min}(I)} (I^m R_P \cap R),$$

while others define them in terms of the *associated primes* of I as

$$I^{(m)_{\mathrm{Ass}}} = \bigcap_{P \in \mathrm{Ass}(I)} (I^m R_P \cap R).$$

The two definitions agree for ideals without embedded primes, in which case we will denote symbolic powers simply as $I^{(m)}$. The second definition has the advantage that it satisfies $I^{(1)_{\mathrm{Ass}}} = I$ for all ideals I, while the first definition is more easily handled and more relevant in geometric contexts. Both notions of symbolic powers can be described as saturations:

$$I^{(m)_{\mathrm{Min}}} = \bigcap_{P \in \mathrm{Min}(I)} (I^m R_P \cap R) = I^m : J^\infty \quad \text{for } J = \bigcap_{P \in \mathrm{Ass}^*(I) \setminus \mathrm{Min}(I)} P,$$

$$I^{(m)_{\mathrm{Ass}}} = \bigcap_{P \in \mathrm{Ass}(I)} (I^m R_P \cap R) = I^m : J^\infty \quad \text{for } J = \bigcap_{P \in \mathrm{Ass}^*(I) \setminus \mathrm{Ass}(I)_\subseteq} P.$$

In the above formulas, $\mathrm{Ass}^*(I)$ denotes the union of the associated prime ideals of I^n for all $n \geq 0$, while $\mathrm{Ass}(I)_\subseteq = \{P : P \subseteq P' \text{ for some } P' \in \mathrm{Ass}(I)\}$. Moreover, the intersections appearing in (2.1) and in (2.1) are finite, as $\mathrm{Ass}^*(I)$ is a finite set [6].

The symbolic powers of a symmetric shifted ideal may not be symmetric shifted.

Example 2.5. Let I be the symmetric shifted ideal with $\Lambda(I) = \{(1,2,2), (0,2,3)\}$, and let $\mathfrak{m} = (x_1, x_2, x_3)$. Note that $\mathfrak{m}$ is symmetric strongly shifted, but I is not, since $(0,2,3) \in P(I)$ but $(1,1,3) \notin P(I)$. Then, the ideal $I^{(1)_{\mathrm{Min}}} = I : \mathfrak{m} = I : \mathfrak{m}^\infty = (x_1^2x_2^2,\ x_1^2x_3^2,\ x_2^2x_3^2)$ is not symmetric shifted and nor is the second symbolic power $I^{(2)_{\mathrm{Min}}} = (x_1^2x_2^2x_3^2,\ x_1^4x_2^4,\ x_1^4x_3^4,\ x_2^4x_3^4)$ in the sense of (2).

We do not currently know of any ssi I admitting a symbolic power $I^{(m)_{\text{Ass}}}$ which is not symmetric shifted. In Proposition 2.10 below, we will prove that the class of symmetric strongly shifted ideals is closed under taking symbolic powers according to both definitions. This generalizes what was proved in [2, Theorem 4.3] for *square-free* symmetric strongly shifted ideals.

Remark 2.6. Any square-free symmetric ideal is the ideal generated by all the square-free monomials of a fixed degree $d \in \mathbb{N}$. Such an ideal is referred to in the literature as the *square-free Veronese ideal* of degree d. It can also be described as the defining ideal of a *monomial star configuration* [21], as

$$I_{n,c} = \bigcap_{1 \le i_1 < \cdots < i_c \le n} (x_{i_1}, \ldots, x_{i_c}),$$

where $c = n - d + 1$ is the height of the ideal. As the ideals $I_{n,c}$ are symmetric strongly shifted, it follows that all square-free symmetric ideals are symmetric strongly shifted. In particular, the radical of any symmetric ideal I is symmetric strongly shifted and can be expressed as

$$\sqrt{I} = \bigcap_{1 \le i_1 < \cdots < i_c \le n} (x_{i_1}, \ldots, x_{i_c}) = I_{n,c}, \quad \text{where } c = \operatorname{ht}(I).$$

A key observation which we will use in the proof of Proposition 2.10 is the fact that, unlike arbitrary symmetric shifted ideals, sssi's can be characterized combinatorially in terms of the so-called dominance order.

Definition 2.7. Partitions $\lambda, \mu \in P_n$ are compared in the *dominance order* $\lhd$ by setting

$$\mu \unlhd \lambda \text{ iff } \Sigma_k(\mu) \le \Sigma_k(\lambda) \quad \text{for all } 1 \le k \le n.$$

Remark 2.8. An $\mathfrak{S}_n$-fixed monomial ideal I is strongly shifted if and only if, for every $\lambda, \mu \in \mathrm{P}_n$ with $|\lambda| = |\mu|$, $\lambda \in P(I)$ and $\mu \unlhd \lambda$ imply $\mu \in \mathrm{P}(I)$. This is because, for λ, μ satisfying $|\lambda| = |\mu|$, the inequality $\mu \unlhd \lambda$ is equivalent to x^μ being obtained from x^λ by a sequence of Borel moves; see [14, Lemma 1.3] for a proof.

The notion of dominance order allows us to describe the saturation of a symmetric strongly shifted ideal with respect to any symmetric monomial ideal.

Proposition 2.9. *Let I be a sssi, and let c be a natural number so that $1 \le c \le n$. For $\lambda \in P_n$, define the truncated partition $\lambda_{<c} = (\lambda_1, \ldots, \lambda_{c-1}) \in P_{c-1}$. Then, we have*

1. $I : I_{n,c}^\infty = (\sigma(x^\mu) \mid \sigma \in \mathfrak{S}_n,\ \mu_{<c} = \lambda_{<c}$ *for some* $\lambda \in P(I))$.
2. $I : I_{n,c}^\infty$ *is symmetric strongly shifted.*
3. *Let J be a symmetric monomial ideal with* $\operatorname{ht}(J) = c$. *Then, the ideal $I : J^\infty$ is symmetric strongly shifted and is described as $I : J^\infty = I : I_{n,c}^\infty$.*

Proof. Denote $L = (\sigma(x^{\mu}) \mid \sigma \in \mathfrak{S}_n,\ \mu_{<c} = \lambda_{<c}$ for some $\lambda \in P(I))$.

(1) Since $I_{n,c}$ is a symmetric monomial ideal, the ideal $I : I_{n,c}^{\infty} = \bigcup_{i \geq 0} I : I_{n,c}^{i}$ is also symmetric and monomial. Thus, it suffices to show that

$$P(I : I_{n,c}^{\infty}) = P(L) = \{\mu \in P_n \mid \mu_{<c} = \lambda_{<c} \text{ for some } \lambda \in P(I)\}.$$

If $x^{\mu} \in I : I_{n,c}^{\infty}$, since $x_c x_{c+1} \cdots x_n \in I_{n,c}$, we have $x^{\mu}(x_c x_{c+1} \cdots x_n)^N = x^{\lambda}$ for some $\lambda \in P(I)$ and $N \gg 0$. This implies that $\mu_{<c} = \lambda_{<c}$ and establishes the containment $I : I_{n,c}^{\infty} \subseteq L$.

Conversely, take $\mu \in P(L)$ and $\lambda \in P(I)$ so that $\mu_{<c} = \lambda_{<c}$ and let $N \geq \lambda_n$. Then, we have $x^{\lambda} \mid x^{\mu}(x_c x_{c+1} \cdots x_n)^N$ and therefore $x^{\mu}(x_c x_{c+1} \cdots x_n)^N \in I$; thus, we conclude $\mu + (0^{c-1}, N^{n-c+1}) \in P(I)$. Now, consider an arbitrary monomial $x^{\alpha} \in I_{n,c}^{N}$. We aim to show that $x^{\mu} x^{\alpha} \in I$. Since I is symmetric strongly shifted, it suffices to prove that $\operatorname{part}(x^{\mu+\alpha}) \trianglelefteq \mu + (0^{c-1}, N^{n-c+1})$. Set $p = \operatorname{part}(x^{\mu+\alpha})$ and observe that $x^{\alpha} \in I_{n,c}^{N}$ implies $\alpha_i \leq N$ for each $1 \leq i \leq n$ so that p satisfies inequalities $p_i \leq \mu_i + N$ for each $1 \leq i \leq n$. Thus, we have

$$\sum_{i=k}^{n} p \leq \sum_{i=k}^{n} (\mu_i + N) = \sum_{i=k}^{n} \left(\mu + (0^{c-1}, N^{n-c+1})\right)_i \quad \text{for } k \geq c.$$

As for the case $k < c$, note that $|\alpha| = N(n - c + 1)$, which implies that

$$\begin{aligned} \sum_{i=k}^{n} p_i &= \sum_{i=k}^{n} \operatorname{part}(x^{\mu+\alpha})_i \leq \left(\sum_{i=k}^{n} \mu_i\right) + |\alpha| \\ &= \sum_{i=k}^{n} \left(\mu + (0^{c-1}, N^{n-c+1})\right)_i \quad \text{for } k < c. \end{aligned}$$

As discussed above, this yields $p \in P(I)$, concluding the proof of the containment $L \subseteq I : I_{n,c}^{\infty}$.

For (2), we will prove the equivalent assertion that L is a sssi. Towards this goal, we identify the minimal generators of L: these are given by the partitions

$$\Lambda(L) = \{\mu \in P_n \mid \mu_{<c} = \lambda_{<c} \text{ for some } \lambda \in \Lambda(I),\ \mu_i = \mu_{c-1} \text{ for } i \geq c\}.$$

Let $\mu \in \Lambda(L)$ and consider $1 \leq i < j \leq n$ so that $\mu_i < \mu_j$. By the description of $\Lambda(L)$, it follows that $1 \leq i < c - 1$ and there exists $\lambda \in P(I)$ with $\mu_{<c} = \lambda_{<c}$. Set $\mu' = \mu + e_i - e_j$ and $\lambda' = \lambda + e_i - e_j$. Since $\mu_{<c} = \lambda_{<c}$ and $1 \leq i < c - 1$, we deduce that $\lambda_i = \mu_i < \mu_j \leq \lambda_j$ as either $j < c$ and $\mu_j = \lambda_j$ or $j \geq c$ and thus

$$\mu_j = \mu_{c-1} = \lambda_{c-1} \leq \lambda_j.$$

Since I is symmetric strongly shifted, we have $\lambda' \in P(I)$ and $\mu'_{<c} = \lambda'_{<c}$, so we obtain $\mu' \in P(L)$, as desired.

Finally, let J be a symmetric monomial ideal as in (3). Notice that

$$I : J^\infty = I : \sqrt{J}^\infty,$$

since every ideal contains a power of its radical. Moreover, $\sqrt{J} = I_{n,c}$ by (2.6). Therefore, $I : J^\infty = I : I_{n,c}^\infty$ is symmetric strongly shifted by part (2). ■

Thanks to Proposition 2.9, we can now understand the symbolic powers of a sssi. More precisely, as symbolic powers can be calculated via saturations as in (2.1) and (2.1), we have the following result.

Proposition 2.10. *The symbolic powers* $I^{(m)_{\mathrm{Min}}}$ *and* $I^{(m)_{\mathrm{Ass}}}$ *of a symmetric strongly shifted ideal* I *are symmetric strongly shifted.*

Proof. Let I be a sssi. Observe that both ideals termed J in (2.1) and (2.1) are square-free. We claim that they are symmetric. Indeed, consider the sets $S = \mathrm{Ass}(I), \mathrm{Min}(I)$, $\mathrm{Ass}(I^m)$, $\mathrm{Ass}^*(I)$. Each of these sets is closed under the action of the symmetric group, that is, $P \in S$ if and only if $\sigma(P) \in S$ for all $\sigma \in \mathfrak{S}_n$. Thus, the set differences $\mathrm{Ass}^*(I) \setminus \mathrm{Min}(I)$ and $\mathrm{Ass}^*(I) \setminus \mathrm{Ass}(I)_{\subseteq}$ are also closed under the action of $\mathfrak{S}_n$, which implies that in both cases J is symmetric.

Since J is symmetric and square-free, Remark 2.6 yields that J is a square-free Veronese ideal. To finish the proof, it then suffices to invoke Proposition 2.9 (2). ■

From the proof of Proposition 2.10, it follows that Proposition 2.9 (3) provides a formula to calculate the symbolic powers of a symmetric strongly shifted ideal I, provided one knows the heights of the associated primes of I. We will see an instance of this in Section 5.1.

3. Partition Borel generators and combinatorial structure

The proofs of Proposition 2.9 and Proposition 2.10, together with Example 2.5, suggest that the combinatorial characterization of sssi's in terms of the dominance order in Remark 2.8 might explain several algebraic differences between sssi's and symmetric shifted ideals which are not strongly shifted.

In this section, we will show that the dominance order not only determines the algebraic structure of symmetric strongly shifted ideals, but also unveils deep analogies between this class and that of strongly stable ideals. In this context, a key notion is that of *partition Borel generators* of a sssi (see Definition 3.3). As this definition is inspired by the notion of Borel generators of a strongly stable ideal, we begin our investigation by recalling relevant background information on strongly stable ideals.

Definition 3.1. A monomial ideal $I \subset R$ is said to be a *strongly stable ideal* if, for every $\alpha \in \mathbb{N}^n$ with $x^\alpha \in I$ and for every pair $i < j$ so that $\alpha_j \neq 0$, one has $x^\alpha x_i/x_j \in I$.

A monomial ideal $I \subset R$ is said to be a *stable ideal* if, for every $\alpha \in \mathbb{N}^n$ with $x^\alpha \in I$, setting

$$\max(\alpha) = \max\{j : \alpha_j \neq 0\},$$

for every $i < \max(\alpha)$, one has $x^\alpha x_i/x_{\max(\alpha)} \in I$.

The monomials $x^\alpha(x_i/x_j)$ and $x^\alpha(x_i/x_{\max(\alpha)})$ occurring above are referred to as being obtained from x^α via a *Borel move*.

The following well-known result conveniently describes the set of monomials obtained from a monomial x^β by performing Borel moves, which we denote by $\mathrm{Borel}(\{x^\beta\})$.

Remark 3.2 ([14, Lemma 1.3]). For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, define

$$\Sigma_k(\alpha) = \alpha_k + \cdots + \alpha_n.$$

Monomials x^α and x^β with $|\alpha| = |\beta|$ satisfy $x^\alpha \in \mathrm{Borel}(\{x^\beta\})$ if and only if $\Sigma_k(\alpha) \leq \Sigma_k(\beta)$ for all $1 \leq k \leq n$. We denote this condition by $x^\alpha \prec_B x^\beta$. Note that $\prec_B$ defines a partial order on the set of n-tuples of nonnegative integers which restricts to the dominance order on P_n (see Remark 2.8).

Given a set of monomials M, the smallest strongly stable ideal which contains M is called the *Borel ideal* generated by M and is denoted by $\mathrm{Borel}(M)$. For a strongly stable ideal I, there is a unique smallest set M of monomials such that $I = \mathrm{Borel}(M)$; the elements of M are called the *Borel generators* of I.

We now introduce a symmetric analogue to the notion of Borel generators.

Definition 3.3. Let $B \subseteq P_n$ be a set of partitions and set $x^B = \{x^\lambda : \lambda \in B\}$.

The *symmetric shifted ideal generated by* B, denoted by $\mathrm{Ss}(B)$, is defined to be the smallest symmetric shifted ideal which contains x^B. Similarly, define the *symmetric strongly shifted ideal generated by* B, denoted by $\mathrm{Sss}(B)$, to be the smallest symmetric strongly shifted ideal which contains x^B.

Conversely, for a symmetric strongly shifted (resp., shifted) ideal I, we define the set of *partition Borel generators* of I, denoted by $B(I)$, to be the smallest $B \subseteq P_n$ so that $I = \mathrm{Sss}(B)$ (resp., $I = \mathrm{Ss}(B)$).

In analogy with Remark 3.2 for strongly stable ideals, the partition Borel generators of a symmetric strongly shifted ideal coincide with the maximal elements in each degree of $\Lambda(I)$ with respect to the dominance order. More precisely, if for a set $C \subseteq P_n$, $\max_{\trianglelefteq}\{C\}$ denotes the set of maximal elements of C in dominance order, we have the following characterization.

Proposition 3.4. *Let $B \subseteq P_n$ be a set of partitions, and let* $\mathrm{Borel}(x^B)$ *denote the strongly stable ideal generated by x^B. Then, the following hold.*

(1) $\mathrm{Sss}(B) = (\{\sigma(x^\mu) : \sigma \in \mathfrak{S}_n,\ \mu \in P_n,\ \exists \lambda \in B \text{ such that } |\lambda| = |\mu| \text{ and } \mu \trianglelefteq \lambda\})$ *and* $\mathrm{Sss}(B) \subset \mathrm{Borel}(x^B)$.

(2) $P(\mathrm{Sss}(B)) = \{\lambda \in P_n : x^\lambda \in \mathrm{Borel}(x^B)\}$.

(3) *Let I be a symmetric strongly shifted ideal. Then, $B(I) = \max_\trianglelefteq\{\Lambda(I)\}$; that is,*

$$B(I) = \{\lambda \in \Lambda(I) : \mu \in \Lambda(I) \text{ with } |\lambda| = |\mu|,\ \lambda \trianglelefteq \mu \text{ implies } \lambda = \mu\}$$

are the partition Borel generators of I.

In particular, for any symmetric strongly shifted ideal I, one has $I = \mathrm{Sss}(B(I))$.

Proof. (1) Denote

$$I = (\{\sigma(x^\mu) : \sigma \in \mathfrak{S}_n,\ \mu \in P_n,\ \exists \lambda \in B \text{ such that } |\lambda| = |\mu| \text{ and } \mu \trianglelefteq \lambda\}).$$

It is clear that I is symmetric and $x^B \subseteq I$. Moreover, notice that $\Lambda(I) \subseteq S$, where

$$S = \{\mu : \mu \in P_n,\ \exists \lambda \in B \text{ such that } |\lambda| = |\mu| \text{ and } \mu \trianglelefteq \lambda\} \subset I,$$

and S is closed under Borel moves. Thus, I is a sssi by Remark 2.8. Finally, I is indeed the smallest symmetric strongly shifted ideal containing x^B, since any sssi I' with $I' \supseteq x^B$ must contain x^S by Remark 2.8; hence, it must contain I by symmetry.

To show that $\mathrm{Sss}(B) \subset \mathrm{Borel}(x^B)$, let $\sigma(x^\lambda) \in \mathrm{Sss}(B)$ for some $\sigma \in \mathfrak{S}_n$, $\lambda \in P_n$. Since λ is a partition, we have

$$\Sigma_k(\sigma(\lambda)) = \sum_{i=k}^{n} \lambda_{\sigma^{-1}(i)} \le \sum_{i=k}^{n} \lambda_i = \Sigma_k(\lambda),$$

which means that $\sigma(x^\lambda) \prec_B x^\lambda$ by Remark 3.2. Furthermore, since $x^\lambda \in \mathrm{Sss}(B)$, Remark 2.8 together with Remark 3.2 implies that $\sigma(x^\lambda) \in \mathrm{Borel}(\{x^\lambda\}) \subseteq \mathrm{Borel}(x^B)$ for each $\sigma \in \mathfrak{S}_n$.

(2) Part (1) yields the containment $P(\mathrm{Sss}(B)) \subseteq \{\lambda \in P_n : x^\lambda \in \mathrm{Borel}(x^B)\}$.

For the opposite containment, let $\lambda \in P_n$ with $x^\lambda \in \mathrm{Borel}(x^B)$. Then, there exists a minimal generator of $\mathrm{Borel}(x^B)$, x^α so that $x^\alpha | x^\lambda$ and $x^\alpha \prec_B x^\beta$ for some $\beta \in B$ with $|\alpha| = |\beta|$. This yields $\lambda_i \ge \alpha_i$ for all $1 \le i \le n$ and $\sum_{i=k}^n \alpha_i \le \sum_{i=k}^n \beta_i$ for all $1 \le k \le n$. Therefore,

$$\sum_{i=1}^{k} \lambda_i \ge \sum_{i=1}^{k} \alpha_i \ge \sum_{i=1}^{k} \beta_i \quad \text{for each } 1 \le k \le n.$$

If $|\lambda| = |\alpha|$, it follows that $\lambda = \alpha$, which yields $\lambda \trianglelefteq \beta$. Thus, $x^\lambda \in \mathrm{Sss}(B)$, i.e., $\lambda \in P(\mathrm{Sss}(B))$.

If instead $|\lambda| > |\alpha| = |\beta|$, set $q = \max\{k : \sum_{i=1}^k \lambda_i \le |\beta|\}$ and $t = |\beta| - \sum_{i=1}^q \lambda_i$. By the assumption, we have $q < n$. Moreover, since $\sum_{i=1}^{q+1} \lambda_i > |\beta|$, we deduce that $\lambda_{q+1} > t$. Now, let

$$\gamma = (\beta_1, \dots, \beta_q, \beta_{q+1} + \lambda_{q+1} - t, \beta_{q+2} + \lambda_{q+2}, \dots, \beta_n + \lambda_n).$$

Note that γ is a partition since $\beta, \lambda \in P_n$ and since $\lambda_{q+1} - t > 0$. Moreover, observe that $x^\beta \mid x^\gamma$, which yields $x^\gamma \in \mathrm{Sss}(B)$ since $x^\beta \in \mathrm{Sss}(B)$. The sum of entries of γ is

$$|\gamma| = \sum_{i=1}^n \beta_i + \sum_{i=q+1}^n \lambda_i - t = |\beta| + \sum_{i=q+1}^n \lambda_i - |\beta| + \sum_{i=1}^q \lambda_i = |\lambda|.$$

We claim that $\lambda \trianglelefteq \gamma$. To see this, compute

$$\sum_{i=k}^n \gamma_i = \begin{cases} \sum_{i=k}^n \beta_i + \sum_{i=k}^n \lambda_i & \text{if } k > q+1, \\ \sum_{i=k}^n \beta_i + \sum_{i=q+1}^n \lambda_i - t & \text{if } k \le q+1. \end{cases}$$

If $k > q+1$, it is clear that $\sum_{i=k}^n \gamma_i \ge \sum_{i=k}^n \lambda_i$. If $k \le q+1$, then inequality (3) yields

$$\sum_{i=k}^n \gamma_i = \sum_{i=k}^n \beta_i + \sum_{i=q+1}^n \lambda_i - |\beta| + \sum_{i=1}^q \lambda_i = |\lambda| - \sum_{i=1}^{k-1} \beta_i \ge |\lambda| - \sum_{i=1}^{k-1} \lambda_i = \sum_{i=k}^n \lambda_i.$$

Since we have shown $\lambda \trianglelefteq \gamma$ and $x^\gamma \in \mathrm{Sss}(B)$, we deduce that $x^\lambda \in \mathrm{Sss}(B)$ by Remark 2.8 and thus $\lambda \in P(\mathrm{Sss}(B))$, as desired.

(3) Set $B = \max_{\trianglelefteq}\{\Lambda(I)\}$. Since $B \subseteq \Lambda(I)$, it is clear that $I \supseteq x^B$, whence $I \supseteq \mathrm{Sss}(B)$ as I is a sssi. Moreover, by the definition of B, for each $\mu \in \Lambda(I)$, we have that $\mu \trianglelefteq \lambda$ for some $\lambda \in B(I)$; hence, $I \subseteq \mathrm{Sss}(B)$ by Remark 2.8, so equality holds. That B is the smallest set of Borel generators of I follows by noting that $\lambda \in B(I)$ and $B' \subset B \setminus \{\lambda\}$ yields $\mathrm{Sss}(B') \subset I \setminus \{x^\lambda\}$. Thus, we conclude that $B = B(I)$. ∎

We emphasize that the conclusion of Proposition 3.4 (1) does not hold if one replaces $\mathrm{Sss}(B)$ with $\mathrm{Ss}(B)$ and $\mathrm{Borel}(x^B)$ with the smallest stable ideal containing x^B, denoted by $\mathrm{St}(x^B)$.

Example 3.5. Let $B = \{(0, 0, 1, 1)\}$, $J = \mathrm{St}(x^B) = (x_3x_4, x_3^2, x_2x_3, x_1x_3)$, and let $I = \mathrm{Ss}(\{\lambda \in P_n : x^\lambda \in J\})$. Observe that $I \not\subseteq J$, since $x_1x_2 \in I \setminus J$.

Building upon the relationship between $\mathrm{Sss}(B)$ and $\mathrm{Borel}(x^B)$ highlighted in Proposition 3.4, the next theorem shows that an ideal I is a sssi if and only if it can be obtained from a strongly stable ideal J by *symmetrization*, i.e., $I = \bigcap_{\sigma \in \mathfrak{S}_n} \sigma(J)$.

Theorem 3.6. (1) *Let J be a strongly stable ideal. Then, the ideal*

$$I = \bigcap_{\sigma \in \mathfrak{S}_n} \sigma(J)$$

is symmetric strongly shifted, with $P(I) = \{\lambda \in P_n : x^\lambda \in J\}$.

(2) *Conversely, every symmetric strongly shifted ideal I has the form in (3.6) for some strongly stable ideal J. In detail, let $B = B(I) \subset P_n$. Then, the ideal $J =$* Borel(x^B) *satisfies*

$$I = \bigcap_{\sigma \in \mathfrak{S}_n} \sigma(J).$$

Moreover, J is the smallest strongly stable ideal with this property.

Proof. (1) It is clear that I is symmetric and that $P(I) \subseteq \{\lambda \in P_n : x^\lambda \in J\}$. Moreover, since J is strongly stable, it follows from Proposition 3.4 that, for each $\sigma \in \mathfrak{S}_n$, $\sigma(x^\lambda) \in \mathrm{Borel}(\{x^\lambda\}) \subseteq J$ and consequently $x^\lambda \in I$ whenever $\lambda \in P_n$ and $x^\lambda \in J$. Therefore, we have $\{\lambda \in P_n : x^\lambda \in J\} \subseteq P(I)$ and thus equality holds. Since J is closed under Borel moves, it then follows that $P(I)$ is also closed under Borel moves; i.e., I is a sssi.

(2) From Proposition 3.4 (1), we know that $\mathrm{Sss}(B) \subseteq \mathrm{Borel}(x^B)$. Thus, for every $\sigma \in \mathfrak{S}_n$, we obtain that $\mathrm{Sss}(B) = \sigma(\mathrm{Sss}(B)) \subseteq \sigma(\mathrm{Borel}(x^B))$. Therefore,

$$\mathrm{Sss}(B) \subseteq \bigcap_{\sigma \in \mathfrak{S}_n} \sigma(\mathrm{Borel}(x^B)).$$

For the opposite containment, notice that $x^\alpha \in \bigcap_{\sigma \in \mathfrak{S}_n} \sigma(\mathrm{Borel}(x^B))$ implies that $\sigma(x^\alpha) \in \mathrm{Borel}(x^B)$ for all $\sigma \in \mathfrak{S}_n$ and in particular that $x^{\mathrm{part}(x^\alpha)} \in \mathrm{Borel}(x^B)$. Setting $\lambda = \mathrm{part}(x^\alpha)$, we deduce by Proposition 3.4 (2) that $\lambda \in P(\mathrm{Sss}(B))$. It follows by the $\mathfrak{S}_n$-invariance of $\mathrm{Sss}(B)$ that also $x^\alpha \in \mathrm{Sss}(B)$, thus finishing the proof. ∎

In light of Theorem 3.6, it is natural to expect that algebraic properties which are well behaved under taking intersections are preserved under symmetrization. Proposition 4.3 and Theorem 5.6 will present instances when this is indeed the case.

Analogously to [19, Propositions 2.15, 2.16, and 2.17] for strongly stable ideals, our next goal is to describe sums, intersections, and products of symmetric strongly shifted ideals in terms of partition Borel generators. To do so, we must consider the lattice structure of P_n.

The following lemma is inspired by a similar description in [8] for the lattice structure on the set of partitions of *fixed sum* $P(d) = \{\lambda \text{ is a partition and } |\lambda| = d\}$. We include the proof here for lack of a specific reference which treats the case of P_n.

Lemma 3.7. *For every n, the set P_n forms a lattice with respect to the dominance order.*

Proof. Notice that there is a one-to-one correspondence between partitions

$$\lambda = (\lambda_1, \dots, \lambda_n) \in P_n$$

and nondecreasing sequences $\widehat{\lambda} \in \mathbb{N}^n$ so that $\widehat{\lambda}_k + \widehat{\lambda}_{k-2} \leq 2\widehat{\lambda}_{k-1}$ for every k. Indeed, given $\lambda = (\lambda_1, \dots, \lambda_n) \in P_n$, one defines $\widehat{\lambda} = (\sum_n \lambda, \sum_{n-1} \lambda, \dots, \sum_1 \lambda)$, where for every $1 \leq k \leq n$, we denote

$$\sum_k \lambda = \lambda_k + \cdots + \lambda_n.$$

Conversely, given a vector $\widehat{\lambda} \in \mathbb{N}^n$, one defines a partition $\lambda = (\lambda_1, \dots, \lambda_n) \in P_n$ by setting $\lambda_n = \widehat{\lambda}_1$ and $\lambda_k = \widehat{\lambda}_{n-k+1} - \widehat{\lambda}_{n-k}$ for $1 \leq k \leq n-1$. Under this correspondence, one has that $\eta \trianglelefteq \lambda$ if and only if $\widehat{\eta}_k \leq \widehat{\lambda}_k$ for each k.

Now, let $\lambda, \mu \in P_n$. We need to prove that λ and μ have a meet and a join in the dominance order. For each $1 \leq k \leq n$, let $v_k = \min\{\widehat{\lambda}_k, \widehat{\mu}_k\}$ and let $v = (v_1, \dots, v_n)$. Observe that, for every k, $v_k + v_{k-2} \leq 2v_{k-1}$, so there exists a partition $\eta \in P_n$ corresponding to v under the identification above. Moreover, $v = \widehat{\lambda} \wedge \widehat{\mu}$ with respect to the componentwise order; hence, $\eta = \lambda \wedge \mu$ with respect to the dominance order. To prove that λ and μ have a join, recall that for partitions $\lambda, \eta \in P_n$ one has that $\lambda \trianglelefteq \eta$ if and only if $\eta^T \trianglelefteq \lambda^T$, where η^T and λ^T denote the partitions corresponding to the transpose of the Young diagrams of η and λ, respectively. Therefore, the join of λ and μ is $\lambda \vee \mu = (\lambda^T \wedge \mu^T)^T$. ∎

Proposition 3.8. *Let $A, B \subseteq P_n$. Then,*

(1) $\mathrm{Sss}(A) + \mathrm{Sss}(B) = \mathrm{Sss}(A \cup B)$ *and*

(2) $\mathrm{Sss}(A) \cap \mathrm{Sss}(B) = \mathrm{Sss}(A \wedge B)$*, where* $A \wedge B = \{\lambda \wedge \mu : \lambda \in A, \mu \in B\}$ *and* $\lambda \wedge \mu$ *denotes the meet of* λ *and* μ *in the dominance order.*

For any two symmetric strongly shifted ideals I and J, one has

$$B(I + J) = \max_{\trianglelefteq}\{B(I) \cup B(J)\} \quad \textit{and} \quad B(I \cap J) = \max_{\trianglelefteq}\{B(I) \wedge B(J)\}.$$

Proof. For (1), it is clear that $x^\lambda \in \mathrm{Sss}(A) + \mathrm{Sss}(B)$ whenever $\lambda \in A \cup B$. Hence, since sssi's are closed under sums by Proposition 2.1, we deduce that $\mathrm{Sss}(A \cup B) \subset \mathrm{Sss}(A) + \mathrm{Sss}(B)$. Conversely, let $p \in P_n$ be so that $x^p \in \mathrm{Sss}(A) + \mathrm{Sss}(B)$. Since x^p is a monomial, we must then have $x^p \in \mathrm{Sss}(A)$ or $x^p \in \mathrm{Sss}(B)$. In either case, there exists a partition $\lambda \in A \cup B$ with $p \trianglelefteq \lambda$. Thus, $x^p \in \mathrm{Sss}(A \cup B)$.

For (2), notice that $x^{\lambda \wedge \mu} \in \mathrm{Sss}(A) \cap \mathrm{Sss}(B)$ whenever $\lambda \in A$ and $\mu \in B$, since $\lambda \wedge \mu \trianglelefteq \lambda$, $\lambda \wedge \mu \trianglelefteq \mu$ and $\mathrm{Sss}(A) \cap \mathrm{Sss}(B)$ is symmetric strongly shifted by Proposition 2.1. Therefore, $\mathrm{Sss}(A \wedge B) \subseteq \mathrm{Sss}(A) \cap \mathrm{Sss}(B)$. Conversely, let $p \in P_n$ be so that $x^p \in \mathrm{Sss}(A) \cap \mathrm{Sss}(B)$. Then, there exist partitions $\lambda \in A, \mu \in B$ so that $p \trianglelefteq \lambda$

and $p \trianglelefteq \mu$. Since λ and μ have a meet by Lemma 3.7, it then follows that $p \trianglelefteq \lambda \wedge \mu$, so $x^p \in \operatorname{Sss}(A \wedge B)$.

The remaining statement follows from the other two, since for any sssi's I and J

$$\operatorname{Sss}(B(I) \cup B(J)) = \operatorname{Sss}(B(I)) + \operatorname{Sss}(B(J)) = I + J$$

and

$$\operatorname{Sss}(B(I) \wedge B(J)) = \operatorname{Sss}(B(I)) \cap \operatorname{Sss}(B(J)) = I \cap J.$$

This implies that

$$\max_{\trianglelefteq}\{B(I) \cup B(J)\} = \max_{\trianglelefteq}\{\Lambda(I + J)\} = B(I + J)$$

and

$$\max_{\trianglelefteq}\{B(I) \wedge B(J)\} = \max_{\trianglelefteq}\{\Lambda(I \cap J)\} = B(I \cap J),$$

which completes the proof. ■

Proposition 3.9. *For $A, B, C \subseteq P_n$, set $A + B = \{\lambda + \mu : \lambda \in A, \mu \in B\}$. Then, we have*

(1) $\operatorname{Sss}(A) \cdot \operatorname{Sss}(B) = \operatorname{Sss}(A + B)$,

(2) $B(IJ) = \max_{\trianglelefteq}\{B(I) + B(J)\}$ *for any symmetric strongly shifted ideals I, J.*

Proof. For (1), it is clear that $x^{\lambda+\mu} \in \operatorname{Sss}(A) \cdot \operatorname{Sss}(B)$ whenever $\lambda \in A$ and $\mu \in B$. Therefore, using the fact that $\operatorname{Sss}(A) \cdot \operatorname{Sss}(B)$ is symmetric strongly shifted by Proposition 2.2, we deduce that $\operatorname{Sss}(A + B) \subseteq \operatorname{Sss}(A) \cdot \operatorname{Sss}(B)$. Conversely, let $p \in P_n$ be so that x^p is a monomial generator of $\operatorname{Sss}(A) \cdot \operatorname{Sss}(B)$. Then, by Proposition 3.4 (1), there exist partitions $\lambda \in A$, $\mu \in B$, and $\lambda', \mu' \in P_n$ and permutations $\sigma, \tau \in \mathfrak{S}_n$ so that $\lambda' \trianglelefteq \lambda$, $\mu' \trianglelefteq \mu$, and $p = \sigma(\lambda') + \tau(\mu')$. Since λ', μ' are ordered increasingly, for all k one has that

$$\begin{aligned} p_k + \cdots + p_n &= \lambda'_{\sigma^{-1}(k)} + \cdots + \lambda'_{\sigma^{-1}(n)} + \mu'_{\tau^{-1}(k)} + \cdots + \mu'_{\tau^{-1}(n)} \\ &\leq \lambda'_k + \cdots + \lambda'_n + \mu'_k + \cdots + \mu'_n \\ &\leq \lambda_k + \cdots + \lambda_n + \mu_k + \cdots + \mu_n, \end{aligned}$$

where the last inequality follows from the fact that $\lambda' \trianglelefteq \lambda$ and $\mu' \trianglelefteq \mu$. Thus, $p \trianglelefteq \lambda + \mu$, whence $x^p \in \operatorname{Sss}(A + B)$, which completes the proof.

For (2), notice that part (1) and the fact that I, J are strongly shifted imply the identity

$$\operatorname{Sss}(B(I) + B(J)) = \operatorname{Sss}(B(I)) \cdot \operatorname{Sss}(B(J)) = IJ.$$

This ensures that $\max_{\trianglelefteq}\{B(I) + B(J)\} = \max_{\trianglelefteq}\{\Lambda(IJ)\} = B(IJ)$, as desired. ■

3.1. Principal Borel sssi's and discrete polymatroids

Strongly stable ideals with one Borel generator are called *principal Borel ideals*. They play a key role in the study of strongly stable ideals, due to their rich combinatorial structure and to the fact that every strongly stable ideal is a sum of principal Borel ideals. Analogously, we introduce the following notion.

Definition 3.10. A *principal Borel symmetric shifted ideal* is any ideal of the form $\mathrm{Sss}(\{\lambda\})$ for some $\lambda \in P_n$, i.e., an ideal whose set of partition Borel generators is a singleton.

Note that a principal Borel sssi is necessarily *equigenerated*; that is, all its minimal generators have the same degree.

Example 3.11. Examples of principal Borel sssi's include the following.

- Powers of the maximal ideal $(x_1, \dots, x_n)^d = \mathrm{Sss}(\{(0^{n-1}, d)\})$.
- The square-free Veronese ideal of degree $n - c + 1$ in n variables (see Remark 2.6)
$$I_{n,c} = \mathrm{Sss}(\{(0^{c-1}, 1^{n-c+1})\}).$$
- Powers $I_{n,c}^m = \mathrm{Sss}(\{(0^{c-1}, m^{n-c+1})\})$ of a square-free Veronese ideal. The formula for the unique partition Borel generator of $I_{n,c}^m$ follows from Proposition 3.9.

It is clear from Proposition 3.8 (1) that every sssi is a sum of principal Borel sssi's and that the class of principal Borel sssi's is not closed under sums. However, it is closed under intersections, products, and powers, thanks to Proposition 3.8 (2) and Proposition 3.9 (1).

A remarkable consequence of Proposition 3.9 is that a principal Borel sssi decomposes as a product of square-free Veronese ideals. Recall that the transpose of λ, denoted by λ^T, is defined as the partition corresponding to the transpose of the Young diagram of λ so that the parts of λ^T record the number of boxes in each row of the Young diagram of λ. More precisely, $\lambda^T \in P_{\lambda_n}$ is defined by

$$\lambda_i^T = |\{j : \lambda_j \geq \lambda_n - i + 1\}| \leq n.$$

Theorem 3.12. *Let $\lambda \in P_n$ be a partition and set $\lambda_0 = 0$. The principal Borel sssi $\mathrm{Sss}(\{\lambda\})$ decomposes as*

$$\mathrm{Sss}(\{\lambda\}) = \prod_{i=1}^{n} I_{n,i}^{\lambda_i - \lambda_{i-1}}.$$

Proof. By the definition of λ^T, λ can be decomposed as

$$\lambda = \sum_{j=1}^{\lambda_n} (0^{n-\lambda_j^T}, 1^{\lambda_j^T}).$$

Therefore, by Proposition 3.9 (1), it follows that

$$\mathrm{Sss}(\{\lambda\}) = \prod_{j=1}^{\lambda_n} \mathrm{Sss}(\{(0^{n-\lambda_j^T}, 1^{\lambda_j^T})\}) = \prod_{j=1}^{\lambda_n} I_{n,n-\lambda_j^T+1}.$$

It remains to observe that, by definition of λ^T, the number of parts of λ^T of size $n-i+1$, that is, the number of rows for the Young diagram of λ which contain exactly $n-i+1$ boxes, is $\lambda_i - \lambda_{i-1}$. Thus, combining the repeated factors of the previous identity yields the claim. ∎

This factorization property will allow us to give another combinatorial characterization of principal Borel sssi's in Theorem 3.14 below. To state this result, we need to recall the following definition, which is due to Herzog and Hibi [26, Definition 2.1 and Remark 6.4].

Definition 3.13. An equigenerated monomial ideal I is called a *polymatroidal ideal* if any two monomial generators $x_1^{u_1}\cdots x_n^{u_n}$ and $x_1^{v_1}\cdots x_n^{v_n}$ of I satisfy the following exchange property:

For every i so that $u_i > v_i$ there exists a j so that $u_j < v_j$ and $(x_1^{u_1}\cdots x_n^{u_n})x_j/x_i \in I$.

The terminology refers to the fact that the exponent vectors $(u_1,\dots,u_n)\in\mathbb{Z}^n$ of the monomials $x_1^{u_1}\cdots x_n^{u_n}$ generating a polymatroidal ideal form a set of bases of a *discrete polymatroid*. The index j in Definition 3.13 can be chosen so that also $(x_1^{v_1}\cdots x_n^{v_n})x_i/x_j \in I$ [26, Theorem 4.1]. This is referred to in the literature as the *symmetric exchange property* and is conjectured to determine the algebraic structure of the toric ring of a polymatroidal ideal [26, 51] (see Conjecture 6.6 and our discussion therein).

Examples of polymatroidal ideals include powers of square-free Veronese ideals and principal Borel ideals by [26, Examples 2.6 (c) and 9.4]. Since the former are principal Borel sssi's and the latter become principal Borel sssi's after symmetrization in the sense of Theorem 3.6, it is then natural to ask whether *every* principal Borel sssi is polymatroidal. The following theorem shows that this is indeed the case.

Theorem 3.14. *A principal Borel sssi is a polymatroidal ideal. In fact, a symmetric monomial ideal is polymatroidal if and only if it is a principal Borel sssi.*

Proof. That every principal Borel sssi is a polymatroidal ideal follows from Theorem 3.12, since square-free Veronese ideals are polymatroidal and products of polymatroidal ideals are polymatroidal by [11, Theorem 5.3].

To prove the converse, we first show that every symmetric polymatroidal ideal is a sssi. Let I be a symmetric polymatroidal ideal. Let $\lambda \in \Lambda(I)$ be a partition with

$\lambda_j < \lambda_i$, and let $\sigma = (ij) \in \mathfrak{S}_n$. Then, as I is symmetric, the monomials $f = x^\lambda$ and $g = \sigma(f)$ are in I. For convenience, in the following, we denote $\deg_i(h)$ the exponent of x_i in a monomial h. By assumption, we have $\lambda_i = \deg_i(f) > \deg_i(g) = \lambda_j$ and j is the only index for which $\lambda_j = \deg_j(f) < \deg_j(g) = \lambda_i$. Since I is polymatroidal, the exchange property then yields that $f x_j/x_i \in I$, so I is symmetric strongly shifted.

We next prove that every polymatroidal sssi must be a principal Borel sssi. Let I be a polymatroidal sssi and suppose that there exist distinct $\lambda, \mu \in B(I)$. After possibly switching the names of λ and μ, we may assume that $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2, \ldots, \lambda_{i-1} = \mu_{i-1}$, and $\lambda_i < \mu_i$. Since I is polymatroidal, there exists an index j so that $\lambda_j > \mu_j$ and $x^\mu x_j/x_i \in I$. Since $\lambda_k = \mu_k$ for $k < i$, and $\lambda_i < \mu_i$, j must satisfy $i < j$. Setting $\mu' := \operatorname{part}(x^{\mu - e_i + e_j})$, it follows that $\mu' \in P(I)$. Note that $\mu' = \mu - e_{i'} + e_{j'}$, where if $\mu_i < \mu_j$, then $i' = \max\{k : \mu_k = \mu_i\}$ and $j' = \min\{k : \mu_k = \mu_j\}$ and if $\mu_i = \mu_j$, then $i' = \min\{k : \mu_k = \mu_j\}$ and $j' = \max\{k : \mu_k = \mu_j\}$ and in both cases $i \le i' < j' \le j$. Hence, $x^\mu = x^{\mu'} x_{i'}/x_{j'}$ is obtained from $x^{\mu'}$ via a Borel move and thus $\mu' \in \Lambda(I)$ and $\mu \trianglelefteq \mu'$. But this is a contradiction, since $\mu \in B(I)$ is a maximal element of $\Lambda(I)$ with respect to dominance by Proposition 3.4 (3). ■

A polymatroidal ideal I is said to satisfy the *strong exchange property* if, for any two distinct monomial generators $x_1^{u_1} \cdots x_n^{u_n}$ and $x_1^{v_1} \cdots x_n^{v_n}$ of I and all indices i and j so that $u_i > v_i$ and $u_j < v_j$, then $(x_1^{u_1} \cdots x_n^{u_n}) x_j/x_i$ is in I (see [26, Definition 2.5]).

Notice that square-free Veronese ideals and their ordinary powers satisfy this property. However, this is not true for arbitrary principal Borel sssi's.

Proposition 3.15. *Let $I = \operatorname{Sss}(\{\lambda\})$ be a principal Borel symmetric strongly shifted ideal. Then, I satisfies the strong exchange property if and only if λ is of one of the following types:*

(1) $\lambda = (a, \ldots, a)$ *for some* $a \neq 0 \in \mathbb{N}$;

(2) $\lambda = (a^s, b^{n-s})$ *for some* $a < b \in \mathbb{N}$, $s > 0$;

(3) $\lambda = (a^s, b, c^{n-s-1})$ *for some* $a < b < c \in \mathbb{N}$, $s > 0$.

Proof. Notice that I satisfies the strong exchange property trivially if (1) holds. Let λ be as in (2), and let $u = x_1^{u_1} \cdots x_n^{u_n}$, $v = x_1^{v_1} \cdots x_n^{v_n}$ be distinct monomial minimal generators of I. Then, if $u_i > v_i$ and $u_j < v_j$, it must be that $u_j = v_i = a$ and $u_i = v_j = b$. Hence, for $u' = u x_j/x_i$, we have

$$\operatorname{part}(u') = \begin{cases} (a^{s-1}, a+1, b-1, b^{n-s-1}) \trianglelefteq \lambda & \text{if } b > a+1, \\ (a^{s-1}, b-1, a+1, b^{n-s-1}) = \lambda & \text{if } b = a+1. \end{cases}$$

Thus, in any case, $\operatorname{part}(u') \in P(I)$, whence $u' \in I$, so I satisfies the strong exchange property. Similarly, if (3) holds, exchanging variables x_i, x_j appearing with distinct exponents among distinct monomial generators u, v of I produces $u' = ux_j/x_i$, where either $\mathrm{part}(u') = \lambda$, or $\mathrm{part}(u') = (a^{s-1}, a+1, b-1, c^{n-s-1})$, or $\mathrm{part}(u') = (a^s, b+1, c-1, c^{n-s-2})$, or

$$\mathrm{part}(u') = (a^{s-1}, a+1, b, c-1, c^{n-s-2}).$$

Since $\mathrm{part}(u') \in P(I)$ in each of these cases, the strong exchange property is satisfied.

We next show that in all other cases the strong exchange property does not hold. Assume first that $\lambda = (a^s, b^t, c^q, d^r, \ldots)$ has at least four distinct parts $a < b < c < d$. Then, there exist monomial generators of I of the form $u = x_1^a x_2^b x_3^c x_4^d \cdots$ and $v = x_1^c x_2^a x_3^d x_4^b \cdots$, respectively. Hence, $u' = ux_3/x_2$ is such that

$$\mathrm{part}(u') = (a^s, b-1, b^{t-1}, c^q, c+1, d^r, \ldots).$$

Since $\lambda \trianglelefteq \mathrm{part}(u')$, $\lambda \neq \mathrm{part}(u')$ and $I = \mathrm{Sss}(\{\lambda\})$, we deduce $\mathrm{part}(u') \notin P(I)$, whence $u' \notin I$. Finally, suppose that $\lambda = (a^s, b^t, c^q)$ with $a < b < c, s, q \geq 1, t \geq 2$ and consider minimal generators for I of the form $u = x_1^a x_2^b x_3^b x_4^c, \ldots, v = x_1^b x_2^a x_3^c x_4^b \cdots$. Then, $u' = ux_3/x_2$ is such that $\mathrm{part}(u') = (a^s, b-1, b^{t-2}, b+1, c^q, \ldots)$. Since $\lambda \trianglelefteq \mathrm{part}(u')$, it follows that $\mathrm{part}(u') \notin P(I)$, which completes the proof. ∎

The factorization of principal Borel sssi's given in Theorem 3.12 parallels the known factorization of principal Borel ideals as products of monomial prime ideals; see, e.g., [20, Propositions 2.7]. Polymatroidal ideals endowed with such a factorization property are called transversal polymatroidal ideals. In detail, a *transversal polymatroidal ideal* is an ideal which can be written as a product of monomial ideals generated by subsets of the variables $x_1, \ldots, x_n$, with repeated factors allowed. The following proposition shows that principal Borel sssi's need not be transversal. In its statement, for $\lambda \in P_n$, we define the discrete difference vectors $\Delta^i \lambda$ inductively by $\Delta^0 \lambda = \lambda$, $(\Delta\lambda)_j = \lambda_{j+1} - \lambda_j$, and $\Delta^i \lambda = \Delta(\Delta^{i-1}\lambda)$.

Proposition 3.16. *The following are equivalent:*

(1) $I = \mathrm{Sss}(\{\lambda\})$ *with* $\lambda \in P_n$ *is a transversal polymatroidal ideal;*

(2) *there exist integers* $a_j \geq 0$ *such that* $\lambda_i = \sum_{j=1}^{i} \binom{i-1}{j-1} a_j$ *for* $1 \leq i \leq n$*;*

(3) *for some (equivalently, for each)* $1 \leq i \leq n-1$ *we have* $\Delta^i \lambda \in P_{n-i}$ *and* $\mathrm{Sss}(\{\Delta^i \lambda\})$ *is transversal polymatroidal;*

(4) $(\Delta^i \lambda)_1 \geq 0$ *for all* $0 \leq i \leq n-1$.

Proof. Notice that a symmetric transversal polymatroidal ideal in n variables is of the form

$$J = \left(\prod_{i=1}^{n} (x_i)^{a_1}\right)\Big(\prod_{1 \leq i < j \leq n} (x_i, x_j)^{a_2}\Big)\Big(\prod_{1 \leq i < j < k \leq n} (x_i, x_j, x_k)^{a_3}\Big) \cdots (x_1, \ldots, x_n)^{a_n}$$

for some integers $a_j \geq 0$. Indeed, if $(x_{i_1}, x_{i_2}, \ldots, x_{i_c})^{a_c}$ is a factor in the product decomposition of J, then for each $\sigma \in \mathfrak{S}_n$ we have $(x_{\sigma(i_1)}, x_{\sigma(i_2)}, \ldots, x_{\sigma(i_c)})^{a_c}$ as a factor in a product decomposition of

$$\sigma(J) = J.$$

Conversely, since the decomposition of J as a product of powers of monomial prime ideals is unique [29, Lemma 4.1], $(x_{i_1}, x_{i_2}, \ldots, x_{i_c})^{a_c}$ must appear in the factorization of J whenever $(x_{\sigma(i_1)}, x_{\sigma(i_2)}, \ldots, x_{\sigma(i_c)})^{a_c}$ does.

To prove the equivalence of (1) and (2), note that if $I = \mathrm{Sss}(\{\lambda\})$, then λ is the largest monomial in $G(I)$ with respect to the monomial order antilex defined by $\alpha <_{\text{antilex}} \beta$ if the leftmost non-zero entry of $\alpha - \beta$ is positive; this is because this order refines the dominance order on partitions. Thus, in order to establish whether $I = J$ for some J as described above, we must identify the largest monomial u in $G(J)$ with respect to antilex. We claim that this monomial u is obtained as follows:

$$u = \left(\prod_{i=1}^{n} x_i^{a_1}\right)\Bigg(\prod_{1 \leq i < j \leq n} x_j^{a_2}\Bigg)\Bigg(\prod_{1 \leq i < j < k \leq n} x_k^{a_3}\Bigg) \cdots x_n^{a_n} = \prod_{i=1}^{n} x_i^{\sum_{j=1}^{i} \binom{i-1}{j-1} a_j}.$$

Indeed, if $i_1 < i_2 < \cdots < i_c$, then the largest monomial in $(x_{i_1}, x_{i_2}, \ldots, x_{i_c})^{a_c}$ with respect to antilex is $x_{i_c}^{a_c}$ and since this is a monomial order, hence compatible with products, it follows that u is indeed the antilex-largest monomial of $G(J)$. The second equality in (3.1) holds as for fixed c and i_c there are $\binom{i-1}{j-1}$ ideals $(x_{i_1}, x_{i_2}, \ldots, x_{i_c})$ with $1 \leq i_1 < i_2 < \cdots < i_c \leq n$. Thus, $\lambda = \mathrm{part}(u)$ yields the desired description of the λ_i in statement (2). Conversely, if the λ_i can be described as in statement (2), then J is a symmetric polymatroidal ideal and thus $J = \mathrm{Sss}(\mathrm{part}(u))$ by Theorem 3.14. Since $\mathrm{part}(u) = \lambda$, we have $I = J$, and so, I is transversal.

We next observe that the system of equations $\lambda_i = \sum_{j=1}^{i} \binom{i-1}{j-1} a_j$ for $1 \leq i \leq n$ is equivalent to the system $\lambda_{i+1} - \lambda_i = \sum_{j=2}^{i-1} \binom{i-1}{j-2} a_j$ for $1 \leq i \leq n$ in the unknowns $a_2, \ldots, a_n$, together with the equation $\lambda_1 = a_1$.

If the latter system admits nonnegative solutions, then $\Delta\lambda$ is a partition, since the functions $\binom{i-1}{j-2}$ are nondecreasing in the argument i. Moreover, this system admits nonnegative solutions if and only if $\mathrm{Sss}(\{\Delta\lambda\})$ is transversal polymatroidal, by the equivalence of (1) and (2). Hence, if (2) holds, it follows by induction on i that, for each $1 \leq i \leq n-1$, $\Delta^i\lambda \in P_{n-i}$ and $\mathrm{Sss}(\{\Delta^i\lambda\})$ is transversal polymatroidal. Thus, (2) implies the stronger form of (3) and hence also the weaker form.

The equivalence of the above systems of equations also shows that if $\Delta\lambda \in P_{n-1}$ and $\mathrm{Sss}(\{\Delta\lambda\})$ is transversal polymatroidal, then $\mathrm{Sss}(\{\lambda\})$ is transversal polymatroidal. Thus, it follows by induction on i that if for some $1 \leq i \leq n$ we have $\Delta^i\lambda \in P_{n-i}$ and $\mathrm{Sss}(\{\Delta^i\lambda\})$ is transversal polymatroidal, then $\mathrm{Sss}(\{\lambda\})$ is transversal polymatroidal. Thus, the weaker (hence also the stronger) form of (3) implies (1).

Since (3) clearly implies (4), we end by showing that (4) implies (2). Note that the solution to the system of identities listed in (2) is $a_i = (\Delta^i \lambda)_1 \geq 0$. This can be seen by induction on i, using the fact that taking the difference of consecutive identities replaces the system in (2) for λ with the similar system for $\Delta\lambda$ and the equation $a_i = \lambda_1$, as noted above. ■

Remark 3.17. Principal Borel ideals can also be regarded as *lattice path polymatroidal ideals*, i.e., polymatroidal ideals whose minimal generators correspond to certain planar lattice paths [45]. On the other hand, principal Borel sssi's are almost never lattice path polymatroidal. Indeed, using Proposition 3.16, one can prove that a lattice path polymatroidal ideal I is symmetric if and only if either $I = \mathrm{Sss}(\{(a, \dots, a)\})$ for some $a \in \mathbb{N}$, or I is a power of the homogeneous maximal ideal.

Proposition 3.15, Proposition 3.16, and Remark 3.17 suggest that the combinatorial properties of principal Borel sssi's might be very different from those of square-free Veronese ideals or principal Borel ideals. In fact, one can easily construct principal Borel sssi's which simultaneously fail to satisfy the strong exchange property and are not transversal.

Example 3.18. Let $I = \mathrm{Sss}(\{(1, 3, 4, 5)\})$. By Proposition 3.15, I does not satisfy the strong exchange property. Moreover, I cannot be transversal, since the equations for the a_j's in Proposition 3.16 have no integer solutions for the given values $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 4$, $\lambda_4 = 5$. Alternatively, $\Delta\lambda = (2, 1, 1)$ is not a partition.

Nevertheless, we will see in later sections that ordinary powers of principal Borel sssi's share similar algebraic properties as powers of polymatroidal ideals that are either transversal or satisfy the strong exchange property.

3.2. Numerical invariants of symmetric shifted ideals

Inspired by the case of strongly stable ideals, in this subsection, we use partition generators to calculate numerical invariants associated with symmetric strongly shifted ideals. In fact, most formulas will hold more generally for arbitrary symmetric ideals.

For $a \in \mathbb{N}^n$, denote $\min(a) = \min\{i : a_i \neq 0\}$ and $\max(a) = \max\{i : a_i \neq 0\}$. If I is a strongly stable ideal, it is known that

$$\mathrm{ht}(I) = \max\{\min(a) : x^a \in G(I)\} \quad \text{and} \quad \mathrm{pd}(R/I) = \max\{\max(a) : x^a \in G(I)\}.$$

The latter formula follows from the Eliahou–Kervaire resolution [17]. Moreover, in both cases, it is enough to only consider the Borel generators of I; see [19, Proposition 2.14].

Replacing exponent vectors with partitions, we derive an analogous formula for the codimension of a symmetric monomial ideal.

Proposition 3.19. *Let I be a symmetric monomial ideal. For each $\lambda \in \Lambda(I)$, denote* $\min(\lambda) = \min\{i : \lambda_i > 0\}$. *The height of I is given by*

$$\mathrm{ht}(I) = \max_{\lambda \in \Lambda(I)} \{\min(\lambda)\}.$$

Moreover, if I is symmetric strongly shifted, then $\mathrm{ht}(I) = \max_{\lambda \in B(I)} \{\min(\lambda)\}$.

Proof. From (2.6), it follows that $\sqrt{I} = I_{n,c}$, where $c = \mathrm{ht}(I)$. To determine the height of I, notice that a monomial $m \in \sqrt{I}$ if and only if $|\mathrm{supp}(m)| > n - c$. Therefore, one has that

$$\begin{aligned} c &= n - \min_{m \in I}\{|\mathrm{supp}(m)|\} + 1 = n - \min_{\lambda \in \Lambda(I)}\{|\mathrm{supp}(\lambda)|\} + 1 \\ &= n - \min_{\lambda \in \Lambda(I)}\{n - \min(\lambda) + 1\} - 1 = \max_{\lambda \in \Lambda(I)}\{\min(\lambda)\}. \end{aligned}$$

In the first line of the displayed equalities, we use the fact that

$$\mathrm{supp}(m) = \mathrm{supp}(\mathrm{part}(m)).$$

Suppose now that I is a sssi. We prove that we can replace $\Lambda(I)$ with $B(I)$ in the formula above by showing that each Borel move can only increase the size of the support. This is because if $\mu_j > \mu_i$ and $\mu' = \mu - e_j + e_i$, then either $\mu_j - 1 > 0$ and then $\mathrm{supp}(\mu) \subseteq \mathrm{supp}(\mu')$ or $\mu_j = 1$ and $\mu_i = 0$, in which case $\mathrm{part}(\mu') = \mu$ and thus $|\mathrm{supp}(\mu')| = |\mathrm{supp}(\mu)|$. ■

On the other hand, the projective dimension of a symmetric strongly shifted ideal cannot be expressed in terms of min/max of the partition Borel generators. In Proposition 3.21 below, we provide a formula for the projective dimension of a symmetric shifted ideal in terms of its partition generators using a different partition statistic, which we call med. Recall from [2, Theorem 3.1] that the Betti numbers of a symmetric shifted ideal are given by

$$\beta_i(I) = \sum_{u \in G(I)} \binom{|G(C(u))|}{i},$$

where, for a monomial $u = \sigma(x^\lambda) \in G(I)$ with $\lambda \in P_n$ and $\sigma \in \mathfrak{S}_n$, the ideal $C(u)$ is constructed as follows (see [2, proof of Theorem 3.2]). First, one defines a total order on the set of monomials in $S = k[x_1, \ldots, x_n]$.

Definition 3.20. Let $\lambda, \mu \in \mathrm{P}_n$, and let $v = \tau(x^\mu)$ and $u = \sigma(x^\lambda)$ be distinct monomials in S for some $\sigma, \tau \in \mathfrak{S}_n$. Define $v \prec u$ if one of the following conditions holds:

(1) $\mu <_{\text{antilex}} \lambda$, that is, either $|\mu| < |\lambda|$ or $|\mu| = |\lambda|$ and the leftmost non-zero entry of $\mu - \lambda$ is positive;

(2) $\mu = \lambda$ and $v <_{\text{antilex}} u$; that is, the leftmost non-zero entry of $\tau(\mu) - \sigma(\lambda)$ is positive.

Next, for a symmetric shifted ideal I and a monomial $u = \sigma(x^\lambda) \in G(I)$, one defines $J = (v \in G(I) : v \prec u)$ and

$$C(u) := J : u = (x_{\sigma(1)}, \ldots, x_{\sigma(p)}) + (x_{\sigma(k)} : p+1 \leq k \leq n-r,\ \sigma(k) < u_{\max}),$$

where

$$p = p(\lambda) = \#\{k : \lambda_k < \lambda_n - 1\},$$
$$r = r(\lambda) = \#\{k : \lambda_k = \lambda_n\},$$
$$u_{\max} = \max\{\sigma(k) : \lambda_k = \lambda_n\}.$$

We are now ready to calculate the projective dimension of a symmetric shifted ideal.

Proposition 3.21. *Let I be a symmetric shifted ideal and define for every $\lambda \in P_n$ the integer* $\operatorname{med}(\lambda) = |\{i : \lambda_i < \lambda_n\}|$. *Then, the projective dimension of I is given by*

$$\operatorname{pd}(R/I) = \max_{\lambda \in \Lambda(I)} \{\operatorname{med}(\lambda)\} + 1.$$

Proof. Utilizing the notation in (3.2), from the formula for the Betti numbers given by (3.2), we deduce that

$$\operatorname{pd}(I) = \max\{i : \exists u \in G(I) \text{ with } |G(C(u))| \geq i\} = \max\{|G(C(u))| : u \in G(I)\}.$$

From (3.2), it follows that $|G(C(u))| \leq G(C(x^{\operatorname{part}(u)}))$. Moreover, it can also be seen from the definitions that $|G(C(x^\lambda))| = \operatorname{med}(\lambda)$ for all $\lambda \in P_n$. Therefore, it follows that

$$\operatorname{pd}(I) = \max\{\operatorname{med}(\lambda) : \lambda \in \Lambda(I)\}. \qquad \blacksquare$$

Unlike for height, when calculating the projective dimension of a symmetric strongly shifted ideal, we cannot replace $\Lambda(I)$ with the partition Borel generators of I.

Example 3.22. Let $I = \operatorname{Sss}(\{(1,5,5)\})$. Notice that $(2,4,5) \in \Lambda(I)$; however,

$$\operatorname{med}((2,4,5)) = 2 > 1 = \operatorname{med}((1,5,5)).$$

Another useful invariant is the *analytic spread*, $\ell(I)$, of I (i.e., the Krull dimension of the fiber cone $\mathcal{F}(I)$ of I, which we define in Section 6). This is well studied, since it controls the asymptotic growth of the powers of I, as we will see in Section 5.

The analytic spread of an arbitrary ideal is usually difficult to calculate. However, if I is an equigenerated monomial ideal, $\ell(I)$ coincides with the rank of the matrix whose rows are the exponent vectors of the monomial generators of I (see [34, Exercise 8.21]). Using this fact, we can compute the analytic spread of any equigenerated symmetric monomial ideal.

Proposition 3.23. *Let I be an equigenerated symmetric monomial ideal. The analytic spread of I is given by*

$$\ell(I) = \begin{cases} n & \text{if } I \neq \operatorname{Sss}(\{(a,\dots,a)\}) \text{ for any } a \in \mathbb{N}, \\ 1 & \text{if } I = \operatorname{Sss}(\{(a,\dots,a)\}) \text{ for some } a \in \mathbb{N}. \end{cases}$$

In particular, if $I = \operatorname{Sss}(B)$ is an equigenerated sssi with $B \neq \{(a,\dots,a)\}$ for any $a \in \mathbb{N}$, then $\ell(I) = \max_{\lambda \in B}\{\max(\lambda)\}$, where for each $\lambda \in B$ we denote

$$\max(\lambda) = \max\{i : \lambda_i > 0\}.$$

Proof. By [34, Exercise 8.21], the analytic spread of an equigenerated monomial ideal I is the rank of the matrix M whose rows are the exponent vectors of the monomial generators of I. Hence, we only need to show that M contains n linearly independent rows.

Towards this end, let V be the row space of M viewed as a $\mathbb{Q}$-vector space. Since I is a symmetric ideal, $V \subseteq \mathbb{Q}^n$ is a representation of $\mathfrak{S}_n$ acting naturally on $\mathbb{Q}^n$. Recall that the natural permutation representation of $\mathfrak{S}_n$ on $\mathbb{Q}^n$ decomposes into two irreducible representations: the trivial representation $T = \{(a,\dots,a) : a \in \mathbb{Q}\}$ and the standard representation $S = \{(q_1,\dots,q_n) : q_i \in \mathbb{Q}, \sum_{i=1}^n q_i = 0\}$. Consider the projections V_T and V_S of V onto T and S, respectively. Since V_T and V_S are subrepresentations of the irreducible representations T and S, respectively, we have $V_T = 0$ or $V_T = T$ and $V_S = 0$ or $V_S = S$, respectively, which yields four possibilities for V: $V = 0$ or $V = T$ or $V = S$ or $V = S \oplus T = \mathbb{Q}^n$.

Since $\Lambda(I) \subseteq V$, $\Lambda(I)$ contains at least one non-zero vector, and no element of $\Lambda(I)$ is in S (as every non-zero element of S must have at least one negative coordinate), we are left with the possibilities $V = T$ or $V = S \oplus T = \mathbb{Q}^n$. Now, the case $V = T$ corresponds to $I = \operatorname{Sss}(\{(a,\dots,a)\})$ for some $a \in \mathbb{N}$, which gives $\ell(I) = \dim(V) = \dim(S) = 1$. The case $V = \mathbb{Q}^n$ corresponds to $\Lambda(I) \cap S \neq \emptyset$ and gives $\ell(I) = \dim(V) = n$. ∎

For $I \neq \operatorname{Sss}(\{(a,\dots,a)\})$, Proposition 3.23 implies that

$$\ell(I) = \max_{\lambda \in \Lambda(I)}\{\max(\lambda)\},$$

where for each $\lambda \in \Lambda(I)$ we denote $\max(\lambda) = \max\{i : \lambda_i > 0\}$. A similar formula for the analytic spread of an equigenerated strongly stable ideal, with partitions replaced by exponent vectors of the monomial generators, was proved in [18, Proposition 6.3] using methods from convex geometry.

In [15, 25], the analytic spread of an equigenerated monomial ideal I has been characterized in terms of the *linear relation graph* $\Gamma = \Gamma(I)$ of I. This graph is defined as follows. If $G(I) = \{u_1, \dots, u_m\}$ denotes a minimal monomial generating set for I, the edge set of Γ is given by

$$E(\Gamma) = \{\{i, j\} : \text{ there exist } u_k, u_l \in G(I) \text{ such that } x_i u_k = x_j u_l\},$$

while the vertex set of Γ is given by $V(\Gamma) = \bigcup_{i:\{i,j\} \in E(\Gamma)} \{i\}$.

Assume that I is equigenerated and let r and s denote the number of vertices and the number of connected components of $\Gamma(I)$, respectively. By [25, Lemma 4.2], one has that

$$\ell(I) \geq r - s + 1,$$

with equality holding if I is a polymatroidal ideal, or, more generally, if I has linear syzygies [15, Lemma 4.3]. Since equigenerated symmetric shifted ideals have a linear resolution by [2, Theorem 3.2], we then deduce the following immediate corollary.

Corollary 3.24. *Let I be an equigenerated symmetric shifted ideal. Then, $\ell(I) = r - s + 1$, where r is the number of vertices in $\Gamma(I)$ and s is the number of its connected components.*

In fact, we can give a full description of the linear relation graph of an equigenerated ssi, which we will use in Section 5 to study the depths of powers of sssi's.

Proposition 3.25. *Let I be an equigenerated symmetric shifted ideal. If*

$$I = \mathrm{Sss}(\{(a, \dots, a)\})$$

for some $a \in \mathbb{N}$, then the linear relation graph $\Gamma = \Gamma(I)$ is the graph with n vertices and no edges. Otherwise, Γ is connected, with $V(\Gamma) = \{1, \dots, n\}$.

Proof. The first claim is clear, since $I = \mathrm{Sss}(\{(a, \dots, a)\}) = (x_1^a \cdots x_n^a)$ is a principal ideal, so no linear relations occur.

Suppose that $I \neq \mathrm{Sss}(\{(a, \dots, a)\})$. Then, there exists a $\lambda \in \Lambda(I)$ with $\lambda_1 < \lambda_n$. Let $\mu := \lambda + e_1 - e_n$; then, $\mu \in \Lambda(I)$ since I is symmetric shifted. Moreover, $x_1 x^\lambda = x_n x^\mu$, that is, $\{1, n\} \in E(\Gamma)$. We show that for any $1 < i < n$ either $\{i, n\} \in E(\Gamma)$ or $\{1, i\} \in E(\Gamma)$; thus, Γ is a connected graph on n vertices. If $\lambda_i < \lambda_n$, since I is symmetric shifted, we have that $\eta := \lambda + e_i - e_n \in \Lambda(I)$. Moreover, $x_i x^\lambda = x_n x^\eta$, that is, $\{i, n\} \in E(\Gamma)$. If instead $\lambda_i = \lambda_n$, let $\tau = (i, n) \in \mathfrak{S}_n$. Then, applying τ to the equality $x_1 x^\lambda = x_n x^\mu$, we get $x_1 x^\lambda = x_i \tau(x^\mu)$. Since $\tau(x^\mu) \in G(I)$, this means that $\{1, i\} \in E(\Gamma)$ and the proof is complete. ∎

4. Integral closure

In the rest of this article, we use the results from Section 3 to study ordinary and symbolic powers of symmetric strongly shifted ideals. In particular, in this section, we study the integral closure and normality property of symmetric strongly shifted ideals.

Our first result parallels an analogous result of Guo [24, Theorem 2.1] for the integral closure of strongly stable ideals.

Proposition 4.1. *The integral closure $\overline{I}$ of a symmetric (strongly) shifted ideal I is also symmetric (strongly) shifted.*

Proof. First, note that the integral closure of a symmetric ideal is symmetric. Indeed, if f is a solution of a monic polynomial

$$x^n + r_1 x^{n-1} + \cdots + r_n = 0$$

with $r_i \in I^i$ and if $\sigma \in \mathfrak{S}_n$, then $\sigma(f)$ is a solution of the equation

$$x^n + \sigma(r_1) x^{n-1} + \cdots + \sigma(r_n) = 0$$

in which each $\sigma(r_i) \in \sigma(I^i) = I^i$ because I^i is stable under the action of $\mathfrak{S}_n$.

Second, recall that since I is a monomial ideal, $x^\alpha \in \overline{I}$ if and only if it satisfies an equation of the form

$$x^{s\alpha} = \prod_{k=1}^{s} x^{\beta^{(k)}} \quad \text{for some } s \in \mathbb{N},$$

where $x^{\beta^{(k)}} \in I$ for each k; see [34, p. 9]. Now, assume that $\alpha \in P(\overline{I})$ and take $i < j$ so that $\alpha_i < \alpha_j$. (If I is shifted but not strongly shifted, assume additionally that $j = n$.) Set $S = \{k \in \{1, \ldots, s\} \mid \beta_i^{(k)} < \beta_j^{(k)}\}$ and $\alpha' = \alpha + e_i - e_j$. From (4), it follows that

$$\sum_{k \in S} (\beta_j^{(k)} - \beta_i^{(k)}) \geq s(\alpha_j - \alpha_i) \geq s.$$

Therefore, there exist nonnegative integers c_k, one for each $k \in S$ so that $\sum_{k \in S} c_k = s$ and $c_k \leq \beta_j^{(k)} - \beta_i^{(k)}$ for each $k \in S$. Define $\alpha' := \alpha + e_i - e_j$ and

$$\gamma^{(k)} = \begin{cases} \beta^{(k)} + c_k(e_i - e_j) & \text{if } k \in S, \\ \beta^{(k)} & \text{otherwise} \end{cases}$$

so that $\sum_{k=1}^{n} \gamma^{(k)} = \sum_{k=1}^{n} \beta^{(k)} + s(e_i - e_j) = s\alpha'$. Then, $\gamma^{(k)} \in I$ by the (strongly) shifted property and the relation $x^{s\alpha'} = \prod_{k=1}^{s} x^{\gamma^{(k)}}$ shows that $x^{\alpha'} \in \overline{I}$, concluding the proof. ∎

An ideal is said to be *integrally closed* if it coincides with its integral closure and is called *normal* if all its powers are integrally closed. To understand normality of an ideal I, a useful object is the *Rees ring* of I, which is defined as the subring

$$\mathcal{R}(I) = \bigoplus_{i \geq 0} I^i t^i \subseteq R[It].$$

Indeed, since $R = K[x_1, \ldots, x_n]$ is a normal domain, I is normal if and only if the Rees algebra $\mathcal{R}(I)$ is a normal domain [34, Propositions 5.2.1 and 5.2.4].

The following example shows that a sssi need not be integrally closed or normal.

Example 4.2. Consider $I = \mathrm{Sss}(\{(2,2,8),(0,6,6)\})$ and observe that for $\lambda = (1,4,7)$ one has $2\lambda = (2,2,8) + (0,6,6)$. Hence, $x^{2\lambda} \in I$ and $x^{\lambda} \in \overline{I}$. However, $\lambda \notin P(I)$ so $x^{\lambda} \notin I$. Hence, I is not integrally closed and therefore not normal.

However, Theorem 3.6 indicates a strategy to construct sssi's which are integrally closed or normal. In particular, in the next proposition, we identify two classes of normal symmetric strongly shifted ideals, obtained by symmetrization of certain normal strongly stable ideals.

To state our result, we recall that a monomial ideal is called a *lex-segment ideal* if, for every degree d, the set of monomials of degree d in I forms a *lex-segment*; namely, for each degree d, there exists a monomial $u \in I$ of degree d so that I contains every degree-d monomial v which is larger than u in the lexicographic order. It is well known that lex-segment ideals are strongly stable, e.g., [27, p. 103].

Proposition 4.3. *Let J be an integrally closed strongly stable ideal, and let $I = \bigcap_{\sigma \in \mathfrak{S}_n} \sigma(J)$ be its symmetrization in the sense of Theorem 3.6. Then, I is an integrally closed sssi.*

Moreover, if I is a sssi such that $\mathrm{Borel}(x^{B(I)})$ *is normal, then I is normal. In particular, I is normal if either of the following conditions hold:*

1. *$B(I) = \{\lambda\}$ for some $\lambda \in P_n$, i.e., I is a principal Borel sssi; or*
2. $\mathrm{Borel}(x^{B(I)})$ *is an equigenerated lex-segment ideal.*

Proof. First, suppose that J is integrally closed. Since every $\sigma \in \mathfrak{S}_n$ acts as an isomorphism on R, then $\sigma(J)$ is integrally closed for every $\sigma \in \mathfrak{S}_n$. As the intersection of integrally closed ideals is integrally closed [34, Remark 1.1.3], it follows that I is integrally closed.

Now, let I be a sssi and $J = \mathrm{Borel}(x^{B(I)})$. Then, for every $k \geq 1$,

$$J^k = \mathrm{Borel}\big(x^{kB(I)}\big)$$

is strongly stable by [24, Proposition 1.2]. Moreover, by Proposition 2.2 and Proposition 3.9, we have that I^k is symmetric strongly shifted, with $I^k = \mathrm{Sss}(kB(I))$.

Hence, Theorem 3.6 (2) implies that $I^k = \bigcap_{\sigma \in \mathfrak{S}_n} \sigma(J^k)$. Therefore, from the first part of the proof, it follows that I^k is integrally closed whenever J^k is; that is, I is normal whenever J is.

Finally, to establish the two particular cases, it remains to note that if I is a principal Borel sssi, then J is a principal Borel ideal. Furthermore, principal Borel ideals and equigenerated lex-segment ideals are normal by [14, Proposition 2.14]. ■

The normality of an ideal I as in Proposition 4.3 has important applications to the study of the asymptotic behavior of the ordinary powers of a sssi which we will explore in Section 5. A key result will be the following corollary, whose proof is well known (see, for instance, [34, Propositions 5.2.1 and 5.2.4] and [27, Theorem B.6.2]).

Corollary 4.4. *Let I be an equigenerated, normal sssi (e.g., I satisfies one of the conditions in Proposition 4.3). Then, the Rees ring $\mathcal{R}(I)$ is a Cohen–Macaulay normal domain.*

4.1. Convex polytopes and minimal reductions

A useful tool to determine the integral closure of a monomial ideal I is its Newton polyhedron. In this subsection, we study the Newton polyhedron of a symmetric strongly shifted ideal, discovering an interesting geometric description of ideals of this kind in terms of well-studied convex polytopes.

We recall that the *Newton polytope* of a monomial ideal I is defined as the convex hull of the exponents of its minimal generators, that is,

$$\operatorname{np}(I) = \operatorname{conv}\{(a_1, \dots, a_n) \mid x^a \in G(I)\}.$$

Moreover, the *Newton polyhedron* of I is defined as the Minkowski sum

$$NP(I) := \operatorname{np}(I) + \mathbb{R}^n_{\geq 0}.$$

(The Minkowski sum of polytopes A, B is defined as $A + B = \{a + b \mid a \in A, b \in B\}$.) It is well known that the integral closure $\overline{I}$ of I can be determined via the formula

$$NP(\overline{I}) = NP(I) \cap \mathbb{N}^n.$$

Moreover, by [3, Theorem 2.3], the analytic spread of I is

$$\ell(I) = \max\{\dim F \mid F \text{ is a compact face of } NP(I)\} + 1.$$

If I is an equigenerated monomial ideal, the Newton polyhedron $NP(I)$ has a unique compact face of maximal dimension, which coincides with the Newton polytope $\operatorname{np}(I)$. As a consequence of Proposition 3.23, we then have the following description of the Newton polytope of an arbitrary equigenerated symmetric monomial ideal.

Corollary 4.5. *Let I be an equigenerated symmetric monomial ideal, and let* $\operatorname{np}(I)$ *denote the Newton polytope of I. Then,*

$$\dim(\operatorname{np}(I)) = \begin{cases} n-1 & \textit{if } I \neq \operatorname{Sss}(\{(a,\dots,a)\}) \textit{ for any } a \in \mathbb{N}, \\ 0 & \textit{if } I = \operatorname{Sss}(\{(a,\dots,a)\}) \textit{ for some } a \in \mathbb{N}. \end{cases}$$

If I is symmetric strongly shifted, we can provide a more detailed description of the Newton polytope of I (see Proposition 4.6 below). Recall that in convex geometry a *permutohedron* is a convex body defined as follows:

$$\mathbf{P}(a_1,\dots,a_n) = \operatorname{conv}\{(a_{\sigma(1)},\dots,a_{\sigma(n)}) \mid \sigma \in \mathfrak{S}_n\},$$

where conv denotes taking the convex hull of a set of points in $\mathbb{R}^n$. Permutohedra have emerged as objects of recent interest in combinatorics [42] and in algebraic geometry [32]. We now show that the convex geometry of sssi's is governed by permutohedra. Some geometric implications of this fact will be discussed in Section 6.2.

Proposition 4.6. *The Newton polytope of a principal Borel sssi is a permutohedron, namely,*

$$\operatorname{np}(\operatorname{Sss}(\{\lambda\})) = \mathbf{P}(\lambda) = \operatorname{conv}\{\sigma(\lambda) \mid \sigma \in \mathfrak{S}_n\}.$$

In general, the Newton polytope of a sssi I is a convex hull of permutohedra, namely,

$$\operatorname{np}(I) = \operatorname{conv}\Big(\bigcup_{\lambda \in B(I)} \mathbf{P}(\lambda)\Big).$$

Proof. It is clear from the definitions that $\mathbf{P}(\lambda) \subseteq \operatorname{np}(\operatorname{Sss}(\{\lambda\}))$ since the vertices of the permutohedron are exponent vectors for some of the monomials in $G(\operatorname{Sss}(\{\lambda\}))$. For the converse, a theorem by Rado [43], as transcribed in [42, Proposition 2.5], states that $\mathbf{P}(\lambda)$ is defined by the following (in)equalities:

$$\mathbf{P}(\lambda) = \left\{ (t_1,\dots,t_n) \in \mathbb{R}^n \;\middle|\; \begin{cases} t_1 + \cdots + t_n = \lambda_1 + \cdots + \lambda_n \\ t_{i_k} + \cdots + t_{i_n} \leq \lambda_k + \cdots + \lambda_n \\ \qquad\qquad \forall 1 \leq k \leq n, \forall 1 \leq i_k \leq \cdots \leq i_n \leq n \end{cases} \right\}.$$

Each exponent vector of a monomial in $G(I)$ satisfies the above system by Remark 2.8; thus, we obtain $\operatorname{np}(\operatorname{Sss}(\{\lambda\})) \subseteq \mathbf{P}(\lambda)$.

To deduce the general statement from that regarding principal Borel sssi's, it suffices to note that by Proposition 3.8 (1) an arbitrary sssi I decomposes as

$$I = \sum_{\lambda \in B(I)} \operatorname{Sss}(\{\lambda\})$$

and that the Newton polytope of a sum of ideals is the convex hull of the union of the Newton polytopes of the summands. ∎

An important example of permutohedron is the *hypersimplex*

$$\Delta_{n,d} = \mathbf{P}(0^{n-d}, 1^d).$$

Our decomposition formula for principal Borel sssi's in Theorem 3.12 recovers a well-known decomposition for the permutohedron as a Minkowski sum of hypersimplices.

Corollary 4.7. *For $\lambda \in P_n$ and $\lambda_0 = 0$, there is an identity*

$$\mathbf{P}(\lambda) = \sum_{i=1}^{n} (\lambda_i - \lambda_{i-1}) \Delta_{i,n}.$$

Proof. Theorem 3.12 yields the following polyhedral identities, as the Newton polytope of a product of ideals is the Minkowski sum of the Newton polytopes of the summands

$$\mathrm{np}(\mathrm{Sss}(\{\lambda\})) = \sum_{i=1}^{n} \mathrm{np}(I_{n,i}^{\lambda_i - \lambda_{i-1}}) = \sum_{i=1}^{n} (\lambda_i - \lambda_{i-1}) \Delta_{i,n}. \qquad \blacksquare$$

An ideal $L \subseteq I$ is called a *reduction* of I if $\overline{L} = \overline{I}$; equivalently, if there exists an integer r so that $LI^r = I^{r+1}$. If this is the case, it follows that $LI^k = I^{k+1}$ for every $k \geq r$. Hence, the reductions of an ideal I give information on the growth of powers of I. A reduction L of I is called a *minimal reduction* if it is minimal with respect to inclusion.

While the reductions of a monomial ideal need not be monomial, in [47, Proposition 2.1], Singla showed that every monomial ideal I admits a unique *minimal monomial reduction*, that is, a reduction which is monomial and contains no other monomial reduction of I. Thanks to Proposition 4.6, we can determine the unique monomial reduction of a principal Borel sssi.

Corollary 4.8. *For $\lambda \in P_n$, define the monomial ideal generated by the $\mathfrak{S}_n$-orbit of x^λ as*

$$L(\lambda) = (\sigma(\lambda) \mid \sigma \in \mathfrak{S}_n).$$

Then, $L = L(\lambda)$ is the minimal monomial reduction of $\mathrm{Sss}(\{\lambda\})$ *and we have*

$$\mathrm{Sss}(\{\lambda\}) = \overline{L}.$$

Proof. Let $I = \mathrm{Sss}(\{\lambda\})$. In [47, Proposition 2.1 and Remark 1.3], it is shown that the exponents for the monomial generators of the minimal monomial reduction of I correspond to the vertices of $NP(I) = \mathrm{np}(I) + \mathbb{R}^n_{\geq 0}$, which coincide with the vertices of $\mathrm{np}(I)$, as the vertices of a Minkowski sum are obtained as sums of vertices from each summand. Therefore, we only need to show that $V = \{\sigma(\lambda) \mid \sigma \in \mathfrak{S}_n\}$ is the set of vertices of $\mathrm{np}(I)$. The latter form a subset of V, because $\mathrm{np}(I) = \mathbf{P}(\lambda)$

by Proposition 4.6. Since V forms an orbit under the $\mathfrak{S}_n$ action on $\mathbb{R}^n$ and since $\operatorname{np}(I)$ and hence its vertex set are $\mathfrak{S}_n$-invariant, we conclude that V coincides with the set of vertices of $\operatorname{np}(I)$. Hence, L is indeed the minimal monomial reduction of I. Now, the fact that L is a reduction of I implies that $\overline{L} = \overline{I}$, while $\overline{I} = I$ follows by Proposition 4.3 (1), finishing the proof. ■

The smallest integer r so that $LI^r = I^{r+1}$ for a minimal reduction L of I is called the reduction number of I with respect to L, denoted by $r_L(I)$. The *reduction number* of I is

$$r(I) = \min\{r_L(I) : L \text{ is a minimal reduction of } I\}.$$

The following corollary estimates the reduction number of any equigenerated normal sssi.

Corollary 4.9. *Let $I \subseteq R = K[x_1, \dots, x_n]$ be equigenerated, normal sssi (e.g., let I satisfy either one of the assumptions in Proposition 4.3) and assume that K is an infinite field. Then, $r(I) \le n-1$, and $r(I) = 0$ if $I = \operatorname{Sss}(\{(a, \dots, a)\})$ for some $a \in \mathbb{N}$.*

Proof. By Corollary 4.4, $\mathcal{R}(I)$ is Cohen–Macaulay, whence [35, Theorem 2.3] implies that $r(I) \le \ell(I) - 1$. The claim now follows from Proposition 3.23. ■

5. Associated primes and primary decomposition

In this section, we continue our study of the ordinary powers of a symmetric strongly shifted ideal I, by examining how the depths and associated primes of I^k depend on the exponent k. In turn, thanks to (2) and (2), this process will sometimes give information on the symbolic powers of I as well.

Our analysis begins by recalling some well-known results which hold for an arbitrary ideal I in a Noetherian ring. In [5, 6], Brodmann proved that, for $k \gg 0$, $\operatorname{Ass}(I^k) = \operatorname{Ass}(I^{k+1})$ and $\operatorname{depth}(R/I^k) = \operatorname{depth}(R/I^{k+1})$; moreover,

$$\lim_{k\to\infty} \operatorname{depth}(R/I^k) \le n - \ell(I).$$

The smallest number k_0 for which $\operatorname{Ass}(I^{k_0}) = \operatorname{Ass}(I^{k_0+1})$ is called the *index of stability* of I and is denoted by $\operatorname{astab}(I)$; $\operatorname{Ass}(I^{k_0})$ is called the *stable set of associated primes* and is denoted by $\operatorname{Ass}^\infty(I)$. Similarly, the smallest integer so that $\operatorname{depth}(R/I^k) = \operatorname{depth}(R/I^{k+1})$ is called the *index of depth stability* of I, $\operatorname{dstab}(I)$. An ideal I is said to satisfy the *persistence property* if $\operatorname{Ass}(I^k) \subseteq \operatorname{Ass}(I^{k+1})$ for every $k \ge 1$.

For monomial ideals in a polynomial ring, these notions have been studied by several authors using various combinatorial techniques; see, for instance, [20, 25, 29, 30, 36]. In particular, in [25, Theorem 3.3], Herzog and Qureshi provide bounds for the depths of powers of an equigenerated monomial ideal I in terms of its linear relation graph $\Gamma = \Gamma(I)$, which is recalled in (3.2). If r and s denote the number of vertices and the number of connected components of Γ, respectively, they show that if $r - s \geq 1$, then

$$\operatorname{depth}(R/I^k) \leq n - k - 1 \quad \text{for } 1 \leq k \leq r - s.$$

Using their result and Proposition 3.25, we next describe the depths of powers of equigenerated sssi's.

Proposition 5.1. *Let I be an equigenerated symmetric strongly shifted ideal of a polynomial ring in n variables. Then,* $\operatorname{depth}(R/I^k) \geq \operatorname{depth}(R/I^{k+1})$ *for all $k \geq 1$. Moreover,*

(1) *if $I = \operatorname{Sss}(\{(a, \dots, a)\})$ for some $a \in \mathbb{N}$, then* $\operatorname{depth}(R/I^k) = n - 1$ *for all $k \geq 1$, and* $\operatorname{dstab}(I) = 1$*;*

(2) *otherwise,* $\operatorname{depth}(R/I^k) \leq n - k - 1$ *for $1 \leq k \leq n - 1$ and* $\operatorname{depth}(R/I^k) = 0$ *for $k \geq n - 1$; in particular,* $\operatorname{dstab}(I) \leq n - 1$ *and $\mathfrak{m} \in \operatorname{Ass}(I^k)$ for every $k \geq n - 1$.*

Proof. Since equigenerated symmetric (strongly) shifted ideals have a linear free resolution by [2, Theorem 3.2], and powers of sssi's are symmetric strongly shifted by Proposition 2.2, it follows from [29, Proposition 2.2] that

$$\operatorname{depth}(R/I^k) \geq \operatorname{depth}(R/I^{k+1}) \quad \text{for all } k \geq 1.$$

If $I = \operatorname{Sss}(\{(a, \dots, a)\})$ for some $a \in \mathbb{N}$, then for all $k \geq 1$ one has

$$I = \operatorname{Sss}(\{(ka, \dots, ka)\})$$

by Proposition 3.9 (2). Hence, for all $k \geq 1$, $\operatorname{depth}(R/I^k) = n - \operatorname{pd}(R/I^k) = n - 1$, where the latter equality follows from Proposition 3.21. This implies that

$$\operatorname{dstab}(I) = 1.$$

Assume now that $I \neq \operatorname{Sss}(\{(a, \dots, a)\})$ for any $a \in \mathbb{N}$. Then, by Proposition 3.25, the linear relation graph Γ of I is connected, with $V(\Gamma) = \{1, \dots, n\}$. Thus, (5) implies that $\operatorname{depth}(R/I^k) \leq n - k - 1$ for $1 \leq k \leq n - 1$. In particular,

$$\operatorname{depth}(R/I^{n-1}) = 0,$$

whence from (5) we obtain that $\operatorname{depth}(R/I^k) = 0$ for all $k \geq n - 1$. The remaining claims now follow immediately (see also [25, Corollary 3.4]). ∎

Remark 5.2. In the previous proof, we only used the assumption that I is symmetric strongly shifted in order to apply Proposition 2.2. Thus, the same statement would hold for arbitrary equigenerated symmetric shifted ideals, should Question 2.4 have an affirmative answer.

In [25, Theorem 1.3], it was proved that an equigenerated graded ideal I satisfies the persistence property if $I^{k+1} : I = I^k$ for all $k \geq 1$. While in general this equality is only known to hold for $k \gg 0$, it is satisfied for every k if I is normal; see [44, Propositions 4.1 and 4.7]. Proposition 4.3 then implies that many equigenerated sssi's, including all principal Borel sssi's, satisfy the persistence property.

Theorem 5.3. *Let $I = \operatorname{Sss}(B)$ be an equigenerated, normal symmetric strongly shifted ideal (e.g., I satisfies one of the assumptions in Proposition 4.3). Then, the following hold.*

(1) *For every $k \geq 1$, $I^{k+1} : I = I^k$.*

(2) *For every $k \geq 1$, $\operatorname{Ass}(I^k) \subseteq \operatorname{Ass}(I^{k+1})$ for every $k \geq 1$. That is, I satisfies the persistence property.*

(3) $\lim_{k\to\infty} \operatorname{depth}(R/I^k) = n - \ell(I)$
$$= \begin{cases} 0 & \text{if } \lambda \neq (a,\dots,a) \text{ for any } a \in \mathbb{N}, \\ n-1 & \text{if } \lambda = (a,\dots,a) \text{ for some } a \in \mathbb{N}. \end{cases}$$

Proof. Since I is a normal ideal, [44, Proposition 4.7] implies (1), whence (2) follows from [25, Theorem 1.3]. Moreover, since the Rees ring $\mathcal{R}(I)$ is Cohen–Macaulay by Corollary 4.4, a well-known result of Huneke implies that equality holds in Brodmann's inequality
$$\lim_{k\to\infty} \operatorname{depth}(R/I^k) \leq n - \ell(I)$$
(see, e.g., [27, Proposition 10.3.2]). This implies the first equality in (3), whence the second equality follows from Proposition 3.23. ∎

While the techniques used to prove [25, Theorem 1.3] do not seem to apply to arbitrary equigenerated symmetric strongly shifted ideals, we currently do not know of any sssi failing to satisfy Theorem 5.3 (1). This motivates the following questions.

Question 5.4. Is it true that $I^{k+1} : I = I^k$ for every $k \geq 1$ if I is an arbitrary symmetric strongly shifted ideal? Does any sssi satisfy the persistence property?

5.1. Stable associated primes of a principal Borel sssi

When I is a principal Borel sssi, its polymatroidal nature complements the information provided by Proposition 5.1 and Theorem 5.3, allowing us to estimate the indices of stability and depth stability of I.

Proposition 5.5. *Let* $I = \operatorname{Sss}(\{\lambda\})$ *be a principal Borel symmetric strongly shifted ideal of a polynomial ring in* n *variables. Then,*

(1) $\operatorname{dstab}(I) = 1$ *if and only if either* $\lambda = (a, \ldots, a)$ *for some* $a \in \mathbb{N}$*, or* $\lambda \neq (a, \ldots, a)$ *for any* $a \in \mathbb{N}$ *and* $\mathfrak{m} = (x_1, \ldots, x_n) \in \operatorname{Ass}(I)$.

(2) $\operatorname{dstab}(I) \leq \operatorname{astab}(I) \leq n - 1$*, provided* $n \geq 2$.

Proof. From Proposition 5.1, it follows that, for all $k \geq 1$,

$$\operatorname{depth}(R/I) \geq \operatorname{depth}(R/I^k) \geq \lim_{k\to\infty} \operatorname{depth}(R/I^k).$$

Then, $\operatorname{dstab}(I) = 1$ if and only if $\operatorname{depth}(R/I) = \lim_{k\to\infty} \operatorname{depth}(R/I^k)$. By Theorem 5.3 (3), the latter equality is equivalent to the condition that $\mathfrak{m} \in \operatorname{Ass}(I)$ if $\lambda \neq (a, \ldots, a)$ for any $a \in \mathbb{N}$ and follows from Proposition 5.1 (1) otherwise. This proves (1).

To prove (2), assume first that $\lambda \neq (a, \ldots, a)$ for any $a \in \mathbb{N}$. Then, from the persistence property and (5.1), it follows that

$$\operatorname{dstab}(I) = \min\{k : \operatorname{depth}(R/I^k) = 0\} = \min\{k : \mathfrak{m} \in \operatorname{Ass}(I^k)\}.$$

Hence, $\operatorname{dstab}(I) \leq \operatorname{astab}(I) \leq n - 1$ by Proposition 5.1 (2) (see also [36, Lemma 2.20] and [25, Theorem 4.1]). If $\lambda = (a, \ldots, a)$ for some $a \in \mathbb{N}$, then I is a transversal polymatroidal ideal, whence $\operatorname{astab}(I) = \operatorname{dstab}(I) = 1 \leq n - 1$ by [29, Corollaries 4.6 and 4.14]. ■

The inequality $\operatorname{dstab}(I) \leq \operatorname{astab}(I)$ in Proposition 5.5 (2) is remarkable, as for an arbitrary monomial ideal I either of the integers $\operatorname{astab}(I)$ and $\operatorname{dstab}(I)$ might be smaller than the other; see [29, p. 295]. It is known that $\operatorname{dstab}(I) = \operatorname{astab}(I)$ if I is a transversal polymatroidal ideal [29, Corollaries 4.6 and 4.14], an ideal of Veronese type [29, Corollary 5.7], or a polymatroidal ideal with the strong exchange property [36, Proposition 2.15]. It thus makes sense to investigate whether $\operatorname{dstab}(I) = \operatorname{astab}(I)$ for symmetric polymatroidal ideals.

In Example 5.7, we show that this is not true in general; however, it follows from Theorem 5.9 below that several principal Borel sssi's satisfy $\operatorname{dstab}(I) = \operatorname{astab}(I) = 1$. A key ingredient in the proof is the following decomposition of a principal Borel sssi as an intersection of symbolic powers of square-free Veronese ideals.

Theorem 5.6. *Let* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *be a partition. For* $1 \leq j \leq n$*, let* $P_j = (x_1, \ldots, x_j)$ *and denote* $a_j = \sum_{i=1}^{j} \lambda_i$*. Then, the principal Borel sssi* $\operatorname{Sss}(\{\lambda\})$ *can be decomposed as*

$$\operatorname{Sss}(\{\lambda\}) = \bigcap_{j=1}^{n} \Big(\bigcap_{\sigma\in\mathfrak{S}_n} \sigma(P_j)^{a_j} \Big) = \bigcap_{j=1}^{n} I_{n,j}^{(a_j)}.$$

Proof. By Theorem 3.6, we know that $I = \bigcap_{\sigma \in \mathfrak{S}_n} \sigma(\mathrm{Borel}(x^\lambda))$. Moreover, from [20, Proposition 2.7 and Theorem 3.1], it follows that

$$\mathrm{Borel}(x^\lambda) = \prod_{j=1}^{n} P_j^{\lambda_j} = \bigcap_{j=1}^{n} P_j^{a_j},$$

where $a_j = \sum_{i : P_i \subseteq P_j} \lambda_i = \sum_{i=1}^{j} \lambda_i$. Therefore, we deduce the identity

$$\mathrm{Sss}(\{\lambda\}) = \bigcap_{\sigma \in \mathfrak{S}_n} \sigma(\mathrm{Borel}(x^\lambda)) = \bigcap_{\sigma \in \mathfrak{S}_n} \sigma\left(\bigcap_{j=1}^{n} P_j^{a_j}\right) = \bigcap_{j=1}^{n} \Big(\bigcap_{\sigma \in \mathfrak{S}_n} \sigma(P_j)^{a_j} \Big).$$

Now, for every j, we can rewrite $\bigcap_{\sigma \in \mathfrak{S}_n} \sigma(P_j)^{a_j} = I_{n,j}^{(a_j)}$, which proves the last equality. ■

Note that the first equality in Theorem 5.6 gives a (possibly redundant) primary decomposition of $I = \mathrm{Sss}(\{\lambda\})$, as for every j and every $\sigma \in \mathfrak{S}_n$ the ideals $\sigma(P_j)^{a_j}$ are P_j-primary. Since the powers of a principal Borel sssi are still principal Borel sssi's, Proposition 3.9 and Theorem 5.6 then together imply that, for every integer $k \geq 1$,

$$I^k = \mathrm{Sss}(\{\lambda\})^k = \mathrm{Sss}(\{k\lambda\}) = \bigcap_{j=1}^{n} I_{n,j}^{(ka_j)}.$$

Using this formula, we can construct a principal Borel sssi I such that $\mathrm{astab}(I) \neq \mathrm{dstab}(I)$.

Example 5.7. Let $I = \mathrm{Sss}(\{1,2,2,4,4\})$. Irredundant primary decompositions of I and I^2 computed using Macaulay2 [22] are the following:

$$I = I_{5,1} \cap I_{5,2}^{(3)} \cap I_{5,4}^{(9)} \cap I_{5,5}^{(13)},$$
$$I^2 = I_{5,1}^{(2)} \cap I_{5,2}^{(6)} \cap I_{5,3}^{(10)} \cap I_{5,4}^{(18)} \cap I_{5,5}^{(26)}.$$

Thus, it follows from (5.1) and the persistence property Theorem 5.3 (1) that

$$\mathrm{astab}(I) = 2.$$

Moreover, $\mathrm{dstab}(I) = 1$ by Theorem 5.3 (3), since $\mathfrak{m} = I_{5,5} \in \mathrm{Ass}(I)$.

Our next goal is to determine $\mathrm{Ass}^\infty(I)$, $\mathrm{astab}(I)$, and $\mathrm{dstab}(I)$ for a principal Borel sssi I. To this end, one needs to be able to predict which components in (5.1) are irredundant in a systematic way. We identify sufficient conditions in Theorem 5.8 below, showing that, for every $k \geq 1$, $\mathrm{Ass}(I^k) \subseteq \{\sigma(P_1), \dots, \sigma(P_n) : \sigma \in \mathfrak{S}_n\}$, and $\sigma(P_j) \in \mathrm{Ass}(I^k)$ if and only if $I_{n,j}^{(ka_j)}$ is needed in (5.1). Our proof relies on the following description of the partition generators of the symbolic powers of square-free sssi's from [2, Proposition 4.1]: for all $j \geq 1$,

$$P\big(I_{n,j}^{(ka_j)}\big) = \left\{\mu \in P_n : \sum_{i=1}^{j} \mu_i \geq ka_j\right\},$$

$$\Lambda\big(I_{n,j}^{(ka_j)}\big) = \left\{\mu \in P_n : \sum_{i=1}^{j} \mu_i = ka_j \text{ and } \mu_i = \mu_j \text{ for all } i > j\right\}.$$

Theorem 5.8. *Let* $I = \operatorname{Sss}(\{\lambda\})$ *and* $j' := \min(\lambda)$. *Adopt the notation of Theorem 5.6. Then, for a fixed* $k \geq 1$,

(1) *all components* $I_{n,j}^{(ka_j)}$ *with* $j < j'$ *are redundant in (5.1);*

(2) $I_{n,j'}^{(ka_{j'})}$ *is not redundant in (5.1) and moreover* $I^{(k)_{\mathrm{Min}}} = I_{n,j'}^{(k\lambda_{j'})}$;

(3) *if* $j > j'$ *and* $\lambda_{j-1} < \lambda_j$, *then the component* $I_{n,j}^{(ka_j)}$ *is not redundant in (5.1);*

(4) *if* $\lambda_1 < \lambda_j$ *and either* $k > j(j-1)$ *or* $\lambda_j > \frac{q+j-r}{k}$, *where* $k\sum_{i=1}^{j-1}\lambda_i = (j-1)q + r$ *and* $0 \leq r \leq j-2$, *then the component* $I_{n,j}^{(ka_j)}$ *is not redundant in (5.1) if and only if* $j \geq j'$.

(5) *if* $\lambda_1 = \lambda_j$ *for some* $j > 1$, *then the component* $I_{n,j}^{(ka_j)}$ *is redundant in (5.1).*

In particular, for $k \gg 0$, *a minimal primary decomposition of* I^k *is*

$$I^k = \bigcap_{j\in\mathcal{J}} \bigcap_{\sigma\in\mathfrak{S}_n} \sigma(P_j)^{ka_j},$$

where

$$\mathcal{J} = \begin{cases} \{j : j \geq j'\} & \text{if } j' > 1,\\ \{1\} \cup \{j : j > 1 \text{ and } \lambda_j \neq \lambda_1\} & \text{if } j' = 1.\end{cases}$$

Proof. For simplicity of notation, we denote

$$\sum_s \lambda := \sum_{r=1}^{s} \lambda_r$$

for any partition $\lambda \in P_n$.

(1) For $j < j'$, one has $a_j = 0$, so the corresponding component $I_{n,j}^{(ka_j)} = R$ of the decomposition (5.1) is redundant.

(2) Since by Proposition 3.19 one has $\operatorname{ht}(I) = \min(\lambda) = j'$, the component $I_{n,j'}^{(ka_{j'})}$ is the intersection of the primary ideals $\sigma(P_{j'}^{ka'_j})$ which belong to minimal primes of I^k, that is, $I^{(k)_{\mathrm{Min}}} = I_{n,j'}^{(ka_{j'})}$. It remains to note that $a_{j'} = \lambda_{j'}$ by definition.

(3) We next establish irredundancy of the components $I_{n,j}^{(a_j)}$ so that $j \geq j'$ and $\lambda_{j-1} < \lambda_j$. This amounts to showing that $\bigcap_{i \neq j} I_{n,i}^{(ka_i)} \not\subseteq I_{n,j}^{(ka_j)}$. Equation (5.1) implies that for $\mu \in P_n$, $x^\mu \in I_{n,i}^{(ka_i)}$ if and only if $\sum_i \mu \geq ka_i = k \sum_i \lambda$. Set

$$\mu_i = \begin{cases} k\lambda_i & \text{for } 1 \leq i < j, \\ k\lambda_{j-1} & \text{for } i = j, \\ k(\lambda_{j+1} + \lambda_j - \lambda_{j-1}) & \text{for } i = j+1, \text{ provided } j+1 \leq n, \\ k\lambda_i & \text{for } j+2 \leq i \leq n. \end{cases}$$

Then, $\mu \in P_n$ and $x^\mu \in I_{n,i}^{(ka_i)}$ for each $i \neq j$ since $\sum_i \mu = k \sum_i \lambda$. By contrast, we have that $\sum_j \mu < k \sum_j \lambda$ since $\lambda_{j-1} < \lambda_j$ and thus $x^\mu \notin I_{n,j}^{(ka_j)}$. Therefore, the component $I_{n,j}^{(ka_j)}$ is not redundant in the decomposition (5.1).

(4) Now, suppose that $j > j'$ and in particular $j > 1$. We construct $\mu \in P_n$ so that $x^\mu \in \bigcap_{i \neq j} I_{n,i}^{(ka_i)}$. By (5.1), μ must satisfy $\sum_i \mu \geq k \sum_i \lambda$ for each $i \neq j$. Set

$$k \sum_{j-1} \lambda = (j-1)q + r \quad \text{with } 0 \leq r \leq j-2$$

and consider the partition given by

$$\mu_i = \begin{cases} q+1 & \text{for } 1 \leq i \leq j, \\ N & \text{for } j+1 \leq i \leq n, \end{cases}$$

where $N \gg 0$ is a sufficiently large integer such that $N \geq q+1$ and $j(q+1) + N(i-j) \geq k \sum_i \lambda$ for each $i > j$. By construction, we have that $\sum_i \mu \geq k \sum_i \lambda$ for $i > j$. Also, if $p \in P_n$ and $a, b \in \mathbb{N}$ with $a \leq b$, then

$$\sum_b p \geq \sum_a p + (b-a)p_a \geq \sum_a p + \frac{b-a}{a} \sum_a p = \frac{b}{a} \sum_a p.$$

Now, for the partition μ defined in (5.1) and for $i < j$, we have that

$$\begin{aligned} \sum_i \mu &= i(q+1) \\ &= i\left[\frac{k \sum_{j-1} \lambda - r}{j-1} + 1\right] \quad \text{by (5.1)} \\ &\geq k \sum_i \lambda + \frac{i(j-r-1)}{j-1} \quad \text{by (5.1)} \quad (\text{with } a = i,\ b = j-1,\ p = \lambda) \\ &\geq k \sum_i \lambda \quad \text{since } j - r - 1 \geq 0. \end{aligned}$$

Similarly, we have

$$\sum_j \mu = j(q+1) = \frac{jk\sum_{j-1}\lambda}{j-1} + \frac{j(j-r-1)}{j-1}$$
$$= k\sum_{j-1}\lambda + k\left[\frac{\sum_{j-1}\lambda}{j-1} + \frac{j(j-r-1)}{k(j-1)}\right],$$

whence $\sum_j \mu \geq k\sum_j \lambda$ if and only if

$$\lambda_j \leq \frac{\sum_{j-1}\lambda}{j-1} + \frac{j(j-r-1)}{k(j-1)} = \frac{q+j-r}{k}.$$

Thus, $x^\mu \in I_{n,j}^{(ka_j)}$ if and only if (5.1) is satisfied.

Whenever the inequality (5.1) fails, we have $\bigcap_{i\neq j} I_{n,i}^{(ka_i)} \not\subseteq I_{n,j}^{(ka_j)}$ and hence $I_{n,j}^{(ka_j)}$ is needed in (5.1). Assume $k > j(j-1)$. Then, we see that

$$\left\lfloor \frac{\sum_{j-1}\lambda}{j-1} + \frac{j(j-r-1)}{k(j-1)} \right\rfloor \leq \frac{\sum_{j-1}\lambda}{j-1}.$$

If (5.1) holds, it implies $\lambda_j \leq \frac{\sum_{j-1}\lambda}{j-1}$. But this is possible if and only if $\lambda_1 = \cdots = \lambda_{j-1} = \lambda_j$, in which case equality holds in (5.1). Since by assumption $\lambda_1 < \lambda_j$, (5.1) must then fail, so $I_{n,j}^{(ka_j)}$ is not redundant.

(5) If $\lambda_1 = \lambda_j$ for some $j > 1$, then $\lambda_1 = \cdots = \lambda_{j-1} = \lambda_j$, whence $\lambda_j = \frac{\sum_{j-1}\lambda}{j-1}$. Let $\mu \in P_n$ be defined as in (5.1). If $x^\mu \in I_{n,j-1}^{(ka_{j-1})}$, (5.1) with $a = j-1$, $b = j$, $p = \mu$, then it yields

$$\sum_j \mu \geq \frac{j}{j-1}\sum_{j-1}\mu \geq \frac{j}{j-1}k\sum_{j-1}\lambda \geq k\sum_j \lambda.$$

This shows that $I_{n,j-1}^{(ka_{j-1})} \subseteq I_{n,j}^{(a_j)}$ and thus the latter ideal is redundant in (5.1).

The formula for the primary decomposition follows by substituting

$$I_{n,j}^{(ka_j)} = \bigcap_{\sigma\in\mathfrak{S}_n} \sigma(P_j)^{ka_j}$$

in (5.1) and removing redundant components. In detail, $\sigma(P_j)^{ka_j}$ is irredundant in

$$I^k = \bigcap_{j=1}^{n}\bigcap_{\sigma\in\mathfrak{S}_n} \sigma(P_j)^{ka_j}$$

if and only if $\sigma(P_j) \in \operatorname{Ass}(I^k)$. As I^k is symmetric, $P_j \in \operatorname{Ass}(I^k)$ if and only if $\sigma(P_j) \in \operatorname{Ass}(I^k)$ for all $\sigma \in \mathfrak{S}_n$. Therefore, the primary components $\sigma(P_j)^{ka_j}$ are either all simultaneously redundant in (5.1), in which case $I_{n,j}^{(ka_j)}$ is redundant in (5.1), or simultaneously irredundant in (5.1), in which case $I_{n,j}^{(ka_j)}$ is irredundant in (5.1). Formula (5.8) then follows from (5.1) by means of statements (1)–(5) and the previous considerations provided $k > j(j-1)$ is satisfied for every j. ■

Combining the previous results together, we can finally determine the stable set of associated primes of a principal Borel symmetric strongly shifted ideal.

Theorem 5.9. *Let* $I = \operatorname{Sss}(\{\lambda\})$. *Adopt the notation of Theorem 5.6, where* $j' = \min(\lambda)$. *Then,*

$$\operatorname{Ass}^\infty(I) = \begin{cases} \{\sigma(P_j) : j' \le j \le n,\ \sigma \in \mathfrak{S}_n\} & \text{if } j' > 1, \\ \{\sigma(P_j) : j = 1 \text{ or } (j > 1 \text{ and } \lambda_j \ne \lambda_1),\ \sigma \in \mathfrak{S}_n\} & \text{if } j' = 1. \end{cases}$$

Moreover,

(1) *if either* $\lambda = (a, \dots, a)$ *for some* $a \in \mathbb{N}$, *or* $\lambda \in P_n$ *has no repeated parts other than possibly allowing for repetitions of* λ_1, *then* $\operatorname{astab}(I) = \operatorname{dstab}(I) = 1$ *and for each* $k \in \mathbb{N}$, $I^k = I^{(k)_{\mathrm{Ass}}}$;

(2) *otherwise, setting* $s = \max\{j : \lambda_1 < \lambda_{j-1} = \lambda_j\}$, *we have*

$$\operatorname{dstab}(I) \le \operatorname{astab}(I) \le \min\{n-1, s(s-1)+1\}.$$

Proof. The claim regarding the stable set of associated primes follows from the primary decomposition (5.8) in Theorem 5.8.

For (1), from Theorem 5.8 (3) and (5), it follows that, in either case, $\operatorname{Ass}(I) = \operatorname{Ass}(I^k)$ for every $k \ge 1$, whence $\operatorname{astab}(I) = 1$. Moreover, $\operatorname{dstab}(I) = 1$ by Proposition 5.5 (1), while (2.1) implies the claim about the symbolic powers $I^{(k)_{\mathrm{Ass}}}$.

For (2), notice that $\mathfrak{m} \in \operatorname{Ass}(I^k)$ for all $k > s(s-1)$ by Theorem 5.8 (3) and (4). Hence, the conclusion follows from the persistence property and Proposition 5.5 (2). ■

Remark 5.10. From Theorem 5.9, it follows that $\operatorname{astab}(I) = \operatorname{dstab}(I) = 1$ for many principal Borel sssi's. While this equality is known to hold for transversal polymatroidal ideals by [29, Corollaries 4.6 and 4.14], Example 3.18 and Theorem 5.9 together show that there exist symmetric polymatroidal ideals with

$$\operatorname{astab}(I) = \operatorname{dstab}(I) = 1$$

which are not transversal.

Remark 5.11. It follows from (2.1) that, for an ideal I, the equalities $I^k = I^{(k)_{\mathrm{Ass}}}$ for $1 \le k \le \operatorname{astab}(I)$ imply that $I^k = I^{(k)_{\mathrm{Ass}}}$ for all $k \ge 1$.

In particular, by Theorem 5.3 (2) and Proposition 5.5 (2), a principal Borel symmetric strongly shifted ideal satisfies $I^k = I^{(k)_{\mathrm{Ass}}}$ for all $k \ge 1$ if and only if $I^k = I^{(k)_{\mathrm{Ass}}}$ for $1 \le k \le n-1$. While the latter statement is true for every polymatroidal ideal by [25, Theorem 4.1], Theorem 5.9 shows that for a principal Borel sssi it suffices to check equality of powers and symbolic powers in a potentially smaller range for $1 \le k \le \min\{n-1, s(s-1)+1\}$. This addresses a question by Huneke, which is open for arbitrary monomial ideals.

In [30], Herzog and Vladoiu define a monomial ideal to be of *intersection type* if it can be decomposed as an intersection of powers of monomial primes. Principal Borel sssi's satisfy this property by Theorem 5.6. In fact, in [30, Proposition 2.1], it is shown that every polymatroidal ideal is of intersection type. Moreover, for every polymatroidal ideal I generated in degree d, one has that $I^{(dk)_{\mathrm{Min}}} \subseteq I^k$ for any $k \ge 1$; see [30, Corollary 3.5]. Our contribution below is to strengthen this containment for principal Borel sssi's.

Proposition 5.12. *Let $I = \operatorname{Sss}(\lambda)$ be a principal Borel sssi, with*

$$d = |\lambda| \quad \text{and} \quad c = \min(\lambda).$$

Then, $I^{(m)_{\mathrm{Min}}} \subseteq I^k$ whenever $m/k \ge d/\lambda_c$.

Proof. Recall from Theorem 5.8 (2) that $I^{(m)_{\mathrm{Min}}} = I_{n,c}^{(\lambda_c m)}$. Let $\mu \in \Lambda(I_{n,c}^{(\lambda_c m)})$ and assume that $m/k \ge d/\lambda_c$, i.e., $\lambda_c m \ge dk$. Then, by (5.1), for each $j \ge c$, we have

$$\begin{aligned}\sum_{i=1}^{j} \mu_i &= \sum_{i=1}^{c} \mu_i + \sum_{i=c+1}^{j} \mu_i \ge \lambda_c m + (j-c)\frac{\lambda_c m}{c} \\ &= \frac{\lambda_c m j}{c} \ge \frac{dkj}{c} \ge dk \ge \left(\sum_{i=1}^{j} \lambda_i\right) k,\end{aligned}$$

since $d = \sum_{i=1}^{n} \lambda_i$. In view of (5.1), the above inequality shows that $I_{n,c}^{(\lambda_c m)} \subseteq I_{n,j}^{(a_j k)}$ for each $c \le j \le n$ and for $a_j = \sum_{i=1}^{j} \lambda_i$. It then follows by Theorem 5.6 that

$$I_{n,c}^{(\lambda_c m)} \subseteq I^k. \qquad \blacksquare$$

The above result is particularly relevant to the *Containment Problem*, which asks for an ideal I to determine the pairs m, k so that $I^{(m)} \subseteq I^k$. This is an important and well-studied problem in commutative algebra, which is open in its full generality. We refer the reader to [12, 23, 40] for known results in the case of monomial ideals.

5.2. The intersection property of a principal Borel sssi

The decomposition formula in Theorem 5.6 mimics analogous decompositions for principal Borel ideals or transversal polymatroidal ideals; see [20, Theorem 3.1] and [29, Corollary 4.10]. Inspired by these results, in [7], Bruns and Conca defined an ideal I to be *P-adically closed* if

$$I = \bigcap_{P \in \mathrm{Ass}(I)} P^{(v(P))}, \quad \text{where } v(P) = \max\{t : I \subseteq P^{(t)}\}.$$

Since the associated primes of monomial ideals are generated by a subset of the variables, that is, a regular sequence, one obtains that a *monomial* ideal I is P-adically closed if

$$I = \bigcap_{P \in \mathrm{Ass}(I)} P^{v(P)}, \quad \text{where } v(P) = \max\{t : I \subseteq P^{t}\}.$$

It thus follows from Theorem 5.6 that principal Borel sssi's are P-adically closed, since for each j the exponent a_j coincides with $v(\sigma(P_j))$ for every $\sigma \in \mathfrak{S}_n$.

We now seek to characterize P-adically closed symmetric monomial ideals.

Proposition 5.13. *A symmetric monomial ideal is P-adically closed if and only if it can be decomposed as*

$$I = \bigcap_{i=1}^{t} I_{n,c_i}^{(v_i)},$$

where $v_i = v(x_1, \ldots, x_{c_i})$. Moreover, all such ideals I are symmetric strongly shifted.

Proof. It is clear from the definition that an ideal admitting such a decomposition is P-adically closed. Moreover, since each $I_{n,c_i}^{(v_i)}$ is symmetric strongly shifted by [2, Theorem 4.3], then I is symmetric strongly shifted by Proposition 2.1.

To prove the converse, let I be a P-adically closed symmetric monomial ideal. We first show that if I is height-unmixed with $\mathrm{ht}(I) = c$, then I is a symbolic power of a square-free Veronese ideal. This is because $\mathrm{Ass}(I)$ is closed under the action of $\mathfrak{S}_n$ and the monomial primes of height equal to $\mathrm{ht}(I)$ form a single orbit under the action of $\mathfrak{S}_n$. Moreover, the symmetry of I yields that for $P = (x_1, \ldots, x_c)$ and each $\sigma \in \mathfrak{S}_n$

$$v(P) = \max\{t : I \subseteq P^t\} = \max\{t : I \subseteq \sigma(P)^t\} = v(\sigma(P)).$$

Thus, setting $v = v(x_1, \ldots, x_c)$, equations (5.2) and (2.6) yield

$$I = \bigcap_{\sigma \in \mathfrak{S}_n} \sigma(x_1, \ldots, x_c)^v = \Big(\bigcap_{\sigma \in \mathfrak{S}_n} \sigma(x_1, \ldots, x_c) \Big)^{(v)} = I_{n,c}^{(v)}.$$

Now, suppose that I is not height-unmixed, with associated primes of distinct heights $c_1, \dots, c_t$. Then, for each i, the ideal $\bigcap_{P \in \mathrm{Ass}(I):\mathrm{ht}(P)=c_i} P^{v(P)}$ is a P-adically closed symmetric ideal that is height-unmixed. Hence, setting $v_i = v(x_1, \dots, x_{c_i})$, equation (5.2) implies that

$$I = \bigcap_{i=1}^{t} \Big(\bigcap_{P \in \mathrm{Ass}(I):\mathrm{ht}(P)=c_i} P^{v(P)} \Big) = \bigcap_{i=1}^{t} I_{n,c_i}^{(v_i)}. \qquad \blacksquare$$

Comparing the previous result with Theorem 5.6, it would be natural to ask whether a P-adically closed sssi must be a principal Borel sssi. The following example shows that this is not necessarily true. This is in sharp contrast with the case of strongly stable ideals, which can only be written as intersections of powers of monomial prime ideals if they are principal Borel; see [30, Proposition 2.8].

Example 5.14. The ideal $I = I_{5,1} \cap I_{5,3}^{(4)}$ is P-adically closed by Proposition 5.13, but it has partition Borel generating set $B(I) = \{(1,1,3,3,3), (1,2,2,2,2)\}$.

We conclude this section by stating another interesting consequence of Theorem 5.6. Recall that an ideal I is said to be *sequentially Cohen–Macaulay* if there exists a filtration of R-modules

$$D_0 = 0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_s = R/I$$

so that, for all $1 \le i \le s$, $\dim(D_{i-1}) < \dim(D_i)$ and the quotient modules $C_i = D_i/D_{i-1}$ are Cohen–Macaulay R-modules. In particular, a Cohen–Macaulay ideal is sequentially Cohen–Macaulay, as it suffices to construct the D_i's by going modulo a maximal regular sequence, one element at a time. The following result can be interpreted as a generalization of the fact that ideals of monomial star configurations are Cohen–Macaulay [21, Proposition 2.9].

Proposition 5.15. *Let I be a principal Borel sssi. Then, I is sequentially Cohen–Macaulay.*

Proof. By Theorem 5.6, I can be decomposed as an intersection of symbolic powers of square-free Veronese ideals (also known as ideals defining monomial star configurations). The conclusion now follows from [38, Proposition 3.1]. $\blacksquare$

6. Toric ideals and Rees algebras

Given a finite set of monomials $G = \{m_1, \dots, m_s\} \subset R$, the toric ring of G is the subring $K[G] = K[m_1, \dots, m_s]$ of R. Let $A = K[T_1, \dots, T_s]$ denote a polynomial ring in s new indeterminates over K and define a surjective homomorphism $\pi : A \twoheadrightarrow K[G]$

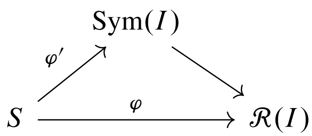
by $\pi(T_i) = m_i$. The *toric ideal* of G is $\mathcal{I}_G = \ker(\pi)$. In other words, the toric ideal of G is the defining ideal of the toric ring $K[G]$. If $G = G(I)$, we often extend the terminology by referring to $K[G]$ and $\mathcal{I}_G$ as the toric ring and toric ideal of I.

Toric rings of equigenerated monomial ideals are coordinate rings of *projective toric varieties*. Indeed, the Zariski closure X_G for the image of the map $\mathbb{P}^{n-1} \to \mathbb{P}^{s-1}$ given by

$$(x_1, \ldots, x_n) \mapsto (m_1, \ldots, m_s)$$

has defining ideal $\mathcal{I}_G$ and coordinate ring $K[G]$.

Toric rings are better understood by considering the blow-up algebras of the ideal I. Recall that the *Rees ring* of an ideal $I \subset R = K[x_1, \ldots, x_n]$, $\mathcal{R}(I) = \bigoplus_{i\geq 0} I^i t^i$, is a quotient of a polynomial ring $S = R[T_1, \ldots, T_s]$ under a ring homomorphism $\varphi : S \twoheadrightarrow \mathcal{R}(I)$ given by $T_i \mapsto f_i t$, with f_i the i-the generator of I. The map φ factors through $\mathrm{Sym}(I)$, the symmetric algebra of I, as indicated in the diagram below.



The *fiber cone*, or *special fiber ring*, of I is the quotient $\mathcal{F}(I) = \mathcal{R}(I)/(x_1, \ldots, x_n) \cong \mathcal{R}(I) \otimes_R K$. If $G = G(I)$, then, in the notation above, $A = S \otimes_R K$ and $\pi = \varphi \otimes_R K$; thus, we recognize by comparing presentations that $\mathcal{F}(I) \cong K[G]$ is the toric ring of G.

From (6), it follows that $\mathrm{Sym}(I)$, $\mathcal{R}(I)$, and $\mathcal{F}(I)$ can be described as quotients of polynomial rings. Understanding the kernels of the maps φ, φ' in (6) provides structure theorems for these algebras, identifying a presentation in terms of generators and relations.

Since $\mathcal{K} := \mathrm{Ker}(\varphi) \supseteq \mathcal{L} := \mathrm{Ker}(\varphi')$, one always has that the relations defining the symmetric algebra of I are also relations for the Rees algebra. In fact, it turns out that $\mathcal{L}$ consists of the elements of $\mathcal{K}$ that are linear in the variables T_i, which we write as $\mathcal{L} = \mathcal{K}_{(*,1)}$ [50]. Moreover, if we denote $\mathcal{J} := \mathcal{K} \otimes_R K \subseteq K[T_1, \ldots, T_s]$, by construction it is clear that $\mathcal{J} S \subseteq \mathcal{K}$. Thus, the relations of the fiber cone are also relations for the Rees algebra, whence $\mathcal{K} \supseteq \mathcal{L} + \mathcal{J} S$. An ideal I is said to be of *fiber type* if the latter containment is an equality. For $G = G(I)$, we have that $\mathcal{J} = \mathcal{I}_G$ is the toric ideal of G. It is well known that every toric ideal is a prime ideal generated by binomials; see, e.g., [27, Proposition 10.1.1].

6.1. Fiber type property

In this section, we prove that every equigenerated symmetric strongly shifted ideal is of fiber type (see Theorem 6.2). This property constitutes yet another similarity between symmetric strongly shifted ideals and strongly stable ideals, which are also of fiber type by [28, Theorem 5.1 and Example 4.2].

A key ingredient in our proof is the fact that, for an equigenerated ideal I, one can define a grading on $S = R[T_1, \dots, T_s]$ by setting

$$\deg(r) = (\deg_R(r), 0) \quad \text{for } r \in R \text{ and } \deg(T_i) = (0, 1).$$

With this grading, $\mathcal{F}(I) \cong [\mathcal{R}(I)]_{(0,*)}$ and $\mathrm{Sym}(I) \cong [\mathcal{R}(I)]_{(*,1)}$. Hence, I is of fiber type if and only if the ideal $\mathcal{K}$ defining the Rees algebra is generated in bidegrees $(0, *)$ and $(*, 1)$. We also crucially use the fact that equigenerated symmetric shifted ideals have *linear resolutions* by [2, Theorem 3.2]. The following lemma yields more information on their syzygies.

Lemma 6.1. *Let I be an equigenerated symmetric shifted ideal. The syzygies on I are generated by relations of the form $x_i u - x_{u_{\max}} v$, where $v \prec u \in G(I)$, $x_i \in C(u)$, and $\prec$ is defined in Definition 3.20 while $C(u)$ and $u_{\max}$ are defined in (3.2).*

Proof. The proof utilizes the notation in (3.2). Set $J = (v \in G(I) : v \prec u)$ and recall that $J : (u) = C(u)$ by [2, proof of Theorem 3.2]. A resolution for I can be constructed as an (iterated) mapping cone from the resolution of J and that of $C(u)$ utilizing the short exact sequence

$$0 \to R/(J : u) \to R/J \to R/(J + (u)) \to 0.$$

In particular, this yields that the relations on I are generated by the relations on J together with the relations of the form $x_i u - w$ with $x_i \in C(u)$ and $w \in J$. Now, take $x_i \in C(u)$ and set $v = u x_i / x_{u_{\max}}$. From the symmetric shifted property of I and the definitions of $C(u)$ and $u_{\max}$ in (3.2), one deduces that $v \in I$ and that $v \prec u$. Moreover, since I is equigenerated and $\deg(v) = \deg(u)$, it follows that $v \in G(I)$ and hence $v \in J$. Since every relation $x_i u - w$ as above can be written as

$$x_i u - w = x_i u - x_{u_{\max}} v + (x_{u_{\max}} v - w),$$

we deduce that the syzygies on I are generated by the syzygies of J and the set of relations $x_i u - x_{u_{\max}} v$. The desired conclusion then follows by induction on the number of monomial generators of I. ∎

Theorem 6.2. *An equigenerated symmetric strongly shifted ideal is of fiber type. The defining relations of its Rees algebra are generated in bidegrees $(0, *)$ and $(1, 1)$ with respect to the grading (6.1).*

Proof. Using the notation in (6), set $\mathcal{K} = \ker(\varphi)$ to be the set of relations of $\mathcal{R}(I)$ and set $\mathcal{L} = \ker(\varphi')$ to be the set of relations of $\mathrm{Sym}(I)$. Let $\mathcal{J}$ be the kernel of the map $\varphi \otimes_R K : K[T_1, \ldots, T_s] \twoheadrightarrow \mathcal{F}(I)$. Our goal is to show that $\mathcal{K} = \mathcal{L} + \mathcal{J}S$. With the containment $\mathcal{L} + \mathcal{J}S \subseteq \mathcal{K}$ being evident, we proceed to establish the opposite containment $\mathcal{K} \subseteq \mathcal{L} + \mathcal{J}S$.

Since I is a monomial ideal, $\mathcal{K}$ is generated by homogeneous binomials. Consider a minimal generator f for $\mathcal{K}$ of bidegree (d, k). If $d \neq 0$, f corresponds to a minimal relation of degree d on I^k. However, since I is symmetric strongly shifted and equigenerated, the same is true of I^k by Proposition 2.2 and thus by [2, Theorem 3.2] I^k has a linear minimal free resolution. It then follows from the minimality of f that $d = 1$. Thanks to Lemma 6.1, we may also assume that f has the form

$$f = x_i T_{i_1} \cdots T_{i_k} - x_{u_{\max}} T_{j_1} \cdots T_{j_k},$$

where $u = \varphi(T_{i_1} \cdots T_{i_k}) = f_{i_1} \cdots f_{i_k}$ with $f_{i_\ell} \in G(I)$. Set

$$v = \varphi(T_{j_1} \cdots T_{j_k}) = f_{j_1} \cdots f_{j_k}$$

and notice that $\varphi(f) = 0$ implies $x_i u = x_{u_{\max}} v$.

By definition of $C(u)$, since $x_i \in C(u)$, the exponent of the variable $x_{u_{\max}}$ in u is larger than the exponent of x_i in u. Thus, the same must be true for at least one f_{i_ℓ}. Set $f_t = f_{i_\ell} x_i / x_{u_{\max}}$. As I is symmetric strongly shifted, we have $f_t \in G(I)$ and $x_i T_{i_\ell} - x_{u_{\max}} T_t \in K$, which yields

$$\begin{aligned} f &= x_i T_{i_1} \cdots T_{i_k} - x_{u_{\max}} T_{j_1} \cdots T_{j_k} \\ &= (x_i T_{i_\ell} - x_{u_{\max}} T_t) T_{i_1} \cdots T_{i_{\ell-1}} T_{i_{\ell+1}} \\ &\quad + x_{u_{\max}} (T_{i_1} \cdots T_{i_{\ell-1}} T_t T_{i_{\ell+1}} \cdots T_{i_k} - T_{j_1} \cdots T_{j_k}) \in \mathcal{L} + \mathcal{J}S. \end{aligned}$$

The above equation implies the fiber-type property and also shows that $\mathcal{K}$ is generated by its elements of bidegrees $(1, 1)$ and $(0, *)$. ■

Remark 6.3. In the proof of Theorem 6.2, the assumption that I is symmetric strongly shifted rather than just symmetric shifted was only used in order to apply Proposition 2.2. This is because we do not currently know whether symmetric (not strongly) shifted ideals are closed under powers. In particular, a positive answer to Question 2.4 would guarantee that equigenerated symmetric shifted ideals are of fiber type.

On the contrary, the following example shows that a symmetric (strongly) shifted ideal which is not equigenerated need not be of fiber type.

Example 6.4. Consider the symmetric strongly shifted ideal with

$$\mathrm{B}(I) = \{(1, 1, 1), (0, 2, 2)\}.$$

It is given by

$$I = (x_1 x_2 x_3,\ x_2^2 x_3^2,\ x_1^2 x_3^2,\ x_1^2 x_2^2)$$

and it is not equigenerated. Calculations on Macaulay2 [22] show that the Rees algebra of I has the following minimal presentation

$$\mathcal{R}(I) = \frac{R[T_1, T_2, T_3, T_4]}{(x_2x_3T_1 - x_1T_2, x_1x_3T_1 - x_2T_3, x_1x_2T_1 - x_3T_4, x_3^2T_1^2 - T_2T_3, x_2^2T_1^2 - T_2T_4, x_1^2T_1^2 - T_3T_4)}.$$

The last three listed relations of $\mathcal{R}(I)$ demonstrate that I is not of fiber type. In particular, the equigeneration hypothesis is needed in Theorem 6.2.

6.2. The toric ideal of a principal Borel sssi

In this subsection, we focus on Rees algebras and fiber cones of principal Borel symmetric strongly shifted ideals. By Theorem 3.14 such an ideal I is a polymatroidal ideal and hence enjoys the symmetric exchange property described in the comments following Definition 3.13. This property yields for each pair $r = x^u$, $s = x^v \in G(I)$ with $u_i > v_i$ an index j, and so that

$$t := x^u x_j/x_i \in G(I), \quad w := x^v x_i/x_j \in G(I), \quad \text{and thus } rs = tw.$$

The last identity implies that $T_rT_s = T_tT_w$ in the toric ring $\mathcal{F}(I)$. Equivalently, the binomial $T_rT_s - T_tT_w$, termed a *symmetric exchange relation*, belongs to the defining ideal $\mathcal{J}$ of $\mathcal{F}(I)$.

Our first result shows that the defining ideals of the fiber cone and Rees algebra of a principal Borel sssi are generated by quadrics. In particular, the toric ideal of a principal Borel sssi is generated by its symmetric exchange relations.

Below we use the notation $\deg_i(m)$ to mean the exponent of x_i in a monomial m.

Theorem 6.5. *Let I be a principal Borel sssi. Then, the toric ideal of $G(I)$, also known as the defining ideal of $\mathcal{F}(I)$, is generated by quadrics, namely, the symmetric exchange relations*

$$T_rT_s - T_tT_w,$$

where $r, s, t, w \in G(I)$ satisfy $\deg_i(r) > \deg_i(s)$, $\deg_j(r) < \deg_j(s)$, $t = rx_j/x_i \in G(I)$, and $w = sx_i/x_j \in G(I)$.

Moreover, the defining ideal of $\mathcal{R}(I)$ is also generated by quadrics, specifically by the exchange relations in (6.5) (viewed as elements of $R[T_1, \ldots, T_s]$) together with the relations

$$x_iT_u - x_{u_{\max}}T_v,$$

where $v \prec u \in G(I)$ cf. Definition 3.20, $x_i \in C(u)$, and $C(u)$ and $u_{\max}$ are as in (3.2).

Proof. It follows from Theorem 3.12 that

$$I = \operatorname{Sss}(\{\lambda\}) = \prod_{i=1}^{n} I_{n,i}^{\lambda_i - \lambda_{i-1}}$$

is a product of square-free Veronese ideals. In particular, each factor is a polymatroidal ideal which satisfies the strong exchange property. Now, notice that for any two polymatroidal ideals J_1 and J_2 with polymatroidal bases $B_1 = G(J_1)$ and $B_2 = G(J_2)$ the set

$$B_1 B_2 = \{b_1 b_2 \mid b_1 \in B_1,\ b_2 \in B_2\},$$

is a polymatroidal base for the polymatroidal ideal $J_1 J_2$, i.e.,

$$G(J_1 J_2) = B_1 B_2$$

(see [11, Theorem 5.3] and [41, p. 4]). Hence, I admits a polymatroidal basis which is a product of polymatroidal bases with the strong exchange property. Therefore, [41, Theorem 3.5] implies that the defining ideal of $\mathcal{F}(I)$ is generated by the symmetric exchange relations.

The claim on the Rees algebra follows from the fiber type property of I, Theorem 6.2, and in particular from the computations in equation (6.1) in the proof of this result. The fact that the defining ideal of $\mathcal{R}(I)$ is quadratic, without the detailed knowledge of the generators (6.5) can also be deduced from [41, Theorem 5.2]. ∎

Since the principal Borel ideals are the symmetric polymatroidal ideals by Theorem 3.14, from Theorem 6.5 we deduce the following result, which answers in the affirmative conjectures of White [51] and Herzog–Hibi [26] in the special case of symmetric polymatroids.

Conjecture 6.6 ([26, 51]). *For a polymatroidal ideal I, the toric ideal of $G(I)$ is generated by the symmetric exchange relations.*

Corollary 6.7. *Every symmetric polymatroidal ideal satisfies Conjecture 6.6.*

Although the proof of Theorem 6.5 heavily utilizes the polymatroidal nature of principal Borel sssi's, we do not have any examples of equigenerated symmetric strongly shifted ideals whose fiber cone cannot be generated by quadrics. Therefore, we ask the following.

Question 6.8. Is the toric ideal of any equigenerated symmetric strongly shifted ideal quadratic?

By contrast, the following example shows that, if I is symmetric shifted but not strongly shifted, the defining ideal of the special fiber ring of $\mathcal{F}(I)$ may not be generated by quadrics.

Example 6.9. In $k[x_1, x_2, x_3, x_4]$, the equigenerated symmetric shifted ideal I with

$$\Lambda(I) = \{(1,1,2,2), (0,2,2,2), (0,1,2,3)\}$$

is not strongly shifted; see [2, Example 2.5]. Moreover, Macaulay2 [22] shows that the defining ideal of $\mathcal{F}(I)$ contains 28 minimal cubic relations.

While Conjecture 6.6 is open for arbitrary polymatroidal ideals, it is indeed satisfied by several classes of polymatroidal ideals. These include, for instance, polymatroidal ideals satisfying the strong exchange property [26, Theorem 5.3 (b)], principal Borel ideals [14, 26], lattice path polymatroidal ideals [46, Theorem 2.10], and polymatroidal ideals satisfying the so-called *one-sided strong exchange property* [39, Theorem 1.2]. A version of Conjecture 6.6 “up to saturation" was settled in [37].

In all of the mentioned cases, the defining ideal of the special fiber ring $\mathcal{F}(I)$ is in fact generated by a Gröbner basis of quadrics. The latter condition is satisfied if the algebra generators of I are *sortable* [49], a condition which unfortunately does not necessarily hold for an arbitrary principal Borel sssi. When $\mathcal{F}(I)$ is generated by a Gröbner basis of quadrics, $\mathcal{F}(I)$ is a Koszul algebra. Recall that a standard graded algebra A over a field K is *Koszul* if the residue class field A/K has a linear A-resolution.

Another class of polymatroidal ideals whose fiber cone $\mathcal{F}(I)$ is Koszul is that of transversal polymatroidal ideals; see [10, Theorem 3.5]. For an ideal I of this kind, in [10, Proposition 3.7] Conca proved that $\mathcal{F}(I)$ is generated by quadratic polynomials, which however need not coincide with the symmetric exchange relations. In fact, to the best of our knowledge, Conjecture 6.6 is open for arbitrary transversal polymatroidal ideals. (We refer the reader to [37] for a proof for transversal matroidal ideals.)

The following result provides classes of principal Borel sssi with Koszul toric ring.

Corollary 6.10. *Let* $I = \mathrm{Sss}(\{\lambda\})$ *be a principal Borel symmetric strongly shifted ideal. Suppose that* λ *is one of the following types:*

(1) $\lambda = (a, \dots, a)$ *for some* $a \neq 0 \in \mathbb{N}$*;*

(2) $\lambda = (a^s, b^{n-s})$ *for some* $a < b \in \mathbb{N}$, $s > 0$*;*

(3) $\lambda = (a^s, b, c^{n-s-1})$ *for some* $a < b < c \in \mathbb{N}$, $s > 0$.

(4) λ *satisfies* $\Delta^i(\lambda)_1 \geq 0$ *for all* $1 \leq i \leq n$.

Then, the toric ring of I*, equivalently, the special fiber ring* $\mathcal{F}(I)$*, is a Koszul algebra.*

Proof. By Proposition 3.15, if λ is of one of the first three given types, then I satisfies the strong exchange property. The conclusion now follows from [26, Theorem 5.3 (b)]. If λ has the fourth listed property, Proposition 3.16 yields that $\mathrm{Sss}(\{\lambda\})$ is transversal, so the desired conclusion follows from [10, Theorem 3.5]. ∎

It is also known that high Veronese subrings of graded rings are Koszul [1, 16]. In this vein, we can establish the Koszul property of principal sssi's up to taking a sufficiently high multiple of the partition Borel generator.

Proposition 6.11. *Let $\lambda \in P_n$ be a partition. Then, for sufficiently large integers k, the toric ring of the ideal $I = \mathrm{Sss}(\{k\lambda\})$ has a quadratic Gröbner basis. In particular, the toric ring of I, $\mathcal{F}(I)$, is a Koszul algebra.*

Proof. Recall that $I = \mathrm{Sss}(\{k\lambda\}) = \mathrm{Sss}(\{\lambda\})^k$ and hence

$$\mathcal{F}(I) = \bigoplus_{i \geq 0} \mathrm{Sss}(\{\lambda\})^{ki}$$

is the k-th Veronese subring of $T = \mathcal{F}(\mathrm{Sss}(\{\lambda\}))$. It is established in [16, Theorem 2] that the defining ideal $\mathcal{J}$ of $\mathcal{F}(I)$ has an initial ideal generated in degree $\leq \max\{\lceil \mathrm{reg}(T)/k \rceil, 2\}$, where reg denotes the Castelnuovo–Mumford regularity. Therefore, the initial ideal of $\mathcal{J}$ has a quadratic Gröbner basis whenever

$$k \geq \mathrm{reg}(T)/2.$$ ∎

While not all principal Borel sssi's satisfy the assumptions of Corollary 6.10, we do not know of any examples of principal Borel sssi's whose fiber cones are not Koszul. Thus, we pose the following question, which can be interpreted as the symmetric case of a similar question raised by Herzog and Hibi for arbitrary polymatroidal ideals in [26, p. 241].

Question 6.12. Is the toric ring of any principal Borel symmetric strongly shifted ideal Koszul?

We conclude this section by describing the geometry of the toric rings associated to principal Borel sssi's. Our first result holds more generally for every equigenerated, normal symmetric strongly shifted ideal, and follows from well-known properties of toric rings of normal ideals; see [27, Theorem B.6.2] and [4, 31].

Corollary 6.13. *Let $I = \mathrm{Sss}(B)$ be an equigenerated, normal sssi (e.g., I satisfies one of the conditions in Proposition 4.3). Then, the Rees ring $\mathcal{R}(I)$ and the special fiber ring $\mathcal{F}(I)$ are Cohen–Macaulay normal domains. Moreover, $\mathcal{F}(I)$ has rational singularities if K is of characteristic 0 and is strongly F-regular if K is of positive characteristic.*

A convex lattice polytope $\mathbf{P}$ is said to be *normal*, or to have the *integer decomposition property*, if it satisfies the following condition: given any positive integer d, every lattice point of the dilation $d \cdot \mathbf{P}$ can be written as the sum of exactly d lattice points in $\mathbf{P}$. Let I be the ideal generated by all monomials with exponents in $\mathbf{P}$. Normality of $\mathbf{P}$ is equivalent to $\overline{I^d} = I^d$ for positive integers d, hence to I being normal. As a consequence of Proposition 4.6 and Proposition 4.3, we thus obtain the following.

Corollary 6.14. *For each $\lambda \in P_n$, the permutohedron $\mathbf{P}(\lambda)$ is a normal polytope.*

A normal lattice polyhedron $\mathbf{P}$ uniquely determines a projective toric variety $X_{\mathbf{P}}$ by means of its normal fan; see [13, Definition 2.3.14]. We term the toric variety defined by the permutohedron $\mathbf{P}(\lambda)$ with respect to the lattice $\mathbb{Z}^n/\text{span}(1,\ldots,1)$ the *permutohedral toric variety* $X_{\mathbf{P}(\lambda)}$. The homogeneous coordinate ring for the image of the projective embedding $X_{\mathbf{P}(\lambda)} \hookrightarrow \mathbb{P}^N$ given by the divisor $D_{\mathbf{P}(\lambda)}$ is in our notation $K[G(\text{Sss}(\lambda))] = \mathcal{F}(\text{Sss}(\lambda))$ and hence the defining equations of $X_{\mathbf{P}(\lambda)}$ in this embedding are given by the toric ideal of $\text{Sss}(\lambda)$. Since $\mathbf{P}(\lambda)$ is normal, $D_{\mathbf{P}(\lambda)}$ is very ample and $X_{\mathbf{P}}$ is projectively normal. Questions 6.8 and 6.12 arise naturally for this class of algebraic sets. Our work yields the following answer.

Corollary 6.15. *For any $\lambda \in P_n$, the defining ideal of the permutohedral toric variety $X_{\mathbf{P}(\lambda)}$ is generated by quadratic polynomials. If λ is of one of the types described in Corollary 6.10, then $X_{\mathbf{P}(\lambda)}$ has a Koszul coordinate ring.*

Proof. The claims follows from Theorem 6.5 and Corollary 6.10, since the coordinate ring of $X_{\mathbf{P}(\lambda)}$ is $\mathcal{F}(\text{Sss}\{\lambda\})$. ∎

The case $\lambda = (0, 1, \ldots, n-1)$, which yields the standard permutohedron and the (standard) *permutohedral variety* X_{A_n}, has been studied extensively from the point of view of its intersection theory [32, 33] in connection with matroid theory. While generalized permutohedral varieties have been considered for various root systems, toric permutohedral varieties in the generality defined above and their coordinate rings seem currently unexplored.

One can extract several numerical invariants for permutohedral toric varieties and hence for toric rings of principal Borel sssi's from related invariants of the permutohedra.

Remark 6.16. Consider a partition $\lambda \in P_n$.

(1) The Hilbert function of $K[G(\text{Sss}(\lambda))] = \mathcal{F}(\text{Sss}(\lambda))$ is the Ehrhart function of $\text{P}(\lambda)$, namely, $d \mapsto H(d) :=$ the number of integer points in $d \cdot P(\lambda)$. If

$$\lambda = (0, 1, \ldots, n-1),$$

then $H(d)$ is the number of forests on n vertices with i edges [48, Example 3.1].

(2) The degree of $X_{\mathbf{P}(\lambda)}$ and the Hilbert-Samuel multiplicity of $K[G(\text{Sss}(\lambda))] = \mathcal{F}(\text{Sss}(\lambda))$ are given by the normalized volume $\frac{\text{Vol}(\mathbf{P}(\lambda))}{(n-1)!}$. Formulas for the volume of a permutohedron can be found in [42]. For instance, if $\lambda = (0^{n-d}, 1^d)$,

then $\frac{\mathrm{Vol}(\mathbf{P}(\lambda))}{(n-1)!}$ is the *Eulerian number*, that is, the number of permutations of size $n-1$ with $d-1$ descents. For an arbitrary principal Borel sssi, the volume of $\mathbf{P}(\lambda)$ is then calculated in terms of the *mixed Eulerian numbers*, i.e., normalized mixed volumes of the hypersimplices [42, Proposition 9.8 and Definition 16.1].

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