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# Liouville-type theorems for a system of elliptic inequalities on weighted graphs

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**Abstract.** Let $(V, E)$ be an infinite, connected, locally finite weighted graph. We are concerned with the system of elliptic inequalities

$$\begin{cases} -\Delta u \geq v^p & \text{in } V, \\ -\Delta v \geq u^q & \text{in } V, \end{cases}$$

where $p$ are $q$ are real numbers and $\Delta$ is the standard graph Laplacian. For $p \leq 0$ or $q \leq 0$ or $p, q > 0$ and $pq \leq 1$, we show that the system has no positive solution. Moreover, we also establish some non-existence results of positive solutions of the system in the case where $p, q > 0$ and $pq > 1$ under some assumptions on the volume growth of graph and weight. We also construct an explicit example to show the existence result in the super-critical case. Our result is, in particular, a natural extension of some results in [Gu, Huang, and Sun, Calc. Var. Partial Differential Equations 62 (2023), no. 2, article no. 42] to the system of inequalities.

## 1. Introduction

In the celebrated paper [11], Gidas and Spruck studied the existence and non-existence of positive solutions of the Lane–Emden equation

$$-\Delta u = u^\sigma \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

They showed that the equation has positive solution if and only if $\sigma \geq \sigma_c := \frac{N+2}{N-2}$, see also [3].

For the Lane-Emden system,

$$\begin{cases} -\Delta u = v^p & \text{in } \mathbb{R}^N, \\ -\Delta v = u^q & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

the Lane–Emden conjecture states that (1.2) has no positive solution if and only if

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}.$$

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This conjecture was confirmed for the radial solutions; see [25, 26, 33]. In general, this conjecture has been only proved when $N \leq 4$, see [31, 33, 34], and it is left open when $N \geq 5$.

Concerning the elliptic inequality

$$-\Delta u \geq u^{\sigma} \quad \text{in } \mathbb{R}^N \tag{1.3}$$

and the system of elliptic inequalities

$$\begin{cases} -\Delta u \geq v^p & \text{in } \mathbb{R}^N, \\ -\Delta v \geq u^q & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

the existence and non-existence of positive solutions were completely established for $p, q > 1$, see [27, 28] and for all $p, q \in \mathbb{R}$, see, e.g., [1]. It was proved that the optimal range for the existence of positive solutions of (1.3) and (1.4) is, respectively, given by $\sigma > \frac{N}{N-2}$ and

$$p, q > 0, \;\; pq > 1 \quad \text{and} \quad \max\left\{\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right\} < N-2.$$

On Riemannian manifolds, the sharp non-existence results for non-negative solutions of the semi-linear elliptic inequality

$$\Delta u + u^{\sigma} \leq 0 \tag{1.5}$$

with $\sigma > 1$ were obtained in the pioneering paper [15]. More precisely, by developing the test function method on the Riemannian manifold $M$, which is different from the one in the Euclidean setting, it was proved in [15] that, under the volume growth condition

$$\operatorname{vol}(B(x_0, r)) \leq C r^{\frac{2\sigma}{\sigma-1}} (\ln r)^{\frac{1}{\sigma-1}}$$

for some $x_0 \in M$ and all $r$ large enough, any non-negative solution of (1.5) must be trivial. Here, $B(x_0, r) \subset M$ denotes the geodesic ball. In addition, an example is also given to show that the volume growth condition is sharp up to the exponent in the logarithmic term. It is also worth mentioning some generalizations of [15] in [36, 37].

In the Euclidean setting, the test function method for an inequality (1.3) with $\sigma > 1$ can be easily generalized to the system of inequalities (1.4) with $p, q \geq 1$; see, e.g., [27, 28]. Nevertheless, it is really a challenging task to extend the test function method from an inequality (1.5) to the system of inequalities on Riemannian manifolds. In [42], the authors studied the existence and non-existence of non-negative solutions of the system of elliptic inequalities

$$\begin{cases} \Delta_{\gamma_1} u + v^p \leq 0 & \text{in } M, \\ \Delta_{\gamma_2} v + u^q \leq 0 & \text{in } M, \end{cases} \tag{1.6}$$

where $M$ is a geodesically complete connected non-compact Riemannian manifold. Here, $p > \gamma_2 - 1 > 0$ and $q > \gamma_1 - 1 > 0$ are given exponents and $\Delta_\gamma u := \operatorname{div}(|\nabla u|^{\gamma-2}\nabla u)$

with $\gamma > 1$. In order to make the test function method work for the system (1.6), they need to assume an extra condition

$$\frac{\gamma_1 q}{q - \gamma_1 + 1} = \frac{\gamma_2 p}{p - \gamma_2 + 1}. \tag{1.7}$$

This condition is very restrictive in comparison with the Euclidean setting; see, e.g., [6–8, 27, 29] for some generalizations of (1.6). In particular, if $\gamma_1 = \gamma_2 = 2$, then (1.7) implies that $p = q$. Up to now, the classification result in [42] has been a unique result for the system (1.6). We also want to mention some results for other systems of inequalities on manifolds [38, 39].

Let us now turn to the graph setting. In this paper, $(V, E)$ is an infinite, connected, locally finite graph, where $V$ is the set of vertices and $E$ is the set of edges. Let

$$w : V \times V \to [0, \infty)$$

be a weight satisfying $w_{xy} > 0$ if and only if $(x, y) \in E$ and $w_{xy} = w_{yx}$. Given a vertex $x \in V$, we write $y \sim x$ if there exists an edge between $y$ and $x$. Let us put

$$\mu(x) = \sum_{y \sim x} w_{xy}.$$

The standard graph Laplace operator is defined by

$$\Delta u(x) = \sum_{y \sim x} \frac{w_{xy}}{\mu(x)}(u(y) - u(x)), \quad x \in V.$$

We refer the readers to the book [12] for the elementary properties of the graph Laplace operator.

In recent years, nonlinear partial differential equations on graph have received considerably attention of many mathematicians [2, 9, 10, 12–14, 16–18, 20–24, 30, 40]. The existence and non-existence of global solutions of the parabolic equation

$$u_t - \Delta u = u^{1+\alpha} \quad \text{in } (0, \infty) \times V$$

were studied in [20, 21]. In [24], the author established Kato's inequality on graphs and applied it to study the boundedness of solutions of the Ginzburg–Landau equation

$$\Delta u + u(1 - u^2) = 0 \quad \text{in } V.$$

By using the monotone iteration scheme, the existence of positive solutions of Yamabe-type equation

$$\Delta u + au = bu^\sigma \quad \text{in } V,$$

where $a, b$ are functions on $V$ was obtained in [22]; see also [14].

In [10], a uniform lower bound on the energy of the solutions to the Liouville equation

$$-\Delta u = e^u$$

on weighted graphs was established under the assumption of the isoperimetric inequality.

Very recently, inspired by the result in [15] on manifolds, the Liouville property, i.e., the existence and non-existence of positive solutions, for the elliptic inequality

$$-\Delta u \geq u^{\sigma} \quad \text{in } V \tag{1.8}$$

has been established in [16]. Here, a positive solution $u$ means $u(x) > 0$ for all $x \in V$. Before presenting the main result in [16], let us recall some notations. Let $d(x, y)$ denote the minimal number of edges among all possible paths connecting $x$ and $y$ in $V$. The ball centered at $o \in V$ of radius $n$ is

$$B(o,n) = \{x \in V; d(o,x) \leq n\},$$

and the volume of $B(o,n)$ is given by

$$\mu(B(o,n)) = \sum_{x \in B(o,n)} \mu(x).$$

**Theorem 1.1** ([16]). *The following statements hold true.*

(1) *Let* $-\infty < \sigma \leq 1$. *Then,* (1.8) *has no positive solution.*

(2) *Let* $\sigma > 1$. *Suppose that there exists* $p_0 > 1$ *such that if* $x \sim y$, *then*

$$\frac{w_{xy}}{\mu(x)} \geq \frac{1}{p_0}. \tag{1.9}$$

*Assume, in addition, that there exists* $o \in V$ *such that*

$$\mu(B(o,n)) \lesssim n^{\frac{2\sigma}{\sigma-1}} (\ln n)^{\frac{1}{\sigma-1}} \quad \textit{for } n \textit{ large};$$

*then* (1.8) *has no positive solution.*

Here and in the sequel, given two non-negative functions $f$ and $g$, $f \lesssim g$ means $f \leq Cg$ and $f \asymp g$ means $cg \leq f \leq Cg$ for some $c, C > 0$.

We would like to mention that in [16, Theorem 1.1], the author considered $\sigma > 0$. However, the proof of [16, Theorem 1.1] still holds true for the non-existence of positive solutions of (1.8) when $\sigma \leq 0$. Furthermore, in this direction, it is worth mentioning a result in [30] on the non-existence of positive solutions of (1.8), where condition (1.9) is replaced by the existence of a pseudo metric on $V$.

Inspired by the interesting results in [16, 42], we address a natural question on the existence and non-existence of positive solutions of the system of elliptic inequalities

$$\begin{cases} -\Delta u \geq v^p & \text{in } V, \\ -\Delta v \geq u^q & \text{in } V, \end{cases} \tag{1.10}$$

where $p, q \in \mathbb{R}$. Here, $(V, E, w)$ is the infinite, connected, locally finite graph given above.

Our purpose is to extend the result in [16] to the system (1.10). More precisely, we obtain the non-existence of positive solutions as follows.

**Theorem 1.2.** *The following statements hold true.*

(1) *The system* (1.10) *has no positive solution if one of the following conditions is satisfied:*

  (i) $p \le 0$ *or* $q \le 0$,

  (ii) $p > 0, q > 0$ *and* $pq \le 1$.

(2) *Suppose that* $p \ge q > 0$ *and* $pq > 1$. *Assume that condition* (1.9) *is satisfied on* $(V, E, w)$. *If there exist a vertex* $o \in V$ *and a positive constant* $\varepsilon$ *such that*

$$\mu(B(o,n)) \lesssim n^{\frac{2(pq+p)}{pq-1}-\varepsilon} \quad \textit{for all large enough } n, \tag{1.11}$$

*then the system* (1.10) *has no positive solution.*

**Remark 1.3.** Some remarks are in order.

(1) To the best of our knowledge, Theorem 1.2 is the first result on the non-existence of positive solutions of the system (1.10) on weighted graphs.

(2) We want to emphasize that when both exponents $p > 1$ and $q > 1$, it is very difficult to generalize the test function method from the inequality (1.8) to the system (1.10) because of the special structure of the test functions; see, e.g., Step 2 in the proof of [16, Theorem 1.2], and also [42]. Moreover, in our paper, the exponents $p$ and $q$ may be smaller than one. This is the second reason why the test function method does not work.

(3) On graphs, condition (1.9) is a replacement for uniform ellipticity. As mentioned in [16], condition (1.9) may be superfluous. It is still left open.

(4) We conjecture that the second assertion in Theorem 1.2 is still true when condition (1.11) is replaced by

$$\mu(B(o,n)) \lesssim n^{\frac{2(pq+p)}{pq-1}} (\ln n)^{\frac{p+1}{pq-1}} \quad \text{for all large enough } n. \tag{1.12}$$

Inspired by [16] and (1.12), for any $\varepsilon > 0$, we will construct a concrete example showing the existence of positive solutions on the homogeneous tree $\mathbb{T}_N$ with

$$\mu(B(o,n)) \asymp n^{\frac{2(pq+p)}{pq-1}} (\ln n)^{\frac{p+1}{pq-1}+\varepsilon} \quad \text{for } n \text{ large enough.}$$

Let $N \ge 2$, we recall the notion of homogeneous tree $\mathbb{T}_N$. It means that $\mathbb{T} = (V, E)$ is a tree where all vertices have degree $N$. Fix an arbitrary vertex $o \in V$ as the root. For $n \ge 0$, we denote by $D_n = \{x \in V; d(o,x) = n\}$ the collection of all the vertices with distance $n$ from $o$, and denote by $E_n$ the collection of all the edges from vertices in $D_n$ to vertices in $D_{n+1}$.

**Theorem 1.4.** *Let* $(VE) = \mathbb{T}_N$, $N \ge 2$. *Suppose that* $p \ge q > 0$ *and* $pq > 1$ *and let* $\varepsilon > 0$. *Then, there exists an edge weight* $w$ *on* $\mathbb{T}_N$ *such that the following hold.*

- $\mu(B(o,n)) \asymp n^{\frac{2(pq+p)}{pq-1}} (\ln n)^{\frac{p+1}{pq-1}+\varepsilon}$ *for* $n \ge 2$.
- *There exists a positive solution* $(u, v)$ *to the system* (1.10).

*More precisely, we can take $w$ and $u, v$ as follows:*

$$w_{xy} = w_n = \frac{(n+n_0)^{\frac{2(p+1)}{pq-1}+1}(\ln(n+n_0))^{\frac{p+1}{pq-1}+\varepsilon}}{(N-1)^n} \quad \text{for any } (x,y) \in E_n, n \geq 0,$$

$$u(x) = u_n = \frac{\delta_1}{(n+n_0)^{\frac{2(p+1)}{pq-1}}(\ln(n+n_0))^{\frac{p+1}{pq-1}}} \quad \text{for any } x \in D_n, n \geq 0,$$

$$v(x) = v_n = \frac{\delta_2}{(n+n_0)^{\frac{2(q+1)}{pq-1}}(\ln(n+n_0))^{\frac{q+1}{pq-1}}} \quad \text{for any } x \in D_n, n \geq 0,$$

*where $n_0 \geq 2$ is sufficiently large and $\delta_1, \delta_2 > 0$ are sufficiently small.*

Let us close the introduction by giving the idea of the proof of our main results. For the system of inequalities when $p \neq q$, the test function method in [16] seems not applicable, see [42]. On the other hand, when $p \leq 1$ or $q \leq 1$, the classification of positive solutions becomes more difficult since we cannot use the Hölder inequality and the test function method. Then, to prove Theorem 1.2, we use instead a simple approach which is completely different from the one in [16]. In fact, we establish a discrete reduction version in order to transform the system of inequalities into an inequality, see [32,35], and also [4] for the related results in the Euclidean setting. Hence, we apply the result for the inequality, see Theorem 1.1 above, to obtain the desired results. We believe that our approach can be used to study the system of elliptic inequalities on manifolds.

The organization of the paper is as follows. In Section 2, we give some auxiliary lemmas. The proof of our main results is provided in Section 3.

## 2. Some auxiliary lemmas

In this section, we provide some lemmas which will be used in the next section. The following result is a kind of maximum principle.

**Lemma 2.1.** *Let $p > 0$ and $q > 0$. If $(u, v)$ is a non-negative solution to the system (1.10), then either $u \equiv 0, v \equiv 0$ or $u > 0, v > 0$ in $V$.*

*Proof.* Assume that there exists some point $x_0 \in V$ such that $u(x_0) = 0$. It follows from the first inequality of the system (1.10) that

$$\sum_{y \sim x_0} \frac{w_{x_0 y}}{\mu(x_0)}(u(y) - u(x_0)) + v(x_0)^p = \sum_{y \sim x_0} \frac{w_{x_0 y}}{\mu(x_0)} u(y) + v(x_0)^p \leq 0.$$

It implies that $v(x_0) = 0$ and $u(y) = 0$ for any $y \sim x_0$. Hence, $u \equiv 0$ by the connected property of $V$. Similarly, from the second inequality of the system (1.10), we have

$$\sum_{y \sim x_0} \frac{w_{x_0 y}}{\mu(x_0)}(v(y) - v(x_0)) + u(x_0)^q = \sum_{y \sim x_0} \frac{w_{x_0 y}}{\mu(x_0)} v(y) \leq 0.$$

Therefore, $v(y) = 0$ for any $y \sim x_0$. Then, $v \equiv 0$ by the connected property of $V$. The proof is complete. ■

From condition (1.9), we prove the boundedness and a Harnack-type inequality for positive solutions of system (1.10) on weighted graphs.

**Lemma 2.2.** *Let $p > 1, q > 1$ and $(V, E, w)$ satisfy condition (1.9). If $(u, v)$ is a positive solution of the system (1.10), then $0 < u < 1$ and $0 < v < 1$ in $V$. Moreover,*

$$\frac{1}{p_0} \le \frac{u(x)}{u(y)} \le p_0 \quad \textit{and} \quad \frac{1}{p_0} \le \frac{v(x)}{v(y)} \le p_0 \quad \textit{for any } x \sim y \textit{ in } V.$$

*Proof.* For $x \in V$, it results from the system (1.10) that

$$\sum_{y\sim x} \frac{w_{xy}}{\mu(x)}(u(y) - u(x)) + v(x)^p = \sum_{y\sim x} \frac{w_{xy}}{\mu(x)} u(y) - u(x) + v(x)^p \le 0$$

and

$$\sum_{y\sim x} \frac{w_{xy}}{\mu(x)}(v(y) - v(x)) + u(x)^q = \sum_{y\sim x} \frac{w_{xy}}{\mu(x)} v(y) - v(x) + u(x)^q \le 0.$$

This combined with the positivity of solutions gives

$$\begin{cases} v(x)^p - u(x) < 0, \\ u(x)^q - v(x) < 0, \end{cases} \tag{2.1}$$

and for $x \sim y$, we have

$$\begin{cases} \frac{w_{xy}}{\mu(x)} u(y) - u(x) < 0, \\ \frac{w_{xy}}{\mu(x)} v(y) - v(x) < 0. \end{cases} \tag{2.2}$$

Hence, (2.1) implies that $0 < u, v < 1$. On the other hand, using conditions (1.9) and (2.2), we obtain that

$$\frac{u(x)}{u(y)} \ge \frac{w_{xy}}{\mu(x)} \ge \frac{1}{p_0} \quad \text{and} \quad \frac{v(x)}{v(y)} \ge \frac{w_{xy}}{\mu(x)} \ge \frac{1}{p_0}.$$

Exchanging $x$ and $y$, we get the other side inequality, and hence, our statement follows. ■

## 3. Proof of main results

In this section, we give the proof of our main results.

### 3.1. Proof of Theorem 1.2

We prove the first assertion in Theorem 1.2.

**Proof of Theorem 1.2 (1).**

**Lemma 3.1.** *If a positive function $u$ satisfies $-\Delta u \geq Cu^\sigma$ for some $C > 0$, then $u = 0$ under the assumptions on $\sigma$ and $\mu(B(o, n))$ made in Theorem 1.1.*

*Proof of Lemma 3.1.* By using a scaling argument $z = C^{\frac{1}{\sigma-1}}u$, we reduce the inequality $-\Delta u \geq Cu^\sigma$ to the inequality $-\Delta z \geq z^\sigma$. Hence, we obtain a conclusion. ■

We now turn to the proof of Theorem 1.2. Notice that $p = 0$ or $q = 0$, we obtain from the system (1.10) that $-\Delta u \geq 1$ or $-\Delta v \geq 1$. However, by Theorem 1.1 ($\sigma = 0$), these inequalities have no positive solution. Then, it is sufficient to consider $p \neq 0$ and $q \neq 0$.

Suppose, on the contrary, that $(u, v)$ is a positive solution of the system (1.10). The following lemma is concerned with $p < 0$ and $q < 0$.

**Lemma 3.2.** *Assume that $p < 0$ and $q < 0$. Then, there exists $C > 0$ such that $-\Delta \tilde{w} \geq C\tilde{w}^{-s}$, where $\tilde{w} = u + v$ and $s = -\frac{2pq}{p+q} > 0$.*

*Proof of Lemma 3.2.* Adding the two inequalities in the system (1.10), we have

$$\begin{aligned} -\Delta \tilde{w} &= -\Delta u - \Delta v \geq u^q + v^p \\ &\geq C(u + v)^{-s} = C\tilde{w}^{-s}, \end{aligned}$$

where in the last inequality, we have used the Young inequality as follows:

$$\frac{1}{(u+v)^s} \leq \frac{1}{2^s(uv)^{s/2}} \leq C(u^{-sm/2(m-1)} + v^{-sm/2}) = C(u^q + v^p)$$

with $m = \frac{p+q}{q} > 1$ and some positive constant $C$. The proof of Lemma 3.2 is complete. ■

The next lemma is concerned with $q < 0 < p$ or $0 < q \leq p$.

**Lemma 3.3.** *Assume that $q < 0 < p$ or $0 < q \leq p$. There exist $C > 0$ and $a, b > 0, a + b = 1$ such that*

$$-\Delta \tilde{w} \geq C\tilde{w}^{1+\frac{pq-1}{a(p+1)+b(q+1)}}, \tag{3.1}$$

*where $\tilde{w} = u^a v^b$.*

Before giving the proof of Lemma 3.3, we note that the change of variable $\tilde{w} = u^a v^b$ was first used in [35] with $a = b = \frac{1}{2}$ and then in [32] with $a + b = 1$ for the existence and symmetry of components of semi-linear elliptic systems on $\mathbb{R}^N$. This technique was also used in [4, 5, 19, 41] for elliptic or parabolic inequalities on the Euclidean spaces. Nevertheless, Lemma 3.3 is the first result, where this technique is exploited on weighted graphs.

*Proof of Lemma 3.3.* We first show that

$$-\Delta \tilde{w} \geq -au^{a-1}v^b \Delta u - bv^{b-1}u^a \Delta v. \tag{3.2}$$

Indeed, using the Young inequality with the pair of conjugate exponents $\frac{1}{a}$ and $\frac{1}{b}$, we have

$$\left(\frac{u(y)}{u(x)}\right)^a \left(\frac{v(y)}{v(x)}\right)^b \le a\frac{u(y)}{u(x)} + b\frac{v(y)}{v(x)}.$$

This is equivalent to

$$u(y)^a v(y)^b \le au(x)^{a-1} v(x)^b u(y) + bv(x)^{b-1} u(x)^a v(y).$$

Consequently,

$$\begin{aligned} &u(y)^a v(y)^b - u(x)^a v(x)^b \\ &\quad \le au(x)^{a-1} v(x)^b u(y) + bv(x)^{b-1} u(x)^a v(y) - (a+b)u(x)^a v(x)^b. \end{aligned}$$

Multiplying both sides by $\frac{w_{xy}}{\mu(x)}$ and taking the sum over $y \sim x$, we obtain

$$\begin{aligned} \sum_{y\sim x} \frac{w_{xy}}{\mu(x)}(u^a(y)v^b(y) - u^a(x)v^b(x)) &\le \sum_{y\sim x} \frac{w_{xy}}{\mu(x)} au(x)^{a-1} v(x)^b (u(y) - u(x)) \\ &\quad + \sum_{y\sim x} \frac{w_{xy}}{\mu(x)} bv(x)^{b-1} u(x)^a (v(y) - v(x)) \end{aligned}$$

or

$$-\Delta\tilde{w} \ge -au^{a-1} v^b \Delta u - bv^{b-1} u^a \Delta v.$$

The inequality (3.2) is proved.

We next use (3.2) to get

$$\begin{aligned} -\Delta\tilde{w} &\ge -au^{a-1} v^b \Delta u - bv^{b-1} u^a \Delta v \\ &\ge au^{a-1} v^{b+p} + bv^{b-1} u^{a+q} = u^a v^b \Big(a\frac{v^p}{u} + b\frac{u^q}{v}\Big). \end{aligned}$$

Applying the Young inequality to the right-hand side of this inequality, we arrive at

$$a\frac{v^p}{u} + b\frac{u^q}{v} \ge C\Big(\frac{v^p}{u}\Big)^{1/m} \Big(\frac{u^q}{v}\Big)^{(m-1)/m},$$

where the constant $C > 0$ is independent in $u, v$ and $m > 0$ will be chosen as below. Hence,

$$-\Delta\tilde{w} \ge Cu^{a-\frac{1}{m}+q\frac{m-1}{m}} v^{b-\frac{m-1}{m}+p\frac{1}{m}}. \tag{3.3}$$

Now, the constant $m$ can be chosen such that

$$\frac{a - \frac{1}{m} + q\frac{m-1}{m}}{b - \frac{m-1}{m} + p\frac{1}{m}} = \frac{a}{b} \quad \text{or equivalently} \quad m = 1 + \frac{ap+b}{bq+a}.$$

It is clear that if $p, q > 0$, then $m > 1$ for all $a, b > 0$. If $q < 0 < p$, we can choose $a < 1$ close to 1 and $b > 0$ close to 0 such that $m > 1$. Then, we obtain (3.1) from (3.3). The proof of Lemma 3.3 is complete. ■

*Proof of Theorem* 1.2. We first remark that the assertion (1) (i) in Theorem 1.2 follows from Lemma 3.1 and Lemma 3.2. Similarly, the assertion (1) (ii) in Theorem 1.2 is a direct consequence of Lemmas 3.1 and 3.3.

The rest of the proof is concerned with the case $p \geq 1 > 0$ and $pq > 1$. Let us put $\widetilde{\sigma} = 1 + \frac{pq-1}{a(p+1)+b(q+1)}$, then (3.1) becomes

$$-\Delta w \geq C w^{\widetilde{\sigma}}. \tag{3.4}$$

Since $q \leq p$, then $\widetilde{\sigma} > 1 + \frac{pq-1}{p+1} = \frac{pq+p}{p+1}$ and

$$\lim_{a \to 1} \widetilde{\sigma} = 1 + \frac{pq-1}{p+1} = \frac{pq+p}{p+1}.$$

Hence, $\frac{2\widetilde{\sigma}}{\widetilde{\sigma}-1} < \frac{2(pq+p)}{pq-1}$ and

$$\lim_{a \to 1} \frac{2\widetilde{\sigma}}{\widetilde{\sigma}-1} = \frac{2(pq+p)}{pq-1}.$$

Therefore, there exists $a$ close to 1 such that

$$\frac{2\widetilde{\sigma}}{\widetilde{\sigma}-1} > \frac{2(pq+p)}{pq-1} - \varepsilon.$$

This and (1.11) imply that

$$\mu(B(o,n)) \lesssim n^{\frac{2(pq+p)}{pq-1} - \varepsilon} \lesssim n^{\frac{2\widetilde{\sigma}}{\widetilde{\sigma}-1}} (\ln n)^{\frac{1}{\widetilde{\sigma}-1}} \text{ for all large enough } n.$$

It allows us to apply Lemma 3.1 to obtain the non-existence of positive solutions of inequality (3.4). ■

### 3.2. Proof of Theorem 1.4

First, we will show that under the weight $w_n$ given in Theorem 1.4, we have

$$\mu(B(o,n)) \asymp n^{\frac{2(pq+p)}{pq-1}} (\ln n)^{\frac{p+1}{pq-1} + \varepsilon}$$

for all $n \geq 2$. In fact,

$$\mu(B(o,n)) = \sum_{k=0}^{n} \mu(D_k) = \sum_{k=0}^{n} (N-1)^k w_k \asymp n^{\frac{2(pq+p)}{pq-1}} (\ln n)^{\frac{p+1}{pq-1} + \varepsilon}.$$

Now, we check that the system (1.10) holds for the $w$ and $(u, v)$ given in Theorem 1.4. It means that

$$u_1 - u_0 + v_0^p \leq 0, \quad \text{i.e., } n = 0, \tag{3.5}$$

$$v_1 - v_0 + u_0^q \leq 0, \quad \text{i.e., } n = 0, \tag{3.6}$$

$$\frac{(N-1) w_n u_{n+1} + w_{n-1} u_{n-1}}{(N-1) w_n + w_{n-1}} - u_n + v_n^p \leq 0 \quad \text{for all } n \geq 1, \tag{3.7}$$

$$\frac{(N-1) w_n v_{n+1} + w_{n-1} v_{n-1}}{(N-1) w_n + w_{n-1}} - v_n + u_n^q \leq 0 \quad \text{for all } n \geq 1. \tag{3.8}$$

The constants $\delta_1$ and $\delta_2$ are closely related to $\varepsilon$ and we will first determine $n_0$ and then $\delta_1, \delta_2$. For simplicity, let $\alpha_1 = \frac{p+1}{pq-1}$ and $\alpha_2 = \frac{q+1}{pq-1}$. One can see that $q\alpha_1 = \alpha_2 + 1$ and $p\alpha_2 = \alpha_1 + 1$.

*Case 1: $n = 0$.* Using the values of $u, v$, the inequalities (3.5) and (3.6) are equivalent to

$$\begin{cases} \frac{\delta_1}{(1+n_0)^{2\alpha_1}(\ln(1+n_0))^{\alpha_1}} - \frac{\delta_1}{n_0^{2\alpha_1}(\ln n_0)^{\alpha_1}} + \big(\frac{\delta_2}{n_0^{2\alpha_2}(\ln n_0)^{\alpha_2}}\big)^p \le 0, \\ \frac{\delta_2}{(1+n_0)^{2\alpha_2}(\ln(1+n_0))^{\alpha_2}} - \frac{\delta_2}{n_0^{2\alpha_2}(\ln n_0)^{\alpha_2}} + \big(\frac{\delta_1}{n_0^{2\alpha_1}(\ln n_0)^{\alpha_1}}\big)^q \le 0. \end{cases}$$

This system is equivalent to

$$\begin{cases} \frac{\delta_2^p}{\delta_1} \le C_1(n_0), \\ \frac{\delta_1^q}{\delta_2} \le C_2(n_0), \end{cases} \tag{3.9}$$

where

$$C_1(n_0) = n_0^{2\alpha_2 p}(\ln n_0)^{\alpha_2 p}(n_0^{-2\alpha_1}(\ln n_0)^{-\alpha_1} - (n_0+1)^{-2\alpha_1}(\ln(n_0+1))^{-\alpha_1})$$

and

$$C_2(n_0) = n_0^{2\alpha_1 q}(\ln n_0)^{\alpha_1 q}(n_0^{-2\alpha_2}(\ln n_0)^{-\alpha_2} - (n_0+1)^{-2\alpha_2}(\ln(n_0+1))^{-\alpha_2}).$$

We can choose $\delta_1$ small enough and then $\delta_2$ satisfying (3.9) as follows:

$$\delta_1 \le (C_2(n_0)^p C_1(n_0))^{\frac{1}{pq-1}} \quad \text{and} \quad \frac{1}{C_2(n_0)}\delta_1^q \le \delta_2 \le C_1(n_0)^{\frac{1}{p}}\delta_1^{\frac{1}{p}}. \tag{3.10}$$

*Case 2: $n \ge 1$.* Using the values of $u, v$ and the weight $w$, the inequalities (3.7) and (3.8) are equivalent to

$$\frac{\delta_2^p}{\delta_1} \le (n+n_0)^2 \ln(n+n_0) - (n+n_0)^{2\alpha_1+2}(\ln(n+n_0))^{\alpha_1+1} F_1 \tag{3.11}$$

and

$$\frac{\delta_1^q}{\delta_2} \le (n+n_0)^2 \ln(n+n_0) - (n+n_0)^{2\alpha_2+2}(\ln(n+n_0))^{\alpha_2+1} F_2, \tag{3.12}$$

where

$$F_1 := \frac{\frac{(n+n_0)^{2\alpha_1+1}(\ln(n+n_0))^{\alpha_1+\varepsilon}}{(n+n_0+1)^{2\alpha_1}(\ln(n+n_0+1))^{\alpha_1}} + \frac{(n+n_0-1)^{2\alpha_1+1}(\ln(n+n_0-1))^{\alpha_1+\varepsilon}}{(n+n_0-1)^{2\alpha_1}(\ln(n+n_0-1))^{\alpha_1}}}{(n+n_0)^{2\alpha_1+1}(\ln(n+n_0))^{\alpha_1+\varepsilon} + (n+n_0-1)^{2\alpha_1+1}(\ln(n+n_0-1))^{\alpha_1+\varepsilon}}$$

and

$$F_2 := \frac{\frac{(n+n_0)^{2\alpha_2+1}(\ln(n+n_0))^{\alpha_2+\varepsilon}}{(n+n_0+1)^{2\alpha_2}(\ln(n+n_0+1))^{\alpha_2}} + \frac{(n+n_0-1)^{2\alpha_2+1}(\ln(n+n_0-1))^{\alpha_2+\varepsilon}}{(n+n_0-1)^{2\alpha_2}(\ln(n+n_0-1))^{\alpha_2}}}{(n+n_0)^{2\alpha_2+1}(\ln(n+n_0))^{\alpha_2+\varepsilon} + (n+n_0-1)^{2\alpha_2+1}(\ln(n+n_0-1))^{2\alpha_2+\varepsilon}}.$$

Using the same argument as in the estimate (4.5) in [16], we also obtain that

$$\lim_{n\to\infty}\left(n^2\ln n - n^{2\alpha_i+2}(\ln(n))^{\alpha_i+1}\frac{\frac{(n)^{2\alpha_i+1}(\ln(n))^{\alpha_i+\varepsilon}}{(n+1)^{2\alpha_i}(\ln(n+1))^{\alpha_i}}+\frac{(n-1)^{2\alpha_i+1}(\ln(n-1))^{\alpha_i+\varepsilon}}{(n-1)^{2\alpha_i}(\ln(n-1))^{\alpha_i}}}{(n)^{2\alpha_i+1}(\ln(n))^{\alpha_i+\varepsilon}+(n-1)^{2\alpha_i+1}(\ln(n-1))^{2\alpha_i+\varepsilon}}\right)=\alpha_i\varepsilon$$

for $i=1,2$. This implies that there exists some large $n_0$ such that for all $n\geq 0$, the right-hand sides of (3.11) and (3.12) are bounded from below by $\alpha_1\varepsilon/2$ and $\alpha_2\varepsilon/2$, respectively. Thus, it is sufficient to choose

$$\begin{cases}\frac{\delta_2^p}{\delta_1}\leq\frac{\alpha_1\varepsilon}{2},\\ \frac{\delta_1^q}{\delta_2}\leq\frac{\alpha_2\varepsilon}{2}.\end{cases}\tag{3.13}$$

As above, we can choose $\delta_1$ small enough and then $\delta_2$ satisfying (3.13) as follows:

$$\delta_1\leq\left(\left(\frac{\alpha_1\varepsilon}{2}\right)^p\frac{\alpha_2\varepsilon}{2}\right)^{\frac{1}{pq-1}}\quad\text{and}\quad\frac{2}{\alpha_2\varepsilon}\delta_1^q\leq\delta_2\leq\left(\frac{\alpha_1\varepsilon}{2}\right)^{\frac{1}{p}}\delta_1^{\frac{1}{p}}.\tag{3.14}$$

It is easy to see that, we can choose $\delta_1$ small enough and then $\delta_2$ satisfying both (3.10) and (3.14). Then, we finish the proof. ∎

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