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Optimal control problems for a parabolic system inspired by a cancer therapy

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Abstract. In this paper, we consider optimal control problems for a parabolic system modeling a therapy, based on oncolytic viruses, for the glioma brain cancer. Using several techniques typical of functional analysis, we prove the global in time well posedness of the control model, the existence of optimal controls for specific objective functionals, which are natural for cancer therapies, and we derive necessary conditions for optimality.

1. Introduction

In this paper, we consider optimal control problems for a 3×3 system of parabolic partial differential equations modeling a therapy in the case of a brain cancer, the glioma one, based on the infusion of oncolytic viruses. They are genetically modified viruses able to infect cancer cells and to replicate inside them, but they are not harmful for healthy cells. With this mechanism, they eventually kill mainly cancer cells. Moreover, when an infected cell dies, it releases many copies of the viruses, which then spread to infect neighboring tumor cells. The main obstacle in the use of oncolytic viruses consists in the fact that the innate immune system recognizes the cells infected by the virus and destroys them before the virus multiplies. In this paper, as in [11, 29], we are neglecting this aspect.

There is a huge mathematical literature for cancer modeling based on differential equations; see [1, 10, 16, 18, 21, 22] and the references therein for a detailed description. This is essentially due to the large variety of diseases commonly named under the word *cancer*. Each tumor has some specific peculiarities and dynamics; hence, it requires an ad-hoc model for a precise mathematical description. Also, therapies vary accordingly. For example, they include chemotherapy, radiotherapy, stem cell transplant, surgery and can be dosed also combined together. This justifies the large number of mathematical papers dealing with this subject. In particular, considering Glioma type cancer, we can distinguish the various models through different categories: based on ODEs [23–25] or on PDEs [4, 6, 11, 12, 29], focusing on controlling aspects [6, 23–25], on therapy calibration [2, 11, 12, 29], or on asymptotic behavior of solutions [4, 11].

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The main results of this paper are the well posedness of the parabolic 3×3 control system, the existence of optimal controls, and first order necessary conditions for optimality. They are obtained through several techniques typical of functional analysis. In particular, different from other papers in the literature, see, for instance, [4, 6], we use the Banach fixed point theorem to prove local in time existence and uniqueness of solution for the parabolic system. Moreover, a combination of a priori estimates and maximum principles for scalar equations permits to extend the solution to arbitrary time intervals obtaining global in time well posedness. Gronwall inequality is used to prove the Lipschitz continuity, in the $\mathbf{L}^2$ topology, of the solution with respect to the controls. Existence of optimal controls is deduced using the direct method in the calculus of variation; see for example [9]. Here, continuous embedding theorems and Ascoli–Arzelà theorem play an important role in the weak and strong convergence of quasi-optimal solutions. Finally, necessary conditions are obtained through the derivative of the input-output map and the adjoint system.

The main novelty of the paper consists in the study of a nonlinear system of parabolic partial differential equations with an open loop control function; see [7, 13, 19, 28] and the references therein for control problems for partial differential equations. Here, we derive necessary and sufficient conditions for optimal controls.

The paper is organized as follows. Section 2 introduces the mathematical model and the definition of solution. In Section 3, we prove the existence and uniqueness of solution both local and global in time. In Section 4, we study the Lipschitz continuous dependence of the solution with respect to the control function, while in Section 5, we deduce the existence of optimal controls for some objective functionals, natural for cancer therapies. Section 6 deals with necessary conditions for optimality. Finally, Appendix A contains classical results about well posedness of scalar parabolic equations, used in Section 3. It is mainly intended to ease the readability of the paper.

2. Basic definitions and notations

In this paper, we consider control problems for the system of partial differential equations

$$\begin{cases} \partial_t \rho_1 = \Delta\rho_1 + (\alpha - \delta_1)\rho_1 - \beta\rho_1 v, \\ \partial_t \rho_2 = \Delta\rho_2 + \beta\rho_1 v - \delta_2\rho_2, \\ \partial_t v = \Delta v + b\delta_2\rho_2 - B\rho_1 v - \delta_v v + u, \end{cases} \tag{2.1}$$

where $t \geq 0$ is the time, $x \in \Omega$ is the spatial variable, $\Omega \subseteq \mathbb{R}^N$ is an open, bounded, and connected set with smooth boundary denoted by $\partial\Omega$, and $N \in \mathbb{N}$, $N \geq 2$ (typically $N = 3$ in applications). Moreover, $\rho_1, \rho_2 : (0, +\infty) \times \Omega \to \mathbb{R}$ describe the density, respectively, of uninfected cancer cells and of infected cancer cells, while $v : (0, +\infty) \times \Omega \to \mathbb{R}$ represents the density of the injected virus. The map $u = u(t, x)$ is the control function modeling the velocity of the virus infusion. Finally, α, β, δ_1, δ_2, δ_v, b, and B are fixed positive constants. In the paper, we consider controls depending also on the spatial variable, although, as detailed in [8], in real situations the viral therapy is administered intravenously so that a control depending only on time should be more realistic.

We augment the system (2.1) with the initial conditions

$$\begin{cases} \rho_1(0,x) = \rho_{1,o}(x), \\ \rho_2(0,x) = \rho_{2,o}(x), \\ v(0,x) = v_o(x), \end{cases} \tag{2.2}$$

where $\rho_{1,o}, \rho_{2,o}, v_o \in \mathbf{L}^2(\Omega)$, and with homogeneous Neumann boundary conditions

$$\begin{cases} \partial_\nu \rho_1(t,\xi) = 0, \\ \partial_\nu \rho_2(t,\xi) = 0, \\ \partial_\nu v(t,\xi) = 0 \end{cases} \tag{2.3}$$

for $\xi \in \partial\Omega$, where the symbol ∂_ν denotes the inner normal derivative.

Throughout the paper, we deal with the following concept of weak solution for system (2.1), (2.2), and (2.3).

Definition 2.1. Given $T > 0$, the triple (ρ_1, ρ_2, v) is a *solution* to the initial-boundary value problem (2.1), (2.2), and (2.3) on the time interval $[0, T]$ if the following statements hold:

(1) $\rho_1, \rho_2, v \in \mathbf{L}^\infty((0,T) \times \Omega; \mathbb{R})$;

(2) $\rho_1, \rho_2, v \in \mathbf{L}^2((0,T); \mathbf{H}^1(\Omega))$;

(3) $\dot\rho_1, \dot\rho_2, \dot v \in \mathbf{L}^2((0,T); \mathbf{H}^1(\Omega)^*)$;

(4) $\rho_1(0,x) = \rho_{1,o}(x)$, $\rho_2(0,x) = \rho_{2,o}(x)$, $v(0,x) = v_o(x)$ in $\mathbf{L}^2(\Omega)$;

(5) for a.e. $t \in [0,T]$ and for any $w_1, w_2, w_3 \in \mathbf{H}^1(\Omega)$,

$$\begin{aligned}
\langle \dot\rho_1(t), w_1 \rangle &= -\int_\Omega \nabla\rho_1(t,x) \cdot \nabla w_1(x)\,\mathrm{d}x + (\alpha - \delta_1) \int_\Omega \rho_1(t,x)\, w_1(x)\,\mathrm{d}x \\
&\quad - \beta \int_\Omega \rho_1(t,x)\, v(t,x)\, w_1(x)\,\mathrm{d}x, \\
\langle \dot\rho_2(t), w_2 \rangle &= -\int_\Omega \nabla\rho_2(t,x) \cdot \nabla w_2(x)\,\mathrm{d}x - \delta_2 \int_\Omega \rho_2(t,x)\, w_2(x)\,\mathrm{d}x \\
&\quad + \beta \int_\Omega \rho_1(t,x)\, v(t,x)\, w_2(x)\,\mathrm{d}x, \\
\langle \dot v(t), w_3 \rangle &= -\int_\Omega \nabla v(t,x) \cdot \nabla w_3(x)\,\mathrm{d}x + b\,\delta_2 \int_\Omega \rho_2(t,x)\, w_3(x)\,\mathrm{d}x \\
&\quad - B \int_\Omega \rho_1(t,x)\, v(t,x)\, w_3(x)\,\mathrm{d}x - \delta_v \int_\Omega v(t,x)\, w_3(x)\,\mathrm{d}x \\
&\quad + \int_\Omega u(t,x)\, w_3(x)\,\mathrm{d}x.
\end{aligned}$$

Remark 1. Note that assumptions 2 and 3 of Definition 2.1 imply that the functions ρ_1, ρ_2, and v belong to the space $\mathbf{C^0}([0,T];\mathbf{L^2}(\Omega))$; see [27, Theorem 7.104]. This justifies the condition 4 of Definition 2.1.

2.1. Model justification

In [23–25], the authors proposed a mathematical model for the therapy of glioma based on oncolytic viruses infusion. Oncolytic viruses are genetically altered viruses able to infect cancer cells but not normal ones. They reproduce in cancer cells and eventually kill them, and when an infected cell dies, many new viruses are released and spread out. The model in [23–25] is given by the following system of nonlinear ordinary differential equations:

$$\begin{cases} \dot{x} = \alpha x - \beta x v - \delta_x x, \\ \dot{y} = \beta x v - \xi y \frac{T}{K+T} - \delta_y y, \\ \dot{M} = A + s y M - \delta_M M, \\ \dot{T} = \frac{\eta}{1+u_2} M - \omega y \frac{T}{K+T} - \delta_T T, \\ \dot{v} = b\delta_y y - \rho x v - \delta_v v + u_1, \end{cases} \tag{2.4}$$

where the unknowns x, y, M, T, and v represent, respectively, the density of uninfected cancer stem cells, the density of infected cancer cells, the density of the macrophages, the concentration of TNF-α inhibitors, and the density of the virus. The control functions $u_1 = u_1(t)$ and $u_2 = u_2(t)$ denote, respectively, the amount of virus and of TNF-α inhibitor that is injected at time t. The descriptions and realistic values of the various parameters appearing in system (2.4) can be found in [25, Table 2]. One can also note in [25] that the dynamics of the unknowns M and T is almost static around the values $M \sim 0.1 \frac{g}{cm^3}$ and $T \sim 5 \times 10^{-6} \frac{g}{cm^3}$. Hence, (2.4) can be approximated by the 3×3 system

$$\begin{cases} \dot{x} = \alpha x - \beta x v - \delta_x x, \\ \dot{y} = \beta x v - \tilde{\xi} y - \delta_y y, \\ \dot{v} = b\delta_y y - \rho x v - \delta_v v + u_1. \end{cases} \tag{2.5}$$

Model (2.1) is the natural generalization of (2.5) once we allow the densities of cancer cells and of the virus to depend also on the spatial coordinate.

3. Local and global existence

In this section, we prove both the local and global well posedness for system (2.1). The local in time result is proved using a fixed point technique, while a priori estimates permit to extend the solution to arbitrary time intervals. In the following, we use the notation $\Omega_T = (0,T) \times \Omega$.

Theorem 3.1. *Assume α, β, δ_1, δ_2, δ_v, b, B, and U fixed positive constants. Let $\Omega \subseteq \mathbb{R}^N$ be an open, connected, and bounded domain with smooth boundary $\partial\Omega$. Fix $\rho_{1,o}, \rho_{2,o}, v_o \in \mathbf{L}^\infty(\Omega; \mathbb{R}_+)$ and $u \in \mathbf{L}^\infty(\mathbb{R} \times \Omega; \mathbb{R}_+)$, with $\|u\|_{\mathbf{L}^\infty(\mathbb{R}\times\Omega)} \le U$. There exist $T > 0$ and a unique solution (ρ_1, ρ_2, v) to (2.1), (2.2), and (2.3) on the time interval $[0, T]$, in the sense of Definition 2.1. Moreover, for a.e. $t \in [0, T]$ and $x \in \Omega$,*

$$\rho_1(t,x) \ge 0, \quad \rho_2(t,x) \ge 0, \quad v(t,x) \ge 0. \tag{3.1}$$

Finally, if moreover $\rho_{1,o}, \rho_{2,o}, v_o \in \mathbf{H}^1(\Omega; \mathbb{R}_+)$, then

$$\rho_1, \rho_2, v \in \mathbf{L}^2((0,T); \mathbf{H}^2(\Omega)).$$

Proof. Define

$$M = 2 \max \left\{ \|\rho_{1,o}\|_{\mathbf{L}^\infty(\Omega)}, \|\rho_{2,o}\|_{\mathbf{L}^\infty(\Omega)}, \|v_o\|_{\mathbf{L}^\infty(\Omega)} \right\} + 1. \tag{3.2}$$

Fix $T > 0$ such that

$$\begin{aligned} T < \min \Bigg\{ & \frac{1}{\alpha + \delta_1 + \frac{3M-1}{2}\beta} \ln\left(\frac{M+1}{M-1}\right), \frac{1}{\delta_2} \ln\left(\frac{M}{M-1}\right), \\ & \frac{1}{\delta_2} \ln\left(1 + \frac{2\,\delta_2}{\beta\,(3M-1)^2}\right), \frac{2}{(3M-1)B + 2\,\delta_v} \ln\left(\frac{M}{M-1}\right), \\ & \frac{2}{(3M-1)B + 2\,\delta_v} \ln\left(1 + \frac{\delta_v}{(3M-1)b\,\delta_2 + 2\,U}\right), \\ & \frac{2}{2\,\alpha + 2\,\delta_1 + (3M-1)\,\beta} \ln\left(\frac{3}{2}\right), \frac{1}{9\beta^2(3M-1)^2}, \\ & \frac{1}{\delta_2} \ln\left(\frac{3}{2}\right) \frac{2}{2\,\delta_v + (3\,M-1)B} \ln\left(\frac{3}{2}\right), \frac{1}{36\,b^2\,\delta_2^2}, \frac{1}{9B^2(3M-1)^2} \Bigg\}. \end{aligned} \tag{3.3}$$

Consider the Banach space $\mathbf{C^0}([0,T]; \mathbf{L}^2(\Omega))$ and the closed subsets

$$X_1 = \left\{ \zeta \in \mathbf{C^0}([0,T]; \mathbf{L}^2(\Omega)) : \sup_{t\in[0,T]} \|\zeta(t) - \rho_{1,o}\|_{\mathbf{L}^\infty(\Omega)} \le M \right\},$$

$$X_2 = \left\{ \zeta \in \mathbf{C^0}([0,T]; \mathbf{L}^2(\Omega)) : \sup_{t\in[0,T]} \|\zeta(t) - \rho_{2,o}\|_{\mathbf{L}^\infty(\Omega)} \le M \right\},$$

$$X_3 = \left\{ \zeta \in \mathbf{C^0}([0,T]; \mathbf{L}^2(\Omega)) : \sup_{t\in[0,T]} \|\zeta(t) - v_o\|_{\mathbf{L}^\infty(\Omega)} \le M \right\},$$

endowed with the norm of $\mathbf{C^0}([0,T]; \mathbf{L}^2(\Omega))$, namely,

$$\|\zeta\|_{X_i} = \sup_{t\in[0,T]} \|\zeta(t)\|_{\mathbf{L}^2(\Omega)}$$

for $i \in \{1, 2, 3\}$.

Define $X = X_1 \times X_2 \times X_3$ with the norm $\|(\zeta_1, \zeta_2, \zeta_3)\|_X = \sum_{i=1}^{3} \|\zeta_i\|_{X_i}$ and the map

$$\mathcal{T} : X \to X$$

such that for every $(r_1, r_2, w) \in X$, $\mathcal{T}(r_1, r_2, w) = (\rho_1, \rho_2, v)$ is the unique weak solution to the decoupled system

$$\begin{cases} \partial_t \rho_1 = \Delta\rho_1 + (\alpha - \delta_1 - \beta\, w)\rho_1, \\ \partial_t \rho_2 = \Delta\rho_2 - \delta_2\rho_2 + \beta r_1 w, \\ \partial_t v = \Delta v + b\delta_2 r_2 + u - (B r_1 + \delta_v) v \end{cases}$$

with initial data $(\rho_{1,o}, \rho_{2,o}, v_o)$ and Neumann homogeneous boundary conditions. Such solution exists by Theorem A.1, since the functions $r_1 w, r_2, u \in \mathbf{L}^2((0,T); \mathbf{H}^1(\Omega)^*)$. Indeed, to comply with Theorem A.1, we have the following:

- $u \in \mathbf{L}^\infty((0,T) \times \Omega) \subseteq \mathbf{L}^2((0,T) \times \Omega)$,
- $r_2 \in \mathbf{L}^2(\Omega_T)$, being in X_2,
- $r_1,\ w \in \mathbf{L}^2(\Omega_T)$, being, respectively, in X_1 and X_3,
- w, r_1 are needed to be in $\mathbf{L}^\infty(\Omega_T)$ and this is true since, by (3.2), for instance,

$$|r_1(t,x)| \le |r_1(t,x) - \rho_{1,o}(x)| + |\rho_{1,o}(x)| \le M + \|\rho_{1,o}\|_{\mathbf{L}^\infty(\Omega)} \le M + \frac{M-1}{2}.$$

We observe that if $r_1 \in X_1$, $r_2 \in X_2$ and $w \in X_3$, then

$$\|r_1\|_{\mathbf{L}^\infty(\Omega)},\ \|r_2\|_{\mathbf{L}^\infty(\Omega)},\ \|w\|_{\mathbf{L}^\infty(\Omega)} \le \frac{3\,M-1}{2}. \tag{3.4}$$

We need to show that $\mathcal{T}$ is well defined, in the sense that $(\rho_1, \rho_2, v) \in X$. First, note that ρ_1, ρ_2, and v belong to $\mathbf{H}^1((0,T); \mathbf{H}^1(\Omega), \mathbf{H}^1(\Omega)^*)$, see (A.2), and so, by [27, Theorem 7.104], to the space $\mathbf{C}^0([0,T]; \mathbf{L}^2(\Omega))$.

Consider first the case of ρ_1. By Proposition A.2, we deduce that

$$0 \le \rho_1(t,x) \le \|\rho_{1,o}\|_{\mathbf{L}^\infty(\Omega)} e^{(\alpha+\delta_1+\frac{3M-1}{2}\beta)t}.$$

Therefore,

$$\begin{aligned} |\rho_1(t,x) - \rho_{1,o}(x)| &\le |\rho_1(t,x)| + |\rho_{1,o}(x)| \\ &= \rho_1(t,x) + \rho_{1,o}(x) \\ &\le \frac{M-1}{2}\left(1 + e^{(\alpha+\delta_1+\frac{3M-1}{2}\beta)t}\right). \end{aligned}$$

By (3.3),

$$\sup_{t\in[0,T]} \|\rho_1(t) - \rho_{1,o}\|_{\mathbf{L}^\infty(\Omega)} \le \frac{M-1}{2}\left(1 + e^{(\alpha+\delta_1+\frac{3M-1}{2}\beta)T}\right) < M,$$

proving that $\rho_1 \in X_1$.

Pass now to ρ_2. Proposition A.2 yields

$$0 \le \rho_2(t,x) \le \left(\|\rho_{2,o}\|_{\mathbf{L}^\infty(\Omega)} + \frac{\beta \, \|r_1 \, w\|_{\mathbf{L}^\infty(\Omega_t)}}{\delta_2} \right) e^{\delta_2 \, t} - \frac{\beta \, \|r_1 \, w\|_{\mathbf{L}^\infty(\Omega_t)}}{\delta_2}.$$

By (3.4), we deduce that

$$\|r_1 \, w\|_{\mathbf{L}^\infty(\Omega)} \le \frac{(3M-1)^2}{4}.$$

Thus,

$$\begin{aligned} |\rho_2(t,x) - \rho_{2,o}(x)| &\le \|\rho_{2,o}\|_{\mathbf{L}^\infty(\Omega)} (1 + e^{\delta_2 \, t}) + \frac{\beta \, \|r_1 \, w\|_{\mathbf{L}^\infty(\Omega_t)}}{\delta_2} (e^{\delta_2 \, t} - 1) \\ &\le \frac{M-1}{2} (1 + e^{\delta_2 \, t}) + \frac{\beta}{\delta_2} \frac{(3M-1)^2}{4} (e^{\delta_2 \, t} - 1). \end{aligned}$$

By (3.3), we obtain

$$\begin{aligned} \sup_{t \in [0,T]} \|\rho_2(t) - \rho_{2,o}\|_{\mathbf{L}^\infty(\Omega)} &\le \frac{M-1}{2} (1 + e^{\delta_2 \, T}) + \frac{\beta}{\delta_2} \frac{(3M-1)^2}{4} (e^{\delta_2 \, T} - 1) \\ &\le \left(M - \frac{1}{2} \right) + \frac{1}{2} = M, \end{aligned}$$

proving that $\rho_2 \in X_2$.

Consider now v. Proposition A.2 yields

$$\begin{aligned} 0 \le v(t,x) \le & \left(\|v_o\|_{\mathbf{L}^\infty(\Omega)} + \frac{b \, \delta_2 \, \|r_2\|_{\mathbf{L}^\infty(\Omega_t)} + U}{B \, \|r_1\|_{\mathbf{L}^\infty(\Omega_t)} + \delta_v} \right) e^{(B \, \|r_1\|_{\mathbf{L}^\infty(\Omega_t)} + \delta_v) \, t} \\ & - \frac{b \, \delta_2 \, \|r_2\|_{\mathbf{L}^\infty(\Omega_t)} + U}{B \, \|r_1\|_{\mathbf{L}^\infty(\Omega_t)} + \delta_v}. \end{aligned}$$

Exploiting (3.2) and (3.4), we obtain

$$\begin{aligned} 0 \le v(t,x) \le & \|v_o\|_{\mathbf{L}^\infty(\Omega)} \, e^{(B \, \|r_1\|_{\mathbf{L}^\infty(\Omega_t)} + \delta_v) \, t} \\ & + \frac{b \, \delta_2 \, \|r_2\|_{\mathbf{L}^\infty(\Omega_t)} + U}{B \, \|r_1\|_{\mathbf{L}^\infty(\Omega_t)} + \delta_v} \left(e^{(B \, \|r_1\|_{\mathbf{L}^\infty(\Omega_t)} + \delta_v) \, t} - 1 \right) \\ \le & \frac{M-1}{2} + \frac{b \, \delta_2 \, (3M-1) + 2 \, U}{2 \, \delta_v} \left(e^{(B \, \frac{3M-1}{2} + \delta_v) \, t} - 1 \right). \end{aligned}$$

Thus,

$$\begin{aligned} |v(t,x) - v_o(x)| \le & \frac{M-1}{2} \left(1 + e^{(B \, \frac{3M-1}{2} + \delta_v) \, t} \right) \\ & + \frac{b \, \delta_2 \, (3M-1) + 2 \, U}{2 \, \delta_v} \left(e^{(B \, \frac{3M-1}{2} + \delta_v) \, t} - 1 \right). \end{aligned}$$

By (3.3), we obtain

$$\begin{aligned}\sup_{t\in[0,T]} \|v(t)-v_o\|_{\mathbf{L}^\infty(\Omega)} \le{}& \frac{M-1}{2}\left(1+e^{(B\,\frac{3M-1}{2}+\delta_v)\,T}\right)\\&+\frac{b\,\delta_2\,(3M-1)+2\,U}{2\,\delta_v}\left(e^{(B\,\frac{3M-1}{2}+\delta_v)\,T}-1\right)\\ \le{}& \left(M-\frac{1}{2}\right)+\frac{1}{2}=M,\end{aligned}$$

proving that $v\in X_3$.

Fix $(\bar r_1,\bar r_2,\bar w)\in X$ and $(\tilde r_1,\tilde r_2,\tilde w)\in X$. Define

$$(\bar\rho_1,\bar\rho_2,\bar v)=\mathcal{T}(\bar r_1,\bar r_2,\bar w)\quad\text{and}\quad(\tilde\rho_1,\tilde\rho_2,\tilde v)=\mathcal{T}(\tilde r_1,\tilde r_2,\tilde w).$$

Note that

$$\begin{cases}\partial_t(\bar\rho_1-\tilde\rho_1)=\Delta(\bar\rho_1-\tilde\rho_1)+(\alpha-\delta_1-\beta\bar w)(\bar\rho_1-\tilde\rho_1)+\beta(\tilde w-\bar w)\tilde\rho_1,\\(\bar\rho_1-\tilde\rho_1)(0,x)=0\partial_\nu(\bar\rho_1-\tilde\rho_1)(t,\xi)=0.\end{cases}$$

Theorem A.1 implies that, for $t\in[0,T]$,

$$\begin{aligned}&\|\bar\rho_1(t)-\tilde\rho_1(t)\|^2_{\mathbf{L}^2(\Omega)}\\&\quad\le e^{2\|\alpha-\delta_1-\beta\bar w\|_{\mathbf{L}^\infty(\Omega_T)}t}big\int_0^t\|\beta(\tilde w(s)-\bar w(s))\tilde\rho_1(s)\|^2_{\mathbf{H}^1(\Omega)^*}\,\mathrm{d}s\\&\quad\le e^{2(\alpha+\delta_1+\beta\,\frac{3M-1}{2})T}\int_0^t\|\beta(\tilde w(s)-\bar w(s))\tilde\rho_1(s)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s\\&\quad\le\beta^2e^{2(\alpha+\delta_1+\beta\,\frac{3M-1}{2})T}\frac{(3M-1)^2}{4}\int_0^t\|\tilde w(s)-\bar w(s)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s\\&\quad\le\beta^2e^{2(\alpha+\delta_1+\beta\,\frac{3M-1}{2})T}\frac{(3M-1)^2}{4}\sup_{s\in[0,T]}\|\tilde w(s)-\bar w(s)\|^2_{\mathbf{L}^2(\Omega)}T,\end{aligned}$$

where we used the fact that $\tilde\rho_1\in X_1$. Therefore, using (3.3),

$$\|\bar\rho_1-\tilde\rho_1\|_{X_1}\le\beta e^{(\alpha+\delta_1+\beta\,\frac{3M-1}{2})T}\frac{3M-1}{2}\sqrt{T}\|\tilde w-\bar w\|_{X_3}\le\frac{1}{4}\|\tilde w-\bar w\|_{X_3}.$$

Proceed analogously for the other two equations. Indeed, since

$$\begin{cases}\partial_t(\bar\rho_2-\tilde\rho_2)=\Delta(\bar\rho_2-\tilde\rho_2)-\delta_2(\bar\rho_2-\tilde\rho_2)-\beta\,\bar r_1\,\bar w+\beta\,\tilde r_1\,\tilde w,\\(\bar\rho_2-\tilde\rho_2)(0,x)=0,\\\partial_\nu(\bar\rho_2-\tilde\rho_2)(t,\xi)=0,\end{cases}$$

Theorem A.1 implies that, for $t\in[0,T]$,

$$\|\bar\rho_2(t)-\tilde\rho_2(t)\|^2_{\mathbf{L}^2(\Omega)}\le e^{2\,\delta_2\,t}\int_0^t\beta^2\|(\bar r_1\,\bar w-\tilde r_1\,\tilde w)(s)\|_{H^1(\Omega)^*}\,\mathrm{d}s.$$

Observe that, for $s \in [0, T]$, we have

$$
\begin{aligned}
&\|(\bar{r}_1\,\bar{w} - \tilde{r}_1\,\tilde{w})(s)\|_{\mathbf{H}^1(\Omega)^*} \\
&\quad \le \|\bar{r}_1(s)(\bar{w}(s) - \tilde{w}(s))\|_{\mathbf{H}^1(\Omega)^*} + \|(\bar{r}_1(s) - \tilde{r}_1(s))\tilde{w}(s)\|_{\mathbf{H}^1(\Omega)^*} \\
&\quad \le \frac{3\,M-1}{2}(\|(\bar{w} - \tilde{w})(s)\|_{\mathbf{L}^2(\Omega)} + \|(\bar{r}_1 - \tilde{r}_1)(s)\|_{\mathbf{L}^2(\Omega)}),
\end{aligned}
$$

since $\bar{r}_1 \in X_1$ and $\tilde{w} \in X_3$ and we exploited (3.4). Therefore, using (3.3),

$$
\begin{aligned}
\|\bar{\rho}_2 - \tilde{\rho}_2\|_{X_2} &\le e^{\delta_2\,T}\,\beta\,\frac{3\,M-1}{2}\,\sqrt{T}(\|\bar{w} - \tilde{w}\|_{X_3} + \|\bar{r}_1 - \tilde{r}_1\|_{X_1}) \\
&\le \frac{1}{4}(\|\bar{w} - \tilde{w}\|_{X_3} + \|\bar{r}_1 - \tilde{r}_1\|_{X_1}).
\end{aligned}
$$

Lastly, we have

$$
\begin{cases}
\partial_t(\bar{v} - \tilde{v}) = \Delta(\bar{v} - \tilde{v}) - (\delta_v + B\,\bar{r}_1)(\bar{v} - \tilde{v}) - B(\bar{r}_1 - \tilde{r}_1)\tilde{v} + b\,\delta_2(\bar{r}_2 - \tilde{r}_2), \\
(\bar{v} - \tilde{v})(0, x) = 0, \\
\partial_\nu(\bar{v} - \tilde{v})(t, \xi) = 0.
\end{cases}
$$

Again, Theorem A.1 implies that, for $t \in [0, T]$,

$$
\begin{aligned}
&\|\bar{v}(t) - \tilde{v}(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\quad \le e^{2\|\delta_v + B\,\bar{r}_1\|_{\mathbf{L}^\infty(\Omega_T)}t} \int_0^t \|b\,\delta_2(\bar{r}_2(s) - \tilde{r}_2(s)) - B\,(\bar{r}_1(s) - \tilde{r}_1(s))\tilde{v}(s)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s \\
&\quad \le e^{2(\delta_v + B\,\frac{3M-1}{2})T} \int_0^t \|b\,\delta_2(\bar{r}_2(s) - \tilde{r}_2(s)) - B\,(\bar{r}_1(s) - \tilde{r}_1(s))\tilde{v}(s)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s \\
&\quad \le e^{2(\delta_v + B\,\frac{3M-1}{2})T}\,T\Bigg(b\,\delta_2 \sup_{s\in[0,T]} \|\bar{r}_2(s) - \tilde{r}_2(s)\|_{\mathbf{L}^2(\Omega)} \\
&\qquad\qquad + B\,\frac{3\,M-1}{2} \sup_{s\in[0,T]} \|\bar{r}_1(s) - \tilde{r}_1(s)\|_{\mathbf{L}^2(\Omega)}\Bigg)^2,
\end{aligned}
$$

where we used the fact that $\tilde{v} \in X_3$. Hence, using (3.3),

$$
\begin{aligned}
&\|\bar{v} - \tilde{v}\|_{X_3} \\
&\quad \le e^{(\delta_v + B\,\frac{3M-1}{2})T}\,\sqrt{T}\max\left\{b\,\delta_2, B\,\frac{3\,M-1}{2}\right\}(\|\bar{r}_1 - \tilde{r}_1\|_{X_1} + \|\bar{r}_2 - \tilde{r}_2\|_{X_2}) \\
&\quad \le \frac{1}{4}(\|\bar{r}_1 - \tilde{r}_1\|_{X_1} + \|\bar{r}_2 - \tilde{r}_2\|_{X_2}).
\end{aligned}
$$

Therefore, for $t \in [0, T]$,

$$
\begin{aligned}
&\|\bar{\rho}_1 - \tilde{\rho}_1\|_{X_1} + \|\bar{\rho}_2 - \tilde{\rho}_2\|_{X_2} + \|\bar{v} - \tilde{v}\|_{X_3} \\
&\quad \le \frac{1}{4}(2\,\|\bar{r}_1 - \tilde{r}_1\|_{X_1} + \|\bar{r}_2 - \tilde{r}_2\|_{X_2} + 2\,\|\bar{w} - \tilde{w}\|_{X_3}) \\
&\quad \le \frac{1}{2}(\|\bar{r}_1 - \tilde{r}_1\|_{X_1} + \|\bar{r}_2 - \tilde{r}_2\|_{X_2} + \|\bar{w} - \tilde{w}\|_{X_3}),
\end{aligned}
$$

proving that $\mathcal{T}$ is a contraction. Banach fixed point theorem implies that the map $\mathcal{T}$ admits a unique fixed point in X, thus ensuring the existence of solutions to (2.1) on the time interval $[0, T]$. Observe that the solution (ρ_1, ρ_2, v) preserves the positivity of the initial data $(\rho_{1,o}, \rho_{2,o}, v_o)$.

If the initial data are in $\mathbf{H^1}(\Omega; \mathbb{R}_+)$, then, due to Proposition A.3, each component of $\mathcal{T}$ maps $\mathbf{L^2}((0, T); \mathbf{H^2}(\Omega))$ into itself. Therefore, repeating the same argument as above yields $\rho_2, \rho_2, v \in \mathbf{L^2}((0, T); \mathbf{H^2}(\Omega))$. ■

The next result deals with the global existence of solutions.

Theorem 3.2. *Assume α, β, δ_1, δ_2, δ_v, b, B, U fixed positive constants. Let $\Omega \subseteq \mathbb{R}^N$ be an open, connected, and bounded domain, with smooth boundary $\partial\Omega$. Fix $\rho_{1,o}, \rho_{2,o}, v_o \in \mathbf{L}^\infty(\Omega; \mathbb{R}_+)$ and $u \in \mathbf{L}^\infty(\mathbb{R} \times \Omega; \mathbb{R}_+)$, with $\|u\|_{\mathbf{L}^\infty(\mathbb{R}\times\Omega)} \leq U$. Then, for every $T > 0$, there exists a unique solution (ρ_1, ρ_2, v) to (2.1), (2.2), and (2.3) on the time interval $[0, T]$, in the sense of Definition 2.1.*

Moreover, for a.e. $(t, x) \in [0, T] \times \Omega$, we have the following estimates:

$$0 \leq \rho_1(t, x) \leq \|\rho_{1,o}\|_{\mathbf{L}^\infty} e^{|\alpha-\delta_1|T}, \tag{3.5}$$

$$0 \leq \rho_2(t, x) \leq (\|\rho_{1,o}\|_{\mathbf{L}^\infty} + \|\rho_{2,o}\|_{\mathbf{L}^\infty}) e^{\alpha T}, \tag{3.6}$$

$$0 \leq v(t, x) \leq \|v_o\|_{\mathbf{L}^\infty} + (b\delta_2(\|\rho_{1,o}\|_{\mathbf{L}^\infty} + \|\rho_{2,o}\|_{\mathbf{L}^\infty}) e^{\alpha T} + U)T. \tag{3.7}$$

Proof. Define

$$\bar{T} = \sup\left\{T > 0 : \text{the solution to (2.1) exists in } [0, T]\right\}.$$

Clearly, Theorem 3.1 implies that $\bar{T} > 0$. Assume by contradiction that $\bar{T} < +\infty$. Since ρ_1 and v are positive by (3.1) and since $\beta > 0$, then ρ_1 is a subsolution to

$$\partial_t \rho_1 \leq \Delta\rho_1 + (\alpha - \delta_1)\rho_1.$$

Hence, Proposition A.2 implies that

$$0 \leq \rho_1(t, x) \leq \|\rho_{1,o}\|_{\mathbf{L}^\infty} e^{|\alpha-\delta_1|\bar{T}} \tag{3.8}$$

for every $t < \bar{T}$ and for a.e. $x \in \Omega$.

Consider now the equation for the sum $\rho_1 + \rho_2$:

$$\partial_t (\rho_1 + \rho_2) = \Delta(\rho_1 + \rho_2) + (\alpha - \delta_1)\rho_1 - \delta_2 \rho_2.$$

Since $\alpha > 0$, $\delta_1 > 0$, $\delta_2 > 0$, and $\rho_2 \geq 0$ by (3.1), $\rho_1 + \rho_2$ is a subsolution to

$$\partial_t (\rho_1 + \rho_2) \leq \Delta(\rho_1 + \rho_2) + \alpha(\rho_1 + \rho_2).$$

Hence, Proposition A.2 implies that

$$0 \leq \rho_1(t, x) + \rho_2(t, x) \leq (\|\rho_{1,o}\|_{\mathbf{L}^\infty} + \|\rho_{2,o}\|_{\mathbf{L}^\infty}) e^{\alpha\bar{T}}$$

for every $t < \bar{T}$ and for a.e. $x \in \Omega$ so that

$$0 \le \rho_2(t,x) \le (\|\rho_{1,o}\|_{\mathbf{L}^\infty} + \|\rho_{2,o}\|_{\mathbf{L}^\infty})e^{\alpha\bar{T}} \tag{3.9}$$

for every $t < \bar{T}$ and for a.e. $x \in \Omega$.

Using the estimates (3.8) and (3.9), we deduce that v is subsolution to

$$\partial_t v \le \Delta v + b\delta_2(\|\rho_{1,o}\|_{\mathbf{L}^\infty} + \|\rho_{2,o}\|_{\mathbf{L}^\infty})e^{\alpha\bar{T}} + U,$$

where we used the fact that $b > 0$, $\delta_2 > 0$, $B > 0$, $\delta_v > 0$, $\|u\|_{\mathbf{L}^\infty} \le U$, and $\rho_1 \ge 0$, $v \ge 0$ by (3.1). Hence, Proposition A.2 implies that

$$0 \le v(t,x) \le \|v_o\|_{\mathbf{L}^\infty} + (b\delta_2(\|\rho_{1,o}\|_{\mathbf{L}^\infty} + \|\rho_{2,o}\|_{\mathbf{L}^\infty})e^{\alpha\bar{T}} + U)\bar{T} \tag{3.10}$$

for every $t < \bar{T}$ and for a.e. $x \in \Omega$.

Standard arguments in Sobolev spaces dependent on time together with the estimates (3.8), (3.9), and (3.10) and the assumptions on Ω permit to extend ρ_1, ρ_2, and v by continuity at time $\bar{T}$. Since $\rho_1(\bar{T}) \in \mathbf{L}^2(\Omega)$, $\rho_2(\bar{T}) \in \mathbf{L}^2(\Omega)$, and $v(\bar{T}) \in \mathbf{L}^2(\Omega)$, Theorem 3.1 implies that the solution (ρ_1, ρ_2, v) exists also for times bigger than $\bar{T}$. This is in contradiction with the definition of $\bar{T}$. Finally, the estimates (3.5), (3.6), and (3.7) easily follow from (3.8), (3.9), and (3.10). ■

4. Dependence from the control *u*

In this part, we show that the solution to (2.1) continuously depends on the control u, viewed as a function in $\mathbf{L}^2((0,T) \times \Omega; [0,U])$ endowed with the strong topology.

Theorem 4.1. *Assume α, β, δ_1, δ_2, δ_v, b, B, and U fixed positive constants. Let $\Omega \subseteq \mathbb{R}^N$ be an open, connected, and bounded domain, with smooth boundary $\partial\Omega$. Fix $T > 0$, $\rho_{1,o}, \rho_{2,o}, v_o \in \mathbf{L}^\infty(\Omega; \mathbb{R}_+)$ and $\bar{u}, \tilde{u} \in \mathbf{L}^\infty(\mathbb{R} \times \Omega; [0,U])$. Define $(\bar{\rho}_1, \bar{\rho}_2, \bar{v})$ and $(\tilde{\rho}_1, \tilde{\rho}_2, \tilde{v})$ the solutions to (2.1), (2.2), and (2.3) on the time interval $[0,T]$ with controls $\bar{u}$ and $\tilde{u}$, respectively.*

Then, there exists a positive constant C, depending on $\|\rho_{1,o}\|_{\mathbf{L}^\infty(\Omega)}$, $\|\rho_{2,o}\|_{\mathbf{L}^\infty(\Omega)}$, and $\|v_o\|_{\mathbf{L}^\infty(\Omega)}$, on T, and on the constants α, δ_1, B, β, b, δ_2, δ_v such that for every $t \in [0,T]$,

$$\|\bar{\rho}_1(t) - \tilde{\rho}_1(t)\|_{\mathbf{L}^2(\Omega)} + \|\bar{\rho}_2(t) - \tilde{\rho}_2(t)\|_{\mathbf{L}^2(\Omega)} + \|\bar{v}(t) - \tilde{v}(t)\|_{\mathbf{L}^2(\Omega)} \le C\|\bar{u} - \tilde{u}\|_{\mathbf{L}^2((0,t)\times\Omega)}, \tag{4.1}$$

and

$$\int_0^t \left(\|\bar{\rho}_1(\tau) - \tilde{\rho}_1(\tau)\|^2_{\mathbf{H}^1(\Omega)} + \|\bar{\rho}_2(\tau) - \tilde{\rho}_2(\tau)\|^2_{\mathbf{H}^1(\Omega)} + \|\bar{v}(\tau) - \tilde{v}(\tau)\|^2_{\mathbf{H}^1(\Omega)}\right) d\tau \le C\|\bar{u} - \tilde{u}\|^2_{\mathbf{L}^2((0,t)\times\Omega)}. \tag{4.2}$$

Proof. Fix two control functions $\bar{u}, \tilde{u} \in \mathbf{L}^\infty((0,T)\times\Omega;[0,U])$ and denote by $(\bar{\rho}_1, \bar{\rho}_2, \bar{v})$ and by $(\tilde{\rho}_1, \tilde{\rho}_2, \tilde{v})$ the corresponding solutions to (2.1), (2.2), and (2.3).

Consider the difference between the equations for $\bar{v}$ and $\tilde{v}$ in (2.1), and rearrange it as follows:

$$\partial_t (\bar{v} - \tilde{v}) = \Delta(\bar{v} - \tilde{v}) - (\delta_v + B\bar{\rho}_1)(\bar{v} - \tilde{v}) + b\delta_2(\bar{\rho}_2 - \tilde{\rho}_2) - B\tilde{v}(\bar{\rho}_1 - \tilde{\rho}_1) + \bar{u} - \tilde{u}.$$

Observe first that the bilinear form appearing above is weakly coercive, so, in particular,

$$\begin{aligned}\int_\Omega \left[\nabla(\bar{v} - \tilde{v}) \cdot \nabla(\bar{v} - \tilde{v}) + (\delta_v + B\bar{\rho}_1)(\bar{v} - \tilde{v})^2\right] \mathrm{d}x \\ + \left(\frac{1}{2} + \delta_v + B\|\bar{\rho}_1(t)\|_{\mathbf{L}^\infty(\Omega)}\right)\|\bar{v} - \tilde{v}\|^2_{\mathbf{L}^2(\Omega)} \geq \frac{1}{2}\|\bar{v} - \tilde{v}\|^2_{\mathbf{H}^1(\Omega)}.\end{aligned} \tag{4.3}$$

By Definition 2.1, point (5), we have that, for a.e. $t \in [0,T]$ and for all $w \in \mathbf{H}^1(\Omega)$,

$$\begin{aligned}&\int_\Omega (\dot{\bar{v}} - \dot{\tilde{v}}) w \,\mathrm{d}x + \int_\Omega \nabla(\bar{v} - \tilde{v}) \cdot \nabla w \,\mathrm{d}x + \int_\Omega (\delta_v + B\bar{\rho}_1)(\bar{v} - \tilde{v}) w \,\mathrm{d}x \\ &\quad = b\delta_2 \int_\Omega (\bar{\rho}_2 - \tilde{\rho}_2) w \,\mathrm{d}x - B\int_\Omega \tilde{v}(\bar{\rho}_1 - \tilde{\rho}_1) w \,\mathrm{d}x + \int_\Omega (\bar{u} - \tilde{u}) w \,\mathrm{d}x.\end{aligned}$$

Taking $w(x) = (\bar{v}(t,x) - \tilde{v}(t,x))$ in the previous equation and exploiting (4.3), we deduce that, for a.e. $t \in [0,T]$,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{v}(t) - \tilde{v}(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\|\bar{v}(t) - \tilde{v}(t)\|^2_{\mathbf{H}^1(\Omega)} \\ &\quad\leq b\delta_2 \int_\Omega (\bar{\rho}_2(t,x) - \tilde{\rho}_2(t,x))(\bar{v}(t,x) - \tilde{v}(t,x))\,\mathrm{d}x \\ &\qquad - B\int_\Omega \tilde{v}(t,x)(\bar{\rho}_1(t,x) - \tilde{\rho}_1(t,x))(\bar{v}(t,x) - \tilde{v}(t,x))\,\mathrm{d}x \\ &\quad + \int_\Omega (\bar{u}(t,x) - \tilde{u}(t,x))(\bar{v}(t,x) - \tilde{v}(t,x))\,\mathrm{d}x \\ &\qquad + \left(\frac{1}{2} + \delta_v + B\|\bar{\rho}_1(t)\|_{\mathbf{L}^\infty(\Omega)}\right)\|\bar{v}(t) - \tilde{v}(t)\|^2_{\mathbf{L}^2(\Omega)} \\ &\quad\leq \frac{b\,\delta_2}{2}\left(\|\bar{\rho}_2(t) - \tilde{\rho}_2(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\bar{v}(t) - \tilde{v}(t)\|^2_{\mathbf{L}^2(\Omega)}\right) \\ &\qquad + \frac{B}{2}\left(\|\tilde{v}(t)\|^2_{\mathbf{L}^\infty(\Omega)}\|\bar{v}(t) - \tilde{v}(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\bar{\rho}_1(t) - \tilde{\rho}_1(t)\|^2_{\mathbf{L}^2(\Omega)}\right) \\ &\qquad + \frac{1}{2}\|\bar{u}(t) - \tilde{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\|\bar{v}(t) - \tilde{v}(t)\|^2_{\mathbf{L}^2(\Omega)} \\ &\qquad + \left(\frac{1}{2} + \delta_v + B\|\bar{\rho}_1(t)\|_{\mathbf{L}^\infty(\Omega)}\right)\|\bar{v}(t) - \tilde{v}(t)\|^2_{\mathbf{L}^2(\Omega)}.\end{aligned}$$

We proceed similarly for $\bar{\rho}_1$ and $\tilde{\rho}_1$: for a.e. $t \in [0, T]$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\rho}_1(t)-\tilde{\rho}_1(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\|\bar{\rho}_1(t)-\tilde{\rho}_1(t)\|^2_{\mathbf{H}^1(\Omega)} \\
&\quad \le -\beta\int_\Omega \tilde{\rho}_1(\bar{\rho}_1(t,x)-\tilde{\rho}_1(t,x))(\bar{v}(t,x)-\tilde{v}(t,x))\,\mathrm{d}x \\
&\qquad + \Big(\frac{1}{2}+\delta_1+\beta\|\bar{v}(t)\|_{\mathbf{L}^\infty(\Omega)}\Big)\|\bar{\rho}_1(t)-\tilde{\rho}_1(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\quad \le \frac{\beta}{2}\|\tilde{\rho}_1(t)\|^2_{\mathbf{L}^\infty(\Omega)}\|\bar{\rho}_1(t)-\tilde{\rho}_1(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\beta}{2}\|\bar{v}(t)-\tilde{v}(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\qquad + \left(\frac{1}{2}+\delta_1+\alpha+\beta\|\bar{v}(t)\|_{\mathbf{L}^\infty(\Omega)}\right)\|\bar{\rho}_1(t)-\tilde{\rho}_1(t)\|^2_{\mathbf{L}^2(\Omega)}.
\end{aligned}$$

Finally, the same arguments can be applied to $\bar{\rho}_2$ and $\tilde{\rho}_2$: for a.e. $t \in [0, T]$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\rho}_2(t)-\tilde{\rho}_2(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\|\bar{\rho}_2(t)-\tilde{\rho}_2(t)\|^2_{\mathbf{H}^1(\Omega)} \\
&\quad \le \beta\int_\Omega \bar{\rho}_1(t,x)(\bar{v}(t,x)-\tilde{v}(t,x))(\bar{\rho}_2(t,x)-\tilde{\rho}_2(t,x))\,\mathrm{d}x \\
&\qquad + \beta\int_\Omega \tilde{v}(t,x)(\bar{\rho}_1(t,x)-\tilde{\rho}_1(t,x))(\bar{\rho}_2(t,x)-\tilde{\rho}_2(t,x))\,\mathrm{d}x \\
&\qquad + \left(\frac{1}{2}+\delta_2\right)\|\bar{\rho}_2(t)-\tilde{\rho}_2(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\quad \le \frac{\beta}{2}\big(\|\bar{\rho}_1(t)\|^2_{\mathbf{L}^\infty(\Omega)}\|\bar{\rho}_2(t)-\tilde{\rho}_2(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\bar{v}(t)-\tilde{v}(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\qquad + \|\tilde{v}(t)\|^2_{\mathbf{L}^\infty(\Omega)}\|\bar{\rho}_1(t)-\tilde{\rho}_1(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\bar{\rho}_2(t)-\tilde{\rho}_2(t)\|^2_{\mathbf{L}^2(\Omega)}\big) \\
&\qquad + \Big(\frac{1}{2}+\delta_2\Big)\|\bar{\rho}_2(t)-\tilde{\rho}_2(t)\|^2_{\mathbf{L}^2(\Omega)}.
\end{aligned}$$

Thus, collecting together the estimates obtained above, for a.e. $t \in [0, T]$, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\bar{\rho}_1(t)-\tilde{\rho}_1(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\bar{\rho}_2(t)-\tilde{\rho}_2(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\bar{v}(t)-\tilde{v}(t)\|^2_{\mathbf{L}^2(\Omega)}\big) \\
&\qquad + \|\bar{\rho}_1(t)-\tilde{\rho}_1(t)\|^2_{\mathbf{H}^1(\Omega)} + \|\bar{\rho}_2(t)-\tilde{\rho}_2(t)\|^2_{\mathbf{H}^1(\Omega)} + \|\bar{v}(t)-\tilde{v}(t)\|^2_{\mathbf{H}^1(\Omega)} \\
&\quad \le \big(B+\beta\|\tilde{\rho}_1(t)\|^2_{\mathbf{L}^\infty(\Omega)} + \beta\|\tilde{v}(t)\|^2_{\mathbf{L}^\infty(\Omega)} + 1 + 2\,\delta_1 + 2\,\alpha + 2\,\beta\|\bar{v}(t)\|_{\mathbf{L}^\infty(\Omega)}\big) \\
&\qquad \times \|\bar{\rho}_1(t)-\tilde{\rho}_1(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\quad + \big(\beta+\beta\|\bar{\rho}_1(t)\|^2_{\mathbf{L}^\infty(\Omega)} + b\,\delta_2 + 1 + 2\,\delta_2\big)\|\bar{\rho}_2(t)-\tilde{\rho}_2(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\qquad + \big(2\,\beta + b\,\delta_2 + B\|\tilde{v}(t)\|^2_{\mathbf{L}^\infty(\Omega)} + 2 + 2\,\delta_v + B\|\bar{\rho}_1(t)\|_{\mathbf{L}^\infty(\Omega)}\big) \\
&\qquad \times \|\bar{v}(t)-\tilde{v}(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\qquad + \|\bar{u}(t)-\tilde{u}(t)\|^2_{\mathbf{L}^2(\Omega)}.
\end{aligned}$$

Set

$$C(t) = \max \Big\{ B + \beta\big(\|\tilde\rho_1(t)\|^2_{\mathbf{L}^\infty(\Omega)} + \|\tilde v(t)\|^2_{\mathbf{L}^\infty(\Omega)} + 2\,\|\bar v(t)\|_{\mathbf{L}^\infty(\Omega)}\big) \\ + 1 + 2\,\delta_1 + 2\alpha,\ \beta + \beta\|\bar\rho_1(t)\|^2_{\mathbf{L}^\infty(\Omega)} + b\,\delta_2 + 1 + 2\,\delta_2, \\ 2\,\beta + b\,\delta_2 + B\|\tilde v(t)\|^2_{\mathbf{L}^\infty(\Omega)} + 2 + 2\,\delta_v + B\|\bar\rho_1(t)\|_{\mathbf{L}^\infty(\Omega)} \Big\}.$$

An application of Gronwall's inequality yields the following bound for the $\mathbf{L}^2$-norm:

$$\|\bar\rho_1(t) - \tilde\rho_1(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\bar\rho_2(t) - \tilde\rho_2(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\bar v(t) - \tilde v(t)\|^2_{\mathbf{L}^2(\Omega)} \\ \le \int_0^t \|\bar u(s) - \tilde u(s)\|^2_{\mathbf{L}^2(\Omega)} \exp\left(\int_s^t C(\tau)\,\mathrm{d}\tau\right) \mathrm{d}s.$$

Moreover, we get

$$\int_0^t \big(\|\bar\rho_1(\tau) - \tilde\rho_1(\tau)\|^2_{\mathbf{H}^1(\Omega)} + \|\bar\rho_2(\tau) - \tilde\rho_2(\tau)\|^2_{\mathbf{H}^1(\Omega)} + \|\bar v(\tau) - \tilde v(\tau)\|^2_{\mathbf{H}^1(\Omega)}\big)\,\mathrm{d}\tau \\ \le \left(1 + \int_0^t C(\tau) \exp\left(\int_\tau^s C(s)\,\mathrm{d}s\right) \mathrm{d}\tau\right) \int_0^t \|\bar u(\tau) - \tilde u(\tau)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}\tau \\ = \exp\left(\int_0^t C(\tau)\,\mathrm{d}\tau\right) \int_0^t \|\bar u(\tau) - \tilde u(\tau)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}\tau.$$

Note that, using the estimates (3.5), (3.6), and (3.7) of Theorem 3.2, the function $C(t)$ can be controlled by a constant depending on T, on the initial conditions, on $|\Omega|$, and on the various constants appearing in system (2.1). Therefore, the estimates (4.1) and (4.2) hold and the proof is finished. ■

5. Optimal control problem

In this section, we consider optimal control problems for the system (2.1), (2.2), and (2.3), obtained through the minimization of a functional, which explicitly depends on the control u and consequently on the solution to (2.1). To this aim, define the (continuous) functions

$$\psi_1 : \mathbb{R}^3 \to \mathbb{R}, \quad \psi_2 : \mathbb{R}^4 \to \mathbb{R},$$

and the functional $J : \mathbf{L}^2((0,T) \times \Omega; [0,U]) \to \mathbb{R}$ as

$$J(u) = \int_\Omega \psi_1(\rho_1(T,x), \rho_2(T,x), v(T,x))\,\mathrm{d}x \\ + \int_0^T \int_\Omega \psi_2(\rho_1(t,x), \rho_2(t,x), v(t,x), u(t,x))\,\mathrm{d}x\,\mathrm{d}t, \tag{5.1}$$

which we aim to minimize. We remark that in this section we consider, as the domain of the functional J, the space $\mathbf{L}^2((0,T) \times \Omega; [0,U])$ (with $0 < U < +\infty$). This implies that, since Ω is a bounded set, every minimizing sequence for J admits a weakly convergent subsequence (see the proof of Theorem 5.1), also in the case some term in (5.1) is missing.

Remark 2. The general form of the functional (5.1) describes in a unified way several possible objective functionals, which are natural in application related to cancer therapies against glioma.

For example, if the main objective is the maximal reduction of the volume of the cancer at a time T, one can consider the functional

$$J(u) = \gamma_1 \int_\Omega \rho_1(T,x)\,\mathrm{d}x + \gamma_2 \int_\Omega \rho_2(T,x)\,\mathrm{d}x$$

for suitable $\gamma_1, \gamma_2 \geq 0$. This is a special version of the functional (5.1), obtained with the choice $\psi_1(\rho_1, \rho_2, v) = \gamma_1\rho_1 + \gamma_2\rho_2$ and $\psi_2(\rho_1, \rho_2, v, u) = 0$.

Another similar example is derived when the objective is the minimal cancer size at a time T, obtained with the least dose of treatment, due to its side effects. In this case, a meaningful functional is

$$J(u) = \gamma_1 \int_\Omega \rho_1(T,x)\,\mathrm{d}x + \gamma_2 \int_\Omega \rho_2(T,x)\,\mathrm{d}x + \int_0^T \int_\Omega u^p(t,x)\,\mathrm{d}x\,\mathrm{d}t,$$

where $\gamma_1, \gamma_2 \geq 0$ and $p \geq 1$. This is a special version of (5.1), obtained with the choice $\psi_1(\rho_1, \rho_2, v) = \gamma_1\rho_1 + \gamma_2\rho_2$ and $\psi_2(\rho_1, \rho_2, v, u) = u^p$.

A third example is related to a possible way to make the glioma a chronic disease. In this case, if $\bar{\rho} \in \mathbf{L}^2(\Omega)$ denotes the distribution of the cancer in a chronic situation, then one aims to minimize

$$\begin{aligned} J(u) = {} & \int_\Omega (\rho_1(T,x) - \bar{\rho}(x))^2\,\mathrm{d}x + \int_0^T \int_\Omega (\rho_1(t,x) - \bar{\rho}(x))^2\,\mathrm{d}x\,\mathrm{d}t \\ & + \int_0^T \int_\Omega u^p(t,x)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

This is (5.1) with $\psi_1(\rho_1, \rho_2, v) = (\rho_1 - \bar{\rho})^2$ and $\psi_2(\rho_1, \rho_2, v, u) = (\rho_1 - \bar{\rho})^2 + u^p$.

Existence of optimal controls is guaranteed by the next result, whose proof is based on the direct method of the calculus of variation; see, for example, [9, 14] and references therein.

Theorem 5.1. *Let α, β, δ_1, δ_2, δ_v, b, B, and U be fixed positive constants. Assume that ψ_1 is continuous, convex and $\psi_1(z) \geq 0$ for every $z \in \mathbb{R}^3$. Suppose moreover that*

$$\psi_2(\rho_1, \rho_2, v, u) = \psi_{2,1}(\rho_1, \rho_2, v) + \psi_{2,2}(\rho_1, \rho_2, v)u^p,$$

where $p \geq 1$ and $\psi_{2,1}, \psi_{2,2} : \mathbb{R}^3 \to \mathbb{R}$ are continuous and positive functions. Let $\mathcal{U} \neq \emptyset$ be a closed (with respect to the strong topology) and convex subset of $\mathbf{L}^2((0,T) \times \Omega; [0,U])$.

Then, there exists $\bar{u} \in \mathcal{U}$ such that

$$J(\bar{u}) = \min_{u \in \mathcal{U}} J(u). \tag{5.2}$$

Proof. Consider a minimizing sequence u_n for the functional J, i.e., a sequence $u_n \in \mathcal{U}$ such that

$$\lim_{n\to+\infty} J(u_n) = \inf_{u\in\mathcal{U}} J(u).$$

This is possible, since $J(u) \geq 0$ for every $u \in \mathcal{U}$ by assumptions on ψ_1, $\psi_{2,1}$, $\psi_{2,2}$, and since $J(0) < +\infty$, which is a consequence of the estimates in Theorem 3.2. We clearly have that

$$\|u_n\|^2_{\mathbf{L}^2((0,T)\times\Omega)} = \int_0^T \int_\Omega |u_n(t,x)|^2 \, \mathrm{d}x \, \mathrm{d}t \leq U^2 \, |\Omega| \, T,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω, so that there exists $\bar{u} \in \mathcal{U}$ (here $\mathcal{U}$ is also closed in the weak topology, since it is convex) and a subsequence u_{n_h} such that $u_{n_h} \rightharpoonup \bar{u}$ in $\mathbf{L}^2((0,T) \times \Omega; [0,U])$. Without loss of generality, we assume that the whole sequence u_n weakly converges in $\mathbf{L}^2((0,T) \times \Omega; [0,U])$ to $\bar{u}$. Note, moreover, that there is weak convergence to the same $\bar{u}$ in every $\mathbf{L}^\mathbf{p}((0,T) \times \Omega; [0,U])$, $p \geq 1$.

For every $n \in \mathbb{N}$, denote with $(\rho_1^n, \rho_2^n, v^n)$ the solution to (2.1) corresponding to the control u_n, in the sense of Definition 2.1, which exists by Theorem 3.2. Moreover, define $(\bar\rho_1, \bar\rho_2, \bar v)$ the solution to (2.1) corresponding to the control $\bar u$.

Define, for every $n \in \mathbb{N}$, $S_n = \rho_1^n v^n$. By (3.5) and (3.7), we deduce that

$$\begin{aligned}|S_n(t,x)| \leq{}& \|\rho_{1,o}\|_{\mathbf{L}^\infty} \|v_o\|_{\mathbf{L}^\infty} e^{|\alpha-\delta_1|T} \\ &+ \|\rho_{1,o}\|_{\mathbf{L}^\infty} \big(b\delta_2(\|\rho_{1,o}\|_{\mathbf{L}^\infty} + \|\rho_{2,o}\|_{\mathbf{L}^\infty})e^{\alpha T} + U\big)T\end{aligned}$$

for a.e. $(t,x) \in [0,T] \times \Omega$. Thus, without loss of generality, there exists $\bar S \in \mathbf{L}^2((0,T) \times \Omega)$ so that S_n weakly converges to $\bar S$ in $\mathbf{L}^2((0,T) \times \Omega)$.

Note that for every $n \in \mathbb{N}$, the triple $(\rho_1^n, \rho_2^n, v^n)$ is a solution to the linear system

$$\begin{cases} \partial_t \rho_1 = \Delta\rho_1 + (\alpha - \delta_1)\rho_1 - \beta S_n, \\ \partial_t \rho_2 = \Delta\rho_2 + \beta S_n - \delta_2\rho_2, \\ \partial_t v = \Delta v + b\delta_2\rho_2 - BS_n - \delta_v v + u_n \end{cases} \tag{5.3}$$

in the sense of Definition 2.1.

We apply Corollary A.1 to each linear equation of (5.3). Since the operator defined in Corollary A.1 is also continuous with respect to the weak topology of both domain and codomain, see [3, Theorem 3.10], there exist $\tilde\rho_1$, $\tilde\rho_2$, and $\tilde v$ in $\mathbf{H}^1((0,T); \mathbf{H}^1(\Omega), \mathbf{H}^1(\Omega)^*)$ such that

$$\rho_1^n \rightharpoonup \tilde\rho_1, \quad \rho_2^n \rightharpoonup \tilde\rho_2, \quad v^n \rightharpoonup \tilde v$$

weakly in $\mathbf{H}^1((0,T); \mathbf{H}^1(\Omega), \mathbf{H}^1(\Omega)^*)$ and the triple $(\tilde\rho_1, \tilde\rho_2, \tilde v)$ satisfies the linear system

$$\begin{cases} \partial_t \tilde\rho_1 = \Delta\tilde\rho_1 + (\alpha - \delta_1)\tilde\rho_1 - \beta\bar S, \\ \partial_t \tilde\rho_2 = \Delta\tilde\rho_2 + \beta\bar S - \delta_2\tilde\rho_2, \\ \partial_t \tilde v = \Delta\tilde v + b\delta_2\tilde\rho_2 - B\bar S - \delta_v\tilde v + \bar u. \end{cases} \tag{5.4}$$

Moreover, since the space $\mathbf{H}^1((0,T);\mathbf{H}^1(\Omega),\mathbf{H}^1(\Omega)^*)$ is continuously embedded in the space $\mathbf{C}^0([0,T];\mathbf{L}^2(\Omega))$ (see [27, Theorem 7.104]), we also deduce that

$$\rho_1^n(T) \rightharpoonup \tilde{\rho}_1(T), \quad \rho_2^n(T) \rightharpoonup \tilde{\rho}_2(T), \quad v^n(T) \rightharpoonup \tilde{v}(T) \tag{5.5}$$

weakly in $\mathbf{L}^2(\Omega)$. This is a consequence that the dual of $\mathbf{C}^0([0,T];\mathbf{L}^2(\Omega))$ can be identified with integral operators from $\mathbf{C}^0([0,T];\mathbb{R})$ to the dual space of $\mathbf{L}^2(\Omega)$ (see [26, Sections 3.2 and 3.5]), which can be described through vector measures on $[0,T]$ over the dual of $\mathbf{L}^2(\Omega)$; see [26, Proposition 5.28].

By [17, Theorem 10.1], for every $n \in \mathbb{N}$, the solution $(\rho_1^n, \rho_2^n, v^n)$ is Hölder continuous in each subset compactly embedded in $(0,T)\times\Omega$, with exponent not depending on n. Therefore, by Ascoli–Arzelà theorem, there exist continuous functions $\hat{\rho}_1$, $\hat{\rho}_2$, and $\hat{v}$ such that, possibly extracting a subsequence,

$$\rho_1^n \to \hat{\rho}_1, \quad \rho_2^n \to \hat{\rho}_2, \quad v^n \to \hat{v}$$

as $n \to +\infty$, and the convergence is uniform. Hence, $\tilde{\rho}_1 = \hat{\rho}_1$, $\tilde{\rho}_2 = \hat{\rho}_2$, $\tilde{v} = \hat{v}$, and so,

$$S_n = \rho_1^n v^n \to \hat{\rho}_1 \hat{v}$$

uniformly and in $\mathbf{L}^2((0,T)\times\Omega)$.

Therefore, the triple $(\hat{\rho}_1, \hat{\rho}_2, \hat{v})$ solves (5.4) with $\bar{S} = \hat{\rho}_1\hat{v}$ and control $\bar{u}$. Since by Theorem 3.2 the solution to such problem is unique, it holds $(\hat{\rho}_1, \hat{\rho}_2, \hat{v}) = (\bar{\rho}_1, \bar{\rho}_2, \bar{v})$.

We now show that the control $\bar{u}$ is indeed optimal. Consider the three terms defining the functional J separately. The functional

$$\begin{aligned} &\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega) \to \mathbb{R} \\ &\qquad (\rho_1,\rho_2,v) \mapsto \int_\Omega \psi_1(\rho_1(x),\rho_2(x),v(x))\,\mathrm{d}x \end{aligned}$$

is sequential lower semicontinuous with respect to the strong topology. Since ψ_1 is a convex function, then it is also sequential lower semicontinuous with respect to the weak topology. Therefore, since (5.5), we deduce that

$$\begin{aligned} \liminf_{n\to+\infty} &\int_\Omega \psi_1(\rho_1^n(T,x),\rho_2^n(T,x),v^n(T,x))\,\mathrm{d}x \\ &\geq \int_\Omega \psi_1(\bar{\rho}_1(T,x),\bar{\rho}_2(T,x),\bar{v}(T,x))\,\mathrm{d}x. \end{aligned}$$

Moreover, since $(\rho_1^n, \rho_2^n, v^n)$ converges to $(\bar{\rho}_1, \bar{\rho}_2, \bar{v})$ for a.e. $(t,x)\in(0,T)\times\Omega$, the estimates (3.5)–(3.7) hold, and $\psi_{2,1}$ and $\psi_{2,2}$ are continuous functions, then the dominated convergence theorem implies that

$$\begin{aligned} \lim_{n\to+\infty} &\int_0^T \int_\Omega \psi_{2,1}(\rho_1^n(t,x),\rho_2^n(t,x),v^n(t,x))\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_\Omega \psi_{2,1}(\bar{\rho}_1(t,x),\bar{\rho}_2(t,x),\bar{v}(t,x))\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

and also that

$$\lim_{n\to+\infty} \psi_{2,2}(\rho_1^n, \rho_2^n, v^n) = \psi_{2,2}(\bar{\rho}_1, \bar{\rho}_2, \bar{v}) \tag{5.6}$$

in $\mathbf{L^p}(\Omega)$ for every $p \in [1, +\infty)$.

Finally, by (5.6) and the fact that $\left|u_n^p(t,x)\right| \le U^p$ for all $(t,x) \in [0,T] \times \Omega$, we deduce that

$$\begin{aligned}
&\liminf_{n\to+\infty} \int_0^T \int_\Omega \psi_{2,2}(\rho_1^n(t,x), \rho_2^n(t,x), v^n(t,x)) u_n^p(t,x)\, \mathrm{d}x\, \mathrm{d}t \\
&\quad = \liminf_{n\to+\infty} \int_0^T \int_\Omega \psi_{2,2}(\bar{\rho}_1(t,x), \bar{\rho}_2(t,x), \bar{v}(t,x)) u_n^p(t,x)\, \mathrm{d}x\, \mathrm{d}t \\
&\quad\quad + \lim_{n\to+\infty} \int_0^T \int_\Omega [\psi_{2,2}(\rho_1^n(t,x), \rho_2^n(t,x), v^n(t,x)) \\
&\quad\quad\quad - \psi_{2,2}(\bar{\rho}_1(t,x), \bar{\rho}_2(t,x), \bar{v}(t,x))] u_n^p(t,x)\, \mathrm{d}x\, \mathrm{d}t \\
&\quad = \liminf_{n\to+\infty} \int_0^T \int_\Omega \psi_{2,2}(\bar{\rho}_1(t,x), \bar{\rho}_2(t,x), \bar{v}(t,x)) u_n^p(t,x)\, \mathrm{d}x\, \mathrm{d}t.
\end{aligned}$$

Since the functional

$$\begin{aligned}
&\mathbf{L^p}((0,T) \times \Omega; [0,U]) \to \mathbb{R} \\
&\qquad u \mapsto \int_0^T \int_\Omega \psi_{2,2}(\bar{\rho}_1(t,x), \bar{\rho}_2(t,x), \bar{v}(t,x)) u^p(t,x)\, \mathrm{d}x\, \mathrm{d}t
\end{aligned}$$

is convex and sequentially lower semicontinuous with respect to the strong topology, then it is also sequentially lower semicontinuous with respect to the weak topology, and so,

$$\begin{aligned}
&\liminf_{n\to+\infty} \int_0^T \int_\Omega \psi_{2,2}(\rho_1^n(t,x), \rho_2^n(t,x), v^n(t,x)) u_n^p(t,x)\, \mathrm{d}x\, \mathrm{d}t \\
&\quad \ge \liminf_{n\to+\infty} \int_0^T \int_\Omega \psi_{2,2}(\bar{\rho}_1(t,x), \bar{\rho}_2(t,x), \bar{v}(t,x)) \bar{u}^p(t,x)\, \mathrm{d}x\, \mathrm{d}t.
\end{aligned}$$

This permits to conclude that

$$\liminf_{n\to+\infty} J(u_n) \ge J(\bar{u}),$$

proving that $\bar{u}$ is an optimal control, i.e., (5.2) holds. This concludes the proof. ■

Remark 3. Theorem 5.1 still holds under the following more general assumption on the function ψ_2 appearing in $J(u)$:

$$\begin{aligned}
\psi_2 &: \mathbb{R}^4 \to \mathbb{R} \\
(\rho_1, \rho_2, v, u) &\mapsto \psi_2(\rho_1, \rho_2, v, u)
\end{aligned}$$

is Lipschitz continuous in all variables, positive and convex in u. Indeed, since $(\rho_1^n, \rho_2^n, v^n)$ converges to $(\bar\rho_1, \bar\rho_2, \bar v)$, and the function ψ_2 is Lipschitz continuous, it holds

$$\lim_{n\to+\infty} \int_0^T \int_\Omega \big[\psi_2(\rho_1^n(t,x), \rho_2^n(t,x), v^n(t,x), u_n(t,x)) \\ - \psi_{2,2}(\bar\rho_1(t,x), \bar\rho_2(t,x), \bar v(t,x), u_n(t,x))\big] \,\mathrm{d}x\,\mathrm{d}t = 0.$$

Therefore,

$$\liminf_{n\to+\infty} \int_0^T \int_\Omega \psi_{2,2}(\rho_1^n(t,x), \rho_2^n(t,x), v^n(t,x), u_n(t,x)) \,\mathrm{d}x\,\mathrm{d}t \\ = \liminf_{n\to+\infty} \int_0^T \int_\Omega \psi_{2,2}(\bar\rho_1(t,x), \bar\rho_2(t,x), \bar v(t,x), u_n(t,x)) \,\mathrm{d}x\,\mathrm{d}t.$$

The convexity of ψ_2 with respect to u ensures, by [28, Theorem 2.12], the weakly lower semicontinuity of the functional

$$\mathbf{L^p}((0,T)\times\Omega; [0,U]) \to \mathbb{R} \\ u \mapsto \int_0^T \int_\Omega \psi_{2,2}(\bar\rho_1(t,x), \bar\rho_2(t,x), \bar v(t,x), u(t,x)) \,\mathrm{d}x\,\mathrm{d}t$$

so that

$$\liminf_{n\to+\infty} \int_0^T \int_\Omega \psi_{2,2}(\rho_1^n(t,x), \rho_2^n(t,x), v^n(t,x), u_n(t,x)) \,\mathrm{d}x\,\mathrm{d}t \\ \geq \liminf_{n\to+\infty} \int_0^T \int_\Omega \psi_{2,2}(\bar\rho_1(t,x), \bar\rho_2(t,x), \bar v(t,x), \bar u(t,x)) \,\mathrm{d}x\,\mathrm{d}t,$$

allowing to conclude that

$$\liminf_{n\to+\infty} J(u_n) \geq J(\bar u).$$

6. Necessary optimality conditions

In this section, given the initial data $\rho_{1,o}, \rho_{2,o}, v_o \in \mathbf{L}^\infty(\Omega; \mathbb{R}_+)$, we aim to prove necessary conditions for optimal controls, i.e., for controls satisfying (5.2) of Theorem 5.1. First, we prove the differentiability of the control-to-state map and then we deduce optimality conditions using the adjoint system.

6.1. Differentiability of the control-to-state map

We follow the line of [6], see also [5]. Consider the control-to-state map G, defined as

$$G : \mathbf{L}^\infty((0,T)\times\Omega; [0,U]) \to \mathbf{C^0}([0,T]; (\mathbf{L^2}(\Omega))^3) \\ u \mapsto (\rho_1, \rho_2, v), \tag{6.1}$$

where the triple (ρ_1, ρ_2, v) is the unique solution to (2.1), (2.2), and (2.3), corresponding to the control u with initial data $\rho_{1,o}, \rho_{2,o}, v_o$.

First, observe that, by Theorem 4.1, the map G is Lipschitz continuous, since it holds $\mathbf{L}^\infty((0,T)\times\Omega;[0,U]) \hookrightarrow \mathbf{L}^2((0,T)\times\Omega;[0,U])$. Let $u^* \in \mathbf{L}^\infty((0,T)\times\Omega;(0,U))$ be fixed and denote the corresponding state by $G(u^*) = (\rho_1^*, \rho_2^*, v^*)$. Given $\bar{u} \in \mathbf{L}^\infty((0,T)\times\Omega;[0,U])$, we introduce the linearized system at $(\rho_1^*, \rho_2^*, v^*)$:

$$\begin{cases} \partial_t X = \Delta X + \partial_{\rho_1} F_1(\rho_1^*, v^*) X + \partial_v F_1(\rho_1^*, v^*) Z, \\ \partial_t Y = \Delta Y + \partial_{\rho_1} F_2(\rho_1^*, \rho_2^*, v^*) X + \partial_{\rho_2} F_2(\rho_1^*, \rho_2^*, v^*) Y \\ \qquad + \partial_v F_2(\rho_1^*, \rho_2^*, v^*) Z, \\ \partial_t Z = \Delta Z + \partial_{\rho_1} F_3(\rho_1^*, \rho_2^*, v^*) X + \partial_{\rho_2} F_3(\rho_1^*, \rho_2^*, v^*) Y \\ \qquad + \partial_v F_3(\rho_1^*, \rho_2^*, v^*) Z + \bar{u} - u^*, \end{cases} \tag{6.2}$$

coupled with zero initial conditions and homogeneous Neumann boundary conditions

$$\begin{cases} X(0,x) = 0, \\ Y(0,x) = 0, \\ Z(0,x) = 0, \end{cases} \qquad \begin{cases} \partial_\nu X(t,\xi) = 0, \\ \partial_\nu Y(t,\xi) = 0, \\ \partial_\nu Z(t,\xi) = 0. \end{cases} \tag{6.3}$$

Above, we used the following notation:

$$\begin{aligned} F_1(\rho_1, v) &= (\alpha - \delta_1)\rho_1 - \beta\rho_1 v, \\ F_2(\rho_1, \rho_2, v) &= \beta\rho_1 v - \delta_2\rho_2, \\ F_3(\rho_1, \rho_2, v) &= b\delta_2\rho_2 - B\rho_1 v - \delta_v v. \end{aligned} \tag{6.4}$$

Lemma 6.1. *The system (6.2)–(6.3) has a unique strong solution $(X, Y, Z) \in \mathcal{X}^3$, satisfying*

$$\|X\|_{\mathcal{X}}^2 + \|Y\|_{\mathcal{X}}^2 + \|Z\|_{\mathcal{X}}^2 \le C\,\|\bar{u}\|^2_{\mathbf{L}^2((0,T)\times\Omega)}, \tag{6.5}$$

where $\mathcal{X} = \mathbf{C^0}([0,T];\mathbf{H^1}(\Omega))$.

Proof. System (6.2)–(6.3) is linear parabolic, the coefficients of X, Y, and Z are functions in $\mathbf{L}^\infty((0,T)\times\Omega)$ by Theorem 3.2, the source term $\bar{u} - u^*$ is by hypothesis in $\mathbf{L}^\infty((0,T)\times\Omega)$, the initial data are zero, then smooth. Hence, by [15, Theorem 3.6] or by [20, Theorem 1.1, Chapter IV], there exists a unique triple $(X, Y, Z) \in \mathcal{X}^3$ that solves (6.2)–(6.3) and satisfies (6.5). ■

Let $\lambda \in (0,1)$, and set

$$u^\lambda = u^* + \lambda(\bar{u} - u^*). \tag{6.6}$$

Clearly, $u^\lambda \in \mathbf{L}^\infty((0,T)\times\Omega;[0,U])$, so that we can define the corresponding state

$$(\rho_1^\lambda, \rho_2^\lambda, v^\lambda) = G(u^\lambda).$$

Note that, as $\lambda \to 0$, by construction we have $u^\lambda \to u^*$ and by the Lipschitz continuity of the control-to-state map G we have $(\rho_1^\lambda, \rho_2^\lambda, v^\lambda) \to (\rho_1^*, \rho_2^*, v^*)$.

The next proposition describes the directional derivative of the control-to-state map G at u^*.

Proposition 6.1. *The directional derivative of the control-to-state map G in the direction $(\bar{u} - u^*)$ is given by*

$$D_{(\bar{u}-u^*)}G(u^*) = (X, Y, Z), \tag{6.7}$$

where the triple (X, Y, Z) is the solution to the linearized system (6.2)–(6.3).

Proof. Set

$$X^\lambda = \frac{\rho_1^\lambda - \rho_1^*}{\lambda} - X, \quad Y^\lambda = \frac{\rho_2^\lambda - \rho_2^*}{\lambda} - Y, \quad Z^\lambda = \frac{v^\lambda - v^*}{\lambda} - Z. \tag{6.8}$$

We claim that the triple $(X^\lambda, Y^\lambda, Z^\lambda)$ converges strongly to the point $(0,0,0)$ in the space $\mathbf{C^0}([0,T]; \mathbf{L^2}(\Omega)^3) \cap \mathbf{L^2}((0,T); \mathbf{H^1}(\Omega)^3)$.

Starting from the definition (6.8), we write the system

$$\begin{cases} \partial_t X^\lambda = \Delta X^\lambda + \partial_{\rho_1} F_1(a_1^\lambda, v^\lambda) X^\lambda + \partial_v F_1(\rho_1^*, c_1^\lambda) Z^\lambda + A_1 X + A_3 Z, \\ \partial_t Y^\lambda = \Delta Y^\lambda + \partial_{\rho_1} F_2(a_2^\lambda, \rho_2^\lambda, v^\lambda) X^\lambda + \partial_{\rho_2} F_2(\rho_1^*, b_2^\lambda, v^\lambda) Y^\lambda \\ \qquad + \partial_v F_2(\rho_1^*, \rho_2^*, c_2^\lambda) Z^\lambda + B_1 X + B_2 Y + B_3 Z, \\ \partial_t Z^\lambda = \Delta Z^\lambda + \partial_{\rho_1} F_3(a_3^\lambda, \rho_2^\lambda, v^\lambda) X^\lambda + \partial_{\rho_2} F_3(\rho_1^*, b_3^\lambda, v^\lambda) Y^\lambda \\ \qquad + \partial_v F_3(\rho_1^*, \rho_2^*, c_3^\lambda) Z^\lambda + C_1 X + C_2 Y + C_3 Z, \end{cases}$$

where F_1, F_2, F_3 are defined as in (6.4), $a_1^\lambda, a_2^\lambda, a_3^\lambda$ are intermediate values between ρ_1^λ and ρ_1^*, b_2^λ, b_3^λ are intermediate values between ρ_2^λ and ρ_2^*, $c_1^\lambda, c_2^\lambda, c_3^\lambda$ are intermediate values between v^λ and v^*, and

$$\begin{aligned} A_1 &= \partial_{\rho_1} F_1(a_1^\lambda, v^\lambda) - \partial_{\rho_1} F_1(\rho_1^*, v^*) = \beta(v^* - v^\lambda), \\ A_3 &= \partial_v F_1(\rho_1^*, c_1^\lambda) - \partial_v F_1(\rho_1^*, v^*) = 0, \\ B_1 &= \partial_{\rho_1} F_2(a_2^\lambda, \rho_2^\lambda, v^\lambda) - \partial_{\rho_1} F_2(\rho_1^*, \rho_2^*, v^*) = \beta(v^\lambda - v^*), \\ B_2 &= \partial_{\rho_2} F_2(\rho_1^*, b_2^\lambda, v^\lambda) - \partial_{\rho_2} F_2(\rho_1^*, \rho_2^*, v^*) = 0, \\ B_3 &= \partial_v F_2(\rho_1^*, \rho_2^*, c_2^\lambda) - \partial_v F_2(\rho_1^*, \rho_2^*, v^*) = 0, \\ C_1 &= \partial_{\rho_1} F_3(a_3^\lambda, \rho_2^\lambda, v^\lambda) - \partial_{\rho_1} F_3(\rho_1^*, \rho_2^*, v^*) = B(v^* - v^\lambda), \\ C_2 &= \partial_{\rho_2} F_3(\rho_1^*, b_3^\lambda, v^\lambda) - \partial_{\rho_2} F_3(\rho_1^*, \rho_2^*, v^*) = 0, \\ C_3 &= \partial_v F_3(\rho_1^*, \rho_2^*, c_3^\lambda) - \partial_v F_3(\rho_1^*, \rho_2^*, v^*) = 0. \end{aligned}$$

We now test the equation for X^λ (respectively, Y^λ and Z^λ) with X^λ (respectively, Y^λ and Z^λ). Exploiting the weakly coercivity of the bilinear forms, as in the proof of Theorem 4.1, since

$$\partial_v F_1(\rho_1^*, c_1^\lambda) = -\beta \rho_1^*,$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|X^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\|X^\lambda(t)\|^2_{\mathbf{H}^1(\Omega)} \\
&\quad \le \int_\Omega \partial_v F_1(\rho_1^*, c_1^\lambda) X^\lambda Z^\lambda + \int_\Omega A_1 X X^\lambda \\
&\qquad + \left(\frac{1}{2} + \delta_1 + \alpha + \beta \|v^\lambda(t)\|_{\mathbf{L}^\infty(\Omega)}\right)\|X^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\quad \le \frac{\beta}{2}\|\rho_1^*(t)\|^2_{\mathbf{L}^\infty(\Omega)}\|X^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\beta}{2}\|Z^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\qquad + \frac{\beta}{2}\|v^*(t) - v^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\beta}{2}\|X(t)\|^2_{\mathbf{L}^\infty(\Omega)}\|X^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\qquad + \left(\frac{1}{2} + \delta_1 + \alpha + \beta \|v^\lambda(t)\|_{\mathbf{L}^\infty(\Omega)}\right)\|X^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)},
\end{aligned}$$

where the $\mathbf{L}^\infty$-norms appearing above can be controlled using (3.5)–(3.7) and the fact that X belongs to $\mathbf{L}^\infty((0,T)\times\Omega)$ by Lemma 6.1.

Proceed similarly for Y^λ, since

$$\partial_{\rho_1} F_2(a_2^\lambda, \rho_2^\lambda, v^\lambda) = \beta v^\lambda \quad \text{and} \quad \partial_v F_2(\rho_1^*, \rho_2^*, c_2^\lambda) = \beta \rho_1^*,$$

we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Y^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\|Y^\lambda(t)\|^2_{\mathbf{H}^1(\Omega)} \\
&\quad \le \int_\Omega \partial_{\rho_1} F_2(a_2^\lambda, \rho_2^\lambda, v^\lambda) X^\lambda Y^\lambda + \int_\Omega \partial_v F_2(\rho_1^*, \rho_2^*, c_2^\lambda) Y^\lambda Z^\lambda + \int_\Omega B_1 X Y^\lambda \\
&\qquad + \left(\frac{1}{2} + \delta_2\right)\|Y^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\quad \le \frac{\beta}{2}\|v^\lambda(t)\|^2_{\mathbf{L}^\infty(\Omega)}\|X^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\beta}{2}\|Y^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\qquad + \frac{\beta}{2}\|\rho_1^*(t)\|^2_{\mathbf{L}^\infty(\Omega)}\|Y^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\beta}{2}\|Z^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\qquad + \frac{\beta}{2}\|v^*(t) - v^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\beta}{2}\|X(t)\|^2_{\mathbf{L}^\infty(\Omega)}\|Y^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\qquad + \left(\frac{1}{2} + \delta_2\right)\|Y^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)},
\end{aligned}$$

where the $\mathbf{L}^\infty$-norms appearing above can be controlled using (3.5)–(3.7) and the fact X belongs to $\mathbf{L}^\infty((0,T)\times\Omega)$ by Lemma 6.1. Lastly, consider Z^λ, since $\partial_{\rho_1} F_3(a_3^\lambda, \rho_2^\lambda, v^\lambda) = -Bv^\lambda$ and $\partial_{\rho_2} F_2(\rho_1^*, b_3^\lambda, v^\lambda) = b\,\delta_2$, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Z^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\|Z^\lambda(t)\|^2_{\mathbf{H}^1(\Omega)} \\
&\quad \le \int_\Omega \partial_{\rho_1} F_3(a_3^\lambda, \rho_2^\lambda, v^\lambda) X^\lambda Z^\lambda + \int_\Omega \partial_{\rho_2} F_2(\rho_1^*, b_3^\lambda, v^\lambda) Y^\lambda Z^\lambda + \int_\Omega C_1 X Z^\lambda \\
&\qquad + \left(\frac{1}{2} + \delta_v + B\|\rho_1^*(t)\|_{\mathbf{L}^\infty(\Omega)}\right)\|Z^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\quad \le \frac{B}{2}\|v^\lambda(t)\|^2_{\mathbf{L}^\infty(\Omega)}\|X^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{B}{2}\|Z^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\qquad + \frac{b\,\delta_2}{2}\|Y^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{b\,\delta_2}{2}\|Z^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\qquad + \frac{B}{2}\|v^*(t) - v^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{B}{2}\|X(t)\|^2_{\mathbf{L}^\infty(\Omega)}\|Z^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\qquad + \left(\frac{1}{2} + \delta_v + B\|\rho_1^*(t)\|_{\mathbf{L}^\infty(\Omega)}\right)\|Z^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)},
\end{aligned}
$$

where the $\mathbf{L}^\infty$-norms appearing above can be controlled using (3.5)–(3.7) and the fact that X belongs to $\mathbf{L}^\infty((0,T)\times\Omega)$ by Lemma 6.1.

Collecting together the estimates above, for a.e. $t \in [0,T]$, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|X^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} + \|Y^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} + \|Z^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)}\big) \\
&\qquad + \|X^\lambda(t)\|^2_{\mathbf{H}^1(\Omega)} + \|Y^\lambda(t)\|^2_{\mathbf{H}^1(\Omega)} + \|Z^\lambda(t)\|^2_{\mathbf{H}^1(\Omega)} \\
&\quad \le C(t)\big(\|X^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} + \|Y^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} + \|Z^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)}\big) \\
&\qquad + (B + 2\beta)\|v^*(t) - v^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)},
\end{aligned}
$$

where we set

$$
\begin{aligned}
C(t) = \max\Big\{&\beta\|\rho_1^*(t)\|^2_{\mathbf{L}^\infty(\Omega)} + \beta\|X(t)\|^2_{\mathbf{L}^\infty(\Omega)} + 1 + 2\delta_1 + 2\alpha \\
&+ 2\beta\|v^\lambda(t)\|_{\mathbf{L}^\infty(\Omega)} + (\beta + B)\|v^\lambda(t)\|^2_{\mathbf{L}^\infty(\Omega)}, \\
&\beta + \beta\|\rho_1^*(t)\|^2_{\mathbf{L}^\infty(\Omega)} + \beta\|X(t)\|^2_{\mathbf{L}^\infty(\Omega)} + 1 + 2\,\delta_2 + b\,\delta_2, \\
&2\,\beta + B + b\,\delta_2 + B\|X(t)\|^2_{\mathbf{L}^\infty(\Omega)} + 1 + 2\,\delta_v + 2B\|\rho_1^*(t)\|^2_{\mathbf{L}^\infty(\Omega)}\Big\}.
\end{aligned}
$$

An application of Gronwall's inequality yields

$$
\begin{aligned}
&\|X^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} + \|Y^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} + \|Z^\lambda(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\qquad + \int_0^t \big(\|X^\lambda(s)\|^2_{\mathbf{H}^1(\Omega)} + \|Y^\lambda(s)\|^2_{\mathbf{H}^1(\Omega)} + \|Z^\lambda(s)\|^2_{\mathbf{H}^1(\Omega)}\big)\,\mathrm{d}s \\
&\quad \le \int_0^t (B + 2\beta)\|v^*(t) - v^\lambda(s)\|^2_{\mathbf{L}^2(\Omega)} \exp\left(\int_s^t C(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\,.
\end{aligned} \tag{6.9}
$$

Since, in the limit $\lambda \to 0$, we have the convergence $v^\lambda \to v^*$, by (6.9) we obtain the thesis, i.e.,

$$(X^\lambda, Y^\lambda, Z^\lambda) \to (0,0,0) \text{ strongly in } \mathbf{C^0}([0,T]; \mathbf{L^2}(\Omega)^3) \cap \mathbf{L^2}((0,T); \mathbf{H^1}(\Omega)^3).$$

Due to the definition (6.8) of the triple $(X^\lambda, Y^\lambda, Z^\lambda)$, this amounts to

$$\left(\frac{\rho_1^\lambda - \rho_1^*}{\lambda}, \frac{\rho_2^\lambda - \rho_2^*}{\lambda}, \frac{v^\lambda - v^*}{\lambda} \right) \xrightarrow{\lambda \to 0} (X, Y, Z),$$

which in terms of the control-to-state operator G gives its directional derivative in the direction $\bar{u} - u^*$:

$$D_{(\bar{u}-u^*)} G(u^*) = (X, Y, Z),$$

concluding the proof. ■

6.2. Adjoint system and necessary conditions

Let us now consider the functional J defined in (5.1). Observe that J is actually a function also of (ρ_1, ρ_2, v), and not only of the control u; thus, it would be more precise to write $J(\rho_1, \rho_2, v, u)$. The control-to-state operator G introduced in (6.1) allows to write $(\rho_1, \rho_2, v) = G(u)$ so that we can define the reduced cost functional f as

$$f(u) := J(\rho_1, \rho_2, v, u) = J(G(u), u). \tag{6.10}$$

We introduce the following assumptions on the cost functions ψ_1 and ψ_2.

($\boldsymbol{\psi}$) $\psi_1 \in \mathbf{C^1}(\mathbb{R}^3; \mathbb{R})$ and $\psi_2 \in \mathbf{C^1}(\mathbb{R}^4; \mathbb{R})$. Moreover, for every $M > 0$, there exists $L_M > 0$ such that

$$\begin{aligned} |\nabla\psi_1(\bar\rho_1, \bar\rho_2, \bar v) - \nabla\psi_1(\hat\rho_1, \hat\rho_2, \hat v)| &\le L_M \left|(\bar\rho_1, \bar\rho_2, \bar v) - (\hat\rho_1, \hat\rho_2, \hat v)\right|, \\ |\nabla\psi_2(\bar\rho_1, \bar\rho_2, \bar v, \bar u) - \nabla\psi_2(\hat\rho_1, \hat\rho_2, \hat v, \hat u)| &\le L_M \left|(\bar\rho_1, \bar\rho_2, \bar v, \bar u) - (\hat\rho_1, \hat\rho_2, \hat v, \hat u)\right| \end{aligned}$$

for every $\bar\rho_1, \bar\rho_2, \bar v, \bar u, \hat\rho_1, \hat\rho_2, \hat v, \hat u \in [0, M]$. Here, the notations $\nabla\psi_1(\rho_1, \rho_2, v)$ and, respectively, $\nabla\psi_2(\rho_1, \rho_2, v, u)$ denote the gradient with respect to the variables (ρ_1, ρ_2, v) and, respectively, (ρ_1, ρ_2, v, u).

Thanks to [28, Lemma 4.12], the functional J admits partial derivatives, while Proposition 6.1 ensures that the differentiability of the control-to-state operator G. Hence, the reduced cost functional f (6.10) is differentiable in $\mathbf{L}^\infty((0,T) \times \Omega)$.

Consider a set of admissible control $\mathcal{U} \neq \emptyset$ that is a closed convex subset of $\mathbf{L}^\infty((0,T) \times \Omega; [0, U])$. Let $u^* \in \mathcal{U}$ be a locally optimal control for problem (2.1), (2.2), and (2.3) subject to the minimization of the functional J (5.1). Then, for any $\bar u \in \mathcal{U}$, defining u^λ as in (6.6) for $\lambda \in (0,1)$, the following inequality holds:

$$f(u^\lambda) - f(u^*) \ge 0.$$

Dividing by λ and passing to the limit as $\lambda \to 0$, we obtain

$$f'(u^*)(\bar{u} - u^*) \geq 0 \quad \forall \bar{u} \in \mathcal{U}. \tag{6.11}$$

Due to the definition of f (6.10), using the chain rule and (6.7), we can compute f' appearing in (6.11) for any $\bar{u} \in \mathcal{U}$ as follows:

$$\begin{aligned}
0 &\leq f'(u^*)(\bar{u} - u^*) \\
&= \nabla_{(\rho_1,\rho_2,v)} J(G(u^*), u^*) \cdot D_{(\bar{u}-u^*)} G(u^*) + \partial_u J(G(u^*), u^*)(\bar{u} - u^*) \\
&= \partial_{\rho_1} J(G(u^*), u^*) X + \partial_{\rho_2} J(G(u^*), u^*) Y + \partial_v J(G(u^*), u^*) Z \\
&\quad + \partial_u J(G(u^*), u^*)(\bar{u} - u^*) \\
&= \int_\Omega \partial_{\rho_1} \psi_1(\rho_1^*(T,x), \rho_2^*(T,x), v^*(T,x)) X(T,x)\, \mathrm{d}x \\
&\quad + \int_\Omega \partial_{\rho_2} \psi_1(\rho_1^*(T,x), \rho_2^*(T,x), v^*(T,x)) Y(T,x)\, \mathrm{d}x \\
&\quad + \int_\Omega \partial_v \psi_1(\rho_1^*(T,x), \rho_2^*(T,x), v^*(T,x)) Z(T,x)\, \mathrm{d}x \\
&\quad + \int_0^T \int_\Omega \partial_{\rho_1} \psi_2(\rho_1^*(t,x), \rho_2^*(t,x), v^*(t,x), u^*(t,x)) X(t,x)\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \partial_{\rho_2} \psi_2(\rho_1^*(t,x), \rho_2^*(t,x), v^*(t,x), u^*(t,x)) Y(t,x)\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \partial_v \psi_2(\rho_1^*(t,x), \rho_2^*(t,x), v^*(t,x), u^*(t,x)) Z(t,x)\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \partial_u \psi_2(\rho_1^*(t,x), \rho_2^*(t,x), v^*(t,x), u^*(t,x))(\bar{u}(t,x) - u^*(t,x))\, \mathrm{d}x\, \mathrm{d}t\,,
\end{aligned} \tag{6.12}$$

where the triple (X, Y, Z) is the solution to the linearized system (6.2)–(6.3).

Introduce the adjoint system, in the variables w, y, z:

$$\begin{cases}
-\partial_t w - \Delta w = \partial_{\rho_1} F_1(\rho_1^*, v^*) w + \partial_v F_1(\rho_1^*, v^*) z + \partial_{\rho_1} \psi_2(\rho_1^*, \rho_2^*, v^*, u^*), \\
-\partial_t y - \Delta y = \partial_{\rho_1} F_2(\rho_1^*, \rho_2^*, v^*) w + \partial_{\rho_2} F_2(\rho_1^*, \rho_2^*, v^*) y \\
\qquad\qquad\qquad + \partial_v F_2(\rho_1^*, \rho_2^*, v^*) z + \partial_{\rho_2} \psi_2(\rho_1^*, \rho_2^*, v^*, u^*), \\
-\partial_t z - \Delta z = \partial_{\rho_1} F_3(\rho_1^*, \rho_2^*, v^*) w + \partial_{\rho_2} F_3(\rho_1^*, \rho_2^*, v^*) y \\
\qquad\qquad\qquad + \partial_v F_3(\rho_1^*, \rho_2^*, v^*) z + \partial_v \psi_2(\rho_1^*, \rho_2^*, v^*, u^*),
\end{cases} \tag{6.13}$$

with the following initial and boundary conditions:

$$\begin{cases}
w(T,x) = \partial_{\rho_1} \psi_1(\rho_1^*(T,x), \rho_2^*(T,x), v^*(T,x)), \\
y(T,x) = \partial_{\rho_2} \psi_1(\rho_1^*(T,x), \rho_2^*(T,x), v^*(T,x)), \\
z(T,x) = \partial_v \psi_1(\rho_1^*(T,x), \rho_2^*(T,x), v^*(T,x)),
\end{cases} \qquad
\begin{cases}
\partial_\nu w(t,\xi) = 0, \\
\partial_\nu y(t,\xi) = 0, \\
\partial_\nu z(t,\xi) = 0.
\end{cases} \tag{6.14}$$

A result similar to Lemma 6.1 holds, implying that there exists a unique solution to (6.13)–(6.14).

Lemma 6.2 ([28, Theorem 3.18]). *Assume* (ψ) *holds. Let* (X, Y, Z) *be the solution to the linearized problem* (6.2)–(6.3)*. Let* (w, y, z) *be the weak solution to the adjoint problem* (6.13)–(6.14)*. Then,*

$$\begin{aligned}
&\int_\Omega \partial_{\rho_1}\psi_1(\rho_1^*(T,x), \rho_2^*(T,x), v^*(T,x))X(T,x)\,\mathrm{d}x \\
&\quad + \int_\Omega \partial_{\rho_2}\psi_1(\rho_1^*(T,x), \rho_2^*(T,x), v^*(T,x))Y(T,x)\,\mathrm{d}x \\
&\quad + \int_\Omega \partial_{v}\psi_1(\rho_1^*(T,x), \rho_2^*(T,x), v^*(T,x))Z(T,x)\,\mathrm{d}x \\
&\quad + \int_0^T\int_\Omega \partial_{\rho_1}\psi_2(\rho_1^*(t,x), \rho_2^*(t,x), v^*(t,x), u^*(t,x))X(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T\int_\Omega \partial_{\rho_2}\psi_2(\rho_1^*(t,x), \rho_2^*(t,x), v^*(t,x), u^*(t,x))Y(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T\int_\Omega \partial_{\rho_2}\psi_2(\rho_1^*(t,x), \rho_2^*(t,x), v^*(t,x), u^*(t,x))Z(t,x)\,\mathrm{d}x\,\mathrm{d}t \qquad (6.15)\\
&= \int_0^T\int_\Omega (\bar{u}(t,x) - u^*(t,x))z(t,x)\,\mathrm{d}x\,\mathrm{d}t\,. \qquad (6.16)
\end{aligned}$$

Theorem 6.1. *Let* $u^* \in \mathcal{U}$ *be a locally optimal control for problem* (2.1)*,* (2.2)*, and* (2.3) *subject to the minimization of the functional* J (5.1)*. If* (w, y, z) *is the associated state solving problem* (6.13)–(6.14)*, then the following variational inequality holds for all* $\bar{u} \in \mathcal{U}$*:*

$$\int_0^T\int_\Omega (z + \partial_u\psi_2(\rho_1^*, \rho_2^*, v^*, u^*))(\bar{u} - u^*)\,\mathrm{d}x\,\mathrm{d}t \geq 0. \tag{6.17}$$

Proof. Inserting (6.16) into (6.12) leads to

$$\begin{aligned}
&\int_0^T\int_\Omega (\bar{u}(t,x) - u^*(t,x))z(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T\int_\Omega \partial_u\psi_2(\rho_1^*(t,x), \rho_2^*(t,x), v^*(t,x), u^*(t,x))(\bar{u}(t,x) - u^*(t,x))\,\mathrm{d}x\,\mathrm{d}t \geq 0,
\end{aligned}$$

concluding the proof. ∎

Remark 4. With reference to the functionals introduced in Remark 2, we deduce the following necessary conditions.

If

$$J(u) = \gamma_1 \int_\Omega \rho_1(T,x)\,\mathrm{d}x + \gamma_2 \int_\Omega \rho_2(T,x)\,\mathrm{d}x$$

for suitable $\gamma_1, \gamma_2 \geq 0$, then $\psi_2 = 0$, and so, (6.17) becomes

$$\int_0^T\int_\Omega z(\bar{u} - u^*)\,\mathrm{d}x\,\mathrm{d}t \geq 0.$$

If

$$J(u) = \gamma_1 \int_\Omega \rho_1(T,x)\,\mathrm{d}x + \gamma_2 \int_\Omega \rho_2(T,x)\,\mathrm{d}x + \int_0^T \int_\Omega u^p(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

or

$$J(u) = \int_\Omega (\rho_1(T,x) - \bar\rho(x))^2\,\mathrm{d}x + \int_0^T \int_\Omega (\rho_1(t,x) - \bar\rho(x))^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega u^p(t,x)\,\mathrm{d}x\,\mathrm{d}t,$$

where $\gamma_1, \gamma_2 \geq 0$, $\bar\rho \in \mathbf{L}^2(\Omega)$, and $p \geq 1$, then $\partial_u \psi_2 = p u^{p-1}$, and so, (6.17) becomes

$$\int_0^T \int_\Omega (z + p(u^*)^{p-1})(\bar u - u^*)\,\mathrm{d}x\,\mathrm{d}t \geq 0.$$

A. Preliminary results on the linear parabolic equation $\partial_t u = \Delta u + c(t,x)u + f(t,x)$

This appendix contains classical results about the well posedness of scalar parabolic equations. It is mainly intended to ease the readability of the paper.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$, fix $T > 0$, and set $\Omega_T = (0,T) \times \Omega$ and $S_T = (0,T) \times \partial\Omega$. Consider the following problem:

$$\begin{cases} \partial_t u - \Delta u + c(t,x)u = f(t,x) & \text{in } \Omega_T, \\ u(0,x) = g(x) & \text{in } \Omega, \\ \partial_\nu u(t,\xi) = 0 & \text{on } S_T, \end{cases} \tag{A.1}$$

where ν is the outward normal on Ω at the boundary $\partial\Omega$, which exists for $\mathcal{H}^{n-1}$ a.e. $\xi \in \partial\Omega$. Assume that $f \in \mathbf{L}^2(\Omega_T;\mathbb{R})$, $g \in \mathbf{L}^2(\Omega;\mathbb{R})$, and $c \in \mathbf{L}^\infty(\Omega_T;\mathbb{R})$. Fix $c_o \geq 1$ such that $\|c\|_{\mathbf{L}^\infty(\Omega_T)} \leq c_o$. We define the Hilbert space

$$\mathbf{H}^1((0,T);\mathbf{H}^1(\Omega),\mathbf{H}^1(\Omega)^*) = \left\{u \in \mathbf{L}^2((0,T);\mathbf{H}^1(\Omega)) : \dot u \in \mathbf{L}^2((0,T);\mathbf{H}^1(\Omega)^*)\right\} \tag{A.2}$$

endowed with inner product

$$(u_1,u_2)_{\mathbf{H}^1((0,T);\mathbf{H}^1(\Omega),\mathbf{H}^1(\Omega)^*)} = \int_0^T (u_1(t),u_2(t))_{\mathbf{H}^1}\,\mathrm{d}t + \int_0^T (\dot u_1(t),\dot u_2(t))_{\mathbf{H}^{1*}}\,\mathrm{d}t$$

and norm

$$\|u\|^2_{\mathbf{H}^1((0,T);\mathbf{H}^1(\Omega),\mathbf{H}^1(\Omega)^*)} = \int_0^T \|u(t)\|^2_{\mathbf{H}^1}\,\mathrm{d}t + \int_0^T \|\dot u(t)\|^2_{\mathbf{H}^{1*}}\,\mathrm{d}t.$$

Following [27, Chapter 10], we introduce the definition of the weak solution to problem (A.1).

Definition A.1. A function $u \in \mathbf{H^1}((0,T);\mathbf{H^1}(\Omega),\mathbf{H^1}(\Omega)^*)$ is a weak solution to (A.1) if $u(0) = g$ and

$$\langle \dot{u}(t), v\rangle_* + B(u(t), v; t) = \langle f(t), v\rangle_*$$

for all $v \in \mathbf{H^1}(\Omega)$ and for a.e. $t \in (0,T)$, where

$$B(u,v;t) = \int_\Omega \left[\nabla u \cdot \nabla v + c(t,x)\, u\, v\right] \mathrm{d}x, \tag{A.3}$$

and $\langle\cdot,\cdot\rangle_*$ denotes the duality between $\mathbf{H^1}(\Omega)^*$ and $\mathbf{H^1}(\Omega)$.

The bilinear form B (A.3) is continuous, with

$$|B(u,v;t)| \le (1+c_o)\|u\|_{\mathbf{H^1}(\Omega)}\|v\|_{\mathbf{H^1}(\Omega)}.$$

Moreover, B is weakly coercive, since, for every $\lambda > c_o$ and $\alpha \in]0,1]$,

$$B(u,u;t) + \lambda\|u\|^2_{\mathbf{L^2}(\Omega)} \ge \alpha\|u\|^2_{\mathbf{H^1}(\Omega)}.$$

Finally, for every $u, v \in \mathbf{H^1}(\Omega)$, the map $t \mapsto B(u,v;t)$ is measurable by Fubini's theorem. Hence, we can apply [27, Theorem 10.6].

Theorem A.1. *There exists a unique weak solution u to problem (A.1) in the sense of Definition A.1. Moreover, for every $t \in [0,T]$, we have*

$$\begin{aligned}
\|u(t)\|^2_{\mathbf{L^2}(\Omega)} &\le e^{2\,c_o\,t}\left\{\|g\|^2_{\mathbf{L^2}(\Omega)} + \int_0^t \|f(s)\|^2_{\mathbf{H^1}(\Omega)^*}\,\mathrm{d}s\right\},\\
\int_0^t \|u(s)\|^2_{\mathbf{H^1}(\Omega)}\,\mathrm{d}s &\le e^{2\,c_o\,t}\left\{\|g\|^2_{\mathbf{L^2}(\Omega)} + \int_0^t \|f(s)\|^2_{\mathbf{H^1}(\Omega)^*}\,\mathrm{d}s\right\},\\
\int_0^t \|\dot{u}(s)\|^2_{\mathbf{H^1}(\Omega)^*}\,\mathrm{d}s &\le 2(1+c_o)^2 e^{2c_o t}\|g\|^2_{\mathbf{L^2}(\Omega)}\\
&\quad + (2(1+c_o)^2 e^{2c_o t} + 2)\int_0^t \|f(s)\|^2_{\mathbf{H^1}(\Omega)^*}\,\mathrm{d}s.
\end{aligned} \tag{A.4}$$

A first simple consequence is the continuity of the solution operator for (A.1).

Corollary A.1. *The operator, which associates to every $f \in \mathbf{L^2}(\Omega_T;\mathbb{R})$ and $g \in \mathbf{L^2}(\Omega;\mathbb{R})$ the unique solution to (A.1), is linear and continuous as a map*

$$\mathbf{L^2}(\Omega_T;\mathbb{R}) \times \mathbf{L^2}(\Omega;\mathbb{R}) \to \mathbf{H^1}((0,T);\mathbf{H^1}(\Omega),\mathbf{H^1}(\Omega)^*).$$

A second consequence of the estimates provided by Theorem A.1 is the stability of solutions to problem (A.1) with respect to the source function f.

Proposition A.1. *Let u_1 and u_2 solve*

$$\begin{cases}
\partial_t u_1 - \Delta u_1 + c(t,x)\,u_1 = f_1(t,x),\\
u_1(0,x) = g(x),\\
\partial_\nu u_1(t,\xi) = 0,
\end{cases}
\qquad
\begin{cases}
\partial_t u_2 - \Delta u_2 + c(t,x)\,u_2 = f_2(t,x),\\
u_2(0,x) = g(x),\\
\partial_\nu u_2(t,\xi) = 0,
\end{cases}$$

with $g \in \mathbf{L}^2(\Omega)$, $c \in \mathbf{L}^\infty(\Omega_T)$ *and* $f_1, f_2 \in \mathbf{L}^2(\Omega_T)$. *Then,*

$$\|u_1(t) - u_2(t)\|^2_{\mathbf{L}^2(\Omega)} \le e^{2c_o t} \int_0^t \|f_1(s) - f_2(s)\|^2_{\mathbf{H}^1(\Omega)^*} \,\mathrm{d}s.$$

Proof. Set $u := u_1 - u_2$. Clearly, u solves

$$\begin{cases} \partial_t u - \Delta u + c(t,x)\, u = f_1(t,x) - f_2(t,x), \\ u(0,x) = 0, \\ \partial_\nu u(t,\xi) = 0. \end{cases}$$

Applying Theorem A.1 to u, we obtain the thesis. ∎

The following a priori $\mathbf{L}^\infty$ estimate holds.

Proposition A.2. *Let* $g \in \mathbf{L}^\infty(\Omega; \mathbb{R}_+)$, $f \in \mathbf{L}^\infty(\Omega_T; \mathbb{R}_+)$, *and* $c \in \mathbf{L}^\infty(\Omega_T; \mathbb{R})$. *Let* u *be the unique weak solution to* (A.1). *Then, for* $t \in [0,T]$,

$$0 \le u(t,x) \le \begin{cases} \big(\|g\|_{\mathbf{L}^\infty} + \frac{\|f\|_{\mathbf{L}^\infty}}{\|c\|_{\mathbf{L}^\infty}}\big) e^{\|c\|_{\mathbf{L}^\infty} t} - \frac{\|f\|_{\mathbf{L}^\infty}}{\|c\|_{\mathbf{L}^\infty}}, & \|c\|_{\mathbf{L}^\infty} > 0, \\ \|g\|_{\mathbf{L}^\infty} + \|f\|_{\mathbf{L}^\infty} t, & \|c\|_{\mathbf{L}^\infty} = 0. \end{cases}$$

Proof. Define the linear operator

$$v \mapsto \mathcal{P} v := \partial_t v - \Delta v - c v.$$

Clearly, $\mathcal{P} u = f \ge 0$ and $u(0) = g \ge 0$; hence, $u(t,x) \ge 0$ by the weak maximum principle; see, for example, [27, Theorem 10.18 and Remark 10.19].

Define

$$w(t,x) = \begin{cases} \big(\|g\|_{\mathbf{L}^\infty} + \frac{\|f\|_{\mathbf{L}^\infty}}{\|c\|_{\mathbf{L}^\infty}}\big) e^{\|c\|_{\mathbf{L}^\infty} t} - \frac{\|f\|_{\mathbf{L}^\infty}}{\|c\|_{\mathbf{L}^\infty}} & \text{if } c \ne 0, \\ \|g\|_{\mathbf{L}^\infty} + \|f\|_{\mathbf{L}^\infty}\, t & \text{if } c \equiv 0. \end{cases}$$

In the case $c \ne 0$, we have

$$\begin{aligned} \mathcal{P}(w - u) &= \mathcal{P} w - \mathcal{P} u \\ &= \|c\|_{\mathbf{L}^\infty} \Big(\|g\|_{\mathbf{L}^\infty} + \frac{\|f\|_{\mathbf{L}^\infty}}{\|c\|_{\mathbf{L}^\infty}}\Big) e^{\|c\|_{\mathbf{L}^\infty} t} \\ &\quad - c(t,x) \Big(\|g\|_{\mathbf{L}^\infty} + \frac{\|f\|_{\mathbf{L}^\infty}}{\|c\|_{\mathbf{L}^\infty}}\Big) e^{\|c\|_{\mathbf{L}^\infty} t} + c(t,x) \frac{\|f\|_{\mathbf{L}^\infty}}{\|c\|_{\mathbf{L}^\infty}} - f(t,x) \\ &= (\|c\|_{\mathbf{L}^\infty} - c(t,x)) \|g\|_{\mathbf{L}^\infty} e^{\|c\|_{\mathbf{L}^\infty} t} + \|f\|_{\mathbf{L}^\infty} e^{\|c\|_{\mathbf{L}^\infty} t} \\ &\quad - \frac{c(t,x)}{\|c\|_{\mathbf{L}^\infty}} \|f\|_{\mathbf{L}^\infty} \big(e^{\|c\|_{\mathbf{L}^\infty} t} - 1\big) - f(t,x) \\ &\ge \|f\|_{\mathbf{L}^\infty} \big(e^{\|c\|_{\mathbf{L}^\infty} t} - 1\big) \Big(1 - \frac{c(t,x)}{\|c\|_{\mathbf{L}^\infty}}\Big) \\ &\ge 0. \end{aligned}$$

On the other hand, in the case $c \equiv 0$, we have

$$\mathcal{P}(w-u) = \|f\|_{\mathbf{L}^\infty} - f(t,x) \geq 0.$$

In both cases, $\partial_\nu (w-u) = 0$ in S_T and

$$(w(0,x) - u(0,x)) = \|g\|_{\mathbf{L}^\infty} - g(x) \geq 0.$$

Hence, the weak maximum principle implies that $w \geq u$ in Ω_T, completing the proof. ■

We briefly recall a regularity result, see [27, Remark 10.17]: the more regular the initial data, the more regular the solution.

Proposition A.3. *Let u be the unique weak solution to problem* (A.1) *in the sense of Definition* A.1. *If $g \in \mathbf{H}^1(\Omega)$, $f \in \mathbf{L}^2((0,T);\mathbf{L}^2(\Omega))$, and $c \in \mathbf{L}^\infty(\Omega_T)$, then u is such that $u \in \mathbf{L}^\infty((0,T);\mathbf{H}^1(\Omega))$ and $\dot u \in \mathbf{L}^2((0,T);\mathbf{L}^2(\Omega))$.*

If, in addition, Ω is a $\mathbf{C}^2$-domain, then $u \in \mathbf{L}^2((0,T);\mathbf{H}^2(\Omega))$.

Proof. The proof is based on Faedo–Galerkin approximation for problem (A.1); see [27, Theorem 10.14] for a similar case.

Take a sequence w_s of eigenvalues of the Laplace operator in Ω with 0 Neumann boundary condition. We select the eigenvalues such that the closure of their span coincides with the space $\mathbf{H}^1(\Omega)$, they are orthogonal in $\mathbf{H}^1(\Omega)$ and orthonormal in $\mathbf{L}^2(\Omega)$. For every $m > 1$, define a Faedo–Galerkin approximation u_m of u as

$$u_m(t) = \sum_{j=1}^{m} c_{jm}(t) w_j,$$

where the coefficients $c_{jm}(t)$ belong to $\mathbf{H}^1((0,T);\mathbb{R})$ for every $j \in \{1,\dots,m\}$, so that for a.e. $t \in [0,T]$, for all m, and for all $v \in \mathbf{H}^1(\Omega)$,

$$\langle \dot u_m(t), v\rangle_* + B(u_m(t), v; t) = \langle f(t), v\rangle_*, \tag{A.5}$$

where the bilinear term B is defined in (A.3). Note that for a.e. $t \in [0,T]$, $u_m(t)$ converges to $u(t)$ in $\mathbf{H}^1(\Omega)$ as $m \to +\infty$ and $\dot u_m(t) \in \mathbf{H}^1(\Omega)$ for every $m \geq 1$. Thus, substituting $v = \dot u_m(t)$ in (A.5) and using (A.3) and the hypothesis that $f \in \mathbf{L}^2((0,T);\mathbf{L}^2(\Omega))$, we get that

$$\begin{aligned} &\|\dot u_m(t)\|^2_{\mathbf{L}^2(\Omega)} + \underbrace{\int_\Omega \nabla u_m(t) \cdot \nabla \dot u_m(t)\,\mathrm{d}x}_{I_1} \\ &= \underbrace{\int_\Omega f(t) \dot u_m(t)\,\mathrm{d}x}_{I_2} - \underbrace{\int_\Omega c(t,x)\, u_m(t)\, \dot u_m(t)\,\mathrm{d}x}_{I_3}. \end{aligned} \tag{A.6}$$

Note that

$$I_1 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_m(t)\|^2_{\mathbf{L}^2(\Omega)}. \tag{A.7}$$

Moreover, for a.e. $t \in (0, T)$,

$$\begin{aligned} I_2 &\leq \int_\Omega |f(t)|\,|\dot{u}_m(t)|\,\mathrm{d}x \leq \|f(t)\|_{\mathbf{L}^2(\Omega)}\|\dot{u}_m(t)\|_{\mathbf{L}^2(\Omega)} \\ &\leq 2\|f(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{8}\|\dot{u}_m(t)\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned} \tag{A.8}$$

Finally, for a.e. $t \in (0, T)$,

$$\begin{aligned} |-I_3| &\leq \int_\Omega |c(t,x)|\,|u_m(t)|\,|\dot{u}_m(t)|\,\mathrm{d}x \\ &\leq \|c\|_{\mathbf{L}^\infty(\Omega_T)}\|u_m(t)\|_{\mathbf{L}^2(\Omega)}\|\dot{u}_m(t)\|_{\mathbf{L}^2(\Omega)} \\ &\leq 2\|c\|^2_{\mathbf{L}^\infty(\Omega_T)}\|u_m(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{8}\|\dot{u}_m(t)\|^2_{\mathbf{L}^2(\Omega)}, \end{aligned} \tag{A.9}$$

provided $\|c\|_{\mathbf{L}^\infty(\Omega_T)} > 0$. Inserting (A.7), (A.8), and (A.9) into (A.6), we deduce that, for a.e. $t \in [0, T]$,

$$\begin{aligned} &\frac{3}{4}\|\dot{u}_m(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_m(t)\|^2_{\mathbf{L}^2(\Omega)} \\ &\qquad \leq 2\|f(t)\|^2_{\mathbf{L}^2(\Omega)} + 2\|c\|^2_{\mathbf{L}^\infty(\Omega_T)}\|u_m(t)\|^2_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

and so, integrating in time, since $\|\nabla u_m(0)\|^2_{\mathbf{L}^2(\Omega)} \leq \|g\|^2_{\mathbf{L}^2(\Omega)}$,

$$\begin{aligned} &\frac{3}{4}\int_0^t \|\dot{u}_m(s)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s + \frac{1}{2}\|\nabla u_m(t)\|^2_{\mathbf{L}^2(\Omega)} \\ &\quad \leq \frac{1}{2}\|\nabla u_m(0)\|^2_{\mathbf{L}^2(\Omega)} + 2\int_0^t \|f(s)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s + 2\|c\|^2_{\mathbf{L}^\infty(\Omega_T)}\int_0^t \|u_m(s)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s \\ &\quad \leq \frac{1}{2}\|\nabla g\|^2_{\mathbf{L}^2(\Omega)} + 2\int_0^t \|f(s)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s + 2\|c\|^2_{\mathbf{L}^\infty(\Omega_T)}\int_0^t \|u_m(s)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s. \end{aligned}$$

Passing to the limit as $m \to +\infty$ and using (A.4), we have

$$\begin{aligned} &\frac{3}{4}\int_0^t \|\dot{u}(s)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s + \frac{1}{2}\|\nabla u(t)\|^2_{\mathbf{L}^2(\Omega)} \\ &\quad \leq \frac{1}{2}\|\nabla g\|^2_{\mathbf{L}^2(\Omega)} + 2\int_0^t \|f(s)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s + 2\|c\|^2_{\mathbf{L}^\infty(\Omega_T)}\int_0^t \|u(s)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s \\ &\quad \leq \frac{1}{2}\|\nabla g\|^2_{\mathbf{L}^2(\Omega)} + 2\int_0^t \|f(s)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s \\ &\qquad + 2\|c\|^2_{\mathbf{L}^\infty(\Omega_T)} e^{2\|c\|_{\mathbf{L}^\infty(\Omega_T)}t}\|g\|^2_{\mathbf{L}^2(\Omega)}t \\ &\qquad + 2\|c\|^2_{\mathbf{L}^\infty(\Omega_T)} e^{2\|c\|_{\mathbf{L}^\infty(\Omega_T)}t}\,t\int_0^t \|f(s)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s. \end{aligned}$$

The previous inequality proves that $u \in \mathbf{L}^\infty((0,T);\mathbf{H}^1(\Omega))$ and $\dot{u} \in \mathbf{L}^2((0,T);\mathbf{L}^2(\Omega))$ since by (A.4), $u \in \mathbf{L}^\infty((0,T);\mathbf{L}^2(\Omega))$. If the boundary $\partial\Omega$ of Ω is of class $\mathbf{C}^2$, then $u(t) \in \mathbf{H}^2(\Omega)$ for a.e. $t \in [0,T]$ (see [27, Theorem 8.28]), proving that $u \in \mathbf{L}^2((0,T);\mathbf{H}^2(\Omega))$. ∎

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