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On a $p(x,.)$-integrodifferential problem with Neumann boundary conditions

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Abstract. This study examines a class of a bi-nonlocal problem involving a generalized integro-differential operator of elliptic type, characterized by a singular kernel, with nonlocal nonlinear Neumann boundary conditions. We establish the existence of infinitely many solutions in a general fractional Sobolev space with variable exponent.

1. Introduction and statement of the problem

The purpose of this study is to investigate the existence of solutions for a bi-nonlocal problem with nonlocal nonlinear Neumann boundary conditions of the following form:

$$\begin{cases} \mathcal{L}_K^{p(x,\cdot)}(u(x)) + |u|^{\bar{p}(x)-2}u(x) = f(x,u)\big(\int_\Omega F(x,u)dx\big)^r & \text{in } \Omega, \\ \mathcal{N}_K^{p(x,\cdot)}u(x) + \beta(x)|u|^{\bar{p}(x)-2}u(x) = 0 & \text{on } \mathbb{R}^N\setminus\Omega, \end{cases} \qquad (\mathcal{P}_K)$$

where Ω is a Lipschitz bounded domain of $\mathbb{R}^N$ with $N > 1$, $s \in (0,1)$, and $sp^+ < N$. Consider the set Q defined as follows:

$$Q := \mathbb{R}^N \times \mathbb{R}^N \setminus (\Omega^c \times \Omega^c), \quad \text{where } \Omega^c = \mathbb{R}^N\setminus\Omega.$$

• $\mathcal{L}_K^{p(x,.)}$ refers to the generalized integro-differential operator with variable exponent which is introduced in [4] as follows:

$$\big[\mathcal{L}_K^{p(x,.)}(u)\big](x) := \text{p.v.} \int_{\mathbb{R}^N} |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dy \quad \text{for all } x \in \mathbb{R}^N,$$

where p.v. is a commonly used abbreviation in the principal value sense.

• $p : \mathbb{R}^N \times \mathbb{R}^N \to (1,+\infty)$ is a continuous bounded function that satisfies

$$1 < p^- = \min_{(x,y)\in\mathbb{R}^{2N}} p(x,y) \leqslant p(x,y) \leqslant p^+ = \max_{(x,y)\in\mathbb{R}^{2N}} p(x,y) < +\infty \qquad (1.1)$$

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and

$$p \text{ is symmetric, that is, } p(x,y) = p(y,x) \text{ for all } (x,y) \in \mathbb{R}^{2N}. \tag{1.2}$$

• The kernel $K : \mathbb{R}^N \times \mathbb{R}^N \to (0,+\infty)$ is a measurable function with the following properties:

$$K(x,y) = K(y,x) \quad \text{for any } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N; \tag{1.3}$$

there exists $k_0 > 0$ such that

$$K(x,y) \geqslant k_0|x-y|^{-(N+sp(x,y))} \quad \text{for any } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \text{ and } x \neq y, \tag{1.4}$$

$$mK \in L^1\big(\mathbb{R}^N \times \mathbb{R}^N\big), \quad \text{where } m(x,y) = \min\big\{1, |x-y|^{p(x,y)}\big\}. \tag{1.5}$$

As a particular case of the singular kernel K, we can take

$$K(x,y) = |x-y|^{-(N+sp(x,y))}. \tag{1.6}$$

In this case, the generalized integro-differential operator $\mathcal{L}_K^{p(x,.)}$ is reduced to the fractional $p(x,.)$-Laplacian operator $(-\Delta)^s_{p(x,.)}$ which is considered as a generalisation of the well-known $p(x)$-Laplacian operator into the fractional case.

• The operator $\mathcal{N}_K^{p(x,.)}$ represents the new generalized nonlocal (nonlinear) normal $p(x,.)$-derivative with singular kernel given by

$$\big[\mathcal{N}_K^{p(x,.)}u\big](x) := \int_\Omega |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dy, \quad x \in \mathbb{R}^N \backslash \overline{\Omega}.$$

If we take the particular case of the kernel K given in (1.6), the generalized nonlocal normal $p(.,.)$-derivative $\mathcal{N}_K^{p(x,.)}$ coincides with the fractional normal $p(.,.)$-derivative $\mathcal{N}_s^{p(x,.)}$ introduced in [7] by (1.9).

• The functions f and β satisfy the following assumptions:

$(\mathcal{B}) : \beta \in L^\infty(\mathbb{R}^N \backslash \Omega)$ and $\beta \geq 0$ in $\mathbb{R}^N \backslash \Omega$.

$(\mathcal{F}) : f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

- f is an odd function with respect to t.
- There exist two positive constants l_1 and l_2 such that

$$l_1 t^{\gamma_1(x)-1} \leqslant f(x,t) \leqslant l_2 t^{\gamma_2(x)-1}, \tag{1.7}$$

  where $\gamma_1(x), \gamma_2(x) \in C(\overline{\Omega})$ are two variable exponents that verify the following condition:

$$1 < \gamma_1(x) \leqslant \gamma_2(x) < p_s^*(x), \quad \forall x \in \overline{\Omega} \text{ and } \gamma_2^+ < \frac{p^-}{r+1}. \tag{1.8}$$

- It is worth noting that $F(x,u) = \int_0^u f(x,\xi)d\xi$.

In the context of PDEs and variational problems with nonstandard $p(x)$-growth conditions, over the past decades, there exists a considerable body of literature in the field of nonlinear analysis. The investigations related to these topics were motivated by a proposed application of variable exponent spaces to modeling electrorheological fluids [26], as well as additional applications to image restoration [11] and the nonlinear Darcy's law application in porous media [2].

On the other hand, over time, extensive literature has been devoted to the study of nonlocal fractional problems regarding their wide range of applications. The investigation of such problems in the presence of Dirichlet boundary conditions has been done; see, for instance, [3, 6, 19, 20, 28] and the references therein. In particular, see the problems related to the fractional Laplacian $(-\Delta)^s$, which arise in a quite natural way in many different contexts both for pure mathematical research and in view of concrete real-world applications, such as [14, 27, 29]. The Dirichlet boundary conditions are used to study the vast bulk of such problems. Nevertheless, the study of Neumann boundary problems for the fractional nonlocal problems requires a different concept. Given what has been said, the inception of nonlocal Neumann boundary conditions was introduced by Valdinoci et al. [15]; This concept emerged to light from a simple probabilistic consideration, which can be summarized as follows:

- The function u represents the probability distribution of the position of a particle making an arbitrary move inside the domain Ω.
- When the particle exists in the domain of study Ω, it straight away comes back into Ω.
- The strategy in which it comes back inside Ω is the following: if the particle has gone to $x \in \mathbb{R}^N \backslash \overline{\Omega}$, it may come back to any point $y \in \Omega$, with the probability density of jumping from x to y being proportional to $|x-y|^{-N-2s}$.

These three situations lead to the introduction to the Neumann problem for the fractional Laplacian operator as follows:

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ \mathcal{N}_s u = 0 & \text{in } \mathbb{R}^N \backslash \overline{\Omega}, \end{cases}$$

where $\mathcal{N}_s$ is the nonlocal normal derivative defined as follows:

$$\mathcal{N}_s u(x) := \int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N \backslash \overline{\Omega}.$$

There exist other problems with nonlocal Neumann boundary conditions [8, 24], but the advantage of Dipierro's approach is that the problem has a variational structure. In particular, they studied the basic properties of this kind of boundary conditions.

In [25], the authors extend the notion of nonlocal Neumann boundary conditions to cover the nonlinear case ($p > 1$), and the nonlocal normal p-derivative is defined as follows:

$$\mathcal{N}_p^s u(x) = \int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dy, \quad \forall x \in \mathbb{R}^N \backslash \Omega.$$

This definition was also introduced in [9], where a nonlocal analog of the divergence theorem (the integration by parts formula) is given.

Very recently, for the Neumann problem involving the fractional $p(.,.)$-Laplacian operator $(-\Delta)^s_{p(.,.)}$, Bahrouni et al. have been introduced the corresponding nonlocal normal $p(.,.)$-derivative for this operator, which is given by

$$\mathcal{N}^s_{p(x,.)}u(x) = \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}dy, \quad \forall x \in \mathbb{R}^N \setminus \Omega. \tag{1.9}$$

In their work, the authors pointed out the fundamental properties associated with this new kind of boundary conditions.

On the other hand, bi-nonlocal problems refer to the fact of the appearance of the nonlocality in both the differential operator and the coefficient of the source term f, whichever makes our study more complicated. There are many works related to bi-nonlocal problems have been published, either the studied problem involving the p-Laplacian operator [1, 10] or the $p(x,.)$-Laplacian operator [12, 13].

Motivated by the above contributions, we consider in this paper a new class of problem involving the generalized $p(x,.)$-integro-differential operator with singular kernel. Up to our knowledge, this is the first attempt to treat bi-nonlocal problems with the generalized nonlocal $p(x,.)$-Neumann boundary conditions $\mathcal{N}^{p(x,.)}_K$. In an adequate new functional framework $(X, \|.\|_X)$ and using Krasnoselkii's genus theory, we establish the existence of infinitely many solutions. More precisely, our main result is stated as follows.

Theorem 1.1. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$, $p : \mathbb{R}^N \times \mathbb{R}^N \to (1,+\infty)$ is continuous variable exponent satisfying (1.1) and (1.2) with $sp^+ < N$. Let $K : \mathbb{R}^N \times \mathbb{R}^N \to (0,+\infty)$ be a measurable function satisfying (1.3)–(1.5). Assume that the assumptions $(\mathcal{B})$ and $(\mathcal{F})$ are fulfilled. Then, problem $(\mathcal{P}_K)$ has infinitely many weak solutions.*

The remainder of this paper is structured in this way. In Section 2, we present some preliminary knowledge about the fractional Sobolev spaces with variable exponent, as well as the basic notions on the Krasnoselkii's genus theory. In Section 3, we will state and construct a functional framework that will be well adapted, and the related properties are given. Using the genus theory, we prove our main result in Section 4. Finally, in order to illustrate our general results, we provide an explicit example and a particular case of our problem.

2. Mathematical background and preliminaries

In this section, we collect some necessary properties and notations about variable exponent Lebesgue and Sobolev spaces and the most properties can be found in [4–6, 16, 18, 21–23]. Furthermore, we present some basic notions on Krasnoselkii's genus that will be used in the proof of our main result.

2.1. Fractional Sobolev spaces with variable exponent

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N > 1)$ with the Lipschitz boundary, and consider the set

$$C^+(\bar{\Omega}) = \{p \in C(\bar{\Omega}) : p(x) > 1 \text{ for all } x \in \bar{\Omega}\}.$$

Let $p \in C^+(\bar{\Omega})$ with $1 < p^- \leqslant p(x) \leqslant p^+ < +\infty$; denote

$$p^+ = \sup_{x\in\bar{\Omega}} p(x) \quad \text{and} \quad p^- = \inf_{x\in\bar{\Omega}} p(x).$$

For any $p \in C^+(\bar{\Omega})$, the variable exponent Lebesgue space $L^{p(.)}(\Omega)$ is defined as follows:

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R}\text{measurable such that: } \int_\Omega |u(x)|^{p(x)} dx < \infty\right\}.$$

The space $L^{p(.)}(\Omega)$ is a Banach space equipped with the well-known Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{\lambda > 0; \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leqslant 1\right\}.$$

Let $L^{\hat{p}(.)}(\Omega)$ be the conjugate space of $L^{p(.)}(\Omega)$, that is, $\frac{1}{p(x)} + \frac{1}{\hat{p}(x)} = 1$, for all $x \in \bar{\Omega}$.

The Hölder's inequality maintains as in the classical Lebesgue spaces, and it is formulated in the following lemma.

Lemma 2.1. *For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{\hat{p}(x)}(\Omega)$, one has the following Hölder-type inequality*

$$\left|\int_\Omega uv\ \mathrm{d}x\right| \leqslant \left(\frac{1}{p^-} + \frac{1}{(\hat{p})^-}\right)\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{\hat{p}(x)}(\Omega)} \leqslant 2\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{\hat{p}(x)}(\Omega)}.$$

A major role in manipulating the generalized Lebesgue spaces with variable exponent is played by the modular of the $L^{p(.)}(\Omega)$ space, which is the mapping $\rho_{p(.)}$ defined by

$$\rho_{p(.)} : L^{p(.)}(\Omega) \to \mathbb{R}$$
$$u \mapsto \rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

Proposition 2.1. *Let $u \in L^{p(x)}(\Omega)$ and $\{u_k\} \subset L^{p(x)}(\Omega)$. Then, we have*

(i) $\|u\|_{L^{p(x)}(\Omega)} < 1$ *(resp.,* $= 1, > 1$*)* $\Leftrightarrow \rho_{p(x)}(u) < 1$ *(resp.,* $= 1, > 1$*),*

(ii) $\|u\|_{L^{p(x)}(\Omega)} < 1 \Rightarrow \|u\|^{p+}_{L^{p(x)}(\Omega)} \leqslant \rho_{p(x)}(u) \leqslant \|u\|^{p-}_{L^{p(x)}(\Omega)}$,

(iii) $\|u\|_{L^{p(x)}(\Omega)} > 1 \Rightarrow \|u\|^{p-}_{L^{p(x)}(\Omega)} \leqslant \rho_{p(x)}(u) \leqslant \|u\|^{p+}_{L^{p(x)}(\Omega)}$,

(iv) $\lim_{k\to+\infty} \|u_k - u\|_{L^{p(x)}(\Omega)} = 0 \Leftrightarrow \lim_{k\to+\infty} \rho_{p(x)}(u_k - u) = 0$.

Lemma 2.2 ([17, Lemma 2.1]). *Let $p \in L^\infty(\mathbb{R}^N)$ be such that $1 \le p(x)q(x) \le \infty$ for a.e. $x \in \mathbb{R}^N$. Let $u \in L^{q(x)}(\mathbb{R}^N)$ with $u \neq 0$. Then,*

(1) *If* $\|u\|_{L^{p(x)q(x)}(\mathbb{R}^N)} \leq 1$, *then*

$$\|u\|^{p^+}_{L^{p(x)q(x)}(\mathbb{R}^N)} \leq \left\||u|^{p(x)}\right\|_{L^{q(x)}(\mathbb{R}^N)} \leq \|u\|^{p^-}_{L^{p(x)q(x)}(\mathbb{R}^N)}.$$

(2) *If* $\|u\|_{L^{p(x)q(x)}(\mathbb{R}^N)} \geq 1$, *then*

$$\|u\|^{p^-}_{L^{p(x)q(x)}(\mathbb{R}^N)} \leq \left\||u|^{p(x)}\right\|_{L^{q(x)}(\mathbb{R}^N)} \leq \|u\|^{p^+}_{L^{p(x)q(x)}(\mathbb{R}^N)}.$$

The natural framework to look for solutions of nonlocal problems involving $(-\Delta)^s_{p(x,.)}$ is the fractional Sobolev space with variable exponent introduced in [5] in this way:

$$W = W^{s,p(x,y)}(Q) = \Big\{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable such that } u_{|\Omega} \in L^{p(x)}(\Omega) \text{ with } \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}dxdy < +\infty \text{ for some } \lambda > 0\Big\}.$$

Since our problem involves the generalized $p(x,.)$-integro-differential operator of elliptic type $\mathcal{L}^{p(x,)}_K$, we have to introduce the functional space related to it. It is the general fractional Sobolev space with variable exponent introduced in [4] as follows:

$$W^{s,p(x,y)}_K(Q) = \Big\{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable such that } u_{|\Omega} \in L^{\bar{p}(x)}(\Omega) \text{ with } \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}}K(x,y)dxdy < +\infty \text{ for some } \lambda > 0\Big\}.$$

The space $W^{s,p(x,y)}_K(Q)$ is equipped with the following norm:

$$\|u\|_{W^{s,p(x,y)}_K(Q)} = \|u\|_{K,p(x,y)} = \|u\|_{L^{p(x)}(\Omega)} + [u]_{K,p(x,y)},$$

where $[u]_{K,p(x,y)} = \inf\{\lambda > 0 : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}}K(x,y)dxdy \leqslant 1\}$.

Corollary 2.1 ([7, Corollary 1]). (i) $(W^{sp(x,y)}_K(Q), \|\cdot\|_{K,p(x,y)})$ *is a separable and reflexive uniformly convex.*

(ii) *If* $\Omega \subset \mathbb{R}^N$ *is a domain of class* $C^{0,1}$, *then* $(W^{s,p(x,y)}_K(Q), \|\cdot\|_{K,p(x,y)})$ *is a Banach space.*

Next, we state a continuous and compact embedding theorem; for the proof, we refer to [4].

Theorem 2.1. *Let* Ω *be a Lipschitz bounded domain in* $\mathbb{R}^N$ *and* $s \in (0,1)$. *Let* $p : \mathbb{R}^N \times \mathbb{R}^N \to (1,+\infty)$ *be a continuous variable exponent satisfying* (1.1) *and* (1.2) *with* $sp^+ < N$. *Let* $r : \overline{\Omega} \to (1,+\infty)$ *be a continuous bounded variable exponent such that* $1 < r^- \leqslant r(x) < p^*_s(x)$ *for all* $x \in \overline{\Omega}$. *Suppose that* $K : \mathbb{R}^N \times \mathbb{R}^N \to (0,+\infty)$ *is a measurable function satisfying* (1.3)–(1.5). *Then, there exists a positive constant* $C = C(N,p,r,s,\Omega) > 0$ *such that, for any* $u \in W^{s,p(x,y)}_K(Q)$, *we have*

$$\|u\|_{L^{r(x)}(\Omega)} \leqslant C\|u\|_{W^{s,p(x,y)}(\Omega)} \leqslant C \max\left\{1, \tilde{k_0}\right\}\|u\|_{K,p(x,y)},$$

$$\textit{where} \quad \tilde{k_0} = \max\left\{k_0^{-\frac{1}{p^-}}, k_0^{-\frac{1}{p^+}}\right\};$$

that is, the space $W_K^{s,p(x,y)}(Q)$ is continuously embedded in $L^{r(x)}(\Omega)$. Moreover, this embedding is compact.

Lemma 2.3 ([4, Lemma 11]). *Let $p : \mathbb{R}^N \times \mathbb{R}^N \to (1, +\infty)$ be a continuous variable exponent satisfying (1.1) and (1.2), and let $K : \mathbb{R}^N \times \mathbb{R}^N \to (0, +\infty)$ be a measurable function satisfying (1.3)–(1.5). Then, the following assertions hold:*

(i) $\mathcal{L}_K^{p(x,.)}$ *is well defined and bounded,*

(ii) $\mathcal{L}_K^{p(x,.)}$ *is a strictly monotone operator,*

(iii) $\mathcal{L}_K^{p(x,.)}$ *is a mapping of type (S_+).*

2.2. Preliminaries on genus theory

Let X be a real Banach space, and denote by $\mathcal{R}$ the class of all closed subsets $A \subset X \backslash \{0\}$, that are symmetric with respect to the origin (i.e., if $u \in A \Rightarrow -u \in A$).

Definition 2.1. Let $A \in \mathcal{R}$ and $X = \mathbb{R}^N$. The genus $\gamma(A)$ of A is defined as follows:

$$\gamma(A) = \min\big\{k \geq 1; \text{there exists an odd continuous mapping } \varphi : A \to \mathbb{R}^k \backslash \{0\}\big\}.$$

If such a k does not exist, we set $\gamma(A) = \infty$.

Remark 2.1. (i) If A is a subset, which consists of finitely many pairs of points, then $\gamma(A) = 1$.

(ii) $\gamma(\varnothing) = 0$ (by definition).

Theorem 2.2. *Let $X = \mathbb{R}^N$, and let Ω be an open, symmetric, and bounded subset of $\mathbb{R}^N$, with $\partial\Omega$ presents its boundary with $0 \in \Omega$. Then, $\gamma(\partial\Omega) = N$.*

Corollary 2.2. *Let us denote by S^{N-1} the $(N-1)$-dimensional sphere in $\mathbb{R}^N$. Then, $\gamma(S^{N-1}) = N$.*

The Clarke's theorem stated below is the main tool used to prove our main result.

Theorem 2.3. *Let $I \in C^1(X, \mathbb{R})$ be a functional satisfying the Palais–Smale condition (PS). Furthermore, we suppose that*

(G_1) *I is even and bounded from below.*

(G_2) *There exists a compact set $K \in \mathcal{R}$ such that $\gamma(K) = k$ and $\sup_{x \in K} I(x) < I(0)$.*

Then, I possesses at least k pairs of distinct critical points and their corresponding critical values are less than $I(0)$.

3. Functional framework of problem ($\mathcal{P}_K$)

In this section, we introduce the appropriate framework to deal with problem ($\mathcal{P}_K$), as inspired by the procedure in the paper of Bahrouni et al. [7]. Moreover, we establish some basic properties of the generalized nonlocal normal $p(x,\cdot)$-derivative $\mathcal{N}_K^{p(x,.)}$ associated with $\mathcal{L}_K^{p(x,)}$ and nonlocal Neumann boundary conditions.

Let $u : \mathbb{R}^N \to \mathbb{R}$ be a measurable function and $p \in C^+(\bar{\Omega})$ satisfying (1.1) and (1.2) such that $\bar{p}(x) = p(x,x)$ for all $x \in \mathbb{R}^{2N}$. We define the space X by

$$X := \left\{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \|u\|_X < +\infty\right\},$$

where

$$\|u\|_X := [u]_{K,p(x,y),Q} + \|u\|_{L^{\bar{p}(x)}(\Omega)} + \left\|\beta^{\frac{1}{\bar{p}(.)}} u\right\|_{L^{\bar{p}(x)}(\Omega^c)},$$

such that

$$[u]_{K,p(x,y),Q} = \inf\left\{\lambda \geq 0 : \frac{1}{2}\int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}} K(x,y)dxdy \leq 1\right\}.$$

Proposition 3.1. *The space $(X, \|u\|_X)$ is a reflexive and separable Banach space.*

Proof. The proof is similar to Proposition 3.1 in [7]. ∎

Theorem 3.1. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. Let*

$$p : \mathbb{R}^N \times \mathbb{R}^N \to (1,+\infty)$$

be a continuous variable exponent satisfying (1.1) and (1.2) with $sp^+ < N$. Let $r \in C^+(\bar{\Omega})$ such that $1 < r^- \leqslant r(x) < p_s^*(x)$ for all $x \in \bar{\Omega}$. Suppose that $K : \mathbb{R}^N \times \mathbb{R}^N \to (0,+\infty)$ is a measurable function satisfying (1.3)–(1.5). Then, there exists a positive constant $C = C(N,p,r,s,\Omega) > 0$ such that, for any $u \in X$, we have

$$\|u\|_{L^{r(x)}(\Omega)} \leqslant C\|u\|_X.$$

Moreover, this embedding is compact.

Proof. It is obvious to see that

$$\|u\|_{W_K^{s,p(x,y)}(Q)} \leq \|u\|_X \quad \text{for all } u \in X.$$

Then, by Theorem 2.1, we get the desired result. ∎

Definition 3.1. Let $p : \mathbb{R}^N \times \mathbb{R}^N \to (1,+\infty)$ be a continuous bounded function satisfying (1.1) and (1.2), and let $K : \mathbb{R}^N \times \mathbb{R}^N \to (0,+\infty)$ be a measurable function satisfying (1.3)–(1.5). For any $u \in X$, we define the modular $\rho_{K,p(.,.)} : X \to \mathbb{R}$ by

$$\rho_{K,p(.,.)}(u) = \int_Q |u(x)-u(y)|^{p(x,y)} K(x,y)dxdy + \int_\Omega |u|^{\bar{p}(x)}dx + \int_{\Omega^c} \beta(x)|u|^{\bar{P}(x)}dx,$$

and

$$\|u\|_{\rho_{K,p(.,.)}} = \inf\left\{\lambda > 0 : \rho_{K,p(.,.)}\left(\frac{u}{\lambda}\right) \leqslant 1\right\}.$$

Remark 3.1. It is clear that $\|u\|_{\rho_{K,p(.,.)}}$ is a norm on X, which is equivalent to the norm $\|u\|_X$.

Lemma 3.1. *Let $p : \mathbb{R}^N \times \mathbb{R}^N \to (1,+\infty)$ be a continuous bounded function satisfying (1.1) and (1.2) with $sp^+ < N$, and let*

$$K : \mathbb{R}^N \times \mathbb{R}^N \to (0,+\infty)$$

be a measurable function satisfying (1.3)–(1.5); moreover, the function β verifies $(\mathcal{B})$. Then, for any $u \in X$, we have

(i) *For $u \neq 0$, we have $\|u\|_X = a$ if and only if $\rho_{K,p(.,.)}(\frac{u}{a}) = 1$.*

(ii) *If $\|u\|_X < 1$, then $\frac{\|u\|_X^{p+}}{3^{p^+-1}} \leq \rho_{K,p(.,.)}(u) \leq 3\|u\|_X^{p^-}$.*

(iii) *If $\|u\|_X > 1$, then $\|u\|_X^{p^-} \leq \rho_{K,p(.,.)}(u)$.*

Following that, we state and prove the following propositions, which are crucial in the study of nonlocal problems with nonlinear nonlocal Neumann boundary conditions.

Proposition 3.2. *Let $p : \mathbb{R}^N \times \mathbb{R}^N \to (1,+\infty)$ be a continuous bounded function satisfying (1.1) and (1.2) with $sp^+ < N$, and let*

$$K : \mathbb{R}^N \times \mathbb{R}^N \to (0,+\infty)$$

be a measurable function satisfying (1.3)–(1.5). Then, for any u a bounded C^2-function in $\mathbb{R}^N$, we have

$$\int_\Omega \mathcal{L}_K^{p(x,.)} u(x)dx = -\int_{\mathbb{R}^N\setminus\Omega} \mathcal{N}_K^{p(x,.)} u(x)dx.$$

Proof. We argue as in [7], since K and p are symmetric, then by direct computations we get

$$\begin{aligned}\int_\Omega\int_\Omega &|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dxdy\\ &= -\int_\Omega\int_\Omega |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dxdy,\end{aligned}$$

which implies that

$$\begin{aligned}\int_\Omega \mathcal{L}_K^{p(x,.)} u(x)dx &= \int_\Omega p.v\int_{\mathbb{R}^N} |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dydx\\ &= \int_\Omega \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus B(x,\varepsilon)} |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dydx\\ &= \int_\Omega\int_{\mathbb{R}^N\setminus\Omega} |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dydx\\ &= \int_{\mathbb{R}^N\setminus\Omega}\int_\Omega |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dxdy\\ &= -\int_{\mathbb{R}^N\setminus\Omega} \mathcal{N}_K^{p(.,y)} u(y)dy.\end{aligned}$$

■

Proposition 3.3. *Let $p : \mathbb{R}^N \times \mathbb{R}^N \to (1, +\infty)$ be a continuous bounded function satisfying (1.1) and (1.2) with $sp^+ < N$, and let*

$$K : \mathbb{R}^N \times \mathbb{R}^N \to (0, +\infty)$$

be a measurable function satisfying (1.3)–(1.5). Then, for any u and v two bounded C^2-functions in $\mathbb{R}^N$, we have the following equality:

$$\begin{aligned}&\frac{1}{2}\int_Q |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy\\&\quad=\int_\Omega v\mathcal{L}_K^{p(x,.)}(u)dx+\int_{\Omega^c} v\mathcal{N}_K^{p(x,.)}(u)dx.\end{aligned}$$

Proof. By conditions (1.2) and (1.3), we have

$$\begin{aligned}&\frac{1}{2}\int_Q |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy\\&\quad=\int_Q v(x)|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dxdy\\&\quad=\int_\Omega v(x)\Big(\int_{\mathbb{R}^N}|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dy\Big)dx\\&\qquad+\int_{\mathcal{C}\Omega} v(x)\Big(\int_\Omega |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)dy\Big)dx\\&\quad=\int_\Omega v\mathcal{L}_K^{p(x,.)}(u)dx+\int_{\mathcal{C}\Omega} v\mathcal{N}_K^{p(x,.)}(u)dx.\end{aligned}$$

Based on the integration by parts formula given by Proposition 3.3, we give the definition of a weak solution for problem $(\mathcal{P}_K)$ in the following form. ∎

Definition 3.2. We say that $u \in X$ is a weak solution of $(\mathcal{P}_K)$ if

$$\begin{aligned}&\frac{1}{2}\int_Q |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy\\&\quad+\int_\Omega |u|^{\bar{p}(x)-2}uvdx+\int_{\Omega^c}\beta(x)|u|^{\bar{p}(x)-2}uvdx\\&\quad=\Big(\int_\Omega F(x,u)dx\Big)^r\Big(\int_\Omega f(x,u)vdx\Big),\end{aligned}\tag{3.1}$$

for every $v \in X$.

As a consequence of the above definition, we obtain the following result.

Proposition 3.4. *Let $u \in X$ be a weak solution of $(\mathcal{P}_K)$; then,*

$$\mathcal{N}_K^{p(x,.)}u(x)+\beta(x)|u|^{\bar{p}(x)-2}u(x)=0 \quad a.e.\ in\ \mathbb{R}^N\backslash\overline{\Omega}.$$

Proof. First, we start by taking $v \in X$ such that $v \equiv 0$ in Ω as a test function in (3.1), and because of the symmetry of the exponent variable $p(.,.)$ and the kernel K, we find

$$\begin{aligned} 0 &= \frac{1}{2}\int_Q |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy \\ &\quad + \int_{\Omega^c} \beta(x)|u|^{\bar{P}(x)-2}uvdx \\ &= \int_{\mathbb{R}^N\setminus\bar{\Omega}} v(x)\int_\Omega |u(x)-u(y)|^{|p(x,y)-2}(u(x)-u(y))K(x,y)dydx \\ &\quad + \int_{\Omega^c} \beta(x)|u|^{\bar{p}(x)-2}uvdx \\ &= \int_{\mathbb{R}^N\setminus\bar{\Omega}} v(x)\mathcal{N}_K^{p(x,.)}u(x)dx + \int_{\Omega^c} \beta(x)|u|^{\bar{p}(x)-2}uvdx \\ &= \int_{\mathbb{R}^N\setminus\bar{\Omega}} \big(\mathcal{N}_K^{p(x,.)}u(x) + \beta(x)|u|^{\bar{p}(x)-2}u\big)v(x)dx. \end{aligned}$$

Consequently,

$$\int_{\mathbb{R}^N\setminus\bar{\Omega}} \big(\mathcal{N}_K^{p(x,.)}u(x) + \beta(x)|u|^{\bar{p}(x)-2}u\big)v(x)dx = 0$$

for every $v \in X$, with $v = 0$ in Ω. In particular, this is true for every $v \in C_c^\infty(\mathbb{R}^N\setminus\bar{\Omega})$; therefore,

$$\mathcal{N}_K^{p(x,.)}u(x) + \beta(x)|u|^{\bar{p}(x)-2}u(x) = 0 \quad \text{a.e. in } \mathbb{R}^N\setminus\bar{\Omega}.$$

∎

4. Genus theory analysis of problem ($\mathcal{P}_K$)

In order to apply the genus theory to our problem, we need to establish some auxiliary results. To this end, we associate to problem ($\mathcal{P}_K$) the energy functional $J : X \to \mathbb{R}$ defined as follows:

$$\begin{aligned} J(u) &= \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{2p(x,y)}K(x,y)dxdy + \int_\Omega \frac{|u|^{\bar{p}(x)}}{\bar{p}(x)}dx \\ &\quad + \int_{\Omega^c} \frac{\beta(x)|u|^{\bar{p}(x)}}{\bar{p}(x)}dx - \frac{1}{r+1}\left(\int_\Omega F(x,u)dx\right)^{r+1} \quad \text{for every } u \in X. \end{aligned}$$

Proposition 4.1. *Under the assumptions* $(\mathcal{B})$ *and* $(\mathcal{F})$*, the functional* J *is well defined; moreover,* $J \in C^1(X,\mathbb{R})$*, and for all* $u, v \in X$*, the derivative of* J *is given by*

$$\begin{aligned} \langle J'(u), v\rangle &= \frac{1}{2}\int_Q |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy \\ &\quad + \int_\Omega |u|^{p(x)-2}uvdx + \int_{\Omega^c} \beta(x)|u|^{\bar{p}(x)-2}uvdx \\ &\quad - \left(\int_\Omega F(x,u)dx\right)^r \int_\Omega f(x,u)vdx. \end{aligned}$$

Proof. Let $u \in X$; by using Lemma 3.1 and condition ($\mathcal{F} - (1.7)$), we get

$$J(u) \leqslant C\|u\|_X, \quad \text{where } C = C(p, \gamma_2, l_2, r).$$

Standard argument as that used in [3, Lemma 3.1] shows that $J \in C^1(X, \mathbb{R})$, and its derivative is given by the following formula:

$$\begin{aligned}\langle J'(u), v\rangle = {} & \frac{1}{2}\int_Q |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy \\ & + \int_\Omega |u|^{p(x)-2}uvdx + \int_{\Omega^c} \beta(x)|u|^{\bar{p}(x)-2}uvdx \\ & - \left(\int_\Omega F(x,u)dx\right)^r \int_\Omega f(x,u)vdx\end{aligned}$$

for any u, $v \in X$. Therefore, we can find weak solutions of ($\mathcal{P}_K$) as the critical points of J. ■

To prove our main result, we will need the following technical lemmas.

Lemma 4.1. *We suppose that hypotheses* ($\mathcal{B}$) *and* ($\mathcal{F}$) *are satisfied; then, the functional* J *is bounded from below on* X.

Proof. Using condition (1.7), Lemma 3.1, and the embedding results, we have, for any $u \in X$,

$$\begin{aligned}J(u) = {} & \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{2p(x,y)}K(x,y)dxdy + \int_\Omega \frac{|u|^{\bar{p}(x)}}{\bar{p}(x)}dx + \int_\Omega \frac{\beta(x)|u|^{\bar{p}(x)}}{\bar{p}(x)}dx \\ & - \frac{1}{r+1}\left(\int_\Omega F(x,u)dx\right)^{r+1} \\ \geqslant {} & \frac{1}{2p^+}\left(\int_Q |u(x)-u(y)|^{p(x,y)}K(x,y)dxdy + \int_\Omega |u|^{\bar{p}(x)}dx\right. \\ & \left. + \int_\Omega \beta(x)|u|^{\bar{p}(x)}dx\right) - \frac{1}{r+1}\left(\frac{l_2}{\gamma_2^-}\right)^{r+1}\left(\int_\Omega u^{\gamma_2(x)}dx\right)^{r+1}.\end{aligned}$$

Taking $\|u\|_X > 1$, we obtain from Lemma 3.1 that

$$J(u) \geqslant \frac{1}{2p^+}\|u\|_X^{p^-} - \frac{1}{r+1}\left(\frac{l_2}{\gamma_2^-}\right)^{r+1}\left(\int_\Omega u^{\gamma_2(x)}dx\right)^{r+1}.$$

Theorem 3.1 and the fact that $\gamma_2(x) < p_s^*(x)$ imply that

$$J(u) \geqslant \frac{1}{2p^+}\|u\|_X^{p^-} - \frac{1}{r+1}\left(\frac{l_2 C^{\gamma_2^+}}{\gamma_2^-}\right)^{r+1}\|u\|_X^{\gamma_2^+(r+1)}, \tag{4.1}$$

where $C = C(N, p, \gamma_2, s, \Omega)$ is a positive constant. ■

Since the upper bound of the exponent variable γ_2 verifies $\gamma_2^+ < \frac{p^-}{r+1}$, inequality (4.1) shows that J is bounded from below.

Proposition 4.2. *Let $\{u_n\} \subset X$ be a sequence such that $u_n \rightharpoonup u \in X$; then, the following assertions hold.*

(i) $\lim_{n\to\infty} \int_\Omega |u_n|^{\bar{p}(x)-2} u_n(u_n - u)dx = 0.$

(ii) $\lim_{n\to\infty} \int_{\mathbb{R}^N\setminus\Omega} \beta(x)|u_n|^{\bar{p}(x)-2} u_n(u_n - u)dx = 0.$

Proof. (i) Applying Hölder's inequality, we get

$$\int_\Omega |u_n|^{\bar{p}(x)-2} u_n(u_n - u)dx \leqslant \||u_n|^{\bar{p}(x)-2} u_n\|_{L^{\hat{p}(x)}(\Omega)} \|u_n - u\|_{L^{\bar{p}(x)}(\Omega)},$$

where $\hat{p}(.)$ is the conjugate of $\bar{p}(.)$.

If $\|u_n\|_{L^{\bar{p}(x)}(\Omega)} \geq 1$, we have

$$\||u_n|^{\bar{p}(x)-2} u_n\|_{L^{\hat{p}(x)}(\Omega)} \leqslant \|u_n\|_{L^{\bar{p}(x)}}^{\bar{p}^+}.$$

If $\|u_n\|_{L^{\bar{p}(x)}(\Omega)} \leq 1$, we have

$$\||u_n|^{\bar{p}(x)-2} u_n\|_{L^{\hat{p}(x)}(\Omega)} \leqslant \|u_n\|_{L^{\bar{p}(x)}}^{\bar{p}^-}.$$

Hence,

$$\||u_n|^{\bar{p}(x)-2} u_n\|_{L^{\hat{p}(x)}(\Omega)} \leqslant \max\left\{\|u_n\|_{L^{\bar{p}(x)}}^{\bar{p}^-}, \|u_n\|_{L^{\bar{p}(x)}}^{\bar{p}^+}\right\}.$$

Since the embedding $X \hookrightarrow L^{\bar{p}(x)}(\Omega)$ is compact, we get the desired result.

(ii) By analogy, using the fact that the embedding $X \hookrightarrow L^{\bar{p}(x)}(\mathbb{R}^N\setminus\Omega)$ is compact and $\beta \in L^\infty(\mathbb{R}^N\setminus\Omega)$, we have that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N\setminus\Omega} \beta(x)|u_n|^{\bar{p}(x)-2} u_n(u_n - u)dx = 0. \qquad \blacksquare$$

Lemma 4.2. *The functional J satisfies the Palais–Smale (PS) condition.*

Proof. Let $\{u_n\} \subset X$ be a sequence such that

$$J(u_n) \to \bar{c} > 0 \quad \text{and} \quad J'(u_n) \to 0 \text{ in } X^*. \tag{4.2}$$

From inequality (4.1), we have

$$J(u_n) \geqslant \frac{1}{2p^+}\|u_n\|_X^{p^-} - \frac{1}{r+1}\left(\frac{l_2 C^{\gamma_2^+}}{\gamma_2^-}\right)^{r+1} \|u_n\|_X^{\gamma_2^+(r+1)}.$$

Then, by the fact that $\{J(u_n)\}_n$ is bounded, we conclude that $\{u_n\}_n$ is bounded in X.

As a consequence, there exists a subsequence, still denoted by $\{u_n\}_n$, such that $u_n \rightharpoonup u$ in X.

By (4.2), we get that $\langle J'(u_n), u_n - u\rangle \to 0$. Therefore,

$$\langle J'(u_n), u_n - u\rangle = \frac{1}{2}\int_Q |u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))[(u_n-u)(x)-(u_n-u)(y)]K(x,y)dxdy + \int_\Omega |u_n|^{\bar{p}(x)-2}u_n(u_n-u)dx + \int_{\Omega^c}\beta(x)|u_n|^{\bar{p}(x)-2}u_n(u_n-u)dx - \left(\int_\Omega F(x,u_n)dx\right)^r \int_\Omega f(x,u_n)(u_n-u)dx \to 0.$$

Since $u_n \rightharpoonup u$ in X, then, by Theorem 3.1, we have

$$\begin{array}{llll} u_n & \to & \text{in} & L^{\gamma_2(x)}(\Omega), \\ u_n & \to & u \quad \text{a.e.} & \Omega. \end{array}$$

On the other hand, by using Hölder's inequality and condition (1.7), we obtain

$$\begin{aligned} \left|\int_\Omega f(x,u_n)(u_n-u)dx\right| &\leqslant \int_\Omega |f(x,u_n)||(u_n-u)|dx \\ &\leqslant l_2\int_\Omega |u_n|^{\gamma_2(x)-1}|(u_n-u)|dx \\ &\leqslant C_1\left\||u_n|^{\gamma_2(x)-1}\right\|_{L^{\frac{\gamma_2(x)}{\gamma_2(x)-1}}(\Omega)}\|u_n-u\|_{L^{\gamma_2(x)}(\Omega)}, \end{aligned}$$

which tends to 0 as $n \to \infty$. Consequently,

$$\left(\int_\Omega F(x,u_n)dx\right)^r \int_\Omega f(x,u_n)(u_n-u)dx \to 0 \quad \text{as } n \to \infty.$$

Using Proposition 4.2, we get

$$\lim_{n\to\infty}\int_Q |u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))[(u_n-u)(x)-(u_n-u)(y)]K(x,y)dxdy = 0,$$

which implies

$$\lim_{n\to\infty}\langle \mathcal{L}_K^{p(x,)}(u_n), u_n-u\rangle = 0;$$

by a similar argument, we have $\lim_{n\to\infty}\langle \mathcal{L}_K^{p(x,)}(u), u_n-u\rangle = 0$. Then,

$$\lim_{n\to\infty}\langle \mathcal{L}_K^{p(x,)}(u_n) - \mathcal{L}_K^{p(x,)}(u), u_n-u\rangle = 0.$$

Thus, from Lemma 2.3, we conclude that $\{u_n\}$ converges strongly to u in X; as a consequence, the functional J satisfies the Palais–Smale condition. ∎

Now, we are in the position to prove our main result by means of Krasnoselskii's genus theory.

Proof of Theorem 1.1. The proof procedure consists of verifying that the functional J satisfies the hypotheses of Theorem 2.3. Indeed, we have that $J \in C^1(X, \mathbb{R})$, and by Lemma 4.2, J satisfies the Palais–Smale compactness condition. Moreover, it is obvious to see that the functional J is even; also, it is bounded from below (because of Lemma 4.1).

It remains to show that condition (G_2) of Theorem 2.3 is fulfilled. Indeed, for each $m \in \mathbb{N}$, consider X_m the linear subspace of X generated by m vectors $e_1, e_2, \dots, e_m$. Since all the norms in the finite-dimensional space are equivalent, then there exists a constant C_m such that

$$\|u\|_{L^{\gamma_1(x)}(\Omega)} \geqslant C_m \|u\|_X \quad \text{for } u \in X_m.$$

For every $u \in X_m$, we obtain for $\|u\|_X$ small enough

$$\begin{aligned} J(u) &\leqslant \frac{3}{p^-}\|u\|_X^{p^-} - \frac{1}{r+1}\left(\frac{l_1}{\gamma_1^+}\right)^{r+1}\left(\int_\Omega u^{\gamma_1(x)}dx\right)^{r+1} \\ &\leqslant \frac{3}{p^-}\|u\|_X^{p^-} - \frac{1}{r+1}\left(\frac{l_1 C_m^{\gamma_1^+}}{\gamma_1^+}\right)^{r+1}\|u\|_X^{\gamma_1^+(r+1)}, \end{aligned}$$

or also

$$J(u) \leqslant \|u\|_X^{\gamma_1^+(r+1)}\left(\frac{3}{p^-}\|u\|_X^{p^- -(r+1)\gamma_1^+} - \frac{1}{r+1}\left(\frac{l_1 C_m^{\gamma_1^+}}{\gamma_1^+}\right)^{r+1}\right).$$

Let ρ_m and $\mathcal{H}$ be two positive constants such that

$$\rho_m < \mathcal{H} < \min\left\{1, \left[\frac{p^-}{3(r+1)}\left(\frac{l_1 C_m^{\gamma_1^+}}{\gamma_1^+}\right)^{r+1}\right]^{\frac{1}{p^- -(r+1)\gamma_1^+}}\right\},$$

and consider $S_{\rho_m}^{(m)} = \{u \in X_m : \|u\|_X = \rho_m\}$; then, by this condition $p^- > (r+1)\gamma_1^+$, we obtain for any $u \in S_{\rho_m}^{(m)}$

$$\begin{aligned} J(u) &\leqslant \rho_m^{\gamma_1^+(r+1)}\left(\frac{3}{p^-}\rho_m^{p^- -(r+1)\gamma_1^+} - \frac{1}{r+1}\left(\frac{l_1 C_m^{\gamma_1^+}}{\gamma_1^+}\right)^{r+1}\right) \\ &< \mathcal{H}^{\gamma_1^+(r+1)}\left(\frac{3}{p^-}\mathcal{H}^{p^- -(r+1)\gamma_1^+} - \frac{1}{r+1}\left(\frac{l_1 C_m^{\gamma_1^+}}{\gamma_1^+}\right)^{r+1}\right) \\ &< 0. \end{aligned}$$

Consequently, $\sup_{u \in S_{\rho_m}^{(m)}} J(u) < 0 = J(0)$.

On the other hand, $S_{\rho_m}^{(m)}$ is homeomorphic to the $(m-1)$-dimensional sphere in $\mathbb{R}^m$. Then, $\gamma(S_{\rho_m}^{(m)}) = m$.

In view of Clarke's theorem, J has at least k pairs of distinct critical points, because k is arbitrary, we infer the existence of infinitely many critical points of J. The proof is completed. ∎

5. Example and particular case

In this section, we give an explicit example in order to illustrate our main results. As a particular case, we can take

- The singular kernel is given by

$$K(x,y) = |x-y|^{-N-sp(x,y)}.$$

- For the nonlinearity function f, we can take

$$f(x,t) = a|t|^{\gamma(x)-2}t,$$

where a is a positive constant and $\gamma(x) < p_s^*(x)\ \forall x \in \overline{\Omega}$, as well as $\gamma^+ < \frac{p^-}{r+1}$.

Therefore, problem ($\mathcal{P}_K$) turns to a bi-nonlocal problem involving the fractional $p(x,.)$-Laplacian operator with nonlocal Neumann boundary conditions:

$$\begin{cases} (-\Delta)^s_{p(x,.)}(u(x)) + |u|^{\bar{p}(x)-2}u(x) = (|u|^{\gamma(x)-2}u)\Big(\int_\Omega \frac{1}{\gamma(x)}|u|^{\gamma(x)-1}u\,dx\Big)^r & \text{in } \Omega, \\ \mathcal{N}_{s,p(x,.)}u(x) + \beta(x)|u|^{\bar{p}(x)-2}u(x) = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{$\mathcal{P}_s$}$$

It easy to see that the kernel K satisfies conditions (1.3)–(1.5). Moreover, $f(x,-t) = -f(x,t)$ and f verifies (1.7)–(1.8). Then, as a direct consequence of Theorem 1.1, we get the following result.

Corollary 5.1. *Problem* ($\mathcal{P}_s$) *has infinitely many weak solutions.*

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