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Topology structure of solution set of fractional non-autonomous evolution inclusions

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Abstract. This paper investigates the topological characteristics of the set of solutions to non-autonomous fractional evolution inclusions. We present the concept of mild solutions for nonautonomous fractional evolution inclusions and prove that the solution set is nonempty and compact. Moreover, when the space is a reflexive Banach space, we demonstrate that the solution set is an R_δ-set, indicating that it may not be a single entity.

1. Introduction

Over the last two decades, fractional calculus has garnered significant interest globally among researchers. Equations incorporating fractional derivatives serve as powerful tools to characterize memory and hereditary attributes in various substances and operations. Models using fractional derivatives typically yield more precise representations than those with traditional integer-order derivatives, thereby providing superior descriptions of certain phenomena. For an in-depth discussion, refer to the works of Kilbas et al., Hilfer, Luchko et al. [2, 15, 18, 23].

Throughout this paper, our focus lies on the subsequent fractional nonautonomous evolution inclusions:

$$\begin{cases} {}^C D_t^\alpha y(t) \in A(t)y(t) + \mathscr{G}(t, y(t)), & t \in [0, T], \\ y(0) + g(y) = y_0. \end{cases} \tag{1}$$

We clarify that ${}^C D_t^\alpha$ embodies the Caputo fractional derivative of order $0 < \alpha < 1$. The operator $A(t) : \mathcal{D}(A(t)) \subset \mathcal{Y} \to \mathcal{Y}$ constitutes the infinitesimal generators of analytic semigroups $\{T_t(\tau)\}_{\tau \geq 0}$ with the domain

$$\mathcal{D}(A(t)) =: \mathcal{D}(\mathcal{A})$$

that is independent of t for all $t \in [0, T]$. The term $\mathscr{G} : [0, T] \times \mathcal{Y} \multimap \mathcal{Y}$ represents a predefined multivalued map that operates on $[0, T] \times \mathcal{Y}$. The function g specified later in context satisfies certain prerequisites and y_0 signifies a member of the Banach space $\mathcal{Y}$.

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The examination of system (1) holds relevance, as illustrated by a parabolic Caputo fractional partial differential equation of the subsequent shape:

$$\begin{cases} {}^C D_t^\alpha y(t,\mu) + \mathcal{A}(t,\mu,D)y(t,\mu) \in \mathcal{G}(t, y(t,\mu)), & t \in [0,T],\ \mu \in \Omega, \\ y(t,\mu) = 0, & t \in [0,T],\ \mu \in \partial\Omega, \\ y(0,z) - \sum_{i=0}^{m} \int_\Omega m(\mu,z) y(a_i,\mu) d\mu = 0, & z \in \Omega. \end{cases} \tag{2}$$

Here, α belongs to $(0, 1)$ and Ω is a subset of $\mathbb{R}^{\mathbb{N}}$, defined as a bounded domain capped by a smooth boundary. We have $0 \le a_0 < a_1 < a_2 < \cdots < a_{m-1} < a_m \le T$ and the function $m(\cdot,\cdot)$ maps $\Omega \times \Omega$ into $\mathbb{R}$. Our multivalued map $\mathcal{G} : [0, T] \times \Omega \to 2^{\mathbb{R}}$ showcases weak upper semicontinuity backed by closed convex values.

The operators $\mathcal{A}(t,\mu,D)$ prove to be closed and densely formed, with their dependence lying upon $t \in [0, T]$ and $\mu \in \overline{\Omega}$. These satisfy the standard ellipticity condition, illustrated when for each $t \in [0, T]$, $\mu \in \overline{\Omega}$, and $\xi \in \mathbb{R}^{\mathbb{N}}$, a constant $k > 0$ is found such that

$$(-1)^m \Re \sum_{|\gamma|=2m} a_\gamma(t,\mu)\xi^\gamma \ge k|\xi|^{2m},$$

where the operator

$$\mathcal{A}(t,\mu,D) = \sum_{|\gamma|\le 2m} a_\gamma(t,\mu) D^\gamma$$

is said to uphold the uniform ellipticity condition in Ω. The coefficients $a_\gamma(t,\mu)$ are smoothly variable functions of μ in $\overline{\Omega}$ for every $t \in [0, T]$ and for some constant $C > 0$ and $0 < \kappa \le 1$ can satisfy

$$\|a_\gamma(t,\mu) - a_\gamma(s,\mu)\| \le C|t-s|^\kappa,$$

with $\mu \in \overline{\Omega}$, $s, t \in [0, T]$. Adopting a method first proposed by Yagi [22], we consider the linear operators family $A(t)$ in $L^p(\Omega)$ given $1 < p < \infty$ and $t \in [0, T]$. These operators correspond to the following domain:

$$\mathcal{D}(A(t)) = H^{2m,p}(\Omega) \cap H_0^{m,p}(\Omega)$$

and can be defined with

$$A(t)y = \mathcal{A}(t,\mu,D)y \quad \text{for } u \in \mathcal{D}(A(t)).$$

Contingent upon t, μ, and the differentiation operator D, the function $\mathcal{A}(t,\mu,D)$ results in system (2) to representatively transform to (1).

Transforming fractional partial differential equations into fractional differential equations in an abstract manner is a crucial tactic, specifically when the densely closed operator $A(t)$ gets simplified to A (not influenced by t) for all $t \ge 0$ [12, 15, 18, 22, 23]. This approach has been found instrumental in resolving scientific and engineering concerns in areas such as anomalous diffusive activities, relaxation phenomena in diverse environments, viscoelastic materials, and the field of finance. For more insights, the reader is encouraged to look into papers [12–14, 18, 19, 23], including the associated references. However, when confronting fractional non-autonomous evolution equations where the operator $A(t)$ varies with time t, the complication levels surge significantly. El-Borai [11] approached the issue by studying the existence and uniqueness of the strong solutions for the Caputo fractional nonautonomous evolution equation. This was followed by an investigation into the L^p-maximal regularity of such non-autonomous evolution equations having zero initial value within Hilbert space [7]. He [13, 14] offered another perspective providing a fresh representation for the classical solution of the non-autonomous fractional evolution equation and exploring its existence, uniqueness, and Hölder regularity. Such studies propel us to extend our research on fractional non-autonomous development dynamics problems, like fractional-order non-autonomous development inclusion issues, and so on.

Growing as an instrumental fragment of nonlinear analysis theory, differential inclusion theory has become indispensable, especially since its inception in the mid-20th century. It was observed that several differential equation models, though efficient in defining systems in engineering, physics, economics, and other fields, fell short in the accurate depiction of real-life dynamic systems. Consequently, the development of differential inclusion theory became necessary to accurately describe unpredictable and discontinuous dynamical processes.Transition to single-valued mappings from multi-valued ones, subject to strong regularity conditions, has facilitated the transformation of solution existence problems in differential inclusions into those for differential equations. An inherent attribute of differential inclusions, the existence of multiple solutions stemming from a specific point, has given rise to new research areas. These include the study of topological properties of solution sets, confirming the significance of differential inclusions as a fascinating field of research [1, 3–5, 9, 12, 25, 26], and others.

Extensive research efforts have been devoted to unraveling the intricate topological properties inherent in the solution sets of differential equations and inclusions. Researchers have delved into examining properties, such as R_δ-sets, acyclicity, and components of connectedness among others [3–6, 9, 17, 20, 26]. The study of topological concepts is foundational for the qualitative examination of differential equations, which focuses on characterizing the nature of solution behaviors independently from precise numerical solutions. Acyclicity, for example, denotes the absence of voids within the solution set. Connectedness, on the other hand, signifies the unification of the solution set into a solitary, uninterrupted whole. Moreover, compactness and the ability to contract (contractibility) are invaluable attributes that benefit the evaluation of existence theorems and the proliferation of solutions' multiplicity. The practical impact of this line of inquiry extends across a plethora of domains, such as physics, engineering, and biology.Within ecology, these topological study outcomes have been pivotal in elucidating population dynamics; similarly, in epidemiology, they offer insights into disease proliferation patterns. The pursuit of understanding the topological structure of solution sets in differential equations remains an endeavor of significant consequence that continues to augment our understanding of various solution phenomena. For an in-depth exploration, the reader is directed to a selection of pertinent literature [3–6, 9, 17, 20, 26].

A wealth of modern publications reveals extensive investigations into the solvability aspect of mild solutions and the topological structure of solution sets pertinent to fractional evolution inclusions (refer to [3, 4, 10, 12, 16, 21, 24, 25] and others). However, it has come to our notice that the fractional nonautonomous evolution equation inclusion research is yet to be initiated. In this manuscript, we establish the existence of a moderate solution for the inclusion system as expressed in (1) and delve into the topological properties of this solution set, specifically by demonstrating it to be a nonempty compact R_δ-set. Compared with fractional autonomous evolution equation, the solution operator for fractional nonautonomous evolution equation, known for its compactness, typically mandates that the closed dense operator families $A(s)$ must generate an analytic semigroup $T_s(t)$ $(t \geq 0)$ for each $s \geq 0$ as seen in [14]. This requirement stands relatively rigid. Acknowledging this issue and drawing inspiration from the autonomous evolution equation, we apply a noncompact measure condition to the nonlinear term. The distinction here is our workflow wherein we apply analytical techniques to validate and present conditions that are less stringent than the non-compact measure condition imposed in [23]. Conversely, an R_δ-set implies an intersection of multiple compact, contractible sets void of emptiness. Nevertheless, asserting a set as contractible often requires the compactness of the solution's operator (see [21, 24, 25]). Taking cue from [16, 24, 25], we also prove the topological properties of the solution set under the condition of non-compactness of the solution operator.

The sections of this article are organized as follows. We recall some definitions, notions, and preliminary facts including multivalued analysis and solution operators in Section 2. Section 3 is assigned for proving the solution set of (1) is a nonempty compact in some assumptions. In Section 4, we consider the topology structure of the solution sets.

2. Preliminaries

This section provides an introduction to the background materials that are relevant throughout the entire paper.

Let $(\mathcal{Y}, \|\cdot\|)$ be a Banach space with the norm $\|\cdot\|$, $\mathcal{L}(\mathcal{Y})$ denote the space of all linear bounded operators on $\mathcal{Y}$ with the norm $\|\cdot\|_{\mathcal{L}}$, and $L^p(I, \mathcal{Y})$ stand for the space of all p-Bochner integrable functions from

$$I = [0, T]$$

to $\mathcal{Y}$ with the norm $\|\cdot\|_p$, and $\mathcal{C}(I, \mathcal{Y})$ be the space of all continuous functions with the sup-norm $\|\cdot\|_c$. We call a bounded integral sequence $\{f_n\} \subset L^p(I, \mathcal{Y})$ semicompact in $L^p([0, T], \mathcal{Y})$, if, for almost every $t \in [0, T]$, the sequence $\{f_n(t)\}_{n=1}^{\infty}$ is relatively compact in $\mathcal{Y}$.

Definition 2.1 ([23]). Let $k : [0, \infty) \to \mathbb{R}$ be a function. The fractional integral of order $0 < \alpha < 1$ is defined by

$$ {}_0I_t^\alpha k(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} k(s)\, ds = g_\alpha(t) * k(t), $$

where $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ denotes a function involving the gamma function.

Definition 2.2 ([18]). Given a function $k : [0, \infty) \to \mathbb{R}$, and for any $\alpha \in (0, 1)$, the α-Caputo fractional derivative of k is delineated as

$$ {}^CD_t^\alpha k(t) = \frac{d}{dt}(g_{1-\alpha}(t) * (k(t) - k(0))). $$

Further, we introduce the Mittag–Leffler function $E_{\alpha,\beta}(z)$ and the Mainardi's Wright-type function $M_\alpha(z)$, $z \in \mathbb{C}$:

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta > 0,\ z \in \mathbb{C} $$

and

$$ M_\alpha(\varrho) = \sum_{n=0}^{\infty} \frac{(-\varrho)^n}{n!\Gamma(1 - \alpha(n+1))}, \quad \alpha \in (0, 1),\ \varrho \in \mathbb{C}. $$

For more details of Riemann–Liouville fractional integral and Caputo fractional derivative, we can refer to [15, 18, 23].

Next, we consider some multivalued results. For the metric space A, Z, define

$$ \begin{aligned} &\mathcal{P}(A) = \{S \subseteq A : S \text{ is nonempty}\}, \quad \mathcal{P}_{\text{cv}}(A) = \{S \in \mathcal{P}(A) : S \text{ is convex}\}, \\ &\mathcal{P}_{\text{cl}}(A) = \{S \in \mathcal{P}(A) : S \text{ is closed}\}, \quad \mathcal{P}_{\text{cp}}(A) = \{S \in \mathcal{P}(A) : S \text{ is compact}\}, \\ &\mathcal{P}_{\text{cl,cv}}(A) = \mathcal{P}_{\text{cl}}(A) \cap \mathcal{P}_{\text{cv}}(A). \end{aligned} $$

A multivalued map denoted by $\eta(\cdot)$ establishes a correspondence between a metric space A and another metric space Z, assigning to each member $y \in A$ a unique, nonempty subset $\eta(y) \subseteq Z$ which is referred to as the image of y. The set symbolized as $\text{Gra}(\eta) = \{(a, b) : a \in A, b \in \eta(a)\} \subseteq A \times Z$ is the graphical representation of $\eta(\cdot)$, whereas $\eta(B) = \bigcup_{a\in B} \eta(a)$ defines the image of the subset B in terms of η. The set

$$ \eta^{-1}(B) = \{a \in A : \eta(a) \cap B \neq \emptyset\} \quad \text{for } B \subset A $$

represents the preimage of the subset B with respect to η.

Definition 2.3. A multivalued map $\eta : A \to \mathcal{P}(Z)$ is defined as the one that

(i) is closed when the graph $\text{Gra}(\eta)$, a closed subset from the Cartesian product $A \times Z$, satisfies $\text{Gra}(\eta) \subseteq A \times Z$;

(ii) is quasicompact when, for every compact subset $B \subset A$, the closure of η image, $\overline{\eta}(B)$, is also compact;

(iii) is upper semi-continuous (u.s.c) when, for each closed subset $B \in \mathcal{P}(A)$, the preimage set $\eta^{-1}(B)$ is closed;

(iv) is weakly upper semi-continuous (w.u.s.c) when, for every weakly closed subset $B \in \mathcal{P}(A)$, the preimage $\eta^{-1}(B)$ remains closed.

The following results can be found in [9, 17].

Lemma 2.4. *Consider a multivalued map $\eta : D \to \mathcal{P}(Y)$.*

(i) *If η is characterized as a closed and quasicompact multivalued map, then η manifests as upper semi-continuous (u.s.c).*

(ii) *If η has weakly compact and convex values, then η corresponds to weakly upper semi-continuous (w.u.s.c) if and only if, for a sequence $x_n \subset D$ converging to $x_0 \in D$ with $y_n \in \eta(x_n)$, there exists a subsequence such that $y_n \rightharpoonup y_0 \in \eta(x_0)$.*

The following exposition concerns the set R_δ and contractible subsets. For a comprehensive explanation, we refer the readers to [10, 17]. Given a metric space Y and a nonempty subset $B \subset Y$, we define B to be contractible if there exist a continuous map $\beta : B \times [0,1] \to B$ and a certain point $z_0 \in B$. This implies that $\beta(z,0) = z_0$, and $\beta(z,1) = z$ for all z belonging to B.

Definition 2.5. A subset B of a metric space is called an R_δ set if there exist a decreasing sequence $\{B_n\}$ of compact and contractible sets such that $B = \bigcap_{n=1}^{\infty} B_n$.

The Kuratowski measure of noncompactness $\mu(\cdot)$ (MNC μ) defined on bounded set S of Banach space $\mathcal{Y}$ is

$$\mu(S) = \inf\left\{\delta > 0 : S = \bigcup_{k=1}^{n} S_k,\ \mathrm{diam}(S_k) \le \delta,\ k = 1, 2, \dots, n\right\}.$$

Lemma 2.6. *Several properties regarding the MNC, denoted as $\mu(\cdot)$, are given as follows.*

(i) *Given that $D \subset \mathcal{C}([0,T],\mathcal{Y})$ is bounded and equicontinuous, $\mu(D(\cdot))$ exhibits continuity over the span $[0,T]$. Moreover, the equality*

$$\mu(D) = \max_{t\in[0,T]} \mu(D(t))$$

holds.

(ii) *Considering a bounded set $D \subset \mathcal{Y}$, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset D$ satisfying the condition $\mu(D) \le 2\mu(\{u_n\}_{n=1}^{\infty}) + \eta$ for any $\eta > 0$.*

(iii) *Assuming a countable set*

$$D = \{u_n\}_{n=1}^{\infty} \subset \mathcal{C}([0,T],\mathcal{Y}),$$

where there exists a function $m \in L^1([0,T],\mathbb{R}^+)$ such that $\|u_n(t)\| \le m(t)$ for almost every $t \in [0,T]$, $\mu(D(t)) = \mu(\{u_n(t)\}_{n=1}^{\infty})$ is Lebesgue integrable on $[0,T]$. Further, the inequality $\mu(\{\int_0^T u_n(t)dt : n \in N\}) \le 2\int_0^T \mu(D(t))dt$ is satisfied.

Remark 2.7. It is commonly understood that the MNC, denoted as μ, possesses the properties of monotonicity, nonsingularity, and regularity. However, in $\mathcal{C}(J,\mathcal{Y})$, the regularity of MNC $\mu(\cdot)$ does not hold. Indeed, for any $B \subset \mathcal{C}(J,\mathcal{Y})$, given that $\mu(B) = \sup_{t\in[0,T]} \mu(B(t))$, if $\mu(B) = 0$, then $\mu(B(t)) = 0$ for every $t \in [0,T]$. As a result, $B(t)$ is relatively compact in $\mathcal{Y}$ for all $t \in [0,T]$. Regardless, the convergence of any subsequence $\{x_n\} \subset B$ is independent of $t \in [0,T]$, implying a lack of regularity for the MNC μ in $\mathcal{C}([0,T],\mathcal{Y})$ when defined over the Banach space $\mathcal{Y}$.

Based on the above facts, we introduce the modulus of equicontinuous to define the MNC ϖ on $\mathcal{C}([0,T],\mathcal{Y})$.

For a bounded set $\mathcal{B} \subset \mathcal{C}([0,T],\mathcal{Y})$, we define

$$\varpi(\mathcal{B}) = \max_{\mathcal{B}\in\Lambda(\mathcal{B})} \Big(\sup_{t\in[0,T]} \mu(\mathcal{B}(t)), \mathrm{mod}_C(\mathcal{B})\Big),$$

where

$$\Lambda(\mathcal{B}) = \{\mathcal{D} \subset \mathcal{B} : \mathcal{D} \text{ is denumerable subset of } \mathcal{B}\},$$

and $\mathrm{mod}_C(\mathcal{B})$ represents the functions $\mathcal{B}$'s equicontinuous modulus in the following form:

$$\mathrm{mod}_C(\mathcal{B}) = \lim_{\delta\to 0} \sup_{y\in\mathcal{B}} \max_{|t_2-t_1|<\delta} \|y(t_2) - y(t_1)\|, \quad t_1, t_2 \in [0,T].$$

On the other hand, we also can define MNC π on $\mathcal{C}([0,T],\mathcal{Y})$; that is,

$$\pi(\mathcal{B}) = \max_{\mathcal{B}\in\Lambda(\mathcal{B})} \Big(\sup_{t\in[0,T]} e^{-\mathcal{L}t}\mu(\mathcal{B}(t)), \mathrm{mod}_C(\mathcal{B})\Big),$$

where $\mathcal{L}$ is a given constant. The MCN ϖ and MCN π each have monotonicity, nonsingularity, and regularity. For more details on MNC, one can refer to [8, 9, 17].

Lemma 2.8 ([17]). *Assume that $\mathcal{Y}$ represents a Banach space and Ψ denotes an operator $\Psi : L^p([0,T],\mathcal{Y}) \to \mathcal{C}([0,T],\mathcal{Y})$, satisfying the following prerequisites.*

(S_1) *Existence of a constant $c_0 > 0$ such that*

$$\|\Psi(f)(\cdot) - \Psi(g)(\cdot)\|_c \le c_0\|f(\cdot) - g(\cdot)\|_p$$

holds for every pair of functions $f, g \in L^p([0,T],\mathcal{Y})$ with $p > 1$.

(S_2) *Suppose that $\{f_n\}_{n=1}^{\infty} \subset L^p([0,T],\mathcal{Y})$ is a sequence of functions and*

$$\{f_n(t)\}_{n=1}^{\infty} \subset S$$

holds for every n, where $S \subset \mathcal{Y}$ is compact. Then, if $f_n \rightharpoonup f_0$ in $L^p([0,T],\mathcal{Y})$, it leads to the convergence $\Psi(f_n) \to \Psi(f_0)$ in $\mathcal{C}([0,T],\mathcal{Y})$.

Under such premise, we have the following.

(i) *Suppose that a sequence of integrably bounded functions* $\{f_n\}_{n=1}^{\infty} \subset L^p([0,T],\mathcal{Y})$ *exists such that* $\mu(f_n(t)) \leq q_1(t)$ *is valid for almost every* $t \in [0,T]$*, with* $q_1 \in L^1([0,T],\mathbb{R}^+)$*. Then, the following inequality holds:*

$$\mu(\{\Psi(f_n)(t)\}_{n=1}^{\infty}) \leq c_0 \int_0^t q_1(s)\mathrm{d}s.$$

(ii) *For any given semicompact sequence* $\{f_n\} \subset L^p([0,T],\mathcal{Y})$*, the sequence* $\{\Psi(f_n)\}$ *results in relative compactness in* $\mathcal{C}([0,T],\mathcal{Y})$*. Furthermore, if* $f_n \rightharpoonup f_0$*, it concludes that* $\Psi(f_n) \to \Psi(f_0)$*.*

In this work, we examine a linear operator that satisfies the Acquistapace–Terreni conditions. For additional information regarding the Acquistapace–Terreni conditions, the reader is referred to [13, 14, 22]. Subsequently, we construct the solution operator detailed in [13, 14].

We analyze the following operators:

$$\phi_s(t) = \int_0^\infty M_\alpha(\theta) T_s(t^\alpha \theta) d\theta, \quad \psi_s(t) = \int_0^\infty \alpha\theta t^{\alpha-1} M_\alpha(\theta) T_s(t^\alpha \theta) d\theta,$$

where $T_s(t)$ denotes the analytic semigroup generated by $A(s)$. We then define the operators:

$$\widetilde{Q}(t,s) = -[A(t) - A(s)]\phi_s(t-s), \quad \widetilde{R}(t,s) = -[A(t) - A(s)]\psi_s(t-s),$$

which are continuous in the operator uniform topology for all $0 \leq s < t \leq T$. According to [22], the subsequent Volterra-type integral equations

$$Q(t,s) = \widetilde{Q}(t,s) + \int_s^t \widetilde{R}(t,\tau) Q(\tau,s) d\tau,$$
$$R(t,s) = \widetilde{R}(t,s) + \int_s^t \widetilde{R}(t,\tau) R(\tau,s) d\tau$$

guarantee that $Q(t,s)$ and $R(t,s)$ are the unique solutions of the respective equations. Furthermore, $Q(t,s)$ and $R(t,s)$ are continuous in the uniform operator topology on $\mathcal{L}(\mathcal{Y})$ for $t > s$.

Definition 2.9. Let $y : [0,T] \to \mathcal{Y}$ be a continuous function. The function y is a mild solution to (1) if it satisfies the initial condition $y(0) + g(y) = y_0$, and there exists a function $f \in L^p([0,T],\mathcal{Y})$ such that, for almost every $t \in [0,T]$, we have $f(t) \in \mathcal{G}(t, y(t))$. Moreover, for each $t \in [0,T]$, $y(t)$ is given by the integral equation

$$y(t) = S_\alpha(t,0)(y_0 - g(y)) + \int_0^t P_\alpha(t,s) f(s) ds,$$

where the operators $S_\alpha(t,s)$ and $P_\alpha(t,s)$ are defined as

$$S_\alpha(t,s) = \phi_s(t-s) + \int_s^t \psi_\tau(t-\tau)Q(\tau,s)d\tau,$$

$$P_\alpha(t,s) = \psi_s(t-s) + \int_s^t \psi_\tau(t-\tau)R(\tau,s)d\tau.$$

Lemma 2.10. *Let $\varepsilon > 0$ and $t, s \in [0,T]$ be given. The operator-valued functions $S_\alpha(t,s)$ and $P_\alpha(t,s)$ exhibit strong continuity on $\mathcal{Y}$ for $t \in [s+\varepsilon, T]$. Furthermore, there exists a constant M dependent on α fulfilling the following conditions:*

$$\|S_\alpha(t,s)\|_{\mathcal{L}} \le M \qquad \text{for all } t \in [s,T],$$

$$\|P_\alpha(t,s)\|_{\mathcal{L}} \le M(t-s)^{\alpha-1} \quad \text{for all } t \in (s,T].$$

Definition 2.11. Let $\mathcal{B}$ be a subset of a Banach space $\mathcal{Y}$. The multimap $\eta : \mathcal{B} \to \mathcal{P}(\mathcal{Y})$ is said to be μ-condensing, if for any bounded set $\mathcal{D} \subset \mathcal{B}$, which is not relatively compact, the condition $\mu(\eta(\mathcal{D})) < \mu(\mathcal{D})$ holds.

The following fixed-point theorem is instrumental in establishing the existence of a mild solution, as detailed subsequently.

Theorem 2.12 ([17]). *Suppose that Ω is a closed, bounded, and convex subset of a Banach space $\mathcal{Y}$. Let $\eta : \Omega \to \mathcal{P}_{\mathrm{cp,cv}}(\Omega)$ be a multivalued map that is upper semicontinuous and μ-condensing. Then, the set of fixed points denoted by $\mathrm{Fix}(\eta) = \{y \in \Omega : y \in \eta(y)\}$ is nonempty and compact.*

3. Existence results

First, we show the following lemma.

Lemma 3.1. *Let $g \in L^1([0,a],\mathbb{R}^+)$ be such that ${}_0I_t^\alpha g(t) \in C([0,a],\mathbb{R}^+)$ and $\lim_{t\to 0^+} {}_0I_t^\alpha g(t) = 0$. Then, there exists a continuous function $\sigma(\cdot)$ such that, for any $t \in [0,a]$ and any $\varepsilon > 0$, it holds that*

$$\int_0^t (t-\theta)^{\alpha-1}|g(\theta) - \sigma(\theta)|d\theta < \varepsilon.$$

Proof. For any $t_0 \in [0,a]$, noting that the space $C[0,t_0]$ is dense in $L^1[0,t_0]$ and by the Weierstrass approximation theorem, one can choose a polynomial function $w_{t_0}(\cdot)$ such that

$$\int_0^{t_0} |(t_0-\theta)^{\alpha-1}g(\theta) - w_{t_0}(\theta)|d\theta < \frac{\varepsilon}{3}.$$

Utilizing the property of uniform continuity for continuous functions on compact sets, we can find a $\delta_{t_0} > 0$ such that for any $t \in (t_0 - \delta_{t_0}, t_0 + \delta_{t_0})$ we have

$$\int_0^t |(t-\theta)^{\alpha-1}g(\theta) - w_t(\theta)|d\theta < \frac{\varepsilon}{3}.$$

The collection $\{(t-\delta_t, t+\delta_t) : t \in [0,a]\}$ forms an open cover of $[0,a]$. By the Heine–Borel theorem, there exists a finite subcover, yielding an $m \in \mathbb{N}^+$ such that

$$[0,a] \subset \bigcup_{i=0}^{m} (t_i - \delta_{t_i}, t_i + \delta_{t_i}).$$

Applying the Cantor intersection theorem, we find a δ_0 which satisfies

$$|(t'-\theta)^{1-\alpha} w_{t'}(\theta) - (t''-\theta)^{1-\alpha} w_{t''}(\theta)| < \frac{\varepsilon\alpha}{3\rho a^\alpha} \tag{3}$$

for any $|t'-t''| < \delta_0$, where ρ is a constant that will be determined later.

Let $\sigma(\theta) = \sup_{\theta \le t \le a} (t-\theta)^{1-\alpha} w_t(\theta)$. Using the previously derived results, we can find a t' such that

$$|\sigma(\theta) - (t'-\theta)^{1-\alpha} w_{t'}(\theta)| < \frac{\varepsilon\alpha}{3a^\alpha}.$$

Since $[0,a]$ is covered by the intervals $(t_i - \delta_{t_i}, t_i + \delta_{t_i})$, we can find a t_j such that $t' \in (t_j - \delta_{t_j}, t_j + \delta_{t_j})$. Thus, we have

$$\int_0^t (t-\theta)^{\alpha-1} |g(\theta) - (t-\theta)^{1-\alpha} w_t(\theta)| d\theta < \frac{\varepsilon}{3},$$
$$\int_0^t (t-\theta)^{\alpha-1} |\sigma(\theta) - (t'-\theta)^{1-\alpha} w_{t'}(\theta)| d\theta < \frac{\varepsilon}{3}.$$

Let $\delta = \min\{\delta_0, \delta_{t_0}, \ldots, \delta_{t_m}\}$. For any $t \in (t_i - \delta_{t_i}, t_i + \delta_{t_i})$ and $t' \in (t_j - \delta_{t_j}, t_j + \delta_{t_j})$, we can select $\vartheta \in \mathbb{N}^+$ such that $|t-t'| < \vartheta\delta$ with $\vartheta = \lceil \frac{t-t'}{\delta} \rceil \le \lceil \frac{t_m - t_0 + \delta_{t_m} + \delta_{t_0}}{\delta} \rceil$. Taking $\rho = \lceil \frac{t_m - t_0 + \delta_{t_m} + \delta_{t_0}}{\delta} \rceil$ ensures that

$$\int_0^t (t-\theta)^{\alpha-1} |(t-\theta)^{1-\alpha} w_t(\theta) - (t'-\theta)^{1-\alpha} w_{t'}(\theta)| d\theta < \frac{\varepsilon}{3}. \tag{4}$$

By combining (3) and (4), we obtain

$$\int_0^t (t-\theta)^{\alpha-1} |g(\theta) - \sigma(\theta)| d\theta \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \rho \frac{a^\alpha}{\alpha} \frac{\varepsilon\alpha}{3\rho a^\alpha} = \varepsilon.$$

This completes the proof. ■

We now turn our attention to an analysis involving a multivalued nonlinearity which is formally described by the operator $\mathscr{G} : [0,T] \times \mathcal{Y} \to \mathscr{P}_{\mathrm{cl,cv}}(\mathcal{Y})$. This operator is subject to the following conditions:

- (H_0) $T_t(\tau)(\tau > 0)$ is equicontinuous; i.e., $T_t(\tau)$ is continuous in the uniform operator topology for $\tau > 0$.
- (H_1) For a given $y \in \mathcal{Y}$, consider the collection

  $$\{f(\cdot) : f(t) \in \mathscr{G}(t, y(t)) \text{ for almost every } t \in [0,T]\}.$$

  This set is nonempty and includes a measurable function $f : [0,T] \to \mathcal{Y}$.

(H_2) The multifunction $\mathcal{G}(t, \cdot)$ exhibits weak upper semicontinuity for almost every $t \in [0, T]$.

(H_3) There exist a constant $p > \frac{1}{\alpha}$ and a function $\alpha(\cdot) \in L^p([0, T], \mathbb{R}^+)$ that fulfil the inequality

$$\|\mathcal{G}(s, y)\| = \sup\{\|f(s)\| : f \in \mathcal{G}(s, y)\} \le \alpha(s)\rho(\|y\|),$$

where $\rho(\cdot) : [0, \infty) \to [0, \infty)$ is a monotonically increasing function satisfying

$$\liminf_{n\to\infty} \frac{\rho(n)}{n} = 0.$$

(H_4) A function $k(\cdot) \in L^1([0, T], \mathbb{R}^+)$ exists such that ${}_0I_t^\alpha k(t) \in \mathcal{C}([0, T], \mathbb{R}^+)$ and $\lim_{t\to0^+} {}_0I_t^\alpha k(t) = 0$. For any bounded subset $\mathcal{B} \subset \mathcal{C}([0, T], \mathcal{Y})$, the inequality $\mu(\mathcal{G}(t, \mathcal{B})) \le k(t)\mu(\mathcal{B})$ holds true.

(H_5) The mapping $g : \mathcal{C}([0, T], \mathcal{Y}) \to \mathcal{Y}$ is compact and continuous. Moreover, there exist constants N_{g1}, N_{g2}, δ such that

$$\|g(y)\| \le N_{g1}\|y\|_{\mathcal{C}}^{\delta} + N_{g2}$$

for $y \in \mathcal{C}([0, T], \mathcal{Y})$, where $0 < \delta < 1$.

For $y \in \mathcal{C}([0, T], \mathcal{Y})$, and almost every $t \in [0, T]$, the set of integrable selections from the multifunction $\mathcal{G}$ with respect to $y \in \mathcal{C}([0, T], \mathcal{Y})$ is defined by

$$\mathcal{S}_{\mathcal{G}}(y) = \{f \in L^p([0, T], \mathcal{Y}) : f(t) \in \mathcal{G}(t, y(t)) \text{ for a.e. } t \in [0, T]\}.$$

Proposition 3.2. *The operator $\mathcal{S}_{\mathcal{G}}: \mathcal{C}([0, T], \mathcal{Y}) \multimap L^p([0, T], \mathcal{Y})$ has weakly upper semicontinuous mappings with convex, weakly compact, and nonempty value sets.*

Proof. Based on assumption (H_3), for every $r > 0$ and for each $y \in \mathcal{Y}$ with $\|y\| \le r$, we can select a function $\mu_r \in L^p([0, T], \mathcal{Y})$ such that, for almost every $s \in [0, T]$, we have

$$\|\mathcal{G}(s, y)\| = \sup\{\|f\|: f \in \mathcal{G}(s, y(s))\} \le \mu_r(s).$$

Indeed, we may take $\mu_r(s) = \alpha(s)\rho(r)$. Employing the technique used in [25], we demonstrate that the set $S_{\mathcal{G}}(y)$ is nonempty for any y in $\mathcal{C}([0, T], \mathcal{Y})$. For any $f_1, f_2 \in \mathcal{S}_{\mathcal{G}}(y)$ and $0 \le \theta \le 1$, it is straightforward to verify $\theta f_1 + (1-\theta) f_2 \in \mathcal{S}_{\mathcal{G}}(y)$ because, for almost every $s \in [0, T]$, we have $f_1(s) \in \mathcal{G}(s, y(s))$ and $f_2(s) \in \mathcal{G}(s, y(s))$, given $\mathcal{G}: [0, T] \times \mathcal{Y} \to \mathcal{P}_{\mathrm{cl,cv}}(\mathcal{Y})$.

We denote

$$\sup_{s\in[0,T]} \|y(s)\| = r;$$

thereby, $\mathcal{S}_{\mathcal{G}}(y)$ is a bounded closed set in $L^p([0, T], \mathcal{Y})$ due to $\|\mathcal{G}(s, y)\| \le \alpha(s)\rho(r)$. Thus, the set $\mathcal{S}_{\mathcal{G}}(y)$ is weakly compact as a result of the Banach–Alaoglu theorem, given $p > 1$. Next, we establish that $\mathcal{S}_{\mathcal{G}}$ is weakly upper semicontinuous. Suppose that we have sequences $f_n \in \mathcal{S}_{\mathcal{G}}(y_n)$ and $f \in \mathcal{S}_{\mathcal{G}}(y)$ such that $f_n \rightharpoonup f$, where $f_n(s) \in \mathcal{G}(s, y_n(s))$ and $f(s) \in \mathcal{G}(s, y(s))$ for almost every $s \in [0, T]$. We assume that $y_n \to y$. By way of contradiction, suppose that there exists $s_0 \in [0, T]$ such that $y_n(s_0) \nrightarrow y(s_0)$. That is, we assume that there exists $\varepsilon_0 > 0$ and $n_0 \in \mathbb{N}_+$ such that $\|y_n(s_0) - y(s_0)\| \geq \varepsilon_0$ for all $n > n_0$. By the persistence property of continuous functions, there exists δ_{s_0} with a positive measure (i.e., $mes(\delta_{s_0}) > 0$) such that

$$\|y_n(s_0) - y(s_0)\| \geq \varepsilon_0/2$$

for any fixed $s \in \delta_{s_0}$. Given $f_n \rightharpoonup f$, there exists $s^* \in \delta_{s_0}$ such that $f_n(s^*) \rightharpoonup f(s^*)$, but $y_n(s^*) \nrightarrow y(s^*)$, which contradicts assumption (H_2).

The proof is thus complete. ∎

Then, we have the following theorem.

Theorem 3.3. *Assume that conditions* (H_0)–(H_5) *are satisfied. Then, the solution set of inclusion* (1) *is a nonempty compact subset of* $C([0, T], \mathcal{Y})$. *In particular, this conclusion holds for the function* $g = 0$ *in* (1).

Proof. For a fixed constant $0 < \varepsilon_0 < 1$, we select an approximate constant $T^* > 0$ such that

$$\varepsilon_0 < \frac{1 - \frac{2MK^*(T^*)^\alpha}{\alpha}}{2M},$$

where $K^* = \max_{\tau \in [0,T]} |\sigma(\tau)|$ and $\sigma(\cdot)$ is a continuous function satisfying Lemma 3.1. For example, by choosing

$$\varepsilon_0 = \frac{1}{2M + 1} \quad \text{and} \quad 0 < T^* < \Big(\frac{\alpha}{2M(2M + 1)K^*}\Big)^{\frac{1}{\alpha}},$$

the inequality holds.

For any $y \in C([0, T^*], \mathcal{Y})$, we define the solution multi-operator

$$\mathcal{K} : C([0, T^*], \mathcal{Y}) \to \mathcal{P}(C([0, T^*], \mathcal{Y}))$$

as follows:

$$\mathcal{K}y(t) = S_\alpha(t, 0)(y_0 - g(y)) + (\Psi \circ S_{\mathcal{G}}(y))(t), \tag{5}$$

where

$$\Psi \circ S_{\mathcal{G}}(y) = \Big\{\Psi(f) : f \in S_{\mathcal{G}}(y), \Psi(f)(t) = \int_0^t P_\alpha(t, s) f(s) ds\Big\}.$$

It is straightforward to verify that $\Psi(f)(\cdot) \in C([0, T^*], \mathcal{Y})$ for any $f \in S_{\mathcal{G}}(y)$, ensuring that multi-operator $\mathcal{K}$ is well defined. Mild solutions to equation (1) are the fixed points of multi-operator $\mathcal{K}$. We employ the following steps to prove the existence of these fixed points.

For convenience, we denote $J = [0, T^*]$ and proceed with the first step.

Step 1. We show that there exists a set $\mathcal{M}$ which is bounded, convex, and closed such that $\mathcal{K}(\mathcal{M}) \subset \mathcal{M}$.

We demonstrate the existence of a constant $r > 0$ such that $\mathcal{K}(M_0) \subset \mathcal{M}_0$, where $\mathcal{M}_0 = \{y \in C(J, \mathcal{Y}) : \|y\|_C \leq r\}$. Assuming the contrary, for any $r > 0$ and $z \in \mathcal{K}(y)$ but $z \notin \mathcal{M}_0$, we obtain

$$z(t) = S_\alpha(t, 0)(y_0 - g(y)) + \int_0^t P_\alpha(t, s) f(s) ds, \quad t \in J,$$

with some $t_0 \in J$ satisfying $\|z(t_0)\| \geq r$. Subsequently, we have

$$\begin{aligned} r \leq \|z(t_0)\| &\leq M\|y_0\| + MN_{g_1} r^\delta + MN_{g_2} + M \int_0^t (t-s)^{\alpha-1} \alpha(s) ds \rho(r) \\ &\leq M\|y_0\| + MN_{g_1} r^\delta + MN_{g_2} + M \left(\frac{p-1}{\alpha p - 1} \right)^{\frac{p-1}{p}} T^{\alpha - \frac{1}{p}} \|\alpha\|_p \rho(r), \end{aligned}$$

which leads to a contradiction. Thus, there must exist a constant $r > 0$ such that $\mathcal{K}(\mathcal{M}_0) \subset \mathcal{M}_0$.

Define $\mathcal{M}_1 = \overline{\text{co}}(\mathcal{K}(\mathcal{M}_0))$. It can be shown that $\mathcal{M}_1 \subset C(J, \mathcal{Y})$ is a bounded, closed, and convex set. Iterating this approach, we construct a descending sequence of sets $\mathcal{M}_1 \supset \mathcal{M}_2 \supset \cdots \supset \mathcal{M}_n$. Define $\mathcal{M} = \bigcap_{n=1}^\infty \mathcal{M}_n$ that is closed, convex, bounded, and nonempty, satisfying

$$\mathcal{K}(\mathcal{M}) \subset \bigcap_{k=1}^\infty \mathcal{K}(\mathcal{M}_k) \subset \bigcap_{k=1}^\infty \mathcal{M}_k = \mathcal{M}.$$

Step 2. We show that operator $\mathcal{K}$ is ϖ-condensing.

Suppose $\mathcal{B} \subset \mathcal{M}$ with $\varpi(\mathcal{B}) \leq \varpi(\mathcal{K}(\mathcal{B}))$. If not, this would imply for any $B \subset M$, $\varpi(\mathcal{K}(\mathcal{B})) < \varpi(\mathcal{B})$; i.e., $\mathcal{K}$ is ϖ-condensing. Otherwise, consider a sequence $\{z_n\}_{n=1}^\infty \subset \mathcal{K}(\mathcal{B})$ such that

$$\varpi(\{z_n\}_{n=1}^\infty) = \max(\sup_{t \in J} \mu(\{z_n(t)\}_{n=1}^\infty), \text{mod}_C(\{z_n\}_{n=1}^\infty)),$$

where, for each n,

$$z_n(t) = S_\alpha(t, 0)(y_0 - g(y_n)) + \int_0^t P_\alpha(t, s) f_n(s) ds, \quad f_n \in S_{\mathcal{g}}(y_n),$$

and $\{y_n\} \subset B$. From assumption (H_5) and utilizing the fact that h is compact and $S_\alpha(t, 0)$ is strongly continuous for $t \geq 0$, we have

$$\mu(\{S_\alpha(t, 0)(y_0 - g(y_n))\}_{n=1}^\infty) = 0.$$

We proceed to show that

$$\mu\left(\left\{ \int_0^t P_\alpha(t, s) f_n(s) ds \right\}_{n=1}^\infty \right) = 0.$$

Given $0 < \varepsilon_0 < 1$, we find that

$$\begin{aligned}
\mu(\{z_n\}_{n=1}^{\infty}) &\leq 2M \sup_{t\in\mathcal{J}} \int_0^t (t-s)^{\alpha-1} k(s)\mu(\{y_n(s)\}_{n=1}^{\infty})ds \\
&\leq 2M \sup_{t\in\mathcal{J}} \int_0^t (t-s)^{\alpha-1} k(s) ds \mu(\{y_n\}_{n=1}^{\infty}) \\
&\leq 2M \sup_{t\in\mathcal{J}} \left(\int_0^t (t-s)^{\alpha-1} |k(s)-\sigma(s)| ds + \int_0^t (t-s)^{\alpha-1}\sigma(s) \right) \mu(\{y_n\}_{n=1}^{\infty}) \\
&\leq 2M \left(\varepsilon_0 + \frac{K^* T^{*\alpha}}{\alpha} \right) \mu(\{y_n\}_{n=1}^{\infty}).
\end{aligned}$$

Since $2M(\varepsilon_0 + \frac{K^*(T^*)^\alpha}{\alpha}) < 1$ and $\mu(\{z_n\}_{n=1}^{\infty}) \geq \mu(\{y_n\}_{n=1}^{\infty})$, it follows that $\mu(\{z_n\}_{n=1}^{\infty}) = 0$.

Next, we establish that $\text{mod}_C(\{z_n\}_{n=1}^{\infty}) = 0$, which confirms that the set B possesses the property of equicontinuity. For any $0 \leq t_1 < t_2 \leq T$ and $n \geq 1$, we have

$$\begin{aligned}
\|z_n(t_2) - z_n(t_1)\| &\leq \|(S_\alpha(t_2,0) - S_\alpha(t_1,0))(y_0 - g(y_n))\| + \int_{t_1}^{t_2} \|P_\alpha(t_2,\tau) f_n(\tau)\| d\tau \\
&\quad + \int_0^{t_1} \|(P_\alpha(t_2,\tau) - P_\alpha(t_1,\tau)) f_n(\tau)\| d\tau \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}$$

As t_2 approaches t_1, it is clear that I_1 tends to zero. Also, as $t_2 \to t_1$, we get

$$\begin{aligned}
\int_{t_1}^{t_2} \|P_\alpha(t,s)\| \alpha(s)\rho(r) ds &\leq M \int_{t_1}^{t_2} (t_2 - t_1)^{\alpha-1} \alpha(s) ds \rho(r) \\
&\leq M\rho(r) \left(\frac{p-1}{\alpha p - 1} \right)^{\frac{p-1}{p}} \|\alpha\|_p (t_2 - t_1)^{\alpha - \frac{1}{p}} \to 0.
\end{aligned}$$

For the term I_3, by choosing a sufficiently small $\delta > 0$, we get

$$\begin{aligned}
I_3 &= \int_0^{t_1} \|(P_\alpha(t_2,\tau) - P_\alpha(t_1,\tau)) f_n(\tau)\| d\tau \\
&\leq \sup_{\tau\in[0,t_1-\delta]} \|P_\alpha(t_2,\tau) - P_\alpha(t_1,\tau)\|_{\mathcal{L}} \int_0^{t_1-\delta} \rho(r)\alpha(\tau) d\tau \\
&\quad + 2M \int_{t_1-\delta}^{t_1} (t_1-\tau)^{\alpha-1} \alpha(\tau) d\tau \rho(r) \\
&\leq \rho(r)(t_1-\delta)^{\frac{p-1}{p}} \|\alpha\|_p \sup_{\tau\in[0,t_1-\delta]} \|P_\alpha(t_2,\tau) - P_\alpha(t_1,\tau)\|_{\mathcal{L}} \\
&\quad + 2M\rho(r) \left(\frac{p-1}{\alpha p - 1} \right)^{\frac{p-1}{p}} \|\alpha\|_p \delta^{\alpha - \frac{1}{p}} \to 0, \quad \text{as } t_2 \to t_1 \text{ and } \delta \to 0.
\end{aligned}$$

Consequently, we conclude that $\text{mod}_C(\{z_n\}_{n=1}^{\infty}) = 0$.

Step 3. For any $y \in \mathcal{M}$, the set $\mathcal{K}(y)$ is compact and convex.

Indeed, take any $z_1, z_2 \in \mathcal{K}(y)$ for each given $y \in \mathcal{M}$. Then, there exist $f_1, f_2 \in S_{\mathcal{G}}(y)$ such that, for $i \in \{1, 2\}$,

$$z_i(t) = S_\alpha(t, 0)(y_0 - g(y)) + \int_0^t P_\alpha(t, \tau) f_i(\tau) d\tau.$$

For any $\theta \in [0, 1]$ and $t \in J$,

$$\theta z_1(t) + (1 - \theta) z_2(t) = S_\alpha(t, 0)(y_0 - g(y)) + \int_0^t P_\alpha(t, \tau)(\theta f_1 + (1 - \theta) f_2)(\tau) d\tau.$$

Applying Proposition 3.2, we deduce that $\theta f_1 + (1 - \theta) f_2 \in S_{\mathcal{G}}(y)$, and therefore,

$$\theta z_1 + (1 - \theta) z_2 \in \mathcal{K}(y).$$

Next, we demonstrate that, given any sequence $\{f_n\}$ in $S_{\mathcal{G}}(y)$ with $y \in \mathcal{M}$, and under the assumptions (H_3) and (H_4), the following inequalities hold:

$$\begin{aligned} \|f_n\|_p &\le \rho(r)\|\alpha(\cdot)\|_p, \\ \mu(\{f_n(t)\}) &\le k(t)\mu(y(t)) = 0 \quad \text{for a.e. } t \in [0, T^*]. \end{aligned}$$

It follows that $\{f_n\} \subset L^p(J, \mathcal{Y})$ is semicompact, and the condition (S_1) is obvious for the operator

$$\begin{aligned} \Psi : L^p([0, T^*], \mathcal{Y}) &\to C([0, T^*], \mathcal{Y}), \\ f &\mapsto \Psi(f), \end{aligned}$$

where

$$\Psi(f)(t) = \int_0^t P_\alpha(t, s) f(s) ds, \quad c_0 = M\left(\frac{p-1}{\alpha p - 1}\right)^{\frac{p-1}{p}} (T^*)^{\frac{\alpha p - 1}{p}},$$

and c_0 is the constant in condition (S_1). Then, $f_n \rightharpoonup f_0$ implies $\Psi(f_n) \to \Psi(f_0)$. We also show that the condition (S_2) is satisfied. Indeed, let a sequence $\{f_n\} \subset L^p([0, T^*], \mathcal{Y})$ and satisfy that $\{f_n(t)\} \subset S$ for a.e. $t \in [0, T^*]$, where $S \subset \mathcal{Y}$ is compact. Define

$$\{f_n(t) : t \in [0, T^*]\} \subset S \cup A,$$

where A satisfies $\mu(\{t : f_n(t) \in A, n \ge 1\}) = 0$. It follows that

$$\begin{aligned} \mu(\{\Psi(f_n(t))\}) &\le \int_0^t (t-s)^{\alpha-1} \mu(\{f_n(s)\}) ds \\ &= \int_{J_1} (t-s)^{\alpha-1} \mu(\{f_n(s)\}) ds + \int_{J_2} (t-s)^{\alpha-1} \mu(\{f_n(s)\}) ds \\ &\le \int_{J_1} (t-s)^{\alpha-1} \mu(\{f_n(s)\}) ds + \int_{J_2} (t-s)^{\alpha-1} \mu(A) ds = 0, \end{aligned}$$

where we split the integral as the sum over sets J_1 and J_2 as defined before. Therefore, $\{\Psi(f_n)\}$ is relatively compact in $\mathcal{C}([0,T],\mathcal{Y})$. Since $\{f_n\}$ is bounded in $L^p([0,T])$, we can extract a subsequence, still denoted by $\{f_n\}$, such that $f_n \rightharpoonup f$. Due to the continuity of Ψ, we also have $\Psi(f_n) \rightharpoonup \Psi(f)$. Thus,

$$\Psi(f_n) \to \Psi(f)$$

because $\{\Psi(f_n)\}$ is relatively compact. Hence, the condition (S_2) is met, and we conclude that $\mathcal{K}(y)$ is compact.

Step 4. The multi-operator $\mathcal{K}$ is u.s.c.

We first establish that the graph of $\mathcal{K}$ is closed. Consider sequences $\{y_n\} \subset M$ with $y_n \to y$ and $z_n \in \mathcal{K}(y_n)$, where $z_n \to z$. It is required to show that $z \in \mathcal{K}(y)$. For each $z_n \in \mathcal{K}(y_n)$, there exists a corresponding $f_n \in S_{\mathcal{G}}(y_n)$ such that

$$z_n(t) = S_\alpha(t,0)(y_0 - g(y_n)) + \int_0^t P_\alpha(t,s) f_n(s)ds.$$

Given that the sequence $\{f_n\}$ is bounded in $L^p([0,T^*],\mathcal{Y})$, there is a function $f \in L^p([0,T^*],\mathcal{Y})$ so that $f_n \rightharpoonup f$ and $f \in S_{\mathcal{G}}(y)$ by virtue of the upper semicontinuous property. Moreover, since $y_n \to y$ in $\mathcal{C}([0,T^*],\mathcal{Y})$ as $n \to \infty$, we have

$$\mu(\{f_n(t)\}) \le k(t)\mu(\{y_n(t)\}) = 0 \quad \text{for a.e. } t \in [0,T^*].$$

The above implies that $\{f_n\}$ is semicompact in $L^p([0,T^*],\mathcal{Y})$ and ensures the relative compactness of $\{\Psi(f_n)\}$, where Ψ is defined by

$$\Psi(f)(t) = \int_0^t P_\alpha(t,s) f(s)ds \quad \text{for } f \in L^p([0,T^*],\mathcal{Y}).$$

The convergence $z_n(t) \to z(t)$ for a.e. $t \in [0,T^*]$ is given by

$$z(t) = S_\alpha(t,0)(y_0 - g(y)) + \int_0^t P_\alpha(t,s) f(s)ds.$$

We proceed to verify that $\mathcal{K}$ is a quasicompact multivalued map. Given a compact subset $\mathcal{D} \subset L^p([0,T^*],\mathcal{Y})$, it is shown that the set $\mathcal{K}(\mathcal{D}) \subset \mathcal{C}([0,T^*],\mathcal{Y})$ is relatively compact. For an arbitrary element $z_n \in \mathcal{K}(\mathcal{D})$, similar arguments infer the relative compactness of the image set $\{\mathcal{K}(f_n)\}$ in $\mathcal{C}([0,T^*],\mathcal{Y})$, thus guaranteeing the convergence of sequence $\{y_n\}$ in $\mathcal{C}([0,T^*],\mathcal{Y})$ and confirming that the map $\mathcal{K}$ is upper semicontinuous.

Therefore, by invoking Theorem 2.12, the solution set of inclusion (1) comprises a nonempty compact subset of $\mathcal{C}([0,T],\mathcal{Y})$. This completes the proof. ∎

We consider the following conditions.

(H_5) There exist a constant $p > \frac{1}{\alpha}$ and a function $\alpha(\cdot) \in L^p([0,T],\mathbb{R}_+)$ such that, for a.e. $s \in [0,T]$,

$$\|\mathcal{G}(s,x)\| = \sup\{\|f\| : f \in \mathcal{G}(s,x)\} \le \alpha(s)(1 + \|x\|).$$

(H_6) There exists a function $k(\cdot) \in L^p([0,T],\mathbb{R}^+)$ such that, for any bounded set $D \subseteq \mathcal{C}([0,T],\mathcal{Y})$ and for a.e. $s \in [0,T]$,

$$\mu(f(s,D)) \le k(s)\mu(D).$$

(H_7) There exists $r > 0$ such that the equation

$$M\|y_0\| + MN_{g_1}r^{\delta} + MN_{g_2} + Mt^{\frac{\alpha p-1}{p-1}}\|\alpha\|_p + Mt^{\frac{\alpha p-1}{p-1}}r \le r$$

has a positive solution T_0.

Theorem 3.4. *Given that conditions* (H_0)*–*(H_2) *and* (H_5)*–*(H_7) *hold, the solution set of inclusion* (1) *is a nonempty compact subset of* $\mathcal{C}([0,T_0],\mathcal{Y})$*. In particular, this conclusion holds for the function* $g = 0$ *in* (1)*.*

Proof. Given the assumptions, Proposition 3.2 is immediately satisfied. Considering the multi-operator (5) and following the argument of Theorem 3.3, it suffices to show that the operator $\mathcal{K}$ maps a bounded closed convex set to another such set and that $\mathcal{K}$ is π-condensing.

Let us define S_0 as the set $\{y \in \mathcal{C}([0,T_0],\mathcal{Y}) : \|y\|_C \le r\}$. For any $y \in S_0$, one can find $z \in \mathcal{K}(y)$ satisfying

$$z(t) = S_\alpha(t,0)(y_0 - g(y)) + \int_0^t P_\alpha(t,s)f(s)ds, \quad t \in [0,T_0],$$

where $f \in S_{\mathcal{G}}(y)$. By hypothesis (H_7),

$$\|z(t)\| \le M\|y_0\| + MN_{g_1}r^{\delta} + MN_{g_2} + MT_0^{\frac{\alpha p-1}{p-1}}\|f\|_p + MT_0^{\frac{\alpha p-1}{p-1}}r \le r.$$

Therefore, $\|z\|_C \le r$, implying that $\mathcal{K}(S_0) \subseteq S_0$.

Let $S_1 = \overline{\text{co}}\mathcal{K}(S_0)$. Since S_1 is a subset of $\mathcal{C}([0,T_0],\mathcal{Y})$, it is bounded, closed, and convex. Employing the previous arguments, for any $y \in S_1$ with $y \in \mathcal{K}(y)$ and $f \in S_{\mathcal{G}}(y)$, we obtain

$$y(t) = S_\alpha(t,0)(y_0 - g(y)) + \int_0^t P_\alpha(t,s)f(s)ds, \quad t \in [0,T_0],$$

and confirm that $\|y(t)\| \le r$. Hence, $\mathcal{K}(S_1) \subseteq S_1$ and $S_1 \subseteq S_0$. Let us define a sequence of sets in the following way. For $n \ge 1$, we define $S_{n+1} = \overline{\text{co}}\,\mathcal{K}(S_n)$. We claim that, by induction, the sequence $\{S_n\}_{n=1}^{\infty}$ has the following properties: for each $n \ge 1$, the set $S_n \subset \mathcal{C}([0,T_0],\mathcal{Y})$ is bounded, convex, and closed. Moreover, the sequence is decreasing in the sense that $S_1 \supseteq S_2 \supseteq \cdots \supseteq S_n$. We then consider the intersection

$$S = \bigcap_{n=1}^{\infty} S_n.$$

It is straightforward to conclude that S is a nonempty, bounded, and closed subset of $\mathscr{C}([0, T_0], \mathscr{Y})$ with the property that $\mathcal{K}(S) \subseteq S$.

Next, we aim to establish that the operator $\mathcal{K}$ is π-condensing. Assume that we have $\mathcal{D} \subset S$ for which

$$\pi(\mathcal{D}) \leq \pi(\mathcal{K}(\mathcal{D})). \tag{6}$$

Suppose, contrarily to (6), that, for any $\mathcal{D} \subset S$,

$$\pi(\mathcal{D}) > \pi(\mathcal{K}(\mathcal{D})),$$

which would imply that $\mathcal{K}$ is indeed π-condensing.

To proceed with the proof, let $\{z_n\}_{n=1}^{\infty} \subset \mathcal{K}(\mathcal{D})$ be an arbitrary sequence. Consider the expression

$$\pi(\{z_n\}_{n=1}^{\infty}) = \max\Big\{ \sup_{t\in[0,T_0]} e^{-\mathcal{L}t} \mu(\{z_n(t)\}_{n=1}^{\infty}), \operatorname{mod}_{\mathscr{C}}(\{z_n\}_{n=1}^{\infty})\Big\},$$

where $\{z_n(t)\}$ is given by the relation

$$z_n(t) = S_\alpha(t, 0)(y_0 - g(y_n)) + \int_0^t P_\alpha(t, s) f_n(s) ds,$$

with $f_n \in S_{\mathscr{G}}(y_n)$ and $\{y_n\} \subset \mathcal{D}$ for $n \geq 1$. We pick a constant $\mathcal{L} > 0$ such that

$$M \sup_{t\in[0,T_0]} \int_0^t e^{-\mathcal{L}\frac{p}{p-1}(t-s)} (t-s)^{\frac{\alpha p - p}{p-1}} ds \|k\|_p < \frac{1}{2}.$$

Thus, we obtain

$$\begin{aligned}
&\sup_{t\in[0,T_0]} e^{-\mathcal{L}t} \mu(\{z_n(t)\}_{n=1}^{\infty}) \\
&\leq 2M \sup_{t\in[0,T_0]} e^{-\mathcal{L}t} \int_0^t (t-s)^{\alpha-1} k(s) \mu(\{y_n(s)\}_{n=1}^{\infty}) ds \\
&\leq 2M \|k\|_p \sup_{t\in[0,T_0]} \int_0^t e^{-\mathcal{L}\frac{p}{p-1}(t-\theta)} (t-\theta)^{\frac{\alpha p - p}{p-1}} d\theta \sup_{\theta\in[0,T_0]} e^{-\mathcal{L}\theta} \mu(\{y_n\}_{n=1}^{\infty}).
\end{aligned}$$

Subsequently, similarly to Step 2 in Theorem 3.3, we obtain $\operatorname{mod}_{\mathscr{C}}(\{z_n\}_{n=1}^{\infty}) = 0$. Hence, from the inequality (6), it follows that

$$\sup_{t\in[0,T_0]} e^{-\mathcal{L}t} \mu(\{z_n(t)\}_{n=1}^{\infty}) \geq \sup_{t\in[0,T_0]} e^{-\mathcal{L}t} \mu(\{y_n(t)\}_{n=1}^{\infty}).$$

Therefore, we conclude that $\mu(\{z_n(t)\}) = 0$, confirming that the multivalued operator $\mathcal{K}$ is π-condensing. By an application of Theorem 2.12, we confirm that the solution set of the inclusion under consideration is a nonempty compact subset of $\mathscr{C}([0, T_0], \mathscr{Y})$. The proof is now complete. ■

4. Topological properties

We now proceed to consider a space $\mathcal{Y}$, assumed reflexive within the Banach context, assigning $g = 0$. This simplification transforms equation (1) into

$$\begin{cases} {}^{C}D_t^{\alpha} y(t) \subseteq A(t)y(t) + \mathcal{G}(t, y(t)), & t \in [0, T], \\ y(0) = y_0. \end{cases} \tag{7}$$

We ascertain the existence of a compact solution set $\Phi(y_0)$ for equation (7), establishing it as a compact R_δ-set which is nonempty. Consistent with Theorem 3.4, $\Phi(y_0)$ also belongs to the continuous function space $\mathcal{C}([0, T_0], \mathcal{Y})$, guaranteeing nonemptiness and compactness. For $s \in [0, a)$, where $a > 0$ is given, the following integral equation is examined:

$$z(t) = \phi(t) + \int_s^t P_\alpha(t, \theta) f(\theta, z(\theta)) d\theta, \quad t \in [s, a], \tag{8}$$

with $P_\alpha(t, \theta)$ as the operator function delineated in Definition 2.9.

Primarily, contingent on compliance of the nonlinear terms with the stipulated non-compactness criteria, the following lemma is established, pivotal for subsequent contractibility affirmation of the solution set.

Lemma 4.1. *Let $p > \frac{1}{\alpha}$. Consider a singular mapping $f : [0, a] \times \mathcal{Y} \to \mathcal{Y}$, and for every $t \in [0, a]$, the operator $A(t)$ is the generator of an analytic semigroup $T_t(s)$ with $s \geq 0$. Assume that there exists a function $h(\cdot) \in L^p([0, a], \mathcal{Y})$ such that, for every subset $B \subseteq \mathcal{C}([0, a], \mathcal{Y})$, it holds that $\mu(f(t, B)) \leq h(t)\mu(B)$. Moreover, the following conditions are satisfied:*

(i) *Given any compact set $K \subset \mathcal{Y}$, there exist $\delta > 0$ and a function*

$$L_K(\cdot) \in L^p([s, a]; \mathbb{R}^+)$$

such that, for almost every $t \in [s, a]$ and all $z_1, z_2 \in B_\delta(K)$, we have

$$\| f(t, z_1) - f(t, z_2) \| \leq L_K(t) \| z_1 - z_2 \|.$$

(ii) *The inequality $\| f(t, z) \| \leq r(t)(d + \| z \|)$ holds for almost every $t \in [s, a]$, where $r(\cdot) \in L^p([s, a], \mathbb{R}^+)$, $z \in \mathcal{Y}$, and d is a fixed constant.*

Under these assumptions, the integral equation (8) has a unique solution that depends continuously on the initial function ψ.

Proof. The proof proceeds in the following steps.

Step 1. Fix a function $\psi(\cdot) \in \mathcal{C}([s, a], \mathcal{Y})$.

Consider the set

$$Q(\psi, \kappa) = \{z \in \mathcal{C}([s, a], \mathcal{Y}), \max_{\theta \in [s, \kappa]} \| z(\theta) - \psi(\theta) \| \leq \zeta\},$$

where $\zeta > 0$ is a constant chosen such that

$$M\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}(\kappa-s)^{\alpha-\frac{1}{p}}\|r(\cdot)\|_p(d+\zeta+\max_{\theta\in[s,\kappa]}\|\psi(\theta)\|)\leq\zeta.$$

Define the operator $\mathcal{F}$ by

$$\mathcal{F}(z)(t)=\psi(t)+\int_s^t P_\alpha(t,\theta)f(\theta,z(\theta))d\theta,$$

which maps $Q(\psi,\kappa)$ into itself. Mimicking the arguments from Step 2 in Theorem 3.3 and using conditions (i) and (ii), we conclude that $\mathcal{F}$ is continuous and condensing. Thus, the fixed point of $\mathcal{F}$ yields the local solution to equation (8), denoted by z.

Using conditions (i) and (ii) and noting that

$$\|\mathcal{F}(z_1)(t)-\mathcal{F}(z_2)(t)\|^{\frac{p}{p-1}}\leq N\int_s^t(t-\theta)^{\frac{\alpha p-p}{p-1}}\|z_1(\theta)-z_2(\theta)\|^{\frac{p}{p-1}}d\theta,$$

where

$$N=(M\|L_K(\cdot)\|)^{\frac{p}{p-1}};$$

uniqueness is also established by a Grönwall-type inequality.

Step 2. Define the extension operator $\Phi(k,z)(\theta)$ by

$$\Phi(k,z)(\theta)=\begin{cases}z(\theta), & \theta\in[s,k],\\ z(k), & \theta\in[k,a],\end{cases}$$

which maps $\mathcal{C}([s,a],\mathcal{Y})\times[s,a]$ onto itself. Let $\Phi(\kappa,z)(\cdot)=z^\kappa(\cdot)$ belong to $\mathcal{C}([s,a],\mathcal{Y})\times[s,a]$ and set

$$A=\{k\in[s,a]:z^k=\Phi(k,\mathcal{F}(z^k))\}.$$

One obtains $A\neq\emptyset$ as $\kappa\in A$, and for $t\in A$, the interval $[s,t]$ is contained within A.

Let $k_0=\sup A$. Considering an increasing sequence $\{k_n\}$ converging to k_0, we assert $k_0\in A$. In fact, for $m\leq n$ and by the definition of Φ, we have

$$\Phi(k_m,z^{k_m})=\Phi(k_m,\mathcal{F}(z^{k_m})),\quad \Phi(k_m,z^{k_n})=\Phi(k_m,\mathcal{F}(z^{k_n}))\text{ on }[s,k_m].$$

Additionally, we arrive at

$$\|z^{k_m}(k_0)-z^{k_n}(k_0)\|=\|z^{k_m}(k_m)-z^{k_n}(k_n)\|.$$

Following the method used in Step 2 of Theorem 3.4 and the continuity of $\psi(\cdot)$, we deduce that

$$\lim_{n,m\to\infty}\|z^{k_m}(k_0)-z^{k_n}(k_0)\|=0,$$

indicating that $\{z^{k_m}(k_0)\}$ is a Cauchy sequence. Define

$$z^{k_0}(\theta) = \begin{cases} z^{k_n}(\theta), & \theta \in [s, k_n], \\ \lim_{n\to\infty} z^{k_n}(k_0), & \theta \in [k_0, a], \end{cases}$$

and as $z^{k_0}(\theta) = \mathcal{F}(z^{k_0})(\theta)$ for $\theta \in [s, k_0)$, continuity implies that z^{k_0} is also continuous, leading to

$$\begin{aligned} z^{k_0}(k_0) &= \lim_{n\to\infty} (\psi(k_n) + \int_s^{k_n} P_\alpha(k_n, \theta) f(\theta, z^{k_n}(\theta)) d\theta) \\ &= \psi(k_0) + \int_s^{k_0} P_\alpha(k_0, \theta) f(\theta, z^{k_0}(\theta)) d\theta, \end{aligned}$$

which confirms that $z^{k_0} = \Phi(k_0, z^{k_0})$ and $k_0 \in A$.

Step 3. We claim that $k_0 = a$. Suppose for contradiction that $k_0 < a$; we then define

$$\psi_1(t) = \psi(t) + \int_s^{k_0} P_\alpha(t, \theta) f(\theta, z^{k_0}(\theta)) d\theta, \quad t \in [k_0, a],$$

and, similar to the proof given above, there exists $\kappa_1 > 0$ such that

$$z(t) = \psi_1(t) + \int_{k_0}^t P_\alpha(t, \theta) g(\theta, z(\theta)) d\theta$$

admits a solution $z \in \mathcal{C}([k_0, k_0 + \kappa_1], \mathcal{Y})$. Define

$$z^{k_0+\kappa_1}(\theta) = \begin{cases} z^{k_0}(\theta), & \theta \in [s, k_0], \\ z(\theta), & \theta \in [k_0, k_0 + \kappa_1], \\ z(k_0 + \kappa_1), & \theta \in [k_0 + \kappa_1, a], \end{cases}$$

which implies that $z^{k_0+\kappa_1} \in \mathcal{C}([s, a], X)$. Consequently,

$$z^{k_0+\kappa_1}(t) = \psi(t) + \int_s^t P_\alpha(t, \theta) f(\theta, z^{k_0+\kappa_1}(\theta)) d\theta, \quad t \in [s, k_0 + \kappa_1],$$

and hence, $z^{k_0+\kappa_1}(\cdot) = \Phi(k_0 + \kappa_1, \mathcal{F}(z^{k_0+\kappa_1}))$. This means that $k_0 + \kappa_1 \in A$, which contradicts the definition of k_0.

Step 4. Consider the solution sequence $\{z_n\}$ corresponding to the sequence of perturbation functions $\{\psi_n\}$, where $\psi_n \to \psi$ in $\mathcal{C}([s, a], \mathcal{Y})$. Specifically,

$$z_n(t) = \psi_n(t) + \int_s^t P_\alpha(t, \theta) f(\theta, z_n(\theta)) d\theta.$$

By an argument similar to that in Theorem 3.4, we deduce that $\text{mod}_c(\{z_n\}) = 0$ and the sequence $\{z_n\}$ is contained within a relatively compact subset of $\mathcal{C}([s, a], \mathcal{Y})$. Consequently, there exists a convergent subsequence, which we continue to denote by $\{z_n\}$, such that as $n \to \infty$ the functions $z_n(\cdot)$ converge to a function $z(\cdot)$. This completes the proof. ∎

Next, we demonstrate that the set R_δ constitutes all solutions to equation (7). We introduce an auxiliary approximation lemma, slightly adapted from the versions in [10, 12], omitting the proofs for conciseness.

Lemma 4.2. *Assume that hypotheses (H_1), (H_2), and (H_5) hold. Then, there exists a sequence $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ from $[0,T_0]\times\mathcal{Y}$ to $\mathcal{P}_{\mathrm{cl,cv}}(\mathcal{Y})$ satisfying the following.*

(i) *For each $s\in[0,T_0]$ and $y\in\mathcal{Y}$, we have the inclusion chain*

$$\mathscr{G}(s,y)\subset\mathscr{G}_{n+1}(s,y)\subset\mathscr{G}_n(s,y)\subset\overline{\mathrm{co}}(\mathscr{G}(s,B_{3^{1-n}}(y)))\quad \text{for all } n\geq 1.$$

(ii) *The norm of $\mathscr{G}_n(s,y)$ is bounded by $\|\mathscr{G}_n(s,y)\|\leq\alpha(s)(2+\|y\|)$ for almost every $s\in[0,T_0]$ and all $y\in\mathcal{Y}$, where $n\geq 1$.*

(iii) *There exists a null set $J\subset[0,T_0]$ in terms of the Lebesgue measure $\vartheta(\cdot)$ such that, for any $e\in\mathcal{Y}^*$, $\delta>0$, and $(s,y)\in[0,T_0]\setminus J\times\mathcal{Y}$, one can find a constant $N(\delta,e)>0$ ensuring that*

$$e(\mathscr{G}_n(t,x))\subset e(\mathscr{G}(t,x))+(-\delta,\delta)\quad \text{for all } n\geq N.$$

(iv) *$\mathscr{G}_n(s,\cdot)$ is a continuous mapping from $\mathcal{Y}$ to $\mathcal{P}_{\mathrm{cl,cv}}(\mathcal{Y})$ for almost every $t\in[0,T_0]$ and all $n\geq 1$.*

(v) *Each $\mathscr{G}_n$ admits a selection z_n such that $z_n(\cdot,y)$ is measurable for any fixed $y\in\mathcal{Y}$, with $z_n(s,y)\in\mathscr{G}_n(s,y)$ for almost every $s\in[0,T_0]$. Additionally, for every compact subset $\mathcal{D}\subset\mathcal{Y}$, there exist a constant $C_{\mathcal{U}}>0$ and $\varepsilon>0$ such that $\mathcal{U}=\mathcal{D}+B_\varepsilon(0)$ and for almost every $s\in[0,T_0]$, any $y_1,y_2\in\mathcal{U}$ satisfy*

$$\|z_n(s,y_1)-z_n(s,y_2)\|\leq C_{\mathcal{U}}\alpha(s)\|y_1-y_2\|.$$

(vi) *If $\mathcal{Y}$ is reflexive, then $\mathscr{G}_n$ satisfies condition (H_2) for each $n\geq 1$.*

Thus, we have the following theorem.

Theorem 4.3. *Let y_0 be given. If the conditions of Theorem 3.4 hold, then the set $\Phi(y_0)$ is a compact R_δ set.*

Proof. To establish the compactness of $\Phi(y_0)$, consider the fractional evolution inclusion

$$\begin{cases} {}^CD_t^\alpha y(t)\in A(t)y(t)+\mathscr{G}_n(t,y(t)), & t\in[0,T_0],\\ y(0)=y_0, \end{cases}\tag{9}$$

where $\Phi^n(y_0)$ denotes the solution set of inclusion (9).

Step 1. Let $\{y_n\}$ be a sequence such that $y_n\in\Phi^n(y_0)$ for all $n\geq 1$. Since $\mathcal{Y}$ is a reflexive Banach space, each $y_n(t)$ for $t\in[0,T_0]$ satisfies

$$y_n(t)=S_\alpha(t,0)y_0+\int_0^t P_\alpha(t,s)z_n(s)\mathrm{d}s,$$

where $z_n(s) \in \mathscr{G}(s, y_n(s))$. Invoking property (i) from Lemma 4.2, for any $N \geq 1$, we establish

$$\begin{aligned}\mu(\{z_n(s)\}_{n\geq 1}) &= \mu(\{z_n(s)\}_{n\geq N})\\ &\leq \mu(\mathscr{G}(s, B(\{y_n(s)\}_{n\geq N}, 3^{1-N})))\\ &\leq k(s)\mu(B(\{y_n(s)\}_{n\geq N}, 3^{1-N}))\\ &\leq k(s)(\mu(\{y_n(s)\}_{n\geq N}) + 3^{1-N}),\end{aligned}$$

and consequently,

$$\mu(\{y_n(t)\}_{n\geq N})^{\frac{p}{p-1}} \leq (2M\|k\|_p)^{\frac{p}{p-1}} \int_0^t (t-s)^{\frac{\alpha p-p}{p-1}} (\mu(\{y_n(s)\}_{n\geq N}))^{\frac{p}{p-1}} \mathrm{d}s.$$

Applying Grönwall's inequality leads to $\mu(\{y_n(t)\}_{n\geq N}) = 0$ for all $t \in [0, T_0]$. Theorem 3.3 provides the equicontinuity of $\{y_n\}$. Thereupon, a subsequence $\{y_{n_k}\}$ converges, denoted as $y_{n_k} \to y$.

Simultaneously, having $\mu(\{z_n\}_{n\geq N}) = 0$ and letting

$$z_n(s) \to z(s) \in L^p([0, T_0], \mathscr{Y}) \quad \text{for } s \in [0, t]$$

yield

$$y(t) = S_\alpha(t, 0)y_0 + \int_0^t P_\alpha(t, s)z(s)\mathrm{d}s.$$

Finally, since $\mathscr{G}_n$ adheres to condition (H_2), it follows that

$$z(t) \in \mathscr{G}(t, y(t)) \subset \mathscr{G}_n(t, y(t)),$$

confirming the compactness of $\Phi^n(y_0)$.

Step 2. We aim to demonstrate that $\Phi(y_0) = \bigcap_{n=1}^\infty \Phi^n(y_0)$.

Firstly, for any $n \geq 1$, it is evident that $\Phi(y_0) \subseteq \Phi^n(y_0)$ since

$$\mathscr{G}(t, y) \subseteq \mathscr{G}_{n+1}(t, y) \subseteq \mathscr{G}_n(t, y).$$

Therefore,

$$\Phi(y_0) \subseteq \bigcap_{n=1}^\infty \Phi^n(y_0).$$

Conversely, given $y \in \bigcap_{n=1}^\infty \Phi^n(y_0)$, there exists a sequence $\{z_n\} \subseteq L^p([0, T_0], \mathscr{Y})$ such that

$$\|z_n(t)\| \leq \alpha(t)(2 + \|y\|) \quad \text{for a.e. } t \in [0, T_0],$$

where $z_n \in S_{\mathscr{G}_n}(y)$ and

$$y(t) = S_\alpha(t, 0)y_0 + \int_0^t P_\alpha(t, s)z_n(s)\mathrm{d}s.$$

Employing the Banach–Alaoglu theorem results in the existence of a subsequence, still denoted by $\{z_n\}$, with $z_n \rightharpoonup z$ in $L^p([0,T_0],\mathcal{Y})$. By Mazur's theorem, we select a sequence $\widetilde{z_n} \in \mathrm{co}\{z_k : k \geq n\}$ for $n \geq 1$ such that $\widetilde{z_n} \to z$ in $L^p([0,T_0],\mathcal{Y})$, where

$$\widetilde{z_n} = \sum_{i=0}^{k_n} a_{n,i} z_{n+i}, \quad a_{n,i} \geq 0, \quad \text{and} \quad \sum_{i=0}^{k_n} a_{n,i} = 1.$$

Therefore, for almost every $t \in [0,T_0]$, $\widetilde{z_n}(t) \to z(t)$ and $z_n(t) \in \mathcal{G}_n(t, y(t))$ for all $n \geq 1$.

For the set

$$A = \{t \in [0,T_0] : \widetilde{z_n}(t) \to z(t) \text{ and } \forall n \geq 1,\ z_n(t) \in \mathcal{G}_n(t, y(t))\},$$

Lemma 4.2 (iii) implies that, for each $t \in ([0,T_0] \setminus J) \cap A$ and each $e \in \mathcal{Y}^*$,

$$e(\widetilde{z_n}(t)) \in \mathrm{co}\{e(z_k(t)) : k \geq n\} \subseteq e \circ \mathcal{G}_n(t, y(t)) \subseteq e \circ \mathcal{G}(t, y(t)) + (-\varepsilon, \varepsilon),$$

which implies $e(z(t)) \in e \circ \mathcal{G}(t, y(t))$ for each $t \in ([0,T_0] \setminus J) \cap A$. Since $\mathcal{F}$ has convex and closed values, we conclude that $z(t) \in \mathcal{G}(t, y(t))$ for each $t \in ([0,T_0] \setminus J) \cap A$, implying that $z \in S_{\mathcal{G}}(y)$. Noting that $y \in \bigcap_{n=1}^{\infty} \Phi^n(y_0)$, and that for each $n \geq 1$, $z_n \in S_{\mathcal{G}_n}(y)$, it follows that $\{z_n\}$ is semicompact in $L^p([0,T_0],\mathcal{Y})$. By Lemma 2.8, we have

$$y(t) = S_\alpha(t,0)y_0 + \int_0^t P_\alpha(t,s) z(s)\mathrm{d}s \in \Phi(y_0).$$

Step 3. For every $n \geq 1$, we prove that $\Phi^n(y_0)$ is a contractible set.

Consider any $y \in \Phi^n(y_0)$ and the following integral equation:

$$b(t) = S_\alpha(t,0)y_0 + \int_0^{\delta T_0} P_\alpha(t,s) z^y(s) ds + \int_{\delta T_0}^t P_\alpha(t,s) f_n(s, z(s)) ds, \tag{10}$$

where $z^y \in Sel_{\mathcal{G}_n}(y)$ and f_n is the measurable selection from $\mathcal{G}_n$ satisfying property (v) in Lemma 4.2. By Lemma 4.1, equation (10) has a unique solution on $[\delta T_0, T_0]$, denoted by $b(t, \delta T_0)$.

We can now define a homotopy $\beta: [0,1] \times \Phi^n(y_0) \to \Phi^n(y_0)$ by

$$\beta(\delta, y)(t) = \begin{cases} y(t), & t \in [0, \delta T_0], \\ b(t, \delta T_0), & t \in [\delta T_0, T_0], \end{cases}$$

for each $(\delta, y) \in [0,1] \times \Phi^n(y_0)$. The mapping $\beta(\cdot)$ is well defined, and it is clear that $\beta(0,y) = y_0$, $\beta(1,y) = y \in \Phi^n(y_0)$. Hence, we deduce that $\Phi(y_0) = \bigcap_{n=1}^{\infty} \Phi^n(y_0)$ is a nonempty compact R_δ-set. The proof is now complete. ∎

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