Z. Anal. Anwend. 44 (2025), 485–500
DOI 10.4171/ZAA/1798
© 2025 European Mathematical Society
Published by EMS Press
This work is licensed under a CC BY 4.0 license

Hardy–Trudinger inequalities in weighted Orlicz spaces

Yoshihiro Mizuta and Tetsu Shimomura

Abstract. We establish Hardy–Trudinger inequalities in weighted Orlicz spaces on $\mathbb{R}^n$ and near the origin.

1. Introduction

Let $B(x,r)$ denote the open ball centered at $x \in \mathbb{R}^n$ with radius r, whose volume is written as $|B(x,r)|$. The Hardy–Sobolev inequalities say that, for nonnegative measurable functions f on $\mathbb{R}^n$,

$$\left(\int_{\mathbb{R}^n}\left(\int_{B(0,|x|)} f(y)dy\right)^q |x|^\alpha dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} f(y)^p |y|^\beta dy\right)^{1/p} \tag{1.1}$$

when $1 \le p \le q$, $\beta < n(p-1)$ and $\alpha = \beta q/p - qn/p' - n$ $(1/p + 1/p' = 1)$, and

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n\setminus B(0,|x|)} f(y)dy\right)^q |x|^\alpha dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} f(y)^p |y|^\beta dy\right)^{1/p} \tag{1.2}$$

when $1 \le p \le q$, $\beta > n(p-1)$ and $\alpha = \beta q/p - qn/p' - n$ (see, e.g., [14,21]). For the classical one-dimensional case, see, e.g., [1,4,5,7,8,18,19]. In [21], Persson and Samko discussed the sharp constant C for the case $p \le q$ for both one-dimensional and multi-dimensional cases. These classical Hardy-type inequalities are given mostly for weighted Lebesgue spaces, but such inequalities are known for other function spaces (see [5, Section 7.6]). See [5, Theorem 7.95] for Hardy-type inequalities for Orlicz spaces. Recently, we studied Hardy–Sobolev inequalities on the unit ball in [11, 13], on the half-space in [9,10,17] and on $\mathbb{R}^n$ in [14,16]. For related results, see, e.g., [2,3,12,15,20].

In this paper, we are concerned with the borderline case $\beta = n(p-1)$ and establish Hardy–Trudinger-type inequalities in weighted Orlicz spaces on $\mathbb{R}^n$. In Section 3, we establish inner Hardy–Trudinger-type inequalities on $\mathbb{R}^n$ (Theorem 3.1) and near the origin (Theorem 3.6). The sharpness of our results will be discussed in Remarks 3.3 and 3.8.

In Section 4, we investigate outer Hardy–Trudinger-type inequalities on $\mathbb{R}^n$ (Theorem 4.1) and near the origin (Theorem 4.8). We give remarks on the best possibility of

Mathematics Subject Classification 2020: 46E30 (primary); 26D15, 47G10 (secondary).
Keywords: Hardy–Trudinger integrability, weighted Orlicz space.

our theorems (Remarks 4.3, 4.4, 4.7, and 4.10). As far as we know, even corollaries in Sections 3 and 4 are new results.

The constants C in all inequalities in this paper depend on some parameters, involved functionals and conditions, etc., but they are independent of the corresponding mappings. Moreover, $f \sim g$ means that $C^{-1}g(r) \le f(r) \le Cg(r)$ for a constant $C > 0$.

2. Preliminaries

Consider a positive convex function φ on $(0, \infty)$ satisfying

($\varphi 0$) $\varphi(0) = \lim_{r\to 0} \varphi(r) = 0$.

The typical examples are

$$\varphi(r) = r^p(\log(c + r))^q, \exp(r^p) - 1, \text{etc.},$$

where $p \ge 1$ and $c \ge e$ is chosen so that $(1 + \log c)(p - 1) + q \ge 0$ [6, Proposition 5.2]. If $\varphi_1(r) = r^p(\log(e + r))^q$, then it may be replaced by

$$\varphi_2(r) = \int_0^r \left\{ \sup_{0<s<t} s^p(\log(e + s))^q \right\} t^{-1}\,dt,$$

which is convex and $\varphi_1(r) \sim \varphi_2(r)$.

Note here that

($\varphi 1$) $s^{-1}\varphi(s) \le t^{-1}\varphi(t)$ whenever $0 < s < t$;

($\varphi 2$) φ^{-1} is concave in $(0, \infty)$;

(φ^{-1}) φ^{-1} is doubling, more precisely,

$$\varphi^{-1}(2r) \le 2\varphi^{-1}(r) \quad \text{for } r > 0.$$

3. Hardy–Trudinger inequalities I

We give the inner Hardy–Trudinger inequality on $\mathbb{R}^n$.

Theorem 3.1. *Let φ and ψ be positive convex functions on $(0, \infty)$ satisfying ($\varphi 0$). For $p \ge 1$ and a positive increasing function ω on $(0, \infty)$,*

$$\omega_p(r) = \int_0^r \omega(t)^{-p'/p} t^{n-1}\,dt.$$

Suppose that $\alpha > 0$ and

($\varphi, \psi, \alpha, \omega_p$) *for $\varepsilon > 0$, there exists a constant $A_1 > 0$ such that*

$$\psi((e + t)^{\alpha p}\omega_p(t)^{p/p'}\varphi^{-1}((e + t)^{-n})/A_1) \le (e + t)^{\varepsilon} \quad \textit{for } t > 0.$$

Then, for $\varepsilon > 0$*, there exists a constant* $C > 1$ *such that*

$$\int_{\mathbb{R}^n} (e+|x|)^{-\varepsilon-n} \psi\left(\left((e+|x|)^{-n/p+\alpha}\left(\int_{B(0,|x|)} |f(y)|dy\right)\Big/ C\right)^p\right) dx \le 1$$

for measurable functions f *on* $\mathbb{R}^n$ *satisfying*

$$\int_{\mathbb{R}^n} \varphi(g(y))dy \le 1, \tag{3.1}$$

where $g(y) = |f(y)|^p \omega(|y|)$.

Note that f satisfying (3.1) is called a weighted Orlicz function.

Proof of Theorem 3.1. Let f be a measurable function on $\mathbb{R}^n$ satisfying (3.1). By using Hölder's inequality, we have

$$\begin{aligned}
&(e+|x|)^{-n/p+\alpha} \int_{B(0,|x|)} |f(y)|dy \\
&\quad \le (e+|x|)^{-n/p+\alpha} \left(\int_{B(0,|x|)} |f(y)|^p \omega(|y|)dy\right)^{1/p} \left(\int_{B(0,|x|)} \omega(|y|)^{-p'/p} dy\right)^{1/p'} \\
&\quad \le C(e+|x|)^{-n/p+\alpha} \omega_p(|x|)^{1/p'} \left(\int_{B(0,|x|)} g(y)dy\right)^{1/p} \\
&\quad \le C(e+|x|)^{\alpha} \omega_p(|x|)^{1/p'} \left((e+|x|)^{-n} \int_{B(0,e+|x|)} g(y)dy\right)^{1/p}.
\end{aligned}$$

In view of Jensen's inequality, (3.1) and (φ^{-1}), we obtain

$$\begin{aligned}
&(e+|x|)^{-n/p+\alpha} \int_{B(0,|x|)} |f(y)|dy \\
&\quad \le C(e+|x|)^{\alpha} \omega_p(|x|)^{1/p'} \left(\varphi^{-1}\left(|B(0,e+|x|)|^{-1} \int_{B(0,e+|x|)} \varphi(g(y))dy\right)\right)^{1/p} \\
&\quad \le C(e+|x|)^{\alpha} \omega_p(|x|)^{1/p'} (\varphi^{-1}(|B(0,e+|x|)|^{-1}))^{1/p} \\
&\quad \le C_1(e+|x|)^{\alpha} \omega_p(|x|)^{1/p'} (\varphi^{-1}((e+|x|)^{-n}))^{1/p}.
\end{aligned}$$

By $(\varphi, \psi, \alpha, \omega_p)$, we obtain

$$\psi\left(\left((e+|x|)^{-n/p+\alpha}\left(\int_{B(0,|x|)} |f(y)|dy\right)\Big/ (C_1 A_1^{1/p})\right)^p\right) \le (e+|x|)^{\varepsilon_1}.$$

Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^n} (e+|x|)^{-n-\varepsilon} \psi\left(\left((e+|x|)^{-n/p+\alpha}\left(\int_{B(0,|x|)} |f(y)|dy\right)\Big/ (C_1 A_1^{1/p})\right)^p\right) dx \\
&\quad \le \int_{\mathbb{R}^n} (e+|x|)^{-n-\varepsilon+\varepsilon_1} dx \le C_2
\end{aligned}$$

when $0 < \varepsilon_1 < \varepsilon$, which yields the result. ■

Example 3.2. Consider $\omega(r) = (e+r)^\beta (\log(e+r))^\gamma$ with $\beta = np/p'$ and $\gamma < p-1$ and $\varphi(r) = \exp r - 1$. Then,

$$\omega_p(r) \sim (\log(e+r))^{-p'\gamma/p+1}$$

and

$$\varphi^{-1}(t) = \log(1+t) \sim t \quad \text{as } t \to 0.$$

Hence, letting $\alpha = n/p$, we find

$$(e+t)^{\alpha p}\omega_p(t)^{p/p'}\varphi^{-1}((e+t)^{-n}) \sim (\log(e+t))^{p(-\gamma/p+1/p')},$$

so that we may consider

$$\psi(t) = \exp t^{1/(p(-\gamma/p+1/p'))} - 1 \tag{3.2}$$

in Theorem 3.1.

Remark 3.3. We see that the exponent $1/p(-\gamma/p+1/p')$ in (3.2) is best suited.

For this, consider $f(y) = (e+|y|)^{-n}(\log(e+|y|))^{-(\gamma+1)/p}(\log\log(e+|y|))^{-\delta/p}$. Then, the following statements hold:

(1) $g(y) = f(y)^p(e+|y|)^\beta(\log(e+|y|))^\gamma = (e+|y|)^{-n}(\log(e+|y|))^{-1}(\log\log(e+|y|))^{-\delta}$ is bounded when $(n+\beta)/p = n$;

(2) $\int_{\mathbb{R}^n}\varphi(g(y))dy \le C\int_{\mathbb{R}^n} g(y)dy = C\int_{\mathbb{R}^n}(e+|y|)^{-n}(\log(e+|y|))^{-1}(\log\log(e+|y|))^{-\delta}dy < \infty$ when $\delta > 1$;

(3) $\int_{B(0,|x|)} f(y)dy \ge C_1(\log(e+|x|))^{1-(\gamma+1)/p}(\log\log(e+|x|))^{-\delta/p}$ when $x \in \mathbb{R}^n$ and $(\gamma+1)/p < 1$;

(4) if $\varepsilon = 0$, then $\int_{\mathbb{R}^n}(e+|x|)^{-\varepsilon-n}\exp((\int_{B(0,|x|)} f(y)dy)/(AC_1))^a dx = \infty$ for $A > 0$ and $a > 0$;

(5) $\int_{\mathbb{R}^n}(e+|x|)^{-\varepsilon-n}\exp((\int_{B(0,|x|)} f(y)dy)/(AC_1))^a dx \ge \int_{\mathbb{R}^n}(e+|x|)^{-\varepsilon-n}$ $\exp(A^{-a}(\log(e+|x|))^{a(1/p'-\gamma/p)}(\log\log(e+|x|))^{-a\delta/p})dx = \infty$ when $a(1/p' - \gamma/p) > 1$, $A > 0$ and $\varepsilon > 0$.

Corollary 3.4. Let $p \ge 1$, $\beta = n(p-1)$, $\gamma < p-1$ and $\varepsilon > 0$. Then, there exist constants $C_1, C_2 > 0$ such that

$$\int_{\mathbb{R}^n}(e+|x|)^{-\varepsilon-n}\exp\left(\left(\int_{B(0,|x|)} f(y)dy\right)\Big/C_1\right)^{1/(-\gamma/p+1/p')} dx \le C_2 \tag{3.3}$$

for nonnegative measurable functions f on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} f(y)^p(e+|y|)^\beta(\log(e+|y|))^\gamma dy \le 1. \tag{3.4}$$

For this, consider $\omega(r) = (e + r)^{\beta}(\log(e + r))^{\gamma}$. Then, in the first considerations in the proof of Theorem 3.1, we find by (3.4) that

$$\begin{aligned}\int_{B(0,|x|)} f(y)dy &\le C\omega_p(|x|)^{1/p'}\left(\int_{B(0,|x|)} g(y)dy\right)^{1/p} \\ &\le C(\log(e + |x|))^{-\gamma/p+1/p'}\left(\int_{B(0,|x|)} g(y)dy\right)^{1/p} \\ &\le C_1(\varepsilon_1 \log(e + |x|))^{-\gamma/p+1/p'}\end{aligned}$$

for $\varepsilon_1 > 0$ since $-\gamma/p + 1/p' > 0$, where $g(y) = f(y)^p(e + |y|)^{\beta}(\log(e + |y|))^{\gamma}$. Hence, we obtain

$$\exp\left(\left(\int_{B(0,|x|)} f(y)dy\right)\Big/C_1\right)^{1/(-\gamma/p+1/p')} \le (e + |x|)^{\varepsilon_1},$$

which gives the result.

Corollary 3.5. Let $p \ge 1$, $\beta = n(p - 1)$, $\gamma < p - 1$ and $\varepsilon > 0$. Then, there exist constants $C_1, C_2 > 0$ such that

$$\int_{\mathbb{R}^n} (e + |x|)^{-\varepsilon-n} \exp\exp\left(\left(\int_{B(0,|x|)} f(y)dy\right)\Big/C_1\right)^{1/(-\gamma/p+1/p')} dx \le C_2$$

for nonnegative measurable functions f on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} f(y)^p(e + |y|)^{\beta}(\log(e + |y|))^{p-1}(\log(e + \log(e + |y|)))^{\gamma} dy \le 1. \tag{3.5}$$

For this, consider $\omega(r) = (e + r)^{\beta}(\log(e + r))^{p-1}(\log(e + \log(e + r)))^{\gamma}$ for $\gamma < p - 1$. Then, as above, we find by (3.5) that

$$\begin{aligned}\int_{B(0,|x|)} f(y)dy &\le C\omega_p(|x|)^{1/p'}\left(\int_{B(0,|x|)} g(y)dy\right)^{1/p} \\ &\le C(\log(e + \log(e + |x|)))^{-\gamma/p+1/p'}\left(\int_{B(0,|x|)} g(y)dy\right)^{1/p} \\ &\le C_1(\log(e + \varepsilon_1 \log(e + |x|)))^{-\gamma/p+1/p'}\end{aligned}$$

for $\varepsilon_1 > 0$ since $-\gamma/p + 1/p' > 0$, where

$$g(y) = f(y)^p(e + |y|)^{\beta}(\log(e + |y|))^{p-1}(\log(e + \log(e + |y|)))^{\gamma}.$$

Hence, we obtain

$$\exp\exp\left(\left(\int_{B(0,|x|)} f(y)dy\right)\Big/C_1\right)^{1/(-\gamma/p+1/p')} \le C(e + |x|)^{\varepsilon_1},$$

which gives the result.

Let $\mathbf{B}$ denote the unit ball in $\mathbb{R}^n$. Next, we study the inner Hardy–Trudinger inequality near the origin.

Theorem 3.6. *Let φ be a positive convex function on $(0, \infty)$ satisfying $(\varphi 0)$. For $p \geq 1$ and positive increasing functions ω, η on $(0, \infty)$, set*

$$\omega_p(r) = \int_0^r \omega(t)^{-p'/p} t^{n-1} dt$$

and

$$\sigma(|x|) = |x|^{n/p} \omega_p(|x|)^{1/p'} (\varphi^{-1}(|x|^{-n}))^{1/p} (\log(e/|x|))^{-1}.$$

Then, for $\varepsilon > 0$, there exists a constant $C > 1$ such that

$$\int_{\mathbf{B}} |x|^{\varepsilon - n} \exp\left(\left(\sigma(|x|)^{-1} \int_{B(0,|x|)} |f(y)| dy\right) \Big/ C\right) dx \leq 1$$

for measurable functions f on $\mathbb{R}^n$ satisfying

$$\int_{\mathbf{B}} \varphi(g(y)) dy \leq 1, \tag{3.6}$$

where $g(y) = |f(y)|^p \omega(|y|)$.

Proof. Let f be a measurable function on $\mathbb{R}^n$ satisfying (3.6). By using Hölder's inequality, we have

$$\begin{aligned}
&\int_{B(0,|x|)} |f(y)| dy \\
&\leq \left(\int_{B(0,|x|)} |f(y)|^p \omega(|y|) dy\right)^{1/p} \left(\int_{B(0,|x|)} \omega(|y|)^{-p'/p} dy\right)^{1/p'} \\
&\leq C \omega_p(|x|)^{1/p'} \left(\int_{B(0,|x|)} g(y) dy\right)^{1/p} \\
&= C |x|^{n/p} \omega_p(|x|)^{1/p'} \left(|x|^{-n} \int_{B(0,|x|)} g(y) dy\right)^{1/p}.
\end{aligned}$$

In view of Jensen's inequality, (3.6) and (φ^{-1}), we obtain

$$\begin{aligned}
&\int_{B(0,|x|)} |f(y)| dy \\
&\leq C |x|^{n/p} \omega_p(|x|)^{1/p'} \left(\varphi^{-1}\left(|x|^{-n} \int_{B(0,|x|)} \varphi(g(y)) dy\right)\right)^{1/p} \\
&\leq C |x|^{n/p} \omega_p(|x|)^{1/p'} (\varphi^{-1}(|x|^{-n}))^{1/p} \\
&= C \sigma(|x|) \log(e/|x|) \\
&\leq C_1 \sigma(|x|) (\varepsilon_1 \log(e/|x|)),
\end{aligned}$$

so that

$$\sigma(|x|)^{-1} \int_{B(0,|x|)} |f(y)|dy \le C_1(\varepsilon_1 \log(e/|x|)).$$

Hence, if $0 < \varepsilon_1 < \varepsilon$, then

$$\int_{\mathbf{B}} |x|^{\varepsilon-n} \exp\left(\left(\sigma(|x|)^{-1} \int_{B(0,|x|)} |f(y)|dy\right)\Big/ C_1\right)dx \le e^{\varepsilon_1} \int_{\mathbf{B}} |x|^{\varepsilon-\varepsilon_1-n}dx \le C,$$

which proves the result. ∎

Example 3.7. Consider $\omega(r) = r^\beta(\log(e+1/r))^\gamma$ with $\beta = np/p'$ and $\gamma > p-1$ and $\varphi(r) = \exp r - 1$. Then, for $0 < r < 1$,

$$\omega_p(r) \sim (\log(e/r))^{-p'\gamma/p+1}.$$

Moreover, if $\varepsilon > 0$, then

$$\varphi^{-1}(t) = \log(1+t) \le Ct^\varepsilon \quad \text{as } t \to \infty.$$

Hence, letting $\alpha = n/p$, we find

$$\sigma(t) = t^{n/p}\omega_p(t)^{1/p'}\{\varphi^{-1}(t^{-n})\}^{1/p}(\log(e/t))^{-1} \le C(\log(e/t))^{-\gamma/p-1/p} \tag{3.7}$$

for $0 < t < 1$.

Remark 3.8. We see that the exponent $-(1+\gamma)/p$ in (3.7) is best suited.

For this, consider $f(y) = |y|^{-n}(\log(e/|y|))^{-(\gamma+\delta_1)/p}$ on $\mathbf{B}$. Then, the following statements hold:

(1) $\int_{\mathbf{B}} f(y)^p|y|^\beta(\log(e/|y|))^\gamma dy = \int_{\mathbf{B}} |y|^{-n}(\log(e/|y|))^{-\delta_1}dy < \infty$ when $\beta = np/p'$ and $\delta_1 > 1$;

(2) $\int_{B(0,|x|)} f(y)dy \ge C_1(\log(e/|x|))^{1-(\gamma+\delta_1)/p}$ when $x \in \mathbf{B}$ and $(\gamma+\delta_1)/p > 1$;

(3) if $\varepsilon = 0$, then

$$\int_{\mathbf{B}} |x|^{\varepsilon-n} \exp\left(\left((\log(e/|x|))^{(\delta+\gamma)/p} \int_{B(0,|x|)} f(y)dy\right)\Big/(AC_1)\right)dx$$
$$\ge \int_{\mathbf{B}} |x|^{\varepsilon-n} \exp((\log(e/|x|))^{1+(\delta-\delta_1)/p}/A)dx = \infty$$

when $A > 0$ and $1 + (\delta-\delta_1)/p > 0$ and $(\gamma+\delta_1)/p > 1$;

(4) $\int_{\mathbf{B}} |x|^{\varepsilon-n} \exp(((\log(e/|x|))^{(\delta+\gamma)/p} \int_{B(0,|x|)} f(y)dy)/(AC_1))dx \ge \int_{\mathbf{B}} |x|^{\varepsilon-n} \exp((\log(e/|x|))^{1+(\delta-\delta_1)/p}/A)dx = \infty$ when $\delta > \delta_1 > 1$, $(\gamma+\delta_1)/p > 1$, $A > 0$ and $\varepsilon > 0$.

Corollary 3.9. Let $\beta/p = n/p'$ and $\gamma > p-1 \ge 0$. Then, for $\varepsilon > 0$, there exist constants $C_1, C_2 > 0$ such that

$$\int_{\mathbf{B}} |x|^{\varepsilon-n} \exp\left((\log(e/|x|))^{(1+\gamma)/p}\left(\int_{B(0,|x|)} f(y)dy\right)\Big/ C_1\right)dx \le C_2 \tag{3.8}$$

for nonnegative measurable functions f on $\mathbf{B}$ satisfying

$$\int_{\mathbf{B}} f(y)^p |y|^\beta (\log(e/|y|))^\gamma dy \le 1. \tag{3.9}$$

For this, consider

$$\omega(r) = r^\beta (\log e/r)^\gamma \quad \text{for } \gamma > p-1 \ge 0.$$

Then, as above, we find by (3.9) that

$$\begin{aligned}\int_{B(0,|x|)} f(y)dy &\le C(\log(e/|x|))^{-\gamma/p+1/p'} \\ &= C_1(\log(e/|x|))^{-\gamma/p+1/p'-1}(\varepsilon_1 \log(e/|x|))\end{aligned}$$

for $x \in \mathbf{B}$ and $\varepsilon_1 > 0$ since $-\gamma/p + 1/p' < 0$. Hence, we obtain

$$\exp\left((\log(e/|x|))^{(1+\gamma)/p}\left(\int_{B(0,|x|)} f(y)dy\right)\Big/C_1\right) \le (e/|x|)^{\varepsilon_1},$$

which gives the result.

Corollary 3.10. Let $\beta/p = n/p'$ and $\gamma > p-1 \ge 0$. Then, for $\varepsilon > 0$, there exist constants $C_1, C_2 > 0$ such that

$$\int_{\mathbf{B}} |x|^{\varepsilon-n} \exp\exp\left((\log(e\log(e/|x|)))^{(1+\gamma)/p}\left(\int_{B(0,|x|)} f(y)dy\right)\Big/C_1\right)dx \le C_2 \tag{3.10}$$

for nonnegative measurable functions f on $\mathbf{B}$ satisfying

$$\int_{\mathbf{B}} f(y)^p |y|^\beta (\log(e/|y|))^{p-1}(\log(e/(\log(e/|y|))))^\gamma dy \le 1. \tag{3.11}$$

For this, consider

$$\omega(r) = r^\beta (\log(e/r))^{p-1}(\log(e/(\log(e/r))))^\gamma \quad \text{for } \gamma > p-1 \ge 0.$$

Then, as above, we find by (3.11) that

$$\begin{aligned}\int_{B(0,|x|)} f(y)dy &\le C(\log(e\log(e/|x|)))^{-\gamma/p+1/p'} \\ &= C_1(\log(e\log(e/|x|)))^{-\gamma/p+1/p'-1}\log(1+\varepsilon_1\log(e/|x|))\end{aligned}$$

for $x \in \mathbf{B}$ and $\varepsilon_1 > 0$ since $-\gamma/p + 1/p' < 0$. Hence, we obtain

$$\exp\exp\left((\log(e\log(e/|x|)))^{(1+\gamma)/p}\left(\int_{B(0,|x|)} f(y)dy\right)\Big/C_1\right) \le C(e/|x|)^{\varepsilon_1},$$

which gives the result.

4. Hardy–Trudinger inequalities II

In this section, we give the outer Hardy–Trudinger inequality on $\mathbb{R}^n$.

Theorem 4.1. *Let φ be a positive convex function on $(0,\infty)$ satisfying $(\varphi 0)$. For $p \ge 1$ and a positive increasing function ω and a positive decreasing function η on $(0,\infty)$, set*

$$\omega_{\eta,p}(r) = \int_r^\infty \{\omega(t)\eta(t)\}^{-p'/p} t^{n-1}\,dt,$$
$$\sigma(r) = \omega_{\eta,p}(r)^{1/p'}(\varphi^{-1}(\eta(r)))^{1/p}(\log(e+r))^{-1}$$

and

$$\eta_1(r) = \int_{\mathbb{R}^n\setminus B(0,r)} \eta(|y|)\,dy.$$

Suppose η_1 is bounded. Then, for $\varepsilon > 0$, there exists a constant $C > 1$ such that

$$\int_{\mathbb{R}^n} (e+|x|)^{-\varepsilon-n} \exp\left(\left(\sigma(|x|)^{-1}\int_{\mathbb{R}^n\setminus B(0,|x|)} |f(y)|\,dy\right)\Big/ C\right) dx \le 1 \tag{4.1}$$

for measurable functions f on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} \varphi(g(y))\,dy \le 1, \tag{4.2}$$

where $g(y) = |f(y)|^p\omega(|y|)$.

Proof. Let f be a measurable function on $\mathbb{R}^n$ satisfying (4.2). By using Hölder's inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B(0,|x|)} |f(y)|\,dy \\
&\le \left(\int_{\mathbb{R}^n\setminus B(0,|x|)} |f(y)|^p\omega(|y|)\eta(|y|)\,dy\right)^{1/p}\left(\int_{\mathbb{R}^n\setminus B(0,|x|)} \{\omega(|y|)\eta(|y|)\}^{-p'/p}\,dy\right)^{1/p'} \\
&\le C\omega_{\eta,p}(|x|)^{1/p'}\left(\int_{\mathbb{R}^n\setminus B(0,|x|)} g(y)\eta(|y|)\,dy\right)^{1/p} \\
&= C\eta_1(|x|)^{1/p}\omega_{\eta,p}(|x|)^{1/p'}\left((\eta_1(|x|))^{-1}\int_{\mathbb{R}^n\setminus B(0,|x|)} g(y)\eta(|y|)\,dy\right)^{1/p}.
\end{aligned}$$

Since

$$C_0 = \sup_{x\in\mathbb{R}^n} \eta_1(|x|) < \infty,$$

we find by $(\varphi 2)$ and (φ^{-1}) that

$$\eta_1(|x|)\varphi^{-1}(t/\eta_1(|x|)) \le C_0\varphi^{-1}(t/C_0) \le C\varphi^{-1}(t) \quad \text{for } t > 0. \tag{4.3}$$

In view of Jensen's inequality, (4.2) and (4.3), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B(0,|x|)} |f(y)|dy \\
&\le C\eta_1(|x|)^{1/p}\omega_{\eta,p}(|x|)^{1/p'}\left(\varphi^{-1}\left(\eta_1(|x|)^{-1}\int_{\mathbb{R}^n\setminus B(0,|x|)}\varphi(g(y))\eta(|y|)dy\right)\right)^{1/p} \\
&\le C\eta_1(|x|)^{1/p}\omega_{\eta,p}(|x|)^{1/p'}(\varphi^{-1}(\eta_1(|x|)^{-1}\eta(|x|)))^{1/p} \\
&\le C\omega_{\eta,p}(|x|)^{1/p'}(\varphi^{-1}(\eta(|x|)))^{1/p} \\
&= C\sigma(|x|)\log(e+|x|) \\
&\le C_1\sigma(|x|)(\varepsilon_1\log(e+|x|)),
\end{aligned}$$

so that

$$\sigma(|x|)^{-1}\int_{\mathbb{R}^n\setminus B(0,|x|)} |f(y)|dy \le C_1(\varepsilon_1\log(e+|x|)).$$

Hence, if $0<\varepsilon_1<\varepsilon$, then

$$\begin{aligned}
&\int_{\mathbb{R}^n}(e+|x|)^{-\varepsilon-n}\exp\left(\left(\sigma(|x|)^{-1}\int_{\mathbb{R}^n\setminus B(0,|x|)}|f(y)|dy\right)\Big/C_1\right)dx \\
&\le \int_{\mathbb{R}^n}(e+|x|)^{\varepsilon_1-\varepsilon-n}dx \\
&\le C
\end{aligned}$$

when $0<\varepsilon_1<\varepsilon$, which proves the result. ∎

Example 4.2. Consider $\omega(r)=(e+r)^{\beta}(\log(e+r))^{\gamma}$ and $\eta(r)=(e+r)^{-n}(\log(e+r))^{-\delta}$. Assume that

$$(\beta-n)(-p'/p)=-n \quad (\text{or } \beta=np), \quad (\gamma-\delta)(-p'/p)+1<0, \quad \delta>1.$$

Then,

$$\omega_{\eta,p}(|x|)\sim(\log(e+|x|))^{(\gamma-\delta)(-p'/p)+1}$$

and

$$\sigma(|x|)\sim(\log(e+|x|))^{-\gamma/p+\delta/p+1/p'-1}(\varphi^{-1}((e+|x|)^{-n}(\log(e+|x|))^{-\delta}))^{1/p}.$$

When $\varphi(r)=\exp r-1$, we see that

$$\sigma(r)\sim(e+r)^{-n/p}(\log(e+r))^{-\gamma/p-1/p}. \tag{4.4}$$

Remark 4.3. We see that σ in (4.4) is best suited.

For this, $\sigma_1(r)=(e+r)^{-n/p}(\log(e+r))^{-\gamma/p-1/p-a}$ with $a>1/p'$ and consider $f(y)=(e+|x_0|)^{-n/p}(e+|y|)^{-n}(\log(e+|y|))^{b}$ on $B_0=B(0,|x_0|)$ with $|x_0|>1$; then,

$$g(y)=f(y)^p\omega(|y|)=(e+|x_0|)^{-n}(\log(e+|y|))^{bp+\gamma}\le C \quad \text{on } B_0,$$

so that

$$\int_{B_0} \varphi(g(y))dy \le C(e+|x_0|)^{-n} \int_{B_0} (\log(e+|y|))^{bp+\gamma} dy \le C(\log(e+|x_0|))^{bp+\gamma} \le C$$

when $bp+\gamma<0$.

If $|x|<|x_0|/2$, then

$$\sigma_1(|x|)^{-1} \int_{\mathbb{R}^n\setminus B(0,|x|)} f(y)dy \ge C(\log(e+|x_0|))^{\gamma/p+1/p+a} \int_{|x|}^{|x_0|} (e+t)^{-n}(\log(e+t))^b t^{n-1} dt \ge C_1(\log(e+|x_0|))^{\gamma/p+1/p+a+b},$$

so that

$$\int_{B(0,|x_0|/2)B(0,|x_0|/4)} (e+|x|)^{-\varepsilon-n} \exp\left(\sigma_1(|x|)^{-1} \int_{\mathbb{R}^n\setminus B(0,|x|)} |f(y)|dy/(AC_1)\right)dx \ge C|x_0|^{-\varepsilon} \exp((\log(e+|x_0|))^{\gamma/p+1/p+a+b}/A) \to \infty \quad \text{as } |x_0|\to\infty$$

when $\varepsilon>0$, $\gamma/p+1/p+a+b>1$ and $A>0$. This holds when

$a>1/p'$ and b is suitably chosen such that $b<-\gamma/p$.

Remark 4.4. We see that $-n/p$ in (4.4) is best suited.

For this, consider $f(y)=(e+|x_0|)^{-n-a}$ on $B_0=B(0,|x_0|^b)$ with $|x_0|>1$ and $b>1$; then,

$$g(y)=f(y)^p\omega(|y|) \le C(e+|x_0|)^{-p(n+a)+bnp}(\log(e+|x_0|))^{\gamma} \le C \quad \text{on } B_0,$$

so that

$$\int_{B_0} \varphi(g(y))dy \le C \int_{B_0} (e+|x_0|)^{-p(n+a)+bnp}(\log(e+|x_0|))^{\gamma} dy \le C$$

when $-p(n+a)+bnp+bn<0$. Moreover,

$$|x_0|^a \int_{\mathbb{R}^n\setminus B(0,|x|)} f(y)dy \ge C_1|x_0|^{-n+bn} \quad \text{for } |x|<|x_0|,$$

so that

$$\int_{B(0,|x_0|)} (e+|x|)^{-\varepsilon-n} \exp\left(\left((e+|x|)^a \int_{\mathbb{R}^n\setminus B(0,|x|)} |f(y)|dy\right)\Big/(AC_1)\right)dx \ge C|x_0|^{-\varepsilon}\exp((C|x_0|^{-n+bn})/A) \to \infty \quad \text{as } |x_0|\to\infty$$

when $\varepsilon>0$, $-p(n+a)+bnp+bn<0$ and $A>0$. This holds when $a>n/p$ and $b>1$ is suitably chosen.

In the same way as Corollaries 3.9 and 3.10, we obtain the following results.

Corollary 4.5. Let $\beta/p = n/p'$ and $\gamma > p-1 \geq 0$. Then, for $\varepsilon > 0$, there exist constants $C_1, C_2 > 0$ such that

$$\int_{\mathbb{R}^n} (e+|x|)^{-\varepsilon-n} \exp\left((\log(e+|x|))^{(1+\gamma)/p} \left(\int_{\mathbb{R}^n \setminus B(0,|x|)} f(y)dy \right) \Big/ C_1 \right) dx \leq C_2 \tag{4.5}$$

for nonnegative measurable functions f on $\mathbb{R}^n$ satisfying (3.4).

For this, consider $\omega(r) = (e+r)^\beta (\log(e+r))^\gamma$ and $\eta(r) = 1$. Then, we find

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus B(0,|x|)} f(y)dy \\
&\quad \leq C\omega_{\eta,p}(|x|)^{1/p'} \left(\int_{\mathbb{R}^n \setminus B(0,|x|)} g(y)(\eta(|y|))dy \right)^{1/p} \\
&\quad \leq C(\log(e+|x|))^{-\gamma/p+1/p'} \left(\int_{\mathbb{R}^n \setminus B(0,|x|)} g(y)dy \right)^{1/p} \\
&\quad \leq C_1(\log(e+|x|))^{-1-\gamma/p+1/p'} (\varepsilon_1 \log(e+|x|)) \left(\int_{\mathbb{R}^n \setminus B(0,|x|)} g(y)dy \right)^{1/p}
\end{aligned}$$

when $-\gamma/p + 1/p' < 0$, where $g(y) = |f(y)|^p (e+|y|)^\beta (\log(e+|y|))^\gamma$. This yields the result.

Corollary 4.6. Let $\beta/p = n/p'$ and $\gamma > p-1 \geq 0$. Then, for $\varepsilon > 0$ there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^n} (e+|x|)^{-\varepsilon-n} \\
&\quad \times \exp\exp\left((\log(e+\log(e+|x|)))^{(1+\gamma)/p} \left(\int_{\mathbb{R}^n \setminus B(0,|x|)} f(y)dy \right) \Big/ C_1 \right) dx \leq C_2
\end{aligned} \tag{4.6}$$

for nonnegative measurable functions f on $\mathbb{R}^n$ satisfying (3.5).

For this, it suffices to consider

$$\omega(r) = (e+r)^\beta (\log(e+r))^{p-1} (\log(e+(\log(e+r))))^\gamma \quad \text{and} \quad \eta(r) = 1.$$

Remark 4.7. We see that the exponent $(1+\gamma)/p$ in (4.5) is best suited.

For this, consider $f(y) = (e+|y|)^{-(n+\beta)/p} (\log(e+|y|))^{-(\gamma+\delta_1)/p}$. Then, the following statements hold:

(1) $\int_{\mathbb{R}^n} f(y)^p (e+|y|)^\beta (\log(e+|y|))^\gamma dy = \int_{\mathbb{R}^n} (e+|y|)^{-n} (\log(e+|y|))^{-\delta_1} dy < \infty$ when $\delta_1 > 1$;

(2) $\int_{\mathbb{R}^n \setminus B(0,|x|)} f(y)dy = \int_{\mathbb{R}^n \setminus B(0,|x|)} (e+|y|)^{-(n+\beta)/p} (\log(e+|y|))^{-(\gamma+\delta_1)/p} dy \geq C_1(\log(e+|x|))^{1-(\gamma+\delta_1)/p}$ when $(n+\beta)/p = n$ and $(\gamma+\delta_1)/p > 1$;

(3) $\int_{\mathbb{R}^n}(e+|x|)^{-\varepsilon-n}\exp(((\log(e+|x|))^{(\delta+\gamma)/p}\int_{\mathbb{R}^n\setminus B(0,|x|)} f(y)dy)/(AC_1))dx \geq \int_{\mathbb{R}^n}(e+|x|)^{-\varepsilon-n}\exp((\log(e+|x|))^{1+(\delta-\delta_1)/p}/A)dx = \infty$ when $\delta > \delta_1 > 1$, $1-(\gamma+\delta_1)/p < 0$, $A > 0$ and $\varepsilon > 0$.

We study the outer Hardy–Trudinger inequality near the origin.

Theorem 4.8. *Let φ and ψ be positive convex functions on $(0,\infty)$ satisfying $(\varphi 0)$. For $p \geq 1$ and a positive increasing function ω on $(0,\infty)$, set*

$$\omega_p(r) = \int_r^1 \omega(t)^{-p'/p} t^{n-1} dt.$$

Suppose that we have

$(\varphi, \psi, \omega_p)$ *for $\varepsilon > 0$ there exists a constant $A_2 > 0$ such that*

$$\psi(\omega_p(t)^{p/p'}/A_2) \leq t^{-\varepsilon} \quad \textit{for } t > 0.$$

Then, for $\varepsilon > 0$ there exists a constant $C > 1$ such that

$$\int_{\mathbf{B}} |x|^{\varepsilon-n}\psi\left(\left(\left(\int_{\mathbf{B}\setminus B(0,|x|)}|f(y)|dy\right)\Big/C\right)^p\right)dx \leq 1$$

for measurable functions f on $\mathbb{R}^n$ satisfying (3.6) with $g(y) = |f(y)|^p\omega(|y|)$.

Proof. Let f be a measurable function on $\mathbf{B}$ satisfying (3.6). By using Hölder's inequality, we have

$$\begin{aligned}
&\int_{\mathbf{B}\setminus B(0,|x|)}|f(y)|dy\\
&\leq \left(\int_{\mathbf{B}\setminus B(0,|x|)}|f(y)|^p\omega(|y|)dy\right)^{1/p}\left(\int_{\mathbf{B}\setminus B(0,|x|)}\omega(|y|)^{-p'/p}dy\right)^{1/p'}\\
&\leq C\omega_p(|x|)^{1/p'}\left(\int_{\mathbf{B}\setminus B(0,|x|)}g(y)dy\right)^{1/p}.
\end{aligned}$$

We set

$$c_4(x) = \int_{\mathbf{B}\setminus B(0,|x|)} dy.$$

Since $\sup_{x\in\mathbf{B}} c_4(x) < \infty$, we find by $(\varphi 2)$ and (φ^{-1}) that

$$c_4(x)\varphi^{-1}(t/c_4(x)) \leq C\varphi^{-1}(t) \quad \text{for } t > 0. \tag{4.7}$$

In view of Jensen's inequality, (3.6) and (4.7), we have

$$\begin{aligned}
&\int_{\mathbf{B}\setminus B(0,|x|)}|f(y)|dy\\
&\leq Cc_4(x)^{1/p}\omega_p(|x|)^{1/p'}\left(c_4(x)^{-1}\int_{\mathbf{B}\setminus B(0,|x|)}g(y)dy\right)^{1/p}\\
&\leq Cc_4(x)^{1/p}\omega_p(|x|)^{1/p'}\left(\varphi^{-1}\left(c_4(x)^{-1}\int_{\mathbf{B}\setminus B(0,|x|)}\varphi(g(y))dy\right)\right)^{1/p} \leq C_1\omega_p(|x|)^{1/p'}.
\end{aligned}$$

By $(\varphi, \psi, \omega_p)$, we obtain

$$\psi\left(\left(\left(\int_{\mathbf{B}\setminus B(0,|x|)} |f(y)|dy\right)\Big/(C_1 A_2^{1/p})\right)^p\right) \le |x|^{-\varepsilon_1}.$$

Hence,

$$\int_{\mathbf{B}} |x|^{\varepsilon-n}\psi\left(\left(\left(\int_{\mathbf{B}\setminus B(0,|x|)} |f(y)|dy\right)\Big/(C_1 A_2^{1/p})\right)^p\right)dx \le \int_{\mathbf{B}} |x|^{-n+\varepsilon-\varepsilon_1}dx \le C_2$$

when $0 < \varepsilon_1 < \varepsilon$, which yields the result. ■

Example 4.9. Consider $\omega(r) = r^\beta(\log(e/r))^\gamma$ with $\beta = np/p'$ and $\gamma < p-1$. Then,

$$\omega_p(r) \sim (\log(e/r))^{-\gamma p'/p+1}.$$

Let $\varphi(r) = \exp r - 1$ and

$$\psi(r) = \exp r^{1/(p(-\gamma/p+1/p'))} - 1. \tag{4.8}$$

Remark 4.10. We see that the exponent $1/(-\gamma/p + 1/p')$ in (4.8) is best suited.

For this, consider $f(y) = |y|^{-n}(\log(e/|y|))^{-(\gamma+\delta_1)/p}$ on $\mathbf{B}$. Then, the following statements hold:

(1) $\int_{\mathbf{B}} f(y)^p|y|^\beta(\log(e/|y|))^\gamma dy = \int_{\mathbf{B}} |y|^{-n}(\log(e/|y|))^{-\delta_1}dy < \infty$ when $\beta = np/p'$ and $\delta_1 > 1$;

(2) $\int_{\mathbf{B}\setminus B(0,|x|)} f(y)dy \ge C_1(\log(e/|x|))^{1-(\gamma+\delta_1)/p}$ when $x \in \mathbf{B}$ and $(\gamma+\delta_1)/p > 1$;

(3) $\int_{\mathbf{B}} |x|^{\varepsilon-n}\exp((\int_{\mathbf{B}\setminus B(0,|x|)} f(y)dy)/(AC_1))^{1/(-\gamma/p+1-\delta/p)}dx \ge \int_{\mathbf{B}} |x|^{\varepsilon-n}$ $\exp((\log(e/|x|))^{(-\gamma/p+1-\delta_1/p)/(-\gamma/p+1-\delta/p)}/A)dx = \infty$ when $1 < \delta_1 < \delta < p-\gamma$, $A > 0$ and $\varepsilon > 0$.

In the same way as Corollaries 3.4 and 3.5, we obtain the following results.

Corollary 4.11. Let $p \ge 1$, $\beta = n(p-1)$, $\gamma < p-1$ and $\varepsilon > 0$. Then, there exist constants $C_1, C_2 > 0$ such that

$$\int_{\mathbf{B}} |x|^{\varepsilon-n}\exp\left(\left(\int_{\mathbf{B}\setminus B(0,|x|)} f(y)dy\right)\Big/C_1\right)^{1/(-\gamma/p+1/p')}dx \le C_2 \tag{4.9}$$

for nonnegative measurable functions f on $\mathbf{B}$ satisfying (3.9).

Corollary 4.12. Let $p \ge 1$, $\beta = n(p-1)$, $\gamma < p-1$ and $\varepsilon > 0$. Then, there exist constants $C_1, C_2 > 0$ such that

$$\int_{\mathbf{B}} |x|^{\varepsilon-n}\exp\exp\left(\left(\int_{\mathbf{B}\setminus B(0,|x|)} f(y)dy\right)\Big/C_1\right)^{1/(-\gamma/p+1/p')}dx \le C_2$$

for nonnegative measurable functions f on $\mathbf{B}$ satisfying (3.10).

References

[1] J. S. Bradley, Hardy inequalities with mixed norms. *Canad. Math. Bull.* **21** (1978), no. 4, 405–408 Zbl 0402.26006 MR 0523580

[2] E. Burtseva, S. Lundberg, L.-E. Persson, and N. Samko, Multi-dimensional Hardy type inequalities in Hölder spaces. *J. Math. Inequal.* **12** (2018), no. 3, 719–729 Zbl 1410.46015 MR 3857358

[3] F. Deringoz, V. S. Guliyev, M. N. Omarova, and M. A. Ragusa, Calderón–Zygmund operators and their commutators on generalized weighted Orlicz–Morrey spaces. *Bull. Math. Sci.* **13** (2023), no. 1, article no. 2250004 Zbl 1537.42022 MR 4581184

[4] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*. 2nd edn., Cambridge University Press, Cambridge, 1952 Zbl 0047.05302 MR 0046395

[5] A. Kufner, L.-E. Persson, and N. Samko, *Weighted inequalities of Hardy type*. 2nd edn., World Scientific, Hackensack, NJ, 2017 Zbl 1380.26001 MR 3676556

[6] F.-Y. Maeda, Y. Mizuta, and T. Ohno, Approximate identities and Young type inequalities in variable Lebesgue–Orlicz spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$. *Ann. Acad. Sci. Fenn. Math.* **35** (2010), no. 2, 405–420 Zbl 1216.46025 MR 2731699

[7] V. Maz'ya, *Sobolev spaces with applications to elliptic partial differential equations*. Grundlehren Math. Wiss. 342, Springer, Heidelberg, 2011 Zbl 1217.46002 MR 2777530

[8] Y. Mizuta, *Potential theory in Euclidean spaces*. GAKUTO Internat. Ser. Math. Sci. Appl. 6, Gakkōtosho, Tokyo, 1996 Zbl 0849.31001 MR 1428685

[9] Y. Mizuta and T. Shimomura, Hardy and Sobolev inequalities in the half space. *Acta Math. Hungar.* **161** (2020), no. 1, 230–244 Zbl 1474.46063 MR 4110368

[10] Y. Mizuta and T. Shimomura, Hardy–Sobolev inequalities in the half-space for double phase functionals. *Rocky Mountain J. Math.* **51** (2021), no. 6, 2159–2169 Zbl 1497.46036 MR 4397671

[11] Y. Mizuta and T. Shimomura, Hardy–Sobolev inequalities in the unit ball for double phase functionals. *J. Math. Anal. Appl.* **501** (2021), no. 1, article no. 124133 Zbl 1478.46037 MR 4258797

[12] Y. Mizuta and T. Shimomura, Hardy–Sobolev inequality for Sobolev functions in central Herz–Morrey spaces. *Math. Nachr.* **294** (2021), no. 6, 1148–1159 Zbl 1535.46033 MR 4288490

[13] Y. Mizuta and T. Shimomura, Hardy and Sobolev inequalities for double phase functionals on the unit ball. *Math. Inequal. Appl.* **25** (2022), no. 2, 319–333 Zbl 1498.46038 MR 4428572

[14] Y. Mizuta and T. Shimomura, Hardy–Sobolev inequalities for double phase functionals. *Hokkaido Math. J.* **52** (2023), no. 2, 331–352 Zbl 1529.46021 MR 4612171

[15] Y. Mizuta and T. Shimomura, Hardy–Sobolev inequalities for Riesz potentials of functions in Orlicz spaces. *Acta Math. Hungar.* **171** (2023), no. 2, 221–240 Zbl 07794380 MR 4682766

[16] Y. Mizuta and T. Shimomura, Integrability for Hardy operators of double phase. *Z. Anal. Anwend.* **42** (2023), no. 3-4, 375–402 Zbl 1548.46024 MR 4699886

[17] Y. Mizuta and T. Shimomura, Hardy–Sobolev inequalities and boundary growth of Sobolev functions for double phase functionals on the half space. *Ric. Mat.* **73** (2024), no. 4, 1707–1723 Zbl 07909902 MR 4780063

[18] B. Muckenhoupt, Hardy's inequality with weights. *Studia Math.* **44** (1972), 31–38 Zbl 0236.26015 MR 0311856

[19] B. Opic and A. Kufner, *Hardy-type inequalities*. Pitman Res. Notes Math. Ser. 219, Longman Scientific & Technical, Harlow, 1990 Zbl 0698.26007 MR 1069756

[20] L.-E. Persson and N. Samko, Weighted Hardy and potential operators in the generalized Morrey spaces. *J. Math. Anal. Appl.* **377** (2011), no. 2, 792–806 Zbl 1211.42018 MR 2769175

[21] L.-E. Persson and S. G. Samko, A note on the best constants in some Hardy inequalities. *J. Math. Inequal.* **9** (2015), no. 2, 437–447 Zbl 1314.26021 MR 3333874

Received 2 March 2024; revised 24 April 2025.

Yoshihiro Mizuta
Department of Mathematics, Graduate School of Advanced Science and Engineering, Hiroshima University, Higashi-Hiroshima 739-8521, Japan; yomizuta@hiroshima-u.ac.jp

Tetsu Shimomura
Department of Mathematics, Graduate School of Humanities and Social Sciences, Hiroshima University, Higashi-Hiroshima 739-8524, Japan; tshimo@hiroshima-u.ac.jp