Z. Anal. Anwend. 44 (2025), 389–411
DOI 10.4171/ZAA/1777

On stability and regularity for subdiffusion equations involving delays

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Abstract. We study a class of nonlocal evolution equations involving time-varying delays which is employed to depict subdiffusion processes. The global solvability, stability and regularity are shown by using the resolvent theory, nonlocal Halanay inequality, fixed point argument and embeddings of fractional Sobolev spaces. Our result is applied to the nonlocal Fokker–Planck model with nonlinear force fields.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded domain with a smooth boundary, and let γ and h be positive numbers. Consider the following equation:

$$\partial_{t,k} u + (-\Delta)^{\gamma} u = G(u, u_{\rho}), \quad t > 0, \tag{1.1}$$

where the unknown $u = u(t,x)$, $t \geq -h$, $x \in \Omega$, obeys the initial condition

$$u(t,x) = \varphi(t,x), \quad t \in [-h, 0],\ x \in \Omega, \tag{1.2}$$

and the Dirichlet boundary condition

$$u(t,x) = 0, \quad t \geq -h,\ x \in \partial\Omega. \tag{1.3}$$

In (1.1), the notation $\partial_{t,k}$ denotes the nonlocal derivative in time of Caputo type with respect to the kernel $k \in L^1_{\text{loc}}(\mathbb{R}^+)$; i.e.,

$$\partial_{t,k} u(t,x) = \frac{\partial}{\partial t} \int_0^t k(t-\tau)[u(\tau,x) - u(0,x)]d\tau.$$

In addition, $u_{\rho}(t) = u(t - \rho(t))$ with $\rho \in C(\mathbb{R}^+)$ such that $-h \leq t - \rho(t) < t$, and G is a given function. In order to deal with (1.1), we use the following standing hypothesis.

(PC) The kernel function $k \in L^1_{\text{loc}}(\mathbb{R}^+)$ is nonnegative and nonincreasing, and there exists a function $l \in L^1_{\text{loc}}(\mathbb{R}^+)$ such that $k \star l(t) := \int_0^t k(t-\tau)l(\tau)d\tau = 1$ for $t \in (0, \infty)$.

Mathematics Subject Classification 2020: 35B40 (primary); 35R11, 35C15, 45D05, 45K05 (secondary).
Keywords: anomalous diffusion, nonlocal delayed PDE, solvability, regularity.

Let $g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$, $\alpha \in (0,1), t > 0$. Then, (1.1) in the well-known case $k(t) = g_{1-\alpha}(t)$ becomes a fractional/slow diffusion equation. For the case $k(t) = \int_0^1 g_\alpha(t)d\alpha$, we have (1.1) as an ultraslow diffusion model. Typical cases of the kernel k obeying (PC) were discussed in an extensive study, in which we refer to [10, 12, 13, 15, 19, 21, 22, 27, 29, 30] for significant results in qualitative theory, e.g., the solvability, regularity, and behavior of solutions. In general, (1.1) with k satisfying (PC) has been employed to describe processes in materials with memory. It is worth noting that the nonlocal derivative $\partial_{t,k}$ was mentioned in [1, 16, 20] as general fractional derivative, where the fundamental calculus, comparison principle, and the associated diffusion equation were studied.

As a matter of fact, various processes in chemistry, biology, and engineering are subject to inheritance, which appears in modeling as delayed terms. It should be noted that the presence of time delays affects remarkably the performance and stability of systems. From the mathematical point of view, the stability and regularity analysis for models subject to time delays is much more complicated. The first attempt was given in [6] to analyze the stability of solutions to (1.1) in the case G takes values in $L^2(\Omega)$; meanwhile, no attempt has been made to address the regularity of solutions, up to our investigation. Our aim in this work is to deal with the question of stability and regularity of solutions to (1.1) in the case G admits weak values, i.e., $G(u, u_\rho)$ belongs to a Hilbert scale $\mathbb{H}^\nu$ with negative order (see the next section for the definition). This allows us to consider various cases of nonlinearity, where G may contain polynomial/gradient terms. It should be mentioned that, this situation was studied for non-delayed model in [3]; however, the analyzing routine in the aforesaid work cannot apply to the delayed case. In addition, the stability result in this note extends nontrivially the one established in [6]. Indeed, the setting in [6] does not allow G to contain polynomial/gradient terms, the important cases in applications. Moreover, the stability statement in [6] does not provide any decay rates, which becomes an issue to address in this work.

The rest of our work is as follows. In the next section, we recall some notions and facts related to Hilbert scales and fractional Sobolev spaces, as well as some estimates for the resolvent families in Hilbert scales. Section 3 is devoted to showing the existence and stability results. In Section 4, we prove the Hölder continuity of solution to (1.1)–(1.3). In the last section, the obtained results are demonstrated for a nonlocal Fokker–Planck equation involving nonlinear force fields.

2. Preliminaries

In this section, we recall some notions and facts on functional spaces, the nonlocal Halanay inequality and the resolvent families, which are the ingredients for our analysis.

2.1. Hilbert scales and fractional Sobolev spaces

Let $\{e_n\}$ be the orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of the operator $-\Delta$ with homogeneous Dirichlet boundary condition, i.e.,

$$-\Delta e_n = \lambda_n e_n \quad \text{in } \Omega, \quad e_n = 0 \quad \text{on } \partial\Omega,$$

where we can assume that $\{\lambda_n\}$ is nondecreasing with $\lambda_1 > 0$. One can define the fractional Laplace operator $(-\Delta)^\gamma$, $\gamma \geq 0$ as follows:

$$D((-\Delta)^\gamma) = \left\{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2\gamma} v_n^2 < \infty \right\},$$

$$(-\Delta)^\gamma v = \sum_{n=1}^{\infty} \lambda_n^\gamma (v, e_n) e_n \quad \text{for } v \in D((-\Delta)^\gamma).$$

Here, the notation $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$.

For $\beta \geq 0$, let $\mathbb{H}^\beta = D((-\Delta)^{\frac{\beta}{2}})$; i.e.,

$$\mathbb{H}^\beta = \left\{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^\beta (v, e_n)^2 < \infty \right\}.$$

Then, $\mathbb{H}^\beta$ is a Hilbert space with the inner product

$$(u, v)_{\mathbb{H}^\beta} = \sum_{n=1}^{\infty} \lambda_n^\beta (u, e_n)(v, e_n)$$

and the norm given by

$$\|v\|_{\mathbb{H}^\beta}^2 = \sum_{n=1}^{\infty} \lambda_n^\beta (v, e_n)^2.$$

Obviously, $\mathbb{H}^0 = L^2(\Omega)$ and $\mathbb{H}^{\beta_1} \hookrightarrow \mathbb{H}^{\beta_2}$ if $\beta_1 \geq \beta_2 \geq 0$.

In addition, the embedding $\mathbb{H}^\beta \subset L^2(\Omega)$ is compact and the dual space of $\mathbb{H}^\beta$ can be identified with the space

$$\mathbb{H}^{-\beta} = \left\{ \phi = \sum_{n=1}^{\infty} \phi_n e_n : \sum_{n=1}^{\infty} \lambda_n^{-\beta} \phi_n^2 < \infty \right\},$$

endowed with norm

$$\|\phi\|_{\mathbb{H}^{-\beta}}^2 := \sum_{n=1}^{\infty} \lambda_n^{-\beta} \phi_n^2.$$

The space $\mathbb{H}^{-\beta}$ is monotonically increasing in β because the embedding $\mathbb{H}^{-\beta_2} \hookrightarrow \mathbb{H}^{-\beta_1}$ holds for $\beta_1 \geq \beta_2 \geq 0$. The Hilbert space $\mathbb{H}^\beta$, $\beta \in \mathbb{R}$, is said to be a scaling of $L^2(\Omega)$.

For $r > 0$ and $p \in [1, +\infty)$, denote by $W^{r,p}(\Omega)$ the fractional Sobolev space (see, e.g., [4, 5]). Let $H^r(\Omega) = W^{r,2}(\Omega)$. Then, we have the embeddings between Sobolev spaces, Hilbert scales, and Lebesgue spaces as follows.

Lemma 2.1 ([3]). *The following statements hold.*

(a) $L^p(\Omega) \hookrightarrow H^r(\Omega) \hookrightarrow \mathbb{H}^r$ *if* $\{-\frac{d}{2} < r \leq 0, p \geq \frac{2d}{d-2r}\}$.

(b) $\mathbb{H}^r \hookrightarrow H^r(\Omega) \hookrightarrow C^{0,\nu}(\Omega \cup \partial\Omega)$ *if* $\{r > \frac{d}{2}, \nu = r - \frac{d}{2}\}$.

(c) $\mathbb{H}^r \hookrightarrow H^r(\Omega) \hookrightarrow L^p(\Omega)$ *if*

$$\left\{ r = \frac{d}{2}, 1 \leq p < \infty \right\} \quad \text{or} \quad \left\{ 0 \leq r < \frac{d}{2}, 1 \leq p \leq \frac{2d}{d-2r} \right\}.$$

2.2. The auxiliary equations

Consider the following scalar integral equations:

$$s(t) + \mu(l \star s)(t) = 1, \qquad t \geq 0, \tag{2.1}$$

$$r(t) + \mu(l \star r)(t) = l(t), \quad t > 0. \tag{2.2}$$

The existence and uniqueness of s and r were analyzed in [25]. Recall that the function l is called a completely positive kernel iff $s(\cdot)$ and $r(\cdot)$ take nonnegative values for every $\mu > 0$. The complete positivity of l is equivalent to that (see [2]), there exist $\alpha \geq 0$ and $k \in L^1_{\text{loc}}(\mathbb{R}^+)$ nonnegative and nonincreasing which satisfy $\alpha l + l \star k = 1$. Particularly, the hypothesis (PC) ensures that l is completely positive.

Denote by $s(\cdot, \mu)$ and $r(\cdot, \mu)$ the solutions of (2.1) and (2.2), respectively. As mentioned in [30], the functions $s(\cdot, \mu)$ and $r(\cdot, \mu)$ take nonnegative values even in the case $\mu \leq 0$. We collect some additional properties of these functions.

Proposition 2.2 ([12, 30]). *Let the hypothesis (PC) hold. Then, for every $\mu > 0$, $s(\cdot, \mu)$, $r(\cdot, \mu) \in L^1_{\text{loc}}(\mathbb{R}^+)$. In addition, we have the following.*

(1) *The function $s(\cdot, \mu)$ is nonnegative and nonincreasing. Moreover,*

$$s(t, \mu)\left[1 + \mu \int_0^t l(\tau)d\tau\right] \leq 1 \quad \forall\, t \geq 0; \tag{2.3}$$

hence, if $l \notin L^1(\mathbb{R}^+)$, then $\lim_{t\to\infty} s(t, \mu) = 0$ for every $\mu > 0$.

(2) *The function $r(\cdot, \mu)$ is nonnegative, and one has*

$$s(t, \mu) = 1 - \mu \int_0^t r(\tau, \mu)d\tau = k \star r(\cdot, \mu)(t), \quad t \geq 0, \tag{2.4}$$

so $\int_0^t r(\tau, \mu)d\tau \leq \mu^{-1}\ \forall\, t > 0$. If $l \notin L^1(\mathbb{R}^+)$, then

$$\int_0^\infty r(\tau, \mu)d\tau = \mu^{-1}$$

for every $\mu > 0$.

(3) *Let $v(t) = s(t, \mu)v_0 + (r(\cdot, \mu) \star g)(t)$; here, $g \in L^1_{\text{loc}}(\mathbb{R}^+)$. Then, v solves the problem*

$$\frac{d}{dt}[k \star (v - v_0)](t) + \mu v(t) = g(t), \quad v(0) = v_0.$$

We now recall a Halanay-type inequality established recently in [11].

Proposition 2.3 ([11]). *Assume that, for a given $\Lambda > 0$, there exists a constant C_k^ρ such that*

$$\limsup_{t\to+\infty} \frac{s(t - \rho(t), \lambda)}{s(t, \lambda)} \leq C_k^\rho$$

for any $\lambda \in [0, \Lambda]$.

Let y be a nonnegative continuous function on $[0, +\infty)$ such that

$$y(t) \le s(t,a)y(0) + \int_0^t r(t-\tau,a)b(\tau)y(\tau-\rho(\tau))d\tau, \quad t > 0,$$
$$y(\theta) = \varphi(\theta), \quad \theta \le 0,$$

where $a > 0$, b and φ are bounded functions. Then, for all numbers μ such that

$$0 < \mu < a - C_k^\rho \limsup_{t\to\infty} b(t),$$

there exists a positive constant $C(\mu)$ independent of φ such that

$$y(t) \le C(\mu)s(t,\mu) \sup_{\theta\le 0} |\varphi(\theta)| \quad \forall\, t \ge 0.$$

Using the last result, we prove the following version of nonlocal Halanay inequality for our stability analysis.

Lemma 2.4. *Let λ, α, and β be positive numbers such that*

$$\lambda > \alpha + \beta C_k^\rho.$$

Assume that v is a nonnegative continuous function on $[0, +\infty)$ satisfying

$$v(t) \le s(t,\lambda)v(0) + \int_0^t r(t-\tau,\lambda)[\alpha v(\tau) + \beta v(\tau-\rho(\tau))]d\tau, \quad t > 0,$$
$$v(\theta) = \varphi(\theta), \quad \theta \le 0. \tag{2.5}$$

Then, for each number μ such that

$$0 < \mu < \lambda - \alpha - \beta C_k^\rho,$$

there exists a constant $C(\mu) > 0$ such that

$$v(t) \le C(\mu)s(t,\mu) \sup_{\theta\le 0} |\varphi(\theta)| \quad \forall\, t \ge 0. \tag{2.6}$$

Proof. Denote by $w(t)$ the right-hand side of (2.5) for $t > 0$ and $w(\xi) = \varphi(\xi)$ for $\xi \in [-h, 0]$. Then, $v(t) \le w(t)$ for all $t \ge -h$. In addition, by Proposition 2.2 (3), $w(\cdot)$ is the solution of the equation

$$\frac{d}{dt}[k \star (w - v(0))] + \lambda w(t) = \alpha v(t) + \beta v(t-\rho(t)), \quad t > 0.$$

It follows that

$$\frac{d}{dt}[k \star (w - v(0))] + (\lambda - \alpha)w(t) = \alpha(v(t) - w(t)) + \beta v(t-\rho(t)), \quad t > 0.$$

Now, using Proposition 2.2 (3) again, we obtain

$$\begin{aligned} w(t) &= s(t,\lambda-\alpha)v(0) + \int_0^t r(t-\tau,\lambda-\alpha)[\alpha(v(\tau)-w(\tau)) + \beta v(\tau-\rho(\tau))]d\tau \\ &\le s(t,\lambda-\alpha)v(0) + \beta\int_0^t r(t-\tau,\lambda-\alpha)v(\tau-\rho(\tau))d\tau, \end{aligned}$$

thanks to the positivity of $r(\cdot,\lambda-\alpha)$ and the fact that $v(\tau)\le w(\tau)$ for $\tau\ge 0$. This implies

$$v(t)\le w(t)\le s(t,\lambda-a)v_0 + \beta\int_0^t r(t-\tau,\lambda-a)v(\tau-\rho(\tau))d\tau,$$

and then, (2.6) follows thanks to Proposition 2.3. The proof is complete. ■

2.3. Resolvent operators

We now define the following families of operators:

$$S(t) = \sum_{n=1}^{\infty} s(t,\lambda_n^\gamma)\langle\cdot,e_n\rangle e_n, \quad t\ge 0, \tag{2.7}$$

$$R(t) = \sum_{n=1}^{\infty} r(t,\lambda_n^\gamma)\langle\cdot,e_n\rangle e_n, \quad t> 0. \tag{2.8}$$

Here, we use the notation $\langle\cdot,\cdot\rangle$ for both the inner product in $L^2(\Omega)$ and the dual pair $\langle\cdot,\cdot\rangle_{\mathbb{H}^{-\beta},\mathbb{H}^\beta}$ for some $\beta>0$, if no confusion arises.

We show some basic properties of these operators in the following lemma.

Lemma 2.5 ([3]). *Let $\{S(t)\}_{t\ge0}$ and $\{R(t)\}_{t>0}$ be the families of linear operators defined by (2.7) and (2.8), respectively. Then, the following statements hold.*

(i) *For $v\in\mathbb{H}^\sigma$ then, we have*

$$\|S(t)v\|_{\mathbb{H}^\sigma}\le s(t,\lambda_1^\gamma)\|v\|_{\mathbb{H}^\sigma}. \tag{2.9}$$

(ii) *For $g\in C([0,T];\mathbb{H}^\nu)$ then, we have $R\star g\in C([0,T];\mathbb{H}^\sigma)$; here, $\nu=\sigma-\gamma(\eta+1)$ and $\eta\in(0,1)$. In addition, we have*

$$\|R\star g(t)\|_{\mathbb{H}^\sigma}\le\left(\int_0^t \frac{l(t-\tau)}{(1\star l)^\eta(t-\tau)}\|g(\tau)\|^2_{\mathbb{H}^\nu}d\tau\right)^{\frac12}. \tag{2.10}$$

(iii) *For $g\in C([0,T];\mathbb{H}^{\sigma-\gamma})$ then $R\star g\in C([0,T];\mathbb{H}^\sigma)$. Moreover, one has*

$$\|R\star g(t)\|_{\mathbb{H}^\sigma}\le\left(\int_0^t r(t-\tau,\lambda_1^\gamma)\|g(\tau)\|^2_{\mathbb{H}^{\sigma-\gamma}}d\tau\right)^{\frac12}.$$

In what follows, we use the notation $u(t)$ for $u(t,\cdot)$ and consider u as a function defined on $[0,T]$, taking values in the space $\mathbb{H}^\beta$ for some $\beta\in\mathbb{R}$.

3. Solvability and stability results

In order to deal with the problem (1.1)–(1.3), we make the following assumption on the nonlinearity.

Assumption (G). Suppose that the nonlinearity function G induces the mapping $G : \mathbb{H}^\sigma \times \mathbb{H}^\sigma \to \mathbb{H}^\nu$ with $\sigma \geq 0$, $\nu = \sigma - \gamma(1+\eta_*) \leq 0$, $\eta_* \in (0,1)$, such that $G(0,0) = 0$ and

$$\|G(u_1, v_1) - G(u_2, v_2)\|_{\mathbb{H}^\nu} \leq L(r)(\|u_1 - u_2\|_{\mathbb{H}^\sigma} + \|v_1 - v_2\|_{\mathbb{H}^\sigma}) \tag{3.1}$$

for each $r > 0$ and for all $u_i, v_i \in \mathbb{H}^\sigma$ such that $\|u_i\|_{\mathbb{H}^\sigma} \leq r$, $\|v_i\|_{\mathbb{H}^\sigma} \leq r$, $i \in \{1; 2\}$; here, $L(\cdot)$ is a nonnegative function.

As stated in [12], we have the following definition.

Definition 3.1. A function $u \in C([-h, T]; \mathbb{H}^\sigma)$, $\sigma \geq 0$, is said to be a mild solution to the problem (1.1)–(1.3) on $[-h, T]$ if and only if

$$u(s) = \varphi(s) \quad \text{for } s \in [-h, 0]$$

and

$$u(t) = S(t)\varphi(0) + \int_0^t R(t-\tau)G(u(\tau), u_\rho(\tau))d\tau \quad \text{for } t \in [0, T].$$

If T can be arbitrary large, then the solution is said to be a global solution.

In what follows, we use $\|\cdot\|_{\sigma,\infty}$ to denote the supremum norm of a continuous function defined on a compact interval taking values in $\mathbb{H}^\sigma$.

Theorem 3.1. *Let $\varphi \in C([-h, 0]; \mathbb{H}^\sigma)$ and the assumption (G) hold. Suppose that*

$$\limsup_{r\to 0} L(r) = L_*$$

with

$$16L_*^2(1-\eta_*)^{-1}(1 \star l)^{1-\eta_*}(T) < 1. \tag{3.2}$$

Then, there exists $r_* > 0$ such that if

$$\|\varphi\|_{\sigma,\infty} \leq \frac{1}{2}r_*,$$

the problem (1.1)–(1.3) has a unique mild solution on $[0, T]$ satisfying

$$\|u(t)\|_{\mathbb{H}^\sigma} \leq r_*.$$

Proof. For $u \in C([0, T]; \mathbb{H}^\sigma)$ and $\varphi \in C([-h, 0]; \mathbb{H}^\sigma)$, denote by $u[\varphi]$ the function given by

$$u[\varphi](t) = \begin{cases} u(t) & \text{if } t > 0, \\ \varphi(t) & \text{if } -h \leq t \leq 0. \end{cases}$$

Let

$$\mathcal{C}_\varphi = \{u \in C([0,T];\mathbb{H}^\sigma) : u(0) = \varphi(0)\}$$

and

$$u_\rho[\varphi](\tau) = u[\varphi](\tau - \rho(\tau)), \quad \tau \geq 0.$$

We make use of the Banach contraction principle for the mapping defined as follows:

$$\mathcal{M} : \mathcal{C}_\varphi \to \mathcal{C}_\varphi,$$
$$\mathcal{M}[u](t) = S(t)\varphi(0) + \int_0^t R(t-\tau)G(u(\tau), u_\rho[\varphi](\tau))d\tau \quad \text{for } t \in [0,T].$$

Due to condition (3.2), we can choose a number $\varepsilon > 0$ such that

$$16(L_*^2 + \varepsilon)(1-\eta_*)^{-1}(1 \star l)^{1-\eta_*}(T) \leq 1. \tag{3.3}$$

By the formulation of L_*, there exists a number $r_* > 0$ such that

$$L(r)^2 \leq L_*^2 + \varepsilon \quad \forall\, r \in (0, r_*].$$

Let $\overline{B}_{r_*}$ be the closed ball centered at the origin with radius r_* in $\mathcal{C}_\varphi$. We will derive some estimates for $\mathcal{M}[u]$ with $u \in \overline{B}_{r_*}$ and

$$\|\varphi\|_{\sigma,\infty} \leq \frac{1}{2} r_*.$$

Firstly, in the light of Lemma 2.5 (ii) with $g(t) = G(u(t), u_\rho[\varphi](t))$, we have

$$\begin{aligned}
&\|R \star G(u, u_\rho[\varphi])(t)\|_{\mathbb{H}^\sigma}^2 \\
&\leq \int_0^t \frac{l(t-\tau)}{(1 \star l)^{\eta_*}(t-\tau)} \|G(u(\tau), u_\rho[\varphi](\tau))\|_{\mathbb{H}^\nu}^2 d\tau \\
&\leq 2\int_0^t \frac{l(t-\tau)}{(1 \star l)^{\eta_*}(t-\tau)} (L_*^2 + \varepsilon)(\|u(\tau)\|_{\mathbb{H}^\sigma}^2 + \|u_\rho[\varphi](\tau)\|_{\mathbb{H}^\sigma}^2) d\tau \\
&\leq 4(L_*^2 + \varepsilon) r_*^2 \int_0^t \frac{l(\tau)}{(1 \star l)^{\eta_*}(\tau)} d\tau \\
&\leq 4(L_*^2 + \varepsilon)(1-\eta_*)^{-1}(1 \star l)^{1-\eta_*}(T) r_*^2 \leq \frac{r_*^2}{4}.
\end{aligned}$$

Here, we used (3.1) and (3.3). Using the formulation of $\mathcal{M}$, we see that

$$\begin{aligned}
\|\mathcal{M}[u](t)\|_{\mathbb{H}^\sigma}^2 &\leq 2\|S(t)\varphi(0)\|_{\mathbb{H}^\sigma}^2 + 2\|R \star G(u, u_\rho[\varphi])(t)\|_{\mathbb{H}^\sigma}^2 \\
&\leq 2\|\varphi\|_{\sigma,\infty}^2 + \frac{r_*^2}{2} \leq r_*^2 \quad \forall\, t \in [0,T].
\end{aligned}$$

That means

$$\mathcal{M}[\overline{B}_{r_*}] \subset \overline{B}_{r_*}.$$

It remains to show that the mapping $\mathcal{M}$ is a contraction map on $\overline{B}_{r_*}$. Indeed, let $u, v \in \overline{B}_{r_*}$. Then, employing Lemma 2.5 (ii) and (3.1), we get

$$
\begin{aligned}
&\|\mathcal{M}[u](t) - \mathcal{M}[v](t)\|_{\mathbb{H}^\sigma}^2 \\
&\leq \int_0^t \frac{l(t-\tau)}{(1 \star l)^{\eta_*}(t-\tau)} \|G(u(\tau), u_\rho[\varphi](\tau)) - G(v(\tau), v_\rho[\varphi](\tau))\|_{\mathbb{H}^\nu}^2 d\tau \\
&\leq 2\int_0^t \frac{l(t-\tau)}{(1 \star l)^{\eta_*}(t-\tau)} (L_*^2 + \varepsilon)(\|u(\tau) - v(\tau)\|_{\mathbb{H}^\sigma}^2 + \|u_\rho[\varphi](\tau) - v_\rho[\varphi](\tau)\|_{\mathbb{H}^\sigma}^2) d\tau \\
&\leq 2(L_*^2 + \varepsilon)(\|u - v\|_{\sigma,\infty}^2 + \|u_\rho[\varphi] - v_\rho[\varphi]\|_{\sigma,\infty}^2) \int_0^t \frac{l(\tau)}{(1 \star l)^{\eta_*}(\tau)} d\tau \\
&= 4(L_*^2 + \varepsilon) \frac{(1 \star l)^{1-\eta_*}(T)}{1 - \eta_*} \|u - v\|_{\sigma,\infty}^2 \\
&\leq \frac{1}{4}\|u - v\|_{\sigma,\infty}^2 \quad \forall\, t \in [0, T],
\end{aligned}
$$

thanks to (3.3), which implies the contraction of $\mathcal{M}$ on $\overline{B}_{r_*}$. The proof is complete. ∎

In the case $G(u, u_\rho)$ takes values in $\mathbb{H}^{\sigma-\gamma}$ instead of $\mathbb{H}^{\sigma-\gamma(1+\eta_*)}$, we are able to prove that the solution of (1.1)–(1.3) exists globally and is asymptotically stable.

Theorem 3.2. *Assume that $G : \mathbb{H}^\sigma \times \mathbb{H}^\sigma \to \mathbb{H}^{\sigma-\gamma}$ obeys the conditions*

$$G(0, 0) = 0,\ \|G(u_1, v_1) - G(u_2, v_2)\|_{\mathbb{H}^{\sigma-\gamma}} \leq L(r)(\|u_1 - u_2\|_{\mathbb{H}^\sigma} + \|v_1 - v_2\|_{\mathbb{H}^\sigma})$$

for all $u_i, v_i \in \mathbb{H}^\sigma$ satisfying $\|u_i\|_{\mathbb{H}^\sigma} \leq r$, $\|v_i\|_{\mathbb{H}^\sigma} \leq r$, $i \in \{1; 2\}$, where $L(\cdot)$ is a non-negative function such that

$$L_* = \limsup_{r \to 0} L(r) < \left(\frac{\lambda_1^\gamma}{8}\right)^{\frac{1}{2}}. \tag{3.4}$$

Then, there exists $\delta > 0$ such that for any initial datum φ with $\|\varphi\|_{\sigma,\infty} \leq \delta$, the problem (1.1)–(1.3) has a unique global solution. Furthermore, if $l \notin L^1(\mathbb{R}^+)$ and

$$L_* < \frac{1}{2}\left(\frac{\lambda_1^\gamma}{1 + C_k^\rho}\right)^{\frac{1}{2}}, \tag{3.5}$$

then the obtained solution is asymptotically stable.

Proof. We first look for $\delta > 0$ and $r_* > 0$ such that

$$\mathcal{M}[\overline{B}_{r_*}] \subset \overline{B}_{r_*} \quad \text{for } \|\varphi\|_{\sigma,\infty} \leq \delta.$$

In view of assumption (3.4), one can take $\varepsilon > 0$ and $r_* > 0$ such that

$$L(r) \leq L_* + \varepsilon \leq \left(\frac{\lambda_1^\gamma}{8}\right)^{\frac{1}{2}} \quad \forall\, r \in (0, r_*]. \tag{3.6}$$

Then, for $u \in \overline{B}_{r_*}$ and $\|\varphi\|_{\sigma,\infty} \le r_*$, we get

$$\begin{aligned}
&\|\mathcal{M}[u](t)\|^2_{\mathbb{H}^\sigma}\\
&\le 2\|S(t)\varphi(0)\|^2_{\mathbb{H}^\sigma} + 2\|R \star G(u, u_\rho[\varphi])(t)\|^2_{\mathbb{H}^\sigma}\\
&\le 2s(t,\lambda_1^\gamma)^2\|\varphi(0)\|^2_{\mathbb{H}^\sigma} + 2\int_0^t r(t-\tau,\lambda_1^\gamma)\|G(u(\tau), u_\rho[\varphi](\tau))\|^2_{\mathbb{H}^{\sigma-\gamma}}d\tau\\
&\le 2s(t,\lambda_1^\gamma)^2\|\varphi(0)\|^2_{\mathbb{H}^\sigma} + 4(L_*+\varepsilon)^2\int_0^t r(t-\tau,\lambda_1^\gamma)(\|u(\tau)\|^2_{\mathbb{H}^\sigma} + \|u_\rho[\varphi](\tau)\|^2_{\mathbb{H}^\sigma})d\tau\\
&\le 2s(t,\lambda_1^\gamma)^2\|\varphi\|^2_{\sigma,\infty} + 8(L_*+\varepsilon)^2 r_*^2\int_0^t r(t-\tau,\lambda_1^\gamma)d\tau.
\end{aligned}$$

Now, using the properties of the relaxation functions stated in Proposition 2.2, we have

$$\begin{aligned}
\|\mathcal{M}[u](t)\|^2_{\mathbb{H}^\sigma} &\le 2s(t,\lambda_1^\gamma)\|\varphi\|^2_{\sigma,\infty} + 8(L_*+\varepsilon)^2 r_*^2\lambda_1^{-\gamma}(1-s(t,\lambda_1^\gamma))\\
&= 2s(t,\lambda_1^\gamma)[\|\varphi\|^2_{\sigma,\infty} - 4(L_*+\varepsilon)^2 r_*^2\lambda_1^{-\gamma}] + 8(L_*+\varepsilon)^2 r_*^2\lambda_1^{-\gamma}\\
&\le 8(L_*+\varepsilon)^2 r_*^2\lambda_1^{-\gamma},
\end{aligned}$$

provided that

$$\|\varphi\|_{\sigma,\infty} \le 2(L_*+\varepsilon)r_*\lambda_1^{-\gamma/2}.$$

Taking (3.6) into account, we obtain

$$\|\mathcal{M}[u](t)\|^2_{\mathbb{H}^\sigma} \le r_*^2 \quad \text{for } t \ge 0 \text{ and } \|\varphi\|_{\sigma,\infty} \le \delta$$

for

$$\delta = \min\left\{r_*; 2(L_*+\varepsilon)r_*\lambda_1^{-\gamma/2}\right\}.$$

This implies $\mathcal{M}[\overline{B}_{r_*}] \subset \overline{B}_{r_*}$. In the next step, we show that $\mathcal{M}$ is a contraction mapping on $\overline{B}_{r_*}$. For $u, v \in \overline{B}_{r_*}$, one gets

$$\begin{aligned}
&\|\mathcal{M}[u](t) - \mathcal{M}[v](t)\|^2_{\mathbb{H}^\sigma}\\
&\quad\le \int_0^t r(t-\tau,\lambda_1^\gamma)\|G(u(\tau), u_\rho[\varphi](\tau)) - G(v(\tau), v_\rho[\varphi](\tau))\|^2_{\mathbb{H}^{\sigma-\gamma}}d\tau\\
&\quad\le 2\int_0^t r(t-\tau,\lambda_1^\gamma)(L_*+\varepsilon)^2(\|u(\tau)-v(\tau)\|^2_{\mathbb{H}^\sigma} + \|u_\rho[\varphi](\tau) - v_\rho[\varphi](\tau)\|^2_{\mathbb{H}^\sigma})d\tau\\
&\quad\le 4(L_*+\varepsilon)^2\|u-v\|^2_\infty\int_0^t r(t-\tau,\lambda_1^\gamma)d\tau \le 4(L_*+\varepsilon)^2\lambda_1^{-\gamma}\|u-v\|^2_{\sigma,\infty}\\
&\quad\le \frac{1}{2}\|u-v\|^2_{\sigma,\infty} \quad \forall\, t \ge 0,
\end{aligned}$$

which implies the contraction of $\mathcal{M}$. We conclude that the problem admits a mild solution on $[0, T]$. Since the assumption of the theorem does not depend on $T > 0$, the obtained solution is global. We now testify the uniqueness. Let u, v be solutions of (1.1)–(1.3). Then, there exists $R > 0$ such that $\|u(t)\|_{H^\sigma}, \|v(t)\|_{\mathbb{H}^\sigma} \le R$ for $t \ge 0$. Hence,

$$\begin{aligned}
&\|u(t) - v(t)\|^2_{\mathbb{H}^\sigma} \\
&\quad \le \int_0^t r(t-\tau, \lambda_1^\gamma)\|G(u(\tau), u_\rho[\varphi](\tau)) - G(v(\tau), v_\rho[\varphi](\tau))\|^2_{\mathbb{H}^{\sigma-\gamma}} d\tau \\
&\quad \le 2\int_0^t r(t-\tau, \lambda_1^\gamma)L(R)^2(\|u(\tau) - v(\tau)\|^2_{\mathbb{H}^\sigma} + \|u_\rho[\varphi](\tau) - v_\rho[\varphi](\tau)\|^2_{\mathbb{H}^\sigma})d\tau
\end{aligned}$$

for all $t \ge 0$. Noting that

$$u_\rho[\varphi](\tau) - v_\rho[\varphi](\tau) = 0, \text{ for } \tau - \rho(\tau) \le 0,$$

we have

$$\begin{aligned}
&\|u(t) - v(t)\|^2_{\mathbb{H}^\sigma} \\
&\quad \le 2\int_0^t r(t-\tau, \lambda_1^\gamma)L(R)^2(\|u(\tau) - v(\tau)\|^2_{\mathbb{H}^\sigma} + \sup_{s\in[0,\tau]} \|u(s) - v(s)\|^2_{\mathbb{H}^\sigma})d\tau \\
&\quad \le 4L(R)^2 \int_0^t r(t-\tau, \lambda_1^\gamma) \sup_{s\in[0,\tau]} \|u(s) - v(s)\|^2_{\mathbb{H}^\sigma} d\tau.
\end{aligned}$$

Observing that the last integral is nondecreasing in t, we get

$$\sup_{s\in[0,t]} \|u(s) - v(s)\|^2_{\mathbb{H}^\sigma} \le 4L(R)^2 \int_0^t r(t-\tau, \lambda_1^\gamma) \sup_{s\in[0,\tau]} \|u(s) - v(s)\|^2_{\mathbb{H}^\sigma} d\tau.$$

It follows that $\sup_{s\in[0,t]} \|u(s) - v(s)\|_{\mathbb{H}^\sigma} = 0$ for all $t \ge 0$, thanks to the Gronwall-type inequality established in [12]. This ensures the uniqueness.

In the final step, we prove that the obtained solution is asymptotically stable, provided (3.5). assumption (3.5) enables us to take $\varepsilon > 0$ such that

$$4(L_* + \varepsilon)^2(1 + C_k^\rho) < \lambda_1^\gamma. \tag{3.7}$$

Let $\tilde{u}$ be the solution of (1.1) with respect to the initial datum $\tilde{\varphi}$ such that $\|\tilde{\varphi}\|_{\mathbb{H}^\sigma} \le \delta$. Then, $\tilde{u} \in \bar{B}_{r_*}$. Moreover, for $t \ge 0$, we have

$$\begin{aligned}
&\|\tilde{u}[\tilde{\varphi}](t) - u[\varphi](t)\|^2_{\mathbb{H}^\sigma} \le 2s^2(t, \lambda_1^\gamma)\|\tilde{\varphi}(0) - \varphi(0)\|^2_{\mathbb{H}^\sigma} \\
&\quad + 2\int_0^t r(t-\tau, \lambda_1^\gamma)\|G(\tilde{u}[\tilde{\varphi}](\tau), \tilde{u}_\rho[\tilde{\varphi}](\tau)) - G(u[\varphi](\tau), u_\rho[\varphi](\tau))\|^2_{\mathbb{H}^{\sigma-\gamma}} d\tau \\
&\le 2s(t, \lambda_1^\gamma)\|\tilde{\varphi}(0) - \varphi(0)\|^2_{\mathbb{H}^\sigma} \\
&\quad + 4(L_* + \varepsilon)^2 \int_0^t r(t-\tau, \lambda_1^\gamma)(\|\tilde{u}[\tilde{\varphi}](\tau) - u[\varphi](\tau)\|^2_{\mathbb{H}^\sigma} + \|\tilde{u}_\rho[\tilde{\varphi}](\tau) - u_\rho[\varphi](\tau)\|^2_{\mathbb{H}^\sigma})d\tau.
\end{aligned}$$

Denote

$$v(t) = \|\tilde{u}[\tilde{\varphi}](t) - u[\varphi](t)\|^2_{\mathbb{H}^\sigma}, \quad t \ge -h.$$

Then, the last estimate reads

$$v(t) \le 2s(t,\lambda_1^\gamma)v(0) + 4(L_* + \varepsilon)^2 \int_0^t r(t-\tau,\lambda_1^\gamma)[v(\tau) + v(\tau-\rho(\tau))]d\tau, \quad t \ge 0.$$

Due to (3.7), one can apply Lemma 2.4 to get

$$v(t) \le C(\mu)s(t,\mu) \sup_{\tau\in[-h,0]} v(\tau), \quad t > 0$$

for $0 < \mu < \lambda_1^\gamma - 4(L_* + \varepsilon)^2(1 + C_k^\rho)$; here, $C(\mu)$ is a positive constant.

Since $l \notin L^1(\mathbb{R}^+)$, we get $s(t,\mu) \to 0$ as $t \to \infty$, which completes the proof. ■

4. Hölder regularity result

We first recall some notions related to the regularity of the resolvent families. For $l \in L^1_{\text{loc}}(\mathbb{R}^+)$, we denote by $\hat{l}$ the Laplace transform of l.

Definition 4.1 ([28]). Let $l \in L^1_{\text{loc}}(\mathbb{R}^+)$ be a function of subexponential growth, i.e.,

$$\int_0^\infty |l(t)|e^{-\varepsilon t}dt < \infty$$

for every $\varepsilon > 0$.

- Suppose that $\hat{l}(\lambda) \ne 0$ for all $\text{Re}(\lambda) > 0$. For $\theta > 0$, l is said to be θ-sectorial if $|\arg \hat{l}(\lambda)| \le \theta$ for all $\text{Re}(\lambda) > 0$.
- For given $m \in \mathbb{N}$, l is called m-regular if there exists a constant $c > 0$ such that

$$|\lambda^n \hat{l}^{(n)}(\lambda)| \le c|\hat{l}(\lambda)| \quad \text{for all } \text{Re}(\lambda) > 0,\ 0 \le n \le m.$$

In this section, we replace (PC) with a stronger one to get the differentiability of the resolvent family.

(K) The assumption (PC) is satisfied with l being 2-regular and θ-sectorial for some $\theta < \pi$.

Lemma 4.1 ([12]). *Let* (K) *hold. Then, the resolvent family* $S(\cdot)$ *is differentiable on* $(0,\infty)$ *and the estimate*

$$\|S'(t)\| \le \frac{M}{t}, \quad t \in (0,\infty), \tag{4.1}$$

hold for some $M \ge 1$.

For given positive numbers σ, r_*, r, and $\eta \in (0, \frac{1}{2})$, denote

$$\mathbb{B}^{\sigma,\eta}_{r_*,r} := \left\{ u \in \mathcal{C}_\varphi : \|u\|_{\sigma,\infty} \le r_*;\ \sup_{0<t<\tilde{t}\le T} \frac{t^\eta \|u(\tilde{t}) - u(t)\|_{\mathbb{H}^\sigma}}{(\tilde{t}-t)^\eta} \le r \right\}.$$

Denote $a \wedge b = \min(a; b)$. Let

$$\ell_\rho(t,\tilde{t}) = \frac{|\tilde{t}-\rho(\tilde{t})|^\eta \wedge |t-\rho(t)|^\eta}{|\tilde{t}-t+\rho(t)-\rho(\tilde{t})|^\eta},$$
$$\ell(t,\tilde{t}) = \frac{t^\eta}{(\tilde{t}-t)^\eta},$$
$$\Psi(t,\tilde{t}) = \frac{1}{\ell(t,\tilde{t})^2} + \frac{1}{\ell_\rho(t,\tilde{t})^2}, \quad 0 < t < \tilde{t}.$$

Theorem 4.2. *Let the hypothesis of Theorem 3.1 and (K) hold. Then, the obtained solution to the problem (1.1)–(1.3) is Hölder continuous on* $(0, T]$*, provided that*

$$\kappa_1 := 6L_*^2 \sup_{0<t<\tilde{t}\le T} \ell(t,\tilde{t})^2 \int_0^t \frac{l(\tau)\Psi(t-\tau,\tilde{t}-\tau)}{(1\star l)^{\eta_*}(\tau)} d\tau < 1,$$
$$\kappa_2 := \sup_{0<t<\tilde{t}\le T} \ell(t,\tilde{t})^2 \int_t^{\tilde{t}} \frac{l(\tau)}{(1\star l)^{\eta_*}(\tau)} d\tau < \infty,$$
$$\|\varphi(\tilde{\tau})-\varphi(\tau)\|_{\mathbb{H}^\sigma} \le L_\varphi|\tilde{\tau}-\tau|^\eta \quad \forall\, \tilde{\tau},\tau \in [-h,0],$$

where $\eta \in (0, \frac{1}{2})$ *and* L_φ *is a positive constant.*

Proof. We will show that the solution map $\mathcal{M}$ is a contraction on $\mathbb{B}^{\sigma,\eta}_{r_*,r}$ for a suitable $r > 0$ and r_* taken from Theorem 3.1. It suffices to show that $\mathcal{M}[\mathbb{B}^{\sigma,\eta}_{r_*,r}] \subset \mathbb{B}^{\sigma,\eta}_{r_*,r}$. Let

$$d_G(u)(t,\tilde{t}) = \|G(u(\tilde{t}), u_\rho[\varphi](\tilde{t})) - G(u(t), u_\rho[\varphi](t))\|^2_{\mathbb{H}^\nu}$$

for $u \in \mathcal{C}_\varphi$, $\nu = \sigma - \gamma(1+\eta^*)$, and $0 < t < \tilde{t} \le T$.

Assume that $u \in \mathbb{B}^{\sigma,\eta}_{r_*,r}$; then, we have

$$d_G(u)(t,\tilde{t}) \le 2(L_*+\varepsilon)^2\big(\|u(\tilde{t})-u(t)\|^2_{\mathbb{H}^\sigma} + \|u_\rho[\varphi](\tilde{t})-u_\rho[\varphi](t)\|^2_{\mathbb{H}^\sigma}\big), \tag{4.2}$$

where $\varepsilon > 0$ is adjusted such that

$$\kappa_1(\varepsilon) := 6(L_*+\varepsilon)^2 \sup_{0<t<\tilde{t}\le T} \ell(t,\tilde{t})^2 \int_0^t \frac{l(\tau)\Psi(t-\tau,\tilde{t}-\tau)}{(1\star l)^{\eta_*}(\tau)} d\tau < 1.$$

Case 1. $\tilde{t}-\rho(\tilde{t}) > 0$ and $t-\rho(t) > 0$. We see that

$$\ell_\rho(t,\tilde{t})^2\|u_\rho[\varphi](\tilde{t})-u_\rho[\varphi](t)\|^2_{\mathbb{H}^\sigma} = \ell_\rho(t,\tilde{t})^2\|u(\tilde{t}-\rho(\tilde{t}))-u(t-\rho(t))\|^2_{\mathbb{H}^\sigma} \le r^2,$$

thanks to the formulation of $\mathbb{B}^{\sigma,\eta}_{r_*,r}$ and ℓ_ρ.

Case 2. $\tilde{t}-\rho(\tilde{t}) > 0$ and $t-\rho(t) \le 0$. One observes that

$$\ell_\rho(t,\tilde{t}) \le \frac{|\tilde{t}-\rho(\tilde{t})|^\eta \wedge |t-\rho(t)|^\eta}{|\tilde{t}-\rho(\tilde{t})|^\eta} \le 1.$$

So, we have

$$\ell_\rho(t,\tilde t)^2\|u_\rho[\varphi](\tilde t)-u_\rho[\varphi](t)\|^2_{\mathbb{H}^\sigma} \le \|u(\tilde t-\rho(\tilde t))-\varphi(t-\rho(t))\|^2_{\mathbb{H}^\sigma} \le 2\|u(\tilde t-\rho(\tilde t))\|^2_{\mathbb{H}^\sigma}+2\|\varphi(t-\rho(t))\|^2_{\mathbb{H}^\sigma}\le 4r_*^2.$$

It is obvious that the same estimate follows in the case $\tilde t-\rho(\tilde t)\le 0$ and $t-\rho(t)>0$.

Case 3. $\tilde t-\rho(\tilde t)\le 0$ and $t-\rho(t)\le 0$. Clearly,

$$\ell_\rho(t,\tilde t)^2\|u_\rho[\varphi](\tilde t)-u_\rho[\varphi](t)\|^2_{\mathbb{H}^\sigma} = \ell_\rho(t,\tilde t)^2\|\varphi(\tilde t-\rho(\tilde t))-\varphi(t-\rho(t))\|^2_{\mathbb{H}^\sigma} \le L_\varphi|\tilde t-\rho(\tilde t)|^\eta\wedge|t-\rho(t)|^\eta\le L_\varphi h.$$

Summing up, we get

$$\ell_\rho(t,\tilde t)^2\|u_\rho[\varphi](\tilde t)-u_\rho[\varphi](t)\|^2_{\mathbb{H}^\sigma}\le \max\{r^2;4r_*^2;L_\varphi h\}. \tag{4.3}$$

Combining (4.2)–(4.3) yields

$$\begin{aligned}d_G(u)(t,\tilde t)&\le 2(L_*+\varepsilon)^2\big(\|u(\tilde t)-u(t)\|^2_{\mathbb{H}^\sigma}+\|u_\rho[\varphi](\tilde t)-u_\rho[\varphi](t)\|^2_{\mathbb{H}^\sigma}\big)\\&\le 2(L_*+\varepsilon)^2\left(\|u(\tilde t)-u(t)\|^2_{\mathbb{H}^\sigma}+\frac{\max\{r^2;4r_*^2;L_\varphi h\}}{\ell_\rho(t,\tilde t)^2}\right)\\&\le 2(L_*+\varepsilon)^2\left(\frac{r^2}{\ell(t,\tilde t)^2}+\frac{\max\{r^2;4r_*^2;L_\varphi h\}}{\ell_\rho(t,\tilde t)^2}\right).\end{aligned}\tag{4.4}$$

We now testify $\mathcal{M}[u]\in\mathbb{B}^{\sigma,\eta}_{r_*,r}$ by showing that

$$\ell(t,\tilde t)\|\mathcal{M}[u](\tilde t)-\mathcal{M}[u](t)\|_{\mathbb{H}^\sigma}\le r\quad \forall\, 0<t<\tilde t\le T.$$

Indeed, it follows from the formulation of $\mathcal{M}$ that

$$\begin{aligned}&\mathcal{M}[u](\tilde t)-\mathcal{M}[u](t)\\&\quad=[S(\tilde t)-S(t)]u\\&\qquad+\int_0^t R(\tau)[G(u(\tilde t-\tau),u_\rho[\varphi](\tilde t-\tau))-G(u(t-\tau),u_\rho[\varphi](t-\tau))]d\tau\\&\qquad+\int_t^{\tilde t}R(\tau)G(u(\tilde t-\tau),u_\rho[\varphi](\tilde t-\tau))d\tau.\end{aligned}$$

Employing Lemma 2.5 (ii), we have

$$\begin{aligned}\|\mathcal{M}[u](\tilde t)-\mathcal{M}[u](t)\|^2_{\mathbb{H}^\sigma}&\le 3\|[S(\tilde t)-S(t)]\varphi(0)\|^2_{\mathbb{H}^\sigma}\\&\quad+3\int_0^t\frac{l(\tau)}{(1\star l)^{\eta_*}(\tau)}d_G(u)(\tilde t-\tau,t-\tau)d\tau\\&\quad+3\int_t^{\tilde t}\frac{l(\tau)}{(1\star l)^{\eta_*}(\tau)}\|G(u(\tilde t-\tau),u_\rho[\varphi](\tilde t-\tau))\|^2_{\mathbb{H}^\nu}d\tau\\&=3(I_1+I_2+I_3).\end{aligned}$$

For I_1, we see that

$$I_1 \leq \left(\int_t^{\tilde{t}} \|S'(\tau)\varphi(0)\|_{\mathbb{H}^\sigma} d\tau\right)^2 \leq M^2\|\varphi(0)\|^2_{\mathbb{H}^\sigma}\left(\int_t^{\tilde{t}} \frac{d\tau}{\tau}\right)^2$$
$$\leq M^2\|\varphi(0)\|^2_{\mathbb{H}^\sigma}\left(\ln\left(1+\frac{\tilde{t}-t}{t}\right)\right)^2 \leq M^2\|\varphi(0)\|^2_{\mathbb{H}^\sigma} t^{-2\eta}(\tilde{t}-t)^{2\eta}, \tag{4.5}$$

where Lemma 4.1 and the inequality $\ln(1+a) \leq \frac{a^\eta}{\eta}$ for $a > 0, \eta \in (0,1)$ have been utilized.

Regarding I_2, we employ (4.4) to get

$$I_2 \leq 2(L_* + \varepsilon)^2 r^2 \int_0^t \frac{l(\tau)d\tau}{(1 \star l)^{\eta_*}(\tau)\ell(t-\tau,\tilde{t}-\tau)^2}$$
$$+ 2(L_* + \varepsilon)^2 \max\{r^2; 4r_*^2; L_\varphi h\} \int_0^t \frac{l(\tau)d\tau}{(1 \star l)^{\eta_*}(\tau)\ell_\rho(t-\tau,\tilde{t}-\tau)^2}.$$

Choosing $r \geq \max\{2r_*, \sqrt{L_\varphi h}\}$, we have

$$I_2 \leq 2(L_* + \varepsilon)^2 r^2 \int_0^t \frac{l(\tau)}{(1 \star l)^{\eta_*}(\tau)}\left[\frac{1}{\ell(t-\tau,\tilde{t}-\tau)^2} + \frac{1}{\ell_\rho(t-\tau,\tilde{t}-\tau)^2}\right]d\tau. \tag{4.6}$$

Concerning I_3, one has

$$I_3 \leq 2(L_* + \varepsilon)^2 \int_t^{\tilde{t}} \frac{l(\tau)}{(1 \star l)^{\eta_*}(\tau)}(\|u(\tilde{t}-\tau)\|^2_{\mathbb{H}^\sigma} + \|u_\rho[\varphi](\tilde{t}-\tau)\|^2_{\mathbb{H}^\sigma})d\tau$$
$$\leq 4(L_* + \varepsilon)^2 r_*^2 \int_t^{\tilde{t}} \frac{l(\tau)}{(1 \star l)^{\eta_*}(\tau)} d\tau. \tag{4.7}$$

It follows from (4.5)–(4.7) that

$$\ell(t,\tilde{t})^2\|\mathcal{M}[u](\tilde{t}) - \mathcal{M}[u](t)\|^2_{\mathbb{H}^\sigma} \leq 3M^2\|\varphi(0)\|^2_{\mathbb{H}^\sigma} + \kappa_1(\varepsilon)r^2 + 12(L_* + \varepsilon)^2\kappa_2 r_*^2.$$

Since $\kappa_1(\varepsilon) < 1$, one can take $r > 0$ large enough such that

$$\ell(t,\tilde{t})^2\|\mathcal{M}[u](\tilde{t}) - \mathcal{M}[u](t)\|^2_{\mathbb{H}^\sigma} \leq r^2 \quad \forall\, 0 < t < \tilde{t} \leq T,$$

which ensures that $\mathcal{M}[u] \in \mathbb{B}^{\sigma,\eta}_{r_*,r}$. The proof is complete. ∎

5. Application to nonlocal Fokker–Planck equations with nonlinear force fields

This section is devoted to a demonstration of the obtained results in a model of nonlocal Fokker–Planck equations.

Consider the problem (1.1)–(1.3) with

$$G(u, u_\rho) = \vec{F}(u_\rho) \cdot \nabla u, \tag{5.1}$$

where $\vec{F} = (f_1, \ldots, f_d)$ is a vector field depending on the history state u_ρ. Moreover, let

$$k(t) = g_{1-\alpha}(t) + \mu g_{1-\beta}(t), \quad 0 < \alpha < \beta < 1, \ \mu > 0,$$
$$\rho(t) = qt + h, \quad q \in (0,1), \ h > 0.$$

In this case, equation (1.1) is a multi-term fractional diffusion model with the delayed term being of pantograph type.

Equation (1.1) with G given by (5.1) becomes a nonlocal Fokker–Planck model, where $\vec{F}$ is the force field subject to a feedback control. In this circumstance, the controller requires an interval of delayed time to sense information. The constitution of fractional Fokker–Planck equations can be found in [23, 24] as a useful approach for the description of transport dynamics in complex systems which are governed by anomalous diffusion and non-exponential relaxation patterns. For recent achievements in qualitative study and numerical analysis for nonlocal Fokker–Planck equations, we refer to [9, 14, 17, 18, 26].

5.1. Hölder regularity

Assume that $f_i : \mathbb{R} \to \mathbb{R}, 1 \le i \le d$, obeys $f_i(0) = 0$ and there exists $L_F > 0$ such that

$$|f_i(\tilde{z}) - f_i(z)| \le L_F |\tilde{z} - z| \quad \forall\, z, \tilde{z} \in \mathbb{R}.$$

Let $1 \le \sigma < \frac{d}{2}, \eta_* \in (0,1)$, and $\nu = \sigma - \gamma(1+\eta_*) \in (-\frac{d}{2}, 0)$. Then, for $v_j, \tilde{v}_j \in \mathbb{H}^\sigma$, $1 \le j \le 2$ and $q_* = \frac{2d}{d-2\nu}$, we have

$$\begin{aligned}
&\|G(v_2, \tilde{v}_2) - G(v_1, \tilde{v}_1)\|_{L^{q_*}} \\
&\quad \le \sum_{k=1}^{d} \|(f_k(\tilde{v}_2) - f_k(\tilde{v}_1))\partial_{x_k} v_1\|_{L^{q_*}} + \sum_{k=1}^{d} \|f_k(\tilde{v}_2)\partial_{x_k}(v_2 - v_1)\|_{L^{q_*}} \\
&\quad \le L_F \sum_{k=1}^{d} \|(\tilde{v}_2 - \tilde{v}_1)\partial_{x_k} v_1\|_{L^{q_*}} + L_F \sum_{k=1}^{d} \|\tilde{v}_2 \partial_{x_k}(v_2 - v_1)\|_{L^{q_*}}.
\end{aligned} \tag{5.2}$$

Let $p = -\frac{d}{\nu} = \frac{d}{\gamma(1+\eta_*)-\sigma}$. Since $\frac{1}{q_*} = \frac{1}{2} + \frac{1}{p}$, applying the generalized Hölder inequality for (5.2) gives

$$\begin{aligned}
\|G(v_2, \tilde{v}_2) - G(v_1, \tilde{v}_1)\|_{L^{q_*}} &\le L_F \sum_{k=1}^{d} \|\tilde{v}_2 - \tilde{v}_1\|_{L^p} \|\partial_{x_k} v_1\|_{L^2} \\
&\quad + L_F \sum_{k=1}^{d} \|\tilde{v}_2\|_{L^p} \|\partial_{x_k}(v_2 - v_1)\|_{L^2} \\
&\le L_F (\|\tilde{v}_2 - \tilde{v}_1\|_{L^p} \|v_1\|_{\mathbb{H}^1} + \|\tilde{v}_2\|_{L^p} \|v_2 - v_1\|_{\mathbb{H}^1}).
\end{aligned}$$

Let $p \le p^* := \frac{2d}{d-2\sigma}$. Then,

$$\mathbb{H}^\sigma \subset L^{p^*}(\Omega) \subset L^p(\Omega).$$

This implies

$$\|G(v_2, \tilde{v}_2) - G(v_1, \tilde{v}_1)\|_{L^{q_*}} \le C(\|\tilde{v}_2 - \tilde{v}_1\|_{\mathbb{H}^\sigma}\|v_1\|_{\mathbb{H}^1} + \|\tilde{v}_2\|_{\mathbb{H}^\sigma}\|v_2 - v_1\|_{\mathbb{H}^1}),$$

where $C > 0$ is a generic constant depending on L_F and constants in Sobolev embeddings. Therefore,

$$\|G(v_2, \tilde{v}_2) - G(v_1, \tilde{v}_1)\|_{\mathbb{H}^\nu} \le C(\|\tilde{v}_2 - \tilde{v}_1\|_{\mathbb{H}^\sigma}\|v_1\|_{\mathbb{H}^\sigma} + \|\tilde{v}_2\|_{\mathbb{H}^\sigma}\|v_2 - v_1\|_{\mathbb{H}^\sigma}),$$

thanks to the fact that $\mathbb{H}^\nu \subset L^{q_*}$ and $\mathbb{H}^\sigma \subset \mathbb{H}^1$. This allows us to state that assumption (G) is fulfilled with $L(r) = Cr$ and

$$L_* = \limsup_{r\to 0} L(r) = 0.$$

We are now in a position to verify the technical conditions imposed in Theorem 4.2.

Since k is completely monotonic, (PC) is satisfied due to [8, Theorem 5.4, p. 159] (see also [25]). In addition, the associated kernel l is nonincreasing (see [2]) and admits the Laplace transform as

$$\hat{l}(\lambda) = \frac{1}{\lambda^\alpha + \mu\lambda^\beta} \quad \forall\, \mathrm{Re}\,\lambda > 0.$$

Hence,

$$\widehat{1 \star l}(\lambda) = \frac{1}{\lambda^{\alpha+1} + \mu\lambda^{\beta+1}} \sim \frac{1}{\mu\lambda^{\beta+1}} \quad \text{as } \lambda \to +\infty.$$

Using the Karamata–Feller Tauberian theorem in [7, Theorem 3, p. 445] with the interchange between 0 and ∞, we get

$$1 \star l(t) \sim \frac{t^\beta}{\mu\Gamma(\beta+1)}, \quad l(t) \sim \frac{t^{\beta-1}}{\mu\Gamma(\beta)} \quad \text{as } t \to 0. \tag{5.3}$$

It was testified in [12] that the kernel l is 2-regular and $\frac{\pi}{2}$-sectorial. So, the assumption (K) takes place.

For given delayed function ρ and $\eta \in (0, \frac{1}{2})$, we have

$$\ell_\rho(t, \tilde{t}) = \frac{|(1-q)\tilde{t} - h|^\eta \wedge |(1-q)t - h|^\eta}{(1-q)^\eta|\tilde{t} - t|^\eta} = \frac{|\tilde{t} - \tilde{h}|^\eta \wedge |t - \tilde{h}|^\eta}{|\tilde{t} - t|^\eta}, \quad \text{with } \tilde{h} = \frac{h}{1-q}. \tag{5.4}$$

Denote

$$\begin{aligned}\Lambda(t, \tilde{t}) &= \ell(t, \tilde{t})^2 \int_0^t \frac{l(\tau)\Psi(t-\tau, \tilde{t}-\tau)}{(1 \star l)^{\eta_*}(\tau)} d\tau \\ &= \ell(t, \tilde{t})^2 \int_0^t \frac{l(\tau)}{(1 \star l)^{\eta_*}(\tau)} \left[\frac{1}{\ell(t-\tau, \tilde{t}-\tau)^2} + \frac{1}{\ell_\rho(t-\tau, \tilde{t}-\tau)^2}\right] d\tau.\end{aligned}$$

Then, our aim is to demonstrate that $\sup_{0<t<\tilde{t}\leq T} \Lambda(t,\tilde{t}) < \infty$. One has

$$\begin{aligned}\Lambda(t,\tilde{t}) &= \frac{t^{2\eta}}{|\tilde{t}-t|^{2\eta}} \int_0^t \frac{l(\tau)|\tilde{t}-t|^{2\eta}}{(1\star l)^{\eta_*}(\tau)(t-\tau)^{2\eta}} d\tau \\ &\quad + \frac{t^{2\eta}}{|\tilde{t}-t|^{2\eta}} \int_0^t \frac{l(\tau)|\tilde{t}-t|^{2\eta}}{(1\star l)^{\eta_*}(\tau)(|\tilde{t}-\tau-\tilde{h}|^\eta \wedge |t-\tau-\tilde{h}|^\eta)^2} d\tau \\ &= t^{2\eta} \int_0^t \frac{l(\tau)}{(1\star l)^{\eta_*}(\tau)(t-\tau)^{2\eta}} d\tau \\ &\quad + t^{2\eta} \int_0^t \frac{l(\tau)}{(1\star l)^{\eta_*}(\tau)(|\tilde{t}-\tau-\tilde{h}|^\eta \wedge |t-\tau-\tilde{h}|^\eta)^2} d\tau \\ &= \Lambda_1(t,\tilde{t}) + \Lambda_2(t,\tilde{t}),\end{aligned}$$

thanks to (5.4). According to (5.3), we get

$$\begin{aligned}\Lambda_1(t,\tilde{t}) &= t^{2\eta} \int_0^t \frac{l(\tau)}{(1\star l)^{\eta_*}(\tau)(t-\tau)^{2\eta}} d\tau \lesssim t^{2\eta} \int_0^t \tau^{(1-\eta_*)\beta-1}(t-\tau)^{-2\eta} d\tau \\ &\lesssim t^{2\eta} g_{(1-\eta_*)\beta} \star g_{1-2\eta}(t) = t^{2\eta} g_{(1-\eta_*)\beta+1-2\eta}(t),\end{aligned}$$

where the property $g_{\eta_1} \star g_{\eta_2} = g_{\eta_1+\eta_2}$ was employed and the notation $A \lesssim B$ means $A \leq mB$ for some $m > 0$. So,

$$\Lambda_1(t,\tilde{t}) \lesssim t^{(1-\eta_*)\beta} \leq T^{(1-\eta_*)\beta} \quad \forall\, 0 < t < \tilde{t} \leq T.$$

Estimating $\Lambda_2(t,\tilde{t})$, we assume that $T < \tilde{h}$. Then,

$$|\tilde{t}-\tau-\tilde{h}|^\eta \wedge |t-\tau-\tilde{h}|^\eta = (\tau+\tilde{h}-\tilde{t})^\eta.$$

It follows that

$$\begin{aligned}\Lambda_2(t,\tilde{t}) &= t^{2\eta} \int_0^t \frac{l(\tau)d\tau}{(1\star l)^{\eta_*}(\tau)(\tau+\tilde{h}-\tilde{t})^{2\eta}} \\ &\leq \frac{T^{2\eta}}{(\tilde{h}-T)^{2\eta}} \int_0^t \frac{l(\tau)d\tau}{(1\star l)^{\eta_*}(\tau)} \leq \frac{T^{2\eta}(1\star l)^{1-\eta_*}(T)}{(1-\eta_*)(\tilde{h}-T)^{2\eta}} \quad \forall\, 0 < t < \tilde{t} \leq T.\end{aligned}$$

We have shown that $\sup_{0<t<\tilde{t}\leq T} \Lambda(t,\tilde{t})$ is finite. It remains to test the finiteness of

$$\Theta(t,\tilde{t}) := \ell(t,\tilde{t})^2 \int_t^{\tilde{t}} \frac{l(\tau)}{(1\star l)^{\eta_*}(\tau)} d\tau, \quad 0 < t < \tilde{t} \leq T.$$

Using the mean value theorem, we get

$$\Theta(t,\tilde{t}) := \ell(t,\tilde{t})^2(\tilde{t}-t)\frac{l(t+\theta(\tilde{t}-t))}{(1\star l)^{\eta_*}(t+\theta(\tilde{t}-t))} \quad \text{for some } \theta \in [0,1].$$

Since the function $t \mapsto \frac{l(t)}{(1\star l)^{\eta_*}(t)}$, $t > 0$, is nonincreasing, we have

$$\Theta(t,\tilde{t}) \leq \ell(t,\tilde{t})^2(\tilde{t}-t)\frac{l(t)}{(1\star l)^{\eta_*}(t)} = t^{2\eta}(\tilde{t}-t)^{1-2\eta}\frac{l(t)}{(1\star l)^{\eta_*}(t)}.$$

In view of (5.3), one has

$$\frac{t^{2\eta}l(t)}{(1\star l)^{\eta_*}(t)} \sim \frac{\beta^{\eta_*}}{[\mu\Gamma(\beta)]^{1-\eta_*}}t^{2\eta+(1-\eta_*)\beta-1} \quad \text{as } t \to 0.$$

Thus, $\Theta(t,\tilde{t})$ is finite if $2\eta + (1-\eta_*)\beta - 1 \geq 0$. In this case, we obtain the Hölder regularity of solution to (1.1)–(1.3).

5.2. Asymptotic stability

We consider a special case of the nonlinearity as follows:

$$G(u,u_\rho) = \vec{F}^*(\|u_\rho\|_{L^p})\cdot\nabla u,$$

where $\vec{F}^* = (f_1^*,\dots,f_d^*)$, $f_i^*, 1 \leq i \leq d$, are smooth functions and $1 < p \leq \frac{2d}{d-2}$. Obviously, $G(u,u_\rho)$ belongs to $L^2(\Omega)$ as long as $u \in \mathbb{H}^1$. In this case, the force field $\vec{F}^*$ is of nonlocal type.

Then, for $v_j, \tilde{v}_j \in \mathbb{H}^1$ such that $\|v_j\|_{\mathbb{H}^1} \leq r$, $\|\tilde{v}_j\|_{\mathbb{H}^1} \leq r$, $j = 1,2$, we observe that

$$\begin{aligned}
\|\tilde{v}_j\|_{L^p(\Omega)} &\leq C_p\|\tilde{v}_j\|_{\mathbb{H}^1} \quad \text{for } C_p > 0,\\
\|G(v_2,\tilde{v}_2)-G(v_1,\tilde{v}_1)\|_{L^2} &\leq \sum_{k=1}^d \left|f_k^*(\|\tilde{v}_2\|_{L^p}) - f_k^*(\|\tilde{v}_1\|_{L^p})\right|\|\partial_{x_k}v_1\|_{L^2}\\
&\quad + \sum_{k=1}^d \left|f_k^*(\|\tilde{v}_2\|_{L^p})\right|\|\partial_{x_k}(v_2-v_1)\|_{L^2}\\
&\leq \sum_{k=1}^d C_k(r)\left|\|\tilde{v}_2\|_{L^p} - \|\tilde{v}_1\|_{L^p}\right|\|\partial_{x_k}v_1\|_{L^2}\\
&\quad + \sum_{k=1}^d D_k(r)\|\partial_{x_k}(v_2-v_1)\|_{L^2},
\end{aligned}$$

where

$$C_k(r) = \sup_{0\leq z\leq C_p r}|f_k^{*\prime}(z)|, \quad D_k(r) = \sup_{0\leq z\leq C_p r}|f_k^*(z)|.$$

Then,

$$\begin{aligned}
\|G(v_2,\tilde{v}_2)-G(v_1,\tilde{v}_1)\|_{L^2} &\leq C(r)\|\tilde{v}_2-\tilde{v}_1\|_{L^p}\|v_1\|_{\mathbb{H}^1} + D(r)\|v_2-v_1\|_{\mathbb{H}^1}\\
&\leq C(r)r\|\tilde{v}_2-\tilde{v}_1\|_{\mathbb{H}^1} + D(r)\|v_2-v_1\|_{\mathbb{H}^1},
\end{aligned}$$

thanks to the embedding $\mathbb{H}^1 \subset L^p(\Omega)$, where

$$C(r) = \max_{1\le k\le d}\{C_k(r)\}, \quad D(r) = \max_{1\le k\le d}\{D_k(r)\}.$$

Therefore, the hypotheses of Theorem 3.2 hold with

$$\gamma = 1, \quad \sigma = 1, \quad L(r) = \max\{C(r)r; D(r)\}, \quad L_* = D(0) = \max_{1\le k\le d}\{|f_k^*(0)|\},$$

which ensure the asymptotic stability of solution to (1.1)–(1.3) in this setting.

Remark 5.1. (i) It should be noted that our approach is able to work with various cases of kernel function satisfying (PC). The case $k(t) = g_{1-\alpha}(t)$, $\alpha \in (0,1)$, is well known. We have demonstrated, in the last section, our results in the case $k(t) = g_{1-\alpha}(t) + \mu g_{1-\beta}(t)$, $0<\alpha<\beta<1, \mu>0$, corresponding to the multi-term fractional diffusion equation. Recall that, the assumption (K) is imposed to ensure the differentiability of $S(t)$ for $t>0$. This assumption is fulfilled in the important case, namely, the tempered fractional diffusion:

$$k(t) = g_{1-\alpha}(t)e^{-\gamma t}, \quad \alpha \in (0,1), \gamma > 0.$$

In this case, we get $\hat{l}(\lambda) = \lambda^{-1}(\lambda+\gamma)^{1-\alpha}$ and it was shown in [3] that l is $\frac{\pi}{2}$-sectorial and 2-regular.

(ii) We now mention another case of kernel function:

$$k(t) = g_\beta(t)E_{\alpha,\beta}(-\omega t^\alpha), \quad 0<\alpha<\beta<1,\ \omega>0,$$

where $E_{\alpha,\beta}(\cdot)$ is the standard Mittag–Leffler function. This case is referred to as the weighted fractional diffusion [22]. In this case, one has $\hat{l}(\lambda) = \frac{\lambda^\alpha+\omega}{\lambda^{1+\alpha-\beta}}$ (see, e.g., [27]). Let Re$\lambda > 0$. Then,

$$|\arg(\lambda^\alpha+\omega)| < |\arg(\lambda^\alpha)| = \alpha|\arg(\lambda)| < \frac{\pi}{2},$$

thanks to the fact that $\omega > 0$. In addition,

$$\left|\arg\Big(\frac{1}{\lambda^{1+\alpha-\beta}}\Big)\right| = |\arg(\lambda^{1+\alpha-\beta})| = (1+\alpha-\beta)|\arg(\lambda)| < \frac{\pi}{2}.$$

The last estimates imply that $|\arg(\hat{l}(\lambda))| < \frac{\pi}{2}$, which guarantees that l is $\frac{\pi}{2}$-sectorial. We are in a position to check that l is 2-regular. By direct computations, we have

$$\begin{aligned}
\lambda\hat{l}'(\lambda) &= \Big(\beta-1-\frac{\alpha\omega}{\lambda^\alpha+\omega}\Big)\hat{l}(\lambda),\\
\lambda^2\hat{l}''(\lambda) &= (\lambda^2\hat{l}')'(\lambda) - 2\lambda\hat{l}'(\lambda)\\
&= \Big(\beta-1-\frac{\alpha\omega}{\lambda^\alpha+\omega}\Big)^2\hat{l}(\lambda) - 2\Big(\beta-1-\frac{\alpha\omega}{\lambda^\alpha+\omega}\Big)\hat{l}(\lambda)\\
&\quad + \Big(\beta-1-\alpha\omega\frac{(1-\alpha)\lambda^\alpha+\omega}{(\lambda^\alpha+\omega)^2}\Big)\hat{l}(\lambda).
\end{aligned}$$

Since Re$\lambda > 0$, one gets

$$|\lambda^\alpha + \omega| > \omega, \quad |(1-\alpha)\lambda^\alpha + \omega| < |\lambda^\alpha + \omega|.$$

Thus,

$$|\lambda \hat{l}'(\lambda)| \le (1-\beta+\alpha)|\hat{l}(\lambda)|,$$
$$|\lambda^2 \hat{l}''(\lambda)| \le [(1-\beta+\alpha)^2 + 3(1-\beta+\alpha)]|\hat{l}(\lambda)|,$$

which implies that l is 2-regular.

(iii) Finally, let us consider the case

$$k(t) = \int_0^1 g_\alpha(t)\,d\alpha, \quad t > 0,$$

which is referred to as the ultra-slow diffusion. In this case, we get the explicit formula for l as follows:

$$l(t) = \int_0^\infty \frac{e^{-st}\,ds}{1+s}, \quad t > 0,$$

and $\hat{l}(\lambda) = \frac{\ln\lambda}{\lambda-1}$. It was shown in [28, Theorem 2.1, Example 2.2] that the resolvent $S(t)$ admits an analytic extension on the sector $\Sigma(0,\theta) = \{\lambda \in \mathbb{C} : |\arg(\lambda)| < \theta\}$ for some $\theta < \pi$. In particular, we obtain the differentiability of $S(t)$ for $t > 0$, regardless of testing the assumption (K).

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Received 24 February 2024; revised 18 August 2024.

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