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A Logvinenko–Sereda theorem for vector-valued functions and application to control theory

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Abstract. We prove a Logvinenko–Sereda theorem for vector-valued functions. That is, for an arbitrary Banach space X, all $p \in [1,\infty]$, all $\lambda \in (0,\infty)^d$, all $f \in L^p(\mathbb{R}^d; X)$ with $\operatorname{supp} \mathcal{F} f \in \times_{i=1}^d (-\lambda_i/2, \lambda_i/2)$, and all thick sets $E \subseteq \mathbb{R}^d$, we have

$$\|\mathbf{1}_E f\|_{L^p(\mathbb{R}^d;X)} \geq C \|f\|_{L^p(\mathbb{R}^d;X)}.$$

The constant is explicitly known independent of the geometric parameters of the thick set and the parameter λ. As an application, we study control theory for normally elliptic operators on Banach spaces whose coefficients of their symbol are given by bounded linear operators. This includes systems of coupled parabolic equations or problems depending on a parameter.

1. Introduction

The paper is split into two parts. The first part concerns a generalization of the classical Logvinenko–Sereda theorem to vector valued functions. The second part then studies an application to control theory.

The Logvinenko–Sereda theorem goes back at least to the papers [42, 43], and has been proven independently in [27, 34]. In order to formulate its result, we introduce some notation. Let $\rho \in (0,1]$ and $L = (L_i)_{i=1}^d \in (0,\infty)^d$. A set $E \subseteq \mathbb{R}^d$ is called (ρ, L)-*thick* if E is measurable, and for all $x \in \mathbb{R}^d$, we have

$$\left| E \cap \left(\bigtimes_{i=1}^d (0, L_i) + x \right) \right| \geq \rho \prod_{i=1}^d L_i. \tag{1.1}$$

Here, $|\cdot|$ denotes the Lebesgue measure. For $\lambda \in (0,\infty)^d$, we use the notation

$$\Pi_\lambda = \times_{i=1}^d (-\lambda_i/2, \lambda_i/2) \tag{1.2}$$

for the parallelepiped with side lengths λ_i, $i \in \{1, 2, \ldots, d\}$. For $f \in L^p(\mathbb{R}^d)$, we denote by $\mathcal{F} f$ its Fourier transform. The results of the above-mentioned papers can be summarized as follows.

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Theorem 1.1. *For all $p \in [1,\infty]$, all $\lambda \in (0,\infty)^d$, all $\rho > 0$, all $L \in (0,\infty)^d$, and all (ρ, L)-thick sets $E \subseteq \mathbb{R}^d$, there exists a constant $C \geq 1$ such that for all $f \in L^p(\mathbb{R}^d)$ with* $\operatorname{supp} \mathcal{F} f \subseteq \Pi_\lambda$, *we have*

$$\|\mathbf{1}_E f\|_{L^p(\mathbb{R}^d)} \geq C \|f\|_{L^p(\mathbb{R}^d)}. \tag{1.3}$$

Thus, the result compares the overall L^p-norm of the function f with its norm only on a thick subset $E \subset \mathbb{R}^d$. The papers [27, 34] also show that the constant C can be chosen as $C = c_1 \mathrm{e}^{c_2|\lambda|}$ with some positive constants c_1 and c_2 depending only on the space dimension and the geometric parameters ρ and L. This result has been significantly improved in [29, 30], in which it is shown that C can be chosen as

$$C = \Big(\frac{\rho}{K}\Big)^{K(1+\lambda\cdot L)}$$

with some positive constant K depending only on the dimension, which appears to be optimal. Subsequently, the classical Logvinenko–Sereda theorem has been adapted to various settings, e.g., to L^2-functions whose Fourier–Bessel transform is supported in an interval [23], or to functions on the torus in [16].

In the case $p = 2$, the condition $\operatorname{supp} \mathcal{F} f \subseteq \Pi_\lambda$ is implied by $f \in \operatorname{ran} P_{\sqrt{\lambda}}(-\Delta)$, where $-\Delta$ denotes the negative Laplacian and $P_{\sqrt{\lambda}}(-\Delta)$ denotes the associated spectral projector on $L^2(\mathbb{R}^d)$ onto energies below $\sqrt{\lambda}$. One can therefore ask whether Theorem 1.1 continues to hold if we assume that $f \in \operatorname{ran} P_\lambda(H)$ for a certain self-adjoint operator H acting on $L^2(\mathbb{R}^d)$. This is indeed the case if $H = -\Delta_g + V$, where g is an analytic perturbation of the flat metric and $V : \mathbb{R}^d \to \mathbb{R}$ is analytic and decays at infinity, as shown in [31]. Moreover, the recent [15] provides a sufficient condition (a Bernstein-like inequality) for $f \in \operatorname{ran} P_\lambda(H)$ such that inequality (1.3) with $p = 2$ holds for thick observation sets E. Examples include the pure Laplacian, which is covered by Theorem 1.1, divergence-type operators, and the harmonic oscillator.

In this paper, we generalize Theorem 1.1 to vector-valued functions $f \in L^p(\mathbb{R}^d; X)$ with values in an arbitrary Banach space X. It is formulated in Theorem 3.1. Let us stress that the substantial novelty of Theorem 3.1 is that X may be of infinite dimension. In particular, this allows to consider infinite-dimensional state spaces in our application to control theory. This is the topic of the second part of our paper which we introduce in the following.

We consider for $T > 0$ the linear control problem

$$\partial_t y(t) + A_p y(t) = \mathbf{1}_E u(t), \quad y(0) = y_0 \in \mathcal{X}^p = L^p(\mathbb{R}^d; X), \quad t \in [0,T], \tag{1.4}$$

where X is an arbitrary Banach space, $p \in [1,\infty)$, A_p is a normally elliptic differential operator in $\mathcal{X}^p$, and where $E \subset \mathbb{R}^d$ is a thick set. We study null-controllability in $L^r([0,T]; \mathcal{X}^p)$ with $r \in [1,\infty]$, that is, for all $y_0 \in \mathcal{X}^p$ there exists a control function $u \in L^r([0,T]; \mathcal{X}^p)$ such that the mild solution y of (1.4) satisfies $y(T) = 0$. A weaker variant of this is approximate null-controllability. This means that for all $\varepsilon > 0$ and all

$y_0 \in \mathcal{X}^p$, there exists a control function $u \in L^r([0,T];\mathcal{X}^p)$ such that the mild solution y of (1.4) satisfies $\|y(T)\| < \varepsilon$.

Null-controllability for heat-like equations is well known in the scalar-valued case $X = \mathbb{C}$ and $p = r = 2$, see, e.g., [18, 19, 21, 32, 36] for bounded regions $\Omega \subset \mathbb{R}^d$, and [8, 9, 28, 37–39, 47] for unbounded regions. We prove in Theorem 4.10 that for arbitrary (possibly infinite-dimensional) Banach spaces X, the system (1.4) is approximately null-controllable if $p = 1$ and null-controllable if $p \in (1,\infty)$. As a special case of our result, one may consider, e.g., a system of n coupled parabolic partial differential equations (if $X = \mathbb{C}^n$), or problems depending on a parameter (here X is a function space). For example, our results apply to strongly elliptic control systems of the form

$$\partial_t y(t) + (-A\nabla\nabla^\top)^m y(t) + By(t) = \mathbf{1}_E u(t), \quad y(0) = y_0 \in L^p(\mathbb{R}^d;\mathbb{C}^n), \quad t \in [0,T],$$

where $A, B \in \mathbb{C}^{n\times n}$ are such that $(-A\nabla\nabla^\top)^m$ is strongly elliptic. For a related result, we refer to [4], where controllability for finite-dimensional systems is studied using a suitable Kalman rank condition. As another example, we consider the following setting: for $\lambda \in [0,1]$, let

$$A_\lambda = \sum_{i,j=1}^{d} a_{i,j}(\lambda)\partial_i\partial_j$$

with $a_{i,j} \in C[0,1]$ and consider the parameter dependent linear control problem

$$\partial_t z(t) + A_\lambda z(t) = \mathbf{1}_E v(t), \quad z(0) = z_0 \in L^p(\mathbb{R}^d), \quad t \in [0,T], \tag{1.5}$$

where we view A_λ as an unbounded operator in $L^p(\mathbb{R}^d)$. Concerning the question of null-controllability, we remark that the control function v may depend on the parameter λ. Next, we reformulate this as a single linear control problem in $L^p(\mathbb{R}^d; C[0,1])$. We write $\mathbf{a}_{i,j} \in \mathcal{L}(C[0,1])$ for the multiplication operator given by $f \mapsto a_{i,j}f$. Consider the operator

$$A = \sum_{i,j=1}^{d} \mathbf{a}_{i,j}\partial_i\partial_j$$

acting on $L^p(\mathbb{R}^d; C[0,1])$. Under certain assumptions on the coefficients $a_{i,j}$, the operator A is normally elliptic and equation (1.4) with A_p replaced by A and $X = C[0,1]$ is well posed. Therefore, the parameter-dependent equation (1.5) can be rewritten in the form (1.4) with, $X = C[0,1]$, $A_p = A$, and $y_0 = z_0 \otimes \mathbf{1}_{[0,1]}$. The thick set $E \subseteq \mathbb{R}^d$ in equation (1.4) may be chosen as in equation (1.5).

For the proof of Theorem 4.10, we employ the classical equivalence between (approximate) null-controllability and final state observability for the adjoint problem. This follows from Douglas' lemma, see [14] in the case of Hilbert spaces, and [10–13, 17, 20, 24] for its generalization to Banach spaces. The observability estimate is formulated in Theorem 4.8. Its proof is based on the classical Lebeau–Robbiano strategy. For Hilbert spaces, it goes back to the papers [26, 32, 33, 40] and was further studied, e.g., in [5, 6, 41, 46, 49].

Recently, it has been adapted to Banach spaces in [7, 22]. The main idea of this strategy is that a so-called spectral inequality and a dissipation estimate implies an observability estimate. While the spectral inequality is provided by our vector-valued version of the Logvinenko–Sereda theorem, the dissipation estimate is derived from representing the semigroup generated by $-A_p$ as a Fourier multiplier with an operator-valued symbol.

2. Preliminaries

The theory of vector-valued distributions was developed by Schwartz in [44, 45]. In [1], this theory was applied to study vector-valued Fourier multipliers. Further results in this direction can be found in [3, 25]. It turns out that we cannot literally apply these results for our purpose, we present in this section some basic properties of vector-valued distributions and Fourier multipliers.

Let X be a Banach space with norm $\|\cdot\|_X$. We denote by $\mathcal{D}(\mathbb{R}^d; X)$, $\mathcal{S}(\mathbb{R}^d; X)$ and $\mathcal{E}(\mathbb{R}^d; X)$ the spaces of X-valued test functions, Schwartz functions, and smooth functions with the usual topologies, and by $\mathcal{D}'(\mathbb{R}^d; X)$, $\mathcal{S}'(\mathbb{R}^d; X)$ and $\mathcal{E}'(\mathbb{R}^d; X)$ the spaces of X-valued distributions, tempered distributions, and compactly supported distributions, respectively. Note that $\mathcal{F}'(\mathbb{R}^d; X) = \mathcal{L}(\mathcal{F}(\mathbb{R}^d); X)$, where $\mathcal{F} \in \{\mathcal{D}, \mathcal{S}, \mathcal{E}\}$. We denote by $\mathcal{O}_M(\mathbb{R}^d; X)$ the space of slowly increasing X-valued functions, that is, $\varphi \in \mathcal{O}_M(\mathbb{R}^d; X)$ if for each multi-index α, there exist constants C_α, m_α such that

$$\|\partial^\alpha \varphi(x)\|_X \leq C_\alpha (1 + |x|)^{m_\alpha}, \quad (x \in \mathbb{R}^d).$$

For $v \in X$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we denote by $\varphi \otimes v$ the element of $\mathcal{D}(\mathbb{R}^d; X)$ given by

$$(\varphi \otimes v)(x) = \varphi(x)v.$$

The set of these functions is called the set of *elementary tensors*. The set of finite linear combinations of elementary tensors is dense in $\mathcal{F}(\mathbb{R}^d; X)$, where $\mathcal{F} \in \{\mathcal{D}, \mathcal{D}', \mathcal{S}, \mathcal{S}', \mathcal{E}, \mathcal{E}', \mathcal{O}_M\}$.

In the usual fashion, we may extend the operations of differentiation, multiplication by smooth functions and Fourier transform to the appropriate classes of distributions by duality. In the case of the Fourier transform, this can be done as follows. We define for $z, x \in \mathbb{C}^d$ the Fourier character $\mathbf{e}_z(x) = \mathrm{e}^{\mathrm{i} z \cdot x}$. Note that $\mathbf{e}_z \in \mathcal{E}(\mathbb{R}^d)$ and that $z \mapsto \mathbf{e}_z(x)$ is entire. We define the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^d; X) \to \mathcal{S}(\mathbb{R}^d; X)$ by

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{R}^d} \mathbf{e}_{-\xi} \varphi \mathrm{d}x.$$

It is an automorphism of $\mathcal{S}(\mathbb{R}^d; X)$ with inverse given by

$$(\mathcal{F}^{-1}\varphi)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbf{e}_x \varphi \mathrm{d}\xi.$$

If $f \in \mathcal{S}'(\mathbb{R}^d; X)$, then we define the Fourier transform $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d; X) \to \mathcal{S}'(\mathbb{R}^d; X)$ by

$$(\mathcal{F} f)(\varphi) = f(\mathcal{F}\varphi), \quad (\varphi \in \mathcal{S}(\mathbb{R}^d))$$

and obtain an automorphism of $\mathcal{S}'(\mathbb{R}^d; X)$.

If $u \in \mathcal{E}'(\mathbb{R}^d; X)$, i.e., u has compact support, then $\mathcal{F}^{-1}u \in \mathcal{E}(\mathbb{R}^d; X)$. Thus, we may define the inverse Fourier–Laplace transform $\mathcal{L} : \mathcal{E}'(\mathbb{R}^d; X) \to C^\infty(\mathbb{C}^d; X)$ by

$$(\mathcal{L}u)(z) = (\mathcal{F}^{-1}(\mathbf{e}_{\mathrm{i}\,\mathrm{Im}\, z} u))(\mathrm{Re}\, z).$$

By checking that the Cauchy–Riemann differential equations hold for $\mathcal{L}u$, it follows that $\mathcal{L}u$ is an entire function. It follows that if $f \in \mathcal{S}'(\mathbb{R}^d; X)$ is such that $\mathcal{F} f \in \mathcal{E}'(\mathbb{R}^d; X)$, then f can be extended to an entire function $f : \mathbb{C}^d \to X$ given by $\mathcal{L}\mathcal{F} f$. In particular, f is analytic on $\mathbb{R}^d$.

For $i = 0, 1, 2$, let X_i be a Banach space with norm $\|\cdot\|_{X_i}$. By a *multiplication* we mean a bilinear continuous map

$$\bullet : X_1 \times X_2 \to X_0, \quad (x_1, x_2) \mapsto x_1 \bullet x_2$$

such that

$$\|x_1 \bullet x_2\|_{X_0} \le \|x_1\|_{X_1} \|x_2\|_{X_2}.$$

In particular, we will be interested in the cases where

(i) $X_1 = \mathbb{C}$, $X_2 = X_0$ and $\lambda \bullet x = \lambda x$,

(ii) $X_1 = X_2'$, $X_0 = \mathbb{C}$ and $x' \bullet x = \langle x', x\rangle$,

(iii) $X_1 = \mathcal{L}(X_2, X_0)$ and $A \bullet x = Ax$.

Note that the first two cases can be seen as special cases of the third case.

From [1], we infer that any multiplication gives rise to a unique hypocontinuous bilinear map

$$B : \mathcal{E}(\mathbb{R}^d; X_1) \times \mathcal{D}'(\mathbb{R}^d; X_2) \to \mathcal{D}'(\mathbb{R}^d; X_0), \quad f_1 \times f_2 \mapsto B(f_1, f_2)$$

such that for all $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^d)$, $x_1 \in X_1$, $x_2 \in X_2$, we have

$$B(\varphi_1 \otimes x_1, \varphi_2 \otimes x_2) = (\varphi_1\varphi_2) \otimes (x_1 \bullet x_2).$$

Here, hypocontinuous means that it is continuous in each variable, and uniformly continuous if one of the variables is restricted a bounded set. Furthermore, the restriction $B|_{\mathcal{O}_M(\mathbb{R}^d; X_1) \times \mathcal{S}'(\mathbb{R}^d; X_2)}$ is hypocontinuous as well. We write $B(f_1, f_2) = f_1 \bullet f_2$ in the following.

Set $D = -\mathrm{i}\nabla$. Given $m \in \mathcal{O}_M(\mathbb{R}^d; X_1)$, we define the Fourier multiplier

$$m(D) : \mathcal{S}'(\mathbb{R}^d; X_2) \to \mathcal{S}'(\mathbb{R}^d; X_0), \quad f \mapsto \mathcal{F}^{-1}(m \bullet \mathcal{F} f).$$

We note that in the special case of $X_0 = X_2 = X$, $X_1 = \mathcal{L}(X)$, we have

$$m_1(D)m_2(D) = (m_1 m_2)(D),$$

with respect to the above multiplication $\bullet$, we define the convolution $*_\bullet$ of two elementary tensors $\varphi_1 \otimes x_1$ and $\varphi_2 \otimes x_2$ (with $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^d)$, $x_1 \in X_1$, and $x_2 \in X_2$) by

$$(\varphi_1 \otimes x_1) *_\bullet (\varphi_2 \otimes x_2) = (\varphi_1 * \varphi_2) \otimes (x_1 \bullet x_2),$$

where $*$ denotes the usual convolution of scalar-valued functions. Theorem 3.1 in [1] implies that $*_\bullet$ extends to bilinear, hypocontinuous maps:

$$*_\bullet : \mathcal{S}(\mathbb{R}^d; X_1) \times \mathcal{S}'(\mathbb{R}^d; X_2) \to \mathcal{S}'(\mathbb{R}^d; X_0),$$
$$*_\bullet : \mathcal{D}'(\mathbb{R}^d; X_1) \times \mathcal{E}'(\mathbb{R}^d; X_2) \to \mathcal{D}'(\mathbb{R}^d; X_0).$$

Moreover, according to [1, Theorem 3.5], for $1 \le p \le \infty$, there is a third extension

$$*_\bullet : L^1(\mathbb{R}^d; X_1) \times L^p(\mathbb{R}^d; X_2) \to L^p(\mathbb{R}^d; X_0), \quad (f, g) \mapsto \int_{\mathbb{R}^d} f(\cdot - y) \bullet g(y) \mathrm{d}y,$$

satisfying Young's inequality

$$\|f *_\bullet g\|_{L^p(\mathbb{R}^d;X_0)} \le \|f\|_{L^1(\mathbb{R}^d;X_1)} \|g\|_{L^p(\mathbb{R}^d;X_2)}.$$

In the following, we will suppress the symbol $\bullet$ if it is clear from the context which multiplication is being employed.

Combining [1, Theorem 4.1] and [1, Corollary 4.4], we obtain the following lemma.

Lemma 2.1. *Let $\varepsilon > 0$. Then, there exists $C > 0$ such that for all $\mu > 0$ and all $m \in W^{d+1,\infty}(\mathbb{R}^d; X_1)$ satisfying*

$$\|m\|_{W^{d+1,\infty}} + \max_{|\alpha| \le d+1} \sup_{\xi \in \mathbb{R}^d} |\xi|^{|\alpha|+\varepsilon} \|\partial^\alpha m(\xi)\|_{X_1} \le \mu < \infty,$$

we have

$$\|\mathcal{F}^{-1} m\|_{L^1(\mathbb{R}^d;X_1)} \le C\mu.$$

In particular, it follows from Young's inequality $m(D) \in \mathcal{L}(L^p(\mathbb{R}^d; X_2), L^p(\mathbb{R}^d; X_0))$ with

$$\|m(D)\| \le C\mu.$$

Following [1, Theorem 2.3], we can define a hypocontinuous bilinear mapping $[\cdot,\cdot]_\bullet : \mathcal{S}'(\mathbb{R}^d; X_1) \times \mathcal{S}(\mathbb{R}^d; X_2) \to X_0$ by setting

$$[f \otimes x_1, \varphi \otimes x_2]_\bullet = \langle f, \varphi \rangle_{\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)} x_1 \bullet x_2$$

for elementary tensors given by $f \in \mathcal{S}'(\mathbb{R}^d), \varphi \in \mathcal{S}(\mathbb{R}^d), x_1 \in X_1, x_2 \in X_2$ and extending by density. As before, we will suppress the notation of $\bullet$ when it is clear from the context which multiplication is being employed. It follows that

$$[\mathcal{F} f, \varphi] = [f, \mathcal{F} \varphi]; \tag{2.1}$$

i.e., the Fourier transform is symmetric with respect to this form. If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d; X_1)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d; X_2)$, we have

$$[f, \varphi] = \int_{\mathbb{R}^d} f(x)\varphi(x)\mathrm{d}x.$$

In the following, we specialize to the case $X_2 = X$, $X_1 = X'$ and $x_1 \bullet x_2 = \langle x_1, x_2\rangle$. Suppose that $m \in \mathcal{O}_M(\mathbb{R}^d; \mathcal{L}(X))$ and consider $m(D) : \mathcal{S}(\mathbb{R}^d; X) \to \mathcal{S}(\mathbb{R}^d; X)$. It is clear that the symbol $m'(-\cdot)$ given by $\mathbb{R}^d \ni \xi \mapsto m(-\xi)' \in \mathcal{L}(X')$ belongs to $\mathcal{O}_M(\mathbb{R}^d; \mathcal{L}(X'))$. Here, $m(-\xi)'$ denotes the adjoint operator of $m(-\xi)$. For any Banach space Y and $f \in \mathcal{S}'(\mathbb{R}^d; Y)$, $\varphi \in \mathcal{S}(\mathbb{R}^d; Y)$, we set $R\varphi = \varphi(-\cdot)$ and define $Rf \in \mathcal{S}'(\mathbb{R}^d; Y)$ by $Rf(\psi) = f(R\psi)$, where $\psi \in \mathcal{S}(\mathbb{R}^d; Y)$. Using that $\mathcal{F}^{-1} f = (2\pi)^{-d}\mathcal{F}Rf$ for $f \in \mathcal{S}'(\mathbb{R}^d; X)$, we deduce from (2.1) that

$$[m'(-D)f, \varphi] = [f, m(D)\varphi].$$

In particular, if $f \in L^p(\mathbb{R}^d; X')$, where $1 \le p < \infty$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we deduce

$$\int_{\mathbb{R}^d} \langle (m'(-D)f(x), \varphi(x))\rangle \mathrm{d}x = \int_{\mathbb{R}^d} \langle f(x), m(D)\varphi(x)\rangle \mathrm{d}x.$$

We may therefore deduce the following result, which will be important when relating our observability estimate to null controllability.

Proposition 2.2. *Let q be such that $p^{-1} + q^{-1} = 1$. Let X be a Banach space such that X' has the Radon–Nikodym property and $1 \le p < \infty$. Let $m \in \mathcal{O}_M(\mathcal{L}(X))$ such that*

$$\|\mathcal{F}^{-1}m\|_{L^1(\mathbb{R}^d;\mathcal{L}(X))} < \infty.$$

Then, $m(D)' = m'(-D) \in \mathcal{L}(L^q(\mathbb{R}^d; X'))$ with

$$\|m(D)'\| \le \|\mathcal{F}^{-1}m\|_{L^1(\mathbb{R}^d;\mathcal{L}(X))}.$$

Before we proceed with the proof, let us recall that, as in the scalar case, we have the convolution identity

$$\mathcal{F}(f * g) = \mathcal{F}f\mathcal{F}g, \quad (f \in L^1(\mathbb{R}^d; \mathcal{L}(X)), g \in L^p(\mathbb{R}^d; X)).$$

This identity can be verified first on elementary tensors and then established in the general case by a density argument. Thus, it follows from Fourier inversion, the above identity, and Young's inequality that

$$\|m(D)\|_{\mathcal{L}(L^p(\mathbb{R}^d;X))} \le \|\mathcal{F}^{-1}m\|_{L^1(\mathbb{R}^d;\mathcal{L}(X))}.$$

Proof. From the Radon–Nikodym property of X', we have that $L^p(\mathbb{R}^d; X)' \simeq L^q(\mathbb{R}^d; X')$ and

$$\langle f, g\rangle_{L^q(\mathbb{R}^d;X')\times L^p(\mathbb{R}^d;X)} = \int_{\mathbb{R}^d} \langle f(x), g(x)\rangle_{X'\times X}\mathrm{d}x, \quad (f, g) \in L^q(\mathbb{R}^d; X') \times L^p(\mathbb{R}^d; X).$$

In particular, if $\varphi \in \mathcal{D}(\mathbb{R}^d; X)$, it holds that

$$\langle m'(-D)f, \varphi\rangle_{L^q(\mathbb{R}^d;X')\times L^p(\mathbb{R}^d;X)} = \langle f, m(D)\varphi\rangle_{L^q(\mathbb{R}^d;X')L^p(\mathbb{R}^d;X)}.$$

Now, let $(f, g) \in L^q(\mathbb{R}^d; X') \times L^p(\mathbb{R}^d; X)$. Since $\mathcal{S}(\mathbb{R}^d; X)$ is dense in $L^p(\mathbb{R}^d; X)$, we can choose a sequence $(\varphi_k)_{k=0}^{\infty} \in \mathcal{S}(\mathbb{R}^d; X)^{\mathbb{N}}$ such that

$$\|\varphi_k - g\|_{L^p(\mathbb{R}^d;X)} \to 0, \quad (k \to \infty).$$

Thus, $m(D)\varphi_k \to m(D)g$ in $L^p(\mathbb{R}^d; X)$ as $k \to \infty$ and since

$$\langle m'(-D)f, g\rangle_{L^q\times L^p} = \langle f, m(D)\varphi_k\rangle_{L^q\times L^p} + \langle m'(-D)f, g-\varphi_k\rangle_{L^q\times L^p},$$

it follows that

$$\langle m'(-D)f, g\rangle_{L^q\times L^p} = \langle f, m(D)g\rangle_{L^q\times L^p},$$

which proves $m(D)' = m'(-D)$.

Let $x \in \mathbb{R}^d$. It follows that for $(\ell, v) \in X' \times X$

$$\begin{aligned}\langle[(\mathcal{F}^{-1}m)(x)]'\ell, v\rangle = \langle \ell, (\mathcal{F}^{-1}m)(x)v\rangle &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{\mathrm{i}\xi\cdot x}\langle \ell, m(\xi)v\rangle \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{\mathrm{i}\xi\cdot x}\langle m(\xi)'\ell, v\rangle \mathrm{d}\xi \\ &= \langle (\mathcal{F}^{-1}m')(x)\ell, v\rangle.\end{aligned}$$

Therefore, since $m(D)' = m'(-D)$, we obtain

$$\|\mathcal{F}^{-1}m'(-\cdot)\|_{L^1(\mathbb{R}^d;\mathcal{L}(X'))} = \|(\mathcal{F}^{-1}m)'\|_{L^1(\mathbb{R}^d;\mathcal{L}(X'))} = \|\mathcal{F}^{-1}m\|_{L^1(\mathbb{R}^d;\mathcal{L}(X))} \le k,$$

and the result follows. ∎

3. Logvinenko–Sereda theorem for vector-valued functions

Let X be a Banach space with norm $\|\cdot\|_X$. In order to formulate our main result, we recall the notion of a (ρ, L)-thick subset E of $\mathbb{R}^d$ and the notation Π_λ for the parallelepiped with side lengths λ_i, $i \in \{1, 2, \ldots, d\}$, cf. equations (1.1) and (1.2) in the introduction. For $f \in L^p(\mathbb{R}^d; X)$, we denote by $\mathcal{F} f$ its Fourier transform, cf. Section 2.

Theorem 3.1. *There exists a constant $C_{\mathrm{LS}} \ge 1$ such that for all $p \in [1, \infty]$, all $\lambda \in (0, \infty)^d$, all $f \in L^p(\mathbb{R}^d; X)$ with $\operatorname{supp} \mathcal{F} f \subseteq \Pi_\lambda$, all $\rho > 0$, all $L \in (0, \infty)^d$, and all (ρ, L)-thick sets $E \subseteq \mathbb{R}^d$, we have*

$$\|\mathbf{1}_E f\|_{L^p(\mathbb{R}^d;X)} \ge \Big(\frac{\rho}{C_{\mathrm{LS}}}\Big)^{C_{\mathrm{LS}}(d+L\cdot\lambda)} \|f\|_{L^p(\mathbb{R}^d;X)}.$$

In the case where $X = \mathbb{C}$, this theorem was originally proven by Logvinenko and Sereda in [34] and significantly improved by Kovrijkine in [29,30]. For further references concerning the case $X = \mathbb{C}$, we refer to the introduction. Let us stress that the essential improvement of Theorem 3.1 is reflected in the (possible) infinite dimensionality of the Banach space X. To this end, let us consider the following example.

Example 3.2. Let I be a countable index set, and consider for $i \in I$ the functions $f_i \in L^p(\mathbb{R}^d)$ with $\operatorname{supp} \mathcal{F} f_i \subset \Pi_\lambda$ for some $\lambda \in (0, \infty)$. Thus, the classical Logvinenko–Sereda theorem (i.e., $X = \mathbb{C}$) applies to each f_i separately. Now, we assume that the pointwise supremum $g : \mathbb{R}^d \to \mathbb{R}$,

$$g(x) = \sup\{|f_i(x)| : i \in I\},$$

is in $L^p(\mathbb{R}^d)$. Then, Theorem 1.1 with $X = \ell^\infty(I)$ applied to the function $f : \mathbb{R}^d \to \ell^\infty(I)$, $(f(x))_i = f_i(x)$, gives

$$\|\mathbf{1}_E g\|_{L^p(\mathbb{R}^d)} = \left(\int_E \|f(x)\|_{\ell^\infty(I)}^p \mathrm{d}x \right)^{1/p} \geq \left(\frac{\rho}{C_{\mathrm{LS}}} \right)^{C_{\mathrm{LS}}(d + L \cdot \lambda)} \|g\|_{L^p(\mathbb{R}^d)}.$$

Indeed, if the index set I is finite, it is feasible to conclude this estimate directly from the classical Logvinenko–Sereda theorem ($X = \mathbb{C}$) with a constant depending on the cardinality of I. If the cardinality of I is infinite, our Theorem 3.1 applies.

For the proof of Theorem 3.1, we will follow the main strategy given in [30]. However, in order to deal with Banach space valued functions instead of $\mathbb{C}$-valued functions, we will need two preparatory results, i.e., Propositions 3.3 and 3.4, which we formulate next. The final proof of Theorem 3.1 is postponed to the appendix.

For $z \in \mathbb{C}$ and $r > 0$, we denote by $D(z, r) \subseteq \mathbb{C}$ the open disc of radius r centered at z. As well, let $B(x, r) \subseteq \mathbb{R}^d$ be the ball of radius r centered at $x \in \mathbb{R}^d$. If $z = 0$ or $x = 0$, respectively, we simply write $D(r)$ or $B(r)$.

Proposition 3.3. *There exists a constant $C_1 \geq 1$ such that for all closed intervals $I \subseteq \mathbb{R}$ with $0 \in I$ and $|I| = 1$, all analytic functions $f : D(6) \to X$ satisfying*

$$\sup_{z \in D(5)} \|f(z)\|_X \leq M \quad \text{and} \quad \sup_{x \in I} \|f(x)\|_X \geq 1$$

for some $M > 0$, and all measurable $A \subseteq I$, we have

$$\sup_{x \in A} \|f(x)\|_X \geq \left(\frac{|A|}{C_1} \right)^{\frac{\ln(M)}{\ln(2)}} \sup_{x \in I} \|f(x)\|_X. \tag{3.1}$$

Proposition 3.4. *There exists a constant $C_2 > 0$ such that for all $\lambda \in (0, \infty)^d$, all $p \in [1, \infty]$, all $f \in L^p(\mathbb{R}^d; X)$ with $\operatorname{supp} \mathcal{F} f \subseteq \Pi_\lambda$ and all $\alpha \in \mathbb{N}_0^d$, we have*

$$\|\partial^\alpha f\|_{L^p(\mathbb{R}^d; X)} \leq C_2^{|\alpha|} \lambda^\alpha \|f\|_{L^p(\mathbb{R}^d; X)}.$$

Proof of Proposition 3.3. Without loss of generality, we assume that $|A| > 0$. Since I is closed and $\|f(\cdot)\|_X$ is continuous on I, there exists $x_0 \in I$ such that $\sup_{x\in I}\|f(x)\|_X = \|f(x_0)\|_X$. By a consequence of the Hahn–Banach theorem, we can find $x' \in X'$ such that $\|x'\|_{X'} = 1$ and

$$\langle x', f(x_0)\rangle = \|f(x_0)\|_X.$$

The function $\varphi : D(5) \to \mathbb{C}$ given by $\varphi = \langle x', f(\cdot + x_0)\rangle$ is analytic and we have

$$|\varphi(0)| = \|f(x_0)\|_X \geq 1$$

as well as

$$|\varphi(z)| \leq \|f(z + x_0)\|_X \leq M$$

for all $z \in D(4)$. Moreover, the sets $I - x_0$ and $A - x_0$ are such that $A - x_0 \subseteq I - x_0$, $A - x_0$ is of positive measure by assumption and $0 \in I - x_0$. Applying Lemma 1 in [29] with φ as above as well as I and A replaced by $I - x_0$ and $A - x_0$, respectively, we obtain that there exists a constant $C_1 > 0$ such that

$$\sup_{x\in A-x_0} |\varphi(x)| \geq \left(\frac{|A|}{C_1}\right)^{\frac{\ln(M)}{\ln(2)}} \sup_{x\in I-x_0} |\varphi(x)|.$$

Inequality (3.1) now follows from

$$\begin{aligned}\sup_{x\in A}\|f(x)\|_X \geq \sup_{x\in A}|\langle x', f(x)\rangle| &= \sup_{x\in A-x_0} |\varphi(x)| \\ &\geq \left(\frac{|A|}{C_1}\right)^{\frac{\ln(M)}{\ln(2)}} |\varphi(0)| = \left(\frac{|A|}{C_1}\right)^{\frac{\ln(M)}{\ln(2)}} \sup_{x\in I}\|f(x)\|_X. \quad \blacksquare\end{aligned}$$

Proof of Proposition 3.4. The proof is an adaption of the classical proof, as it can be found, for example, in [50], to the vector-valued setting. We only prove the assertion in the case $|\alpha| = 1$. The case $|\alpha| = 0$ is trivial, and the case $|\alpha| > 1$ follows by induction. We choose a real-valued function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $0 \leq \varphi \leq 1$ as well as $\varphi = 1$ on $[-1/2, 1/2]^d$ and define $\varphi_\lambda = \varphi(T_\lambda \cdot)$, where

$$T_\lambda : \mathbb{R}^d \to \mathbb{R}^d, \quad (x_1, \ldots, x_d) \mapsto (x_1/\lambda_1, \ldots, x_d/\lambda_d).$$

Clearly, $\varphi_\lambda = 1$ on Π_λ, $\mathcal{F}^{-1}\varphi_\lambda = \lambda_1\lambda_2\cdots\lambda_d(\mathcal{F}^{-1}\varphi)(T_\lambda^{-1}\cdot)$. Moreover, since the usual convolution identity also holds in the vector-valued setting, we have $f = \mathcal{F}^{-1}(\varphi_\lambda \mathcal{F} f) = (\mathcal{F}^{-1}\varphi_\lambda) * f$. From Young's inequality, we conclude for all $j \in \{1, 2, \ldots, d\}$ that

$$\|\partial_j f\|_{L^p(\mathbb{R}^d;X)} = \|(\partial_j \mathcal{F}^{-1}\varphi_\lambda) * f\|_{L^p(\mathbb{R}^d;X)} \leq \|\partial_j \mathcal{F}^{-1}\varphi_\lambda\|_{L^1(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d;X)}.$$

Since

$$\|\partial_j \mathcal{F}^{-1}\varphi_\lambda\|_{L^1(\mathbb{R}^d)} = \lambda_j\|\lambda_1\lambda_2\cdots\lambda_d(\partial_j\mathcal{F}^{-1}\varphi)(T_\lambda^{-1}\cdot)\|_{L^1(\mathbb{R}^d)} = \lambda_j\|\partial_j\mathcal{F}^{-1}\varphi\|_{L^1(\mathbb{R}^d)},$$

the assertion (in the case $|\alpha| = 1$) follows with $C_2 = \sup_{j=1,\ldots,d}\|\partial_j\mathcal{F}^{-1}\varphi\|_{L^1(\mathbb{R}^d)}$. $\blacksquare$

4. Control theory for normally elliptic operators on Banach spaces

4.1. Normally elliptic operators and their semigroups

In [2], the notion of normal ellipticity has been introduced for operators with variable, $\mathcal{L}(X)$-valued, non-smooth coefficients and it was shown that their negatives generate analytic semigroups on $L^p(\mathbb{R}^d; X)$. This general framework is technically challenging and involves, for example, Besov spaces of vector-valued functions. In what follows, we consider normally elliptic operator A with *constant* coefficients only. As a consequence, certain proofs of [2] simplify and we obtain stronger results. In particular, using ideas from [1–3], we show the following.

(i) The operator $-A_p$, the part of $-A$ in $L^p(\mathbb{R}^d; X)$, is a semigroup generator and one can represent the resulting semigroup as a Fourier multiplier. This is suggested by [1, Remark 7.5]. Here, we give a full proof of this result.

(ii) The derivatives of the symbol of this multiplier decay exponentially. This is the content of Lemma 4.4 which is the crucial result of this section for our application to control theory. In Proposition 3.5.7 of [3], a similar estimate is given, but with polynomial decay.

Let X be a Banach space and $d, m \in \mathbb{N}$. For given coefficients $a_\alpha \in \mathcal{L}(X)$, where α ranges over all multi-indices with $|\alpha| \leq m$, consider the polynomial $a : \mathbb{R}^d \to \mathcal{L}(X)$,

$$a(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha.$$

We suppose that a has degree m, meaning that there exists a multi-index $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| = m$ and $a_\alpha \neq 0$. The set of all polynomials of this type is denoted by $\mathcal{P}_m(\mathbb{R}^d; \mathcal{L}(X))$. The associated Fourier multiplier $A = a(D)$ is a differential operator acting on $\mathcal{S}'(\mathbb{R}^d; X)$, see Section 2. The principal symbol of A is the polynomial $a_m : \mathbb{R}^d \to \mathcal{L}(X)$,

$$a_m(\xi) = \sum_{|\alpha| = m} a_\alpha \xi^\alpha.$$

Let $\kappa \geq 1$, $\vartheta \in [0, \pi)$ and $\omega \in \mathbb{R}$. We write

$$\Sigma_{\vartheta,\omega} = \left\{ z \in \mathbb{C} : |\arg(z - \omega)| \leq \vartheta \right\} \cup \{0\}.$$

Given a linear operator $T \in \mathcal{L}(X)$, we denote its resolvent set by $\rho(T)$. We say that a differential operator A is $(\kappa, \vartheta, \omega)$-elliptic if for all $\xi \in \mathbb{R}^d$ with $|\xi| = 1$ it holds that

$$\rho(-a_m(\xi)) \supseteq \Sigma_{\vartheta,\omega}$$

and for all $\lambda \in \Sigma_{\vartheta,\omega}$,

$$\|(\lambda + a_m(\xi))^{-1}\| \leq \frac{\kappa}{1 + |\lambda - \omega|}.$$

We say that A is normally elliptic (with symbol a) if it is $(\kappa, \pi/2, 0)$-elliptic and call κ a ellipticity constant of A.

Let $1 \le p < \infty$. We denote by A_p the part of A in $L^p(\mathbb{R}^d; X)$, that is,

$$\operatorname{dom}(A_p) = \{f \in L^p(\mathbb{R}^d; X) : Af \in L^p(\mathbb{R}^d; X)\}, \quad A_p f = Af.$$

Remark 4.1. Suppose that A is $(\kappa, \vartheta, \omega)$-elliptic. By homogeneity, we obtain for all $\xi \neq 0$ and $\lambda \in \rho(-a_m(\xi))$

$$(\lambda + a_m(\xi))^{-1} = (\lambda + |\xi|^m a_m(\xi/|\xi|))^{-1} = |\xi|^{-m}(|\xi|^{-m}\lambda + a_m(\xi/|\xi|))^{-1}.$$

Therefore, if $\xi \neq 0$, and $\lambda \in |\xi|^m \Sigma_{\vartheta,\omega} = \Sigma_{\vartheta,\omega|\xi|^m}$, then

$$\|(\lambda + a_m(\xi))^{-1}\| \le \frac{\kappa}{|\xi|^m + |\lambda - \omega|\xi|^m|}.$$

Proposition 4.2. *If A is normally elliptic with ellipticity constant κ, there exist $\varphi > \pi/2$ and $M > 0$ as well as $\mu < 0$ such that A is (M, φ, μ)-elliptic. Moreover, we can choose $(M, \varphi, \mu) = (2\kappa + 1, \pi - \arctan(2\kappa), -1/(2\kappa))$.*

Proof. Suppose that $T \in \mathcal{L}(X)$ and $K > 0$, $z \in \rho(-T)$ are such that

$$\|(z + T)^{-1}\| \le K.$$

Then, it follows from the usual Neumann series argument that $D(z, K^{-1}) \subseteq \rho(-T)$ and we have for all $w \in D(z, K^{-1})$ that

$$(w + T)^{-1} = \sum_{n=0}^{\infty} (z - w)^n (z + T)^{-1-n},$$

which leads to the estimate

$$\|(w + T)^{-1}\| \le K \sum_{n=0}^{\infty} |z - w|^n K^n = \frac{K}{1 - |z - w|K}.$$

In particular, if $w \in \bar{D}(z, (2K)^{-1})$, we get

$$\|(w + T)^{-1}\| \le 2K.$$

Now, let A be normally elliptic. Fix $\sigma \in \mathbb{R}$ and $\tau \in [0, (1 + |\sigma|)/(2\kappa)]$. Clearly, we have $-\tau + \mathrm{i}\sigma$ in $\bar{D}(\mathrm{i}\sigma, (1 + |\sigma|)/(2\kappa))$. Let $|\xi| = 1$. Applying the above considerations to $T = a_m(\xi)$, we obtain

$$\|(-\tau + \mathrm{i}\sigma + a_m(\xi))^{-1}\| \le \frac{2\kappa}{1 + |\sigma|}.$$

Furthermore, since

$$1 + |-\tau + \mathrm{i}\sigma + \frac{1}{2\kappa}| \le 1 + |-\tau + \frac{1}{2\kappa}| + |\sigma| \le 1 + \frac{1}{2\kappa}(1 + |\sigma|) + |\sigma| \le \frac{2\kappa + 1}{2\kappa}(1 + |\sigma|),$$

we obtain

$$\frac{2\kappa}{1+|\sigma|} \le \frac{2\kappa+1}{1+|-\tau+\mathrm{i}\sigma+\frac{1}{2\kappa}|}.$$

Moreover, we have

$$\left|\arg\left(-\tau+\mathrm{i}\sigma+\frac{1}{2\kappa}\right)\right| \le \left|\arg\left(-\frac{|\sigma|}{2\kappa}+\mathrm{i}\sigma\right)\right| \le \pi-\arctan(2\kappa),$$

where the argument of a complex number has to be understood as an element of $[-\pi,\pi)$. Since

$$\Sigma_{\pi-\arctan(2\kappa),-\frac{1}{2\kappa}} \cap \{\lambda\in\mathbb{C} : \mathrm{Re}(\lambda)\le 0\} = \left\{-\tau+\mathrm{i}\sigma : \sigma\in\mathbb{R}, \tau\in\left[0,\frac{1+|\sigma|}{2\kappa}\right]\right\},$$

we conclude that, for all $\lambda \in \Sigma_{\pi-\arctan(2\kappa),-1/(2\kappa)} \cap \{\lambda\in\mathbb{C} : \mathrm{Re}(\lambda)\le 0\}$, we have

$$\|(\lambda+a_m(\xi))^{-1}\| \le \frac{2\kappa+1}{1+|\lambda+1/(2\kappa)|}.$$

It is easy to see that this estimate also holds if $\mathrm{Re}(\lambda)>0$. The latter inequality implies that A is $(2\kappa+1,\pi-\arctan(2\kappa),-1/(2\kappa))$-elliptic. ∎

Let $n\ge 0$ and $p\in\mathcal{P}_n(\mathbb{R}^d;\mathcal{L}(X))$. For all multi-indices $\alpha\in\mathbb{N}_0^d$, we define

$$N_\alpha(p) = \max_{\beta\le\alpha}\sup_{\xi\in\mathbb{R}^d}\frac{\|\partial^\beta p(\xi)\|}{(1+|\xi|)^{n-|\beta|}},$$

where for multi-indices $\alpha,\beta\in\mathbb{N}_0^d$ we write $\beta\le\alpha$ if $\beta_i\le\alpha_i$ for all $i\in\{1,2,\dots,d\}$.

Proposition 4.3. *Suppose that A is normally elliptic with ellipticity constant κ. Then, there exist $\varphi,\gamma,\omega,M>0$ such that for all $\xi\in\mathbb{R}^d$, and all $\lambda\in\Sigma_{\varphi,-\gamma|\xi|^m+\omega}$, we have*

$$\|(\lambda+a(\xi))^{-1}\| \le \frac{M}{|\xi|^m+|\lambda+\gamma|\xi|^m|}.$$

The parameters φ,γ depend only on a_m while ω depends on a_m and $N_0(a-a_m)$. Moreover, we can choose

$$M = 4\kappa+2.$$

Proof. We employ the following well-known perturbation result based on the Neumann series: if $T,S\in\mathcal{L}(X)$ such that

$$\|ST^{-1}\| \le \frac{1}{2},$$

then $T+S$ is invertible and

$$\|(T+S)^{-1}\| \le 2\|T\|.$$

We infer from Proposition 4.2 that there exist constants $C, \varphi, \gamma > 0$ depending only on a_m such that for all $\lambda \in \Sigma_{\varphi, -\gamma|\xi|^m}$,

$$\|(\lambda + a_m(\xi))^{-1}\| \leq \frac{C}{|\xi|^m + |\lambda + \gamma|\xi|^m|}.$$

We note that $a - a_m$ has degree $m-1$. For a sufficiently large $\omega > 0$, we obtain for all $\lambda \in \Sigma_{\varphi, -\gamma|\xi|^m + \omega}$,

$$\|(a(\xi) - a_m(\xi))(\lambda + a_m(\xi))^{-1}\| \leq \frac{C N_0(a - a_m)(1 + |\xi|)^{m-1}}{|\xi|^m + |\lambda + \gamma|\xi|^m|} \leq \frac{1}{2}.$$

From the perturbation result and Proposition 4.2, we obtain the claimed inequality. ■

Let A be a normally elliptic operator. The above proposition implies that for all $\xi \in \mathbb{R}^d$ and all $\lambda \in \Sigma_{\varphi, -\gamma|\xi|^m + \omega}$, we have

$$\|(\lambda + a(\xi))^{-1}\| \leq \frac{1}{\sin(\varphi)} \frac{M}{|\lambda + \gamma|\xi|^m - \omega|}. \tag{4.1}$$

This can be seen as follows: using the notation $\lambda + \gamma|\xi|^m - \omega = r\mathrm{e}^{\mathrm{i}\psi}$, where $r > 0$ and $\psi \in [-\varphi, \varphi]$, we find

$$\begin{aligned} \frac{|\lambda + \gamma|\xi|^m - \omega|}{|\xi|^m + |\lambda + \gamma|\xi|^m} &\leq \frac{|\lambda + \gamma|\xi|^m - \omega|}{|\lambda + \gamma|\xi|^m - \omega + |\xi|^m + \omega|} \\ &\leq \sup_{r>0} \sup_{\psi \in [-\varphi, \varphi]} \frac{\left|r\mathrm{e}^{\mathrm{i}\psi}\right|}{|r\mathrm{e}^{\mathrm{i}\psi} + |\xi|^m + \omega|} \\ &\leq \frac{r}{\operatorname{Im}(|r\mathrm{e}^{\mathrm{i}\psi} + |\xi|^m + \omega|)} \leq \frac{1}{\sin(\varphi)}. \end{aligned}$$

This implies inequality (4.1). Thus, $-a(\xi)$ is a sectorial operator in the sense of [35, Definition 2.0.1]. Hence, $-a(\xi)$ generates for all $\xi \in \mathbb{R}^d$ an analytic semigroup on X which we denote by $(S_t(\xi))_{t \geq 0}$. Consequently, there exists a $C > 0$ such that for all $\xi \in \mathbb{R}^d$ and all $t \geq 0$, we have

$$\|S_t(\xi)\| \leq C\mathrm{e}^{\omega t - \gamma|\xi|^m t}. \tag{4.2}$$

Note that the constant C is independent of ξ since M and φ in inequality (4.1) are independent of ξ.

Lemma 4.4. *Let A be a normally elliptic operator with symbol a and denote for each $\xi \in \mathbb{R}^d$ the semigroup generated by $-a(\xi)$ by $(S_t(\xi))_{t \geq 0}$. Then, there exist $\mu, \omega > 0$ depending only on a_m such that for each multi-index α there exists a constant $K_\alpha > 0$ such that for all $\xi \in \mathbb{R}^d$ and $t \geq 0$ it holds that*

$$\|\partial^\alpha S_t(\xi)\| \leq K_\alpha \mathrm{e}^{\omega t - \mu|\xi|^m t}. \tag{4.3}$$

The constant K_α can be chosen to depend only on the principal symbol a_m and the constant $N_\alpha(a)$.

Proof. Let $\xi \in \mathbb{R}^d$. Since A is normally elliptic, Proposition 4.2 implies that there exist $\widetilde{M}, \lambda, \gamma > 0$ and $\varphi \in (\pi/2, \pi)$ such that

$$\|(\lambda - \gamma|\xi|^m + \omega + a(\xi))^{-1}\| \leq \frac{\widetilde{M}}{|\xi|^m + |\lambda + \omega|}, \quad (\lambda \in \Sigma_{\varphi,0}). \tag{4.4}$$

We set $b(\xi) = -a(\xi) + \gamma|\xi|^m - \omega$. Due to $|\lambda + \omega| \geq \sin(\varphi)|\lambda|$ for $\lambda \in \Sigma_{\varphi,0}$ and setting $M = \widetilde{M}(\sin(\varphi))^{-1}$, it follows that

$$\|(\lambda - b(\xi))^{-1}\| = \|(\lambda - \gamma|\xi|^m + \omega + a(\xi))^{-1}\| \leq \frac{M}{|\xi|^m + |\lambda|}, \quad (\lambda \in \Sigma_{\varphi,0}).$$

Write $(T_t(\xi))_{t\geq 0}$ for the semigroup generated by $b(\xi)$. It is clear that

$$T_t(\xi) = \mathrm{e}^{-\omega t + \gamma t|\xi|^m} S_t(\xi). \tag{4.5}$$

Let α be a multi-index. We show that there exists a constant $\widetilde{M}_\alpha > 0$ such that

$$\|\partial^\alpha T_t(\xi)\| \leq \widetilde{M}_\alpha (t(1 + |\xi|)^{m-1} + t^{|\alpha|}(1 + |\xi|)^{(m-1)|\alpha|}) \quad (\xi \in \mathbb{R}^d, t \geq 0) \tag{4.6}$$

holds. For $|\alpha| = 0$, this is straightforward by inequality (4.2). Therefore, we assume that $|\alpha| \geq 1$ in the following.

Let $r > 0$. Consider the contour

$$\Gamma = \mathrm{e}^{i\varphi}[r, \infty) \cup (r\mathbb{T} \cap \Sigma_{\varphi,0}) \cup \mathrm{e}^{-i\varphi}[r, \infty)$$

with positive orientation, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$. Let α be a multi-index. For every $t \geq 0$, we consider the functions

$$T_t^{(\alpha)} : \mathbb{R}^d \to \mathcal{L}(X), \quad T_t^{(\alpha)}(\xi) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{t\lambda} \partial^\alpha (\lambda - b(\xi))^{-1} \mathrm{d}\lambda.$$

For the sake of simplicity, we will write b instead of $b(\xi)$. Since

$$\partial_j (\lambda - b)^{-1} = (\lambda - b)^{-1}(\partial_j b)(\lambda - b)^{-1},$$

it follows by induction on the length of α that $\partial^\alpha(\lambda - b)^{-1}$ is a finite sum of terms having the form

$$Q(\beta_1, \beta_2, \ldots, \beta_\nu, b, \lambda) = (\lambda - b)^{-1}(\partial^{\beta_1} b)(\lambda - b)^{-1}(\partial^{\beta_2} b) \cdots (\lambda - b)^{-1}(\partial^{\beta_\nu} b)(\lambda - b)^{-1},$$

where $1 \leq \nu \leq |\alpha|$ and $\beta_1, \beta_2, \ldots, \beta_\nu$ are nonzero multi-indices of length $\leq m$ such that $\beta_1 + \beta_2 + \cdots + \beta_\nu = \alpha$, see [1, equation (7.4)]. We have the estimate

$$\begin{aligned}\|Q(\beta_1, \beta_2, \ldots, \beta_\nu, b, \lambda)\| &\leq \|(\lambda - b)^{-1}\|^{\nu+1} \prod_{\mu=1}^{\nu} \|\partial^{\beta_\mu} b\| \\ &\leq \frac{N_\alpha(b) M^{\nu+1}}{(|\xi|^m + |\lambda|)^{\nu+1}} \prod_{\mu=1}^{\nu} (1 + |\xi|)^{m - |\beta_\mu|} \\ &\leq \frac{N_\alpha(b) M^{\nu+1}(1 + |\xi|)^{\nu m - |\alpha|}}{(|\xi|^m + |\lambda|)^{\nu+1}}.\end{aligned} \tag{4.7}$$

Now, for $\lambda = \rho e^{\pm i\psi}$ with $\rho > 0$ and $\psi \in [-\varphi, \varphi]$, it follows that

$$\|e^{t\lambda} Q(\beta_1, \beta_2, \dots, \beta_\nu, b, \lambda)\| \leq \frac{N_\alpha(b) M^{\nu+1}(1+|\xi|)^{\nu m-|\alpha|}}{(|\xi|^m + \rho)^{\nu+1}} e^{t\rho\cos(\psi)}.$$

Thus, it follows that

$$\begin{aligned}
&\left\| \int_\Gamma e^{t\lambda} Q(\alpha_1, \alpha_2, \dots, \alpha_\nu, b, \lambda) d\lambda \right\| \\
&\quad \leq N_\alpha(b) M^{\nu+1}(1+|\xi|)^{\nu m-|\alpha|} \left(2\int_r^\infty \frac{e^{t\rho\cos(\varphi)}}{(|\xi|^m+\rho)^{\nu+1}} d\rho + \frac{2\varphi r e^{tr}}{(|\xi|^m + r)^{\nu+1}} \right) \\
&\quad \leq N_\alpha(b) M^{\nu+1}(1+|\xi|)^{\nu m-|\alpha|} \left(\frac{2e^{tr\cos(\varphi)}}{t|\cos(\varphi)|(|\xi|^m + r)^{\nu+1}} + \frac{2\varphi r e^{tr}}{(|\xi|^m + r)^{\nu+1}} \right).
\end{aligned}$$

Choosing $r = 1/t$ and noting that

$$\frac{1}{t(|\xi|^m + \frac{1}{t})^{\nu+1}} = \frac{t^\nu}{t^{\nu+1}(|\xi|^m + \frac{1}{t})^{\nu+1}} = \frac{t^\nu}{(t|\xi|^m + 1)^{\nu+1}},$$

we obtain that there exists a constant $C_\varphi > 0$ depending only on φ such that

$$\begin{aligned}
\left\| \int_\Gamma e^{t\lambda} Q(\alpha_1, \alpha_2, \dots, \alpha_\nu, b, \lambda) d\lambda \right\| &\leq C_\varphi N_\alpha(b) M^{\nu+1} \frac{t^\nu(1+|\xi|)^{\nu m-|\alpha|}}{(t|\xi|^m + 1)^{\nu+1}} \\
&\leq C_\varphi N_\alpha(b) M^{\nu+1} t^\nu (1+|\xi|)^{\nu m-|\alpha|}.
\end{aligned}$$

Denote by C a generic constant depending only on d and m whose value may change from line to line. Since $T_t^{(\alpha)}(\xi)$ is a finite sum of terms such as the one above with $1 \leq \nu \leq |\alpha|$ it follows that there exists a constant C such that if we set

$$K_0 = C_\varphi M^{|\alpha|+1} N_\alpha(b),$$

we obtain for all $\xi \in \mathbb{R}^d$ and $t \geq 0$

$$\|T_t^{(\alpha)}(\xi)\| \leq CK_0(t(1+|\xi|)^{m-1} + t^{|\alpha|}(1+|\xi|)^{(m-1)|\alpha|}). \tag{4.8}$$

In particular, in view of the Dunford–Riesz representation,

$$T_t(\xi) = \frac{1}{2\pi i} \int_\Gamma e^{t\lambda} (\lambda - b(\xi))^{-1} d\lambda.$$

the above calculations imply that we may differentiate under the integral sign and obtain

$$T_t^{(\alpha)} = \partial^\alpha T_t.$$

Thus, (4.6) follows. To deduce (4.3) from (4.6), we merely need to observe that by (4.5) and the Leibniz rule, we obtain that there exists a constant $C_\gamma > 0$ such that if we set

$K_1 = C_\gamma K_0$, we obtain

$$\begin{aligned}
&\|\partial^\alpha S_t(\xi)\| \\
&\quad \leq Ce^{\omega t} \sum_{\beta \leq \alpha} \big|\partial^\beta(\mathrm{e}^{-\gamma t|\xi|^m})\big| \|\partial^{\alpha-\beta} T_t(\xi)\| \\
&\quad \leq CK_0 \mathrm{e}^{\omega t - \gamma t|\xi|^m} \sum_{\beta \leq \alpha} (1 + (\gamma t|\xi|^{(m-1)|\beta|})(t(1+|\xi|)^{m-1} + t^{|\alpha|}(1+|\xi|)^{(m-1)|\alpha|}) \\
&\quad \leq CK_1 \mathrm{e}^{\omega t - \gamma t|\xi|^m/2}.
\end{aligned}$$

By the triangle inequality, we have that $N_\alpha(b) \leq C_{\gamma,\omega} N_\alpha(a)$. Thus, we obtain the statement of the lemma with $\mu = \gamma/2$ and

$$K_\alpha = C_{\varphi,\gamma,\omega,d,m} M^{|\alpha|+1} N_\alpha(a). \qquad \blacksquare$$

Remark 4.5. By inspecting the proof of Lemma 4.4, in particular, the estimate (4.7), we note that the constant M_α appearing in (4.3) may be chosen such that it depends only on the parameters appearing in (4.4) and

$$\max_{|\alpha| \leq m} \|a_\alpha\|,$$

where $a_\alpha \in \mathcal{L}(X)$ are the coefficients of a. From this, we see that the estimate (4.3) is stable under certain perturbations. Let, for example, $(A_\tau)_{\tau \in [0,1]}$ be a family of differential operators such that their symbols $(a_\tau)_{\tau \in [0,1]}$ take the form

$$a_\tau(\xi) = a_m(\xi) + \sum_{|\alpha|<m} a_{\alpha,\tau} \xi^\alpha, \qquad (\xi \in \mathbb{R}^d, \tau \in [0,1]),$$

where $a_m(\xi)$ is homogeneous of degree m and satisfies the normal ellipticity condition and there exists a constant K such that

$$\|a_{\alpha,\tau}\| \leq K, \qquad (|\alpha| \leq m, \tau \in [0,1]).$$

Applying the perturbation argument of Lemma 4.3, we see that there exist $\varphi, \gamma, \omega, M > 0$ independent of τ such that

$$\|(\lambda + a_\tau(\xi))^{-1}\| \leq \frac{M}{|\xi|^m + \big|\lambda + \gamma|\xi|^m\big|} \qquad (\xi \in \mathbb{R}^d,\ \lambda \in \Sigma_{\varphi,-\gamma|\xi|^m+\omega}).$$

Let $(S_{t,\tau}(\xi))_{t\geq 0}$ be the semigroup generated by $-a_\tau(\xi)$. Under these conditions, it follows that for each multi-index α there exists a constant M_α independent of τ such that

$$\|\partial^\alpha S_{t,\tau}(\xi)\| \leq M_\alpha \mathrm{e}^{\omega t - \mu|\xi|^m t}.$$

Lemma 4.6. *Let A be a normally elliptic differential operator with symbol a, denote for each $\xi \in \mathbb{R}^d$ the semigroup generated by $-a(\xi)$ by $(S_t(\xi))_{t\geq 0}$, and let $f \in \mathcal{S}(\mathbb{R}^d; X)$.*

For all $t \geq 0$, we define $S_t f : \mathbb{R}^d \to X, \xi \mapsto S_t(\xi) f(\xi)$. Then, we have $S_t f \in \mathcal{S}(\mathbb{R}^d; X)$ and

$$(S_t f - f) \to 0, \tag{4.9}$$

and

$$\frac{1}{t}(S_t f - f) \to -af \tag{4.10}$$

in the topology of $\mathcal{S}(\mathbb{R}^d; X)$ as $t \to 0$.

Proof. To show (4.9), we need to prove that for all multi-indices α and β, we have

$$\sup_{\xi \in \mathbb{R}^d} \|\xi^\beta \partial^\alpha (S_t(\xi) f(\xi) - f(\xi))\| \to 0 \quad (t \to 0).$$

Using the Leibniz rule, it is easy to see that we need to show that for each multi-index α and $t \geq 0$, there exist $\Phi_\alpha(t) \geq 0$ and $N_\alpha > 0$ such that $\Phi_\alpha(t) \to 0$ as $t \to 0$ and

$$\|\partial^\alpha (S_t(\xi) - 1)\| \leq \Phi_\alpha(t)(1 + |\xi|)^{N_\alpha}, \quad (\xi \in \mathbb{R}^d).$$

In fact, by another application of the Leibniz rule, we may reduce matters to proving

$$\|\partial^\alpha (T_t(\xi) - 1)\| \leq \Phi_\alpha(t)(1 + |\xi|)^{N_\alpha}, \quad (\xi \in \mathbb{R}^d),$$

where $T_t(\xi)$ is (as in the proof of Lemma 4.4) the semigroup generated by $b(\xi) = -a(\xi) + \gamma|\xi|^m - \omega$ with γ as in Proposition 4.3. Suppose that B is a sectorial operator on X and $(V_t)_{t\geq 0}$ the associated semigroup. Then, we have

$$V_t - 1 = B \int_0^t V_\tau \mathrm{d}\tau, \quad (t \geq 0).$$

Applying this with $B = b(\xi)$, where $\xi \in \mathbb{R}^d$, we obtain by the Leibniz rule and (4.3) that there exist $C > 0$ and $C_\alpha > 0$ such that

$$\begin{aligned} \|\partial^\alpha (T_t(\xi) - 1)\| &\leq C \sum_{\beta \leq \alpha} \|\partial^{\alpha-\beta} b(\xi)\| \int_0^t \|T_\tau^{(\beta)}\| \mathrm{d}\tau \\ &\leq C_\alpha (1 + |\xi|)^m \int_0^t \mathrm{d}\tau \leq C_\alpha t (1 + |\xi|)^m. \end{aligned}$$

To show (4.10), we need to prove that for all multi-indices α and β, we have

$$\sup_{\xi \in \mathbb{R}^d} \left\| \xi^\beta \partial^\alpha \left[\frac{1}{t}(S_t(\xi) f(\xi) - f(\xi)) + a(\xi) f(\xi) \right] \right\| \to 0$$

as t tends to zero. Again, we may reduce matters to proving that for each multi-index α there exist $\Phi_\alpha(t) \geq 0$ and $N_\alpha > 0$ such that for all $\xi \in \mathbb{R}^d$ we have

$$\left\| \frac{1}{t} \partial^\alpha (T_t(\xi) - 1) - \partial^\alpha b(\xi) \right\| \leq \Phi_\alpha(t)(1 + |\xi|)^{N_\alpha}. \tag{4.11}$$

Since

$$\frac{1}{t}(T_t(\xi)-1)-b(\xi)=b(\xi)\frac{1}{t}\int_0^t(T_\tau(\xi)-1)\mathrm{d}\tau,$$

by the mean value theorem for integrals, we have that

$$\left\|\frac{1}{t}\int_0^t T_\tau(\xi)-1\mathrm{d}\tau\right\|\leq\sup_{0\leq s\leq t}\|T_s(\xi)-1\|\leq\sup_{0\leq s\leq t}\left\|b(\xi)\int_0^s T_\tau(\xi)\mathrm{d}\tau\right\|\leq Ct\|b(\xi)\|. \tag{4.12}$$

Therefore, we obtain

$$\left\|\frac{1}{t}(T_t(\xi)-1)-b(\xi)\right\|\leq Ct\|b(\xi)\|^2\leq Ct(1+|\xi|)^{2m}.$$

This proves (4.11) in the case that $\alpha=0$. If $\alpha>0$, we may write

$$\begin{aligned}&\frac{1}{t}\partial^\alpha(T_t(\xi)-1)-\partial^\alpha b(\xi)\\&\quad=(\partial^\alpha b)(\xi)\frac{1}{t}\int_0^t(T_\tau(\xi)-1)\mathrm{d}\tau+\sum_{\beta<\alpha}\binom{\alpha}{\beta}(\partial^{\alpha-\beta}b)(\xi)\frac{1}{t}\int_0^t T_\tau^{(\beta)}(\xi)\mathrm{d}\tau.\end{aligned}$$

We obtain from (4.12) that

$$\left\|(\partial^\alpha b)(\xi)\frac{1}{t}\int_0^t(T_\tau(\xi)-1)\mathrm{d}\tau\right\|\leq C_\alpha t\|\partial^\alpha b(\xi)\|\|b(\xi)\|\leq C_\alpha t(1+|\xi|)^{2m-|\alpha|}.$$

If $0\leq t\leq 1$, then it follows from (4.8) that

$$\|T_\tau^{(\beta)}(\xi)\|\leq C_\beta t(1+|\xi|)^{(m-1)|\beta|},$$

which shows that

$$\begin{aligned}\left\|(\partial^{\alpha-\beta}b)(\xi)\frac{1}{t}\int_0^t T_\tau^{(\beta)}(\xi)\mathrm{d}\tau\right\|&\leq C_\beta(1+|\xi|)^{(m-1)|\beta|}\|\partial^{\alpha-\beta}b(\xi)\|\int_0^t\mathrm{d}\tau\\&\leq C_{\alpha,\beta}t(1+|\xi|)^{(m-1)|\beta|}(1+|\xi|)^{|\beta|},\end{aligned}$$

where we have used in the second line that $m-|\alpha-\beta|=m-m+|\beta|=|\beta|$. Summing up, we obtain

$$\left\|\frac{1}{t}\partial^\alpha(T_t(\xi)-1)-\partial^\alpha b(\xi)\right\|\leq C_\alpha t(1+|\xi|)^{2m-|\alpha|},$$

which concludes the proof. ■

Let $A:\mathcal{S}'(\mathbb{R}^d;X)\to\mathcal{S}'(\mathbb{R}^d;X)$ be a normally elliptic differential operator with symbol a and for each $\xi\in\mathbb{R}^d$, denote by $(S_t(\xi))_{t\geq 0}$ the semigroup generated by $-a(\xi)$ and by $S_t:\mathbb{R}^d\to\mathcal{L}(X)$ the mapping $\xi\mapsto S_t(\xi)$. As a consequence of (4.3), we obtain that for all $t\geq 0$ we have that $S_t\in\mathcal{S}(\mathbb{R}^d;\mathcal{L}(X))\subseteq\mathcal{O}_M(\mathbb{R}^d;\mathcal{L}(X))$. Therefore, the Fourier multiplier

$$V_t=S_t(D):\mathcal{S}'(\mathbb{R}^d;X)\to\mathcal{S}'(\mathbb{R}^d;X),\quad f\mapsto\mathcal{F}^{-1}S_t\mathcal{F}f$$

is well defined. Let $1 \leq p \leq \infty$. From Lemma 2.1 with $m = S_t$, we obtain that there exist constants K and ω such that

$$\|V_t\|_{L^p(\mathbb{R}^d;X)\to L^p(\mathbb{R}^d;X)} \leq Ke^{t\omega},$$

and by checking on elementary tensors, we see that the semigroup property

$$V_t V_s = V_{t+s}, \quad (s,t \geq 0)$$

holds. Thus,

$$(V_t^{(p)})_{t\geq 0} = (V_t|_{L^p(\mathbb{R}^d;X)})_{t\geq 0}$$

is a bounded semigroup. If $p < \infty$, then it follows from the density of $\mathcal{S}(\mathbb{R}^d; X)$ in $L^p(\mathbb{R}^d; X)$ and the first statement of Lemma 4.6 that $V_t^{(p)}$ is a C_0-semigroup. We denote the negative of the generator of $V_t^{(p)}$ by $\widetilde{A}_p$.

Lemma 4.7. *We have $A_p = \widetilde{A}_p$. In particular, $-A_p$ generates a semigroup given by $S_t(D)|_{L^p(\mathbb{R}^d)}$.*

Proof. Let us start by showing the inclusion $\widetilde{A}_p \subseteq A_p$. Using the second statement of Lemma 4.6, we have

$$\frac{1}{t}(V_t^{(p)} f - f) \to -Af$$

in the topology of $\mathcal{S}(\mathbb{R}^d; X)$ as $t \to 0$, and thus, $\widetilde{A}_p f = Af = A_p f$ for $f \in \mathcal{S}(\mathbb{R}^d; X)$. Moreover, $\mathcal{S}(\mathbb{R}^d; X)$ is dense in $\operatorname{dom}(\widetilde{A}_p)$ since $\mathcal{S}(\mathbb{R}^d; X)$ is dense in $L^p(\mathbb{R}^d; X)$ and $\mathcal{S}(\mathbb{R}^d; X)$ is invariant under $V_t^{(p)}$. Hence, using the notation $\mathcal{X}^p = L^p(\mathbb{R}^d; X)$, we conclude

$$\begin{aligned}\operatorname{Graph}(\widetilde{A}_p) &= \overline{\{(f, \widetilde{A}_p f) : f \in \mathcal{S}(\mathbb{R}^d; X)\}}^{\mathcal{X}^p\times\mathcal{X}^p} \\ &= \overline{\{(f, A_p f) : f \in \mathcal{S}(\mathbb{R}^d; X)\}}^{\mathcal{X}^p\times\mathcal{X}^p} \subseteq \overline{\operatorname{Graph}(A_p)}^{\mathcal{X}^p\times\mathcal{X}^p}.\end{aligned}$$

Since the embedding $J : L^p(\mathbb{R}^d; X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; X)$ is continuous and $\operatorname{Graph}(A)$ is closed, $\operatorname{Graph}(A_p) = (J \times J)^{-1}\operatorname{Graph}(A)$ is closed.

Now, observe that it follows directly from Lemma 2.5.5 in [25] that

$$\operatorname{Graph}(A_p) = \overline{\{(f, Af) : f \in \mathcal{D}(\mathbb{R}^d; X)\}}^{\mathcal{X}^p\times\mathcal{X}^p}.$$

Note that, in [25], it is assumed that the coefficients of A are scalar. However, the proof given there generalizes to operator coefficients without change. Since

$$\overline{\{(f, Af) : f \in \mathcal{D}(\mathbb{R}^d; X)\}}^{\mathcal{X}^p\times\mathcal{X}^p} \subseteq \overline{\{(f, Af) : f \in \mathcal{S}(\mathbb{R}^d; X)\}}^{\mathcal{X}^p\times\mathcal{X}^p} = \operatorname{Graph}(A_p),$$

we obtain $\widetilde{A}_p = A_p$. ■

4.2. Observability estimate

Let $m \in \mathbb{N}$ and $A : \mathcal{S}'(\mathbb{R}^d; X) \to \mathcal{S}'(\mathbb{R}^d; X)$ is a normally elliptic differential operator of order m with symbol $a \in \mathcal{P}_m(\mathbb{R}^d; X)$. Set

$$S_t : \mathbb{R}^d \to \mathcal{L}(X), \quad \xi \mapsto S_t(\xi),$$

where $(S_t(\xi))_{t\geq 0}$ denotes the analytic semigroup generated by $-a(\xi)$. Furthermore, for $t \geq 0$, we define $V_t = S_t(D) : \mathcal{S}'(\mathbb{R}^d; X) \to \mathcal{S}'(\mathbb{R}^d; X)$ the Fourier multiplier with symbol S_t. Let $p \in [1, \infty]$. Then, the restriction $(V_t^{(p)})_{t\geq 0} = (V_t|_{L^p(\mathbb{R}^d;X)})_{t\geq 0}$ is a bounded semigroup on $L^p(\mathbb{R}^d; X)$. If $p < \infty$, the semigroup $(V_t^{(p)})_{t\geq 0}$ is strongly continuous and we denote its generator by A_p. In the following, we will write $V_t = V_t^{(p)}$ when there is no risk of confusion.

Theorem 4.8. *Let $\rho, T > 0$, $L \in (0, \infty)^d$, $E \subseteq \mathbb{R}^d$ a (ρ, L)-thick set, and $1 \leq p, r \leq \infty$. Then, there exists a constant $C_{\mathrm{obs}} > 0$ such that for all $f \in L^p(\mathbb{R}^d; X)$ it holds that*

$$\|V_T f\|_{L^p(\mathbb{R}^d;X)} \leq C_{\mathrm{obs}} \|V_{(\cdot)} f\|_{L^r([0,T];L^p(E;X))}.$$

We choose a function $\varphi \in C_c^\infty(\mathbb{R})$ such that $0 \leq \varphi \leq 1$, $\operatorname{supp} \varphi \subseteq B(0, 1)$ and $\varphi = 1$ on $B(0, 1/2)$. For $\xi \in \mathbb{R}^d$, we set $\chi_\lambda(\xi) = \varphi(|\xi|/\lambda)$ and define $P_\lambda = \chi_\lambda(D)$.

Lemma 4.9. *There exist constants $c_1, c_2, \lambda_0 > 0$ depending only on a such that for all $t \geq 0$ and $\lambda \geq \lambda_0$, we have*

$$\|\mathcal{F}^{-1}(1 - \chi_\lambda) S_t\|_{L^1(\mathbb{R}^d;\mathcal{L}(X))} \leq c_1 e^{-c_2 t \lambda^m}.$$

Moreover, for all $p \in [1, \infty]$, $t \geq 0$ and $\lambda \geq \lambda_0$, we have

$$\|(I - P_\lambda) V_t\|_{L^p(\mathbb{R}^d;X)\to L^p(\mathbb{R}^d;X)} \leq c_1 e^{-c_2 t \lambda^m}.$$

Proof. We consider 3 separate cases.

Case 1. $t > 1$. Let $\varepsilon > 0$. By Lemma 2.1, it suffices to show that

$$\|(1 - \chi_\lambda) S_t\|_{W^{d+1,\infty}} + \max_{|\alpha|\leq d+1} \sup_{\xi\in\mathbb{R}^d} |\xi|^{|\alpha|+\varepsilon} \|\partial^\alpha((1 - \chi_\lambda)(\xi) S_t(\xi))\| \leq c_1 e^{-c_2 \lambda^m t} \tag{4.13}$$

for some constants c_1, c_2. For this, we observe that by the Leibniz rule, for each multi-index α, there exists a constant C_α such that

$$\|\partial^\alpha((1 - \chi_\lambda(\xi)) S_t(\xi))\| \leq C_\alpha \sum_{\beta\leq\alpha} |\partial^\beta(1 - \chi_\lambda)(\xi)| \|\partial^{\alpha-\beta} S_t(\xi)\|.$$

Observe that if $\lambda \geq 1$, there exists an absolute constant $C_\beta > 0$ such that

$$|\partial^\beta(1 - \chi_\lambda)(\xi)| \leq C_\beta \mathbf{1}_{|\xi|\geq\lambda/2}.$$

Therefore, by (4.3), it follows that there exist K_α, ω, μ such that

$$\|\partial^\alpha(1-\chi_\lambda(\xi))S_t(\xi)\| \le K_\alpha \mathbf{1}_{|\xi|\ge\lambda/2} e^{\omega t-\mu|\xi|^m t}.$$

Choosing $\lambda_0^m = \max\{1, 2^{m+1}\mu^{-1}\omega\}$, we obtain for all multi-indices α such that $|\alpha| \le d+1$ and $\lambda \ge \lambda_0$

$$\|\partial^\alpha((1-\chi_\lambda(\xi))S_t(\xi))\| \le K_\alpha e^{-\mu 2^{-m-1}\lambda^m t}.$$

This shows that there exist constants c_1', c_2' such that

$$\|(1-\chi_\lambda)S_t\|_{W^{d+1,\infty}} \le c_1' e^{-c_2'\lambda^m t}.$$

Moreover, observe that

$$|\xi|^{|\alpha|+\varepsilon}\|\partial^\alpha((1-\chi_\lambda)(\xi)S_t(\xi))\| \le K_\alpha|\xi|^{|\alpha|+\varepsilon}\mathbf{1}_{|\xi|\ge\lambda/2}e^{\omega t-\mu|\xi|^m t},$$

and thus, employing that $t>1$, it follows that there exists K_α' such that

$$|\xi|^{|\alpha|+\varepsilon}\|\partial^\alpha((1-\chi_\lambda)(\xi)S_t(\xi))\| \le K_\alpha'\mathbf{1}_{|\xi|\ge\lambda/2}e^{\omega t-(\mu/2)|\xi|^m t}.$$

Arguing as before, we find c_1'', c_2'' such that

$$|\xi|^{|\alpha|+\varepsilon}\|\partial^\alpha(1-\chi_\lambda(\xi))S_t(\xi)\| \le c_1'' e^{-c_2''\lambda^m t}.$$

We now obtain (4.13) by summing up.

Case 2. $0 \le t \le 1$, $t^{1/m}\lambda > 1$. We begin with two easy observations. Firstly, if $m : \mathbb{R}^d \to \mathcal{L}(X)$ is such that $\|\mathcal{F}^{-1}m\|_{L^1(\mathbb{R}^d;\mathcal{L}(X))} < \infty$, then for any $\mu > 0$, we have

$$\|\mathcal{F}^{-1}[m(\mu\cdot)]\|_{L^1(\mathbb{R}^d)} = \mu^{-d}\|(\mathcal{F}^{-1}m)(\mu^{-1}\cdot)\|_{L^1(\mathbb{R}^d)} = \|\mathcal{F}^{-1}m\|_{L^1(\mathbb{R}^d)}. \tag{4.14}$$

Secondly, if $(W_t)_{t\ge0}$ is a C_0-semigroup with generator B, then for any $\mu > 0$ the rescaled semigroup defined by $(\widetilde{W}_t)_{t\ge0} = (W_{\mu t})_{t\ge0}$ is associated to μB. Denote by $(T_{\tau,t})_{\tau\ge0}$ the semigroup on X associated to $-ta(t^{-1/m}\cdot) \in \mathcal{L}(X)$. We consider the rescaled symbol

$$\sigma_{t,\lambda} = ((1-\chi_\lambda)S_t)(t^{-1/m}\cdot) = (1-\chi_{t^{1/m}\lambda})S_t(t^{-1/m}\cdot) = (1-\chi_{t^{1/m}\lambda})T_{1,t}.$$

It follows from (4.14) that it suffices to show that there exist constants $c_1, c_2 > 0$ such that

$$\|\mathcal{F}^{-1}\sigma_{t,\lambda}\| \le c_1 e^{-c_2 t\lambda^m}.$$

Observe that

$$ta(t^{-1/m}\xi) = a_m(\xi) + \sum_{|\alpha|<m} t^{1-\frac{|\alpha|}{m}} a_\alpha \xi^\alpha,$$

and therefore, $N_0(a_m(\xi) - ta(t^{-1/m}\xi)) \le K$ for some constant K independent of t. It thus follows from Lemma 4.4 and Remark 4.5 that for each multi-index α there exist constants $K_\alpha, \mu > 0$ such that

$$\|\partial^\alpha T_{1,t}(\xi)\| \le K_\alpha e^{-\mu|\xi|^m}.$$

Moreover, since $t^{1/m}\lambda > 1$, we have that for each multi-index β there exist constants $C_\beta > 0$ such that

$$|\partial^\beta(1 - \chi_{t^{1/m}\lambda})(\xi)| \le C_\beta \mathbf{1}_{|\xi| \ge t^{1/m}\lambda/2}.$$

By the Leibniz rule, it therefore follows that there exist constants $C_\alpha > 0$ such that

$$\|\partial^\alpha \sigma_{t,\lambda}(\xi)\| \le C_\alpha \mathbf{1}_{|\xi| \ge t^{1/m}\lambda/2} e^{\omega t - \mu|\xi|^m}.$$

Let $\varepsilon > 0$. Arguing as in Case 1, we see that there exist $\lambda_0 > 0$ and constants c_1', c_2' and c_1'', c_2'' such that for all $\lambda \ge \lambda_0$,

$$\|\partial^\alpha \sigma_{t,\lambda}(\xi)\| \le c_1' e^{-c_2' t \lambda^m}$$

and

$$|\xi|^{m+\varepsilon} \|\partial^\alpha \sigma_{t,\lambda}\| \le c_1'' e^{-c_2'' t \lambda^m}$$

for all multi-indices α with $|\alpha| \le d + 1$. We can thus apply Lemma 2.1 also in this case.

Case 3. $0 \le t \le 1$, $0 \le t^{1/m}\lambda \le 1$. Employing the notation of Case 2, we see from (4.2) and Lemma 2.1 that there exists $A > 0$ such that

$$\|\mathcal{F}^{-1} T_{1,t}\| \le A.$$

Again by (4.14) it follows that there exists $B > 0$ such that

$$\|\mathcal{F}^{-1}(1 - \chi_{t^{1/m}\lambda})\| \le B.$$

It thus follows from Young's inequality that

$$\|\mathcal{F}^{-1}\sigma_{t,\lambda}\| \le AB.$$

Since we have due to the restriction $0 \le t^{1/m}\lambda \le 1$ for any $c > 0$ that

$$AB \le ABe^c e^{-ct\lambda^m},$$

the result also follows in this case. ∎

Proof of Theorem 4.8. We apply Theorem A.1 from [7] to the semigroup $(V_t^{(p)})_{t \ge 0}$ acting on the Banach space $L^p(\mathbb{R}^d; X)$ and the family of quasi-projections $(P_\lambda)_{\lambda > 0}$. We only need to verify that there exist positive constants $\lambda_0, d_0, d_1, d_2, d_3$ such that for all $f \in L^p(\mathbb{R}^d; X)$, all $\lambda > \lambda_0$ and all $t \in [0, T/2]$, we have

$$\|P_\lambda f\|_{L^p(\mathbb{R}^d;X)} \le d_0 e^{d_1 \lambda} \|\mathbf{1}_E P_\lambda f\|_{L^p(\mathbb{R}^d;X)}$$

and

$$\|(I - P_\lambda) V_t f\|_{L^p(\mathbb{R}^d;X)} \le d_2 e^{-d_3 \lambda^m t} \|f\|_{L^p(\mathbb{R}^d;X)},$$

and that the mapping $\Phi : [0,T] \ni t \mapsto \|\mathbf{1}_E V_t f\|_{L^p(\mathbb{R}^d;E)}$ is measurable. The first inequality is satisfied by Theorem 3.1, whereas the second inequality follows from Lemma 4.9. Measurability of Φ follows from the strong continuity of V_t if $p < \infty$. Suppose now that $p = \infty$. By Proposition 1.3.1 of [25], we have that the linear subspace

$$\left\{ f \mapsto \int_{\mathbb{R}^d} \langle g(x), f(x) \rangle_{X' \times X} \mathrm{d}x \colon g \in L^1(\mathbb{R}^d; X') \right\} \subseteq (L^\infty(\mathbb{R}^d; X))'$$

is norming for $L^\infty(\mathbb{R}^d; X)$, meaning that

$$\Phi(t) = \|\mathbf{1}_E V_t f\|_{L^\infty(\mathbb{R}^d;X)} = \sup \left\{ \int_{\mathbb{R}^d} \langle g(x), \mathbf{1}_E V_t f(x) \rangle_{X' \times X} \mathrm{d}x \colon \|g\|_{L^1(\mathbb{R}^d;X')} = 1 \right\}.$$

By the strong continuity of V_t, the map

$$t \mapsto \int_{\mathbb{R}^d} \langle g(x), \mathbf{1}_E V_t f(x) \rangle_{X' \times X} \mathrm{d}x$$

is continuous for each $g \in L^1(\mathbb{R}^d; X')$. Thus, Φ is lower semicontinuous as it is the supremum of continuous functions and therefore measurable. ■

4.3. Null-controllability

Let $E \subseteq \mathbb{R}^d$ be measurable, $p \in [1, \infty)$ and $T > 0$. Set $\mathcal{X}^p = L^p(\mathbb{R}^d; X)$ and consider the controlled system

$$\partial_t y(t) + A_p y(t) = \mathbf{1}_E u(t), \quad y(0) = y_0 \in \mathcal{X}^p, \quad t \in [0,T]. \tag{4.15}$$

Let $r \in [1, \infty]$. Given a control function $u \in L^r([0,T]; \mathcal{X}^p)$, the mild solution of (4.15) is given by

$$y(t) = V_t y_0 + \int_0^t V_{t-s} \mathbf{1}_E u(s) \mathrm{d}s.$$

We say that the system (4.15) is null-controllable in $L^r([0,T]; \mathcal{X}^p)$ in time T if for any $y_0 \in \mathcal{X}^p$ there exists an $u \in L^r([0,T]; \mathcal{X}^p)$ such that $y(T) = 0$. Setting

$$\mathcal{B}_T : L^r([0,T]; \mathcal{X}^p) \to \mathcal{X}^p, \quad u \mapsto \int_0^T V_{t-s} \mathbf{1}_E u(s) \mathrm{d}s,$$

we see that (4.15) is null-controllable in $L^r([0,T]; \mathcal{X}^p)$ at time T if and only if $\operatorname{ran}(V_T) \subseteq \operatorname{ran}(\mathcal{B}_T)$. Moreover, we define (4.15) to be *approximately null-controllable* at time T if

$$\operatorname{ran}(V_T) \subseteq \overline{\operatorname{ran}(\mathcal{B}_T)}$$

with the bar denoting the norm closure of the set $\operatorname{ran}(\mathcal{B}_T)$ in $\mathcal{X}^p$. Thus, (4.15) is *approximately null-controllable* at time T if and only if for all $\varepsilon > 0$ and all $y_0 \in \mathcal{X}^p$ there exists $u \in L^r([0,T]; \mathcal{X}^p)$ such that $\|y(T)\|_{\mathcal{X}^p} < \varepsilon$.

Theorem 4.10. *Let $\rho > 0$, $L \in (0,\infty)^d$ and E (ρ, L)-thick, and assume that X' has the Radon–Nikodym property. Then, the following statements hold.*

(a) *If $p \in (1,\infty)$, the system (4.15) is null-controllable in $L^r([0,T];\mathcal{X}^p)$ at time T.*

(b) *If $p = 1$, the system (4.15) is approximately null-controllable in $L^r([0,T];\mathcal{X}^p)$ at time T.*

Proof. Let q be such that $p^{-1} + q^{-1} = 1$ and s such that $r^{-1} + s^{-1} = 1$. Write $\mathcal{Y}^q = (\mathcal{X}^p)'$. It holds that $\mathcal{Y}^q = L^q(\mathbb{R}^d; X')$ due to the Radon–Nikodym property of X'. For $t \geq 0$, we set $W_t = V_t'$. By Douglas' lemma, the statement of the theorem is equivalent to the fact that there exists a constant C_{obs} such for every $f \in \mathcal{Y}^q$ we have the observability estimate

$$\|W_T f\|_{\mathcal{Y}^q} \leq C_{\mathrm{obs}} \|\mathcal{B}_T' f\|_{L^r([0,T];\mathcal{X}^p)'}. \tag{4.16}$$

By [48, Theorem 2.1], it holds

$$\|\mathcal{B}_T' f\|_{L^r([0,T];\mathcal{X}^p)'} = \|W_{(\cdot)} f\|_{L^s([0,T];\mathcal{Y}^q)}.$$

Recall that S_t is the symbol of V_t. To obtain (4.16), we note that due to Proposition 2.2 and Lemma 4.9, we obtain for all $\lambda \geq \lambda_0$ the dissipation estimate

$$\|(I - P_\lambda) W_t\| \leq \|\mathcal{F}^{-1} S_t (1 - \chi_\lambda)\|_{L^1(\mathbb{R}^d;\mathcal{L}(X))} \leq c_1 e^{-c_2 t \lambda^m}.$$

Since the uncertainty principle also holds for functions with values in X', we obtain the observability estimate as in the proof of Theorem 4.8. ∎

A. Proof of Theorem 3.1

First, we assume $L = (1,1,\ldots,1)$, and fix $\lambda \in (0,\infty)^d$, $\rho > 0$, a $(\rho,1)$-thick set E, and $f \in L^p(\mathbb{R}^d; X)$ with $\operatorname{supp} \mathcal{F} f \subseteq \Pi_\lambda$ as in the assumptions of the theorem. Note that f is analytic since $\operatorname{supp} \mathcal{F} f$ is compact, see Section 2. For $k \in \mathbb{Z}^d$, we denote by $\Lambda_k = (-1/2, 1/2)^d + k \subseteq \mathbb{R}^d$ the open unit cube centered at k. Let

$$A > \frac{1}{1 - (2^d + 1)^{-1/d}} \in (3/2, 2), \tag{A.1}$$

and let $C_2 > 0$ be the absolute constant from Proposition 3.4. We call $k \in \mathbb{Z}^d$ *bad* if there exists $\alpha \in \mathbb{N}_0^d$ with $\alpha \neq 0$ such that

$$\|\mathbf{1}_{\Lambda_k} \partial^\alpha f\|_{L^p(\mathbb{R}^d;X)} \geq 2^d A^{|\alpha|} (C_2 \lambda)^\alpha \|\mathbf{1}_{\Lambda_k} f\|_{L^p(\mathbb{R}^d;X)}.$$

Otherwise, we call $k \in \mathbb{Z}^d$ *good*. Moreover, we will use the notation

$$\Lambda_{\mathrm{bad}} = \bigcup_{\substack{k\in\mathbb{Z}^d: \\ k \text{ is bad}}} \Lambda_k \quad \text{and} \quad \Lambda_{\mathrm{good}} = \bigcup_{\substack{k\in\mathbb{Z}^d: \\ k \text{ is good}}} \Lambda_k.$$

Lemma A.1. (i) *We have* $\|\mathbf{1}_{\Lambda_{\mathrm{good}}} f\|_{L^p(\mathbb{R}^d;X)} \geq C_3 \|f\|_{L^p(\mathbb{R}^d;X)}$, *where*

$$C_3 := C_3(A) := 1 - \left(\frac{1}{2^d}\left[\left(\frac{1}{1-1/A}\right)^d - 1\right]\right)^{1/p} \in (0,1)$$

if $p \in [1,\infty)$, *and* $C_3 = 1$ *if* $p = \infty$.

(ii) *There exists* $B > A$ *such that for all good* $k \in \mathbb{Z}^d$, *there exists* $x \in \Lambda_k$ *such that for all* $\alpha \in \mathbb{N}_0^d$, *we have*

$$\|\partial^\alpha f(x)\|_X \leq 4^d B^{|\alpha|} (C_2\lambda)^\alpha \|\mathbf{1}_{\Lambda_k} f\|_{L^p(\mathbb{R}^d;X)}.$$

Proof. It follows by definition that for all $p \in [1,\infty)$

$$\begin{aligned} \|\mathbf{1}_{\Lambda_{\mathrm{bad}}} f\|_{L^p(\mathbb{R}^d;X)}^p &= \sum_{k\in\mathbb{Z}^d\cap\Lambda_{\mathrm{bad}}} \|\mathbf{1}_{\Lambda_k} f\|_{L^p(\mathbb{R}^d;X)}^p \leq \sum_{k\in\mathbb{Z}^d\cap\Lambda_{\mathrm{bad}}} \sum_{\substack{\alpha\in\mathbb{N}_0^d:\\ \alpha\neq 0}} \frac{\|\mathbf{1}_{\Lambda_k}\partial^\alpha f\|_{L^p(\mathbb{R}^d;X)}^p}{2^{dp}A^{p|\alpha|}(C_2\lambda)^{p\alpha}} \\ &= \sum_{\substack{\alpha\in\mathbb{N}_0^d:\\ \alpha\neq 0}} \frac{\|\mathbf{1}_{\Lambda_{\mathrm{bad}}}\partial^\alpha f\|_{L^p(\mathbb{R}^d;X)}^p}{2^{dp}A^{p|\alpha|}(C_2\lambda)^{p\alpha}} \leq \sum_{\substack{\alpha\in\mathbb{N}_0^d:\\ \alpha\neq 0}} \frac{\|\partial^\alpha f\|_{L^p(\mathbb{R}^d;X)}^p}{2^{dp}A^{p|\alpha|}(C_2\lambda)^{p\alpha}}. \end{aligned}$$

By Proposition 3.4, and since $A \geq 1$, we conclude for all $p \in [1,\infty)$ that

$$\begin{aligned} \|\mathbf{1}_{\Lambda_{\mathrm{bad}}} f\|_{L^p(\mathbb{R}^d;X)}^p &\leq \sum_{\substack{\alpha\in\mathbb{N}_0^d:\\ \alpha\neq 0}} \frac{\|f\|_{L^p(\mathbb{R}^d;X)}^p}{2^{dp}A^{p|\alpha|}} = \frac{1}{2^{dp}}\left[\left(\frac{1}{1-1/A^p}\right)^d - 1\right]\|f\|_{L^p(\mathbb{R}^d;X)}^p \\ &\leq \frac{1}{2^d}\left[\left(\frac{1}{1-1/A}\right)^d - 1\right]\|f\|_{L^p(\mathbb{R}^d;X)}^p = (1-C_3)^p\|f\|_{L^p(\mathbb{R}^d;X)}^p. \end{aligned}$$

For $p \in [1,\infty)$, it follows that

$$\|\mathbf{1}_{\Lambda_{\mathrm{good}}} f\|_{L^p(\mathbb{R}^d;X)} \geq C_3\|f\|_{L^p(\mathbb{R}^d;x)}.$$

By (A.1), we have $C_3 \in (0,1)$. This proves the first claim in the case $p \in [1,\infty)$. If $p = \infty$, the proof is even easier. By the definition of bad and Proposition 3.4, we have

$$\begin{aligned} \|\mathbf{1}_{\Lambda_{\mathrm{bad}}} f\|_{L^\infty(\mathbb{R}^d;X)} &\leq \sup_{k\in\mathbb{Z}^d:k \text{ bad}} \sum_{\substack{\alpha\in\mathbb{N}_0^d:\\ \alpha\neq 0}} \frac{\|\mathbf{1}_{\Lambda_k}\partial^\alpha f\|_{L^\infty(\mathbb{R}^d;X)}}{2^d A^{|\alpha|}(C_2\lambda)^\alpha} \\ &\leq \frac{1}{2^d}\left[\left(\frac{1}{1-1/A}\right)^d - 1\right]\|f\|_{L^\infty(\mathbb{R}^d;X)}. \end{aligned}$$

Since the prefactor in the last inequality is strictly smaller than one, we conclude that $\|\mathbf{1}_{\Lambda_{\mathrm{good}}} f\|_{L^\infty(\mathbb{R}^d;X)} = \|f\|_{L^\infty(\mathbb{R}^d;X)}$.

In order to prove part (ii), we consider the contraposition, that is, for all $B > A$ there exists a good $k \in \mathbb{Z}^d$ such that for all $x \in \Lambda_k$ there is $\alpha \in \mathbb{N}_0^d$ with

$$\|\partial^\alpha f(x)\|_X > 4^d B^{|\alpha|} (C_2\lambda)^\alpha \|\mathbf{1}_{\Lambda_k} f\|_{L^p(\mathbb{R}^d;X)}.$$

This and the definition of good implies that there exists a good $k \in \mathbb{Z}^d$ such that we have

$$2^d \|\mathbf{1}_{\Lambda_k} f\|_{L^p(\mathbb{R}^d;X)} < \sum_{\alpha\in\mathbb{N}_0^d} \frac{\|\mathbf{1}_{\Lambda_k}\partial^\alpha f\|_{L^p(\mathbb{R}^d;X)}}{2^d B^{|\alpha|}(C_2\lambda)^\alpha} \leq \sum_{\alpha\in\mathbb{N}_0^d} \left(\frac{A}{B}\right)^{|\alpha|} \|\mathbf{1}_{\Lambda_k} f\|_{L^p(\mathbb{R}^d;X)}.$$

Choosing, for instance, $B = 3A$, we obtain

$$2^d \|\mathbf{1}_{\Lambda_k} f\|_{L^p(\mathbb{R}^d;X)} \leq \left(\frac{1}{1-(1/3)}\right)^d \|\mathbf{1}_{\Lambda_k} f\|_{L^p(\mathbb{R}^d;X)},$$

which is a contradiction. ■

Let $s = 1$ if $p \in [1,\infty)$ or some arbitrary number $s \in (0,1)$ if $p = \infty$, $k \in \mathbb{Z}^d$ be good and $y \in \Lambda_k$ be such that $\|f(y)\|_X \geq s\|\mathbf{1}_{\Lambda_k} f\|_{L^p(\mathbb{R}^d;X)}$. Furthermore, let $\Omega \subseteq \Lambda_k$ be a measurable set to be chosen later. Then, using spherical coordinates, we have

$$|\Omega| = \int_{\Lambda_k} \mathbf{1}_\Omega(x)\mathrm{d}x = \int_{S^{d-1}} \int_{r=0}^{r(\vartheta)} \mathbf{1}_\Omega(y + r\vartheta) r^{d-1} \mathrm{d}r\,\mathrm{d}\sigma(\vartheta),$$

where $r(\vartheta) = \sup\{t > 0 \colon y + t\vartheta \subseteq \Lambda_k\}$, and where σ denotes the surface measure. There exists a $\vartheta_0 \in S^{d-1}$ such that

$$|\Omega| \leq \sigma(S^{d-1}) \int_0^{r(\vartheta_0)} \mathbf{1}_\Omega(y + r\vartheta_0) r^{d-1} \mathrm{d}r. \tag{A.2}$$

Indeed, if the converse inequality to (A.2) would hold for all $\vartheta \in S^{d-1}$, then averaging over S^{d-1} would give a contradiction. Let now $I_0 = \{y + r\vartheta_0 \colon r > 0,\ y + r\vartheta_0 \in \Lambda_k\}$ be the largest line segment in Λ_k starting in y in the direction of ϑ_0. Since $r(\vartheta_0) \leq d^{1/2}$, we conclude from (A.2) that $|\Omega| \leq \sigma(S^{d-1}) d^{(d-1)/2} |\Omega \cap I_0|$, where, with some abuse of notation, we use the notation $|\Omega \cap I_0| = \int_0^{r(\vartheta_0)} \mathbf{1}_\Omega(y + r\vartheta_0)\mathrm{d}r$.

Now, we define the function $F \colon \mathbb{C}^d \to X$ by

$$F(w) = \frac{1}{N}(\mathcal{L}\mathcal{F} f)(y + w|I_0|\vartheta_0),$$

where $\mathcal{L}$ denotes the inverse Fourier–Laplace transform, cf. Section 2, and where N denotes the normalization $N = s\|\mathbf{1}_{\Lambda_k} f\|_{L^p(\mathbb{R}^d;X)}$. Note that F is an entire function which extends $(1/N) f(y + \cdot|I_0|\vartheta_0)$ to $\mathbb{C}^d$, see Section 2. Thus, we have for all $w \in \mathbb{C}^d$ and $x \in \mathbb{R}^d$,

$$\|F(w)\|_X \leq \frac{1}{N} \sum_{\alpha\in\mathbb{N}_0^d} \frac{\|f^{(\alpha)}(x)\|_X}{\alpha!} \prod_{i=1}^d |(y + w|I_0|\vartheta_0 - x)_i|^{\alpha_i}.$$

By Lemma A.1, there exists $x_0 \in \Lambda_k$ such that for all $w \in \mathbb{C}^d$,

$$\|F(w)\|_X \le \frac{4^d}{N} \sum_{\alpha \in \mathbb{N}_0^d} \frac{B^{|\alpha|}(C_2\lambda)^\alpha \|\mathbf{1}_{\Lambda_k} f\|_{L^p(\mathbb{R}^d;X)}}{\alpha!} \prod_{i=1}^d |(y + w|I_0|\vartheta_0 - x_0)_i|^{\alpha_i}.$$

Since for all $w \in D(5)$ we have

$$y - x_0 + w|I_0|\vartheta_0 \in \bigtimes_{i=1}^d D(6\sqrt{d}),$$

we conclude for all $w \in D(5)$ that

$$\begin{aligned}\|F(w)\|_X &\le \frac{4^d}{N} \sum_{\alpha \in \mathbb{N}_0^d} \frac{B^{|\alpha|}(C_2\lambda)^\alpha \|\mathbf{1}_{\Lambda_k} f\|_{L^p(\mathbb{R}^d;X)}}{\alpha!} (6\sqrt{d})^{|\alpha|} \\ &= \frac{4^d}{N} \|\mathbf{1}_{\Lambda_k} f\|_{L^p(\mathbb{R}^d;X)} \exp(6d^{1/2}BC_2|\lambda|) = 4^d \exp(6d^{1/2}BC_2|\lambda|) =: M.\end{aligned}$$

We recall that by assumption on y we have $\|F(0)\|_X = N^{-1}\|f(y)\|_X \ge 1$. By Proposition 3.3, we have for all closed intervals $I \subseteq \mathbb{R}$ with $0 \in I$ and $|I| = 1$, and all measurable $A \subseteq I$ that

$$\sup_{x \in A} \|F(x)\|_X \ge \left(\frac{|A|}{C_1}\right)^{\frac{\ln(M)}{\ln(2)}} \sup_{x \in I} \|F(x)\|_X$$

with some absolute constant $C_1 \ge 1$. Choose $I = [0, 1]$ and $A = \{t \in [0, 1] \colon y + t\vartheta_0 \in \Omega \cap I_0\}$; then,

$$\sup_{x \in \Omega \cap I_0} \|f(x)\|_X \ge \left(\frac{|\Omega \cap I_0|}{C_1}\right)^{\frac{\ln(M)}{\ln(2)}} \sup_{x \in I_0} \|f(x)\|_X.$$

By our choice of y, we have that $\sup_{x \in I_0} \|f(x)\|_X \ge \|f(y)\|_X \ge s\|\mathbf{1}_{\Lambda_k} f\|_{L^p(\mathbb{R}^d;X)}$. Moreover, we have shown above that $|\Omega| \le \sigma(S^{d-1})d^{(d-1)/2}|\Omega \cap I_0|$. Hence, we conclude

$$\sup_{x \in \Omega} \|f(x)\|_X \ge \left(\frac{|\Omega|}{C_1\sigma(S^{d-1})d^{(d-1)/2}}\right)^{\frac{\ln(M)}{\ln(2)}} s\|\mathbf{1}_{\Lambda_k} f\|_{L^p(\mathbb{R}^d;X)}.$$

Recall that $s = 1$ if $p \in [1, \infty)$, and that the above inequality holds for arbitrary $s \in (0, 1)$ if $p = \infty$. By taking limits, we obtain

$$\sup_{x \in \Omega} \|f(x)\|_X \ge \left(\frac{|\Omega|}{C_1\sigma(S^{d-1})d^{(d-1)/2}}\right)^{\frac{\ln(M)}{\ln(2)}} \|\mathbf{1}_{\Lambda_k} f\|_{L^p(\mathbb{R}^d;X)}. \tag{A.3}$$

Now, we choose

$$\Omega = \left\{x \in \Lambda_k \colon \left(\frac{|E \cap \Lambda_k|}{2C_1\sigma(S^{d-1})d^{(d-1)/2}}\right)^{\frac{\ln(M)}{\ln(2)}} \|\mathbf{1}_{\Lambda_k} f\|_{L^p(\mathbb{R}^d;X)} > \|f(x)\|_X\right\}.$$

By inequality (A.3) and the definition of Ω, we obtain

$$\left(\frac{|E\cap\Lambda_k|}{2|\Omega|}\right)^{\frac{\ln(M)}{\ln(2)}}\sup_{x\in\Omega}\|f(x)\|_X \geq \left(\frac{|E\cap\Lambda_k|}{2C_1\sigma(S^{d-1})d^{(d-1)/2}}\right)^{\frac{\ln(M)}{\ln(2)}}\|\mathbf{1}_{\Lambda_k}f\|_{L^p(\mathbb{R}^d;X)}$$
$$\geq \sup_{x\in\Omega}\|f(x)\|,$$

and thus, $|\Omega|\leq|E\cap\Lambda_k|/2$. The definition of Ω implies that

$$\begin{aligned}&\|\mathbf{1}_{E\cap\Lambda_k}f\|_{L^p(\mathbb{R}^d;X)}\\&\quad\geq\|\mathbf{1}_{E\cap\Lambda_k}\mathbf{1}_{\Omega^c}f\|_{L^p(\mathbb{R}^d;X)}\\&\quad\geq\left(\frac{|E\cap\Lambda_k|}{2C_1|S^{d-1}|d^{(d-1)/2}}\right)^{\frac{\ln(M)}{\ln(2)}}\|\mathbf{1}_{\Lambda_k}f\|_{L^p(\mathbb{R}^d;X)}\|\mathbf{1}_{E\cap\Lambda_k\cap\Omega^c}\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

Moreover, since $|\Omega|\leq|E\cap\Lambda_k|/2$, we have

$$|E\cap\Lambda_k\cap\Omega^c|=|E\cap\Lambda_k|-|E\cap\Lambda_k\cap\Omega|\geq|E\cap\Lambda_k|-|\Omega|\geq\frac{|E\cap\Lambda_k|}{2}.$$

Since E is thick, we have that $1\geq|E\cap\Lambda_k|>0$; thus, $E\cap\Lambda_k\cap\Omega^c$ has positive measure as well. We conclude that

$$\|\mathbf{1}_{E\cap\Lambda_k\cap\Omega^c}\|_{L^p(\mathbb{R}^d)}\geq\frac{|E\cap\Lambda_k|}{2}.$$

Hence, using $C_4:=2C_1|S^{d-1}|d^{(d-1)/2}\geq 2$, the fact that $|E\cap\Lambda_k|\geq\rho$ by the definition of the thick set E, and $\rho/C_4\leq 1$, we can conclude that

$$\begin{aligned}\|\mathbf{1}_{E\cap\Lambda_k}f\|_{L^p(\mathbb{R}^d;X)}&\geq\left(\frac{|E\cap\Lambda_k|}{2C_1|S^{d-1}|d^{(d-1)/2}}\right)^{\frac{\ln(M)}{\ln(2)}}\|\mathbf{1}_{\Lambda_k}f\|_{L^p(\mathbb{R}^d;X)}\left(\frac{|E\cap\Lambda_k|}{2}\right)\\&\geq\left(\frac{\rho}{C_4}\right)^{\frac{\ln(M)}{\ln(2)}+1}\|\mathbf{1}_{\Lambda_k}f\|_{L^p(\mathbb{R}^d;X)}.\end{aligned}$$

Since $k\in\mathbb{Z}^d$ was arbitrary but good, we can either sum over all good cubes (if $p\in[1,\infty)$), or take the supremum over all good cubes (if $p=\infty$), and obtain by using Lemma A.1

$$\|\mathbf{1}_E f\|_{L^p(\mathbb{R}^d;X)}\geq\|\mathbf{1}_{E\cap\Lambda_{\text{good}}}f\|_{L^p(\mathbb{R}^d;X)}\geq C_3\left(\frac{\rho}{C_4}\right)^{\frac{\ln(M)}{\ln(2)}+1}\|f\|_{L^p(\mathbb{R}^d;X)}.$$

By the definitions of M, C_3, and C_4 and using that $\rho\leq 1$, we find that there exists a constant $C_d\geq 1$ depending only on the dimension d such that for all $p\in[1,\infty]$, we have

$$\|\mathbf{1}_E f\|_{L^p(\mathbb{R}^d;X)}\geq\left(\frac{\rho}{C_d}\right)^{C_d(1+|\lambda|_1)}\|f\|_{L^p(\mathbb{R}^d;X)}.$$

This proves the statement in the case $L=(1,1,\ldots,1)$. Let now $L\in[0,\infty)^d$ be arbitrary. Theorem 3.1 follows by applying the result for $L=(1,1,\ldots,1)$ to the function $f\circ T_L$, where $T_L:\mathbb{R}^d\to\mathbb{R}^d$ is given by $T_L x=(L_k x_k)_{k=1}^d$.

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