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L^p-estimates for the wave equation on the Heisenberg group

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Abstract. Let $\mathcal{L}$ denote the sub-Laplacian on the Heisenberg group $\mathbb{H}_m$. We prove that $e^{i\sqrt{-\mathcal{L}}}/(1-\mathcal{L})^{\alpha/2}$ extends to a bounded operator on $L^p(\mathbb{H}_m)$, for $1 \le p \le \infty$, when $\alpha > (d-1)\,|1/p - 1/2|$.

0. Introduction.

On the Heisenberg group $\mathbb{H}_m$, which is $\mathbb{C}^m \times \mathbb{R}$ endowed with the group law

$$(z,t)\cdot(z',t') := \Big(z+z', t+t' - \frac{1}{2}\operatorname{Im} z\cdot \overline{z'}\Big),$$

the vector fields

$$X_j := \frac{\partial}{\partial x_j} - \frac{1}{2}\, y_j\, \frac{\partial}{\partial t}\,, \qquad Y_j := \frac{\partial}{\partial y_j} + \frac{1}{2}\, x_j\, \frac{\partial}{\partial t}\,,$$

$j = 1, \dots, m$, and $T := \partial/\partial t$ form a natural basis for the Lie algebra of left-invariant vector fields. The only non-trivial commutation relations among those are $[X_j, Y_j] = T$, $j = 1, \dots, m$. Due to these relations, the non-elliptic *sub-Laplacian*

$$\mathcal{L} := \sum_{j=1}^{m} (X_j^2 + Y_j^2)$$

on $\mathbb{H}_m$ is still hypoelliptic, and provides one of the simplest examples of a non-elliptic "sum of squares operator" in the sense of Hörmander (see *e.g.* [K], [Hö]). Moreover, $\mathcal{L}$ takes over in many respects of analysis on $\mathbb{H}_m$ the role which the Laplacian plays on Euclidian space.

Consider the following Cauchy problem for the wave equation on $\mathbb{H}_m \times \mathbb{R}$ associated to $\mathcal{L}$

$$\text{(CP)} \qquad \frac{\partial^2 u}{\partial \tau^2} - \mathcal{L}u = 0\,, \qquad u|_{\tau=0} = f\,, \qquad \frac{\partial u}{\partial \tau}\Big|_{\tau=0} = g\,,$$

where $\tau \in \mathbb{R}$ denotes time.

If we put $L := -\mathcal{L}$, then the solution to this problem is formally given by

$$u(x,\tau) = \Big(\frac{\sin(\tau\sqrt{L})}{\sqrt{L}}\, g\Big)(x) + (\cos(\tau\sqrt{L})f)(x)\,, \qquad (x,\tau) \in \mathbb{H}_m \times \mathbb{R}.$$

In fact, if $L^p(\mathbb{H}_m), 1 \le p \le \infty$, denotes the L^p-Lebesgue space on $\mathbb{H}_m$ with respect to the bi-invariant Haar measure (which incidentally agrees with the Lebesgue measure on $\mathbb{C}^m \times \mathbb{R}$), then the above expression for u makes perfect sense at least for $f, g \in L^2(\mathbb{H}_m)$, if one defines the functions of L involved by the spectral theorem (notice that L is essentially selfadjoint on $C_0^\infty(\mathbb{H}_m)$).

If one decides to measure smoothness properties of the solution $u(x,\tau)$ to (CP) for fixed time τ in terms of Sobolev norms of the form $\|f\|_{L^p_\alpha} := \|(1+L)^{\alpha/2} f\|_{L^p}$, one is naturally led to study the mapping properties of operators such as

$$\frac{e^{i\tau\sqrt{L}}}{(1+L)^{\alpha/2}}$$

or

$$\frac{\sin(\tau\sqrt{L})}{\sqrt{L}(1+L)^{\alpha/2}}$$

as operators on $L^p(\mathbb{H}_m)$ into itself.

For the classical wave equation on Euclidian space, sharp estimates for the corresponding operators have been established by Peral [P] and Miyachi [Mi].

In particular, if Δ denotes the Laplacian on $\mathbb{R}^d$, then

$$(1-\Delta)^{-\alpha/2}\, e^{i\tau\sqrt{-\Delta}}$$

is bounded on $L^p(\mathbb{R}^d)$, if $\alpha \geq \alpha(d,p) := (d-1)\,|1/p - 1/2|$, for $1 < p < \infty$. Moreover, $(1-\Delta)^{-((d-1)/2)/2}\, e^{i\tau\sqrt{-\Delta}}$ is bounded from the real Hardy space $H^1(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.

Local analogues of these results hold true for solutions to strictly hyperbolic differential equations (see *e.g.* [CF], [P], [B], [Mi], [SSS]).

Indeed, as has been shown in [B] and [SSS], the estimates in [P] and [Mi] locally hold true more generally for large classes of Fourier integral operators, and solutions to strictly hyperbolic equations can be expressed in terms of such operators.

The problem in studying the wave equation associated to the sub-Laplacian on the Heisenberg group is the lack of strict hyperbolicity, since $\mathcal{L}$ is degenerate-elliptic, and Fourier integral operator technics do not seem to be available any more.

Interesting information about solutions to (CP) have been obtained by Nachman [N]. Among other things, Nachman showed that the wave operator on $\mathbb{H}_m$ admits a fundamental solution supported in a "forward light cone", whose singularities lie along the cone Γ formed by the bicharacteristics through the origin. Moreover, he computed the asymptotic behaviour of the fundamental solution as one approaches a generic singular point on Γ. His method does, however, not provide uniform estimates on these singularities, so that it cannot be used to prove L^p-estimates for solutions to (CP). What his results do reveal, however, is that Γ is by far more complex for $\mathbb{H}_m$ than the corresponding cone in the Euclidian case. This is related to the underlying, more complex sub-Riemannian geometry.

Nevertheless, in this article we shall prove the following theorem: Let $d = m+1$ denote the Euclidian dimension of $\mathbb{H}_m$.

Theorem. $e^{i\sqrt{L}}/(1+L)^{\alpha/2}$ *extends to a bounded operator on* $L^p(\mathbb{H}_m)$, *for* $1 \leq p \leq \infty$, *when* $\alpha > (d-1)\,|1/p - 1/2|$.

REMARK. One can see below that the same result holds for

$$\frac{\sin\sqrt{L}}{\sqrt{L}\,(1+L)^{(\alpha-1)/2}}\ ,$$

or with the factors $(1+L)^{-\alpha/2}$ (respectively $(1+L)^{-(\alpha-1)/2}$) replaced by $(1+\sqrt{L})^{-\alpha}$, (respectively $(1+\sqrt{L})^{-(\alpha-1)}$).

Notice that the restriction to time $\tau = 1$ in our theorem is inessential, since L is homogeneous of degree 2 with respect to the automorphic dilations $(z,t) \longmapsto (r\,z, r^2\,t)$, $r > 0$.

Our theorem is slightly weaker than what one would expect in direct analogy with the afore mentioned result of Peral and Miyachi. It would be interesting to know whether the condition $\alpha = \alpha(d,p)$ does already suffice, if $1 < p < \infty$, and if there is an endpoint result for $p = 1$.

Finally we would like to mention that the spectral multiplier theorem for L in [MS] (see also [H]) can easily be deduced from our theorem by means of the method of subordination.

Our approach to the theorem is based on harmonic analysis on $\mathbb{H}_m$, in the sense expressed by Strichartz in [St], as the joint spectral theory of the two operators L and iT. We shall closely follow the notation in [St], and freely make use of the results of that paper, as well as of those in [MRS1,2].

1. Basic reductions and dyadic decomposition of $e^{i\sqrt{L}}$.

In order to prove the theorem and the subsequent remark, we first observe that it suffices to prove the case $p = 1$. This follows from a standard interpolation argument. Namely, if we assume that the case $p = 1$ was true, and define the analytic family of operators $T_\alpha := e^{i\sqrt{L}}/(1+L)^{\alpha/2}$, then we had

$$\|T_\alpha f\|_2 \le C_\alpha \|f\|_2 \,, \qquad \text{if } \operatorname{Re}\alpha = 0 \,,$$

$$\|T_\alpha f\|_1 \le C_\alpha \|f\|_1 \,, \qquad \text{if } \operatorname{Re}\alpha > \frac{d-1}{2} \,.$$

The latter inequality remains true even if $\operatorname{Im}\alpha \neq 0$, since the operators $(1+L)^{-\varepsilon+i\gamma}$, γ real, are known to be bounded on L^1, for any $\varepsilon > 0$, with norm growing at most polynomially in γ. This can in fact also be seen by a slight modification of the proof of Corollary 1.2 to follow. Hence one can use the analytic interpolation theorem in [S1] and a standard duality argument to deduce the theorem for arbitrary p (the results in the remark can be obtained similarly).

For any bounded function ψ on $\mathbb{R}^+$ we define the operator $\psi(L)$ by the spectral theorem, and denote by $M_\psi \in \mathcal{S}'(\mathbb{H}_m)$ the corresponding Schwartz convolution kernel, so that $\psi(L) = f * M_\psi$ whenever $f \in \mathcal{S}$. We also write $M_\psi = \psi(L)\,\delta_0$, where δ_0 is the Dirac measure at the origin.

The results for the case $p = 1$ are proved by showing that the corresponding convolution kernels belong to $L^1(\mathbb{H}_m)$.

1.1. Reduction to an estimate for the local part of the convolution kernel.

Let η be an even $C_0^\infty(\mathbb{R})$ function, so that $\eta(\xi) = 1$ for small $|\xi|$, and $\eta(\xi) = 0$, if $|\xi| \geq 1$. For some large constant $N > 1$, to be chosen later, put $\eta_N(\xi) := \eta(\xi/N)$. Consider the function

$$h(\xi) := (1 - \eta_N)(\xi)\, \xi^{-\alpha/2}\, e^{i\sqrt{\xi}}\,, \qquad \xi > 0\,. \tag{1.1}$$

We let M denote the corresponding convolution kernel, so that $h(L)f = f * M$.

Proposition 1.1. *To prove the theorem, it suffices to show that* $\chi_{B_2} M$ *belongs to* $L^1(\mathbb{H}_m)$.

Here B_r denotes the ball of radius r centered at 0 with respect to the optimal control distance on $\mathbb{H}_m$. (For the definition of this distance see *e.g.* [VCS]).

The proof of the proposition is based on the following two facts. The first deals with the speed of propagation of the wave equation and can be found in [Me].

(1.2). *The support of the distribution* $\cos(t\sqrt{L})\,\delta_0$ *is contained in* $B_{|t|}$.

The second fact guarantees that for certain multipliers ψ the corresponding kernel M_ψ is in $L^1(\mathbb{H}_m)$.

(1.3). *Suppose* $\psi \in C^{(k)}(\mathbb{R}^+)$, *with* k *assumed to be sufficiently large. If* ψ *satisfies the inequalities*

$$\begin{cases} |\xi^\ell\, \psi^{(\ell)}(\xi)| \leq A\, \xi^{1/2}\,, & \text{when } 0 < \xi \leq 1\,, \\ |\xi^\ell\, \psi^{(\ell)}(\xi)| \leq A\, \xi^{-1/2}\,, & \text{when } 1 \leq \xi < \infty\,, \end{cases}$$

for $0 \leq \ell \leq k$, *then* $M_\psi = \psi(L)\,\delta_0$ *is in* $L^1(\mathbb{H}_m)$.

REMARKS. 1) It actually suffices to take $k > (d-1)/2$; also the exponent $1/2$ can be reduced to $\varepsilon > 0$. However the above special case suffices for our purposes.

2) The proof gives the bound $\|M_\psi\|_{L^1(\mathbb{H}_m)} \leq \text{constant}\, A$, with A as in (1.3).

To prove (1.3), we let

$$1 = \sum_{j=-\infty}^{\infty} \chi_j(x)$$

be a standard dyadic partition of unity for $\mathbb{R}^+$, with $\chi_j(x) := \chi(2^{-j}x)$, where $\chi \in C_0^\infty(\mathbb{R})$ is non-negative and supported in $[1/2, 2]$.

We write $\psi_j := 2^{|j|/2}\chi_j\psi$. Then $\psi(\xi) = \sum_j 2^{-|j|/2}\psi_j(\xi)$, and $\|M_{\psi_j}\|_{L^1(\mathbb{H}_m)} \le A$, uniformly in j.

In fact, with $\tilde\psi_j(\xi) := \psi_j(2^j\xi)$, each $\tilde\psi_j$ is supported in $[1/2, 2]$, and the $\tilde\psi_j$ satisfy the inequalities

$$\sup_{j,\xi} |\tilde\psi_j^{(\ell)}(\xi)| \le A, \qquad \text{for } 0 \le \ell \le k.$$

Thus the key step in the proof of the Marcinkiewicz-Mikhlin-Hörmander multiplier theorem for $\mathbb{H}_m$ (for which see *e.g.* [FoS], [C], [MM]; also [MS], [H], [MRS1,2]) shows that

$$\sup_j \|M_{\psi_j}\|_{L^1(\mathbb{H}_m)} = \sup_j \|M_{\tilde\psi_j}\|_{L^1(\mathbb{H}_m)} < \infty,$$

and the assertion (1.3) is proved, since $M_\psi = \sum 2^{-|j|/2} M_{\psi_j}$.

Now for $\alpha > 0$ and $|t| \le 1$ set

$$f_{\alpha,t}(\xi) := (1-\eta_N)(\xi)\,\xi^{-\alpha/2}\cos(t\sqrt{\xi}),$$

$$g_{\alpha,t}(\xi) := (1-\eta_N)(\xi)\,\xi^{-\alpha/2}\sin(t\sqrt{\xi}),$$

so that by (1.1) $h(\xi) = f_{\alpha,1}(\xi) + i\,g_{\alpha,1}(\xi)$.

It is easily seen that for $\xi > 0$

$$(1-\eta_N)(\xi)\,|\xi|^{-\alpha/2} = \int \varphi(\tau)\cos(\tau\sqrt{\xi})\,d\tau,$$

where φ is such that $\eta\,\varphi \in L^1$ and $(1-\eta)\,\varphi \in \mathcal{S}$.

Hence

$$f_{\alpha,t}(\xi) = \int (\eta\,\varphi)(\tau)\cos(\tau\sqrt{\xi})\cos(t\sqrt{\xi})\,d\tau + \Phi(\sqrt{\xi})\cos(t\sqrt{\xi}),$$

with $\Phi \in \mathcal{S}$.

Now the support of the distribution corresponding to

$$\int (\eta\,\varphi)(\tau)\cos(\tau\sqrt{L})\cos(t\sqrt{L})\,d\tau$$

lies in B_2. This is because $2\cos(\tau\sqrt{L})\cos(t\sqrt{L}) = \cos((\tau+t)\sqrt{L}) + \cos((\tau-t)\sqrt{L})$ and the result of fact (1.2). However the kernel corresponding to $\Phi(\sqrt{L})\cos(t\sqrt{L})$ is in $L^1(\mathbb{H}_m)$, uniformly for $|t| \le 1$, as long as $\Phi \in \mathcal{S}$.

This can be seen by applying the result (1.3) to the function $\psi(\xi) := \Phi(\sqrt{\xi})\cos(t\sqrt{\xi}) - \Phi(0)\,e^{-\xi}$, and recalling that the kernel corresponding to $e^{-\xi}$ is the heat-kernel, which is in $L^1(\mathbb{H}_m)$.

Thus we have that the $f_{\alpha,t}(L)\,\delta_0$ are uniformly in $L^1(\mathbb{H}_m)$ in the complement of the ball B_2.

As for $g_{\alpha,1}$, we observe that

$$g_{\alpha+1,1}(\xi) = \int_0^1 f_{\alpha,t}(\xi)\,dt\,, \tag{1.4}$$

and thus $g_{\alpha+1,1}(L)(\delta_0)$ is in $L^1(\mathbb{H}_m)$ outside the ball B_2, if $\alpha > 0$. As a result

$$h(L)\,\delta_0 = f_{\alpha,1}(L)\,\delta_0 + i\,g_{\alpha,1}(L)\,\delta_0 = M$$

is in $L^1(\mathbb{H}_m)$ outside the ball B_2, if $\alpha > 1$. Thus, if we knew that $\chi_{B_2} M$ was in $L^1(\mathbb{H}_m)$, we could conclude that $M \in L^1(\mathbb{H}_m)$.

The conditional assertion for $(1+L)^{-\alpha/2}\,e^{i\sqrt{L}}$ can now be obtained as follows. We write $(1+\xi)^{-\alpha/2}\,e^{i\sqrt{L}}$ as

$$\eta_N(\xi)\,(1+\xi)^{-\alpha/2}\,e^{i\sqrt{\xi}} + \frac{\xi^{\alpha/2}}{(1+\xi)^{\alpha/2}}(1-\eta_N)(\xi)\,\xi^{-\alpha/2}\,e^{i\sqrt{\xi}}\,.$$

The function $\eta_N(\xi)\,(1+\xi)^{-\alpha/2}\,e^{i\sqrt{\xi}} - e^{-\xi}$ satisfies the hypothesis of (1.3), and $e^{-\xi}$ corresponds to the heat kernel, thus

$$\eta_N(L)\,(1+L)^{-\alpha/2}\,e^{i\sqrt{L}}$$

has an $L^1(\mathbb{H}_m)$ kernel. Next the function $\xi^{\alpha/2}/(1+\xi)^{\alpha/2} - 1 + e^{-\xi}$ satisfies the hypothesis of (1.3), thus $L^{\alpha/2}/(1+L)^{\alpha/2}$ is the identity operator plus a convolution operator whose kernel is in $L^1(\mathbb{H}_m)$. Combining this with the previous assertion about $(1-\eta_N)(L)\,L^{-\alpha/2}\,e^{i\sqrt{L}}$ proves the proposition. The further conclusions in the remark are proved similarly.

We have reduced the proof of our theorem to showing that

$$\|\chi_{B_2} M\|_{L^1(\mathbb{H}_m)} < \infty ,$$

with $M = M_h$ and h given by (1.1), provided $\alpha > m = (d-1)/2$.

A further reduction is given as follows: We let $\tilde{\chi}_{B_2}$ be a smooth variant of χ_{B_2}; that is, $\tilde{\chi}_{B_2}$ is in $C_0^\infty(\mathbb{H}_m)$, with $\tilde{\chi}_{B_2}(x) = 1$, if $x \in B_2$.

Corollary 1.2. *To prove the theorem, it suffices to prove that the operator $f \longmapsto f * (\tilde{\chi}_{B_2} M)$ is bounded on $L^p(\mathbb{H}_m)$ to itself, for all p, $1 < p < \infty$.*

PROOF. Write $M = M_\alpha$ to indicate the dependence on α. Now, if $\alpha > (d-1)/2$, we can write $\alpha = \alpha' + \varepsilon$, $\varepsilon > 0$, $\alpha' > (d-1)/2$. We know from the above that $(1 - \tilde{\chi}_{B_2})M_{\alpha'}$ is in $L^1(\mathbb{H}_m)$, so if $f \longmapsto f * (\tilde{\chi}_{B_2} M_{\alpha'})$ is bounded on L^p, $1 < p < \infty$, so is $f \longmapsto f * M_{\alpha'}$. But $M_\alpha = M_{\alpha'} * ((1+L)^{-\varepsilon}\,\delta_0)$.

However, $(1+L)^{-\varepsilon}\,\delta_0$ is in $L^p(\mathbb{H}_m)$ for some $p > 1$, if $\varepsilon > 0$ (we shall prove this momentarily). We would then have $M_\alpha \in L^p(\mathbb{H}_m)$, and hence $\chi_{B_2} M_\alpha \in L^1(\mathbb{H}_m)$. However, $(1 - \chi_{B_2})M_\alpha$ was already shown to be in $L^1(\mathbb{H}_m)$, and so this would imply that $M_\alpha \in L^1(\mathbb{H}_m)$.

To see that $(1+L)^{-\varepsilon}\,\delta_0 \in L^p(\mathbb{H}_m)$ for some $p > 1$, we write

$$(1+L)^{-\varepsilon}\,\delta_0 = \frac{1}{\Gamma(\varepsilon)} \int_0^\infty e^{-s}\, s^{-m-1}\, p(s^{-1/2}z, s^{-1}t)\, s^{-1+\varepsilon}\, ds ,$$

where $p(z,t)$ denotes the heat kernel associated to L at unit time. As is well-known, $p(z,t) = O\left((1 + |z|^2 + |t|)^{-N}\right)$, for every $N \geq 0$, and as a result

$$\begin{aligned} &(1+L)^{-\varepsilon}\,\delta_0(z,t) \\ &\quad = \begin{cases} O\left((|z|^2 + |t|)^{-m-1+\varepsilon}\right), & \text{if } |z|^2 + |t| \leq 1 , \\ O\left((|z|^2 + |t|)^{-N}\right), & \text{if } |z|^2 + |t| > 1 , \text{ for all } N \geq 0 . \end{cases} \end{aligned}$$

From this it follows that $(1+L)^{-\varepsilon}\delta_0 \in L^p(\mathbb{H}_m)$, if $(-m-1+\varepsilon)\,p > -m-1$.

In order to verify the assumption in the corollary, we shall invoke the Gelfand transform $\mathcal{G}$ for the algebra of radial functions on $\mathbb{H}_m$ (compare [MRS1]). For $f \in L^1(\mathbb{H}_m)$ radial, we have

$$\mathcal{G}(h(L)f)(\lambda, n) = h((m + 2\,n)\,|\lambda|)\,\mathcal{G}f(\lambda, n)\,,$$

$\lambda \in \mathbb{R}^\times := \mathbb{R} \setminus \{0\}$, $n \in \mathbb{N}$.

If χ_j, $j \in \mathbb{Z}$, denotes again our dyadic decomposition of unity on $\mathbb{R}^+$, we put

$$\varphi^\varepsilon_{k,j}(\lambda, n) := h((m + 2\,n)|\,\lambda|)\,\chi_{2k-j}(\varepsilon\,\lambda)\,\chi_j(m + 2\,n)\,,$$

for $j \geq 0$, $k \in \mathbb{Z}$, $\varepsilon \in \{-1, 1\}$. We also set

$$K^\varepsilon_{k,j} = \mathcal{G}^{-1}(\varphi^\varepsilon_{k,j})\,.$$

By [MRS1], since $\varphi^\varepsilon_{k,j}$ is smooth and supported away from the axes, one has $K^\varepsilon_{k,j} \in \mathcal{S}$. Observe also that

$$2^{k-1} \leq \sqrt{(m + 2\,n)\,|\lambda|} \leq 2^{k+1} \text{ on } \operatorname{supp} \varphi^\varepsilon_{k,j}\,, \tag{1.5}$$

so that $\varphi^\varepsilon_{k,j} = 0$, unless $2^k \geq N/4$. So, if we fix any $k_0 \gg 1$, we may choose N sufficiently large so that

$$M = \sum_{\substack{\varepsilon=\pm1 \\ k\geq k_0 \\ j\geq 0}} K^\varepsilon_{k,j}\,, \tag{1.6}$$

for instance in the sense of distributions.

The proof of the theorem is then reduced to showing the following

Proposition 1.3. *If $\alpha > m$, then*

$$\sum_{\substack{\varepsilon=\pm1 \\ k\geq k_0 \\ j\geq 0}} \|\tilde{\chi}_{B_2} K_{k,j}\|_{(p,p)} < \infty\,,$$

for every p, $1 < p < \infty$, where $\|K\|_{(p,p)}$ denotes the norm of the convolution operator $f \longmapsto f * K$ on $L^p(\mathbb{H}_m)$.

1.2. Formulas for $K_{k,j}$.

In order to compute $K^{\varepsilon}_{k,j}$, we first observe that

$$\varphi^{-1}_{k,j}(\lambda, n) = \varphi^{1}_{k,\varepsilon}(-\lambda, n)\,,$$

hence $K^{-1}_{k,j}(z,u) = K^{1}_{k,j}(z,-u)$ (compare [MRS2]). This allows us to reduce to the case $\varepsilon = 1$, and we shall from now on suppress the suffix ε, assuming that it is 1.

Next, observe that $\mathcal{G}(i\,\mathcal{L}\,T^{-1}f)(\lambda,n) = (m+2\,n)\,\mathcal{G}f(\lambda,n)$, if $\lambda > 0$. Therefore, by [St, Corollary 2.5],

$$\varphi(i\,\mathcal{L}\,T^{-1})\,f = \sum_n \varphi(m+2\,n)\,f * \tilde{P}_n\,,$$

for any bounded multiplier φ, if $\mathcal{G}f$ is supported in $\lambda > 0$, where $\tilde{P}_n = c_n\,\delta_0 + \text{p.v.}\,P_n$, and where P_n is the Calderón-Zygmund kernel

$$P_n(z,t) = 2^{m-1}\,\pi^{-m-1}\,(-1)^n\,\frac{(m+n)!}{n!} \cdot \frac{(|z|^2 - 4\,i\,t)^n}{(|z|^2+4\,i\,t)^{m+1+n}}\Big(1 + \frac{n}{m+n}\,\frac{|z|^2+4\,i\,t}{|z|^2-4\,i\,t}\Big)$$

(see [St, Lemma 2.1 and (2.25)]).

If we define "polar coordinates" by putting

$$r := (|z|^4 + 16\,t^2)^{1/4}\,, \qquad 4\,t + i\,|z|^2 =: r^2\,e^{i\theta/2}\,, \qquad 0 \le \theta < 2\pi\,,$$

then we have

$$P_n = c_m\,(Q_n - Q_{n-1})\,, \tag{1.7.a}$$

with

$$\begin{aligned} Q_n &:= \frac{(m+n)!}{n!}\,\frac{e^{i(m+2\,n+1)\theta/2}}{r^{2m+2}} \\ &= \frac{(m+n)!}{n!}\,\frac{(4\,t+i\,|z|^2)^n}{(4\,t-i\,|z|^2)^{m+n+1}} \\ &= i^{m+1}\,(-1)^n\,\frac{(m+n)!}{n!}\,\frac{(|z|^2-4\,i\,t)^n}{(|z|^2+4\,i\,t)^{m+1+n}}\,. \end{aligned} \tag{1.7.b}$$

Next, observe that

$$\begin{aligned}\varphi_{k,j}(\lambda, n) &:= \varphi^1_{k,j}(\lambda, n)\\ &= ((m+2\,n)\,|\lambda|)^{-\alpha/2}\,\chi_{2k-j}(\lambda)\,\chi_j(m+2\,n)\,e^{i\sqrt{(m+2n)|\lambda|}}\,,\end{aligned}$$

if k is sufficiently large (in that case we may in fact delete the factor $(1-\eta)\,(\cdot/N)$ in h), which we may assume. Putting

$$\tilde\chi(x) := x^{-\alpha/2}\,\chi(x)\,,$$

this may be written as

$$\varphi_{k,j}(\lambda, n) = 2^{-\alpha k}\,\tilde\chi_{2k-j}(\lambda)\,\tilde\chi_j(m+2\,n)\,e^{i\sqrt{(m+2n)|\lambda|}}\,.$$

Since $\tilde\chi$ is of similar type as χ, we shall again write χ in place of $\tilde\chi$, and then get, for k, j fixed,

$$\varphi_{k,j}(\lambda, n') = 2^{-\alpha k}\sum_n \chi_j(m+2\,n)\,\gamma_n(\lambda)\,\delta_n(n')\,,$$

where $\delta_n(n') = 1$ if $n = n'$ and $\delta_n(n') = 0$ otherwise, with

$$\gamma_n(\lambda) := \chi_{2k-j}(\lambda)\,e^{i\sqrt{(m+2n)|\lambda|}}\,.$$

This implies

$$\begin{aligned}f * K_{k,j} &= 2^{-\alpha k}\sum_n \chi_j(m+2\,n)\,\gamma_n(-i\,T)\,\delta_n\Big(\frac{i\,\mathcal{L}\,T^{-1}-m}{2}\Big)f\\ &= 2^{-\alpha k}\sum_n \chi_j(m+2\,n)\,\gamma_n(-i\,T)\,(f * \tilde P_n)\,.\end{aligned}$$

In the sequel, we shall often use the following abbreviation

$$\ell := 2\,k - j\,.$$

We put

$$\Phi_{\ell,n}(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\sqrt{(m+2n)\lambda}}\,\chi_\ell(\lambda)\,e^{i\lambda t}\,d\lambda\,.$$

Then, away from $z = 0$, $K_{k,j}$ is given by

$$(1.8) \quad K_{k,j}(z,t) = 2^{-\alpha k} \sum_n \chi_j(m + 2\,n) \int P_n(z, t-s)\, \Phi_{2k-j,n}(s)\, ds$$

(we do know that $K_{k,j} \in \mathcal{S}$, although this is not evident from this formula).

Putting $\lambda = 2^\ell x^2$ in the integral defining $\Phi_{\ell,n}$, we write

$$\Phi_{\ell,n}(t) = \frac{2^\ell}{\pi} \int_{-\infty}^{\infty} e^{i(\sqrt{(m+2n)\,2^\ell}x - 2^\ell t x^2)}\, \chi(x)\, x\, dx\,,$$

which shows that the asymptotics of $\Phi_{\ell,n}$ can be computed by the stationary phase method of

Lemma 1.4. *Let $f \in C_0^\infty(\mathbb{R})$ be supported in $[1/2, 2]$. For every $N \in \mathbb{N}$ there exist functions $f_0, \ldots, f_N \in C_0^\infty(\mathbb{R})$ supported in $[1/4, 4]$ and $E_N \in C^\infty(\mathbb{R}^2)$, such that for $(a,b) \in \mathbb{R}^2$ with $|(a,b)| > 1$*

$$\int_{-\infty}^{\infty} e^{i(ax - bx^2/2)} f(x)\, dx = e^{ia^2/(2b)} \sum_{\nu=0}^{N} b^{-1/2-\nu} f_\nu\Big(\frac{a}{b}\Big) + E_N(a,b)\,,$$

where E_N satisfies

$$E_N^{(\alpha)}(a,b) = O\left(|(a,b)|^{-N/2-1}\right),$$

for every $\alpha \in \mathbb{N}^2$.

Proof. We have

$$\int_{-\infty}^{\infty} e^{i(ax - bx^2/2)}\, f(x)\, dx = e^{ia^2/(2b)} \int_{-\infty}^{\infty} e^{-ibx^2/2}\, f\Big(x + \frac{a}{b}\Big)\, dx\,.$$

Now, in the region where $1/4 \le a/b \le 4$, the result follows easily from the proof of [S2, Proposition 3, Chapter VIII] and the remarks in [S2, Chapter VIII, 1.3.4], since the critical point in the integral on the right is $x = 0$. The functions f_ν do in fact arise as linear combinations of derivatives of f.

In the remaining region, the result is obtained by integrating by parts in the integral on the left.

We may apply the Lemma to $\Phi_{\ell,n}(t)$, since $\sqrt{(m+2\,n)\,2^\ell} \sim 2^k \gg 1$, and obtain

$$(1.9)\qquad \begin{aligned}\Phi_{\ell,n}(t) = e^{i(m+2n)/(4t)} \sum_{\nu=0}^{N} (2^{\ell+1}\,t)^{-1/2-\nu}\, 2^\ell\, f_\nu\Big(\sqrt{\frac{m+2\,n}{2^\ell}}\,\frac{1}{2\,t}\Big)\\ + 2^\ell\, E_N\Big(\sqrt{(m+2\,n)\,2^\ell}, 2^{\ell+1}\,t\Big)\,,\end{aligned}$$

with f_ν and E_N as in the lemma.

Put

$$a_{n,\ell} := \sqrt{(m+2\,n)\,2^\ell}\;.$$

Since $a_{n,\ell} \sim 2^k$, and since

$$(2^{\ell+1}\,t)^{-1/2-\nu} = a_{n,\ell}^{-1/2-\nu}\Big(\sqrt{\frac{m+2\,n}{2^\ell}}\,\frac{1}{2\,t}\Big)^{1/2+\nu}\,,$$

the ν-th term in (1.9) is of the form

$$c_\nu\,\tilde{f}_\nu\Big(\sqrt{\frac{m+2\,n}{2^\ell}}\,\frac{1}{2\,t}\Big)\,,$$

with $c_\nu = O\,(2^{-(1/2+\nu)k})$, $\tilde{f}_\nu(x) = x^{1/2+\nu}\,f_\nu(x)$.

Consequently, we may reduce in (1.8) to the case where $\Phi_{\ell,n}$ is either of the form

$$\text{(a)}\qquad \Phi_{\ell,n}(t) = 2^\ell\, a_{n,\ell}^{-1/2-\nu}\, f\Big(\sqrt{\frac{m+2\,n}{2^\ell}}\,\frac{1}{4\,t}\Big)\, e^{i(m+2n)/(4t)}$$

with $f \in C_0^\infty(\mathbb{R})$ supported in [1/8,2] and $\nu \geq 0$, or of the form

$$\text{(b)}\qquad \Phi_{\ell,n}(t) = 2^\ell\, E_N(a_{n,\ell}, 2^{\ell+1}t)$$

with E_N as in the lemma and N sufficiently large.

Case (b) is easily dealt with by the Marcinkiewicz multiplier theorem in [MRS1, Theorem 2.2].

If $\Phi_{\ell,n}$ is of the form (b), then its inverse Fourier transform $\eta_{\ell,n}$ is of the form

$$\eta_{\ell,n}(\lambda) = \Psi(a_{n,\ell}, 2^{-\ell}\,\lambda)\,,$$

where

$$\Psi(a,\lambda) := \int_{-\infty}^{\infty} E_N(a,t)\, e^{-i\lambda t}\, dt$$

is of class C^M for $M=[N/2]-1$ and satisfies

$$\partial_a^\alpha \partial_\lambda^\beta \Psi(a,\lambda) = O\left(a^{-M+\beta}(1+|\lambda|)^{-K}\right), \tag{1.10}$$

for every α, $K \in \mathbb{N}$ and every $\beta \le M$, as can easily be seen by integration by parts.

Now, since our "original" function $\varphi_{\ell,n}$ had its Fourier transform supported where $\lambda \sim 2^\ell$, we may also localize the support of $\eta_{\ell,n}$ in this region. And, in the region where $n \sim 2^j$ and $\lambda \sim 2^\ell = 2^{2k-j}$, we see from (1.10) that $\eta_{\ell,n}(\lambda) = \psi(\sqrt{(m+2\,n)\,2^{2k-j}}, 2^{j-2k}\lambda)$ satisfies estimates of the form

$$|\partial_n^\alpha \partial_\lambda^\beta \eta_{\ell,n}(\lambda)| \le C_{\alpha,\beta}\, 2^{-Mk+(\beta-\alpha)(j-k)} \le C_{\alpha,\beta}\, 2^{-\alpha j}\, 2^{-\beta(2k-j)},$$

if $\alpha+\beta \le M$.

Thus, if we define $K_{k,j}$ by (1.8), with $\Phi_{\ell,n}$ as in (b), but Fourier transform localized in $\lambda \sim 2^\ell$, and choose N sufficiently large, we see that for any $\alpha \ge 0$ in (1.8)

$$K := \sum_{\substack{k \ge k_0 \\ j \ge 0}} K_{k,j}$$

is a kernel whose Gelfand transform satisfies the multiplier condition in [MRS1, Theorem 2.2], and thus satisfies the kernel estimates of [MRS1, Theorem 3.1], for sufficiently many derivatives. But then one checks easily that the same is true of the truncated kernel $\tilde{\chi}_{B_2} K$, and consequently the operator $f \longmapsto f * (\tilde{\chi}_{B_2} K)$ is L^p-bounded for $1 < p < \infty$ by [MRS1, Theorem 4.4].

Moreover, since

$$a_{n,\ell}^{-1/2}\, \chi_j(m+2\,n) = 2^{-k/2}\, \tilde{\chi}_j(m+2\,n)$$

with $\tilde{\chi}(x) = x^{-1/2}\chi(x)$, by modifying χ we may assume that the factor $a_{n,\ell}$ in (a) equals 2^k. We thus find that, in order to prove Proposition 1.3, it suffices to prove the following

Proposition 1.5. *Suppose that $K_{k,j}$ is given by*

$$K_{k,j}(z,t) = 2^{-mk} \sum_n \chi_j(m+2\,n) \int P_n\Big(z, \frac{t-s}{4}\Big)\, \Phi_{k,j,n}(s)\, ds$$

with

$$\Phi_{k,j,n}(t) := 2^{k/2+k-j} f\Big(\sqrt{\frac{m+2n}{2^{2k-j}}}\ \frac{1}{t}\Big)\, e^{i(m+2n)/t},$$

$f \in C_0^\infty(\mathbb{R})$ *supported in* $[1/8,2]$, $k \geq 0$ *sufficiently large. Then*

$$\sum_{j\geq 0} \|K_{k,j}\|_{L^1(B_2)} = O(k^2).$$

2. Integral formulas for $K_{k,j}$.

In order to sum the series for $K_{k,j}$ in Proposition 1.5, we first observe that $\chi_j(m+2n)\,\Phi_{k,j,n}(s) = 0$, unless $1/4 \leq 2^{k-j}\, s \leq 16$. Thus, if we choose $\tilde\chi \in C_0^\infty(\mathbb{R})$ such that $\tilde\chi(x) = 1$ for $1/4 \leq x \leq 16$ and $\operatorname{supp}\tilde\chi \subset [1/8, 32]$, then we may replace $\Phi_{k,j,n}(t)$ by $\tilde\chi(2^{k-j}\,t)\,\Phi_{k,j,n}(t)$.

Moreover, writing

$$f\Big(\sqrt{\frac{m+2n}{2^{2k-j}}}\,\frac{1}{t}\Big) = g\Big(\log\Big(\frac{m+2n}{2^{2k-j}}\,t^{-2}\Big)\Big),$$

with g smooth on $[\log 1/16, \log 2] \subset [-\pi,\pi]$ and, say, supported in $]-\pi,\pi[$, and developping g into a Fourier series on $[-\pi,\pi]$, we see that

$$(2.1)\qquad \begin{aligned}\Phi_{k,j,n}(t) &= \sum_{\nu\in\mathbb{Z}} a_\nu \Big(\frac{m+2n}{2^{2k-j}\,t^2}\Big)^{i\nu} 2^{3k/2-j}\,\tilde\chi(2^{k-j}t)\,e^{i(m+2n)/t}\\ &=: \sum_{\nu\in\mathbb{Z}} a_\nu\,\Phi_{k,j,n,\nu}(t),\end{aligned}$$

where

$$(2.2)\qquad a_\nu = O(|\nu|^{-N}), \qquad \text{for every } N\in\mathbb{N}.$$

We also put

$$K_{k,j,\nu} := 2^{-mk}\sum_n \chi_j(m+2n)\int P_n\Big(z, \frac{t-s}{4}\Big)\Phi_{k,j,n,\nu}(s)\,ds,$$

so that

$$K_{k,j} = \sum_\nu a_\nu\, K_{k,j,\nu}\,.$$

Writing

$$\tilde{\chi}_{(\nu)}(t) := t^{-2i\nu}\,\tilde{\chi}(t)\,,$$

we still have $\tilde{\chi}_{(\nu)} \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\tilde{\chi}_{(\nu)} \subset [1/8, 32]$.

Moreover,

$$\|\tilde{\chi}_{(\nu)}^{(\alpha)}\|_\infty = O\left((|\nu|+1)^\alpha\right), \qquad \alpha \in \mathbb{N}, \tag{2.3}$$

and

$$\Phi_{k,j,n,\nu}(t) = 2^{3k/2-j}\,2^{ij}\,(m+2\,n)^{i\nu}\,\tilde{\chi}_{(\nu)}(2^{k-j}\,t)\,e^{i(m+2n)/t}\,.$$

Consequently, by (1.7.b),

$$\begin{aligned}\sum_n \chi_j(m+2\,n)\,Q_n\Big(z,\frac{t-s}{4}\Big)\Phi_{k,j,n,\nu}(s) &= 2^{ij}\,2^{3k/2-j}\sum_n \chi_{(\nu),j}(m+2\,n)\,\frac{(m+n)!}{n!}\,e^{i(m+1+2n)(\theta'/2+1/s)}\\ &\qquad\cdot\frac{e^{-i/s}}{r'^{2m+2}}\,\tilde{\chi}_{(\nu)}(2^{k-j}\,s)\,,\end{aligned}$$

where r' and θ' are defined by

$$t-s+i\,|z|^2 = r'^2\,e^{i\theta'/2}\,,$$

and where $\chi_{(\nu),j}(x) = \chi_{(\nu)}(2^{-j}\,x)$, with

$$\chi_{(\nu)}(x) := x^{i\nu}\,\chi(x)\,.$$

Let us put

$$\zeta_{\nu,j}(\omega) := \sum_n \chi_{(\nu),j}(m+2\,n)\,\frac{(m+n)!}{n!}\,2^{-mj}\,e^{i(2n+m+1)\omega}\,.$$

For fixed ν, $\varrho_j := |\zeta_{\nu,j}|$ has the following properties, as can easily be seen by applying Poisson's summation formula:

$$\begin{aligned}&\text{i) } \varrho_j \text{ is } \pi\text{-periodic}\,,\\ &\text{ii) } \varrho_j(\omega) \le C_N\,\frac{2^j}{(1+2^j\,|\omega|)^N}\,, \qquad \text{for } |\omega| \le \frac{\pi}{2}\,,\end{aligned} \tag{2.4}$$

for every $N \in \mathbb{N}$, $j \in \mathbb{N}$. ii) means that $\zeta_{\nu,j}$ is essentially supported in $\{|\omega| \leq 2^{-j}\}$, and implies that $\|\zeta_{\nu,j}\|_1$ is uniformly bounded in j. Notice also that the constants C_N in (2.4.ii) will grow with ν, however, only polynomially, namely

$$C_N = O\left((|\nu|+1)^{N+1}\right). \tag{2.5}$$

With $\zeta_{\nu,j}$ as above, we have

$$\begin{aligned}\sum_n \chi_j(m+2n)\, Q_n\Big(z, \frac{t-s}{4}\Big)\, \Phi_{k,j,n,\nu}(s) \\ = 2^{ij}\, 2^{3k/2+(m-1)j}\, \zeta_{\nu,j}\Big(\frac{\theta'}{2}+\frac{1}{s}\Big)\frac{e^{-i/s}}{r'^{2m+2}}\, \tilde{\chi}_{(\nu)}(2^{k-j}\, s)\,.\end{aligned}$$

And, since

$$\frac{\theta'}{2} = \arctan\Big(\frac{|z|^2}{t-s}\Big),$$

where arctan denotes the branch of $\tan^{-1}$ taking values in $[0,\pi]$, we obtain

$$\begin{aligned}&2^{-2mk}\sum_n \chi_j(m+2n)\int Q_n\Big(z, \frac{t-s}{4}\Big)\, \Phi_{k,j,n,\nu}(s)\, ds \\ &= 2^{ij}\, 2^{k/2}\, 2^{(m-1)(j-k)} \\ &\quad\cdot \int \frac{\zeta_{\nu,j}\Big(\arctan\Big(\frac{|z|^2}{t-s}\Big)+\frac{1}{s}\Big)}{(|z|^4+(t-s)^2)^{m+1/2}}\, e^{-i/s}\, \tilde{\chi}_{(\nu)}(2^{k-j}s)\, ds\,.\end{aligned} \tag{2.6'}$$

Since $P_n = c_m(Q_n - Q_{n-1})$, this allows to establish an integral formula for $K_{k,j,\nu}$.

In order to simplify the notation, we shall do this only for the case $\nu = 0$. In fact, we shall see that the estimates of $K_{k,j,\nu}$ will only depend on the constants C_N in (2.4) for a finite number of N's and on the norms of a finite number of derivatives of $\tilde{\chi}_{(\nu)}$. Therefore, in view of (2.3) and (2.5), we shall get the same type of estimate for $\|K_{k,j,\nu}\|_1$ as for $\|K_{k,j,0}\|_1$, except possibly for a factor which grows like a power of $|\nu|+1$. But, because of (2.2), it will then be clear that

$$\|K_{k,j}\|_1 \leq \sum_\nu |a_\nu|\, \|K_{k,j,\nu}\|_1\,,$$

which leads to an estimate of the same type as for $\|K_{k,j,0}\|_1$.

So, from now on we shall assume that $\Phi_{k,j,n} = \Phi_{k,j,n,0}$, *i.e.* that

$$\Phi_{k,j,n}(t) = 2^{3k/2-j}\,\chi(2^{k-j}\,t)\,e^{i(m+2n)/t}\,, \tag{2.7}$$

with $\chi \in C_0^\infty(\mathbb{R})$ supported in $[1/8, 32]$. Then, by (2.6′),

$$\begin{aligned} &2^{-mk}\sum_n \chi_j(m+2\,n)\int Q_n\Big(z,\frac{t-s}{4}\Big)\,\Phi_{k,j,n}(s)\,ds \\ &\quad = 2^{ij}\,2^{k/2}\,2^{(m-1)(j-k)} \\ &\qquad \cdot \int \frac{\zeta_j\Big(\arctan\Big(\dfrac{R}{t-s}\Big)+\dfrac{1}{s}\Big)}{(R^2+(t-s)^2)^{(m+1)/2}}\,e^{-i/s}\,\tilde\chi(2^{k-j}s)\,ds\,, \end{aligned} \tag{2.6}$$

where we have used the abbreviations $\zeta_j = \zeta_{0,j}$ and

$$R := |z|^2\,.$$

Now, observe that if we replace Q_n by Q_{n-1} in the left hand side of (2.6), we have to sum

$$\begin{aligned} &\sum_n \chi_j(m+2\,n)\,Q_{n-1}\Big(z,\frac{t-s}{4}\Big)\,\Phi_{k,j,n}(s) \\ &\quad = 2^{ij}\,2^{3k/2-j}\sum_n \chi_j(m+2\,n)\,\frac{(m+n-1)!}{(n-1)!}\,e^{i(m+1+2(n-1))(\theta'/2+1/s)} \\ &\qquad \cdot \frac{e^{i/s}}{r'^{2m+2}}\,\chi(2^{k-j}s)\,. \end{aligned}$$

Replacing $\chi_j(m+2\,n)$ in this sum by $\chi_j(m+2\,(n-1)) + 2^{-j}\,\tilde\chi_j(m+2\,(n-1))$, with

$$\begin{aligned} \tilde\chi_j(m+2\,(n-1)) &= 2^j\Big(\chi\Big(\frac{m+2\,n}{2^j}\Big)-\chi\Big(\frac{m+2\,n-2}{2^j}\Big)\Big) \\ &= -2\int_0^1 \chi'\Big(\frac{m+2\,n-2\,t}{2^j}\Big)\,dt \end{aligned}$$

having similar properties as χ_j, we find that

$$\sum_n \chi_j(m+2\,n)\, Q_{n-1}\Big(z, \frac{t-s}{4}\Big)\Phi_{k,j,n}(s)$$
$$= 2^{ij}\, 2^{k/2}\, 2^{(m-1)(j-k)}$$
$$\cdot \left(\int \frac{\zeta_j\Big(\arctan\Big(\dfrac{R}{t-s}\Big)+\dfrac{1}{s}\Big)}{(R^2+(t-s)^2)^{(m+1)/2}}\, e^{i/s}\,\chi(2^{k-j}s)\, ds \right.$$
$$\left. + 2^{-j}\int \frac{\tilde{\zeta}_j\Big(\arctan\Big(\dfrac{R}{t-s}\Big)+\dfrac{1}{s}\Big)}{(R^2+(t-s)^2)^{(m+1)/2}}\, e^{i/s}\,\chi(2^{k-j}s)\, ds \right),$$

with the same ζ_j as in (2.6), and $\tilde{\zeta}_j$ of the same type as ζ_j.

Writing $K_{k,j}(R,t)$ instead of $K_{k,j}(z,t)$, we then get

$$K_{k,j}(R,t) = C_m\, 2^{ij}\, 2^{k/2}\, 2^{(m-1)(j-k)}$$
$$\cdot \left(\int \frac{\zeta_j\Big(\arctan\Big(\dfrac{R}{t-s}\Big)+\dfrac{1}{s}\Big)}{(R^2+(t-s)^2)^{(m+1)/2}}\, (e^{-i/s}-e^{i/s})\,\chi(2^{k-j}s)\, ds \right. \tag{2.8}$$
$$\left. + 2^{-j}\int \frac{\tilde{\zeta}_j\Big(\arctan\Big(\dfrac{R}{t-s}+\dfrac{1}{s}\Big)\Big)}{(R^2+(t-s)^2)^{(m+1)/2}}\, e^{i/s}\,\chi(2^{k-j}s)\, ds \right).$$

Formula (2.8) will be useful in the region where $R^2+(t-s)^2$ is large. To deal with the region where $R^2+(t-s)^2$ is small, we establish a second formula for $K_{k,j}$.

To this end, we put

$$R_n(z,t) := \frac{(m+n-1)!}{n!}\,\frac{(4\,t+i\,R)^n}{(4\,t-i\,R)^{m+n}} = \frac{(m+n-1)!}{n!}\,\frac{e^{i(m+2n)\theta/2}}{r^{2m}}\ ,$$

and observe that

$$\partial_t R_n = 4\,Q_{n-1} - 4\,Q_n\ ,$$

so that by (1.7)

$$P_n = -\frac{c_m}{4}\,\partial_t R_n\ . \tag{2.9}$$

Integrating by parts in the formula for $K_{k,j}$ in Proposition 1.5, we thus obtain

$$K_{k,j}(z,t) = -c_m\, 2^{-mk} \sum_n \chi_j(m+2\,n) \int R_n\Big(z, \frac{t-s}{4}\Big) \Phi'_{k,j,n}(s)\, ds\,.$$

And, one realizes easily that

$$\Phi'_{k,j,n} = 2^{\ell}\, \tilde{\Phi}_{k,j,n}\ ,$$

with $\tilde{\Phi}_{k,j,n}$ similar to $\Phi_{k,j,n}$ (only $\tilde{\chi}$ has to be modified in the definition of $\tilde{\Phi}_{k,j,n}$).

Arguing now similarly as before, we find that

$$\begin{aligned} K_{k,j}(R,t) &= C_m\, 2^{ij}\, 2^{k/2}\, 2^{(m-3)(j-k)} \\ &\quad \cdot \int \frac{\zeta_j\Big(\arctan\Big(\frac{R}{t-s}\Big) + \frac{1}{s}\Big)}{(R^2+(t-s)^2)^{m/2}}\, \chi(2^{k-j}s)\, ds\,, \end{aligned} \tag{2.10}$$

with functions ζ_j and χ similar as in (2.8), but not necessarily identical.

Notice that in passing from (2.8) to (2.10) we "gain" a factor $2^{2(k-j)}\,(R^2+(t-s)^2)^{1/2}$. In addition we should point out that the right-side of (2.8) contains factors of $e^{\pm i/s}$, which do not appear in (2.10); this is due to the extra factor $e^{i\theta/2}$ occuring in the formula (1.7.b) for Q_n, which is not present in the formula for R_n.

We shall now specialize these formulas in the cases $j \le k$ and $j > k$.

A) *The case* $j \le k + M$. Fix $M \in \mathbb{N}$ to be chosen later. If $j \le k + M$, then the variable s in (2.8) is of the order $2^{j-k} \le 2^M$, so that no cancellation in the factor $e^{-i/s} - e^{i/s}$ can be expected. We therefore estimate each of the terms appearing in (2.8) , which are all of similar type, seperately.

In order to exploit formulas (2.8) as well as (2.10), we choose a cut-off function $\varrho \in C_0^\infty(\mathbb{R})$ such that $\varrho(x) = 1$ for $|x| \le 2^{11}$, $\varrho(x) = 0$ for $|x| \ge 2^{12}$, and split $K_{k,j}$ into

$$\begin{aligned} &K_{k,j}(z,t) \\ &\qquad = 2^{-mk} \sum_n \chi_j(m+2\,n) \int (1-\varrho)\,(2^{4(k-j)}\,(R^2+(t-s)^2)) \end{aligned}$$

$$\cdot P_n\Big(z, \frac{t-s}{4}\Big)\, \Phi_{k,j,n}(s)\, ds$$

$$+ 2^{-mk} \sum_n \chi_j(m+2n) \int \varrho(2^{4(k-j)}\, (R^2 + (t-s)^2))$$

$$\cdot P_n\Big(z, \frac{t-s}{4}\Big) \Phi_{k,j,n}(s)\, ds\,.$$

Using (2.9), and performing an integration by parts in the second term, we then find that $K_{k,j}$ will be made up of a finite number of terms of the following types

$$\tilde F_{k,j}(z,t) := 2^{k/2} 2^{(m-1)(j-k)}$$

$$\cdot \int \frac{\zeta_j\Big(\arctan\Big(\dfrac{R}{t-s}\Big) + \dfrac{1}{s}\Big)}{(R^2+(t-s)^2)^{(m+1)/2}} \tag{Ã.1}$$

$$\cdot\, (1-\varrho)\, (2^{4(k-j)}\, (R^2+(t-s)^2))\, e^{i/s}\, \chi(2^{k-j} s)\, ds\,,$$

the complex conjugate of $\tilde F_{k,j}$,

$$\tilde G_{k,j}(z,t) := 2^{k/2}\, 2^{(m-3)(j-k)}$$

$$\cdot \int \frac{\zeta_j\Big(\arctan\Big(\dfrac{R}{t-s}\Big) + \dfrac{1}{s}\Big)}{(R^2+(t-s)^2)^{m/2}} \tag{Ã.2}$$

$$\cdot\, \varrho(2^{4(k-j)} (R^2+(t-s)^2))\, \chi(2^{k-j} s)\, ds\,,$$

and

$$\tilde H_{k,j}(z,t) := 2^{k/2} 2^{(m-5)(j-k)-j}$$

$$\cdot \int \frac{\zeta_j\Big(\arctan\Big(\dfrac{R}{t-s}\Big) + \dfrac{1}{s}\Big)}{(R^2+(t-s)^2)^{m/2}} (t-s) \tag{Ã.3}$$

$$\cdot\, \varrho'(2^{4(k-j)}\, (R^2+(t-s)^2))\, \chi(2^{k-j} s)\, ds\,.$$

Notice also that $\zeta_j(\omega)\, e^{-i\omega} =: \tilde\zeta_j(\omega)$ is a function of the same type as ζ_j, and that

$$\zeta_j\Big(\arctan\Big(\frac{R}{t-s}\Big) + \frac{1}{s}\Big) e^{i/s}$$

$$= \tilde\zeta_j\Big(\arctan\Big(\frac{R}{t-s}\Big) + \frac{1}{s}\Big) \frac{t-s-i\,R}{(R^2+(t-s)^2)^{1/2}}\,.$$

This shows that we may put $\tilde{F}_{k,j}$ also into the form

$$
\begin{aligned}
\tilde{F}_{k,j}(z,t) = {} & 2^{k/2}\, 2^{(m-1)(j-k)} \\
& \cdot \int \tilde{\zeta}_j \Big(\arctan\Big(\frac{R}{t-s}\Big) + \frac{1}{s}\Big) \frac{t-s-i\,R}{(R^2+(t-s)^2)^{(m+2)/2}} \\
& \cdot (1-\varrho)\,(2^{4(k-j)}\,(R^2+(t-s)^2))\,\chi(2^{k-j}s)\,ds\,.
\end{aligned} \tag{Ã.1'}
$$

Now observe that there is some $A \in \mathbb{N}$ such that

$$B_2 \subset \{(z,t) \in \mathbb{H}_m :\ |z|^2 \le 2^A,\ |t| \le 2^A\} =: Q\,.$$

This is clear since $|(z,t)| := (\max\{|z|^2, |t|\})^{1/2}$ is a homogeneous norm on $\mathbb{H}_m$, hence equivalent to the optimal control norm. Thus $\|f\|_{L^1(B_2)} \le \|f\|_{L^1(Q)}$.

Moreover, replacing R by $2^{j-k}R$, t by $2^{j-k}t$ and s by $2^{j-k}s$, we see that $\|\tilde{F}_{k,j}\|_{L^1(B_2)} \le \|F_{k,j}\|_{L^1(2^{k-j}Q,dRdt)}$, with

$$
\begin{aligned}
F_{k,j}(R,t) := {} & 2^{k/2+m(j-k)} \\
& \cdot \int \zeta_j \Big(\arctan\Big(\frac{R}{t-s}\Big) + \frac{2^{k-j}}{s}\Big) \frac{R^{m-1}(t-s-i\,R)}{(R^2+(t-s)^2)^{(m+2)/2}} \\
& \cdot (1-\varrho)\,(2^{2(k-j)}\,(R^2+(t-s)^2))\,\chi(s)\,ds\,.
\end{aligned} \tag{A.1}
$$

Similarly, instead of estimating the $L^1(B_2)$-norms of $\tilde{G}_{k,j}$ and $\tilde{H}_{k,j}$, we may estimate the $L^1(2^{k-j}Q)$-norms of $G_{k,j}$ and $H_{k,j}$, defined by

$$
\begin{aligned}
G_{k,j}(R,t) := {} & 2^{k/2}\, 2^{(m-1)(j-k)} \\
& \cdot \int \zeta_j \Big(\arctan\Big(\frac{R}{t-s}\Big) + \frac{2^{k-j}}{s}\Big) \frac{R^{m-1}}{(R^2+(t-s)^2)^{m/2}} \\
& \cdot \varrho(2^{2(k-j)}(R^2+(t-s)^2))\,\chi(s)\,ds\,,
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
H_{k,j}(R,t) := {} & 2^{k/2}\, 2^{(m-2)(j-k)-j} \\
& \cdot \int \zeta_j \Big(\arctan(\frac{R}{t-s}\Big) + \frac{2^{k-j}}{s}\Big) \frac{(t-s)R^{m-1}}{(R^2+(t-s)^2)^{m/2}} \\
& \cdot \varrho'(2^{2(k-j)}\,(R^2+(t-s)^2))\,\chi(s)\,ds\,.
\end{aligned} \tag{A.3}
$$

Notice that we are interested in these functions in the region

$$0 \le R \le 2^{k-j+A}\,, \qquad |t| \le 2^{k-j+A}\,. \tag{2.11}$$

By $\|f\|$ we shall denote the L^1-norm of f restricted to this domain.

B) *The case* $j > k+M$. In this case, we have $s \leq 2^{j-k+5}$ in the integral for $\tilde{F}_{k,j}$, and so, if $(R,t) \in Q$, then

$$2^{4(k-j)}\,(R^2 + (t-s)^2) \leq C\, 2^{2(k-j)} \leq 1\,,$$

if M is choosen sufficiently large. Thus, in this case $\tilde{F}_{k,j} = 0$, and consequently we shall here entirely make use of formula (2.10).

After scaling, we are thus lead to estimating the L^1-norm of

$$G_{k,j}(R,t) := 2^{k/2}\, 2^{(m-1)(j-k)}$$

$$\text{(B)} \qquad \cdot \int \zeta_j \Big(\arctan\Big(\frac{R}{t-s}\Big) + \frac{2^{k-j}}{s}\Big) \frac{R^{m-1}}{(R^2+(t-s)^2)^{m/2}}\, \chi(s)\, ds$$

on the region given by (2.11).

3. The change of coordinates.

In the estimates to come, the following change of coordinates turns out to be useful

$$\text{(3.1)} \qquad x := \frac{R}{t-s}\,, \qquad y := \frac{R^2+(t-s)^2}{t-s}\,, \qquad s := s\,,$$

with inverse transformation

$$\text{(3.2)} \qquad t = \frac{y}{\langle x\rangle^2} + s\,, \qquad R = \frac{x\,y}{\langle x\rangle^2}\,, \qquad s = s\,,$$

where we have put

$$\langle x\rangle := (1+x^2)^{1/2}\,.$$

Then one verifies easily the following formulas

$$\text{(3.3)} \qquad dR\, dt\, ds = \frac{|y|}{\langle x\rangle^4}\, dy\, ds\, dx\,,$$

$$\text{(3.4)} \qquad R^2 + (t-s)^2 = \frac{y^2}{\langle x\rangle^2}\,, \qquad t-s = \frac{y}{\langle x\rangle^2}\,.$$

Put

$$\psi_{k,j}(x,s) := \Big|\zeta_j\Big(\arctan x + \frac{2^{k-j}}{s}\Big)\Big| .$$

Let $A \in \mathbb{N}$ be as in (2.11), and fix $M \in \mathbb{N}$ such that $M \ge A + 20$.

We shall frequently make use of the following

Lemma 3.1. a) *If* $j \le k + M$, *then*

$$\int \psi_{k,j}(x,s)\,|\chi(s)|\,ds \le C\,,$$

with C *independent of* j, k *and* x.

b) *If* $j \ge k + M$, *then*

$$\int_{|x| \le 2^{10+A+k-j}} \psi_{k,j}(x,s)\,dx \le C\,,$$

with C *independent of* k, j *and* s.

PROOF. Put $u := \arctan x \in [0,\pi]$. Since $\operatorname{supp}\chi \subset [1/8, 32]$, we have

$$\int \Big|\zeta_j\Big(u + \frac{2^{k-j}}{s}\Big)\chi(s)\Big|\,ds \le C_1\,2^{j-k} \int_{2^{k-j-4}}^{2^{k-j+3}} |\zeta_j(u+s)|\,ds\,.$$

And, if $k - j \ge -M$, then it follows easily from (2.4) that

$$\int_{2^{k-j-4}}^{2^{k-j+3}} |\zeta_j(u+s)|\,ds \le C_2\,2^{k-j}\,.$$

This covers parts a) of the lemma.

As for b), assume now that $16 + A + k - j \le 10 + A - M \le -10$, and $|x| \le 2^{10+A+k-j} =: L \ll 1$.

Since $\arctan(-x) = \pi - \arctan x$, and since $|\zeta_j|$ is π-periodic, we have

$$\int_{-L}^{0} \psi_{k,j}(x,s)\,dx = \int_0^L \Big|\zeta_j\Big(-\arctan x + \frac{2^{k-j}}{s}\Big)\Big|\,dx\,.$$

Thus, choosing $u = \tan^{-1} x$ with $|u| \le \tan^{-1}(L) \le L$ as variable of integration in place of x, where $\tan^{-1}$ denotes the branch with values in $[-\pi/2, \pi/2]$, then we get

$$\int_{|x|\le L} \psi_{k,j}(x,s)\,dx \le C' \int_{|u| \le L} \Big|\zeta_j\Big(u + \frac{2^{k-j}}{s}\Big)\Big|\,du \le C\,,$$

again by (2.4).

4. Estimates for $j \le k + M$.

4.1. Estimation of $H_{k,j}$.

Since $\tau \sim 1$ for $\tau \in \operatorname{supp} \varrho'$, we get by (A.3) and Lemma 3.1 that

$$\begin{aligned} \|H_{k,j}\| &\le C\, 2^{k/2+(m-2)(j-k)-j} \\ &\quad \cdot \int_{y\sim 2^{j-k}\langle x\rangle} |\psi_{k,j}(x,s)| \left| \frac{\frac{y}{\langle x\rangle^2}\left(\frac{x\,y}{\langle x\rangle^2}\right)^{m-1}}{\left(\frac{y}{\langle x\rangle}\right)^m} \right| \frac{|y|}{\langle x\rangle^4}\, dy\, ds\, dx \\ &\le C\, 2^{k/2+m(j-k)-j} \int \frac{|x|^{m-1}}{\langle x\rangle^{m+2}}\, dx \\ &\le C\, 2^{k/2-mk+(m-1)j}\,. \end{aligned}$$

Consequently,

$$\sum_{j\le k+M} \|H_{k,j}\|_1 \le C\, 2^{-k/2}\,. \tag{4.1}$$

REMARK. In the above estimate, we did not make use of the condition (2.11). Notice that in the (x, y, s)-coordinates, this condition is equivalent to

$$\frac{|y|}{\langle x\rangle} \le C\, 2^{k-j}\,. \tag{4.2}$$

This condition will be of importance in the estimations of $\|F_{k,j}\|$ and $\|G_{k,j}\|$.

In fact, arguing similarly as for $H_{k,j}$ and using (4.2), one finds that

$$\|F_{k,j}\| \le C\,((k-j)+M)\, 2^{k/2+m(j-k)}\,, \qquad \|G_{k,j}\| \le C\, 2^{k/2+m(j-k)}\,.$$

Thus, if one choose any $\varepsilon > 0$, one finds that

$$\sum_{j<(1-(\varepsilon+1/2)/m)k} (\|F_{k,j}\| + \|G_{k,j}\|) \le C_\varepsilon\,. \tag{4.3}$$

In order to deal with the remaining values of $j \leq k + M$, we shall have to perform another integration by parts.

Notice, however, that by (4.3) we may from now on assume that j is sufficiently large, so that $2^{-j} \ll 1$.

4.2. Estimation of $F_{k,j}$.

For R and t fixed, let us put

$$a(s) := (R^2 + (t-s)^2)^{1/2} = |t - s - i\,R|\,.$$

Notice that (4.2) just means that

$$a(s) \leq C\,2^{k-j}\,, \tag{4.2'}$$

and that for s in the support of the integrand of (A.1) we have $a(s) \geq 2^{j-k}$, so that

$$2^{j-k} \leq a(s) \leq C\,2^{k-j}\,. \tag{4.4}$$

In the discussion to follow, we shall always assume that the estimates we shall establish are valid for s in the support of the integrals under examination without further mentioning.

We shall say that a function h of R, t and s is a *symbol of order* β, if it satisfies estimates of the form

$$|\partial_s^{(j)} h| \leq C_j\, a^{\beta - j}\,, \qquad j \in \mathbb{N},$$

at least for $j = 0, 1, 2$, where C_j is independent of R, t and s. Evidently a is a symbol of order 1. Similarly, $\arctan(R/(t-s))$ is a symbol of order 0. For R and t fixed, let us write

$$\varphi(s) := \arctan\Big(\frac{R}{t-s}\Big) + \frac{2^{k-j}}{s}\,,$$

$$\kappa(s) := \frac{t - s - i\,R}{(R^2 + (t-s)^2)^{(m+2)/2}}\,(1 - \varrho)\,((2^{k-j}\,a(s))^2)\,.$$

One checks easily that κ is a symbol of order $-m-1$. We may then write

$$F_{k,j}(R,t) = 2^{k/2 + m(j-k)}\,R^{m-1} \int \zeta_j \circ \varphi(s)\,\kappa(s)\,\chi(s)\,ds\,. \tag{4.5}$$

In the latter integral, we can perform an integration by parts, if we observe that

$$\zeta_j = -2^{-j}\,\tilde\zeta_j' \,, \tag{4.6}$$

where $\tilde\zeta_j$ is a function of the same type as ζ_j, so that $\varrho_j = |\tilde\zeta_j|$ satisfies in particular (2.4). (4.6) can in fact easily be obtained by going back to the definition of ζ_j. Since

$$\zeta_j(\varphi(s)) = -2^{-j}\,\frac{d}{ds}\,(\tilde\zeta_j\circ\varphi)(s)\,\frac{1}{\varphi'(s)}\,,$$

we may write $F_{k,j}$ also in the form

$$F_{k,j}(R,t) = 2^{k/2+m(j-k)}\,R^{m-1}\int \tilde\zeta_j\circ\varphi(s)\,2^{-j}\Big(\frac{\kappa\,\chi}{\varphi'}\Big)'(s)\,ds\,. \tag{4.7}$$

Since

$$\frac{2^{-j}\Big(\frac{\kappa\,\chi}{\varphi'}\Big)'}{\kappa\,\chi} = \frac{2^{-j}\,\kappa'}{\varphi'\,\kappa} + \frac{2^{-j}\,\chi'}{\varphi'\,\chi} - 2^{-j}\,\frac{\varphi''}{\varphi'^2}\,,$$

we shall gain by the integration by parts if these terms are bounded, say by $1/3$. Now, if $\tilde\kappa$ is formed as κ, only with ρ replaced by a function $\tilde\rho$ of slightly smaller support, then

$$\Big|\frac{\kappa'}{\varphi'\,\tilde\kappa}\Big| \le C\,\frac{a^{-1}}{|\varphi'|}\,, \tag{4.8.a}$$

and similarly, since $\arctan\,(R/(t-s))$ is a symbol of order 0,

$$|\varphi''| \le C'\Big(a^{-2}+\frac{2^{k-j}}{s^3}\Big) \le C\,(2^{(k-j)/2}+a^{-1})^2\,,$$

hence

$$\Big|\frac{\varphi''}{\varphi'^2}\Big| \le C\,\Big(\frac{2^{(k-j)/2}+a^{-1}}{\varphi'}\Big)^2\,. \tag{4.8.b}$$

Finally, if $\tilde\chi$ is similar to χ, only with a slightly larger support, then

$$\Big|\frac{\chi'}{\varphi'\,\tilde\chi}\Big| \le C\,\frac{1}{|\varphi'|}\,. \tag{4.8.c}$$

The natural condition in order to gain by the integration by parts is thus

$$\sigma := 2^{-j}\Big(\frac{2^{(k-j)/2}+a^{-1}}{\varphi'}\Big)^2 \le 1\,. \tag{4.9}$$

Under this condition, we get

$$\Big|2^{-j}\Big(\frac{\kappa\,\chi}{\varphi'}\Big)'\Big| \le C\,|(\sigma+(2^{-j}\,\sigma)^{1/2})\,\tilde\kappa\,\tilde\chi|\,. \tag{4.10}$$

In fact, we have

$$2^{-j}\,\sigma \ge \Big(\frac{2^{-j}\,a^{-1}}{\varphi'}\Big)^2 , \tag{4.11}$$

so that

$$\Big|2^{-j}\,\frac{\kappa'}{\varphi'\,\tilde\kappa}\Big| \le C\,(2^{-j}\,\sigma)^{1/2}\,.$$

Moreover,

$$\varphi'(s) = \frac{R}{R^2+(t-s)^2} - \frac{2^{k-j}}{s^2}\ ,$$

so that by (4.4)

$$|\varphi'(s)| \le C'\,(a^{-1}(s)+2^{k-j}) \le C\,2^{k-j}\,.$$

But then

$$\sigma \ge \frac{2^{k-2j}}{\varphi'^2} \ge C\,\frac{2^{-j}}{|\varphi'|}\ ,$$

so that

$$\Big|2^{-j}\,\frac{\chi'}{\varphi'\,\tilde\chi}\Big| \le C\,\sigma\,.$$

In order to simplify the notation, we shall often replace again $\tilde\kappa$ by κ and $\tilde\chi$ by χ in the estimates to follow.

Let us now express the relevant functions in the coordinates (x, y, s) of Section 3. Some easy computations based on (3.4) yield

$$a(s) = \frac{|y|}{\langle x\rangle}\,,$$

$$\varphi(s) = \arctan x + \frac{2^{k-j}}{s}\,,$$

$$\varphi'(s) = \frac{x}{y} - \frac{2^{k-j}}{s^2}\,,$$

$$R^{m-1}\,\kappa(s) \le C\,\frac{|x|^{m-1}}{\langle x\rangle^{m-3}}\,\frac{1}{y^2}\,(1-\varrho)\Big(\Big(2^{k-j}\,\frac{y}{\langle x\rangle}\Big)^2\Big)\,, \tag{4.12}$$

$$\sigma(s) = 2^{-j}\left(\frac{2^{(k-j)/2} + \dfrac{\langle x\rangle}{|y|}}{\dfrac{x}{y} - \dfrac{2^{k-j}}{s^2}}\right)^2$$

$$= s^4\Big(2^{-j/2}\,\frac{2^{(j-k)/2}\,|y| + 2^{j-k}\,\langle x\rangle}{y - 2^{j-k}s^2\,x}\Big)^2\,.$$

Observe now that, due to the choice of ϱ, $|y| \ge 2\cdot 2^{j-k}s^2\langle x\rangle$ for any s with $(\chi\,\kappa)(s) \ne 0$, so that

$$\sigma(s) \sim 2^{-j}\Big(2^{(j-k)/2} + \frac{2^{k-j}\,\langle x\rangle}{|y|}\Big)^2 \le C\,2^{-j} \ll 1\,, \tag{4.13}$$

hence

$$\sigma(s) + (2^{-j}\,\sigma(s)^{1/2}) \le C\,2^{-j}\,. \tag{4.14}$$

We shall therefore estimate $\|F_{k,j}\|$ by means of formula (4.7), which, in combination with (4.10), (4.12), (4.14) and (4.4) yields

$$\begin{aligned}\|F_{k,j}\| &\le C\,2^{k/2+m(j-k)} \\ &\qquad\cdot\int_{2^{j-k}\le a\le C2^{k-j}} \psi_{k,j}(x,s)\,R^{m-1}\,|(\sigma + (2^{-j}\,\sigma)^{1/2})\,\kappa\,\chi|(s)\,ds \\ &\le C\,2^{k/2+m(j-k)-j} \\ &\qquad\cdot\int_{2^{j-k}\langle x\rangle\le|y|\le C2^{k-j}} \psi_{k,j}(x,s)\,\frac{|x|^{m-1}}{\langle x\rangle^{m+1}}\,\frac{1}{|y|}\,dy\,ds\,dx \\ &\le C\,2^{k/2+m(j-k)-j}\,k\,,\end{aligned}$$

again by Lemma 3.1.a). This implies

$$\sum_{j\le k+M} \|F_{k,j}\| \le C\,k^2\,2^{-k/2}\,. \tag{4.15}$$

4.3. Estimation of $G_{k,j}$.

We shall proceed similarly as in the preceding section. We define $\varphi(s)$ and $a(s)$ as before, only κ has to be replaced here by

$$\kappa(s) := (R^2+(t-s)^2)^{-m/2}\,\varrho(2^{2(k-j)}\,(R^2+(t-s)^2))\,,$$

so that now κ is a symbol of order $-m$. Notice also that for $\kappa(s)\neq 0$ we have

$$a(s) \le C\,2^{j-k} \tag{4.16}$$

in place of (4.4).

Then

$$G_{k,j}(R,t) = 2^{k/2+(m-1)(j-k)}\,R^{m-1}\int \zeta_j\circ\varphi(s)\,\kappa(s)\,\chi(s)\,ds\,.$$

We may perform an integration by parts as in the preceding section, and the gain by this can be estimated by the same function σ defined in (4.9). However, here we may have $\sigma \gg 1$. Therefore we fix a cut-off function ϱ_1 supported in $|x|\le 2$ and with $\varrho_1(x)=1$ for $|x|\le 1$, and write

$$\begin{aligned} G_{k,j}(R,t) = 2^{k/2+(m-1)(j-k)}\,R^{m-1} & \\ \cdot\Big(\int \zeta_j\circ\varphi(s)\,\kappa(s)\,\chi(s)\,\varrho_1(\sigma(s))\,ds & \\ +\int \zeta_j\circ\varphi(s)\,\kappa(s)\,\chi(s)\,(1-\varrho_1)\,(\sigma(s))\,ds\Big)\,. & \end{aligned}$$

Performing the integration by parts in the first integral, we find that

$$|G_{k,j}| \le C\,(G^1_{k,j}+G^2_{k,j}+G^3_{k,j})\,, \tag{4.17}$$

with

$$G^1_{k,j}(R,t) := 2^{k/2+(m-1)(j-k)-j}\, R^{m-1} \int_{1\le\sigma\le 2} \Big|\tilde\zeta_j\circ\varphi(s)\, \frac{\kappa\,\chi}{\varphi'}(s)\,\sigma'(s)\Big|\, ds\,,$$

$$G^2_{k,j}(R,t) := 2^{k/2+(m-1)(j-k)}\, R^{m-1} \int_{\sigma\le 2} |\tilde\zeta_j\circ\varphi(s)\,((\sigma+(2^{-j}\sigma)^{1/2})\,\kappa\,\chi)(s)|\, ds\,,$$

$$G^3_{k,j}(R,t) := 2^{k/2+(m-1)(j-k)}\, R^{m-1} \int_{\sigma\ge 1} |\zeta_j\circ\varphi(s)\,(\kappa\,\chi)(s)|\, ds\,.$$

Notice that in the second term we have already estimated $|2^{-j}\,(\kappa\,\chi/\varphi)'|$ by $C\,|(\sigma+(2^{-j}\,\sigma)^{1/2})\,\tilde\kappa\,\tilde\chi|$. This is justified, since (4.16) remains valid here – the only property of κ made use of here is that $|\kappa'/\tilde\kappa|\le C\,a^{-1}$.

If one expresses the functions arising in these integrals in the (x,y,s)-coordinates, formulas (4.12) remain the same except for the estimate for $R^{m-1}\kappa(s)$, which here is to be replaced by

$$R^{m-1}\,\kappa(s) \le C\,\frac{|x|^{m-1}}{\langle x\rangle^{m-2}}\,\frac{1}{|y|}\,\varrho\Big(\Big(2^{k-j}\,\frac{y}{\langle x\rangle}\Big)^2\Big)\,. \tag{4.12'}$$

Observe also that

$$|\sigma'| \le C\,2^j\,|\varphi'|\,, \qquad \text{if } \sigma(s)\sim 1\,. \tag{4.18}$$

In fact, if $\sigma(s)\sim 1$, then by (4.8.b)

$$\begin{aligned} |\sigma'| &\le C\,2^{-j/2}\,|\sigma|^{1/2}\Big(\frac{a^{-2}}{|\varphi'|} + (2^{(k-j)/2}+a^{-1})\,\Big|\frac{\varphi''}{\varphi'^2}\Big|\Big) \\ &\le C\Big(\Big(2^{-j}\,\frac{a^{-2}}{\varphi'^2}\Big)2^{j/2} + 2^{-j/2}\Big(\frac{2^{(k-j)/2}+a^{-1}}{|\varphi'|}\Big)^3\Big)|\varphi'| \\ &\le C\,(2^{j/2}\,\sigma + 2^j\,\sigma^{3/2})\,|\varphi'|\,. \end{aligned}$$

And, since $|y|\le C\,2^{j-k}\,\langle x\rangle$, by (4.16), we see that

$$\sigma(s) \sim \Big(\frac{2^{-j/2+j-k}\,\langle x\rangle}{y-2^{j-k}\,s^2\,x}\Big)^2\,. \tag{4.19}$$

This implies

$$|y - 2^{j-k}\, s^2\, x| \leq C\, 2^{-j/2+j-k}\, \langle x\rangle\,, \qquad \text{if } \sigma(s) \geq 1\,. \tag{4.20}$$

Now, observe that, due to (4.18), we have $|G^1_{k,j}| \leq C\,|G^3_{k,j}|$. Therefore

$$\|G^1_{k,j}\| + \|G^3_{k,j}\| \leq C\, 2^{k/2+(m-1)(j-k)} \cdot \int_{\sigma\geq 1, s\sim 1} \psi_{k,j}(x,s)\, \frac{|x|^{m-1}}{\langle x\rangle^{m-1}}\, \frac{1}{|y|}\, \frac{|y|}{\langle x\rangle^4}\, dy\, ds\, dx\,, \tag{4.21}$$

which, by (4.20) and Lemma 3.1.a), can be estimated by

$$C\, 2^{k/2+(m-1)(j-k)-j/2+j-k} \int \frac{|x|^{m-1}}{\langle x\rangle^{m+1}}\, dx\,.$$

This yields

$$\|G^1_{k,j}\| + \|G^3_{k,j}\| \leq C\, 2^{(m-1/2)(j-k)}\,. \tag{4.22}$$

Finally, putting $B := 2^{-j/2+j-k}\langle x\rangle$, by (4.19) we have

$$\sigma(s) \sim \Big(\frac{B}{y - 2^{j-k}\, s^2\, x}\Big)^2\,.$$

Thus, if $\sigma(s) \leq C^2$, then this and (4.16) imply

$$\frac{B}{C} \leq |y - 2^{j-k}\, s^2\, x| \leq C'\, 2^{j-k}\, \langle x\rangle\,,$$

and then

$$\begin{aligned}
\int_{\sigma\leq 2, a\leq C2^{j-k}} |\sigma|\, dy
&\leq \sqrt{2} \int_{\sigma\leq 2, a\leq C2^{j-k}} |\sigma|^{1/2}\, dy \\
&\leq \sqrt{2} \int_{B/C\leq |y-2^{j-k}s^2x|\leq C' 2^{j-k}\langle x\rangle} \frac{B}{|y - 2^{j-k}\, s^2\, x|}\, dy \\
&\leq CB \log\Big(\frac{c\, 2^{j-k}\, \langle x\rangle}{B}\Big) \\
&\leq C\, 2^{-j/2+j-k}\, j\, \langle x\rangle\,.
\end{aligned}$$

Similarly,

$$\int_{\sigma\le 2, a\le C2^{j-k}} |2^{-j}\,\sigma|^{1/2}\,dy \le C\,2^{-j/2}\,2^{-j/2+j-k}\,j\,\langle x\rangle\,.$$

This implies

$$\begin{aligned}\|G^2_{k,j}\| &\le C\,2^{k/2+(m-1)(j-k)}\\ &\quad\cdot\int_{s\sim 1}\int_{\sigma\le 2, a\le 2^{j-k}} \psi_{k,j}(x,s)\,(|\sigma(x,y,s)| + |2^{-j}\,\sigma(x,y,s)|^{1/2})\\ &\qquad\cdot dy\,\frac{|x|^{m-1}}{\langle x\rangle^{m+2}}\,dx\,ds\\ &\le C\,k\,2^{(m-1/2)(j-k)}\int\frac{|x|^{m-1}}{\langle x\rangle^{m+1}}\,dx\\ &\le C\,k\,2^{(m-1/2)(j-k)}\,.\end{aligned}$$

In combination with (4.22) and (4.17) we thus find

$$\sum_{j\le k+M}\|G_{k,j}\| \le C\,k\,.$$

Put together, the estimates of this section yield

$$\sum_{j\le k+m}\|K_{k,j}\|_{L^1(B_2)} \le C\,k\,. \tag{4.23}$$

5. Estimates for $j > k + M$.

In order to estimate the norm of $G_{k,j}$, now given by (B), we follow the same scheme as in the preceding Section 4.3 and split G_{kj} as in (4.17) by performing an integration by parts on the region where $\sigma \le C$. Notice, however, the following differences compared to the case $j \le k + M$:

Firstly, since $1/8 \le s \le 32$ in (B), by (2.11) we have

$$2^{-5} \le a(s) \le 2^6 \tag{5.1}$$

in place of (4.16). Moreover, since

$$\varphi'(s) = \frac{R}{R^2+(t-s)^2} - \frac{2^{k-j}}{s^2}\,, \qquad \varphi''(s) = \frac{R\,(t-s)}{(R^2+(t-s)^2)^2} + \frac{2^{k-j+1}}{s^3}\,,$$

by (2.11) and (5.1) we now have

$$\Big|\frac{\varphi''}{\varphi'^2}\Big| \le C\,\frac{2^{k-j}}{\varphi'^2}\,, \qquad |\varphi'| \le C\,2^{k-j}\,,$$

so that

$$2^{-j}\Big(\Big|\frac{\kappa'}{\varphi'\,\tilde\kappa}\Big| + \Big|\frac{\chi'}{\varphi'\,\tilde\chi}\Big| + \Big|\frac{\varphi''}{\varphi'^2}\Big|\Big) \le C\,\frac{2^{k-2j}}{\varphi'^2}\,. \tag{5.2}$$

We shall therefore put

$$\sigma := \frac{2^{k-2j}}{\varphi'^2}$$

here. Then (4.10) remains valid.

With this function σ, and with $\kappa = (R^2+(t-s)^2)^{-m/2} \sim 1$ here, we may define $G^\ell_{k,j}$, $\ell = 1,2,3$, as before, where, because of (5.2), we may even assume that $G^2_{k,j}$ is given by

$$2^{k/2+(m-1)(j-k)}\,R^{m-1}\int_{\sigma\le 2} |\zeta_j\circ\varphi(s)\,(\sigma\,\kappa\,\chi)(s)|\,ds\,.$$

Then (4.17) remains true.

Since

$$|\sigma'| = \Big|\frac{2^{k-2j+1}}{\varphi'^3}\,\varphi''\Big| \le \Big|\frac{2^{2k-3j}}{\varphi'^3}\Big| \le C\,2^{k/2}\,\sigma^{3/2} \le C\,2^j\,|\varphi'|\,,$$

if $\sigma \sim 1$, also (4.18) remains true, so that again $|G^1_{k,j}| \le C\,|G^3_{k,j}|$. We thus only have to estimate $\|G^3_{k,j}\|$ and $\|G^2_{k,j}\|$.

Now, by (5.1),

$$2^{-5}\langle x\rangle \le |y| \le 2^6\,\langle x\rangle\,.$$

Given this, (2.11) implies $|x/\langle x\rangle| \le 2^{5+A+k-j}$, hence

$$|x| \le 2^{10+A+k-j} \ll 1\,, \tag{5.3}$$

as well as

$$\Big|\frac{y}{\langle x\rangle^2}+s\Big|\le 2^{A+k-j}\,,$$

hence

$$|y+s\,\langle x\rangle^2|\le C\,2^{k-j}\,. \tag{5.4}$$

In particular, we find that

$$|y|\sim\langle x\rangle\sim 1\,. \tag{5.5}$$

In view of the definition of σ, this implies that, in place of (4.19),

$$\sigma\sim\frac{2^{-k}}{(y-2^{j-k}\,s^2\,x)^2}\,. \tag{5.6}$$

Now, if $\sigma\ge 1$, then

$$|y-2^{j-k}\,s^2\,x|\le C\,2^{-k/2}\,. \tag{5.7}$$

Let $\mathcal{D}$ denote the domain given by (5.3), (5.4), (5.7) and $s\sim 1$. Then, similarly as in (4.21), we get

$$\begin{aligned}\|G^3_{k,j}\|&\le C\,2^{k/2+(m-1)(j-k)}\int_{\mathcal{D}}\psi_{k,j}(x,s)\,|x|^{m-1}\,dy\,dx\,ds\\&\le C\,2^{k/2}\int_{\mathcal{D}}\psi_{k,j}(x,s)\,dy\,dx\,ds\,.\end{aligned}$$

And, by (5.4), (5.7), we have

$$\int_{(x,y,s)\in\mathcal{D}}dy\le C\min\{2^{k-j},2^{-k/2}\}\,.$$

Moreover, by Lemma 3.1.b),

$$\int_{|x|\le 2^{10+A+k-j},s\sim 1}\psi_{k,j}(x,s)\,dx\,ds\le C\,,$$

hence

$$\|G^1_{k,j}\|+\|G^3_{k,j}\|\le C\min\{2^{3k/2-j},1\}\,. \tag{5.8}$$

There remains to estimate $\|G^2_{k,j}\|$, which can be done similarly as in the preceding section:

If $\sigma \le 2$, then we have

$$\frac{1}{c}\, 2^{-k/2} \le |y - 2^{j-k}\, s^2\, x| \le c\,, \tag{5.9}$$

for some $c \ge 1$, hence

$$\int_{\sigma \le 2, |y| \sim 1} |\sigma|\, dy \le \sqrt{2} \int_{2^{-k/2}/c \le |y - 2^{j-k} s^2 x| \le c} \frac{2^{-k/2}}{|y - 2^{j-k}\, s^2\, x|}\, dy$$
$$\le C\, k\, 2^{-k/2}\,.$$

Moreover, by (5.4),

$$\int_{\sigma \le 2, |y + s\langle x\rangle^2| \le C 2^{k-j}} |\sigma|\, dy \le \int_{|y + s\langle x\rangle^2| \le C 2^{k-j}} 2\, dy \le C\, 2^{k-j}\,.$$

Thus, if $\mathcal{E}$ denotes the domain given by (5.3), (5.4), (5.5) and (5.9), then

$$\|G^2_{k,j}\| \le C\, 2^{k/2 + (m-1)(j-k)} \int_{\mathcal{E}} \psi_{k,j}(x,s)\, |\sigma|\, |x|^{m-1}\, dy\, dx\, ds$$
$$\le C\, k 2^{k/2} \min\{2^{-k/2}, 2^{k-j}\} \int_{|x| \le 2^{10+A+k-j}, s \sim 1} \psi_{k,j}(x,s)\, dx\, ds$$
$$\le C\, k \min\{1, 2^{3k/2 - j}\}\,.$$

In combination with (5.8) and (4.17) we thus obtain

$$\sum_{j > k+M} \|K_{k,j}\| \le C\, k^2\,. \tag{5.10}$$

Together with (4.23), this proves Proposition 1.5, which completes the proof of the theorem.

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