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The Euclidean ϕ_2^4 theory as a limit of an interacting Bose gas

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Abstract. We prove that the complex Euclidean field theory with local quartic self-interaction in two dimensions arises as a limit of an interacting Bose gas at positive temperature, when the density of the gas becomes large and the range of the interaction becomes small. The field theory is supported on distributions of negative regularity, which requires a renormalization by divergent mass and energy counterterms. We obtain convergence of the relative partition function and uniform convergence of the renormalized reduced density matrices. The proof is based on three main ingredients: (a) a quantitative analysis of the infinite-dimensional saddle point argument for the functional integral introduced by Fröhlich et al. (2022) using continuity properties of Brownian paths, (b) a Nelson-type estimate for a general nonlocal field theory in two dimensions, and (c) repeated Gaussian integration by parts in field space to obtain uniform control on the renormalized correlation functions. As a byproduct of our proof, in two and three dimensions we also extend the results on the mean-field limit by Fröhlich et al. (2022) and Lewin et al. (2021) to unbounded interaction potentials satisfying the optimal integrability conditions proposed by Bourgain (1997).

Keywords: interacting Bose gas, Euclidean field theory.

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1. Introduction

1.1. Overview of Euclidean field theory

A Euclidean field theory of a scalar field on a domain $\Lambda \subset \mathbb{R}^d$ is specified by a formal probability measure on a space of fields[1] $\phi : \Lambda \to \mathbb{R}^K$ given by

$$\mu(\mathrm{d}\phi) = \frac{1}{c}\mathrm{e}^{-S(\phi)}\mathrm{D}\phi, \tag{1.1}$$

where $\mathrm{D}\phi = \prod_{x\in\Lambda} \mathrm{d}\phi(x)$ is the formal uniform measure on the space of fields, and S is the action. The latter is typically the integral over Λ of a local function of the field ϕ and its gradient. One of the simplest field theories with nontrivial interaction is the K-component *Euclidean* ϕ^4_d *theory*, whose action is given by

$$S(\phi) := -\int_\Lambda \mathrm{d}x\, \phi(x)\cdot(\theta + \Delta/2)\phi(x) + \frac{\lambda}{2}\int_\Lambda \mathrm{d}x\, |\phi(x)|^4, \tag{1.2}$$

where θ is a constant, λ is a coupling constant, Δ is the Laplacian on Λ with appropriate boundary conditions, and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^K$.

Euclidean field theories originally arose in high-energy physics in $d = 4$ space-time dimensions, through an analytic continuation of the time variable of the quantum field ϕ, which replaces the Minkowski space-time metric with a Euclidean one [61, 71]. Subsequently, Euclidean field theories have proven of great importance in statistical mechanics in $d \leqslant 3$ dimensions, in particular through their connection with the theory of phase transitions and critical phenomena. The works [74, 75] recognized the analogy between Euclidean field theories and classical statistical mechanics, which was followed by a purely probabilistic formulation of Euclidean field theories in [63,64]. The rigorous study of field theories of the form (1.1) has been a major topic in mathematical physics since the late sixties; see e.g. [43, 72] as well as the more recent [47] for reviews.

Euclidean field theories also play a central role in the theory of stochastic nonlinear partial differential equations. Formally, (1.1) is the stationary measure of the stochastic nonlinear heat equation

$$\partial_t\phi = -\tfrac{1}{2}\nabla S(\phi) + \xi = (\theta + \Delta/2)\phi - \lambda|\phi|^2\phi + \xi$$

with space-time white noise ξ, which can be regarded as the Langevin equation for a time-dependent field ϕ with potential given by the action S in (1.2). Constructing measures of the form (1.1) by exhibiting them as stationary measures of stochastic nonlinear partial differential equations is the goal of stochastic quantization developed in [26, 52, 62, 66]. See for instance [23, 45, 46, 51] for recent developments.

[1] Rigorously, the space of fields is the Schwartz distribution space $\mathcal{S}'(\Lambda, \mathbb{R}^K)$.

In addition, Euclidean field theories are of great importance in the probabilistic Cauchy theory of nonlinear dispersive equations. For $K = 2$ and identifying $\mathbb{R}^2 \equiv \mathbb{C}$, the measure (1.1) is formally invariant under the nonlinear Schrödinger (NLS) equation

$$\mathrm{i}\partial_t \phi = \tfrac{1}{2}\nabla S(\phi) = -(\theta + \Delta/2)\phi + \lambda|\phi|^2\phi. \tag{1.3}$$

Gibbs measures (1.1) for the NLS (1.3) have proven a powerful tool for constructing almost sure global solutions with random initial data of low regularity. One considers the flow of the NLS (1.3) with random initial data distributed according to (1.1). The invariance of the measure (1.1) under the NLS flow (in low dimensions) serves as a substitute for energy conservation, which is not available owing to the low regularity of the solutions. See for instance the seminal works [10–14,52] as well as [15,18–21,25,39,40,58–60,76] and references given there for later developments.

The main difficulty in all of the works cited above is that, in dimensions larger than 1, under the measure (1.1) the field ϕ is almost surely a distribution of negative regularity, and hence the interaction term

$$V(\phi) := \frac{\lambda}{2}\int_\Lambda \mathrm{d}x\, |\phi(x)|^4$$

in (1.2) is ill-defined. This is an *ultraviolet* problem: a divergence for large wave vectors (i.e. spatial frequencies) producing small-scale singularities in the field. As the dimension $d = 1, 2, 3$ increases, the difficulty of making sense of the measure in (1.1) increases significantly.

To outline the rigorous construction of the measure in (1.1), we introduce an ($\mathbb{R}^K$-valued) Gaussian free field on Λ whose law $\mathbb{P}$ is the Gaussian measure on the space of fields with mean zero and covariance $(2\kappa - \Delta)^{-1}$, where $\kappa > 0$ is some positive constant. Then we write

$$\mu(\mathrm{d}\phi) = \frac{1}{\zeta}\mathrm{e}^{-V(\phi)}\,\mathbb{P}(\mathrm{d}\phi) \tag{1.4}$$

for some normalization constant $\zeta > 0$. For $d = 1$, the right-hand side of (1.4) makes sense as is, since, under $\mathbb{P}$, the field ϕ is almost surely a continuous function and hence $V(\phi)$ is almost surely nonnegative and finite. This provides a simple construction of (1.1) for $d = 1$ and $\kappa = -\theta$. See e.g. [72] for a careful treatment.

For $d > 1$, the simple approach just sketched no longer works, since ϕ is almost surely of negative regularity, and the interaction term $V(\phi)$ has to be renormalized by subtracting suitably chosen infinite counterterms. The most elementary renormalization is Wick ordering of $V(\phi)$ with respect to the Gaussian measure $\mathbb{P}$, denoted by $:\cdot:$ (see Appendix A). After Wick ordering, the interaction term becomes

$$\begin{aligned} V(\phi) &= \frac{\lambda}{2}\int_\Lambda \mathrm{d}x :|\phi(x)|^4: \\ &= \frac{\lambda}{2}\int_\Lambda \mathrm{d}x \left(|\phi(x)|^4 - \frac{4+2K}{K}\mathbb{E}[|\phi(x)|^2]|\phi(x)|^2 + \frac{K+2}{K}\mathbb{E}[|\phi(x)|^2]^2\right), \end{aligned} \tag{1.5}$$

where $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$. The second and third terms on the right-hand side of (1.5) are infinite counterterms, which may be regarded as mass and energy renormalizations, respectively. Hence, for $d > 1$, the constant θ in (1.2) is formally $-\infty$. To make rigorous sense of (1.5) in dimension $d = 2$, one has to mollify ϕ by convolving it with an approximate delta function, and then show that as the mollifier is removed, the right-hand side of (1.5) converges in $L^2(\mathbb{P})$ (see Section 2.1 below for more details). It is not hard to show that for $d = 2$ the renormalization on the right-hand side of (1.5) yields a well-defined interaction term $V(\phi) \in L^2(\mathbb{P})$. However, owing to the mass renormalization in (1.5), after Wick ordering, $V(\phi)$ is unbounded from below, and the integrability of $\mathrm{e}^{-V(\phi)}$ with respect to $\mathbb{P}$ represents a nontrivial problem, which was successfully solved in the landmark work of Nelson [63, 64].

For $d = 3$, it is easy to see that, even after Wick ordering, $V(\phi)$ almost surely does not exist in $L^2(\mathbb{P})$. Further, a simple expansion of the exponential $\mathrm{e}^{-V(\phi)}$ in the two-point correlation function, $\frac{1}{\zeta}\int \phi_i(x)\,\phi_j(y)\mathrm{e}^{-V(\phi)}\,\mathbb{P}(\mathrm{d}\phi)$, yields a divergent term already at second order, associated with the so-called sunset diagram of quantum field theory. Hence, a further mass renormalization of $V(\phi)$ is required, which results in a measure μ that is mutually singular with respect to the free-field Gaussian measure $\mathbb{P}$. The mathematically rigorous construction of the Euclidean ϕ_3^4 theory, first achieved in the seminal work of Glimm and Jaffe [42], is one of the major successes of the constructive field theory programme started in the sixties. By now, several different constructions of this theory have been developed, based on, first, phase cell expansions [27, 42, 43, 67], then on renormalization group methods [8, 16, 37], later on correlation inequalities [17], and, most recently, on paracontrolled calculus [22, 44], as well as variational methods [5].

For $d \geqslant 4$, it is expected, and indeed proven in some cases, that the ϕ_d^4 theory is trivial: any renormalization of $V(\phi)$ resulting in a well-defined measure μ yields a (generalized free-field) Gaussian measure. For $d \geqslant 5$, this triviality was proven in [2, 29]. Recently, the triviality of ϕ_4^4 for $K = 1$ was established in [3].

1.2. The ϕ_2^4 theory as a limit of a Bose gas

In this paper, we establish for the first time a relationship beween a local Euclidean field theory in dimension larger than 1 and an interacting quantum gas. We show that the complex Euclidean ϕ_2^4 theory describes the limiting behaviour of an interacting Bose gas at positive temperature. The limiting regime is a high-density limit in a box[2] of fixed size, where the range of the interaction is much smaller than the diameter of the box. This result provides a rigorous derivation of the ϕ_2^4 theory starting from a realistic model of statistical mechanics. Viewed differently, we introduce a new regularization of the ϕ_2^4 theory in terms of an interacting Bose gas, in addition to the commonly used smooth mollifiers or lattice approximations.

[2]For conciseness, in this paper we assume that Λ is the unit torus, although the actual shape of Λ and the boundary conditions are not essential for our proof; see Remark 2.3 below.

To explain our result more precisely, we recall that a quantum system of n spinless nonrelativistic bosons of mass m in Λ is described by the Hamiltonian

$$\mathbb{H}_n := -\sum_{i=1}^{n} \frac{\Delta_i}{2m} + \frac{g}{2} \sum_{i,j=1}^{n} v(x_i - x_j)$$

acting on the space $\mathcal{H}_n$ of square-integrable wave functions that are symmetric in their arguments $x_1, \dots, x_n$ and supported in Λ^n. Here Δ_i is the Laplacian in the variable x_i, g is a coupling constant, and v is a repulsive (i.e. with nonnegative Fourier transform) two-body interaction potential. We consider a system in the grand canonical ensemble at positive temperature, characterized by the density matrix

$$\frac{1}{Z} \bigoplus_{n \in \mathbb{N}} \mathrm{e}^{-\beta(\mathbb{H}_n - \theta n)} \tag{1.6}$$

acting on Fock space $\mathcal{F} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n$, where $\beta < \infty$ is the inverse temperature, θ is the chemical potential, and Z is a normalization factor. We refer to e.g. [7,57] for reviews on interacting Bose gases.

The limiting regime of this paper is obtained by introducing two parameters, $\nu, \varepsilon > 0$, where $\nu = \beta/m = \sqrt{\beta g}$, and the potential v is taken to be an approximate delta function of range ε. We suppose that $\nu, \varepsilon \to 0$ under the technical constraint $\varepsilon \geqslant \exp(-(\log \nu^{-1})^{1/2-c})$ for some constant $c > 0$. We show that there exists a suitable renormalization of the chemical potential $\theta \equiv \theta_\nu^\varepsilon$ such that the reduced density matrices of the quantum state (1.6) converge to the correlation functions of the field theory (1.1), (1.2).

Previously, this result was obtained for $d = 1$ in [31,53,55], where, as explained in Section 1.1, no renormalization is required. In higher dimensions $d = 2, 3$, the *mean-field limit* was investigated in [31–33,54,56,73], where the parameter ε was fixed as $\nu \to 0$. The resulting limiting field theory differs from ϕ_d^4 in that the interaction term $V(\phi)$ is nonlocal, given by a convolution with a bounded two-body interaction potential v. This nonlocal interaction term is considerably less singular than the local one of ϕ_2^4 theory. The stronger singularity of $V(\phi)$ requires additional renormalization as compared to the nonlocal potential. This makes the local problem significantly more difficult than the nonlocal one. In particular, the renormalized interaction term $V(\phi)$ is unbounded from below, whereas in the nonlocal regime it is almost surely nonnegative.

The above lower bound on the range of the interaction ε is technical in nature (see Remark 2.5 below for a more detailed discussion on its origin). We expect that it can be improved, but at the cost of a considerably more complicated argument. In this paper we wish to emphasize that, using relatively simple methods, one can establish a connection between local Euclidean field theories and interacting quantum gases. We leave quantitative improvements of such results to future work.

Using our methods, we also extend the results on the mean-field limit for a nonlocal interaction term $V(\phi)$ in [33,56] from bounded two-body interaction potentials, v, to unbounded ones. Our integrability assumptions on the function v are optimal, as given in [13]. We refer to Section 5.3 below for details.

1.3. Outlook

The close relationship between Euclidean field theory and interacting Bose gases established in this paper leads to a web of conjectures concerning properties of ϕ^4_d theories inspired by results on Bose gases and, conversely and perhaps more interestingly, properties of interacting Bose gases inspired by known results on ϕ^4_d theories. In the following, we outline some of these conjectures.

We remark that an analysis very similar to the one in this paper yields an analogous relationship between the ϕ^4_2 theory with N complex components (that is, with $K = 2N$ real components) and an interacting Bose gas with N species of identical Bosons; see Remark 2.4 below.

(1) It is known (see [4, 9, 38]) that systems of nonrelativisitic quantum particles moving in d-dimensional Euclidean space and interacting through delta function potentials are equivalent to systems of *free* (i.e. noninteracting) particles, provided that $d \geqslant 4$. Given the connection between interacting Bose gases and ϕ^4_d theories exhibited in this paper, this suggests that the latter theories are equivalent to free (i.e. Gaussian) field theories in dimensions $d \geqslant 4$, for a field ϕ with an arbitrary number of complex components.

(2) In $d = 3$ dimensions, ϕ^4_d theories with N complex components are known to undergo a phase transition accompanied by spontaneous $O(2N)$-symmetry breaking and the emergence of Goldstone bosons [34] (see also [30], as well as [36] for recent results on related lattice models with disorder). Given our results for $d = 2$, as well as analogous results for $d = 3$ to appear in a future paper, the existence of a phase transition in the Euclidean field theory strongly suggests that translation-invariant Bose gases with repulsive two-body interactions in three dimensions exhibit Bose–Einstein condensation accompanied by the appearance of massless quasi-particles with approximately relativistic dispersion at small wave vectors.

In two dimensions, the Mermin–Wagner theorem implies that such phase transitions do not exist, and the $O(2N)$-symmetry remains unbroken for arbitrary values of the coupling constant λ. A similar result is expected to hold for two-dimensional interacting Bose gases (and easy to see for ideal Bose gases).

(3) The *one-component* complex ϕ^4_d theory in dimension $d = 2$ is expected to exhibit a *Berezinskii–Kosterlitz–Thouless transition.* This is rigorously known for the classical XY-model on a square lattice, which is the limiting theory of lattice ϕ^4_2 theory, as λ tends to ∞, with $\kappa = 2\lambda$; see [30, 35]. In view of the results proven in this paper, this suggests that two-dimensional Bose gases of *one species* of particles might exhibit a transition to a low-temperature phase where reduced density matrices exhibit *slow decay*, analogous to the Berezinskii–Kosterlitz–Thouless transition.

In contrast, for a two-dimensional ϕ^4_d theory with two or more complex components, with an $O(2N)$-symmetry, it is expected that connected correlations exhibit exponential decay for arbitrary values of the coupling constant λ; see [68]. This sug-

gests that two-dimensional Bose gases of several species of identical particles exhibit rapidly decaying correlations at all temperatures and densities.

(4) For ϕ_d^4 theories with N complex components, there exists a systematic $1/N$-expansion; see [49, 50] and [78, Chapter 30]. The model obtained in the limit, as $N \to \infty$, is the spherical model, which is exactly solved. It is tempting to extend the method of the $1/N$-expansion to Bose gases of N species of identical particles interacting through two-body interactions of strength $O(1/N)$. The model obtained in the limit, as $N \to \infty$, appears to be equivalent to an ideal Bose gas, but with a renormalized chemical potential. In attempting to prove Bose–Einstein condensation for translation-invariant interacting Bose gases, therefore, it seems judicious to begin by studying Bose gases with a large number of species of identical particles. The connection between Bose–Einstein condensation and phase transitions in classical field theory has been discussed in e.g. [6, 48].

2. Setup and results

2.1. *Classical field theory*

In this subsection we define the Euclidean field theory and its correlation functions. We note that the measure μ from (1.1) can be formally viewed as the thermal equilibrium measure of a *classical* field theory with Hamilton function given by $S(\phi)$ from (1.2). We work on the d-dimensional torus $\Lambda := [-1/2, 1/2)^d$. We use the Euclidean norm $|\cdot|$ for elements of Λ regarded as a subset of $\mathbb{R}^d$. We use the shorthand $\int \mathrm{d}x\,(\cdot) := \int_\Lambda \mathrm{d}x\,(\cdot)$ to denote integration over Λ with respect to Lebesgue measure. We abbreviate $\mathcal{H} := L^2(\Lambda; \mathbb{C})$ and denote by $\langle \cdot, \cdot \rangle$ the inner product of the space $\mathcal{H}$, which is by definition linear in the second argument. On $\mathcal{H}$ we use the standard Laplacian Δ with periodic boundary conditions.

The classical free field ϕ is by definition the complex-valued Gaussian field with covariance $(\kappa - \Delta/2)^{-1}$, where $\kappa > 0$ is a constant. Explicitly, the free field may be constructed as follows. We use the spectral decomposition $\kappa - \Delta/2 = \sum_{k \in \mathbb{Z}^d} \lambda_k u_k u_k^*$, with eigenvalues $\lambda_k > 0$ and normalized eigenfunctions $u_k \in \mathcal{H}$ (see also (4.5) below). Let $X = (X_k)_{k \in \mathbb{Z}^d}$ be a family of independent standard complex Gaussian random variables,[3] whose law and associated expectation are denoted by $\mathbb{P}$ and $\mathbb{E}$, respectively. The *classical free field* is then given by

$$\phi = \sum_{k \in \mathbb{Z}^d} \frac{X_k}{\sqrt{\lambda_k}} u_k,$$

[3] We recall that Z is a standard complex Gaussian if it is Gaussian and satisfies $\mathbb{E}Z = 0$, $\mathbb{E}Z^2 = 0$, and $\mathbb{E}|Z^2| = 1$, or, equivalently, if it has law $\pi^{-1}\mathrm{e}^{-|z|^2}\mathrm{d}z$ on $\mathbb{C}$, where $\mathrm{d}z$ denotes Lebesgue measure.

which is easily seen to converge[4] in $L^2(\mathbb{P})$ of the L^2-Sobolev space $H^{1-d/2-c}$ for any $c > 0$.

In order to define the interacting theory, it is necessary to regularize the field ϕ by convolving it with a smooth mollifier. To that end, choose a nonnegative function $\vartheta : \mathbb{R}^d \to \mathbb{R}_+$ of rapid decay satisfying $\vartheta(0) = 1$, and for $0 < N < \infty$ define the regularized field

$$\phi_N := \sum_{k \in \mathbb{Z}^d} \frac{X_k}{\sqrt{\lambda_k}} \sqrt{\vartheta(k/N)}\, u_k, \tag{2.1}$$

which is almost surely a smooth function on Λ. We define the regularized interaction

$$V_N := \frac{1}{2} \int \mathrm{d}x : |\phi_N(x)|^4 :,$$

where $:\cdot:$ denotes Wick ordering with respect to the Gaussian measure $\mathbb{P}$ (see Appendix A.1). Explicitly,

$$: |\phi_N(x)|^4 : = |\phi_N(x)|^4 - 4\mathbb{E}[|\phi_N(x)|^2]\, |\phi_N(x)|^2 + 2\mathbb{E}[|\phi_N(x)|^2]^2.$$

Here, the deterministic factor $\mathbb{E}[|\phi_N(x)|^2] = \sum_{k \in \mathbb{Z}^d} \vartheta(k/N)/\lambda_k$ diverges as $N \to \infty$ for $d > 1$.

For $d = 2$, using Wick's theorem, it is easy to see that V_N converges as $N \to \infty$ in $L^2(\mathbb{P})$ to a random variable, denoted by V, which does not depend on the choice of ϑ. See e.g. [31, Lemma 1.5] for details. The interacting field theory is given as the probability measure

$$\frac{1}{\zeta} \mathrm{e}^{-V}\, \mathrm{d}\mathbb{P}, \quad \zeta := \mathbb{E}[\mathrm{e}^{-V}]. \tag{2.2}$$

By the well-known Nelson bounds [63, 64] mentioned in Section 1.1, e^{-V} is integrable with respect to $\mathbb{P}$.

We characterize the interacting field theory through its correlation functions, defined as follows. For $p \in \mathbb{N}$ and $\mathbf{x}, \widetilde{\mathbf{x}} \in \Lambda^p$, we define the *p-point correlation function* as

$$(\gamma_p)_{\mathbf{x},\widetilde{\mathbf{x}}} := \frac{1}{\zeta} \mathbb{E}[\bar{\phi}(\widetilde{x}_1) \cdots \bar{\phi}(\widetilde{x}_p)\, \phi(x_1) \cdots \phi(x_p) \mathrm{e}^{-V}], \tag{2.3}$$

which is the $2p$th moment of the field ϕ under the probability measure (2.2). This measure is sub-Gaussian, and is hence determined by its moments $(\gamma_p)_{p \in \mathbb{N}^*}$. (Note that any moment containing a different number of $\bar{\phi}$s and ϕs vanishes by invariance of the measure (2.2) under the gauge transformation $\phi \mapsto \alpha\phi$, where $|\alpha| = 1$.)

As explained in [33, Section 1.5], the correlation function γ_p is divergent on the diagonal, even for the free field. Hence, for instance, it cannot be used to analyse the distribution

[4] In fact, an application of Wick's rule shows that the convergence holds in L^m for any $m < \infty$.

of the mass density $|\phi(x)|^2$. As in [33, Section 1.5], we remedy this by introducing the *Wick-ordered p-point correlation function*

$$(\hat{\gamma}_p)_{\mathbf{x},\tilde{\mathbf{x}}} := \frac{1}{\zeta}\mathbb{E}[:\bar{\phi}(\tilde{x}_1)\cdots\bar{\phi}(\tilde{x}_p)\,\phi(x_1)\cdots\phi(x_p):\mathrm{e}^{-V}], \tag{2.4}$$

which has a regular behaviour on the diagonal. The Wick-ordered correlation function (2.4) can be expressed explicitly in terms of the correlation functions (2.3) and the correlation functions of the free field; see (2.20) below.

2.2. *Quantum many-body system*

In this subsection we define the quantum many-body system and its reduced density matrices. For $n \in \mathbb{N}$, we denote by P_n the orthogonal projection onto the symmetric subspace of $\mathcal{H}^{\otimes n}$; explicitly, for $\Psi_n \in \mathcal{H}^{\otimes n}$,

$$P_n\Psi_n(x_1,\dots,x_n) := \frac{1}{n!}\sum_{\pi\in S_n}\Psi_n(x_{\pi(1)},\dots,x_{\pi(n)}), \tag{2.5}$$

where S_n is the group of permutations of $\{1,\dots,n\}$. For $n \in \mathbb{N}^*$, we define the n-particle space as $\mathcal{H}_n := P_n\mathcal{H}^{\otimes n}$. We define Fock space as the Hilbert space $\mathcal{F} \equiv \mathcal{F}(\mathcal{H}) := \bigoplus_{n\in\mathbb{N}}\mathcal{H}_n$. We denote by $\mathrm{Tr}_{\mathcal{F}}(X)$ the trace of an operator X acting on $\mathcal{F}$. For $f \in \mathcal{H}$ we define the bosonic annihilation and creation operators $a(f)$ and $a^*(f)$ on $\mathcal{F}$ through their action on a dense set of vectors $\Psi = (\Psi_n)_{n\in\mathbb{N}} \in \mathcal{F}$ as

$$(a(f)\Psi)_n(x_1,\dots,x_n) = \sqrt{n+1}\int \mathrm{d}x\ \bar{f}(x)\Psi_{n+1}(x,x_1,\dots,x_n), \tag{2.6}$$

$$(a^*(f)\Psi)_n(x_1,\dots,x_n) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} f(x_i)\Psi_{n-1}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n). \tag{2.7}$$

The operators $a(f)$ and $a^*(f)$ are unbounded closed operators on $\mathcal{F}$, and are each other's adjoints. They satisfy the canonical commutation relations

$$[a(f),a^*(g)] = \langle f\,,g\rangle\,1, \quad [a(f),a(g)] = [a^*(f),a^*(g)] = 0, \tag{2.8}$$

where $[X,Y] := XY - YX$ denotes the commutator. We regard a and a^* as operator-valued distributions and use the notations

$$a(f) = \int \mathrm{d}x\ \bar{f}(x)a(x), \quad a^*(f) = \int \mathrm{d}x\ f(x)a^*(x). \tag{2.9}$$

The distribution kernels $a^*(x)$ and $a(x)$ satisfy the canonical commutation relations

$$[a(x),a^*(\tilde{x})] = \delta(x-\tilde{x}), \quad [a(x),a(\tilde{x})] = [a^*(x),a^*(\tilde{x})] = 0. \tag{2.10}$$

For $\nu > 0$, we define the free quantum Hamiltonian $H^{(0)} \equiv H_\nu^{(0)}$ through

$$H^{(0)} := \nu\int \mathrm{d}x\, a^*(x)((\kappa - \Delta/2)a)(x). \tag{2.11}$$

To describe the interaction potential of the Bose gas, we choose $v : \mathbb{R}^d \to \mathbb{R}$ to be an even, smooth, compactly supported function of positive type[5] whose integral is equal to 1. For $\varepsilon > 0$ we define the rescaled interaction potential on Λ as

$$v^\varepsilon(x) = \sum_{n \in \mathbb{Z}^d} \frac{1}{\varepsilon^d} v\left(\frac{x-n}{\varepsilon}\right). \tag{2.12}$$

For $\varepsilon, \nu > 0$ we define the interacting quantum Hamiltonian $H \equiv H_\nu^\varepsilon$ through

$$H := H^{(0)} + \frac{\nu^2}{2} \int \mathrm{d}x \, \mathrm{d}\widetilde{x} \, a^*(x)a(x)v^\varepsilon(x-\widetilde{x})a^*(\widetilde{x})a(\widetilde{x}) - \nu\alpha_\nu^\varepsilon \int \mathrm{d}x \, a^*(x)a(x) + \theta_\nu^\varepsilon, \tag{2.13}$$

where α_ν^ε and θ_ν^ε are real renormalization parameters that we shall define shortly in (2.16) below.

Using (2.13), the quantum grand canonical density matrix from (1.6) can be expressed as the operator

$$(1.6) = \mathrm{e}^{-H}/Z, \quad Z := \mathrm{Tr}_{\mathcal{F}}(\mathrm{e}^{-H}),$$

where Z is the grand canonical partition function. Analogously, the free grand canonical partition function is

$$Z^{(0)} := \mathrm{Tr}_{\mathcal{F}}(\mathrm{e}^{-H^{(0)}}).$$

We shall also use the relative partition function

$$\mathcal{Z} := Z/Z^{(0)}. \tag{2.14}$$

In order to define the renormalization parameters α_ν^ε and θ_ν^ε, we introduce the Green function G of the free field ϕ, i.e. the integral kernel of the operator $(\kappa - \Delta/2)^{-1}$. Since $\kappa - \Delta/2$ is invariant under translations, we can write $G_{x,y} = G(x-y)$. Explicitly, in the sense of distributions,

$$G(x-y) = \mathbb{E}[\phi(x)\bar{\phi}(y)].$$

The Green function G exhibits a logarithmic singularity at the origin (see Lemma B.1 below). Moreover, we denote by

$$\varrho_\nu := \nu \, \mathrm{Tr}_{\mathcal{F}}\left(a^*(0)a(0)\frac{\mathrm{e}^{-H^{(0)}}}{Z^{(0)}}\right) \tag{2.15}$$

the expected rescaled particle density in the free quantum state. Then we set

$$\alpha_\nu^\varepsilon := \varrho_\nu + \tau^\varepsilon, \quad \theta_\nu^\varepsilon := \tfrac{1}{2}\varrho_\nu^2 + \tau^\varepsilon\varrho_\nu - E^\varepsilon, \tag{2.16}$$

where

$$\tau^\varepsilon := \int \mathrm{d}x \, v^\varepsilon(x)G(x), \quad E^\varepsilon := \frac{1}{2}\int \mathrm{d}x \, v^\varepsilon(x)G(x)^2. \tag{2.17}$$

[5]This means that the Fourier transform of v is a positive measure. Note that we do not assume that v is pointwise nonnegative.

The parameter α_ν^ε describes a renormalization of the chemical potential, and θ_ν^ε corresponds to an energy renormalization. As $\varepsilon, \nu \to 0$, the renormalization of the chemical potential behaves as $\alpha_\nu^\varepsilon \to +\infty$. We remark that, using the quantities (2.15) and (2.17), we can rewrite the Hamiltonian (2.13) in the form

$$H = H^{(0)} + \frac{1}{2}\int \mathrm{d}x\, \mathrm{d}\widetilde{x}\, \big(\nu a^*(x)a(x) - \varrho_\nu\big)v^\varepsilon(x-\widetilde{x})\big(\nu a^*(\widetilde{x})a(\widetilde{x}) - \varrho_\nu\big) - \tau^\varepsilon \int \mathrm{d}x\, \big(\nu a^*(x)a(x) - \varrho_\nu\big) - E^\varepsilon. \tag{2.18}$$

Next, we define the *p-particle reduced density matrix* as

$$(\Gamma_p)_{\mathbf{x},\widetilde{\mathbf{x}}} := \mathrm{Tr}_{\mathscr{F}}\left(a^*(\widetilde{x}_1)\cdots a^*(\widetilde{x}_p)a(x_1)\cdots a(x_p)\frac{\mathrm{e}^{-H}}{Z}\right). \tag{2.19}$$

As with the correlation function (2.3) and its Wick-ordered version (2.4), we would like to replace (2.19) with its Wick-ordered version. To that end, we regard the expressions (2.3) and (2.4) as integral kernels of operators acting on $\mathscr{H}_p$, and observe that (see [33, Lemma A.4])

$$\widehat{\gamma}_p = \sum_{k=0}^{p}\binom{p}{k}^2 (-1)^{p-k} P_p(\gamma_k \otimes \gamma_{p-k}^{(0)})P_p, \tag{2.20}$$

where $\gamma_m^{(0)}$ denotes the m-point correlation function from (2.3) with $V = 0$. In analogy with (2.20), we therefore define the *Wick-ordered p-particle reduced density matrix* as

$$(\widehat{\Gamma}_p)_{\mathbf{x},\widetilde{\mathbf{x}}} := \sum_{k=0}^{p}\binom{p}{k}^2 (-1)^{p-k} P_p(\Gamma_k \otimes \Gamma_{p-k}^{(0)})P_p, \tag{2.21}$$

where $\Gamma_m^{(0)}$ denotes the m-particle reduced density matrix of the free grand canonical density matrix $\mathrm{e}^{-H^{(0)}}/Z^{(0)}$. (For an interpretation of (2.21) as a result of Wick ordering (2.19) with respect to the free field in the functional integral representation of quantum many-body theory, we refer the reader to the discussion in [33, Section 1.7]).

2.3. *Results*

We may now state our main result.

Theorem 2.1. *Suppose that $d = 2$ and $\varepsilon \equiv \varepsilon(\nu)$ satisfies*

$$\varepsilon \geqslant \exp\big(-(\log \nu^{-1})^{1/2-c}\big) \tag{2.22}$$

for some constant $c > 0$. Then as $\varepsilon, \nu \to 0$ we have the convergence of the partition function

$$\mathcal{Z} \to \zeta \tag{2.23}$$

and of the Wick-ordered correlation functions

$$\nu^p\, \widehat{\Gamma}_p \xrightarrow{C} \widehat{\gamma}_p \tag{2.24}$$

for all $p \in \mathbb{N}$, where $\xrightarrow{C}$ denotes convergence in the space of continuous functions on $\Lambda^p \times \Lambda^p$ with respect to the supremum norm.

We refer to [33, Section 1.5] for an in-depth discussion of applications of Theorem 2.1. In particular, Theorem 2.1 yields the following result for unrenormalized correlation functions.

Corollary 2.2. *Under the assumptions of Theorem 2.1,*

$$\nu^p \Gamma_p \xrightarrow{L^r} \gamma_p$$

for all $p \in \mathbb{N}$ and $r < \infty$, where $\xrightarrow{L^r}$ denotes convergence in the $L^r(\Lambda^p \times \Lambda^p)$-norm.

Another application of Theorem 2.1 is the convergence of the joint distribution of the Wick-ordered quantum particle densities $a^*(x)a(x)$ to those of the Wick-ordered mass densities $|\phi(x)|^2$; see [33, Theorem 1.4].

Remark 2.3. In this paper we set Λ to be the unit torus for definiteness, but our methods extend without complications to more general domains and boundary conditions. In particular, they also apply to the full space $\mathbb{R}^2$ with one-body Hamiltonian $-\Delta/2 + U(x)$, where the particles are confined by a suitable external potential $U : \mathbb{R}^2 \to \mathbb{R}$ satisfying $U(x) \geqslant b|x|^\theta$ for some $b > 0$ and $\theta > 2$. We refer to [33, Sections 1.6 and 7] and [56, Section 3.2] for an in-depth discussion of the analogous extension for the mean-field scaling. The corresponding counterterm problem, which relates the bare and renormalized external potentials, was formulated and solved in [31] for the mean-field scaling. It, along with the arguments of [31, Section 5], can be adapted to the local scaling of the current paper; we omit further details.

Remark 2.4. The proof of Theorem 2.1 can be extended to establish the convergence of the interacting Bose gas of N species of identical Bosons to the ϕ_2^4 theory with N complex components. (Theorem 2.1 corresponds to $N = 1$.) More precisely, we introduce the species index $i = 1, \dots, N$, and augment the creation and annihilation operators to $a_i^*(x), a_i(x)$ satisfying the canonical commutation relations

$$[a_i(x), a_j^*(\widetilde{x})] = \delta_{ij}\delta(x - \widetilde{x}), \quad [a_i(x), a_j(\widetilde{x})] = [a_i^*(x), a_j^*(\widetilde{x})] = 0, \tag{2.25}$$

which generalize (2.10). The Hamiltonian from (2.11) and (2.13) is generalized to

$$H^{(0)} := \nu \sum_{i=1}^N \int \mathrm{d}x\, a_i^*(x)((\kappa - \Delta/2)a_i)(x)$$

and

$$H := H^{(0)} + \frac{\nu^2}{2} \sum_{i=1}^N \int \mathrm{d}x\, \mathrm{d}\widetilde{x}\, a_i^*(x)a_i(x)v^\varepsilon(x - \widetilde{x})a_i^*(\widetilde{x})a_i(\widetilde{x})$$
$$- \nu\alpha_\nu^\varepsilon \sum_{i=1}^N \int \mathrm{d}x\, a_i^*(x)a_i(x) + \theta_\nu^\varepsilon.$$

Then we find that the reduced density matrices of the N-species quantum Bose gas converge to the correlation functions of ϕ_2^4 theory with N complex components, in the sense of Theorem 2.1.

Remark 2.5. We conclude this section with a discussion on the technical condition (2.22) on the range ε of the interaction potential. It is instructive to compare the right-hand side of (2.22) to the typical inter-particle distance, which we claim is of order

$$\ell = \nu^{1/2}(\log \nu^{-1})^{-1/2}. \tag{2.26}$$

To show (2.26), it suffices to show that the expected number of particles, given by $\mathrm{Tr}(\Gamma_1)$, is of order $\nu^{-1}\log\nu^{-1}$. By Theorem 2.1 and the definition (2.21), we find $\mathrm{Tr}(\Gamma_1) = \mathrm{Tr}(\Gamma_1^{(0)}) + O(\nu^{-1})$. Hence, it suffices to show that $\mathrm{Tr}(\Gamma_1^{(0)})$ is of order $\nu^{-1}\log\nu^{-1}$, which follows using the Wick theorem for quasi-free quantum states (see e.g. [33, Remark 1.5 and Lemma 2.9]), since $\mathrm{Tr}(\Gamma_1^{(0)}) = \mathrm{Tr}(\frac{1}{\mathrm{e}^{\nu(\kappa-\Delta/2)}-1}) \asymp \nu^{-1}\log\nu^{-1}$, as claimed.

Comparing (2.26) and the right-hand side of (2.22), we conclude that the range of the interaction ε may vanish much faster than any power of $(\log\ell^{-1})^{-1}$ but much slower than any power of ℓ. As stated in Section 1.2, we expect that it can be improved, but at the cost of a considerably more complicated argument. We leave such quantitative improvements to future work, focusing here on the first result of this kind while aiming for a relatively simple proof.

The origin of condition (2.22) in our proof arises from controlling oscillatory integrals. It can be traced to the functional Fourier representation from Lemmas 5.6 and 5.7 below. There, the partition function is expressed in terms of an integral over a Gaussian field, where the integrand includes a phase with a diverging prefactor. To compensate this rapidly oscillating phase and obtain a quantity of order 1, the integral is multiplied by the large factor

$$\mathrm{e}^{(\tau^\varepsilon)^2/2+E^\varepsilon},$$

(see (5.15)) which is propagated throughout the estimates of Section 5, for example in Lemma 5.9. This large factor needs to be compensated by powers of ν which arise from our error estimates, leading to the lower bound on ε in terms of ν.

3. Structure of the proof

The rest of this paper is devoted to the proof of Theorem 2.1. We begin with a short section that lays out the general strategy. We use c, C to denote generic positive constants, which may change from one expression to the next, and may depend on fixed parameters. We write $x \lesssim y$ or $x = O(y)$ to mean $x \leqslant Cy$. If C depends on a parameter α, we write $x \lesssim_\alpha y$, $x \leqslant C_\alpha y$, or $x = O_\alpha(y)$. We abbreviate $[n] = \{1, \dots, n\}$.

We shall need two different interacting field theories approximating (2.2), obtained by replacing the interaction V with regularized variants, denoted by W^ε and V^ε, respectively.

They are defined by

$$W^\varepsilon := \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}\widetilde{x} : |\phi(x)|^2 : v^\varepsilon(x-\widetilde{x}) : |\phi(\widetilde{x})|^2 : - \tau^\varepsilon \int \mathrm{d}x : |\phi(x)|^2 : - E^\varepsilon, \tag{3.1}$$

$$V^\varepsilon := \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\, v^\varepsilon(x-\widetilde{x}) : |\phi(x)|^2\, |\phi(\widetilde{x})|^2 :. \tag{3.2}$$

The rigorous construction of the random variables $W^\varepsilon, V^\varepsilon$ proceeds exactly like that of V explained in Section 2.1: one introduces truncated versions $W^\varepsilon_N, V^\varepsilon_N$ defined in terms of the truncated free field ϕ_N (see e.g. (4.7) below), and proves using Wick's theorem that as $N \to \infty$ they converge in $L^2(\mathbb{P})$ to their respective limits $W^\varepsilon, V^\varepsilon$. Throughout the following, we shall make use of such constructions of Wick-ordered functions of the free field without further comment. The integrability of $\mathrm{e}^{-W^\varepsilon}$ and $\mathrm{e}^{-V^\varepsilon}$ is established in Section 4 below.

To emphasize the dependence of the quantities (2.2) and (2.4) on the interaction V, we sometimes include the interaction V in our notation as a superscript, writing ζ^V and $\widehat{\gamma}^V_p$, respectively.

The proof consists of two main steps.

Step 1. We compare $\mathcal{Z}$ and $\nu^p\,\widehat{\Gamma}_p$ with ζ^{W^ε} and $\widehat{\gamma}^{W^\varepsilon}_p$, respectively, in the limit $\nu, \varepsilon \to 0$ under condition (2.22).

Step 2. We compare ζ^{W^ε} and $\widehat{\gamma}^{W^\varepsilon}_p$ with ζ^V and $\widehat{\gamma}^V_p$, respectively, in the limit $\varepsilon \to 0$. This step is done by passing via the further intermediate interaction V^ε.

Step 1 relies on a quantitative analysis of the infinite-dimensional saddle point argument for the functional integral introduced in [33].

Step 2 relies on three main ingredients. First, we show integrability of $\mathrm{e}^{-V^\varepsilon}$, uniformly in ε. Second, we use the fact that $V^\varepsilon - W^\varepsilon$ is small in $L^2(\mathbb{P})$ and it lies in the second polynomial chaos (see Section A.2 below), which allows us to deduce integrability of $\mathrm{e}^{-W^\varepsilon}$ by expansion in $V^\varepsilon - W^\varepsilon$ and hypercontractive moment bounds. Third, to obtain uniform control on the Wick-ordered correlation functions, we use Gaussian integration by parts, analogous to Malliavin calculus, to derive a representation of the correlation functions in terms of expectations of derivatives of the interaction potential.

The results of these two steps are summarized in the following two propositions.

Proposition 3.1. *Suppose that $d = 2$ and that $\nu, \varepsilon \to 0$ under the constraint (2.22). Then $\mathcal{Z} - \zeta^{W^\varepsilon} \to 0$. Moreover, for all $p \in \mathbb{N}^*$,*

$$\big\| \nu^p \widehat{\Gamma}_p - \widehat{\gamma}^{W^\varepsilon}_p \big\|_C \to 0. \tag{3.3}$$

Proposition 3.2. *Suppose that $d = 2$ and $\varepsilon \to 0$. Then $\zeta^{W^\varepsilon} \to \zeta^V$. Moreover, for all $p \in \mathbb{N}^*$,*

$$\big\| \widehat{\gamma}^{W^\varepsilon}_p - \widehat{\gamma}^V_p \big\|_C \to 0. \tag{3.4}$$

We remark that Proposition 3.1 holds also for $d = 3$, with the same proof, provided that condition (2.22) is suitably adjusted. We refer to Section 5 for more details and for the proof.

4. Proof of Proposition 3.2

In this section we prove Proposition 3.2. We set $d = 2$ throughout.

4.1. L^2-estimates

In this subsection we derive L^2-estimates for the differences $V^\varepsilon - V$ and $V^\varepsilon - W^\varepsilon$.

Lemma 4.1. *We have* $\|V^\varepsilon - W^\varepsilon\|_{L^2(\mathbb{P})} \to 0$ *as* $\varepsilon \to 0$.

Proof. A straightforward calculation using (A.2) below yields

$$\begin{aligned} V^\varepsilon &= \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,{:}|\phi(x)|^2{:}\,v^\varepsilon(x-\widetilde{x})\,{:}|\phi(\widetilde{x})|^2{:} \\ &\quad - \int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,v^\varepsilon(x-\widetilde{x})G(x-\widetilde{x})\,\bar{\phi}(x)\phi(\widetilde{x}) + E^\varepsilon \\ &= \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,{:}|\phi(x)|^2{:}\,v^\varepsilon(x-\widetilde{x})\,{:}|\phi(\widetilde{x})|^2{:} \\ &\quad - \int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,v^\varepsilon(x-\widetilde{x})G(x-\widetilde{x})\,{:}\bar{\phi}(x)\phi(\widetilde{x}){:} - E^\varepsilon \\ &= W^\varepsilon - \int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,v^\varepsilon(x-\widetilde{x})G(x-\widetilde{x})\big({:}\bar{\phi}(x)\phi(\widetilde{x}){:} - {:}|\phi(x)|^2{:}\big). \end{aligned} \tag{4.1}$$

By Lemma A.1 below (see also Example A.2), we find

$$\begin{aligned} \|V^\varepsilon - W^\varepsilon\|_{L^2(\mathbb{P})}^2 = \int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,v^\varepsilon(x-\widetilde{x})v^\varepsilon(y-\widetilde{y})G(\widetilde{x}-x)G(\widetilde{y}-y) \\ \times \big(G(x-\widetilde{y}) - G(x-y)\big)\big(G(\widetilde{x}-y) - G(x-y)\big). \end{aligned} \tag{4.2}$$

We emphasize the crucial double cancellation on the second line of (4.2), which will ensure convergence of the right-hand side, even though the first line of the right-hand side on its own is divergent as $\varepsilon \to 0$. From Lemma B.1 below, we find

$$|G(x-\widetilde{y}) - G(x-y)| \lesssim |y-\widetilde{y}| + \left|\log\frac{|x-\widetilde{y}|}{|x-y|}\right| \tag{4.3}$$

and similarly for $|G(\widetilde{x}-y) - G(x-y)|$. Switching to new integration variables $h = (\widetilde{x}-x)/\varepsilon$, $k = (y-\widetilde{y})/\varepsilon$, and $z = x-y$, we obtain

$$\begin{aligned} &\|V^\varepsilon - W^\varepsilon\|_{L^2(\mathbb{P})}^2 \\ &\lesssim \int \mathrm{d}h\,\mathrm{d}k\,\mathrm{d}z\,v(h)v(k)G(\varepsilon h)G(\varepsilon k)\left(\varepsilon|h| + \left|\log\frac{|z+\varepsilon h|}{|z|}\right|\right)\left(\varepsilon|k| + \left|\log\frac{|z+\varepsilon k|}{|z|}\right|\right) \end{aligned}$$

We multiply out the two parentheses on the right-hand side and treat each of the four terms separately. The term arising from $\varepsilon|h|\varepsilon|k|$ is easily estimated by $O_\alpha(\varepsilon^{2\alpha})$ for any $\alpha \in (0,1)$, using $G(x)|x| \lesssim_\alpha |x|^\alpha$, by Lemma B.1.

For the other three terms containing the logarithmic factor, we use

$$\left|\log\frac{|x+y|}{|x|}\right| \lesssim \begin{cases} \big|\log|x+y|\big| + \big|\log|x|\big| & \text{if } |x| \leqslant 2|y|, \\ |y|/|x| & \text{if } |x| > 2|y|, \end{cases} \tag{4.4}$$

We estimate the mixed terms, for any $\alpha \in [0,1)$, as

$$\begin{aligned} &\int \mathrm{d}h\,\mathrm{d}k\,\mathrm{d}z\; v(h)v(k)G(\varepsilon h)G(\varepsilon k)\,\varepsilon|k|\left|\log\frac{|z+\varepsilon h|}{|z|}\right| \\ &\qquad\lesssim \int \mathrm{d}k\; v(k)G(\varepsilon k)\varepsilon|k| \int \mathrm{d}h\; v(h)G(\varepsilon h) \\ &\qquad\quad\times \int \mathrm{d}z \left(\big(\big|\log|z+\varepsilon h|\big| + \big|\log|z|\big|\big)\mathbf{1}_{|z|\leqslant 2\varepsilon|h|} + \frac{\varepsilon|h|}{|z|}\mathbf{1}_{|z|>2\varepsilon|h|}\right) \\ &\qquad\lesssim_\alpha \int \mathrm{d}k\; v(k)G(\varepsilon k)\varepsilon|k| \int \mathrm{d}h\; v(h)G(\varepsilon h)(\varepsilon^{2\alpha}|h|^{2\alpha} + \varepsilon|h|) \lesssim_\alpha \varepsilon^{2\alpha}. \end{aligned}$$

Finally, we estimate, for any $\alpha' < \alpha < 1$,

$$\begin{aligned} &\int \mathrm{d}h\,\mathrm{d}k\,\mathrm{d}z\; v(h)v(k)G(\varepsilon h)G(\varepsilon k)\left|\log\frac{|z+\varepsilon h|}{|z|}\right|\left|\log\frac{|z+\varepsilon k|}{|z|}\right| \\ &\lesssim \int \mathrm{d}h\,\mathrm{d}k\; v(h)v(k)G(\varepsilon h)G(\varepsilon k) \\ &\quad\times \int \mathrm{d}z \left(\big(\big|\log|z+\varepsilon h|\big| + \big|\log|z|\big|\big)\mathbf{1}_{|z|\leqslant 2\varepsilon|h|} + \frac{\varepsilon|h|}{|z|}\mathbf{1}_{|z|>2\varepsilon|h|}\right) \\ &\quad\times \left(\big(\big|\log|z+\varepsilon k|\big| + \big|\log|z|\big|\big)\mathbf{1}_{|z|\leqslant 2\varepsilon|k|} + \frac{\varepsilon|k|}{|z|}\mathbf{1}_{|z|>2\varepsilon|k|}\right) \\ &\lesssim_\alpha \int \mathrm{d}h\,\mathrm{d}k\; v(h)v(k)G(\varepsilon h)G(\varepsilon k)\big([\varepsilon(|h|\wedge|k|)]^{2\alpha} + 2\,(\varepsilon|h|)^\alpha\,\varepsilon|k| + (\varepsilon|h|)^\alpha(\varepsilon|k|)^\alpha\big) \\ &\lesssim \int \mathrm{d}h\,\mathrm{d}k\; v(h)v(k)G(\varepsilon h)G(\varepsilon k)\,(\varepsilon|h|)^\alpha(\varepsilon|k|)^\alpha \lesssim_{\alpha'} \varepsilon^{2\alpha'}, \end{aligned}$$

where in the second step we have used the estimates

$$\int \mathrm{d}z\,\big(\big|\log|z+\varepsilon h|\big|^2 + \big|\log|z|\big|^2\big)\mathbf{1}_{|z|\leqslant 2\varepsilon|h|}\mathbf{1}_{|z|\leqslant 2\varepsilon|k|} \lesssim_\alpha [\varepsilon(|h|\wedge|k|)]^{2\alpha},$$

$$\int \mathrm{d}z\,\frac{\big|\log|z+\varepsilon h|\big| + \big|\log|z|\big|}{|z|}\mathbf{1}_{|z|\leqslant 2\varepsilon|h|} \lesssim_\alpha (\varepsilon|h|)^\alpha$$

and $\varepsilon|h|/|z| \leqslant (\varepsilon|h|/|z|)^\alpha$ for $|z| > 2\varepsilon|h|$. We conclude that $\|V^\varepsilon - W^\varepsilon\|_{L^2(\mathbb{P})} \lesssim_\alpha \varepsilon^\alpha$ for any $\alpha \in (0,1)$. ∎

Lemma 4.2. *We have* $\|V^\varepsilon - V\|_{L^2(\mathbb{P})} \to 0$ *as* $\varepsilon \to 0$.

Proof. Clearly,

$$V^\varepsilon - V = \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,\big(v^\varepsilon(x-\widetilde{x}) - \delta(x-\widetilde{x})\big) : |\phi(x)|^2\,|\phi(\widetilde{x})|^2 :$$

Using Lemma A.1 below (see also Example A.2) we therefore find

$$\mathbb{E}(V^\varepsilon - V)^2 = \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,(v^\varepsilon(x-\widetilde{x})-\delta(x-\widetilde{x}))\,(v^\varepsilon(y-\widetilde{y})-\delta(y-\widetilde{y}))$$
$$\times G(x-y)G(\widetilde{x}-\widetilde{y})\,(G(x-y)G(\widetilde{x}-\widetilde{y})+G(\widetilde{x}-y)G(x-\widetilde{y})).$$

The right-hand side splits into two terms. We only consider the first one; the analysis of the second one is analogous. With the change of variables $\widetilde{x}-x=h$, $\widetilde{y}-y=k$, and $z=x-y$, the first term reads

$$\frac{1}{2}\int \mathrm{d}z\,\mathrm{d}h\,\mathrm{d}k\,\big(v^\varepsilon(h)-\delta(h)\big)\big(v^\varepsilon(k)-\delta(k)\big)G(z)^2G(z+h-k)^2$$
$$= \frac{1}{2}\int \mathrm{d}z\,G(z)^2\int \mathrm{d}h\,\mathrm{d}k\,v^\varepsilon(h)v^\varepsilon(k)$$
$$\times\big(G(z+h-k)^2-G(z-k)^2-G(z+h)^2+G(z)^2\big).$$

We now estimate the first two terms and the last two terms separately. (A more careful second order analysis could be done to take into account a further cancellation between all four terms, yielding a bound $\varepsilon^{2\alpha}$ for any $\alpha<1$ instead of ε, but we shall not need it.) The sum of the first two terms on the right-hand side is estimated by

$$\frac{1}{2}\int \mathrm{d}z G(z)^2\int \mathrm{d}h\,\mathrm{d}k\,v^\varepsilon(h)v^\varepsilon(k)|G(z+h-k)^2-G(z-k)^2|$$
$$\lesssim \frac{1}{2}\int \mathrm{d}h\,\mathrm{d}k\,v^\varepsilon(h)v^\varepsilon(k)\int \mathrm{d}z\,G(z)^2\big(G(z+h-k)+G(z-k)\big)\Big(|h|+\Big|\log\frac{|z+h-k|}{|z-k|}\Big|\Big)$$
$$\lesssim \frac{1}{2}\int \mathrm{d}h\,\mathrm{d}k\,v^\varepsilon(h)v^\varepsilon(k)\int \mathrm{d}z\,G(z)^2\big(G(z+h-k)+G(z-k)\big)$$
$$\times\Big(|h|+\big(\big|\log|z+h-k|\big|+\big|\log|z-k|\big|\big)\mathbf{1}_{|z-k|\leqslant 2|h|}+\frac{|h|}{|z-k|}\mathbf{1}_{|z-k|>2|h|}\Big),$$

where in the second step we have used estimate (4.3), and in the third, estimate (4.4). Since on the support of the integral over h we have $|h|\lesssim\varepsilon$, we may perform integration over z, followed by integration over h and k, to deduce that the above expression is bounded by $O(\varepsilon)$. Here we also use the fact that G has a logarithmic singularity at the origin, as established in Lemma B.1 below and the fact that $\big\|\log|x|\,\mathbf{1}_{|x|\lesssim\varepsilon}\big\|_{L^p}\lesssim_p \varepsilon$ for $p\in(1,2)$, combined with Hölder's inequality. This concludes the proof. ∎

4.2. *Integrability of* $\mathrm{e}^{-V^\varepsilon}$

In this subsection we establish the integrability of $\mathrm{e}^{-V^\varepsilon}$, uniformly in ε. This is an adaptation of Nelson's argument [64] (see also [47] for a recent pedagogical account) to a nonlocal interaction.

Proposition 4.3. *There is a constant $c>0$ such that for all $\varepsilon>0$ and $t\geqslant 1$ we have*

$$\mathbb{P}(\mathrm{e}^{-V^\varepsilon}>t)\lesssim \exp(-\mathrm{e}^{c\sqrt{\log t}}).$$

The same estimate holds for V^ε replaced with V.

In particular, $\mathrm{e}^{-V^\varepsilon}$ is uniformly integrable in $\varepsilon > 0$.

The rest of this subsection is devoted to the proof of Proposition 4.3. We start by noting that $\kappa - \Delta/2$ has eigenfunctions $u_k \in \mathcal{H}$ and eigenvalues λ_k indexed by $k \in \mathbb{Z}^d$ and given by

$$\lambda_k = \kappa + 2\pi^2|k|^2, \quad u_k = \mathrm{e}^{2\pi \mathrm{i} k\cdot x}. \tag{4.5}$$

We shall use the truncated field ϕ_N from (2.1) with a suitable truncation ϑ, which is smooth in Fourier space. To that end, we fix ρ to be a smooth, nonnegative, rotation invariant function, that has integral 1 and is supported in the unit ball. We suppose that its Fourier transform

$$\vartheta(\xi) := \mathfrak{F}\rho(\xi) := \int_{\mathbb{R}^2} \mathrm{d}x\, \mathrm{e}^{-2\pi \mathrm{i}\xi\cdot x}\rho(x) \tag{4.6}$$

is nonnegative and radially nonincreasing (this can always be achieved by taking ρ as a convolution of two nonnegative functions).

We define the truncated version of V^ε from (3.2) through

$$V_N^\varepsilon := \frac{1}{2}\int \mathrm{d}x\, \mathrm{d}\widetilde{x}\, v^\varepsilon(x-\widetilde{x}) :|\phi_N(x)|^2\, |\phi_N(\widetilde{x})|^2 :, \tag{4.7}$$

which converges in $L^2(\mathbb{P})$ to V^ε as $N \to \infty$.

Next, let $(Y_k)_{k\in\mathbb{Z}^d}$ be a family of i.i.d. standard complex Gaussian random variables, which is independent of the family $(X_k)_{k\in\mathbb{Z}^d}$. For $0 < N \leqslant M \leqslant \infty$ we define the field

$$\psi_{N,M} := \sum_{k\in\mathbb{Z}^d} \frac{Y_k}{\sqrt{\lambda_k}} \sqrt{\vartheta(k/M) - \vartheta(k/N)}\, u_k.$$

By construction, ϕ_N and $\psi_{N,M}$ are independent. For $M < \infty$, they are almost surely smooth on Λ. We define the truncated Green function

$$G_N := G * \rho_N, \quad \rho_N(x) := \sum_{n\in\mathbb{Z}^d} N^2\rho(N(x+n)), \tag{4.8}$$

and find by Poisson summation that, for $N \leqslant M$,

$$\mathbb{E}[\phi_N(x)\bar{\phi}_N(y)] = \sum_{k\in\mathbb{Z}^d} \frac{1}{\lambda_k}\mathrm{e}^{2\pi \mathrm{i}k\cdot(x-y)}\vartheta(k/N) = G_N(x-y), \tag{4.9}$$

$$\begin{aligned}\mathbb{E}[\psi_{N,M}(x)\bar{\psi}_{N,M}(y)] &= \sum_{k\in\mathbb{Z}^d} \frac{1}{\lambda_k}\mathrm{e}^{2\pi \mathrm{i}k\cdot(x-y)}\big(\vartheta(k/M) - \vartheta(k/N)\big) \\ &= G_M(x-y) - G_N(x-y).\end{aligned} \tag{4.10}$$

By independence of ϕ_N and $\psi_{N,M}$, we therefore find that for any $N \leqslant M$ we have the decomposition into low and high frequencies

$$\phi_N + \psi_{N,M} \overset{\mathrm{d}}{=} \phi_M, \tag{4.11}$$

and in particular setting $M = \infty$ we get

$$\phi_N + \psi_{N,\infty} \overset{\mathrm{d}}{=} \phi.$$

Here $\overset{\mathrm{d}}{=}$ denotes equality in law.

By (4.11) we have, for any $N \leqslant M$,

$$V_M^\varepsilon \stackrel{\mathrm{d}}{=} \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\, v^\varepsilon(x-\widetilde{x}) :|\phi_N(x)+\psi_{N,M}(x)|^2\,|\phi_N(\widetilde{x})+\psi_{N,M}(\widetilde{x})|^2 :.$$

For any $N \leqslant M$ we therefore have

$$V_M^\varepsilon \stackrel{\mathrm{d}}{=} \sum_{a,\widetilde{a},b,\widetilde{b}\in\{0,1\}} V_{N,M}^\varepsilon(a,\widetilde{a},b,\widetilde{b}),$$

where

$$\begin{aligned} V_{N,M}^\varepsilon(a,\widetilde{a},b,\widetilde{b}) :=& \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\, v^\varepsilon(x-\widetilde{x}) \\ &\times :\phi_N(x)^{1-a}\phi_N(\widetilde{x})^{1-\widetilde{a}}\bar{\phi}_N(x)^{1-b}\bar{\phi}_N(\widetilde{x})^{1-\widetilde{b}} \\ &\times \psi_{N,M}(x)^a\psi_{N,M}(\widetilde{x})^{\widetilde{a}}\bar{\psi}_{N,M}(x)^b\bar{\psi}_{N,M}(\widetilde{x})^{\widetilde{b}} :. \end{aligned} \tag{4.12}$$

Hence, for $N \leqslant M$ we have

$$V_M^\varepsilon - V_N^\varepsilon = \sum_{a,\widetilde{a},b,\widetilde{b}\in\{0,1\}} \mathbf{1}_{a+\widetilde{a}+b+\widetilde{b}>0}\, V_{N,M}^\varepsilon(a,\widetilde{a},b,\widetilde{b}). \tag{4.13}$$

Lemma 4.4. *There is a constant C depending on v such that almost surely*

$$V_N^\varepsilon \geqslant -C(\log N)^2 \quad \textit{for all } \varepsilon > 0.$$

Proof. Abbreviate $S = 1 + \int_{\mathbb{R}^2} \mathrm{d}x\,|v(x)|$. Using the explicit form (A.2) of the Wick power in (4.7) as well as (4.9), we find

$$\begin{aligned} V_N^\varepsilon &= \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\, v^\varepsilon(x-\widetilde{x})\big(|\phi_N(x)|^2|\phi_N(\widetilde{x})|^2 - G_N(0)|\phi_N(x)|^2 - G_N(0)|\phi_N(\widetilde{x})|^2 \\ &\qquad - 2\,\mathrm{Re}\, G_N(x-\widetilde{x})\phi_N(x)\bar{\phi}_N(\widetilde{x}) + G_N(0)^2 + G_N(x-\widetilde{x})^2\big) \\ &= \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\, v^\varepsilon(x-\widetilde{x})\big[\big(|\phi_N(x)|^2 - SG_N(0)\big)\big(|\phi_N(\widetilde{x})|^2 - SG_N(0)\big) \\ &\qquad + (S-1)G_N(0)\big(|\phi_N(x)|^2 + |\phi_N(\widetilde{x})|^2\big) - 2\,\mathrm{Re}\, G_N(x-\widetilde{x})\phi_N(x)\bar{\phi}_N(\widetilde{x}) \\ &\qquad - (S^2-1)G_N(0)^2 + G_N(x-\widetilde{x})^2\big] \\ &\geqslant (S-1)G_N(0)\int \mathrm{d}x\,|\phi_N(x)|^2 \\ &\quad - \mathrm{Re}\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\, v^\varepsilon(x-\widetilde{x})G_N(x-\widetilde{x})\phi_N(x)\bar{\phi}_N(\widetilde{x}) - \frac{S^2}{2}G_N(0)^2, \end{aligned}$$

where in the last step we have used the fact that v (and hence also v^ε) is of positive type with integral 1. Using $|G_N(x)| \leqslant G_N(0)$ by (4.9) and Cauchy–Schwarz combined with Young's inequality, we find

$$\left|\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\, v^\varepsilon(x-\widetilde{x})G_N(x-\widetilde{x})\phi_N(x)\bar{\phi}_N(\widetilde{x})\right| \leqslant (S-1)G_N(0)\int \mathrm{d}x\,|\phi_N(x)|^2,$$

and the claim follows from Lemma B.3. ∎

Next, we derive an estimate for the L^2-norm of $V_M^\varepsilon - V_N^\varepsilon$.

Lemma 4.5. *For any fixed $\delta > 0$ and for any $0 < N \leqslant M < \infty$ we have*

$$\|V_M^\varepsilon - V_N^\varepsilon\|_{L^2(\mathbb{P})} \lesssim N^{-1+\delta}$$

Proof. By (4.13) and Minkowski's inequality, it suffices to estimate

$$\mathcal{R} := \mathbb{E}[|V_{N,M}^\varepsilon(a,\widetilde{a},b,\widetilde{b})|^2] = \mathbb{E}\big[V_{N,M}^\varepsilon(a,\widetilde{a},b,\widetilde{b})\,\overline{V_{N,M}^\varepsilon(a,\widetilde{a},b,\widetilde{b})}\big] \tag{4.14}$$

for any fixed $a,\widetilde{a},b,\widetilde{b} \in \{0,1\}$ satisfying $a+\widetilde{a}+b+\widetilde{b} > 0$.

Using Lemma B.3 below we find the bounds $G_N(x) \lesssim p(x)$ and $|G_M(x) - G_N(x)| \lesssim q(x)$, where

$$p(x) := 1 + \big|\log|x|\big|, \quad q(x) \equiv q_N(x) := \big(1 + |\log(N|x|)|\big) \wedge \frac{1}{N^2|x|^2}.$$

Note that $q(x) \lesssim p(x)$. Using Wick's theorem, Lemma A.1, and Young's inequality, we find

$$\begin{aligned}
\mathcal{R} &\lesssim \int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,|v^\varepsilon(x-\widetilde{x})|\,|v^\varepsilon(y-\widetilde{y})|q(x-y)p(x-y)p(\widetilde{x}-\widetilde{y})^2 \\
&\quad + \int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,|v^\varepsilon(x-\widetilde{x})|\,|v^\varepsilon(y-\widetilde{y})|q(x-y)p(\widetilde{x}-\widetilde{y})p(x-\widetilde{y})p(\widetilde{x}-y) \\
&\lesssim \sup_{y\in\Lambda}\int \mathrm{d}x\,q(x)p(x-y)^3 + \int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,q(x)p(\widetilde{x})^3 \\
&\lesssim \sup_{y\in\Lambda}\int \mathrm{d}x\,q(x)p(x-y)^3 \\
&\lesssim \sup_{y\in\Lambda}\bigg[\int_{|x|\leqslant C/N}\mathrm{d}x\,\big(1+\big|\log|x|\big|\big)^4 + \frac{1}{N^2}\int_{|x|>C/N}\mathrm{d}x\,\big(1+\big|\log|x-y|\big|\big)^3\frac{1}{|x|^2}\bigg] \\
&\lesssim_\delta N^{-2+\delta}
\end{aligned}$$

for any $\delta > 0$, where in the third step we have used $\int \mathrm{d}\widetilde{x}\,p(\widetilde{x})^3 \lesssim 1 \leqslant p(x)^3$, and in the last step we have used Hölder's inequality. ∎

Proof of Proposition 4.3. For any $N \geqslant 1$ we have, by Lemma 4.4,

$$\mathbb{P}(\mathrm{e}^{-V^\varepsilon} > t) = \mathbb{P}(V^\varepsilon - V_N^\varepsilon < -\log t - V_N^\varepsilon) \leqslant \mathbb{P}\big(V^\varepsilon - V_N^\varepsilon < -(\log t - C(\log N)^2)\big).$$

Now choose $N \geqslant 1$ such that

$$\log t - C(\log N)^2 = 1,$$

which is always possible for t large enough.

Next, we find that V_M^ε (or more precisely its real and imaginary parts) is in the 4th polynomial chaos (see Section A.2), by using Lemma A.3 and the easy fact that V_M^ε is orthogonal to the nth chaos for $n \neq 4$, which is a consequence of Wick's theorem in

Lemma A.1. Hence, from Remark A.5 and Lemma 4.5 we deduce that for any $0 < N \leqslant M < \infty$ and $p \in 2\mathbb{N}$ we have

$$\|V_M^\varepsilon - V_N^\varepsilon\|_{L^p(\mathbb{P})} \lesssim \frac{p^2}{N^{2/3}}. \tag{4.15}$$

Since $V_N^\varepsilon \to V^\varepsilon$ in $L^2(\mathbb{P})$ as $N \to \infty$, by Lemma A.4 we find that (4.15) holds also for $M = \infty$ (i.e. replacing V_M^ε with V^ε). Hence from Chebyshev's inequality, for any $p \in 2\mathbb{N}$ we get

$$\begin{aligned}\mathbb{P}(\mathrm{e}^{-V^\varepsilon} > t) &\leqslant \mathbb{E}[|V^\varepsilon - V_N^\varepsilon|^p] \leqslant \left(\frac{Cp^2}{N^{2/3}}\right)^p \leqslant \left(\frac{p^2}{\sqrt{N}}\right)^p \\ &\leqslant (p^2 \mathrm{e}^{-c\sqrt{\log t}})^p,\end{aligned}$$

for large enough t (and hence N). Choosing p to be the largest element of $2\mathbb{N}$ smaller than $\mathrm{e}^{c/2\sqrt{\log t}-1/2}$ yields the claim for V^ε.

Finally, the claim for V easily follows from the one for V^ε and Lemma 4.2. ∎

4.3. *Convergence of the partition function*

The first claim of Proposition 3.2, the convergence $\zeta^{W^\varepsilon} \to \zeta^V$, follows immediately from the following result.

Lemma 4.6. *For any $1 \leqslant p < \infty$ we have $\|\mathrm{e}^{-V} - \mathrm{e}^{-W^\varepsilon}\|_{L^p(\mathbb{P})} \to 0$ as $\varepsilon \to 0$.*

Proof. We begin by estimating $\|\mathrm{e}^{-W^\varepsilon}\|_{L^p(\mathbb{P})}$ by comparing it to $\|\mathrm{e}^{-V^\varepsilon}\|_{L^p(\mathbb{P})}$ and recalling Proposition 4.3. To that end, we note that $V^\varepsilon - W^\varepsilon$ is in the second polynomial chaos by (4.1) (more precisely, by Lemma A.1, $V^\varepsilon - W^\varepsilon$ is orthogonal to the nth polynomial chaos for $n \neq 2$ and the claim hence follows from Lemma A.3). Hence, by the hypercontractive bound from Lemma A.4 we obtain

$$\begin{aligned}\|\mathrm{e}^{V^\varepsilon - W^\varepsilon} - 1\|_{L^p(\mathbb{P})} &\leqslant \sum_{k\geqslant 1} \frac{1}{k!} \|(V^\varepsilon - W^\varepsilon)^k\|_{L^p(\mathbb{P})} = \sum_{k\geqslant 1} \frac{1}{k!} \|V^\varepsilon - W^\varepsilon\|_{L^{pk}(\mathbb{P})}^k \\ &\lesssim \sum_{k\geqslant 1} \frac{1}{k!} (pk)^k \|V^\varepsilon - W^\varepsilon\|_{L^2(\mathbb{P})}^k \leqslant \sum_{k\geqslant 1} (Cp\|V^\varepsilon - W^\varepsilon\|_{L^2(\mathbb{P})})^k\end{aligned}$$

for some constant $C > 0$, by Stirling's approximation for $k!$. Using Lemma 4.1 and Proposition 4.3, we conclude that for small enough ε (depending on p), $\|\mathrm{e}^{-W^\varepsilon}\|_{L^p(\mathbb{P})}$ is uniformly bounded in ε. The claim now follows by writing

$$\left\|\mathrm{e}^{-V} - \mathrm{e}^{-W^\varepsilon}\right\|_{L^p(\mathbb{P})} \leqslant \int_0^1 \mathrm{d}t \, \|(V - W^\varepsilon)\mathrm{e}^{-tW^\varepsilon - (1-t)V}\|_{L^p(\mathbb{P})},$$

applying Hölder's inequality to the right-hand side, and combining Lemmas 4.1 and 4.2 with Lemma A.4 and the observation that $V - W^\varepsilon$ lies in the span of the polynomial chaoses up to order four. ∎

4.4. *Convergence of correlation functions*

In this subsection we prove the second claim of Proposition 3.2, the convergence of the correlation functions in (3.4). In order to obtain the uniform convergence of the Wick-ordered correlation functions $(\widehat{\gamma}_p^{W^\varepsilon})_{\mathbf{x},\tilde{\mathbf{x}}}$ to $(\widehat{\gamma}_p^{V})_{\mathbf{x},\tilde{\mathbf{x}}}$, we use a representation obtained by repeated Gaussian integration by parts. To that end, we shall introduce a differential operator, denoted by $L_{N,x}$, such that

$$L_{N,x}\bar{\phi}(y) = G_N(x-y) \tag{4.16}$$

and hence, formally,

$$L_{N,x} = \int \mathrm{d}y\, G_N(x-y)\frac{\delta}{\delta\bar{\phi}(y)}. \tag{4.17}$$

Our argument may be viewed as an instance of Malliavin calculus, with $L_{N,x}$ playing the role of the Malliavin derivative.

We choose the regularizing function ϑ to have compact support. Recall (see (2.1)) that the underlying probability space consists of elements $X = (X_k)_{k\in\mathbb{Z}^d}$ with $X_k \in \mathbb{C}$. Define $\mathcal{T}$ to be the space of random variables of the form $f(X)$, where f is smooth in the sense that all of its partial derivatives exist. We denote by ∂_{X_k} and $\partial_{\bar{X}_k}$ the usual holomorphic and antiholomorphic partial derivatives in the complex variable X_k. On the space $\mathcal{T}$ we define the first order differential operators

$$L_{N,x} := \sum_{k\in\mathbb{Z}^d} \frac{1}{\sqrt{\lambda_k}}\sqrt{\vartheta(k/N)}\,u_k(x)\partial_{\bar{X}_k}, \quad \bar{L}_{N,x} := \sum_{k\in\mathbb{Z}^d} \frac{1}{\sqrt{\lambda_k}}\sqrt{\vartheta(k/N)}\,\bar{u}_k(x)\partial_{X_k}.$$

where $x \in \Lambda$ and $0 < N < \infty$. Here we recall the definitions (4.5). Note that, owing to our choice of ϑ, each sum is finite. That this definition indeed satisfies (4.16) is verified in Lemma 4.8 below.

We record a few simple properties of $L_{N,x}$. The first property is Gaussian integration by parts for the operator $L_{N,x}$.

Lemma 4.7. *Let $f(X) \in \mathcal{T} \cap L^1(\mathbb{P})$. Then*

$$\mathbb{E}[L_{N,x}f(X)] = \mathbb{E}[\phi_N(x)f(X)], \quad \mathbb{E}[\bar{L}_{N,x}f(X)] = \mathbb{E}[\bar{\phi}_N(x)f(X)].$$

Proof. We only prove the first identity. We use the fact that if Z is a standard complex Gaussian random variable, then $\mathbb{E}[Zf(Z)] = \mathbb{E}[\partial_{\bar{Z}}f(Z)]$, as can be seen by integration by parts. Thus, since each X_k is a standard complex Gaussian random variable independent of the others, we get

$$\begin{aligned}\mathbb{E}[L_{N,x}f(X)] &= \sum_{k\in\mathbb{Z}^d} \frac{1}{\sqrt{\lambda_k}}\sqrt{\vartheta(k/N)}\,u_k(x)\mathbb{E}[\partial_{\bar{X}_k}f(X)] \\ &= \sum_{k\in\mathbb{Z}^d} \frac{1}{\sqrt{\lambda_k}}\sqrt{\vartheta(k/N)}\,u_k(x)\mathbb{E}[X_k f(X)] \\ &= \mathbb{E}[\phi_N(x)f(X)].\end{aligned}$$

∎

The second property that we verify is condition (4.16).

Lemma 4.8. *For $x, y \in \Lambda$, we have*

$$L_{N,x}\,\phi(y) = 0, \quad L_{N,x}\bar{\phi}(y) = G_N(x-y)$$

in the sense of distributions in the variable y. Similar identities hold for $\bar{L}_{N,x}$.

Proof. We only prove the second identity. We compute

$$L_{N,x}\,\bar{\phi}(y) = \sum_{k\in\mathbb{Z}^d} \frac{1}{\sqrt{\lambda_k}}\vartheta(k/N)u_k(x)\frac{1}{\sqrt{\lambda_k}}\bar{u}_k(y) = G_N(x-y),$$

where the last step follows from (4.9). ∎

We may now prove the representation of (2.4) underlying our proof. We denote the regularized Wick-ordered correlation function by

$$(\hat{\gamma}^V_{N,p})_{\mathbf{x},\tilde{\mathbf{x}}} := \frac{1}{\zeta^V}\mathbb{E}[\mathrm{e}^{-V}{:}\bar{\phi}_N(\tilde{x}_1)\cdots\bar{\phi}_N(\tilde{x}_p)\,\phi_N(x_1)\cdots\phi_N(x_p){:}].$$

Lemma 4.9. *We have*

$$(\hat{\gamma}^V_{N,p})_{\mathbf{x},\tilde{\mathbf{x}}} = \frac{1}{\zeta^V}\mathbb{E}[\bar{L}_{N,\tilde{x}_1}\cdots\bar{L}_{N,\tilde{x}_p}L_{N,x_1}\cdots L_{N,x_p}\mathrm{e}^{-V}].$$

The same holds for V replaced with W^ε.

Proof. Using the recursive characterization of Wick ordering from (A.3), we find

$$\begin{aligned}\mathbb{E}\Big[\mathrm{e}^{-V}{:}\prod_{i\in[p]}\bar{\phi}_N(\tilde{x}_i)\prod_{i\in[p]}\phi_N(x_i){:}\Big] &= \mathbb{E}\Big[\mathrm{e}^{-V}{:}\prod_{i\in[p]}\bar{\phi}_N(\tilde{x}_i)\prod_{i\in[p-1]}\phi_N(x_i)\,{:}\,\phi_N(x_p)\Big]\\ &\quad - \sum_{j\in[p]}G_N(\tilde{x}_j-x_p)\mathbb{E}\Big[\mathrm{e}^{-V}{:}\prod_{i\in[p]\setminus\{j\}}\bar{\phi}_N(\tilde{x}_i)\prod_{i\in[p-1]}\phi_N(x_i){:}\Big].\end{aligned}$$

Here we have used (4.9) and $\mathbb{E}[\phi_N(x)\phi_N(y)] = 0$. Using Lemmas 4.7 and 4.8 as well as the Leibniz rule for the operator $L_{N,x}$, we write the first term on the right-hand side as

$$\begin{aligned}&\mathbb{E}\Big[L_{N,x_p}\Big(\mathrm{e}^{-V}{:}\prod_{i\in[p]}\bar{\phi}_N(\tilde{x}_i)\prod_{i\in[p-1]}\phi_N(x_i){:}\Big)\Big]\\ &\quad= \mathbb{E}\Big[L_{N,x_p}(\mathrm{e}^{-V}){:}\prod_{i\in[p]}\bar{\phi}_N(\tilde{x}_i)\prod_{i\in[p-1]}\phi_N(x_i){:}\Big]\\ &\qquad+\sum_{j\in[p]}G_N(\tilde{x}_j-x_p)\mathbb{E}\Big[\mathrm{e}^{-V}{:}\prod_{i\in[p]\setminus\{j\}}\bar{\phi}_N(\tilde{x}_i)\prod_{i\in[p-1]}\phi_N(x_i){:}\Big].\end{aligned}$$

We conclude that

$$\mathbb{E}\Big[\mathrm{e}^{-V}{:}\prod_{i\in[p]}\bar{\phi}_N(\tilde{x}_i)\prod_{i\in[p]}\phi_N(x_i){:}\Big] = \mathbb{E}\Big[L_{N,x_p}(\mathrm{e}^{-V}){:}\prod_{i\in[p]}\bar{\phi}_N(\tilde{x}_i)\prod_{i\in[p-1]}\phi_N(x_i){:}\Big].$$

Repeating this argument $2p$ times yields the claim. ∎

The following result is the main analytical tool behind the proof of (3.4).

Lemma 4.10. *Let $\ell \geqslant 0$ and $\mathbf{z} = (z_1, \dots, z_\ell) \in \Lambda^\ell$. Abbreviate*

$$\mathcal{L}_{N,\mathbf{z}} := L^{\#}_{N,z_1} \cdots L^{\#}_{N,z_\ell},$$

where each $L^{\#}$ stands for either L or $\bar{L}$. Then the following hold for any $r \geqslant 1$:

(i) $\sup_N \sup_{\mathbf{z}} \|\mathbb{L}_{N,\mathbf{z}} V\|_{L^r(\mathbb{P})} < \infty$.

(ii) *As $\varepsilon \to 0$ we have* $\sup_N \sup_{\mathbf{z}} \|\mathbb{L}_{N,\mathbf{z}} W^\varepsilon - \mathbb{L}_{N,\mathbf{z}} V\|_{L^r(\mathbb{P})} \to 0$.

(iii) *As $M, N \to \infty$ we have* $\sup_{\mathbf{z}} \|(\mathbb{L}_{N,\mathbf{z}} - \mathbb{L}_{M,\mathbf{z}}) V\|_{L^r(\mathbb{P})} \to 0$.

(iv) *For any $\varepsilon > 0$, as $M, N \to \infty$ we have* $\sup_{\mathbf{z}} \|(\mathbb{L}_{N,\mathbf{z}} - \mathbb{L}_{M,\mathbf{z}}) W^\varepsilon\|_{L^r(\mathbb{P})} \to 0$.

Before proving Lemma 4.10, we use it to conclude the proof of (3.4), and hence also of Proposition 3.2.

Proof of (3.4). We begin by showing that $(\hat{\gamma}^V_{N,p})_{N>0}$ is a Cauchy sequence in $\|\cdot\|_C$, and hence it converges to a limit that is by definition $\hat{\gamma}^V_p$. In the notation of Lemma 4.10, we deduce from Lemma 4.9 that

$$\zeta^V \big\| \hat{\gamma}^V_{N,p} - \hat{\gamma}^V_{M,p} \big\|_C = \sup_{\mathbf{z}} |\mathcal{E}[(\mathcal{L}_{N,\mathbf{z}} - \mathcal{L}_{M,\mathbf{z}}) \mathrm{e}^{-V}]|, \tag{4.18}$$

where we abbreviate $\mathbf{z} = (\mathbf{x}, \tilde{\mathbf{x}}) \in \Lambda^{2p}$ and choose the supercripts # in the definition of $\mathcal{L}_{N,\mathbf{z}}$ appropriately. Applying the chain rule and the Leibniz rule to the $2p$ derivatives in $\mathcal{L}_{N,\mathbf{z}}$ and $\mathcal{L}_{M,\mathbf{z}}$, we estimate the right-hand side by a finite number of terms of the form

$$\sup_{\mathbf{z}_1,\dots,\mathbf{z}_k} \mathcal{E}\Big| \Big(\prod_{i=1}^k (\mathcal{L}_{N,\mathbf{z}_i} V) - \prod_{i=1}^k (\mathcal{L}_{M,\mathbf{z}_i} V) \Big) \mathrm{e}^{-V} \Big|, \tag{4.19}$$

where $k \leqslant 2p$ and, for all $i = 1, \dots, k$, $\mathbf{z}_i \in \Lambda^{\ell_i}$ for some $\ell_i \leqslant 4$. Using the identity $\prod_i a_i - \prod_i b_i = \sum_i (a_i - b_i) \prod_{j<i} a_j \prod_{j>i} b_j$, we estimate (4.19) by

$$\sum_{i=1}^k \sup_{\mathbf{z}_1,\dots,\mathbf{z}_k} \mathcal{E}\Big| (\mathcal{L}_{N,\mathbf{z}_i} V - \mathcal{L}_{M,\mathbf{z}_i} V) \prod_{j<i} (\mathcal{L}_{N,\mathbf{z}_j} V) \prod_{j>i} (\mathcal{L}_{M,\mathbf{z}_j} V) \mathrm{e}^{-V} \Big|. \tag{4.20}$$

Applying Hölder's inequality, we conclude using Lemma 4.10 (i, iii), as well as Proposition 4.3, that (4.18) converges to 0 as $M, N \to \infty$. We conclude that $(\hat{\gamma}^V_{N,p})_{N>0}$ is a Cauchy sequence in $\|\cdot\|_C$.

The same argument with V replaced with W^ε, using Lemma 4.10 (iv), shows that, for any $\varepsilon > 0$, $\hat{\gamma}^{W^\varepsilon}_{N,p}$ converges in $\|\cdot\|_C$ to $\hat{\gamma}^{W^\varepsilon}_p$. Here, we also use the observation that $\sup_N \sup_{\mathbf{z}} \|\mathbb{L}_{N,\mathbf{z}} W^\varepsilon\|_{L^r(\mathbb{P})} < \infty$, which follows from Lemma 4.10 (i, ii).

Now writing

$$\begin{aligned} &\|\zeta^{W^\varepsilon} \hat{\gamma}^{W^\varepsilon}_p - \zeta^V \hat{\gamma}^V_p\|_C \\ &\qquad \leqslant \|\zeta^{W^\varepsilon} \hat{\gamma}^{W^\varepsilon}_p - \zeta^{W^\varepsilon} \hat{\gamma}^{W^\varepsilon}_{N,p}\|_C + \|\zeta^{W^\varepsilon} \hat{\gamma}^{W^\varepsilon}_{N,p} - \zeta^V \hat{\gamma}^V_{N,p}\|_C + \|\zeta^V \hat{\gamma}^V_{N,p} - \zeta^V \hat{\gamma}^V_p\|_C, \end{aligned}$$

we conclude, using $\lim_{\varepsilon\to 0}\zeta^{W^\varepsilon}=\zeta^V$ by Lemma 4.6, that the convergence (3.4) holds provided that

$$\lim_{\varepsilon\to 0}\sup_N \|\zeta^{W^\varepsilon}\widehat{\gamma}_{N,p}^{W^\varepsilon}-\zeta^V\widehat{\gamma}_{N,p}^{V}\|_C=0. \tag{4.21}$$

To prove (4.21), we write, using Lemma 4.9,

$$\|\zeta^{W^\varepsilon}\widehat{\gamma}_{N,p}^{W^\varepsilon}-\zeta^V\widehat{\gamma}_{N,p}^{V}\|_C=\sup_{\mathbf{z}}|\mathbb{E}[\mathcal{L}_{N,\mathbf{z}}(\mathrm{e}^{-W^\varepsilon}-\mathrm{e}^{V})]|.$$

Similarly to (4.18)–(4.19), we estimate the right-hand side by a sum of terms of the form

$$\sup_{\mathbf{z}_1,\dots,\mathbf{z}_k}\mathcal{E}\Big|\prod_{i=1}^k(\mathcal{L}_{N,\mathbf{z}_i}W^\varepsilon)\mathrm{e}^{-W^\varepsilon}-\prod_{i=1}^k(\mathcal{L}_{N,\mathbf{z}_i}V)\mathrm{e}^{-V}\Big|,$$

which converges to 0 as $\varepsilon\to 0$, uniformly in N, by telescoping (analogously to (4.20)), Hölder's inequality, as well as Lemma 4.10 (i, ii) and Proposition 4.6. This concludes the proof. ∎

Proof of Lemma 4.10. Since all quantities appearing inside the $L^r(\mathbb{P})$-norms in Lemma 4.10 are superpositions of random variables in polynomial chaoses of order at most four (see Section A.2), from the hypercontractivity estimate of Remark A.5 we find that it suffices to consider $r=2$.

The proof of (i) is similar to that of (ii), and we omit it. To prove (ii), we proceed by telescoping via V^ε, writing

$$W^\varepsilon-V=(W^\varepsilon-V^\varepsilon)+(V^\varepsilon-V).$$

Thus, we have to differentiate the random variables

$$W^\varepsilon-V^\varepsilon=\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,v^\varepsilon(x-\widetilde{x})G(x-\widetilde{x})\big(:\bar{\phi}(x)\phi(\widetilde{x}):-:|\phi(x)|^2:\big),$$
$$V^\varepsilon-V=\frac{1}{2}\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,\big(v^\varepsilon(x-\widetilde{x})-\delta(x-\widetilde{x})\big):|\phi(x)|^2\,|\phi(\widetilde{x})|^2:$$

by $\mathcal{L}_{N,\mathbf{v}}$.

The zeroth order derivatives, $\ell=0$, were estimated in Lemmas 4.1 and 4.2. For the higher order derivatives, let us start with $W^\varepsilon-V^\varepsilon$. For the first order derivative, from (A.4) and Lemma 4.8, we find

$$L_{N,z}(W^\varepsilon-V^\varepsilon)=\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,v^\varepsilon(x-\widetilde{x})G(x-\widetilde{x})G_N(z-x)\,(\phi(\widetilde{x})-\phi(x)),$$

so that Lemma A.1 (see also Example A.2) yields

$$\begin{aligned}\|L_{N,z}(W^\varepsilon-V^\varepsilon)\|^2_{L^2(\mathbb{P})}&\\ =\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,v^\varepsilon(x-\widetilde{x})G(x-\widetilde{x})G_N(z-x)v^\varepsilon(y-\widetilde{y})G(y-\widetilde{y})G_N(z-y)&\\ \times\big(G(x-y)+G(\widetilde{x}-\widetilde{y})-G(x-\widetilde{y})-G(\widetilde{x}-y)\big).&\end{aligned}$$

The right-hand side is estimated similarly to the proof of Lemma 4.1, uniformly in N and $z \in \Lambda$, using the fact that G_N is uniformly bounded in $L^2(\Lambda)$, and the bound

$$|G(x-y)+G(\widetilde{x}-\widetilde{y})-G(x-\widetilde{y})-G(\widetilde{x}-y)| \leqslant |G(x-y)-G(x-\widetilde{y})|+|G(\widetilde{x}-\widetilde{y})-G(\widetilde{x}-y)|.$$

For the second order derivative, we obtain

$$\bar{L}_{N,\widetilde{z}} L_{N,z}(W^\varepsilon - V^\varepsilon) = \int \mathrm{d}x\, \mathrm{d}\widetilde{x}\, v^\varepsilon(x-\widetilde{x}) G(x-\widetilde{x}) G_N(z-x)\big(G_N(\widetilde{z}-\widetilde{x})-G_N(\widetilde{z}-x)\big),$$

which can again be estimated as in the proof of Lemma 4.1, uniformly in N and $z, \widetilde{z} \in \Lambda$. This concludes the estimate of $\|\mathbb{L}_{N,\mathbf{z}} W^\varepsilon - \mathbb{L}_{N,\mathbf{z}} V^\varepsilon\|_{L^2(\mathbb{P})}$.

To estimate $\|\mathbb{L}_{N,\mathbf{z}} V^\varepsilon - \mathbb{L}_{N,\mathbf{z}} V\|_{L^2(\mathbb{P})}$, we compute, using (A.4) and Lemma 4.8,

$$L_{N,z}(V^\varepsilon - V) = \int \mathrm{d}x\, \mathrm{d}\widetilde{x}\, \big(v^\varepsilon(x-\widetilde{x})-\delta(x-\widetilde{x})\big) G_N(z-x) :\phi(x)\,|\phi(\widetilde{x})|^2:,$$

so that Lemma A.1 (see also Example A.2) yields

$$\begin{aligned}\|L_{N,z}(V^\varepsilon - V)\|^2_{L^2(\mathbb{P})} &= \int \mathrm{d}x\, \mathrm{d}\widetilde{x}\, \mathrm{d}y\, \mathrm{d}\widetilde{y}\, (v^\varepsilon(x-\widetilde{x})-\delta(x-\widetilde{x}))(v^\varepsilon(y-\widetilde{y})-\delta(y-\widetilde{y})) G_N(z-x) G_N(z-y) \\ &\quad \times \big(G(x-y)G(\widetilde{x}-\widetilde{y})^2 + G(x-\widetilde{y})G(\widetilde{x}-y)G(\widetilde{x}-\widetilde{y})\big).\end{aligned}$$

The right-hand side is estimated as in the proof of Lemma 4.2, uniformly in N and $z \in \Lambda$. The higher order derivatives are estimated analogously. This concludes the proof of (ii).

The proofs of (iii) and (iv) are similar, and we focus on (iii). For the first order derivative, we find

$$(L_{N,z} - L_{M,z})V = \int \mathrm{d}x\, \big(G_N(x-z)-G_M(x-z)\big) :\phi(x)|\phi(x)|^2:,$$

so that Lemma A.1 (see also Example A.2) yields

$$\|(L_{N,z}-L_{M,z})V\|^2_{L^2(\mathbb{P})} = 2\int \mathrm{d}x\, \mathrm{d}y\, \big(G_N(x-z)-G_M(x-z)\big)\big(G_N(y-z)-G_M(y-z)\big) G(x-y)^3.$$

Telescoping $G_N - G_M = (G_N - G) - (G_M - G)$ and using Lemma B.1 and Hölder's inequality, we conclude that

$$\|(L_{N,z}-L_{M,z})V\|_{L^2(\mathbb{P})} \lesssim \left(\int \mathrm{d}x\, |G_N(x)-G(x)|^3\right)^{1/3} + \left(\int \mathrm{d}x\, |G_M(x)-G(x)|^3\right)^{1/3}.$$

Using (B.2) and splitting the integration domain into $|x| \leqslant 1/N$ and $|x| > 1/N$, we easily deduce that the right-hand side vanishes as $N, M \to \infty$. The higher order derivatives are estimated in exactly the same way. This concludes the proof. ∎

5. Proof of Proposition 3.1

We study the rate of convergence of the relative partition function and the correlation functions in the mean-field limit, while keeping track of the parameter ε. This amounts to a quantitative analysis of the infinite-dimensional saddle point argument for the functional integral introduced in [33]. Note that, without the τ^ε and E^ε correction terms in (2.18), (3.1), and with $\varepsilon = 1$, this convergence was obtained in a qualitative way in [33, Section 5]. The main ingredients that enable a quantitative analysis are: (a) the Lipschitz continuity of the interaction potential v^ε, with Lipschitz constant depending on ε (see Lemma 5.5 (ii) below), and (b) quantitative L^p-Hölder continuity properties of Brownian motion (see Lemma 5.4 (ii)). As a result, we can find a suitable choice of ε as a function of ν such that we get the wanted convergence as $\nu \to 0$; see (5.1). Our methods work for $d = 2, 3$, and hence all results of this section are stated for both dimensions.

We now state the explicit lower bound on $\varepsilon \equiv \varepsilon(\nu)$. Namely, throughout we assume that

$$\varepsilon(\nu) \gtrsim \begin{cases} \exp\big(-(\log \nu^{-1})^{(1-a)/2}\big) & \text{if } d = 2, \\ (\log \nu^{-1})^{-(1-a)/2} & \text{if } d = 3, \end{cases} \tag{5.1}$$

for some $a \in (0, 1)$. Let us define $\chi : \mathbb{R}^+ \to \mathbb{R}$ by

$$\chi(t) := \begin{cases} \log t^{-1} & \text{if } d = 2, \\ t^{-1} & \text{if } d = 3. \end{cases} \tag{5.2}$$

By (2.12), (2.17), (5.2), Lemma B.1 (when $d = 2$), and Remark B.2 (when $d = 3$), we note that[6]

$$|\tau^\varepsilon| \lesssim_{\kappa,v} \chi(\varepsilon), \quad |E^\varepsilon| \lesssim_\kappa \chi(\varepsilon)^2. \tag{5.3}$$

Furthermore, by (5.2) and (5.1), it follows that for all $C, b > 0$, we have

$$\lim_{\nu \to 0} \mathrm{e}^{C\chi(\varepsilon)^2} \nu^b = 0. \tag{5.4}$$

We now state the main results which, in light of (5.4), imply Proposition 3.1.

Proposition 5.1. *There exists $C_1 > 0$ depending on κ, v such that*

$$\big|\mathcal{Z} - \zeta^{W^\varepsilon}\big| \lesssim_{\kappa,v} \begin{cases} \mathrm{e}^{C_1\chi(\varepsilon)^2} \nu^{1/4} & \text{if } d = 2, \\ \mathrm{e}^{C_1\chi(\varepsilon)^2} \nu^{1/4} \log \nu^{-1} & \text{if } d = 3. \end{cases}$$

In statements of results, we use the notation $b-$ to mean that the statement holds for $b - c$ for any constant $c > 0$.

[6] Throughout, we do not emphasize the dependence of the implied constants on $d = 2, 3$.

Proposition 5.2. *For $p \in \mathbb{N}$, we define*

$$\theta(d,p) := \begin{cases} \frac{1}{4p+4} & \text{if } d = 2, \\ \frac{1}{12p+8}- & \text{if } d = 3. \end{cases} \tag{5.5}$$

There exists $C_2 > 0$ depending on κ, v such that

$$\left\| v^p \, \widehat{\Gamma}_p - \widehat{\gamma}_p^{W^\varepsilon} \right\|_C \lesssim_{p,\kappa,v} \mathrm{e}^{C_2 \chi(\varepsilon)^2} v^{\theta(d,p)}.$$

Remark 5.3. We note that, in order to obtain Propositions 5.1 and 5.2 above, and hence Proposition 3.1, we only need to use the bounds from Lemma 5.5 below. In light of this observation, we can consider more general v, which are not smooth. We always assume that $v : \mathbb{R}^d \to \mathbb{R}$ is even, L^1 with integral 1, and of positive type. Since v is not smooth, we need to consider suitable regularizations v_η of v. For a motivation and detailed description of the regularization, we refer the reader to [33, Sections 3.1 and 4.1]. A summary is also given in the study of the nonlocal problem in Section 5.3 below.

In the first generalization, we consider v Lipschitz and compactly supported. Then the result of Lemma 5.5 holds for v_η^ε (given by (2.12) with v replaced by v_η), uniformly in η. This follows immediately from the proof of Lemma 5.5 below.

In the second generalization, we assume that v is differentiable and that uniformly in $\varepsilon \in (0,1)$, we have

$$\sup_{x \in \Lambda} \sum_{n \in \mathbb{Z}^d} \left| v\left(\frac{x-n}{\varepsilon} \right) \right| + \sup_{x \in \Lambda} \sum_{n \in \mathbb{Z}^d} \left| \nabla v\left(\frac{x-n}{\varepsilon} \right) \right| \lesssim 1. \tag{5.6}$$

Let us note that (5.6) holds if we assume that there exists $a > d$ such that for all $x \in \mathbb{R}^d$,

$$|v(x)| + |\nabla v(x)| \lesssim \frac{1}{(1+|x|)^a}.$$

In particular, under the latter conditions, we do not need to assume that v is compactly supported.

5.1. The partition function

In this subsection, we prove Proposition 5.1. Before proceeding with the proof, we make several observations and review the functional integral representation from [33]. See also [41]. We first recall some basic notions for Brownian paths. Given $0 \leqslant \widetilde{\tau} < \tau$, we denote by $\Omega^{\tau,\widetilde{\tau}}$ the space of continuous paths $\omega : [\widetilde{\tau}, \tau] \to \Lambda$. Given $\widetilde{x} \in \Lambda$ and $0 \leqslant \widetilde{\tau} < \tau$, $\mathbb{P}_{\widetilde{x}}^{\tau,\widetilde{\tau}}(\mathrm{d}\omega)$ denotes the law on $\Omega^{\tau,\widetilde{\tau}}$ of standard Brownian motion with periodic boundary conditions on Λ that equals $\widetilde{x}$ at time $\widetilde{\tau}$. Given $x, \widetilde{x} \in \Lambda$ and $0 \leqslant \widetilde{\tau} < \tau$, $\mathbb{P}_{x,\widetilde{x}}^{\tau,\widetilde{\tau}}(\mathrm{d}\omega)$ denotes the law of the Brownian bridge $\Omega^{\tau,\widetilde{\tau}}$ with periodic boundary conditions on Λ that equals $\widetilde{x}$ at time $\widetilde{\tau}$ and x at time τ. For $t > 0$, we write the heat kernel on Λ as

$$\psi^t(x) := \mathrm{e}^{t\Delta/2}(x) = \sum_{n \in \mathbb{Z}^d} \frac{1}{(2\pi t)^{d/2}} \mathrm{e}^{-\frac{|x-n|^2}{2t}}.$$

For $x, \widetilde{x} \in \Lambda$ and $0 \leqslant \widetilde{\tau} < \tau$, we define the positive measure

$$\mathbb{W}_{x,\widetilde{x}}^{\tau,\widetilde{\tau}}(\mathrm{d}\omega) := \psi^{\tau-\widetilde{\tau}}(x-\widetilde{x})\, \mathbb{P}_{x,\widetilde{x}}^{\tau,\widetilde{\tau}}(\mathrm{d}\omega). \tag{5.7}$$

Given $n \in \mathbb{N}^*$, $\widetilde{\tau} < t_1 < \cdots < t_n < \tau$, and $f : \Lambda^n \to \mathbb{R}$ continuous, the measure (5.7) satisfies

$$\begin{aligned}\int \mathbb{W}_{x,\widetilde{x}}^{\tau,\widetilde{\tau}}(\mathrm{d}\omega)\, f(\omega(t_1), \dots, \omega(t_n)) \\ = \int \mathrm{d}x_1 \cdots \mathrm{d}x_n\, \psi^{t_1-\widetilde{\tau}}(x_1-\widetilde{x})\, \psi^{t_2-t_1}(x_2-x_1) \cdots \psi^{t_n-t_{n-1}}(x_n-x_{n-1}) \\ \times \psi^{\tau-t_n}(x-x_n)\, f(x_1, \dots, x_n).\end{aligned}$$

We note several useful estimates for the above quantities.

Lemma 5.4. *The following estimates hold:*

(i) *There exists a constant $C > 0$ such that for all $0 \leqslant \widetilde{\tau} < \tau$, we have*

$$\sup_{x,\widetilde{x}} \int \mathbb{W}_{x,\widetilde{x}}^{\tau,\widetilde{\tau}}(\mathrm{d}\omega) = \sup_{x,\widetilde{x}} \psi^{\tau,\widetilde{\tau}}(x-\widetilde{x}) \leqslant C\left(1 + \frac{1}{(\tau-\widetilde{\tau})^{d/2}}\right).$$

(ii) *There exists a constant $C > 0$ such that for all $\widetilde{\tau} \leqslant s \leqslant t \leqslant \tau$, we have*

$$\begin{aligned}\int \mathbb{W}_{x,\widetilde{x}}^{\tau,\widetilde{\tau}}(\mathrm{d}\omega)\, |\omega(t)-\omega(s)|_\Lambda^2 \\ \leqslant C\left(1 + \frac{1}{(\tau-\widetilde{\tau})^{d/2}}\right)\left((t-s) + |x-\widetilde{x}|_\Lambda^2\, \frac{(t-s)^2}{(\tau-\widetilde{\tau})^2}\right).\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}\int \mathbb{W}_{x,\widetilde{x}}^{\tau,\widetilde{\tau}}(\mathrm{d}\omega)\, |\omega(t)-\omega(s)|_\Lambda \\ \leqslant C\left(1 + \frac{1}{(\tau-\widetilde{\tau})^{d/2}}\right)\left((t-s)^{1/2} + |x-\widetilde{x}|_\Lambda\, \frac{t-s}{\tau-\widetilde{\tau}}\right).\end{aligned} \tag{5.9}$$

Here $|x|_\Lambda := \min_{n\in\mathbb{Z}^d} |x-n|$ denotes the periodic Euclidean norm of $x \in \Lambda$.

(iii) *For $0 < s \leqslant t$, we have*

$$\|\psi^t - \psi^s\|_{L^1(\Lambda)} \leqslant d\, \log(t/s).$$

The results of Lemma 5.4 are contained in [33]. Part (i) is given in [33, Lemma 2.2]. Estimate (5.8) is proved in [33, Lemma 2.3]. Estimate (5.9) then follows from the Cauchy–Schwarz inequality and part (i). Part (iii) follows since

$$\|\psi^t - \psi^s\|_{L^1} \leqslant \int_s^t \mathrm{d}u\, \|\partial_u \psi^u\|_{L^1} \leqslant \int_s^t \mathrm{d}u \int_{\mathbb{R}^d} \mathrm{d}x \left(\frac{d}{2u} + \frac{|x|^2}{2u^2}\right) \widetilde{\psi}^u(x) = d\, \log(t/s),$$

where $\widetilde{\psi}^u(x) = \frac{1}{(2\pi u)^{d/2}} \mathrm{e}^{-\frac{|x|^2}{2u}}$ is the heat kernel on $\mathbb{R}^d$, as was noted in the proof of [33, Lemma 5.18].

Let us fix a function $\varphi \in C_c^\infty(\mathbb{R})$ which is even, nonnegative, of positive type, and which satisfies $\varphi(0) = 1$. For fixed $L > 0$, given $\eta > 0$, we define the ν-periodic function

$$\delta_\eta(\tau) := \frac{1}{\eta} \sum_{y \in \mathbb{Z}} (\mathfrak{F}^{-1}\varphi)\left(\frac{\tau - \nu y}{\eta}\right), \tag{5.10}$$

where $\mathfrak{F}$ denotes Fourier transform (see (4.6)). Here, (5.10) can be interpreted as an approximate delta function on $[-\nu/2, \nu/2)$. By construction, we have

$$\int_0^\nu \mathrm{d}\tau\, \delta_\eta(\tau) = 1, \quad \delta_\eta \geqslant 0. \tag{5.11}$$

For simplicity of notation, we suppress the dependence on ε and ν in the quantum objects. We only emphasize the η dependence through a subscript when appropriate. We write the ε dependence as a superscript in the classical objects. Let us note several properties of v^ε that follow from (2.12).

Lemma 5.5. *There exists $C > 0$, depending only on v, such that the following properties hold:*

(i) $\|v^\varepsilon\|_{L^\infty(\Lambda)} \leqslant C/\varepsilon^d$.

(ii) *We have*

$$|v^\varepsilon(x) - v^\varepsilon(y)| \leqslant \frac{C}{\varepsilon^{d+1}} |x - y|_\Lambda \quad \textit{for all } x, y \in \Lambda.$$

For $\eta > 0$, recalling (5.10), we let

$$(\mathcal{C}_\eta)^{\tau,\widetilde{\tau}}_{x,\widetilde{x}} := \nu \delta_\eta(\tau - \widetilde{\tau}) v^\varepsilon(x - \widetilde{x})$$

and define $\mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma)$ to be the real Gaussian measure with mean zero and covariance

$$\int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma)\, \sigma(\tau, x)\sigma(\widetilde{\tau}, \widetilde{x}) = (\mathcal{C}_\eta)^{\tau,\widetilde{\tau}}_{x,\widetilde{x}}. \tag{5.12}$$

Since $v \in C_c^\infty(\mathbb{R}^d)$, under the law $\mu_{\mathcal{C}_\eta}$, σ is almost surely a smooth periodic function on $[0, \nu] \times \Lambda$. Let us note that we can rewrite (2.18) as

$$H = \bigoplus_{n \in \mathbb{N}} (H_{n,\varrho^\varepsilon} - (\tau^\varepsilon)^2/2 - E^\varepsilon), \tag{5.13}$$

where

$$H_{n,\varrho} := \left[\nu \sum_{i=1}^n (\kappa - \Delta/2)_i + \frac{\nu^2}{2} \sum_{i,j=1}^n v^\varepsilon(x_i - x_j)\right] - \varrho\nu^2 n + \frac{\varrho^2 \nu^2}{2} \tag{5.14}$$

and

$$\varrho^\varepsilon := \varrho_\nu/\nu + \tau^\varepsilon/\nu,$$

where we recall the definition (2.15) of ϱ_ν. Namely, from (2.18), using notation as in (5.14) as well as $\widehat{v}_\varepsilon(0) = 0$, it follows that H acts on the nth sector of Fock space as

$$\left[\nu\sum_{i=1}^{n}(\kappa-\Delta/2)_i+\frac{\nu^2}{2}\sum_{i,j=1}^{n}v_\varepsilon(x_i-x_j)\right]-\tau_\varepsilon(n\nu-\varrho_\nu)-\nu\varrho_\nu\widehat{v}_\varepsilon(0)n+\frac{1}{2}\widehat{v}_\varepsilon(0)\varrho_\nu^2-E^\varepsilon$$
$$=H_{n,0}-\left(\frac{\varrho_\nu}{\nu}+\frac{\tau^\varepsilon}{\nu}\right)\nu^2 n+\left(\frac{\varrho_\nu}{\nu}+\frac{\tau^\varepsilon}{\nu}\right)^2\frac{\nu^2}{2}-\frac{(\tau^\varepsilon)^2}{2}-E^\varepsilon,$$

which gives us (5.13). We denote

$$T^\varepsilon := (\tau^\varepsilon)^2/2 + E^\varepsilon. \tag{5.15}$$

In particular, from (5.15) and (5.3), it follows that

$$|T^\varepsilon| \lesssim_{\kappa,v} \chi(\varepsilon)^2. \tag{5.16}$$

In Lemmas 5.6 and 5.7 below, we give the functional integral representation of the (quantum) relative partition function (2.14) and the (classical) relative partition function ζ^{W^ε} corresponding to the interaction W^ε given by (3.1). In both cases, the representation is based on the use of the Hubbard–Stratonovich transformation [33].

Lemma 5.6. *The relative partition function* (2.14) *can be written as* $\mathcal{Z} = \lim_{\eta\to 0}\mathcal{Z}_\eta$, *where*

$$\mathcal{Z}_\eta := \int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma)\,\mathrm{e}^{T^\varepsilon-\frac{\mathrm{i}\tau^\varepsilon[\sigma]}{\nu}}\,\mathrm{e}^{F_2(\sigma)}, \tag{5.17}$$

for

$$F_2(\sigma) = -\sum_{\mathbf{r}\in(\nu\mathbb{N})^3}\frac{\mathbf{1}_{|\mathbf{r}|>0}\mathrm{e}^{-\kappa|\mathbf{r}|}}{|\mathbf{r}|}\int_{[0,\nu]^3}\mathrm{d}\boldsymbol{\tau}\int\mathrm{d}\mathbf{x}\,\sigma(\tau_2,x_2)\,\sigma(\tau_3,x_3)$$
$$\times\int\mathbb{W}_{x_1,x_3}^{\tau_1+r_3,\tau_3}(\mathrm{d}\omega_3)\,\mathbb{W}_{x_3,x_2}^{\tau_3+r_2,\tau_2}(\mathrm{d}\omega_2)\,\mathbb{W}_{x_2,x_1}^{\tau_2+r_1,\tau_1}(\mathrm{d}\omega_1)\,\mathrm{e}^{\mathrm{i}\int\mathrm{d}s\,\sigma([s]_\nu,\omega_1(s))}, \tag{5.18}$$

which satisfies

$$\operatorname{Re}F_2 \leqslant 0. \tag{5.19}$$

In (5.17), *we write*

$$[\sigma] := \int_0^\nu \mathrm{d}\tau\int\mathrm{d}x\,\sigma(\tau,x) \tag{5.20}$$

and in (5.18), *we write*

$$[t]_\nu := (t \bmod \nu) \in [0,\nu). \tag{5.21}$$

Proof. The proof follows from that of [33, Proposition 3.12] by setting

$$f(\tau,x) = \sum_{i=1}^{n}\delta(x-\omega_i(\tau)) - \frac{\varrho_\nu}{\nu} - \frac{\tau^\varepsilon}{\nu}. \tag{5.22}$$

Note that the only difference then is the factor $\mathrm{e}^{T^\varepsilon}$ coming from the constant term (5.15) in (5.13) and the extra term $-\tau^\varepsilon/\nu$ in (5.22) above. By applying the Hubbard–Stratonovich transformation

$$\mathrm{e}^{-\frac{1}{2}\langle f,\mathcal{C}_\eta f\rangle} = \int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma)\,\mathrm{e}^{\mathrm{i}\langle f,\sigma\rangle},$$

this new term adds a factor of

$$\mathrm{e}^{-\frac{\mathrm{i}\tau^\varepsilon}{\nu}\int_0^\nu \mathrm{d}\tau\int \mathrm{d}x\,\sigma(\tau,x)} = \mathrm{e}^{-\frac{\mathrm{i}\tau^\varepsilon[\sigma]}{\nu}}$$

to the integrand of the functional integral (5.17). The formula (5.18) is given in [33, Lemma 5.4]. ∎

We now explain how to obtain the functional integral representation for the classical partition function in our setting. We define $\mu_{v^\varepsilon}(\mathrm{d}\xi)$ to be the real Gaussian measure with covariance

$$\int \mu_{v^\varepsilon}(\mathrm{d}\xi)\,\xi(x)\xi(\widetilde{x}) = v^\varepsilon(x-\widetilde{x}). \tag{5.23}$$

Since $v\in C_c^\infty(\mathbb{R}^d)$, under the law μ_{v^ε}, ξ is almost surely a smooth periodic function on Λ. We have the following representation.

Lemma 5.7. *We have*

$$\zeta^{W^\varepsilon} = \int \mu_{v^\varepsilon}(\mathrm{d}\xi)\,\mathrm{e}^{T^\varepsilon-\mathrm{i}\tau^\varepsilon\langle\xi,1\rangle_{L^2}}\,\mathrm{e}^{f_2(\xi)} \tag{5.24}$$

for

$$\begin{aligned} f_2(\xi) = -\int_{[0,\infty)^3}\mathrm{d}\mathbf{r}\,\frac{\mathrm{e}^{-\kappa|\mathbf{r}|}}{|\mathbf{r}|}\int \mathrm{d}\mathbf{x}\,\xi(x_2)\xi(x_3) \\ \times\int \mathbb{W}^{r_3,0}_{x_1,x_3}(\mathrm{d}\omega_3)\,\mathbb{W}^{r_2,0}_{x_3,x_2}(\mathrm{d}\omega_2)\,\mathbb{W}^{r_1,0}_{x_2,x_1}(\mathrm{d}\omega_1)\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}s\,\xi(\omega_1(s))}, \end{aligned} \tag{5.25}$$

which satisfies

$$\operatorname{Re} f_2 \leqslant 0. \tag{5.26}$$

Note that in (5.24), $\langle\xi,1\rangle_{L^2} = \int \mathrm{d}x\,\xi(x)$ *denotes the* L^2 *inner product.*

Proof. We note that for fixed $0<N<\infty$, we have

$$\begin{aligned} W_N^\varepsilon &:= \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,\big(|\phi_N(x)|^2-\mathbb{E}[|\phi_N(x)|^2]\big)v(x-\widetilde{x})\big(|\phi_N(\widetilde{x})|^2-\mathbb{E}[|\phi_N(\widetilde{x})|^2]\big) \\ &\quad -\tau^\varepsilon\int \mathrm{d}x\,\big(|\phi_N(x)|^2-\mathbb{E}[|\phi_N(x)|^2]\big)-E^\varepsilon \\ &= \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,\big(|\phi_N(x)|^2-\{\mathbb{E}[|\phi_N(x)|^2]+\tau^\varepsilon\}\big)v(x-\widetilde{x}) \\ &\qquad\qquad \times\big(|\phi_N(\widetilde{x})|^2-\{\mathbb{E}[|\phi_N(\widetilde{x})|^2]+\tau^\varepsilon\}\big)-T^\varepsilon. \end{aligned} \tag{5.27}$$

Here, we recall (5.15). The claim then follows by using (5.27) in the proof of [33, Proposition 4.1]. In particular, by applying the Hubbard–Stratonovich transformation

$$\mathrm{e}^{-\frac{1}{2}\langle f, v^\varepsilon f\rangle} = \int \mu_{v^\varepsilon}(\mathrm{d}\xi)\, \mathrm{e}^{\mathrm{i}\langle f,\xi\rangle}$$

for

$$f(x) = |\phi_N(x)|^2 - \mathbb{E}[|\phi_N(x)|^2] + \tau^\varepsilon,$$

the τ^ε terms in (5.27) add a factor of $\mathrm{e}^{-\mathrm{i}\tau^\varepsilon\langle\xi,1\rangle_{L^2}}$ to the integrand of the functional integral (5.24). Here, we also recall (5.15). Formula (5.25) is given in [33, Lemma 5.4]. ∎

Recalling (5.11), let us note the following useful result.

Lemma 5.8. *If $\sigma = \sigma(\tau, x)$ has law $\mu_{\mathcal{C}_\eta}$ with covariance (5.12), then its time average*

$$\langle\sigma\rangle(x) := \frac{1}{\nu}\int_0^\nu \mathrm{d}\tau\, \sigma(\tau, x) \tag{5.28}$$

has law μ_{v^ε} with covariance (5.23).

We observe that

$$\langle\langle\sigma\rangle, 1\rangle_{L^2} = \frac{1}{\nu}[\sigma] \tag{5.29}$$

for $[\sigma]$ as in (5.20). Therefore, we can rewrite (5.24) as

$$\zeta^{W^\varepsilon} = \int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma)\, \mathrm{e}^{T^\varepsilon - \frac{\mathrm{i}\tau^\varepsilon[\sigma]}{\nu}} \mathrm{e}^{f_2(\langle\sigma\rangle)}. \tag{5.30}$$

We note the following result.

Lemma 5.9. *Uniformly in $\eta > 0$, we have*

$$\big|\mathcal{Z}_\eta - \zeta^{W^\varepsilon}\big| \leqslant \mathrm{e}^{T^\varepsilon}\left(\int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma)\, |F_2(\sigma) - f_2(\langle\sigma\rangle)|^2\right)^{1/2}.$$

Proof. By (5.17) and (5.30), we have

$$\mathcal{Z}_\eta - \zeta^{W^\varepsilon} = \int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma)\, \mathrm{e}^{T^\varepsilon - \frac{\mathrm{i}\tau^\varepsilon[\sigma]}{\nu}} (\mathrm{e}^{F_2(\sigma)} - \mathrm{e}^{f_2(\langle\sigma\rangle)}). \tag{5.31}$$

By (5.19) and (5.26) and the elementary inequality $|\mathrm{e}^a - \mathrm{e}^b| \leqslant |a - b|$ for $a, b \in \mathbb{C}$ with $\operatorname{Re} a, \operatorname{Re} b \leqslant 0$, we see that for all σ,

$$|\mathrm{e}^{F_2(\sigma)} - \mathrm{e}^{f_2(\langle\sigma\rangle)}| \leqslant \big|F_2(\sigma) - f_2(\langle\sigma\rangle)\big|. \tag{5.32}$$

The claim follows from (5.31), (5.32), and the Cauchy–Schwarz inequality. ∎

In order to simplify notation, we define the function $\Theta : \mathbb{R}^+ \to \mathbb{R}$ by

$$\Theta(t) := \begin{cases} \sqrt{t} & \text{if } d = 2, \\ \sqrt{t}\log t^{-1} & \text{if } d = 3. \end{cases} \tag{5.33}$$

Note that the upper bound in Proposition 5.1 can then be rewritten as $e^{C_1\chi(\varepsilon)^2}\Theta(v)^{1/2}$, with χ given by (5.2). We prove the following two estimates, which correspond to quantitative versions of [33, Lemmas 5.9 and 5.10] respectively, and in turn let us use the bound from Lemma 5.9 to prove Proposition 5.1.

Lemma 5.10. *Uniformly in $\eta > 0$, we have*

$$\left| \int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma)\, \overline{F_2(\sigma)} F_2(\sigma) - \int \mu_{v^\varepsilon}(\mathrm{d}\xi)\, \overline{f_2(\xi)} f_2(\xi) \right| \lesssim_{\kappa,v} \frac{\Theta(v)}{\varepsilon^{5d+1}}.$$

Lemma 5.11. *Uniformly in $\eta > 0$, we have*

$$\left| \int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma)\, \overline{f_2(\langle\sigma\rangle)} F_2(\sigma) - \int \mu_{v^\varepsilon}(\mathrm{d}\xi)\, \overline{f_2(\xi)} f_2(\xi) \right| \lesssim_{\kappa,v} \frac{\Theta(v)}{\varepsilon^{5d+1}}.$$

Assuming Lemmas 5.10 and 5.11 for now, we can prove the convergence rate given in Proposition 5.1.

Proof of Proposition 5.1. This follows from Lemmas 5.6–5.11 by recalling (5.16). ∎

The rest of this section is devoted to showing Lemmas 5.10 and 5.11. Before proceeding with the proofs, we need to introduce some notation and definitions. Throughout we use the convention that, given a path $\omega \in \Omega^{\tau,\widetilde{\tau}}$ and a function f, we write

$$\int \mathrm{d}s\, f(\omega(s)) \equiv \int_{\widetilde{\tau}}^{\tau} \mathrm{d}s\, f(\omega(s)).$$

We define the following quantities that will allow us to rewrite the terms that arise.

Definition 5.12 (Classical interactions). Let $x, \widetilde{x} \in \Lambda$ and $\omega \in \Omega^{\tau_1,\widetilde{\tau}_1}, \widetilde{\omega} \in \Omega^{\tau_2,\widetilde{\tau}_2}$ be continuous paths. We then define the *point-point interaction*

$$(\mathbb{V}^\varepsilon)_{x,\widetilde{x}} := \int \mu_{v^\varepsilon}(\mathrm{d}\xi)\, \xi(x)\xi(\widetilde{x}) = v^\varepsilon(x - \widetilde{x}),$$

the *point-path interaction*

$$(\mathbb{V}^\varepsilon)_x(\omega) := \int \mu_{v^\varepsilon}(\mathrm{d}\xi) \int \mathrm{d}s\, \xi(\omega(s)) = \int \mathrm{d}s\, v^\varepsilon(x - \omega(s)),$$

and the *path-path interaction*

$$\mathbb{V}^\varepsilon(\omega, \widetilde{\omega}) := \int \mu_{v^\varepsilon}(\mathrm{d}\xi) \int \mathrm{d}s\, \xi(\omega(s)) \int \mathrm{d}\widetilde{s}\, \xi(\widetilde{\omega}(\widetilde{s})) = \int \mathrm{d}s \int \mathrm{d}\widetilde{s}\, v^\varepsilon(\omega(s) - \widetilde{\omega}(\widetilde{s})).$$

In what follows, we use the notation $x_{i,0} \equiv x_i, \widetilde{x}_{i,1} \equiv \widetilde{x}_i$ for $i = 1, 2$ and we write

$$A := \{2, 3\} \times \{0, 1\}. \tag{5.34}$$

Arguing analogously to [33, (5.8)–(5.9)], we get

$$\int \mu_{v^\varepsilon}(\mathrm{d}\xi)\, \overline{f_2(\xi)} f_2(\xi) = \int_{[0,\infty)^3} \mathrm{d}\mathbf{r}\, \frac{\mathrm{e}^{-\kappa|\mathbf{r}|}}{|\mathbf{r}|} \int_{[0,\infty)^3} \mathrm{d}\widetilde{\mathbf{r}}\, \frac{\mathrm{e}^{-\kappa|\widetilde{\mathbf{r}}|}}{|\widetilde{\mathbf{r}}|} I^\varepsilon(\mathbf{r}, \widetilde{\mathbf{r}}), \tag{5.35}$$

where

$$I^\varepsilon(\mathbf{r},\tilde{\mathbf{r}}) := \int \mathrm{d}\mathbf{x} \int \mathrm{d}\tilde{\mathbf{x}} \int \mathbb{W}^{r_3,0}_{x_1,x_3}(\mathrm{d}\omega_3)\, \mathbb{W}^{r_2,0}_{x_3,x_2}(\mathrm{d}\omega_2)\, \mathbb{W}^{r_1,0}_{x_2,x_1}(\mathrm{d}\omega_1)$$
$$\times\, \mathbb{W}^{\tilde{r}_3,0}_{\tilde{x}_1,\tilde{x}_3}(\mathrm{d}\tilde{\omega}_3)\, \mathbb{W}^{\tilde{r}_2,0}_{\tilde{x}_3,\tilde{x}_2}(\mathrm{d}\tilde{\omega}_2)\, \mathbb{W}^{\tilde{r}_1,0}_{\tilde{x}_2,\tilde{x}_1}(\mathrm{d}\tilde{\omega}_1)\, \mathrm{e}^{-\frac{1}{2}(\mathbb{V}^\varepsilon(\omega_1,\omega_1)+\mathbb{V}^\varepsilon(\tilde{\omega}_1,\tilde{\omega}_1)-2\mathbb{V}^\varepsilon(\omega_1,\tilde{\omega}_1))}$$
$$\times \sum_{\Pi\in\mathbb{M}(A)} \prod_{\{a,b\}\in\Pi} (\mathbb{V}^\varepsilon)_{x_a,x_b} \prod_{a\in A\setminus[\Pi]} \mathrm{i}\big((\mathbb{V}^\varepsilon)_{x_a}(\omega_1) - (\mathbb{V}^\varepsilon)_{x_a}(\tilde{\omega}_1)\big). \tag{5.36}$$

Here, we write $|\mathbf{r}| = r_1 + r_2 + r_3$, $|\tilde{\mathbf{r}}| = \tilde{r}_1 + \tilde{r}_2 + \tilde{r}_3$. Moreover, we denote by $\mathfrak{M}(A)$ the set of partial pairings on the set A.

Definition 5.13 (Quantum interactions). Let $(\tau, x), (\tilde{\tau}, \tilde{x}) \in [0,\nu] \times \Lambda$ and let $\omega \in \Omega^{\tau_1,\tilde{\tau}_1}$ and $\tilde{\omega} \in \Omega^{\tau_2,\tilde{\tau}_2}$ be continuous paths. With δ_η given by (5.10), we define the *point-point interaction*

$$(\mathbb{V}_\eta)^{\tau,\tilde{\tau}}_{x,\tilde{x}} := \int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma)\, \sigma(\tau,x)\sigma(\tilde{\tau},\tilde{x}) = \nu\delta_\eta(\tau-\tilde{\tau})v^\varepsilon(x-\tilde{x}),$$

the *point-path interaction*

$$(\mathbb{V}_\eta)^\tau_x(\omega) := \int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma)\, \sigma(\tau,x) \int_0^\nu \mathrm{d}t \int \mathrm{d}s\, \delta(t-[s]_\nu)\, \sigma(t,\omega(s))$$
$$= \nu \int \mathrm{d}s\, \delta_\eta(\tau-[s]_\nu)\, v^\varepsilon(x-\omega(s)),$$

and the *path-path interaction*

$$\mathbb{V}_\eta(\omega,\tilde{\omega}) := \int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma) \int_0^\nu \mathrm{d}t \int \mathrm{d}s\, \sigma(\tau,x)\, \delta(t-[s]_\nu)\, \sigma(t,\omega(s))$$
$$\times \int_0^\nu \mathrm{d}\tilde{t} \int \mathrm{d}\tilde{s}\, \delta(\tilde{t}-[\tilde{s}]_\nu)\sigma(\tilde{t},\tilde{\omega}(\tilde{s}))$$
$$= \nu \int \mathrm{d}s \int \mathrm{d}\tilde{s}\, \delta_\eta([s]_\nu - [\tilde{s}]_\nu)v^\varepsilon(\omega(s)-\tilde{\omega}(\tilde{s})).$$

Here, we recall (5.21).

Arguing analogously to [33, (5.17)–(5.18), (5.21)], we have

$$\int \mu_{\mathbb{C}_\eta}(\mathrm{d}\sigma)\, \overline{F_2(\sigma)} F_2(\sigma) = \sum_{\mathbf{r}\in(\nu\mathbb{N})^3} \frac{\mathbf{1}_{|\mathbf{r}|>0}\, \mathrm{e}^{-\kappa|\mathbf{r}|}}{|\mathbf{r}|} \sum_{\tilde{\mathbf{r}}\in(\nu\mathbb{N})^3} \frac{\mathbf{1}_{|\tilde{\mathbf{r}}|>0}\, \mathrm{e}^{-\kappa|\tilde{\mathbf{r}}|}}{|\tilde{\mathbf{r}}|}\, J(\mathbf{r},\tilde{\mathbf{r}}), \tag{5.37}$$

where

$$J(\mathbf{r},\tilde{\mathbf{r}})$$
$$:= \int_{[0,\nu]^3} \mathrm{d}\boldsymbol{\tau} \int_{[0,\nu]^3} \mathrm{d}\tilde{\boldsymbol{\tau}} \int \mathrm{d}\mathbf{x} \int \mathrm{d}\tilde{\mathbf{x}} \int \mathbb{W}^{\tau_1+r_3,\tau_3}_{x_1,x_3}(\mathrm{d}\omega_3)\, \mathbb{W}^{\tau_3+r_2,\tau_2}_{x_3,x_2}(\mathrm{d}\omega_2)\, \mathbb{W}^{\tau_2+r_1,\tau_1}_{x_2,x_1}(\mathrm{d}\omega_1)$$
$$\times\, \mathbb{W}^{\tilde{\tau}_1+\tilde{r}_3,\tilde{\tau}_3}_{\tilde{x}_1,\tilde{x}_3}(\mathrm{d}\tilde{\omega}_3)\, \mathbb{W}^{\tilde{\tau}_3+\tilde{r}_2,\tilde{\tau}_2}_{\tilde{x}_3,\tilde{x}_2}(\mathrm{d}\tilde{\omega}_2)\, \mathbb{W}^{\tilde{\tau}_2+\tilde{r}_1,\tilde{\tau}_1}_{\tilde{x}_2,\tilde{x}_1}(\mathrm{d}\tilde{\omega}_1)\, \mathrm{e}^{-\frac{1}{2}(\mathbb{V}_\eta(\omega_1,\omega_1)+\mathbb{V}_\eta(\tilde{\omega}_1,\tilde{\omega}_1)-2\mathbb{V}_\eta(\omega_1,\tilde{\omega}_1))}$$
$$\times \sum_{\Pi\in\mathbb{M}(A)} \prod_{\{a,b\}\in\Pi} (\mathbb{V}_\eta)^{\tau_a,\tau_b}_{x_a,x_b} \prod_{a\in A\setminus[\Pi]} \mathrm{i}\big((\mathbb{V}_\eta)^{\tau_a}_{x_a}(\omega_1) - (\mathbb{V}_\eta)^{\tau_a}_{x_a}(\tilde{\omega}_1)\big). \tag{5.38}$$

The first step in the proof of Lemma 5.10 is to compare (5.38) with $\nu^6 I^\varepsilon(\mathbf{r},\widetilde{\mathbf{r}})$, which appears in a Riemann sum of mesh size ν for (5.35). We show the following quantitative estimate.

Lemma 5.14 (Approximation of $J(\mathbf{r},\widetilde{\mathbf{r}})$). *For all $\mathbf{r},\widetilde{\mathbf{r}} \in (\nu\mathbb{N})^3$ with $|\mathbf{r}|, |\widetilde{\mathbf{r}}| > 0$, we have, uniformly in $\eta > 0$,*

$$J(\mathbf{r},\widetilde{\mathbf{r}}) = \nu^6 I^\varepsilon(\mathbf{r},\widetilde{\mathbf{r}}) + O\bigg(\frac{\nu^{13/2}}{\varepsilon^{5d+1}}\bigg(1+\frac{1}{|\mathbf{r}|^{d/2}}\bigg)\bigg(1+\frac{1}{|\widetilde{\mathbf{r}}|^{d/2}}\bigg)(1+|\mathbf{r}|+|\widetilde{\mathbf{r}}|)^6\bigg). \tag{5.39}$$

The second step in the proof of Lemma 5.10 consists in giving a quantitative estimate on the error obtained by approximating the integral (5.35) with the above Riemann sum. To this end, we prove the following estimate.

Lemma 5.15 (Quantitative Riemann sum approximation for (5.35)). *Recalling (5.33), we have, uniformly in $\eta > 0$,*

$$\bigg|\int_{[0,\infty)^3} \mathrm{d}\mathbf{r}\,\frac{\mathrm{e}^{-\kappa|\mathbf{r}|}}{|\mathbf{r}|}\int_{[0,\infty)^3}\mathrm{d}\widetilde{\mathbf{r}}\,\frac{\mathrm{e}^{-\kappa|\widetilde{\mathbf{r}}|}}{|\widetilde{\mathbf{r}}|}I^\varepsilon(\mathbf{r},\widetilde{\mathbf{r}}) - \nu^6\sum_{\mathbf{r}\in(\nu\mathbb{N})^3}\frac{\mathbf{1}_{|\mathbf{r}|>0}\mathrm{e}^{-\kappa|\mathbf{r}|}}{|\mathbf{r}|}\sum_{\widetilde{\mathbf{r}}\in(\nu\mathbb{N})^3}\frac{\mathbf{1}_{|\widetilde{\mathbf{r}}|>0}\mathrm{e}^{-\kappa|\widetilde{\mathbf{r}}|}}{|\widetilde{\mathbf{r}}|}I^\varepsilon(\mathbf{r},\widetilde{\mathbf{r}})\bigg| \lesssim_{\kappa,\nu} \frac{\Theta(\nu)}{\varepsilon^{5d+1}}.$$

With the above two results, we have all of the necessary tools to prove Lemma 5.10.

Proof of Lemma 5.10. The claim follows from Lemmas 5.14 and 5.15 by using (5.35) and (5.37). ∎

We now give the proofs of Lemmas 5.14 and 5.15. In the proof of Lemma 5.14, we use the following estimates that are obtained from Definition 5.13 and Lemma 5.5 (i), and (5.11):

$$|(\mathbb{V}_\eta)^{\tau,\widetilde{\tau}}_{x,\widetilde{x}}| \leqslant \frac{C\nu}{\varepsilon^d}\delta_\eta(\tau-\widetilde{\tau}), \quad |(\mathbb{V}_\eta)^{\tau}_{x}(\omega')| \leqslant \frac{C}{\varepsilon^d}[(\tau'-\widetilde{\tau}')+\nu]. \tag{5.40}$$

In the second estimate in (5.40), we take $\omega' \in \Omega^{\tau',\widetilde{\tau}'}$. We note that for paths $\omega', \widetilde{\omega}'$ with

$$f(\tau,x) = \int \mathrm{d}s\,\delta(\tau-[s]_\nu)\delta(x-\omega'(s)) - \int \mathrm{d}\widetilde{s}\,\delta(\tau-[\widetilde{s}]_\nu)\delta(x-\widetilde{\omega}'(\widetilde{s}))$$

we have

$$\mathbb{V}_\eta(\omega',\omega') + \mathbb{V}_\eta(\widetilde{\omega}',\widetilde{\omega}') - 2\mathbb{V}_\eta(\omega',\widetilde{\omega}') = \langle f, \mathcal{C}_\eta f\rangle \geqslant 0, \tag{5.41}$$

since δ_η and v^ε are of positive type.

Proof of Lemma 5.14. We recall the definition (5.34) of the set A. For fixed $\boldsymbol{\tau},\widetilde{\boldsymbol{\tau}} \in [0,\nu]^3$, arguing analogously to [33, (5.21)], we apply a time translation in the integrand and rewrite (5.38) as

$$\int_{[0,\nu]^3} \mathrm{d}\boldsymbol{\tau} \int_{[0,\nu]^3} \mathrm{d}\widetilde{\boldsymbol{\tau}} \int \mathrm{d}\mathbf{x} \int \mathrm{d}\widetilde{\mathbf{x}} \int \mathbb{W}_{x_1,x_3}^{\tau_1+|\mathbf{r}|,\tau_3+r_1+r_2}(\mathrm{d}\omega_3)\, \mathbb{W}_{x_3,x_2}^{\tau_3+r_1+r_2,\tau_2+r_1}(\mathrm{d}\omega_2)$$
$$\times\, \mathbb{W}_{x_2,x_1}^{\tau_2+r_1,\tau_1}(\mathrm{d}\omega_1)\, \mathbb{W}_{\widetilde{x}_1,\widetilde{x}_3}^{\widetilde{\tau}_1+|\widetilde{\mathbf{r}}|,\widetilde{\tau}_3+\widetilde{r}_1+\widetilde{r}_2}(\mathrm{d}\widetilde{\omega}_3)\, \mathbb{W}_{\widetilde{x}_3,\widetilde{x}_2}^{\widetilde{\tau}_3+\widetilde{r}_1+\widetilde{r}_2,\widetilde{\tau}_2+\widetilde{r}_1}(\mathrm{d}\widetilde{\omega}_2)\, \mathbb{W}_{\widetilde{x}_2,\widetilde{x}_1}^{\widetilde{\tau}_2+\widetilde{r}_1,\widetilde{\tau}_1}(\mathrm{d}\widetilde{\omega}_1)$$
$$\times\, \mathrm{e}^{-\frac{1}{2}(\mathbb{V}_\eta(\omega_1,\omega_1)+\mathbb{V}_\eta(\widetilde{\omega}_1,\widetilde{\omega}_1)-2\mathbb{V}_\eta(\omega_1,\widetilde{\omega}_1))}$$
$$\times \sum_{\Pi\in\mathbb{M}(A)} \prod_{\{a,b\}\in\Pi} (\mathbb{V}_\eta)_{x_a,x_b}^{\tau_a,\tau_b} \prod_{a\in A\setminus[\Pi]} \mathrm{i}\big((\mathbb{V}_\eta)_{x_a}^{\tau_a}(\omega_1) - (\mathbb{V}_\eta)_{x_a}^{\tau_a}(\widetilde{\omega}_1)\big) = J(\mathbf{r},\widetilde{\mathbf{r}}). \tag{5.42}$$

In what follows, we denote by $\widehat{k}$ either the quantity k or $\widetilde{k}$. We hence consider paths $\widehat{\omega}: [\widehat{\tau}_1, \widehat{\tau}_1 + |\widehat{\mathbf{r}}|] \to \Lambda$ obtained by concatenating $\widehat{\omega}_1, \widehat{\omega}_2, \widehat{\omega}_3$ occurring in (5.42). In particular,

$$\widehat{\omega}_1 = \widehat{\omega}|_{[\widehat{\tau}_1,\widehat{\tau}_2+\widehat{r}_1]}, \quad \widehat{\omega}_2 = \widehat{\omega}|_{[\widehat{\tau}_2+\widehat{r}_1,\widehat{\tau}_3+\widehat{r}_1+\widehat{r}_2]}, \quad \widehat{\omega}_3 = \widehat{\omega}|_{[\widehat{\tau}_3+\widehat{r}_1+\widehat{r}_2,\widehat{\tau}_1+|\widehat{\mathbf{r}}|]}. \tag{5.43}$$

We define the following parameters for $a \in A$:

$$\begin{cases} t_{(2,0)} := \tau_2 + r_1, \;\; t_{(3,0)} := \tau_3 + r_1 + r_2, \;\; t_{(2,1)} := \widetilde{\tau}_2 + \widetilde{r}_1, \;\; t_{(3,1)} := \widetilde{\tau}_3 + \widetilde{r}_1 + \widetilde{r}_2, \\ s_{(2,0)} := \tau_1 + r_1, \;\; s_{(3,0)} := \tau_1 + r_1 + r_2, \;\; s_{(2,1)} := \widetilde{\tau}_1 + \widetilde{r}_1, \;\; s_{(3,1)} := \widetilde{\tau}_1 + \widetilde{r}_1 + \widetilde{r}_2, \\ \omega_{(2,0)} = \omega_{(3,0)} := \omega, \;\; \omega_{(2,1)} = \omega_{(3,1)} := \widetilde{\omega}. \end{cases} \tag{5.44}$$

Note that the times t_a in (5.44) were chosen as initial and final times of the paths $\widehat{\omega}_j$ in (5.43). The s_a satisfy $|s_a - t_a| \leqslant \nu$ and are used as approximations of the t_a which help us remove the $\widehat{\tau}_2, \widehat{\tau}_3$ dependence.

Using (5.41), (5.43) and Lemma C.1 (i) for $\mathrm{e}^{-\frac{1}{2}(\mathbb{V}_\eta(\omega_1,\omega_1)+\mathbb{V}_\eta(\widetilde{\omega}_1,\widetilde{\omega}_1)-2\mathbb{V}_\eta(\omega_1,\widetilde{\omega}_1))}$ occurring in the integrand of (5.42), recalling (5.44) and integrating in the $\widehat{x}_2, \widehat{x}_3$ variables, we deduce from (5.42) that

$$J(\mathbf{r},\widetilde{\mathbf{r}}) = \int_{[0,\nu]^3} \mathrm{d}\boldsymbol{\tau} \int_{[0,\nu]^3} \mathrm{d}\widetilde{\boldsymbol{\tau}} \int \mathrm{d}x_1 \int \mathrm{d}\widetilde{x}_1 \int \mathbb{W}_{x_1,x_1}^{\tau_1+|\mathbf{r}|,\tau_1}(\mathrm{d}\omega)\, \mathbb{W}_{\widetilde{x}_1,\widetilde{x}_1}^{\widetilde{\tau}_1+|\widetilde{\mathbf{r}}|,\widetilde{\tau}_1}(\mathrm{d}\widetilde{\omega})$$
$$\Big[\mathrm{e}^{-\frac{1}{2}(\mathbb{V}^\varepsilon(\omega_1,\omega_1)+\mathbb{V}^\varepsilon(\widetilde{\omega}_1,\widetilde{\omega}_1)-2\mathbb{V}^\varepsilon(\omega_1,\widetilde{\omega}_1))} + O\big(|\widehat{\mathbb{V}}_\eta(\omega_1,\omega_1)| + |\widehat{\mathbb{V}}_\eta(\widetilde{\omega}_1,\widetilde{\omega}_1)| + |\widehat{\mathbb{V}}_\eta(\omega_1,\widetilde{\omega}_1)|\big)$$
$$+ O\bigg(\frac{\nu(r_1+\widetilde{r}_1+\nu)}{\varepsilon^d}\bigg)\Big]$$
$$\times \sum_{\Pi\in\mathfrak{M}(A)} \prod_{\{a,b\}\in\Pi} (\mathbb{V}_\eta)_{\omega_a(t_a),\omega_b(t_b)}^{\tau_a,\tau_b} \prod_{a\in A\setminus[\Pi]} \mathrm{i}((\mathbb{V}_\eta)_{\omega_a(t_a)}^{\tau_a}(\omega_1) - (\mathbb{V}_\eta)_{\omega_a(t_a)}^{\tau_a}(\widetilde{\omega}_1)). \tag{5.45}$$

We first estimate the contributions to (5.45) coming from the two error terms occurring in the square brackets. We recall (C.2), (C.3), (5.43), and apply Fubini's theorem, Lemma 5.4 (i)–(ii), and (5.11) to deduce that

$$\int \mathbb{W}_{x_1,x_1}^{\tau_1+|\mathbf{r}|,\tau_1}(\mathrm{d}\omega)\, \mathbb{W}_{\widetilde{x}_1,\widetilde{x}_1}^{\widetilde{\tau}_1+|\widetilde{\mathbf{r}}|,\widetilde{\tau}_1}(\mathrm{d}\widetilde{\omega}) \left(|\widehat{\mathbb{V}}_\eta(\omega_1,\omega_1)| + |\widehat{\mathbb{V}}_\eta(\widetilde{\omega}_1,\widetilde{\omega}_1)| + |\widehat{\mathbb{V}}_\eta(\omega_1,\widetilde{\omega}_1)|\right)$$
$$\leqslant \frac{C\sqrt{\nu}}{\varepsilon^{d+1}} \bigg(1 + \frac{1}{|\mathbf{r}|^{d/2}}\bigg)\bigg(1 + \frac{1}{|\widetilde{\mathbf{r}}|^{d/2}}\bigg)(1 + |\mathbf{r}| + |\widetilde{\mathbf{r}}|)^2. \tag{5.46}$$

Note that here we have used estimate (5.9) when applying Lemma 5.4 (ii). Combining (5.46) with (5.40), and using (5.11) for the $\mathrm{d}\widehat{\boldsymbol{\tau}}$ integration, we deduce that the first error term in the square brackets in (5.45) gives a contribution which is

$$O\bigg(\frac{\nu^{13/2}}{\varepsilon^{5d+1}}\bigg(1+\frac{1}{|\mathbf{r}|^{d/2}}\bigg)\bigg(1+\frac{1}{|\widetilde{\mathbf{r}}|^{d/2}}\bigg)(1+|\mathbf{r}|+|\widetilde{\mathbf{r}}|)^6\bigg). \tag{5.47}$$

Here, we note that there are at most $|A|=4$ factors of $1/\varepsilon^d$ coming from (5.40). Likewise, there are at most 4 additional factors of $1+|\mathbf{r}|+|\widetilde{\mathbf{r}}|$. Similarly, the second error term also gives a contribution which is bounded from above by (5.47) (note that now, we do not need to use (5.46)). Therefore, we can write (5.45) as

$$J(\mathbf{r},\widetilde{\mathbf{r}})=\bar{J}(\mathbf{r},\widetilde{\mathbf{r}})+O\bigg(\frac{\nu^{13/2}}{\varepsilon^{5d+1}}\bigg(1+\frac{1}{|\mathbf{r}|^{d/2}}\bigg)\bigg(1+\frac{1}{|\widetilde{\mathbf{r}}|^{d/2}}\bigg)(1+|\mathbf{r}|+|\widetilde{\mathbf{r}}|)^6\bigg), \tag{5.48}$$

where

$$\begin{aligned}\bar{J}(\mathbf{r},\widetilde{\mathbf{r}}):=\int_{[0,\nu]^3}\mathrm{d}\boldsymbol{\tau}\int_{[0,\nu]^3}\mathrm{d}\widetilde{\boldsymbol{\tau}}\int\mathrm{d}x_1\int\mathrm{d}\widetilde{x}_1\int\mathbb{W}^{\tau_1+|\mathbf{r}|,\tau_1}_{x_1,x_1}(\mathrm{d}\omega)\,\mathbb{W}^{\widetilde{\tau}_1+|\widetilde{\mathbf{r}}|,\widetilde{\tau}_1}_{\widetilde{x}_1,\widetilde{x}_1}(\mathrm{d}\widetilde{\omega})\\ \mathrm{e}^{-\frac{1}{2}(\mathbb{V}^\varepsilon(\omega_1,\omega_1)+\mathbb{V}^\varepsilon(\widetilde{\omega}_1,\widetilde{\omega}_1)-2\mathbb{V}^\varepsilon(\omega_1,\widetilde{\omega}_1))}\sum_{\Pi\in\mathfrak{M}(A)}\prod_{\{a,b\}\in\Pi}(\mathbb{V}_\eta)^{\tau_a,\tau_b}_{\omega_a(t_a),\omega_b(t_b)}\\ \times\prod_{a\in A\setminus[\Pi]}\mathrm{i}\big((\mathbb{V}_\eta)^{\tau_a}_{\omega_a(t_a)}(\omega_1)-(\mathbb{V}_\eta)^{\tau_a}_{\omega_a(t_a)}(\widetilde{\omega}_1)\big).\end{aligned} \tag{5.49}$$

In order to analyse (5.49), we note the following three estimates.

(i) We have

$$\begin{aligned}&\mathrm{e}^{-\frac{1}{2}(\mathbb{V}^\varepsilon(\omega_1,\omega_1)+\mathbb{V}^\varepsilon(\widetilde{\omega}_1,\widetilde{\omega}_1)-2\mathbb{V}^\varepsilon(\omega_1,\widetilde{\omega}_1))}\\ &\quad=\mathrm{e}^{-\frac{1}{2}(\mathbb{V}^\varepsilon(\omega|_{I_1},\omega|_{I_1})+\mathbb{V}^\varepsilon(\widetilde{\omega}|_{\widetilde{I}_1},\widetilde{\omega}|_{\widetilde{I}_1})-2\mathbb{V}^\varepsilon(\omega|_{I_1},\widetilde{\omega}|_{\widetilde{I}_1}))}+O\bigg(\frac{\nu}{\varepsilon^d}(1+r_1+\widetilde{r}_1)\bigg),\end{aligned} \tag{5.50}$$

where $I_1=[\tau_1,\tau_1+r_1]$ and $\widetilde{I}_1=[\widetilde{\tau}_1,\widetilde{\tau}_1+\widetilde{r}_1]$.

(ii) For all $a,b\in A$, we have

$$\begin{aligned}&\int_0^\nu\mathrm{d}\tau_a\int_0^\nu\mathrm{d}\tau_b\,(\mathbb{V}_\eta)^{\tau_a,\tau_b}_{\omega_a(t_a),\omega_b(t_b)}=\nu^2\,(\mathbb{V}^\varepsilon)_{\omega_a(s_a),\omega_b(s_b)}\\ &+O\bigg(\frac{\nu}{\varepsilon^{d+1}}\int_0^\nu\mathrm{d}\tau_a\int_0^\nu\mathrm{d}\tau_b\,\delta_\eta(\tau_a-\tau_b)\big(|\omega_a(t_a)-\omega_a(s_a)|_\Lambda+|\omega_b(t_b)-\omega_b(s_b)|_\Lambda\big)\bigg).\end{aligned} \tag{5.51}$$

(iii) For all $a\in A$, we have

$$\begin{aligned}\int_0^\nu\mathrm{d}\tau_a\,(\mathbb{V}_\eta)^{\tau_a}_{\omega_a(t_a)}(\widehat{\omega}_1)&=\nu\,(\mathbb{V}^\varepsilon)_{\omega_a(s_a)}(\widehat{\omega}|_{[\widehat{\tau}_1,\widehat{\tau}_1+\widehat{r}_1]})\\ &+O\bigg(\frac{\widehat{r}_1}{\varepsilon^{d+1}}\int_0^\nu\mathrm{d}\tau_a\,|\omega_a(t_a)-\omega_a(s_a)|_\Lambda\bigg)+O\bigg(\frac{\nu^2}{\varepsilon^d}\bigg).\end{aligned} \tag{5.52}$$

In order to show (i), we note that for paths $\omega', \widetilde{\omega}'$ and

$$f(x) = \int \mathrm{d}s\, \delta(x - \omega(s)) - \int \mathrm{d}\widetilde{s}\, \delta(x - \omega'(\widetilde{s})),$$

we obtain

$$\mathbb{V}^\varepsilon(\omega', \omega') + \mathbb{V}^\varepsilon(\widetilde{\omega}', \widetilde{\omega}') - 2\mathbb{V}^\varepsilon(\omega', \widetilde{\omega}') = \int \mathrm{d}x\, \mathrm{d}y\; f(x) v^\varepsilon(x - y) f(y) \geqslant 0, \tag{5.53}$$

since v^ε is of positive type.

Claim (i) then follows by using Definition 5.12, (5.53), and Lemma C.1 (ii). In order to prove claim (ii), we use Definition 5.13, (5.44) and (5.11) to rewrite the left-hand side of (5.51) as

$$\begin{aligned}
&\nu^2\, v^\varepsilon(\omega_a(s_a) - \omega_b(s_b)) \\
&\qquad + \nu \int_0^\nu \mathrm{d}\tau_a \int_0^\nu \mathrm{d}\tau_b\, \delta_\eta(\tau_a - \tau_b)[v^\varepsilon(\omega_a(t_a) - \omega_b(t_b)) - v^\varepsilon(\omega_a(s_a) - \omega_b(s_b))],
\end{aligned}$$

which by Lemma 5.5 (ii) is of the form given by the right-hand side of (5.51). In order to prove claim (iii), we first recall (5.43) and use $|\widehat{\tau}_1 - \widehat{\tau}_2| \leqslant \nu$, Definition 5.13, (5.11), and Lemma 5.5 (i) to rewrite the left-hand side of (5.52) as

$$\begin{aligned}
&\nu \int_0^\nu \mathrm{d}\tau_a \int_{\widehat{\tau}_1}^{\widehat{\tau}_1 + \widehat{r}_1} \mathrm{d}s\, \delta_\eta(\tau_a - [s]_\nu) v^\varepsilon(\omega_a(s_a) - \widehat{\omega}(s)) \\
&+ \nu \int_0^\nu \mathrm{d}\tau_a \int_{\widehat{\tau}_1}^{\widehat{\tau}_1 + \widehat{r}_1} \mathrm{d}s\, \delta_\eta(\tau_a - [s]_\nu)[v^\varepsilon(\omega_a(t_a) - \widehat{\omega}(s)) - v^\varepsilon(\omega_a(s_a) - \widehat{\omega}(s))] + O\left(\frac{\nu^2}{\varepsilon^d}\right).
\end{aligned} \tag{5.54}$$

The last error term in (5.54) comes from replacing $\widehat{\tau}_2 + \widehat{r}_2$ by $\widehat{\tau}_1 + \widehat{r}_1$ in the upper limit of the s integral. For the first term, we use (5.11) to integrate in τ_a. For the second term, we use Lemma 5.5 (ii) followed by (5.11) to integrate in s. It follows that (5.54) is of the form given by the right-hand side of (5.52).

We recall (5.36), (5.43), (5.44). Then, we use estimates (i)–(iii) above together with a telescoping argument in (5.49), and argue analogously to the proof of (5.48) to deduce that

$$\bar{J}(\mathbf{r}, \widetilde{\mathbf{r}}) = \nu^6 I^\varepsilon(\mathbf{r}, \widetilde{\mathbf{r}}) + O\left(\frac{\nu^{13/2}}{\varepsilon^{5d+1}} \left(1 + \frac{1}{|\mathbf{r}|^{d/2}}\right)\left(1 + \frac{1}{|\widetilde{\mathbf{r}}|^{d/2}}\right)(1 + |\mathbf{r}| + |\widetilde{\mathbf{r}}|)^6\right). \tag{5.55}$$

We now deduce (5.39) from (5.48) and (5.55). ∎

In the proof of Lemma 5.15, we use the following estimates that follow from Definition 5.12 and Lemma 5.5 (i):

$$|(\mathbb{V}^\varepsilon)_{x,\widetilde{x}}| \leqslant \frac{C}{\varepsilon^d}, \quad |(\mathbb{V}^\varepsilon)_x(\omega)| \leqslant \frac{C(\tau_1 - \widetilde{\tau}_1)}{\varepsilon^d}. \tag{5.56}$$

Proof of Lemma 5.15. We recall (5.36) and note that for $\mathbf{r}, \widetilde{\mathbf{r}} \in [0,\infty)^3 \times [0,\infty)^3$, we have

$$|I^\varepsilon(\mathbf{r},\widetilde{\mathbf{r}})| \leqslant \frac{C}{\varepsilon^{4d}} \left(1 + \frac{1}{|\mathbf{r}|^{d/2}}\right)\left(1 + \frac{1}{|\widetilde{\mathbf{r}}|^{d/2}}\right)(1 + |\mathbf{r}| + |\widetilde{\mathbf{r}}|)^4, \tag{5.57}$$

by using (5.56), (5.53), and Lemma 5.4 (i).

Let $C_0 > 0$ be given. Recalling (5.33), we first prove the following two estimates:

$$\left| \int_{[0,\infty)^3} \mathrm{d}\mathbf{r}\, \frac{\mathrm{e}^{-\kappa|\mathbf{r}|}}{|\mathbf{r}|} \int_{[0,\infty)^3} \mathrm{d}\widetilde{\mathbf{r}}\, \frac{\mathrm{e}^{-\kappa|\widetilde{\mathbf{r}}|}}{|\widetilde{\mathbf{r}}|} I^\varepsilon(\mathbf{r},\widetilde{\mathbf{r}})\, \mathbf{1}_{\min(r_j,\widetilde{r}_j)\leqslant C_0\nu} \right| \lesssim_{C_0,\kappa,\upsilon} \frac{\sqrt{\nu}}{\varepsilon^{4d}}, \tag{5.58}$$

$$\left| \nu^6 \sum_{\mathbf{r}\in(\nu\mathbb{N})^3} \frac{\mathbf{1}_{|\mathbf{r}|>0}\mathrm{e}^{-\kappa|\mathbf{r}|}}{|\mathbf{r}|} \sum_{\widetilde{\mathbf{r}}\in(\nu\mathbb{N})^3} \frac{\mathbf{1}_{|\widetilde{\mathbf{r}}|>0}\mathrm{e}^{-\kappa|\widetilde{\mathbf{r}}|}}{|\widetilde{\mathbf{r}}|} I^\varepsilon(\mathbf{r},\widetilde{\mathbf{r}})\, \mathbf{1}_{\min(r_j,\widetilde{r}_j)\leqslant C_0\nu} \right| \lesssim_{C_0,\kappa,\upsilon} \frac{\Theta(\nu)}{\varepsilon^{4d}}. \tag{5.59}$$

By using (5.57) and symmetry, we note that (5.58) follows from[7]

$$\int_{[0,\infty)^3} \mathrm{d}\mathbf{r}\, \mathbf{1}_{r_3\leqslant C_0\nu} \frac{\mathrm{e}^{-\kappa|\mathbf{r}|/2}}{|\mathbf{r}|^{1+d/2}} \lesssim_{C_0,\kappa} \nu^{2-d/2}. \tag{5.60}$$

The estimate (5.60) follows by using spherical coordinates and considering the contributions $C_0\nu/r \geqslant 1$ and $C_0\nu/r < 1$ separately. We omit the details.

By using (5.57) and symmetry, we deduce that (5.59) follows from

$$\nu^3 \sum_{r_3\in\nu\mathbb{N}} \mathbf{1}_{r_3>0} \frac{\mathrm{e}^{-\kappa r_3}}{r_3^{1+d/2}} + \nu^3 \sum_{\mathbf{r}\in(\nu\mathbb{N})^3} \mathbf{1}_{r_3\leqslant C_0\nu} \frac{\mathbf{1}_{|(r_1,r_2)|>0}\mathrm{e}^{-\kappa|\mathbf{r}|}}{|\mathbf{r}|^{1+d/2}} \lesssim_{C_0,\kappa} \Theta(\nu). \tag{5.61}$$

The first term on the left-hand side of (5.61) is

$$\leqslant \nu^{3-d/2} \sum_{r_3\in\nu\mathbb{N}} \mathbf{1}_{r_3>0} \frac{\mathrm{e}^{-\kappa r_3}}{r_3} \lesssim_\kappa \nu^{2-d/2} \log \nu^{-1} \lesssim \Theta(\nu)$$

by considering Riemann sums in one dimension. The second term on the left-hand side of (5.61) is

$$\lesssim_{C_0} \nu^3 \sum_{\mathbf{r}\in(\nu\mathbb{N})^2} \frac{\mathbf{1}_{|\mathbf{r}|>0}\mathrm{e}^{-\kappa|\mathbf{r}|}}{|\mathbf{r}|^{1+d/2}} \lesssim_\kappa \sqrt{\nu},$$

where the latter inequality follows by considering Riemann sums in two dimensions. Here, we use the convention that for $\mathbf{r} = (r_1, r_2) \in (\nu\mathbb{N})^2$, we take $|\mathbf{r}| = r_1 + r_2$. We deduce (5.61), and (5.59) then follows.

We henceforth fix $C_0 > 0$ large and consider $\mathbf{r}, \widetilde{\mathbf{r}} \in [0,\infty)^3$, $\boldsymbol{\alpha}, \widetilde{\boldsymbol{\alpha}} \in \mathbb{R}^3$ with

$$|r_i|, |\widetilde{r}_i| \geqslant C_0\nu, \quad |\alpha_i|, |\widetilde{\alpha}_i| \leqslant \nu. \tag{5.62}$$

[7] We observe that this is a slightly stronger bound than what we need when $d = 2$. The bound (5.58) is sufficient for the rest of the argument. An analogous observation holds for the bound (5.59).

Under the assumption (5.62), we show that

$$|I^\varepsilon(\mathbf{r}+\boldsymbol{\alpha},\tilde{\mathbf{r}}+\tilde{\boldsymbol{\alpha}})-I^\varepsilon(\mathbf{r},\tilde{\mathbf{r}})| = O\bigg(\frac{\sqrt{\nu}}{\varepsilon^{5d+1}}\bigg(1+\frac{1}{|\mathbf{r}|^{d/2}}\bigg)\bigg(1+\frac{1}{|\tilde{\mathbf{r}}|^{d/2}}\bigg)(1+|\mathbf{r}|+|\tilde{\mathbf{r}}|)^5\bigg)$$
$$+O\bigg(\frac{\nu}{\varepsilon^{4d}}\,\frac{1}{r_3}\bigg(1+\frac{1}{(r_1+r_2)^{d/2}}\bigg)\bigg(1+\frac{1}{|\tilde{\mathbf{r}}|^{d/2}}\bigg)(1+|\mathbf{r}|+|\tilde{\mathbf{r}}|)^4\bigg)$$
$$+O\bigg(\frac{\nu}{\varepsilon^{4d}}\bigg(1+\frac{1}{|\mathbf{r}|^{d/2}}\bigg)\frac{1}{\tilde{r}_3}\bigg(1+\frac{1}{(\tilde{r}_1+\tilde{r}_2)^{d/2}}\bigg)(1+|\mathbf{r}|+|\tilde{\mathbf{r}}|)^4\bigg) =: \mathcal{E}_1(\mathbf{r},\tilde{\mathbf{r}}). \tag{5.63}$$

By using (5.57), (5.62), and the notation in (5.63) we have by a direct calculation that

$$\frac{\mathrm{e}^{-\kappa|\mathbf{r}+\boldsymbol{\alpha}|}}{|\mathbf{r}+\boldsymbol{\alpha}|}\,\frac{\mathrm{e}^{-\kappa|\tilde{\mathbf{r}}+\tilde{\boldsymbol{\alpha}}|}}{|\tilde{\mathbf{r}}+\tilde{\boldsymbol{\alpha}}|}I^\varepsilon(\mathbf{r}+\boldsymbol{\alpha},\tilde{\mathbf{r}}+\tilde{\boldsymbol{\alpha}})-\frac{\mathrm{e}^{-\kappa|\mathbf{r}|}}{|\mathbf{r}|}\,\frac{\mathrm{e}^{-\kappa|\tilde{\mathbf{r}}|}}{|\tilde{\mathbf{r}}|}I^\varepsilon(\mathbf{r},\tilde{\mathbf{r}})$$
$$= O\bigg(\frac{\mathrm{e}^{-\kappa|\mathbf{r}|}}{|\mathbf{r}|}\,\frac{\mathrm{e}^{-\kappa|\tilde{\mathbf{r}}|}}{|\tilde{\mathbf{r}}|}\mathcal{E}_1(\mathbf{r},\tilde{\mathbf{r}})\bigg)$$
$$+O_\kappa\bigg(\frac{\nu}{\varepsilon^{4d}}\bigg(1+\frac{1}{|\mathbf{r}|^{(d+3)/2}}\bigg)\bigg(1+\frac{1}{|\tilde{\mathbf{r}}|^{(d+3)/2}}\bigg)(1+|\mathbf{r}|+|\tilde{\mathbf{r}}|)^4\,\mathrm{e}^{-\kappa(|\mathbf{r}|+|\tilde{\mathbf{r}}|)}\bigg)$$
$$=: \mathcal{E}_2(\mathbf{r},\tilde{\mathbf{r}}). \tag{5.64}$$

In order to obtain the second error term in (5.64), we note that, by (5.62), we have[8]

$$\frac{\mathrm{e}^{-\kappa|\hat{\mathbf{r}}+\hat{\boldsymbol{\alpha}}|}}{|\hat{\mathbf{r}}+\hat{\boldsymbol{\alpha}}|}-\frac{\mathrm{e}^{-\kappa|\hat{\mathbf{r}}|}}{|\hat{\mathbf{r}}|} = O_\kappa\bigg(\frac{\mathrm{e}^{-\kappa|\hat{\mathbf{r}}|}\,\nu}{|\hat{\mathbf{r}}|}+\frac{\mathrm{e}^{-\kappa|\hat{\mathbf{r}}|}\,\nu^{1/2}}{|\hat{\mathbf{r}}|^{3/2}}\bigg). \tag{5.65}$$

We then deduce the lemma by using (5.58), (5.59), (5.62), (5.63), (5.64), and considering Riemann sums for (5.35). Indeed, for all $(\mathbf{s},\tilde{\mathbf{s}})\in[0,\infty)^3\times[0,\infty)^3$, we take $(\mathbf{r},\tilde{\mathbf{r}})\in(\nu\mathbb{N})^3\times(\nu\mathbb{N})^3$ such that $\hat{r}_j=\lfloor\hat{s}_j\rfloor_\nu$, where

$$\lfloor s\rfloor_\nu := \max\{u\in\nu\mathbb{N} : u\leqslant s\}. \tag{5.66}$$

Then we automatically have $|\hat{r}_j-\hat{s}_j|\leqslant\nu$ for all $j=1,2,3$. We then use (5.63), (5.64) with $\hat{\mathbf{r}}+\hat{\boldsymbol{\alpha}}\equiv\hat{\mathbf{s}}$ and we reduce the claim to showing

$$\nu^6\sum_{\mathbf{r},\tilde{\mathbf{r}}\in(\nu\mathbb{N})^3\cap[C_0\nu,\infty)^3}\mathcal{E}_2(\mathbf{r},\tilde{\mathbf{r}})\lesssim\Theta(\nu),$$

which follows from (5.63)–(5.64). Let us note that when estimating the contributions from the last two error terms in (5.63), we use

$$\sum_{r_3\in(\nu\mathbb{N})\cap[C_0\nu,\infty)}\frac{\nu}{r_3}\mathrm{e}^{-\kappa r_3/2}\lesssim_\kappa\log\nu^{-1}, \tag{5.67}$$

$$\sum_{\mathbf{r}\in(\nu\mathbb{N})^2\cap[C_0\nu,\infty)^2}\nu^2\bigg(1+\frac{1}{|\mathbf{r}|^{1+d/2}}\bigg)\mathrm{e}^{-\kappa|\mathbf{r}|/2}\lesssim_{C_0,\kappa}\log\nu^{-1}\,\mathbf{1}_{d=2}+\nu^{-1/2}\mathbf{1}_{d=3}, \tag{5.68}$$

[8] In order to obtain the second error term in (5.65), we interpolate between the estimates

$$\bigg|\frac{1}{|\hat{\mathbf{r}}+\hat{\boldsymbol{\alpha}}|}-\frac{1}{|\hat{\mathbf{r}}|}\bigg|\lesssim\frac{1}{|\hat{\mathbf{r}}|}\quad\text{and}\quad\bigg|\frac{1}{|\hat{\mathbf{r}}+\hat{\boldsymbol{\alpha}}|}-\frac{1}{|\hat{\mathbf{r}}|}\bigg|\lesssim\frac{\nu}{|\hat{\mathbf{r}}|^2}.$$

which follow by considering Riemann sums in one and two dimensions respectively. Here, the term on the left-hand side of (5.68) comes from estimating $\frac{1}{|(r_1,r_2,r_3)|}(1+\frac{1}{(r_1+r_2)^{d/2}}) \lesssim 1+\frac{1}{(r_1+r_2)^{1+d/2}}$. Finally, let us note that when $d=3$, we have $(d+3)/2=3$ and therefore the second error term in (5.64) yields the logarithmic factor in the error term. Similarly, in light of (5.67)–(5.68), the same is true for the last two error terms in (5.63). This requires the necessary modification in (5.33) when $d=3$.

The rest of the proof is devoted to showing (5.63). Let $\mathbf{r},\widetilde{\mathbf{r}},\boldsymbol{\alpha},\widetilde{\boldsymbol{\alpha}}$ be as in (5.62). Recalling (5.36) and using an appropriate time translation of the paths,[9] we can write

$$
\begin{aligned}
&I^{\varepsilon}(\mathbf{r}+\boldsymbol{\alpha},\widetilde{\mathbf{r}}+\widetilde{\boldsymbol{\alpha}}) = \\
&\int \mathrm{d}\mathbf{x}\int \mathrm{d}\widetilde{\mathbf{x}}\int \mathbb{W}_{x_1,x_3}^{|\mathbf{r}|+|\boldsymbol{\alpha}|,r_1+r_2+\alpha_1+\alpha_2}(\mathrm{d}\omega_3)\,\mathbb{W}_{x_3,x_2}^{r_1+r_2+\alpha_1+\alpha_2,r_1+\alpha_1}(\mathrm{d}\omega_2)\,\mathbb{W}_{x_2,x_1}^{r_1+\alpha_1,0}(\mathrm{d}\omega_1)\\
&\quad\times \mathbb{W}_{\widetilde{x}_1,\widetilde{x}_3}^{|\widetilde{\mathbf{r}}|+|\widetilde{\boldsymbol{\alpha}}|,\widetilde{r}_1+\widetilde{r}_2+\widetilde{\alpha}_1+\widetilde{\alpha}_2}(\mathrm{d}\widetilde{\omega}_3)\,\mathbb{W}_{\widetilde{x}_3,\widetilde{x}_2}^{\widetilde{r}_1+\widetilde{r}_2+\widetilde{\alpha}_1+\widetilde{\alpha}_2,\widetilde{r}_1+\widetilde{\alpha}_1}(\mathrm{d}\widetilde{\omega}_2)\,\mathbb{W}_{\widetilde{x}_2,\widetilde{x}_1}^{\widetilde{r}_1+\widetilde{\alpha}_1,0}(\mathrm{d}\widetilde{\omega}_1)\\
&\quad\times \mathrm{e}^{-\frac{1}{2}(\mathbb{V}^{\varepsilon}(\omega_1,\omega_1)+\mathbb{V}^{\varepsilon}(\widetilde{\omega}_1,\widetilde{\omega}_1)-2\mathbb{V}^{\varepsilon}(\omega_1,\widetilde{\omega}_1))}\sum_{\Pi\in\mathfrak{M}(A)}\prod_{\{a,b\}\in\Pi}(\mathbb{V}^{\varepsilon})_{x_a,x_b}\\
&\qquad\times\prod_{a\in A\setminus[\Pi]}\mathrm{i}\big((\mathbb{V}^{\varepsilon})_{x_a}(\omega_1)-(\mathbb{V}^{\varepsilon})_{x_a}(\widetilde{\omega}_1)\big). \qquad (5.69)
\end{aligned}
$$

Note that, by (5.62), we indeed have $|\widetilde{\mathbf{r}}|+|\widetilde{\boldsymbol{\alpha}}|>\widetilde{r}_1+\widetilde{r}_2+\widetilde{\alpha}_1+\widetilde{\alpha}_2$ and $|\widetilde{\mathbf{r}}|+|\widetilde{\boldsymbol{\alpha}}|>\widetilde{r}_1+\widetilde{r}_2+\widetilde{\alpha}_1+\widetilde{\alpha}_2$, hence the above expression is well-defined.

We now show that

$$
\begin{aligned}
(5.69) &= \bar{I}^{\varepsilon}(\mathbf{r}+\boldsymbol{\alpha},\widetilde{\mathbf{r}}+\widetilde{\boldsymbol{\alpha}})\\
&\quad + O\bigg(\frac{\nu}{\varepsilon^{4d}}\frac{1}{r_3}\bigg(1+\frac{1}{(r_1+r_2)^{d/2}}\bigg)\bigg(1+\frac{1}{|\widetilde{\mathbf{r}}|^{d/2}}\bigg)(1+|\mathbf{r}|+|\widetilde{\mathbf{r}}|)^4\bigg)\\
&\quad + O\bigg(\frac{\nu}{\varepsilon^{4d}}\bigg(1+\frac{1}{|\mathbf{r}|^{d/2}}\bigg)\frac{1}{\widetilde{r}_3}\bigg(1+\frac{1}{(\widetilde{r}_1+\widetilde{r}_2)^{d/2}}\bigg)(1+|\mathbf{r}|+|\widetilde{\mathbf{r}}|)^4\bigg), \qquad (5.70)
\end{aligned}
$$

where

$$
\begin{aligned}
&\bar{I}^{\varepsilon}(\mathbf{r}+\boldsymbol{\alpha},\widetilde{\mathbf{r}}+\widetilde{\boldsymbol{\alpha}})\\
&:= \int \mathrm{d}\mathbf{x}\int \mathrm{d}\widetilde{\mathbf{x}}\int \mathbb{W}_{x_1,x_3}^{|\mathbf{r}|,r_1+r_2+\alpha_1+\alpha_2}(\mathrm{d}\omega_3)\,\mathbb{W}_{x_3,x_2}^{r_1+r_2+\alpha_1+\alpha_2,r_1+\alpha_1}(\mathrm{d}\omega_2)\,\mathbb{W}_{x_2,x_1}^{r_1+\alpha_1,0}(\mathrm{d}\omega_1)\\
&\quad\times \mathbb{W}_{\widetilde{x}_1,\widetilde{x}_3}^{|\widetilde{\mathbf{r}}|,\widetilde{r}_1+\widetilde{r}_2+\widetilde{\alpha}_1+\widetilde{\alpha}_2}(\mathrm{d}\widetilde{\omega}_3)\,\mathbb{W}_{\widetilde{x}_3,\widetilde{x}_2}^{\widetilde{r}_1+\widetilde{r}_2+\widetilde{\alpha}_1+\widetilde{\alpha}_2,\widetilde{r}_1+\widetilde{\alpha}_1}(\mathrm{d}\widetilde{\omega}_2)\,\mathbb{W}_{\widetilde{x}_2,\widetilde{x}_1}^{\widetilde{r}_1+\widetilde{\alpha}_1,0}(\mathrm{d}\widetilde{\omega}_1)\\
&\quad\times \mathrm{e}^{-\frac{1}{2}(\mathbb{V}^{\varepsilon}(\omega_1,\omega_1)+\mathbb{V}^{\varepsilon}(\widetilde{\omega}_1,\widetilde{\omega}_1)-2\mathbb{V}^{\varepsilon}(\omega_1,\widetilde{\omega}_1))}\sum_{\Pi\in\mathfrak{M}(A)}\prod_{\{a,b\}\in\Pi}(\mathbb{V}^{\varepsilon})_{x_a,x_b}\\
&\qquad\times\prod_{a\in A\setminus[\Pi]}\mathrm{i}\big((\mathbb{V}^{\varepsilon})_{x_a}(\omega_1)-(\mathbb{V}^{\varepsilon})_{x_a}(\widetilde{\omega}_1)\big) \qquad (5.71)
\end{aligned}
$$

[9] The time-translation is analogous to that used to in order to rewrite (5.38) as (5.42) above.

is obtained by modifying (5.69) to replace the final time of ω_3 and $\widetilde{\omega}_3$ by $|\mathbf{r}|$ and $|\widetilde{\mathbf{r}}|$ respectively.[10]

In order to deduce the estimate (5.70), let us first consider the case $\widetilde{\boldsymbol{\alpha}} = \mathbf{0}$. We note that neither of the integrands in (5.69), (5.71) depends on ω_3. Using (5.53), (5.56), integrating in $x_2, \widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3$ and recalling Lemma 5.4 (i) as well as (5.62), we deduce that

$$\begin{aligned}
&|\bar{I}^{\varepsilon}(\mathbf{r}+\boldsymbol{\alpha},\widetilde{\mathbf{r}}) - I^{\varepsilon}(\mathbf{r}+\boldsymbol{\alpha},\widetilde{\mathbf{r}})| \\
&\qquad \lesssim_{d,v} \frac{1}{\varepsilon^{4d}} \int \mathrm{d}x_1\, \mathrm{d}x_3\, |\psi^{r_3+\alpha_3-\alpha_1-\alpha_2}(x_1-x_3) - \psi^{r_3-\alpha_1-\alpha_2}(x_1-x_3)| \\
&\qquad\qquad \times \left(1+\frac{1}{(r_1+r_2)^{d/2}}\right)\left(1+\frac{1}{|\widetilde{\mathbf{r}}|^{d/2}}\right)(1+|\mathbf{r}|+|\widetilde{\mathbf{r}}|)^4 \\
&\qquad \lesssim_{d,v} \frac{|\alpha_3|}{\varepsilon^{4d}}\,\frac{1}{r_3}\left(1+\frac{1}{(r_1+r_2)^{d/2}}\right)\left(1+\frac{1}{|\widetilde{\mathbf{r}}|^{d/2}}\right)(1+|\mathbf{r}|+|\widetilde{\mathbf{r}}|)^4.
\end{aligned}$$

For the last inequality, we have used Lemma 5.4 (iii) and (5.62). We hence deduce (5.70) when $\widetilde{\boldsymbol{\alpha}} = \mathbf{0}$. The general claim of (5.70) follows by a telescoping argument.

We now show that $|\bar{I}^{\varepsilon}(\mathbf{r}+\boldsymbol{\alpha},\widetilde{\mathbf{r}}+\widetilde{\boldsymbol{\alpha}}) - I^{\varepsilon}(\mathbf{r},\widetilde{\mathbf{r}})|$ is bounded by the first error term on the right-hand side of (5.63). In order to do this, we rewrite $\bar{I}^{\varepsilon}(\mathbf{r}+\boldsymbol{\alpha},\widetilde{\mathbf{r}}+\widetilde{\boldsymbol{\alpha}})$ by arguing as in the proof of Lemma 5.14. Similarly to (5.43), consider loops $\widehat{\omega}:[0,|\widehat{\mathbf{r}}|]\to\Lambda$ which are obtained by concatenating $\widehat{\omega}_1, \widehat{\omega}_2, \widehat{\omega}_3$. In particular,

$$\widehat{\omega}_1 = \widehat{\omega}|_{[0,\hat{r}_1+\hat{\alpha}_1]}, \quad \widehat{\omega}_2 = \widehat{\omega}|_{[\hat{r}_1+\hat{\alpha}_1,\hat{r}_1+\hat{r}_2+\hat{\alpha}_1+\hat{\alpha}_2]}, \quad \widehat{\omega}_3 = \widehat{\omega}|_{[\hat{r}_1+\hat{r}_2+\hat{\alpha}_1+\hat{\alpha}_2,|\widehat{\mathbf{r}}|+|\widehat{\boldsymbol{\alpha}}|]}. \tag{5.72}$$

Given $a \in A$, we define the following quantities, similarly to (5.44):

$$\begin{cases}
t_{(2,0)} := r_1+\alpha_1, \quad t_{(3,0)} := r_1+r_2+\alpha_1+\alpha_2, \\
t_{(2,1)} := \widetilde{r}_1+\widetilde{\alpha}_1, \quad t_{(3,1)} := \widetilde{r}_1+\widetilde{r}_2+\widetilde{\alpha}_1+\widetilde{\alpha}_2, \\
s_{(2,0)} := r_1, \quad s_{(3,0)} := r_1+r_2, \quad s_{(2,1)} := \widetilde{r}_1, \quad s_{(3,1)} := \widetilde{r}_1+\widetilde{r}_2, \\
\omega_{(2,0)} = \omega_{(3,0)} := \omega, \quad \omega_{(2,1)} = \omega_{(3,1)} := \widetilde{\omega}.
\end{cases} \tag{5.73}$$

Note that, in (5.73), the s_a are the approximations of the t_a which have been decoupled from the variables $\widehat{\alpha}_j$. By construction, we have $\omega_a(t_a) = x_a$ and $|s_a - t_a| \lesssim \nu$ for all $a \in A$.

Therefore, integrating in $\widehat{x}_2, \widehat{x}_3$, we get

$$\begin{aligned}
&\bar{I}^{\varepsilon}(\mathbf{r}+\boldsymbol{\alpha},\widetilde{\mathbf{r}}+\widetilde{\boldsymbol{\alpha}}) \\
&= \int \mathrm{d}x_1 \int \mathrm{d}\widetilde{x}_1 \int \mathbb{W}^{|\mathbf{r}|,0}_{x_1,x_1}(\mathrm{d}\omega)\, \mathbb{W}^{|\widetilde{\mathbf{r}}|,0}_{\widetilde{x}_1,\widetilde{x}_1}(\mathrm{d}\widetilde{\omega})\, \mathrm{e}^{-\frac{1}{2}(\mathbb{V}^{\varepsilon}(\omega_1,\omega_1)+\mathbb{V}^{\varepsilon}(\widetilde{\omega}_1,\widetilde{\omega}_1)-2\mathbb{V}^{\varepsilon}(\omega_1,\widetilde{\omega}_1))} \\
&\quad \times \sum_{\Pi\in\mathbb{M}(A)} \prod_{\{a,b\}\in\Pi} (\mathbb{V}^{\varepsilon})_{\omega_a(t_a),\omega_b(t_b)} \prod_{a\in A\setminus[\Pi]} \mathrm{i}\big((\mathbb{V}^{\varepsilon})_{\omega_a(t_a)}(\omega_1) - (\mathbb{V}^{\varepsilon})_{\omega_a(t_a)}(\widetilde{\omega}_1)\big).
\end{aligned} \tag{5.74}$$

[10] $\bar{I}^{\varepsilon}(\mathbf{r}+\boldsymbol{\alpha},\widetilde{\mathbf{r}}+\widetilde{\boldsymbol{\alpha}})$ can easily be written as a function of $(\mathbf{r},\widetilde{\mathbf{r}})$. This notation is more convenient for the purposes of our argument.

We set $\boldsymbol{\alpha} = \widetilde{\boldsymbol{\alpha}} = \mathbf{0}$ in (5.69) and recall (5.72)–(5.73) to write

$$I^{\varepsilon}(\mathbf{r},\widetilde{\mathbf{r}}) = \int \mathrm{d}x_1 \int \mathrm{d}\widetilde{x}_1 \int \mathbb{W}^{|\mathbf{r}|,0}_{x_1,x_1}(\mathrm{d}\omega)\, \mathbb{W}^{|\widetilde{\mathbf{r}}|,0}_{\widetilde{x}_1,\widetilde{x}_1}(\mathrm{d}\widetilde{\omega})\, \mathrm{e}^{-\frac{1}{2}(\mathbb{V}^{\varepsilon}(\omega_1,\omega_1)+\mathbb{V}^{\varepsilon}(\widetilde{\omega}_1,\widetilde{\omega}_1)-2\mathbb{V}^{\varepsilon}(\omega_1,\widetilde{\omega}_1))} \times \sum_{\Pi\in\mathbb{M}(A)} \prod_{\{a,b\}\in\Pi} (\mathbb{V}^{\varepsilon})_{\omega_a(s_a),\omega_b(s_b)} \prod_{a\in A\setminus[\Pi]} \mathrm{i}\big((\mathbb{V}^{\varepsilon})_{\omega_a(s_a)}(\omega_1) - (\mathbb{V}^{\varepsilon})_{\omega_a(s_a)}(\widetilde{\omega}_1)\big). \tag{5.75}$$

In order to compare (5.74) and (5.75), we note the following three estimates, which hold uniformly in $\eta > 0$.

(i) We have

$$\mathrm{e}^{-\frac{1}{2}(\mathbb{V}^{\varepsilon}(\omega_1,\omega_1)+\mathbb{V}^{\varepsilon}(\widetilde{\omega}_1,\widetilde{\omega}_1)-2\mathbb{V}^{\varepsilon}(\omega_1,\widetilde{\omega}_1))} = \mathrm{e}^{-\frac{1}{2}(\mathbb{V}^{\varepsilon}(\omega|_{[0,r_1]},\omega_{[0,r_1]})+\mathbb{V}^{\varepsilon}(\widetilde{\omega}_{[0,\widetilde{r}_1]},\widetilde{\omega}_{[0,\widetilde{r}_1]})-2\mathbb{V}^{\varepsilon}(\omega_{[0,r_1]},\widetilde{\omega}_{[0,\widetilde{r}_1]}))} + O\bigg(\frac{\nu}{\varepsilon^d}(1+r_1+\widetilde{r}_1)\bigg).$$

(ii) For all $a, b \in A$, we have

$$|(\mathbb{V}^{\varepsilon})_{\omega_a(t_a),\omega_b(t_b)} - (\mathbb{V}^{\varepsilon})_{\omega_a(s_a),\omega_b(s_b)}| \leqslant \frac{C}{\varepsilon^{d+1}}\big(|\omega_a(t_a)-\omega_a(s_a)|_{\Lambda} + |\omega_b(t_b)-\omega_b(s_b)|_{\Lambda}\big).$$

(iii) For all $a \in A$, we have

$$\big|(\mathbb{V}^{\varepsilon})_{\omega_a(t_a)}(\widehat{\omega}_1) - (\mathbb{V}^{\varepsilon})_{\omega_a(s_a)}(\widehat{\omega}_1)\big| \leqslant \frac{C}{\varepsilon^{d+1}}\,\widehat{r}_1\,|\omega_a(t_a)-\omega_a(s_a)|_{\Lambda}.$$

Claim (i) follows by arguing as for (5.50). Claim (ii) follows by using Definition 5.12 and Lemma 5.5 (ii). Claim (iii) is shown analogously.

Starting from the identities (5.74), (5.75), using (5.56), estimates (i)–(iii) above, and recalling Lemma 5.4 (i)–(ii), we deduce that

$$|\bar{I}^{\varepsilon}(\mathbf{r}+\boldsymbol{\alpha},\widetilde{\mathbf{r}}+\widetilde{\boldsymbol{\alpha}}) - I^{\varepsilon}(\mathbf{r},\widetilde{\mathbf{r}})| = O\bigg(\frac{\sqrt{\nu}}{\varepsilon^{5d+1}}\bigg(1+\frac{1}{|\mathbf{r}|^{d/2}}\bigg)\bigg(1+\frac{1}{|\widetilde{\mathbf{r}}|^{d/2}}\bigg)(1+|\mathbf{r}|+|\widetilde{\mathbf{r}}|)^5\bigg). \tag{5.76}$$

Note that here we have again used estimate (5.9) when applying Lemma 5.4 (ii). Combining (5.69)–(5.70) and (5.76), we obtain (5.63) and the lemma follows. ∎

In order to prove Lemma 5.11, we need to make some minor modifications. Arguing analogously to [33, (5.31)–(5.33)], we have

$$\int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma)\,\overline{f_2(\langle\sigma\rangle)}F_2(\sigma) = \frac{1}{\nu^3}\sum_{\mathbf{r}\in(\nu\mathbb{N})^3} \frac{\mathbf{1}_{|\mathbf{r}|>0}\mathrm{e}^{-\kappa|\mathbf{r}|}}{|\mathbf{r}|}\int_{[0,\infty)^3}\mathrm{d}\widetilde{\mathbf{r}}\,\frac{\mathrm{e}^{-\kappa|\widetilde{\mathbf{r}}|}}{|\widetilde{\mathbf{r}}|}\widetilde{J}^{\varepsilon}(\mathbf{r},\widetilde{\mathbf{r}}), \tag{5.77}$$

where

$$\begin{aligned}
&\widetilde{J}^{\varepsilon}(\mathbf{r},\widetilde{\mathbf{r}}) \\
&:= \int_{[0,\nu]^3} \mathrm{d}\boldsymbol{\tau} \int_{[0,\nu]^3} \mathrm{d}\widetilde{\boldsymbol{\tau}} \int \mathrm{d}\mathbf{x} \int \mathrm{d}\widetilde{\mathbf{x}} \int \mathbb{W}^{\tau_1+r_3,\tau_3}_{x_1,x_3}(\mathrm{d}\omega_3)\, \mathbb{W}^{\tau_3+r_2,\tau_2}_{x_3,x_2}(\mathrm{d}\omega_2)\, \mathbb{W}^{\tau_2+r_1,\tau_1}_{x_2,x_1}(\mathrm{d}\omega_1) \\
&\quad \times \mathbb{W}^{\widetilde{r}_3,0}_{\widetilde{x}_1,\widetilde{x}_3}(\mathrm{d}\widetilde{\omega}_3)\, \mathbb{W}^{\widetilde{r}_2,0}_{\widetilde{x}_3,\widetilde{x}_2}(\mathrm{d}\widetilde{\omega}_2)\, \mathbb{W}^{\widetilde{r}_1,0}_{\widetilde{x}_2,\widetilde{x}_1}(\mathrm{d}\widetilde{\omega}_1)\, \mathrm{e}^{-\frac{1}{2}(\mathbb{V}_\eta(\omega_1,\omega_1)+\mathbb{V}^{\varepsilon}(\widetilde{\omega}_1,\widetilde{\omega}_1)-2\mathbb{V}^{\varepsilon}(\omega_1,\widetilde{\omega}_1))} \\
&\quad \times \sum_{\Pi\in\mathbb{M}(A)} \prod_{\{a,b\}\in\Pi} (\mathbb{V}_\eta)^{\tau_a,\tau_b}_{x_a,x_b} \prod_{a\in A\setminus[\Pi]} \mathrm{i}\big((\mathbb{V}_\eta)^{\tau_a}_{x_a}(\omega_1) - (\mathbb{V}^{\varepsilon})_{x_a}(\widetilde{\omega}_1)\big). \qquad (5.78)
\end{aligned}$$

The following analogues of Lemmas 5.14 and 5.15 hold.

Lemma 5.16. *For all* $\mathbf{r} \in (\nu\mathbb{N})^3$ *and* $\widetilde{\mathbf{r}} \in [0,\infty)^3$ *with* $|\mathbf{r}|, |\widetilde{\mathbf{r}}| > 0$, *we have, uniformly in* $\eta > 0$,

$$J(\mathbf{r},\widetilde{\mathbf{r}}) = \nu^6 I^{\varepsilon}(\mathbf{r},\widetilde{\mathbf{r}}) + O\bigg(\frac{\nu^{13/2}}{\varepsilon^{5d+1}}\bigg(1+\frac{1}{|\mathbf{r}|^{d/2}}\bigg)\bigg(1+\frac{1}{|\widetilde{\mathbf{r}}|^{d/2}}\bigg)(1+|\mathbf{r}|+|\widetilde{\mathbf{r}}|)^6\bigg).$$

Lemma 5.17. *Recalling* (5.33), *we have, uniformly in* $\eta > 0$,

$$\begin{aligned}
&\bigg|\int_{[0,\infty)^3} \mathrm{d}\mathbf{r}\, \frac{\mathrm{e}^{-\kappa|\mathbf{r}|}}{|\mathbf{r}|} \int_{[0,\infty)^3} \mathrm{d}\widetilde{\mathbf{r}}\, \frac{\mathrm{e}^{-\kappa|\widetilde{\mathbf{r}}|}}{|\widetilde{\mathbf{r}}|} I^{\varepsilon}(\mathbf{r},\widetilde{\mathbf{r}}) \\
&\qquad - \nu^3 \sum_{\mathbf{r}\in(\nu\mathbb{N})^3} \frac{\mathbf{1}_{|\mathbf{r}|>0}\mathrm{e}^{-\kappa|\mathbf{r}|}}{|\mathbf{r}|} \int_{[0,\infty)^3} \mathrm{d}\widetilde{\mathbf{r}}\, \frac{\mathrm{e}^{-\kappa|\widetilde{\mathbf{r}}|}}{|\widetilde{\mathbf{r}}|} I^{\varepsilon}(\mathbf{r},\widetilde{\mathbf{r}})\bigg| \lesssim_{\kappa,\nu} \frac{\Theta(\nu)}{\varepsilon^{5d+1}}.
\end{aligned}$$

The proofs of Lemmas 5.16 and 5.17 are very similar to those of Lemmas 5.14 and 5.15; see Appendix C.

Proof of Lemma 5.16. The proof is analogous to that of Lemma 5.14 given above. The only difference is that for $\widehat{\omega}_1 = \widetilde{\omega}_1$, we replace (5.52) by the estimate

$$(\mathbb{V}^{\varepsilon})_{\omega_a(t_a)}(\widetilde{\omega}_1) = (\mathbb{V}^{\varepsilon})_{\omega_a(s_a)}(\widetilde{\omega}|_{[\widetilde{\tau}_1,\widetilde{\tau}_1+\widetilde{r}_1]}) + O\bigg(\frac{\widetilde{r}_1}{\varepsilon^{d+1}}|\omega_a(t_a)-\omega_a(s_a)|_\Lambda\bigg) + O\bigg(\frac{\nu}{\varepsilon^d}\bigg),$$

which follows from Lemma 5.5 (i)–(ii) and (5.43)–(5.44). ∎

Proof of Lemma 5.17. The result follows directly from the proof of Lemma 5.15. ∎

Proof of Lemma 5.11. The claim follows from Lemmas 5.16 and 5.17 by using (5.35) and (5.77). ∎

5.2. *Correlation functions*

We note the following analogues of Lemmas 5.6 and 5.7 that give us functional integral representations for (2.21) and (2.20) respectively.

Lemma 5.18. *For all $p \in \mathbb{N}$, we have $\widehat{\Gamma}_{p,\eta} \xrightarrow{C} \widehat{\Gamma}_p$ as $\eta \to 0$, where $\widehat{\Gamma}_p$ satisfies*

$$\nu^p \, \widehat{\Gamma}_{p,\eta} = \frac{p!}{\mathcal{Z}_\eta^\varepsilon} P_p \, Q_{p,\eta} \tag{5.79}$$

for

$$(Q_{p,\eta})_{\mathbf{x},\widetilde{\mathbf{x}}} := \int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma) \, \mathrm{e}^{T^\varepsilon - \frac{\mathrm{i}\tau^\varepsilon[\sigma]}{\nu}} \, \mathrm{e}^{F_2(\sigma)} \times \prod_{i=1}^{p} \Bigg[\nu \sum_{r_i \in \nu\mathbb{N}^*} \mathrm{e}^{-\kappa r_i} \int \mathbb{W}_{x_i,\widetilde{x}_i}^{r_i,0}(\mathrm{d}\omega_i) \left(\mathrm{e}^{\mathrm{i}\int_0^{r_i} \mathrm{d}t \, \sigma([t]_\nu, \omega_i(t))} - 1 \right) \Bigg]. \tag{5.80}$$

Here, F_2 is given as in (5.18). Moreover, we recall that P_p denotes the projection given by (2.5) and $\xrightarrow{C}$ convergence in the space of continuous functions on $\Lambda^p \times \Lambda^p$ with respect to the supremum norm.

Proof. We obtain (5.79)–(5.80) by arguing analogously to [33, (5.36)–(5.37)]. The only difference is that, in the σ integral, we have to add the extra factor of $\mathrm{e}^{T^\varepsilon - \mathrm{i}\tau^\varepsilon[\sigma]/\nu}$ due to the ε-dependent corrections in (2.18). This change is justified by arguing analogously to the proof of Proposition 5.6. ∎

Lemma 5.19. *For all $p \in \mathbb{N}$, we have*

$$\hat{\gamma}_p^{W^\varepsilon} = \frac{p!}{\xi^{W^\varepsilon}} P_p q_p^\varepsilon \tag{5.81}$$

for

$$(q_p^\varepsilon)_{\mathbf{x},\widetilde{\mathbf{x}}} := \int \mu_{v^\varepsilon}(\mathrm{d}\xi) \, \mathrm{e}^{T^\varepsilon - \mathrm{i}\tau^\varepsilon \langle \xi, 1 \rangle_{L^2}} \, \mathrm{e}^{f_2(\xi)} \times \prod_{i=1}^{p} \Bigg[\int_0^\infty \mathrm{d}r_i \, \mathrm{e}^{-\kappa r_i} \int \mathbb{W}_{x_i,\widetilde{x}_i}^{r_i,0}(\mathrm{d}\omega_i) \left(\mathrm{e}^{\mathrm{i}\int_0^{r_i} \mathrm{d}t \, \xi(\omega_i(t))} - 1 \right) \Bigg]. \tag{5.82}$$

Here, f_2 is given as in (5.25).

Proof. We obtain (5.81)–(5.82) analogously to [33, (5.38)]. The only difference is that, in the ξ integral, we have to add the extra factor of $\mathrm{e}^{T^\varepsilon - \mathrm{i}\tau^\varepsilon \langle \xi, 1 \rangle_{L^2}}$ due to the second and third terms in (3.1). This change is justified by arguing analogously to the proof of Proposition 5.7. Moreover, since v^ε is already assumed to be smooth, there is no need to regularize it with the parameter η, which was used in [33]. ∎

Recalling (5.2), we now compare the quantities (5.80) and (5.82).

Lemma 5.20. *There exists $c_3 > 0$ depending on κ, v such that the following results hold for all $p \in \mathbb{N}$:*

(i) *We have*

$$\| Q_{p,\eta} \|_C \leq C_{\kappa,v}^p \mathrm{e}^{c_3 \chi(\varepsilon)^2} \left(\frac{1}{\varepsilon^d} \right)^{p/2}.$$

(ii) *For $\theta(d,p)$ as in (5.5), we have, uniformly in $\eta > 0$,*

$$\|Q_{p,\eta} - q_p^\varepsilon\|_C \leqslant C_{\kappa,v}^p \mathrm{e}^{c_3\chi(\varepsilon)^2}\left(\left(\frac{1}{\varepsilon^d}\right)^{p/2} + \left(\frac{1}{\varepsilon^{5d+1}}\right)^{1/2}\right)\nu^{\theta(d,p)}. \tag{5.83}$$

Furthermore, we show the following lower bound on the classical and quantum relative partition functions.

Lemma 5.21 (Lower bound on the relative partition function). *The following estimates hold for some constant $c_4 > 0$ depending on κ:*

(i) $\zeta^{W^\varepsilon} \geqslant \exp[-c_4\chi(\varepsilon)^2]$.

(ii) $\mathcal{Z} \geqslant \exp[-c_4\chi(\varepsilon)^2]$.

We prove Lemmas 5.20 and 5.21 in Appendix C. Using Lemmas 5.20 and 5.21, we now prove Proposition 5.2.

Proof of Proposition 5.2. By Lemmas 5.6, 5.18, and 5.19 it suffices to estimate

$$\lim_{\eta\to 0}\left\|\frac{Q_{p,\eta}}{\mathcal{Z}_\eta} - \frac{q_p^\varepsilon}{\zeta^{W^\varepsilon}}\right\|_C \leqslant \frac{\left|\mathcal{Z} - \zeta^{W^\varepsilon}\right|}{\mathcal{Z}\,\zeta^{W^\varepsilon}} \lim_{\eta\to 0}\|Q_{p,\eta}\|_C + \frac{1}{\zeta^{W^\varepsilon}} \lim_{\eta\to 0}\|Q_{p,\eta} - q_p^\varepsilon\|_C. \tag{5.84}$$

The claim now follows from (5.84) by using Proposition 5.1, Lemma 5.20 (i) and Lemma 5.21 to estimate the first term, and Lemma 5.20 (ii) combined with Lemma 5.21 (i) to estimate the second term. Throughout we recall (5.1), and we obtain the claim if we take $C_2 > C_1 + c_3 + 2c_4$. ∎

5.3. The mean-field limit for unbounded nonlocal interactions in dimensions $d = 2, 3$

We conclude this section by using the techniques developed above to extend the mean-field limit of [33, 56], with nonlocal interaction, from bounded interaction potentials to unbounded interaction potentials. Our assumptions on the potential are the same as in the seminal work [13]. Previously, the mean-field limit with unbounded interaction potentials was considered in [73], however with a modified, regularized, quantum many-body state instead of the grand canonical state (1.6). We remark that the results in [13] are originally stated in a setting that does not assume any positivity of the interaction, and hence require a truncation in the Wick-ordered mass of the field. However, when the setting is restricted to positive (defocusing) interactions, the truncation can be removed.

Assumption 5.22. In the classical setting, we consider $v \in L^q(\Lambda)$ which is even, real-valued, and of positive type, such that

$$\begin{cases} q > 1 & \text{if } d = 2, \\ q > 3 & \text{if } d = 3. \end{cases} \tag{5.85}$$

Note that, in terms of L^q integrability, (5.85) is the optimal range for q. We refer the reader to [13] and [73, Section 1.4] for a further discussion. In particular, for $d = 2$ we

can take v to be the Coulomb potential. For q as in (5.85), one verifies that $G \in L^{2q'}(\Lambda)$ (where q' denotes the Hölder conjugate of q) by writing G as a Fourier series and using Sobolev embedding.

With v as above, we study the interacting field theory (2.2), where now

$$V = \frac{1}{2}\int \mathrm{d}x\, \mathrm{d}y : |\phi(x)|^2 : v(x-y) : |\phi(y)|^2 :. \tag{5.86}$$

The interaction V is rigorously defined by using a frequency truncation, as in Section 2.1. We refer the reader to [73, Lemma 1.4] for a precise summary. We also make the appropriate modifications in the definition of the correlation functions.

Similarly to (5.10), we consider $\psi \in C_c^\infty(\mathbb{R}^d)$ even, nonnegative and satisfying $\psi(0) = 1$ and for $\varepsilon > 0$, we define $\delta_\varepsilon : \Lambda \to \mathbb{C}$ by

$$\delta_{\varepsilon,\Lambda}(x) := \frac{1}{\varepsilon^d} \sum_{y \in \mathbb{Z}^d} (\mathfrak{F}^{-1}\psi)\left(\frac{x-y}{\varepsilon}\right), \tag{5.87}$$

where $\mathfrak{F}$ denotes Fourier transform (see (4.6)). With notation as in (5.87) and v as in Assumption 5.22, we now write

$$v^\varepsilon := v * \delta_{\varepsilon,\Lambda} \in C^\infty(\Lambda). \tag{5.88}$$

From (5.88), we deduce that

$$\|v^\varepsilon\|_{L^q} = \|v\|_{L^q}, \quad v^\varepsilon \xrightarrow{L^q} v. \tag{5.89}$$

When working in the quantum setting, we hence set $\tau^\varepsilon = 0$ and $E^\varepsilon = 0$ in (2.18). In this section, instead of (5.1), we take

$$\varepsilon(\nu) \gtrsim \frac{1}{\log \nu^{-1}}. \tag{5.90}$$

Theorem 5.23. *Let $d \leqslant 3$. With V as in (5.86), v as in Assumption 5.22, and $\varepsilon > 0$ as in (5.90), we have the following results as $\varepsilon, \nu \to 0$:*

(i) $\mathcal{Z} \to \zeta^V$.

(ii) $\nu^p \widehat{\Gamma}_p \xrightarrow{C} \widehat{\gamma}_p^V$.

Proof. We first prove (i). If $v \in L^q(\Lambda)$, we use (5.88) and Young's inequality, to deduce that for $\varepsilon > 0$ sufficiently small, we have

$$\|v^\varepsilon\|_{L^\infty(\Lambda)} \lesssim_{\psi,q,v} \varepsilon^{-d/q}, \quad \|\nabla v^\varepsilon\|_{L^\infty(\Lambda)} \lesssim_{\psi,q,v} \varepsilon^{-d/q-1}. \tag{5.91}$$

Furthermore, v^ε is of positive type and has compactly supported Fourier transform. We use the functional integral setup as before and start from appropriate analogues of Lemmas 5.6 and 5.7. Using (5.91) instead of Lemma 5.5, and arguing as for Lemmas 5.9–5.11, we deduce that for $\varepsilon > 0$ sufficiently small and all $\nu > 0$, we have

$$\left|\mathcal{Z} - \zeta^{V^\varepsilon}\right| \lesssim \left(\frac{\Theta(\nu)}{\varepsilon^{5d/q+1}}\right)^{1/2}, \tag{5.92}$$

where V^ε denotes the interaction as in (5.86) with v replaced by v^ε, and $\Theta(v)$ is as in (5.33). By (5.90), this is an acceptable upper bound and we reduce the claim to showing that

$$\lim_{\varepsilon\to 0} \zeta^{V^\varepsilon} = \zeta^V. \tag{5.93}$$

Using the assumption that v, v^ε are of positive type and the Cauchy–Schwarz inequality, we obtain

$$|\zeta^\varepsilon - \zeta| \leqslant \|V^\varepsilon - V\|_{L^2(\mathbb{P})}. \tag{5.94}$$

We now show that

$$\|V^\varepsilon - V\|_{L^2(\mathbb{P})} \lesssim \|v^\varepsilon - v\|_{L^q}. \tag{5.95}$$

By (5.89), we see that (5.95) indeed implies (5.93).

By using Fubini's theorem followed by Wick's theorem, we can rewrite the right-hand side of (5.94) as

$$\frac{1}{2}\bigg(\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,F^\varepsilon(x,\widetilde{x},y,\widetilde{y})\bigg)^{1/2}, \tag{5.96}$$

where

$$\begin{aligned} &F^\varepsilon(x,\widetilde{x},y,\widetilde{y}) \\ &:= \sum_{\Pi\in\mathcal{M}_c^{\mathrm{Wick}}(B)} \prod_{\{i,j\}\in\Pi} G(x_i - x_j)\big(v^\varepsilon(x-y) - v(x-y)\big)\big(v^\varepsilon(\widetilde{x}-\widetilde{y}) - v(\widetilde{x}-\widetilde{y})\big). \end{aligned} \tag{5.97}$$

In (5.97), we let $B := \{1,2\}\times\{1,2\}\times\{+,-\}$. Furthermore, we use the variables $x_{(1,1,\pm)} \equiv x$, $x_{(2,1,\pm)} \equiv \widetilde{x}$, $x_{(1,2,\pm)} \equiv y$, $x_{(2,2,\pm)} \equiv \widetilde{y}$ and denote by $\mathcal{M}_c^{\mathrm{Wick}}(B)$ the set of all complete pairings Π of B where $\{(a,b,+),(a,b,-)\} \notin \Pi$ for all $a,b\in\{1,2\}$.

We note that each integration variable in (5.96) appears exactly once as part of the argument of $v^\varepsilon - v$ and exactly twice as part of an argument of a Green function. We can therefore apply Hölder's inequality and deduce (5.95). Here, we use the fact that, uniformly in $c, d \in \Lambda$,

$$\int \mathrm{d}x\,|w(x)|(1+G(x-c))(1+G(x-d)) \leqslant \|w\|_{L^q}(1+\|G\|_{L^{2q'}}^2) \lesssim \|w\|_{L^q}, \tag{5.98}$$

with $w = v^\varepsilon - v$, since $G \in L^{2q'}(\Lambda)$. When applying (5.98), we recall (5.89).

We now prove (ii). With notation defined analogously to (5.79)–(5.82), we use (5.91) and argue as in the proof of Lemma 5.20 to deduce that for all $\varepsilon > 0$ and uniformly in $\eta > 0$,

$$\begin{aligned} \|Q_{p,\eta}\|_C &\leqslant C_{\kappa,v}^p\bigg(\frac{1}{\varepsilon^{d/q}}\bigg)^{p/2}, \\ \|Q_{p,\eta} - q_p^\varepsilon\|_C &\leqslant C_{\kappa,v}^p\bigg(\bigg(\frac{1}{\varepsilon^{d/q}}\bigg)^{p/2} + \bigg(\frac{1}{\varepsilon^{5d/q+1}}\bigg)^{1/2}\bigg)\nu^{\theta(d,p)}, \end{aligned} \tag{5.99}$$

with $\theta(d,p)$ given as in (5.5). Similarly, arguing as in the proof of Lemma 5.21, using

Wick's theorem we find that for all $\varepsilon > 0$,

$$\zeta^{V^\varepsilon} \gtrsim 1, \quad \mathcal{Z} \geqslant \exp\left[-c\left(1 + \frac{\nu\chi(\sqrt{\nu})}{\varepsilon^{d/q}}\right)\right], \tag{5.100}$$

with χ as in (5.2). In order to obtain (5.100) we have used (5.98). Using (5.99)–(5.100), arguing as in the proof of Proposition 5.2, and recalling (5.90), we deduce that

$$\left\| \nu^p \widehat{\Gamma}_p - \widehat{\gamma}_p^{V^\varepsilon} \right\|_C \lesssim \nu^\alpha \tag{5.101}$$

for $0 < \alpha < \theta(d, p)$. Hence, we reduce to showing

$$\lim_{\varepsilon \to 0} \|\widehat{\gamma}_p^{V^\varepsilon} - \widehat{\gamma}_p^{V}\|_C = 0. \tag{5.102}$$

By (5.95), we have $\|\mathrm{e}^{-V^\varepsilon} - \mathrm{e}^{-V}\|_{L^2(\mathbb{P})} \to 0$ as $\varepsilon \to 0$. Therefore, we obtain (5.102) if we prove bounds analogous to those in Lemma 4.10 (with W^ε replaced by V^ε). More precisely, we note that the following claims hold, with V as in (5.86) and V^ε as in (5.86) with v replaced by v^ε and notation as in Lemma 4.10:

(i) $\sup_N \sup_{\mathbf{z}} \|\mathbb{L}_{N,\mathbf{z}} V\|_{L^r(\mathbb{P})} < \infty$.

(ii) As $\varepsilon \to 0$ we have $\sup_N \sup_{\mathbf{z}} \|\mathbb{L}_{N,\mathbf{z}} V^\varepsilon - \mathbb{L}_{N,\mathbf{z}} V\|_{L^r(\mathbb{P})} \to 0$.

(iii) As $M, N \to \infty$ we have $\sup_{\mathbf{z}} \|(\mathbb{L}_{N,\mathbf{z}} - \mathbb{L}_{M,\mathbf{z}}) V\|_{L^r(\mathbb{P})} \to 0$.

(iv) For any $\varepsilon > 0$, as $M, N \to \infty$ we have $\sup_{\mathbf{z}} \|(\mathbb{L}_{N,\mathbf{z}} - \mathbb{L}_{M,\mathbf{z}}) V^\varepsilon\|_{L^r(\mathbb{P})} \to 0$.

We first show (ii). For fixed $0 < N < \infty$ and $z \in \Lambda$, we note that by Lemma 4.8,

$$L_{N,z}(V^\varepsilon - V) = \int \mathrm{d}x \, \mathrm{d}\widetilde{x} \left(v^\varepsilon(x - \widetilde{x}) - v(x - \widetilde{x})\right) G_N(z - x) \phi(x) : |\phi(\widetilde{x})|^2 :$$

and hence by Lemma A.1,

$$\begin{aligned} &\|L_{N,z}(V^\varepsilon - V)\|_{L^2(\mathbb{P})}^2 \\ &= \int \mathrm{d}x \, \mathrm{d}\widetilde{x} \, \mathrm{d}y \, \mathrm{d}\widetilde{y} \, [v^\varepsilon(x - \widetilde{x}) - v(x - \widetilde{x})][v^\varepsilon(y - \widetilde{y}) - v(y - \widetilde{y})] \, G_N(z - x) \, G_N(z - y) \\ &\times [G(x - y) \, G(\widetilde{x} - \widetilde{y})^2 + G(x - \widetilde{x}) \, G(\widetilde{x} - \widetilde{y}) G(y - \widetilde{y}) + G(x - \widetilde{y}) G(\widetilde{x} - \widetilde{y}) G(y - \widetilde{x})]. \end{aligned} \tag{5.103}$$

Analogously to (5.96), each integration variable in (5.103) appears exactly once as part of the argument of $v^\varepsilon - v$ and exactly twice as part of the argument of G or G_N. We then use Hölder's inequality as in (5.98) where some of the factors of G can be replaced by G_N and note that $\|G_N\|_{L^{2q'}} \leqslant \|G\|_{L^{2q'}} < \infty$, which holds by (4.8) and Young's inequality. Putting everything together, we get

$$(5.103) \lesssim \|v^\varepsilon - v\|_{L^q}^2,$$

which is an acceptable bound by (5.89). The other terms for (ii) are treated similarly. We omit the details. The proof of (i) is analogous, except that now we apply Hölder's inequality similarly to (5.98) with $w = v$.

We now show (iii). We first compute

$$(L_{N,z} - L_{M,z})V = \int \mathrm{d}x\, \mathrm{d}\widetilde{x}\, v(x-\widetilde{x})(G_N(x-z) - G_M(x-z))\phi(x){:}|\phi(\widetilde{x})|^2{:}.$$

Hence, by Lemma A.1, we have

$$\begin{aligned}\|(L_{N,z} - L_{M,z})V\|^2_{L^2(\mathbb{P})} = \int \mathrm{d}x\, \mathrm{d}\widetilde{x}\, \mathrm{d}y\, \mathrm{d}\widetilde{y}\, v(x-\widetilde{x})\, v(y-\widetilde{y})\, [G_N(z-x) - G_M(z-x)] \\ \times [G_N(z-y) - G_M(z-y)][G(x-y)G(\widetilde{x}-\widetilde{y})^2 + G(x-\widetilde{x})G(\widetilde{x}-\widetilde{y})G(y-\widetilde{y}) \\ + G(x-\widetilde{y})G(\widetilde{x}-\widetilde{y})G(y-\widetilde{x})] \qquad (5.104)\end{aligned}$$

Since each integration variable occurs exactly once as part of an argument of v and exactly twice as a part of an argument of G or $G_N - G_M$, we can use Hölder's inequality as earlier to deduce that

$$(5.104) \lesssim \|v\|^2_{L^q} \|G_N - G_M\|^2_{L^{2q'}} (1 + \|G\|^3_{L^{2q'}}),$$

which is an acceptable upper bound by using (B.2) as in the proof of Lemma 4.10 (iii). The higher order derivatives are estimated in the same way. The proof of (iv) is analogous, except that we now apply Hölder's inequality similarly to (5.98) with $w = v^\varepsilon$ and recall (5.89). We hence obtain (5.102). ∎

Remark 5.24. We note that if we relax the topology of convergence in (5.102) (and hence in Theorem 5.23 (ii)) to the weak operator topology, we can obtain the result by using the first bound in (5.100), and the Cauchy–Schwarz inequality, similarly to (5.94). This applies to the case $(d, q) = 1$ as well. We refer the reader to the proof of [33, Proposition 4.4] for details.

Remark 5.25. One can also consider $v \in L^1(\Lambda)$ which is even, real-valued, and of positive type with suitable decay on its Fourier coefficients (see [13, (16)–(17)]). For $d = 3$, the assumption in [13] is that $\widehat{v}(k) \leqslant C/\langle k\rangle^{2+\delta}$ for some $\delta > 0$, which is covered by Theorem 5.23 above by the Hausdorff–Young inequality (in the classical setting, the decay assumption was recently relaxed in [24]). For $d = 2$, the assumption in [13] is that $\widehat{v}(k) \leqslant C/\langle k\rangle^{\delta}$ for some $\delta > 0$. Note that this corresponds to the *endpoint admissible* regime in the terminology of [73, Definition 1.2 and Section 4], except that we do not assume pointwise nonnegativity of v. Here, it is possible to prove convergence of the partition function (and consequently the convergence of the correlation functions in the weak operator topology as in Remark 5.24 above). We present the details in Appendix C.

Appendix A. Wick ordering and hypercontractive moment bounds

In this appendix we recall some standard facts about Wick ordering and hypercontractive estimates. We refer e.g. to [65] for a comprehensive account. For the convenience of the reader, we keep this appendix self-contained.

A.1. Wick ordering

Let $\xi = (\xi_1, \dots, \xi_n)$ be a real[11] Gaussian vector with mean zero. We define the *Wick ordering of the monomial* $\xi_1 \cdots \xi_n$ through

$$:\xi_1 \cdots \xi_n: = \frac{\partial^n}{\partial\lambda_1 \cdots \partial\lambda_n} \frac{\mathrm{e}^{\lambda\cdot\xi}}{\mathbb{E}[\mathrm{e}^{\lambda\cdot\xi}]}\bigg|_{\lambda=0}. \tag{A.1}$$

The expectation is equal to

$$\mathbb{E}[\mathrm{e}^{\lambda\cdot\xi}] = \mathrm{e}^{\frac{1}{2}\lambda\cdot\mathcal{C}\lambda},$$

where

$$\mathcal{C}_{ij} := \mathbb{E}[\xi_i\xi_j]$$

is the covariance matrix of ξ. Computing the derivatives explicitly, we hence find

$$:\xi_1 \cdots \xi_n: = \sum_{\Pi\in\mathfrak{M}([n])} \prod_{i\in[n]\setminus[\Pi]} \xi_i \prod_{\{i,j\}\in\Pi} (-\mathbb{E}[\xi_i\xi_j]), \tag{A.2}$$

where we defined $[n] := \{1, \dots, n\}$, and $\mathfrak{M}([n])$ is the set of partial pairings of the set $[n]$ (i.e. a set of disjoint unordered pairs of elements of $[n]$) with $[\Pi] := \bigcup_{\{i,j\}\in\Pi}\{i, j\}$. Since both sides of (A.2) are linear in $\xi_1, \dots, \xi_n$, we can extend (A.2) to a complex Gaussian vector $(\xi_1, \dots, \xi_n)$.

By splitting the summation in (A.2) over Π satisfying $n \notin [\Pi]$ and $n \in [\Pi]$, we obtain the recursion

$$:\xi_1 \cdots \xi_n: = :\xi_1 \cdots \xi_{n-1}:\xi_n - \sum_{i=1}^{n-1} \mathbb{E}[\xi_i\xi_n]:\xi_1 \cdots \xi_{i-1}\xi_{i+1} \cdots \xi_{n-1}:. \tag{A.3}$$

Moreover, from the definition (A.1) we find that Wick ordering commutes with differentiation, since

$$\frac{\partial}{\partial\xi_1}:\xi_1 \cdots \xi_n: = \frac{\partial^n}{\partial\lambda_1 \cdots \partial\lambda_n} \frac{\lambda_1\mathrm{e}^{\lambda\cdot\xi}}{\mathbb{E}[\mathrm{e}^{\lambda\cdot\xi}]}\bigg|_{\lambda=0} = \frac{\partial^{n-1}}{\partial\lambda_2 \cdots \partial\lambda_n} \frac{\mathrm{e}^{\sum_{i=2}^n \lambda_i\xi_i}}{\mathbb{E}[\mathrm{e}^{\sum_{i=2}^n \lambda_i\xi_i}]}\bigg|_{\lambda=0} = :\xi_2 \cdots \xi_n:. \tag{A.4}$$

Note that the variables $\xi_1, \dots, \xi_n$ are treated as independent for the differentiation, although they need not be stochastically independent. For instance, (A.4) implies that $\frac{\mathrm{d}}{\mathrm{d}\xi}:\xi^n: = :n\xi^{n-1}:$.

The following is a generalization of Wick's rule to moments of Wick-ordered monomials.

Lemma A.1. *Let $\xi = (\xi_1, \dots, \xi_n)$ be a complex Gaussian vector with mean zero. Let Q be a partition of $[n]$. Then*

$$\mathbb{E}\Big[\prod_{q\in Q} :\prod_{i\in q} \xi_i:\Big] = \sum_{\Pi\in\mathfrak{M}_c([n],Q)} \prod_{\{i,j\}\in\Pi} \mathbb{E}[\xi_i\xi_j],$$

where $\mathfrak{M}_c([n], Q)$ is the set of complete pairings Π of the set $[n]$ such that no pair $\{i, j\}$ in Π satisfies $i, j \in q$ for some $q \in Q$.

[11] When dealing with complex vectors, we split them into their real and imaginary parts.

Proof. By linearity, we may assume that $(\xi_1, \dots, \xi_n)$ is real. From the definition (A.1) we find

$$\prod_{q\in Q} :\prod_{i\in q} \xi_i : = \frac{\partial^n}{\partial\lambda_1 \cdots \partial\lambda_n} e^{\lambda\cdot\xi} \prod_{q\in Q} \exp\Big(-\frac{1}{2}\sum_{i,j\in q} \mathcal{C}_{ij}\lambda_i\lambda_j\Big)\Big|_{\lambda=0},$$

so that taking the expectation yields

$$\mathcal{E}\Big[\prod_{q\in Q} :\prod_{i\in q} \xi_i :\Big] = \frac{\partial^n}{\partial\lambda_1 \cdots \partial\lambda_n} \exp\Big(\frac{1}{2}\sum_{i,j\in[n]} \mathcal{C}_{ij}^{Q}\lambda_i\lambda_j\Big)\Big|_{\lambda=0},$$

where $\mathcal{C}_{ij}^{Q} := (1 - \sum_{q\in Q} \mathbf{1}_{i,j\in q})\mathcal{C}_{ij}$. The claim now follows by differentiation. ∎

Example A.2. If Q has two blocks, then Lemma A.1 takes on the following form. If $(\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_n)$ is a complex Gaussian vector with mean zero, then

$$\mathbb{E}[:\xi_1 \cdots \xi_n :: \zeta_1 \cdots \zeta_n :] = \sum_{\sigma\in S_n} \prod_{i=1}^{n} \mathbb{E}[\xi_i \zeta_{\sigma(i)}].$$

A.2. Hypercontractive moment bounds

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a separable real Hilbert space, and let $(\phi(f))_{f\in\mathcal{H}}$ be the abstract Gaussian process indexed by $\mathcal{H}$. For an explicit definition, let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$, and let $\Omega = \mathbb{R}^{\mathbb{N}}$ be equipped with the product sigma-algebra $\mathcal{F}$ and the probability measure $\mathbb{P}$, which is an infinite product of standard Gaussians. With the notation $\omega = (\omega_k)_{k\in\mathbb{N}} \in \Omega$, we define

$$\phi(f) := \sum_{k\in\mathbb{N}} \omega_k \langle e_k, f\rangle,$$

which converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, since $\mathbb{E}[\phi(f)\phi(g)] = \langle f, g\rangle$, the map $\phi : \mathcal{H} \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ is an isometry.

For $n \in \mathbb{N}$ we define the nth *polynomial chaos*, denoted by $\mathcal{B}_n$, as the closure of the subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ spanned by random variables of the form

$$:\phi(f_1)\cdots\phi(f_n):, \quad f_1, \dots, f_n \in \mathcal{H}.$$

Lemma A.3. *We have*

$$L^2(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{n\in\mathbb{N}} \mathcal{B}_n.$$

Proof. The orthogonality of the spaces $(\mathcal{B}_n)_{n\in\mathbb{N}}$ is easy to deduce from the definition of Wick ordering. It remains to show that $L^2(\Omega, \mathcal{F}, \mathbb{P}) \subset \bigoplus_{n\in\mathbb{N}} \mathcal{B}_n$.

For $K \in \mathbb{N}$, let $\mathcal{H}_K = \mathrm{Span}(e_k : k \leqslant K) \subset \mathcal{H}$. Let $\mathcal{B}_n^{(K)}$ be the subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ spanned by random variables of the form $:\phi(f_1)\cdots\phi(f_n):$ with $f_1, \dots, f_n \in \mathcal{H}_K$. Let $\mathcal{F}_K$ be the sigma-algebra generated by $(\phi(f) : f \in \mathcal{H}_K)$. Note that $\mathcal{B}_n^{(K)} \subset L^2(\Omega, \mathcal{F}_K, \mathbb{P})$. We now claim that

$$L^2(\Omega, \mathcal{F}_K, \mathbb{P}) = \bigoplus_{n\in\mathbb{N}} \mathcal{B}_n^{(K)}. \tag{A.5}$$

To see this, it suffices to show that if $\xi \in L^2(\Omega, \mathcal{F}_K, \mathbb{P})$ satisfies $\mathbb{E}[\xi\zeta] = 0$ for every $\zeta \in \mathcal{B}_n^{(K)}$ and every $n \in \mathbb{N}$ then $\xi = 0$. We can write $\xi = f(\phi(e_1), \dots, \phi(e_K))$ for some measurable function $f : \mathbb{R}^K \to \mathbb{R}$, which is square integrable with respect to the standard Gaussian measure on $\mathbb{R}^K$, which we denote by μ_K. By assumption, $\int \mu_K(\mathrm{d}x)\, f(x)P(x) = 0$ for all polynomials P on $\mathbb{R}^K$. Since μ_K has Gaussian tails, we easily deduce that $f = 0$.

To deduce the claim from (A.5), we choose $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and set $\xi_K := \mathbb{E}[\xi|\mathcal{F}_K]$ so that $(\xi_K)_{K\in\mathbb{N}}$ is a martingale. By Doob's martingale convergence theorem (see e.g. [77, Chapter 12]), we have $\xi_K \to \xi$ in L^2, which concludes the proof. ∎

Next, we state and prove a hypercontractive moment bound. Such an estimate is usually derived as a consequence of the hypercontractive property of the Ornstein–Uhlenbeck semigroup associated with ϕ. Here we give an elementary and very simple argument, relying only on Lemma A.1 and the Cauchy–Schwarz inequality.

Lemma A.4. *Let $n \in \mathbb{N}$ and $\xi \in \mathcal{B}_n$. Then for any $p \in 2\mathbb{N}^*$ and some universal constant C, we have*

$$\mathbb{E}[\xi^p] \leqslant C p^{np/2} \left(\mathbb{E}[\xi^2]\right)^{p/2}.$$

Proof. Let

$$\xi = \sum_{k_1,\dots,k_n\in\mathbb{N}} a_{k_1\cdots k_n} {:}\phi(e_{k_1})\cdots\phi(e_{k_n}){:},$$

where $a_{k_1\cdots k_n}$ is symmetric under permutations. By Lemma A.1 (see also Example A.2) we have

$$\mathbb{E}[\xi^2] = n! \sum_{k_1,\dots,k_n} a_{k_1\cdots k_n}^2. \tag{A.6}$$

To estimate the pth moment, it is convenient to introduce the index sets $I = [p] \times [n]$ and $I_i = \{i\} \times [n]$ for $i \in [p]$. For any $A \subset I$ we use the notation $\mathbf{k}_A = (k_{ij} : (i, j) \in A) \in \mathbb{N}^A$ for the summation variables indexed by the set A. In this notation we can write $\xi = \sum_{\mathbf{k}_{I_i}} a_{\mathbf{k}_{I_i}} {:}\prod_{j=1}^n \phi(e_{k_{ij}}){:}$ for each $i \in [n]$, and hence we get, using Lemma A.1,

$$\mathbb{E}[\xi^p] = \sum_{\mathbf{k}_I} \prod_{i=1}^p a_{\mathbf{k}_{I_i}} \mathbb{E}\Big[\prod_{i=1}^p {:}\prod_{j=1}^n \phi(e_{k_{ij}}){:}\Big] = \sum_{\Pi} \sum_{\mathbf{k}_I} \prod_{i=1}^p a_{\mathbf{k}_{I_i}} \prod_{\{(i,j),(i',j')\}\in\Pi} \mathbf{1}_{k_{ij}=k_{i'j'}}, \tag{A.7}$$

where the summation ranges over all complete pairings Π of $[p] \times [n]$ such that for all $\{(i, j), (i', j')\} \in \Pi$ we have $i \neq i'$.

The idea is to fix Π and to sum over the variables k_{ij} in pairs connected by the delta function on the right-hand side of (A.7), by using a simple repeated application of Cauchy–Schwarz. To that end, we introduce an inductive summation of the edges of Π one by one; the order of summation is immaterial. After summing out a number of edges, we obtain a partial pairing $\Sigma \subset \Pi$, whose blocks contain those summation variables that have not yet been summed out. We refer to Figure A.1 for an illustration. For the example Π of Figure A.1, the complete estimate can be explicitly written out:

$$
\begin{aligned}
&\sum_{k,l,m,u,v,w} a_{klm}a_{klu}a_{vwu}a_{vwm} \\
&\qquad \leqslant \sum_{l,m,u,v,w} \Big(\sum_k a_{klm}^2\Big)^{1/2}\Big(\sum_k a_{klu}^2\Big)^{1/2} a_{vwu}a_{vwm} \\
&\qquad \leqslant \sum_{l,m,u,v} \Big(\sum_k a_{klm}^2\Big)^{1/2}\Big(\sum_k a_{klu}^2\Big)^{1/2}\Big(\sum_w a_{vwu}^2\Big)^{1/2}\Big(\sum_w a_{vwm}^2\Big)^{1/2} \\
&\qquad \leqslant \sum_{l,u,v} \Big(\sum_{k,m} a_{klm}^2\Big)^{1/2}\Big(\sum_k a_{klu}^2\Big)^{1/2}\Big(\sum_w a_{vwu}^2\Big)^{1/2}\Big(\sum_{m,w} a_{vwm}^2\Big)^{1/2} \\
&\qquad \leqslant \sum_{u,v} \Big(\sum_{k,l,m} a_{klm}^2\Big)^{1/2}\Big(\sum_{k,l} a_{klu}^2\Big)^{1/2}\Big(\sum_w a_{vwu}^2\Big)^{1/2}\Big(\sum_{m,w} a_{vwm}^2\Big)^{1/2} \\
&\qquad \leqslant \sum_{v} \Big(\sum_{k,l,m} a_{klm}^2\Big)^{1/2}\Big(\sum_{k,l,u} a_{klu}^2\Big)^{1/2}\Big(\sum_{u,w} a_{vwu}^2\Big)^{1/2}\Big(\sum_{m,w} a_{vwm}^2\Big)^{1/2} \\
&\qquad \leqslant \Big(\sum_{k,l,m} a_{klm}^2\Big)^{1/2}\Big(\sum_{k,l,u} a_{klu}^2\Big)^{1/2}\Big(\sum_{u,v,w} a_{vwu}^2\Big)^{1/2}\Big(\sum_{m,v,w} a_{vwm}^2\Big)^{1/2}.
\end{aligned}
$$

The estimate has six steps, corresponding to the six edges to be summed out. The third step corresponds to the step illustrated in the right half of Figure A.1.

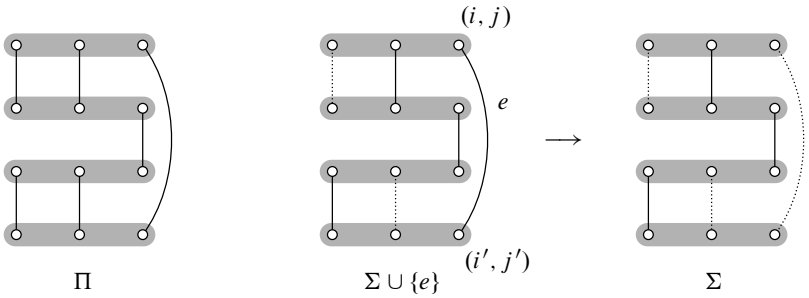

Fig. A.1. An illustration of the inductive algorithm for successively summing out edges of the pairing Π. Here $p = 4$ and $n = 3$. The pairing Π is illustrated in the figure on the left-hand side. The four grey blocks correspond to the four factors $a_{\mathbf{k}_{I_1}}, \dots, a_{\mathbf{k}_{I_4}}$, and each vertex $(i, j) \in [p] \times [n]$ corresponds to a summation variable k_{ij}. Note that each edge connects vertices from different grey blocks, as is required by Lemma A.1. A partial pairing $\Sigma \subset \Pi$ is represented in the figure on the right-hand side, where the edges of Σ are drawn using solid lines. The dashed lines, incident to vertices corresponding to the summation variables $\mathbf{k}_{I\setminus[\Sigma]}$, have been summed out at this point. This summation contributed a factor $\prod_{i=1}^{p}(\sum_{\mathbf{k}_{I_i}\setminus[\Sigma]} a_{\mathbf{k}_{I_i}}^2)^{1/2}$, which depends on the remaining summation variables $\mathbf{k}_{[\Sigma]}$ that correspond to the vertices incident to the edges of Π. The middle figure corresponds to the partial pairing $\Sigma \cup \{e\}$ with an edge $e \in \Pi \setminus \Sigma$ given by $e = \{(i, j), (i', j')\}$. The induction step underlying the argument, going from $\Sigma \cup \{e\}$ to Σ, is the summation of a single edge e, which amounts to summing over the variables $k_{ij} = k_{i',j'}$, and using Cauchy–Schwarz. For this, it is crucial that $i \neq i'$, i.e. e connects vertices in different grey blocks.

To describe the general procedure more formally, we denote, for any partial pairing $\Sigma \subset \Pi$,

$$V_\Pi(\Sigma) := \sum_{\mathbf{k}_{[\Sigma]}} \prod_{i=1}^{p} \Big(\sum_{\mathbf{k}_{I_i \setminus [\Sigma]}} a^2_{\mathbf{k}_{I_i}} \Big)^{1/2} \prod_{\{(i,j),(i',j')\} \in \Sigma} \mathbf{1}_{k_{ij} = k_{i'j'}}, \tag{A.8}$$

where $[\Sigma] := \bigcup_{\sigma \in \Sigma} \sigma$. This expression has three crucial properties. First, by (A.7),

$$\mathbb{E}[\xi^p] = \sum_{\Pi} V_\Pi(\Pi). \tag{A.9}$$

Second, by the definition (A.8),

$$V_\Pi(\emptyset) = \Big(\sum_{k_1, \dots, k_n} a^2_{k_1 \cdots k_n} \Big)^{p/2}. \tag{A.10}$$

Third, for any partial pairing $\Sigma \subset \Pi$ and any edge $e \in \Pi \setminus \Sigma$,

$$V_\Pi(\Sigma \cup \{e\}) \leqslant V_\Pi(\Sigma). \tag{A.11}$$

To show (A.11), we suppose that $e = \{(i,j),(i',j')\}$ with $i \neq i'$ (see Figure A.1), and estimate the sum over $\mathbf{k}_e = (k_{ij}, k_{i'j'})$ in the expression (A.8) for $V_\Pi(\Sigma \cup \{e\})$ as

$$\begin{aligned} \sum_{\mathbf{k}_e} \Big(\sum_{\mathbf{k}_{I_i \setminus [\Sigma \cup \{e\}]}} a^2_{\mathbf{k}_{I_i}} \Big)^{1/2} \Big(\sum_{\mathbf{k}_{I_{i'} \setminus [\Sigma \cup \{e\}]}} a^2_{\mathbf{k}_{I_{i'}}} \Big)^{1/2} \mathbf{1}_{k_{ij} = k_{i'j'}} \\ \leqslant \Big(\sum_{\mathbf{k}_{I_i \setminus [\Sigma]}} a^2_{\mathbf{k}_{I_i}} \Big)^{1/2} \Big(\sum_{\mathbf{k}_{I_{i'} \setminus [\Sigma]}} a^2_{\mathbf{k}_{I_{i'}}} \Big)^{1/2}, \end{aligned}$$

by Cauchy–Schwarz and the fact that the summation variable $k_{ij} = k_{i'j'}$ appears exactly once in each factor.

From (A.9)–(A.11), we conclude that

$$\mathbb{E}[\xi^p] \leqslant \sum_{\Pi} \Big(\sum_{k_1, \dots, k_n} a^2_{k_1 \cdots k_n} \Big)^{p/2}. \tag{A.12}$$

Since the number of pairings Π is bounded by $\frac{(np)!}{(np/2)! 2^{np/2}}$, we conclude from (A.12) and (A.6) that

$$\mathbb{E}[\xi^p] \leqslant \frac{(np)!}{(np/2)! 2^{np/2}} \frac{1}{(n!)^{p/2}} (\mathbb{E}[\xi^2])^{p/2} \leqslant C p^{np/2} (\mathbb{E}[\xi^2])^{p/2},$$

by Stirling's approximation. This concludes the proof. ∎

Remark A.5. If ξ is complex-valued such that $\operatorname{Re} \xi$ and $\operatorname{Im} \xi$ both belong to $\mathcal{B}_n$, then by Minkowski's inequality we deduce from Lemma A.4 that for any $p \in 2\mathbb{N}^*$ we have

$$\mathbb{E}[|\xi|^p] \leqslant C p^{np/2} (\mathbb{E}[|\xi|^2])^{p/2}.$$

Appendix B. Basic estimates for the Green function

In this appendix we prove some basic analytic properties of the Green function G on the torus. All of these results are well known, and we collect and prove them here for the reader's convenience. The following lemma is similar to e.g. [69, Lemma 5.4] or [1, Section III.3].

Lemma B.1. *Let $d = 2$. There exists $\widetilde{G} \in C^\infty(\Lambda \setminus \{0\})$ such that*

$$G(x) = -\frac{1}{\pi} \log|x| + \widetilde{G}(x),$$

as well as $|\widetilde{G}(x)| + |\nabla \widetilde{G}(x)| \lesssim 1$ and $|\nabla^2 \widetilde{G}(x)| \lesssim 1 + \big|\log|x|\big|$.

Proof. Let g_κ denote the Green function of the operator $\kappa - \Delta/2$ on $\mathbb{R}^2$, which for $\kappa > 0$ has Fourier transform

$$\widehat{g}_\kappa(\xi) = \int \mathrm{d}x\, g_\kappa(x) \mathrm{e}^{-2\pi \mathrm{i}\xi \cdot x} = \frac{1}{\kappa + 2\pi^2|\xi|^2}.$$

Moreover, $g_0(x) = \frac{1}{\pi} \log|x|^{-1}$. Let $0 < r \leqslant 1/4$, and choose $\varphi \in [0,1]$ to be a smooth compactly supported function equal to 1 in the ball of radius r around the origin and zero outside the ball of radius $2r$ around the origin.

First we note that $(1-\varphi)g_\kappa \in C^\infty(\Lambda)$. This is a manifestation of elliptic regularity (see for instance [28, Theorem 8.6.1]), which can be seen directly by applying the operator $(\kappa - \Delta/2)^k$ to $(1-\varphi)g_\kappa$, using the Leibniz rule, and then estimating the decay of $\widehat{g}_\kappa$ for large enough k using the a priori bound $|\widehat{g}_\kappa(\xi)| \leqslant 1/\kappa$.

Using h to denote some generic smooth function which may change from one expression to the next, we therefore find that

$$(\kappa - \Delta/2)\varphi(g_\kappa - g_0) = -\kappa\varphi g_0 + h.$$

We conclude that

$$\varphi g_\kappa = \varphi g_0 - \kappa g_\kappa * (\varphi g_0) + h = \varphi g_0 - \kappa(\varphi g_\kappa) * (\varphi g_0) + h.$$

By plugging this equation into itself, we conclude that

$$\varphi g_\kappa - \varphi g_0 + \kappa(\varphi g_0) * (\varphi g_0) \in C^2(\Lambda),$$

since $(\varphi g_\kappa) * (\varphi g_0) * (\varphi g_0) \in C^2(\Lambda)$ because its Fourier transform decays as $|\xi|^{-6}$ for $|\xi| \to \infty$.

It is elementary that $\widetilde{g} := \kappa(\varphi g_0) * (\varphi g_0)$ satisfies $|\widetilde{g}(x)| + |\nabla \widetilde{g}(x)| \lesssim 1$ and $|\nabla^2 \widetilde{g}(x)| \lesssim 1 + \big|\log|x|\big|$ for $|x| \leqslant 1$. The claim now follows from the observation that

$$G(x) = \sum_{n \in \mathbb{Z}^d} g_\kappa(x+n),$$

by Poisson summation. ∎

Remark B.2. An analogue of the result of Lemma B.1 holds in three dimensions, with the same proof. Namely, if G denotes the Green function of the operator $\kappa - \Delta/2$ on Λ for $d = 3$, then

$$G(x) = \frac{1}{4\pi|x|} + \widetilde{G}(x),$$

where $|\widetilde{G}(x)| \lesssim 1$.

Lemma B.3. *Let $d = 2$. The truncated Green function from (4.8) satisfies the bounds*

$$|G_N(x)| \lesssim \big(1 + \big|\log|x|\big|\big) \wedge \log N \tag{B.1}$$

and

$$|G_N(x) - G(x)| \lesssim \big(1 + |\log(N|x|)|\big) \wedge \frac{1}{N^2|x|^2}. \tag{B.2}$$

Proof. The bound (B.1) follows easily from Lemma B.1 and the definition (4.8), by considering the cases $|x| > 2/N$ and $|x| \leqslant 2/N$ separately.

To prove (B.2), we write

$$G_N(x) - G(x) = \int \mathrm{d}y\, (G(x+y) - G(x))\rho_N(y). \tag{B.3}$$

For $|x| \leqslant 1/8$, we deduce from Lemma B.1 that

$$\begin{aligned} |G_N(x) - G(x)| \lesssim{}& \left| \int \mathrm{d}y\, (\widetilde{G}(x+y) - \widetilde{G}(x))\rho_N(y) \right| \\ &+ \left| \int_{\mathbb{R}^2} \mathrm{d}y \left(\log\left| x + \frac{y}{N} \right|^2 - \log|x|^2 \right) \rho(y) \right|. \end{aligned} \tag{B.4}$$

If $|x| \leqslant 8/N$ then we estimate the right-hand side of (B.4) by

$$C + 2\int_{\mathbb{R}^2} \mathrm{d}y \left| \log\left| \frac{x}{|x|} + \frac{y}{N|x|} \right| \right| \rho(y) \lesssim 1 + \left| \log \frac{1}{N|x|} \right|.$$

If $8/N < |x| \leqslant 1/8$ we estimate the right-hand side of (B.4) as

$$\frac{1 + \big|\log|x|\big|}{N^2} + \left| \int_{\mathbb{R}^2} \mathrm{d}y\, \log\left(1 + 2\frac{x\cdot y}{N|x|^2} + \frac{|y|^2}{N^2|x|^2} \right) \rho(y) \right| \lesssim \frac{1}{N^2|x|^2},$$

by Taylor expansion and the fact that ρ is an even function. Similarly, for $|x| > 1/8$ we easily deduce from Lemma B.1, a Taylor expansion and the evenness of ρ that the right-hand side of (B.3) is bounded by $1/N^2$. This concludes the proof. ∎

Appendix C. Proofs of auxiliary claims from Section 5

Throughout this appendix, we recall that ε satisfies (5.1). Before proceeding to the proof of Lemma 5.14, we note a quantitative analogue of [33, Lemma 5.12] concerning path-path interactions.

Lemma C.1. *For continuous paths $\omega \in \Omega^{\tau_1,\widetilde{\tau}_1}, \widetilde{\omega} \in \Omega^{\tau_2,\widetilde{\tau}_2}$, the following properties hold:*

(i) *We have, uniformly in $\eta > 0$,*

$$\mathbb{V}_\eta(\omega,\widetilde{\omega}) = \mathbb{V}^\varepsilon(\omega,\widetilde{\omega}) + \widehat{\mathbb{V}}_\eta(\omega,\widetilde{\omega}) + O\bigg(\frac{\nu}{\varepsilon^d}(\tau_1-\widetilde{\tau}_1)\bigg), \tag{C.1}$$

where $\widehat{\mathbb{V}}_\eta(\omega,\widetilde{\omega})$ satisfies

$$\big|\widehat{\mathbb{V}}_\eta(\omega,\widetilde{\omega})\big| \leqslant \frac{C}{\varepsilon^{d+1}} \int \mathrm{d}s \sum_{i=1}^N \int_{I_i} \mathrm{d}\widetilde{s} \int_{I_i} \mathrm{d}\widehat{s}\, |\widetilde{\omega}(\widetilde{s}) - \widetilde{\omega}(\widehat{s})|_\Lambda\, \delta_\eta([s]_\nu - [\widehat{s}]_\nu) \tag{C.2}$$

for $N := \lceil \frac{\tau_2-\widetilde{\tau}_2}{\nu} \rceil$ and

$$I_i := \begin{cases} [\widetilde{\tau}_2 + (i-1)\nu, \widetilde{\tau}_2 + i\nu] & \text{for } 1 \leqslant i \leqslant N-1, \\ [\widetilde{\tau}_2 + (N-1)\nu, \tau_2] & \text{for } i = N. \end{cases} \tag{C.3}$$

(ii) *Consider continuous paths $\omega' \in \Omega^{\tau_1',\widetilde{\tau}_1'}, \widetilde{\omega}' \in \Omega^{\tau_2',\widetilde{\tau}_2'}$ that agree with the paths $\omega, \widetilde{\omega}$ respectively on the intersection of their domains. Suppose that $|\tau_j - \tau_j'| \leqslant \nu$ and $|\widetilde{\tau}_j - \widetilde{\tau}_j'| \leqslant \nu$ for $j = 1, 2$. Then*

$$\mathbb{V}^\varepsilon(\omega,\widetilde{\omega}) = \mathbb{V}^\varepsilon(\omega',\widetilde{\omega}') + O\bigg(\frac{\nu}{\varepsilon^d}(\tau_1-\widetilde{\tau}_1) + \frac{\nu}{\varepsilon^d}(\tau_2-\widetilde{\tau}_2) + \frac{\nu^2}{\varepsilon^d}\bigg).$$

Proof of Lemma C.1. Let us first show (i). By using Definition 5.12, (5.11), and (C.3) we write

$$\mathbb{V}_\eta(\omega,\widetilde{\omega}) = \int \mathrm{d}s \sum_{i=1}^N \int_{I_i} \mathrm{d}\widetilde{s} \int_{\widehat{I}_i} \mathrm{d}\widehat{s}\, v^\varepsilon(\omega(s) - \widetilde{\omega}(\widehat{s}))\delta_\eta([s]_\nu - [\widehat{s}]_\nu), \tag{C.4}$$

where $\widehat{I}_i := [\widetilde{\tau}_2 + (i-1)\nu, \widetilde{\tau}_2 + i\nu]$ for $1 \leqslant i \leqslant N$. By Lemma 5.5 (i), it follows that

$$\int \mathrm{d}s \int_{I_N} \mathrm{d}\widetilde{s} \int_{I_N \setminus \widehat{I}_N} \mathrm{d}\widehat{s}\, v^\varepsilon(\omega(s) - \widetilde{\omega}(\widehat{s}))\delta_\eta([s]_\nu - [\widehat{s}]_\nu) = O\bigg(\frac{\nu}{\varepsilon^d}(\tau_1-\widetilde{\tau}_1)\bigg). \tag{C.5}$$

For (C.5), we have also used (5.11) to deduce that

$$0 \leqslant \int_{\widehat{I}_N \setminus I_N} \mathrm{d}\widehat{s}\, \delta_{\eta,\nu}([s]_\nu - [\widehat{s}]_\nu) \leqslant 1.$$

Combining (C.4) and (C.4), it follows that

$$\mathbb{V}_\eta(\omega,\widetilde{\omega}) = \int \mathrm{d}s \sum_{i=1}^N \int_{I_i} \mathrm{d}\widetilde{s} \int_{I_i} \mathrm{d}\widehat{s}\, v^\varepsilon(\omega(s) - \widetilde{\omega}(\widehat{s}))\delta_\eta([s]_\nu - [\widehat{s}]_\nu) + O\bigg(\frac{\nu}{\varepsilon^d}(\tau_1-\widetilde{\tau}_1)\bigg). \tag{C.6}$$

From (C.6) and Definition 5.13, we obtain (C.1) with

$$\widehat{\mathbb{V}}_\eta(\omega,\widetilde{\omega}) := \int \mathrm{d}s \sum_{i=1}^N \int_{I_i} \mathrm{d}\widetilde{s} \int_{I_i} \mathrm{d}\widehat{s}\, [v^\varepsilon(\omega(s) - \widetilde{\omega}(\widehat{s})) - v^\varepsilon\big(\omega(s) - \widetilde{\omega}(\widetilde{s})\big)]\delta_\eta([s]_\nu - [\widehat{s}]_\nu). \tag{C.7}$$

By using Lemma 5.5 (ii), we find that (C.7) satisfies (C.2). Claim (ii) follows from Definition 5.12 by using Lemma 5.5 (i). ∎

We now give the proofs of Lemmas 5.20 and 5.21, which are used in the analysis of the rate of convergence for correlation functions.

Proof of Lemma 5.20. In order to prove (i), we rewrite (5.80) as

$$(Q_{p,\eta})_{\mathbf{x},\widetilde{\mathbf{x}}} = \nu^p \sum_{\mathbf{r}\in(\nu\mathbb{N}^*)^p} \mathrm{e}^{-\kappa|\mathbf{r}|}\, Y_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r}), \tag{C.8}$$

where

$$Y_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r}) := \int \mu_{\mathbb{C}_\eta}(\mathrm{d}\sigma)\, \mathrm{e}^{T^\varepsilon - \frac{\mathrm{i}\tau^\varepsilon[\sigma]}{\nu}}\, \mathrm{e}^{F_2(\sigma)} \prod_{i=1}^{p}\Big[\int \mathbb{W}^{r_i,0}_{x_i,\widetilde{x}_i}(\mathrm{d}\omega_i)\big(\mathrm{e}^{\mathrm{i}\int_0^{r_i}\mathrm{d}t\,\sigma([t]_\nu,\omega_i(t))}-1\big)\Big]. \tag{C.9}$$

Here we recall the definition of T^ε in (5.15) and the definition of F_2 in (5.18). By arguing analogously to [33, (5.41)], we deduce that

$$|Y_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r})| \leqslant C_d^p \mathrm{e}^{T^\varepsilon}\|v^\varepsilon\|_{L^\infty}^{p/2}\prod_{i=1}^{p}(r_i + r_i^{1-d/2}) \leqslant C_{d,v}^p \mathrm{e}^{T^\varepsilon}\left(\frac{1}{\varepsilon^d}\right)^{p/2}\prod_{i=1}^{p}(r_i + r_i^{1-d/2}), \tag{C.10}$$

where in the second inequality we have used Lemma 5.5 (i). Substituting (C.10) into (C.8), considering Riemann sums, and recalling (5.16), we deduce claim (i).

We now prove claim (ii). Analogously to (C.8), we rewrite (5.82) as

$$(q_p^\varepsilon)_{\mathbf{x},\widetilde{\mathbf{x}}} = \int_{[0,\infty)^p} \mathrm{d}\mathbf{r}\, \mathrm{e}^{-\kappa|\mathbf{r}|}\, y^\varepsilon_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r},\mathbf{r}), \tag{C.11}$$

where for $\mathbf{r},\mathbf{s}\in[0,\infty)^p$ with $\mathbf{s}\leqslant\mathbf{r}$ (meaning that $s_i\leqslant r_i$ for all i), we define

$$y^\varepsilon_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r},\mathbf{s}) := \int \mu_{v^\varepsilon}(\mathrm{d}\xi)\, \mathrm{e}^{T^\varepsilon - \mathrm{i}\tau^\varepsilon\langle\xi,1\rangle_{L^2}}\, \mathrm{e}^{f_2(\xi)} \prod_{i=1}^{p}\Big[\int \mathbb{W}^{r_i,0}_{x_i,\widetilde{x}_i}(\mathrm{d}\omega_i)\big(\mathrm{e}^{\mathrm{i}\int_0^{s_i}\mathrm{d}t\,\xi(\omega_i(t))}-1\big)\Big]. \tag{C.12}$$

Here, we recall the definition of f_2 in (5.25). Similarly to [33, Section 5.2], we now divide the proof into the following three steps.

Step 1. We truncate the variables r_i occurring in (C.8), (C.11) to lie in some interval $[\delta, 1/\delta]$ for $\delta > 0$ small.

Step 2. In the expression obtained by using the above truncation in q_p^ε given by (5.82), we compare the integral $\int_\delta^{1/\delta}\mathrm{d}r_i\,(\cdots)$ with the corresponding Riemann sum

$$\nu \sum_{r_i\in\nu\mathbb{N}^*} \mathbf{1}_{\delta\leqslant r_i\leqslant 1/\delta}(\cdots).$$

Step 3. We replace ξ by $\langle\sigma\rangle$ given by (5.28) in the functional integral and compare the resulting approximations of q_p^ε and Q_p^ε.

We now carry out the details of each of the steps above. At every stage, we keep explicit track of the error terms.

Step 1. Denoting $[a,b]_\nu := [a,b] \cap \nu\mathbb{Z}$, we observe the following estimate.

$$\left|(Q_{p,\eta})_{\mathbf{x},\widetilde{\mathbf{x}}} - \nu^p \sum_{\mathbf{r}\in[\delta,1/\delta]_\nu^p} \mathrm{e}^{-\kappa|\mathbf{r}|} Y_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r})\right| + \left|(q_p^\varepsilon)_{\mathbf{x},\widetilde{\mathbf{x}}} - \int_{[\delta,1/\delta]^p} \mathrm{d}\mathbf{r}\, \mathrm{e}^{-\kappa|\mathbf{r}|}\, y^\varepsilon_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r},\mathbf{r})\right| \leqslant C^p_{\kappa,\nu} \mathrm{e}^{T^\varepsilon} \varepsilon^{-dp/2} \delta^{2-d/2}. \tag{C.13}$$

The estimate for the first term on the left-hand side of (C.13) follows from (C.8) and (C.10) by considering Riemann sums. Similarly, the estimate for the second term on the left-hand side of (C.13) follows from (C.11) and from the observation that $y^\varepsilon_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r})$ satisfies the same bound as in (C.10) by an analogous proof. We omit the details.

Step 2. We show that for $\delta = \delta(\nu)$ sufficiently small (for the precise bound, see (C.20) below), the following estimate holds:

$$\left|\int_{[\delta,1/\delta]^p} \mathrm{d}\mathbf{r}\, \mathrm{e}^{-\kappa|\mathbf{r}|} y^\varepsilon_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r},\mathbf{r}) - \nu^p \sum_{\mathbf{r}\in[\delta,1/\delta]_\nu^p} \mathrm{e}^{-\kappa|\mathbf{r}|} y^\varepsilon_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r},\mathbf{r})\right| \leqslant C^p_{\kappa,\nu} \mathrm{e}^{T^\varepsilon} \varepsilon^{-d/2} \delta^{-dp/2} \sqrt{\nu} + C^d_p \mathrm{e}^{T^\varepsilon} \delta^{-dp/2-(d-1)}\, \nu. \tag{C.14}$$

By arguing analogously to the proof of [33, Lemma 5.17], we find that for all $\mathbf{r} \in [\delta, 1/\delta]_\nu^p$ and for all $\mathbf{s} \in (0,\infty)^p$ with $\mathbf{s} \leqslant \mathbf{r}$ we have

$$|y^\varepsilon_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r},\mathbf{r}) - y^\varepsilon_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r},\mathbf{s})| \leqslant C^p_{d,\nu} \mathrm{e}^{T^\varepsilon} \varepsilon^{-d/2} \delta^{-dp/2} \max_i (r_i - s_i). \tag{C.15}$$

We note that the δ dependence in (C.15) is obtained from Lemma 5.4 (i). Furthermore, the ε dependence is obtained from Lemma 5.5 (i).

By arguing analogously to the proof of [33, Lemma 5.18], we deduce that

$$|\mathrm{e}^{-\kappa|\mathbf{r}|} y^\varepsilon_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r},\mathbf{s}) - \mathrm{e}^{-\kappa|\widetilde{\mathbf{r}}|} y^\varepsilon_{\mathbf{x},\widetilde{\mathbf{x}}}(\widetilde{\mathbf{r}},\mathbf{s})| \leqslant C^p_\kappa \mathrm{e}^{T^\varepsilon} \delta^{-dp/2} \frac{\nu}{\min_i (r_i - s_i)}, \tag{C.16}$$

provided that $\mathbf{r}, \widetilde{\mathbf{r}} \in [\delta, 1/\delta]^p$ are such that $|\mathbf{r} - \widetilde{\mathbf{r}}| \leqslant \nu$ and $\mathbf{s} \in (0,\infty)^p$ satisfies

$$\mathbf{s} \leqslant \mathbf{r}, \quad \mathbf{s} \leqslant \widetilde{\mathbf{r}}, \quad \min_i s_i \geqslant \delta/2, \quad 1 \geqslant \min_i (r_i - s_i) \geqslant 2\nu. \tag{C.17}$$

We now explain how (C.15) and (C.16) imply (C.14) for a suitable choice of δ. We recall (5.66) and write $\lfloor \mathbf{r} \rfloor_\nu := (\lfloor r_i \rfloor_\nu)_{i=1}^p$. By using (5.26) and Lemma 5.4 (i) in (C.12), we deduce that for all $\mathbf{r} \in [\delta, 1/\delta]^p$ and $\mathbf{s} \in (0,\infty)^p$ with $\mathbf{s} \leqslant \mathbf{r}$,

$$|y^\varepsilon_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r},\mathbf{s})| \leqslant C^p_d \mathrm{e}^{T^\varepsilon} \delta^{-dp/2}. \tag{C.18}$$

Using (C.18), we get

$$\left|\nu^p \sum_{\mathbf{r}\in[\delta,1/\delta]_\nu^p} \mathrm{e}^{-\kappa|\mathbf{r}|} y^\varepsilon_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r},\mathbf{r}) - \int_{[\delta,1/\delta]^p} \mathrm{d}\mathbf{r}\, \mathrm{e}^{-\kappa|\lfloor\mathbf{r}\rfloor_\nu|}\, y^\varepsilon_{\mathbf{x},\widetilde{\mathbf{x}}}(\lfloor\mathbf{r}\rfloor_\nu, \lfloor\mathbf{r}\rfloor_\nu)\right| \leqslant C^p_d \mathrm{e}^{T^\varepsilon} \delta^{-dp/2-(d-1)}\, \nu. \tag{C.19}$$

We henceforth assume

$$\delta \sim \nu^a, \quad a \in (0, 1/2). \tag{C.20}$$

For fixed $\mathbf{r} \in [\delta, 1/\delta]^p$, and ν sufficiently small, we define $\mathbf{s} \equiv \mathbf{s}(\mathbf{r}) \in (0,\infty)^p$ by $s_i := r_i - \sqrt{\nu}$. Note that $s_i > 0$ for small ν by (C.20). By (C.15), we have

$$\left| \int_{[\delta,1/\delta]^p} \mathrm{d}\mathbf{r}\, \mathrm{e}^{-\kappa|\mathbf{r}|}\, y^{\varepsilon}_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r},\mathbf{r}) - \int_{[\delta,1/\delta]^p} \mathrm{d}\mathbf{r}\, \mathrm{e}^{-\kappa|\mathbf{r}|}\, y^{\varepsilon}_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r},\mathbf{s}(\mathbf{r})) \right| \leqslant C^p_{\kappa,\nu}\, \mathrm{e}^{T^\varepsilon} \varepsilon^{-d/2} \delta^{-dp/2} \sqrt{\nu} \tag{C.21}$$

and

$$\left| \int_{[\delta,1/\delta]^p} \mathrm{d}\mathbf{r}\, \mathrm{e}^{-\kappa|\lfloor \mathbf{r} \rfloor_\nu|}\, y^{\varepsilon}_{\mathbf{x},\widetilde{\mathbf{x}}}(\lfloor \mathbf{r} \rfloor_\nu, \lfloor \mathbf{r} \rfloor_\nu) - \int_{[\delta,1/\delta]^p} \mathrm{d}\mathbf{r}\, \mathrm{e}^{-\kappa|\lfloor \mathbf{r} \rfloor_\nu|}\, y^{\varepsilon}_{\mathbf{x},\widetilde{\mathbf{x}}}(\lfloor \mathbf{r} \rfloor_\nu, \mathbf{s}(\mathbf{r})) \right| \leqslant C^p_{\kappa,\nu} \mathrm{e}^{T^\varepsilon} \varepsilon^{-d/2} \delta^{-dp/2} \sqrt{\nu}. \tag{C.22}$$

In (C.22), we have used $|\lfloor r_i \rfloor_\nu - s_i| \lesssim \sqrt{\nu}$. By (C.16), we have

$$\left| \int_{[\delta,1/\delta]^p} \mathrm{d}\mathbf{r}\, \mathrm{e}^{-\kappa|\mathbf{r}|}\, y^{\varepsilon}_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r},\mathbf{s}(\mathbf{r})) - \int_{[\delta,1/\delta]^p} \mathrm{d}\mathbf{r}\, \mathrm{e}^{-\kappa|\lfloor \mathbf{r} \rfloor_\nu|}\, y^{\varepsilon}_{\mathbf{x},\widetilde{\mathbf{x}}}(\lfloor \mathbf{r} \rfloor_\nu,\mathbf{s}(\mathbf{r})) \right| \leqslant C^p_{\kappa} \mathrm{e}^{T^\varepsilon} \varepsilon^{-d/2} \delta^{-dp/2} \sqrt{\nu}. \tag{C.23}$$

Here, we have used the fact that for small ν, $\mathbf{s}(\mathbf{r})$ satisfies (C.17) with $\widetilde{\mathbf{r}} = \lfloor \mathbf{r} \rfloor_\nu$. We now deduce (C.14) from (C.19), (C.21), (C.22), and (C.23).

Step 3. Let us define

$$\mathcal{I}^\varepsilon := \nu^p \sum_{\mathbf{r} \in [\delta,1/\delta]^p_\nu} \mathrm{e}^{-\kappa|\mathbf{r}|}\, y^{\varepsilon}_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r},\mathbf{r}), \tag{C.24}$$

$$\mathcal{J} := \nu^p \sum_{\mathbf{r} \in [\delta,1/\delta]^p_\nu} \mathrm{e}^{-\kappa|\mathbf{r}|}\, Y_{\mathbf{x},\widetilde{\mathbf{x}}}(\mathbf{r}). \tag{C.25}$$

Recalling (5.33), we show that

$$|\mathcal{I}^\varepsilon - \mathcal{J}| \leqslant C^p_{\kappa,\nu} \mathrm{e}^{T^\varepsilon} \varepsilon^{-5d+1/2} \delta^{-dp/2} \Theta(\nu)^{1/2}. \tag{C.26}$$

To do so, we first recall (C.12), Lemma 5.8, and (5.29), and compare (C.24) with

$$\widetilde{\mathcal{I}} := \nu^p \sum_{\mathbf{r} \in [\delta,1/\delta]^p_\nu} \mathrm{e}^{-\kappa|\mathbf{r}|} \int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma)\, \mathrm{e}^{T^\varepsilon - \frac{\mathrm{i}\tau^\varepsilon[\sigma]}{\nu}} \mathrm{e}^{F_2(\sigma)} \times \prod_{i=1}^p \left[\int \mathbb{W}^{r_i,0}_{x_i,\widetilde{x}_i}(\mathrm{d}\omega_i) \left(\mathrm{e}^{\mathrm{i}\int_0^{r_i} \mathrm{d}t\, \langle\sigma\rangle(\omega_i(t))} - 1 \right) \right]. \tag{C.27}$$

In other words, (C.27) is obtained by replacing the factor by $\mathrm{e}^{f_2(\langle\sigma\rangle)}$ by $\mathrm{e}^{F_2(\sigma)}$ in the σ integral representation of (C.24). Using (5.19), (5.26), the Cauchy–Schwarz inequality,

Lemmas 5.10–5.11, and Lemma 5.4 (i), we deduce that

$$\big|\mathcal{I}^{\varepsilon} - \widetilde{\mathcal{I}}\big| \leqslant C^{p}_{\kappa,v} \mathrm{e}^{T^{\varepsilon}} \varepsilon^{-5d+1/2} \delta^{-dp/2} \Theta(v)^{1/2}. \tag{C.28}$$

We now compare (C.25) and (C.27). By (5.19) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |\widetilde{\mathbb{I}} - \mathbb{J}| &\leqslant \mathrm{e}^{T^{\varepsilon}} v^{p} \sum_{\mathfrak{r} \in [\delta, 1/\delta]_v^p} \mathrm{e}^{-\kappa|\mathfrak{r}|} \prod_{i=1}^{p} \int \mathbb{W}^{r_i,0}_{x_i,\widetilde{x}_i}(\mathrm{d}\omega_i) \\ &\times \bigg(\int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma) \Big| \prod_{i=1}^{p} \big(\mathrm{e}^{\mathrm{i}\int_0^{r_i} \mathrm{d}t\, \langle\sigma\rangle(\omega_i(t))} - 1 \big) - \prod_{i=1}^{p} \big(\mathrm{e}^{\mathrm{i}\int_0^{r_i} \mathrm{d}t\, \sigma([t]_v, \omega_i(t))} - 1 \big) \Big|^2 \bigg)^{1/2}. \end{aligned} \tag{C.29}$$

By arguing analogously to the proof of [33, Lemma 5.20], we find that for fixed $1 \leqslant i \leqslant p$,

$$\begin{aligned} &\int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma) \big| \mathrm{e}^{\mathrm{i}\int_0^{r_i} \mathrm{d}t\, \langle\sigma\rangle(\omega_i(t))} - \mathrm{e}^{\mathrm{i}\int_0^{r_i} \mathrm{d}t\, \sigma([t]_v, \omega_i(t))} \big|^2 \\ &\qquad \leqslant \frac{C}{v} \sum_{s,\widetilde{s} \in [0,r_i)_v} \int_0^{v} \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}\widetilde{u}_1\, \mathrm{d}\widetilde{u}_2\, \delta_\eta(u_1 - \widetilde{u}_1) \\ &\qquad\quad \times \big(v^{\varepsilon}(\omega_i(s+u_1) - \omega_i(\widetilde{s} + \widetilde{u}_1)) - v^{\varepsilon}(\omega_i(s+u_2) - \omega_i(\widetilde{s} + \widetilde{u}_2)) \big), \end{aligned} \tag{C.30}$$

where $[a,b)_v := [a, b-v]_v$. Using Lemma 5.5 (ii) and $\delta_\eta \geqslant 0$, it follows that

$$\begin{aligned} \text{(C.30)} &\leqslant \frac{C}{\varepsilon^{d+1} v} \sum_{s,\widetilde{s} \in [0,r_i)_v} \int_0^{v} \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}\widetilde{u}_1\, \mathrm{d}\widetilde{u}_2\, \delta_\eta(u_1 - \widetilde{u}_1) \\ &\qquad \times [|\omega_i(s+u_1) - \omega_i(s+u_2)|_\Lambda + |\omega_i(\widetilde{s} + \widetilde{u}_1) - \omega_i(\widetilde{s} + \widetilde{u}_2)|_\Lambda]. \end{aligned} \tag{C.31}$$

We now use (C.31) and a telescoping argument in (C.29). In particular, by using the Cauchy–Schwarz inequality in $\mathbb{W}^{r_i,0}_{x_i,\widetilde{x}_i}(\mathrm{d}\omega_i)$, applying Lemma 5.4 (i)–(ii) and (5.11), we deduce that

$$|\widetilde{\mathcal{I}} - \mathcal{J}| \leqslant C^{p}_{\kappa,v} \mathrm{e}^{T^{\varepsilon}} \varepsilon^{-\frac{d+1}{2}} \delta^{-dp/2} \sqrt{v}. \tag{C.32}$$

Above we have applied (5.8) when using Lemma 5.4 (ii). We hence deduce (C.26) from (C.28) and (C.32). Here, we also recall (5.33).

We combine (C.13), (C.14), (C.26), and optimize in δ, keeping (C.20) in mind. We hence take $\delta \sim v^{\frac{1}{4p+4}}$ when $d = 2$ and $\delta \sim v^{\frac{1}{6p+4}-}$ when $d = 3$. Note that δ then indeed satisfies (C.20). Putting everything together, and recalling (5.16), we obtain (5.83), with $\theta(d,p)$ as in (5.5), which concludes the proof of claim (ii). ∎

Proof of Lemma 5.21. We first prove (i). We recall (3.1). From Jensen's inequality we get

$$\zeta^{W^{\varepsilon}} = \mathbb{E}[\mathrm{e}^{-W^{\varepsilon}}] \geqslant \mathrm{e}^{-\mathbb{E}[W^{\varepsilon}(\phi)]}. \tag{C.33}$$

We note that

$$\mathbb{E}\bigg[\int \mathrm{d}x : |\phi(x)|^2 : \bigg] = 0. \tag{C.34}$$

Furthermore, by using Wick's theorem and arguing analogously to [31, Section 3.1], we have

$$\mathbb{E}\bigg[\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,{:}|\phi(x)|^2{:}v^\varepsilon(x-\widetilde{x}){:}|\phi(\widetilde{x})|^2{:}\bigg] = \int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,v^\varepsilon(x-\widetilde{x})G(x-\widetilde{x})G(\widetilde{x}-x).$$

By using the fact that G is even, followed by (2.12) and Lemma 5.5 (i), the above expression is in absolute value

$$\leqslant \int \mathrm{d}x\,|v^\varepsilon(x)|G(x)^2 \lesssim_v \frac{1}{\varepsilon^d}\int_{|x|\lesssim\varepsilon}\mathrm{d}x\,G(x)^2. \tag{C.35}$$

Using Lemma B.1 when $d=2$ and Remark B.2 when $d=3$, a direct calculation shows that

$$(\text{C.35}) \lesssim_{\kappa,v} \chi(\varepsilon)^2. \tag{C.36}$$

Here, we recall (5.2). Claim (i) now follows by recalling (3.1) and substituting (C.34) and (C.36) into (C.33).

We now show (ii). We show that for $\alpha\in(0,1-d/4)$, we have

$$\mathcal{Z} \geqslant \exp\bigg[-c\bigg(\chi(\varepsilon)^2+\frac{\nu^\alpha}{\varepsilon^d}\bigg)\bigg] \tag{C.37}$$

for some $c>0$ depending on κ. Note that (C.37) implies claim (i), since by (5.1)–(5.2), we have $\chi(\varepsilon)^2 \gtrsim_{\alpha,d} \nu^\alpha/\varepsilon^d$.

We recall (2.18) and (2.11), and set $\mathcal{W} := H^\varepsilon - H^{(0)}$. By using the Peierls–Bogolyubov inequality (see [70, Section 2.5]) in (2.14), we have

$$\mathcal{Z} \geqslant \exp\bigg[\mathrm{Tr}_{\mathcal{F}}\bigg(-\mathcal{W}\frac{\mathrm{e}^{-H^{(0)}}}{\mathrm{Tr}_{\mathcal{F}}(\mathrm{e}^{-H^{(0)}})}\bigg)\bigg] = \exp[-\varrho_\nu^{(0)}(\mathcal{W})], \tag{C.38}$$

where for a closed operator $\mathcal{A}$ on $\mathcal{F}$, we define

$$\varrho_\nu^{(0)}(\mathcal{A}) := \frac{\mathrm{Tr}_{\mathcal{F}}(\mathcal{A}\,\mathrm{e}^{-H^{(0)}})}{\mathrm{Tr}_{\mathcal{F}}(\mathrm{e}^{-H^{(0)}})}.$$

Since $\varrho_\nu^{(0)}[\int \mathrm{d}x\,(\nu a^*(x)a(x)-\varrho_\nu)]=0$, we get

$$\varrho_\nu^{(0)}(\mathcal{W}) = \frac{1}{2}\varrho_\nu^{(0)}\bigg[\int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,\big(\nu a^*(x)a_\nu(x)-\varrho_\nu\big)v^\varepsilon(x-\widetilde{x})\big(\nu a^*(\widetilde{x})a(\widetilde{x})-\varrho_\nu\big)\bigg], \tag{C.39}$$

By using the quantum Wick theorem (see [31, Lemma B1]) and using [31, Lemma 2.10], the right-hand side of (C.39) is

$$\sim \int \mathrm{d}x\,\mathrm{d}\widetilde{x}\,G_{[\nu]}(x-\widetilde{x})[G_{[\nu]}(\widetilde{x}-x)+\nu\delta(\widetilde{x}-x)]v^\varepsilon(x-\widetilde{x}), \tag{C.40}$$

where the quantum Green function $G_{[\nu]}$ is given by

$$G_{[\nu]}(x) := \sum_{k\in\mathbb{Z}^d}\frac{\nu}{\mathrm{e}^{\lambda_k\nu}-1}u_k, \tag{C.41}$$

with notation as in (4.5). From (C.41), we see that $G_{[\nu]}$ is even and therefore by Lemma 5.5 (i), we have

$$|(C.40)| \lesssim \int \mathrm{d}x\, |v^\varepsilon(x)| G_{[\nu]}(x)^2 + \frac{\nu}{\varepsilon^d} G_{[\nu]}(0). \tag{C.42}$$

We now bound each of the two terms on the right-hand side of (C.42) separately. For the first term, we note that for $\alpha \in (0, 1-d/4)$, $G = \sum_{k\in\mathbb{Z}^d} \frac{1}{\lambda_k} \mathrm{e}^{2\pi \mathrm{i} k\cdot x} \in L^2(\Lambda)$ satisfies

$$\|G_{[\nu]} - G\|_{L^2} \lesssim_\kappa \nu^\alpha. \tag{C.43}$$

In order to obtain (C.43), we note that for all $\delta \in [0,1]$,

$$\left| \frac{\nu}{\mathrm{e}^{\lambda_k \nu} - 1} - \frac{1}{\lambda_k} \right| \leqslant \nu^{1-\delta} \frac{1}{\lambda_k^\delta}. \tag{C.44}$$

Using Plancherel's theorem and taking $\delta > d/4$ in (C.44), we deduce (C.43). We now use Hölder's inequality, Lemma 5.5 (i), (C.43), and recall (C.35)–(C.36) to deduce that the first term on the right-hand side of (C.42) is

$$\lesssim_\kappa \int \mathrm{d}x\, |v^\varepsilon(x)| G(x)^2 + \frac{\nu^\alpha}{\varepsilon^d} \lesssim_{\kappa,v} \chi(\varepsilon)^2 + \frac{\nu^\alpha}{\varepsilon^d}. \tag{C.45}$$

From (C.41), by considering the terms with $\lambda_k \leqslant 1/\nu$ and $\lambda_k > 1/\nu$ separately, we deduce that the second term on the right-hand side of (C.42) is

$$\lesssim_\kappa \frac{\nu \chi(\sqrt{\nu})}{\varepsilon^d} \lesssim_\alpha \frac{\nu^\alpha}{\varepsilon^d}, \tag{C.46}$$

since $\alpha < 1/2$. Estimate (C.37) now follows from (C.42), (C.45), and (C.46). ∎

We now give the details of the proof of the claim from Remark 5.25.

Proof of claim from Remark 5.25. The claim follows if, instead of (5.95), we show

$$\|V^\varepsilon - V\|_{L^2(\mathbb{P})} \lesssim_v \|v^\varepsilon - v\|_{H^{-1+\delta/2}}^{1/2}. \tag{C.47}$$

Let us note that the x dependence in (5.97) is of the form

$$[v^\varepsilon(x-y) - v(x-y)] G(x-x_a)\, G(x-x_b)$$

for some distinct $a, b \in B \setminus \{(1,1,\pm)\}$. We consider the different possibilities for x_a and x_b.

Case 1: $x_a = x_b = \widetilde{x}$. This case is easy because the variables $(x, \widetilde{x})$ and $(y, \widetilde{y})$ decouple. In particular, the contribution to (5.96) is

$$\begin{aligned} &\lesssim \left| \int \mathrm{d}x\, \mathrm{d}\widetilde{x}\, [v^\varepsilon(x-\widetilde{x}) - v(x-\widetilde{x})] G(x-\widetilde{x})^2 \right| \\ &\lesssim \|v^\varepsilon - v\|_{H^{-1+\delta/2}} \|G\|_{H^{-1+\delta/4}}^2 \lesssim \|v^\varepsilon - v\|_{H^{-1+\delta/2}}. \end{aligned} \tag{C.48}$$

In order to prove (C.48), we have used duality, and the bilinear estimate given in [73, Lemma 4.2], which implies that $\|G^2\|_{H^{-1+\delta/2}} \lesssim_\delta \|G\|_{H^{-1+\delta/4}}^2 < \infty$.

Case 2: $x_a = x_b \neq \widetilde{x}$. We can assume without loss of generality that $x_a = y$. The integrand that we then consider is

$$[v^\varepsilon(x-\widetilde{x}) - v(x-\widetilde{x})]G(x-y)^2[v^\varepsilon(y-\widetilde{y}) - v(y-\widetilde{y})]G(\widetilde{x}-\widetilde{y})^2.$$

We first integrate in x and use an estimate analogous to (C.48). We then integrate in $\widetilde{x}$ and use $G \in L^2(\Lambda)$. Finally, we integrate in $y, \widetilde{y}$ and recall (5.88) to deduce that the contribution to (5.96) is $\lesssim \|v^\varepsilon - v\|_{H^{-1+\delta/2}}^{1/2} \|v\|_{L^1}^{1/2}$.

Case 3: $x_a = \widetilde{x}, x_b \neq \widetilde{x}$ *or* $x_a \neq \widetilde{x}, x_b = \widetilde{x}$. We can assume without loss of generality that $x_b = y$. Then, we consider the integrand

$$[v^\varepsilon(x-\widetilde{x}) - v(x-\widetilde{x})]G(x-\widetilde{x})G(x-y)[v^\varepsilon(y-\widetilde{y}) - v(y-\widetilde{y})]G(\widetilde{x}-\widetilde{y})G(y-\widetilde{y}).$$

We first fix $x, \widetilde{y}$ and integrate in $\widetilde{x}$ and y, using estimates analogous to (C.48). We then integrate in $x, \widetilde{y}$ to deduce that the contribution to (5.96) is $\lesssim \|v^\varepsilon - v\|_{H^{-1+\delta/2}}$.

Case 4: $x_a \neq x_b, x_a \neq \widetilde{x}, x_b \neq \widetilde{x}$. This case is similar to Case 2 and we get the same upper bound by an analogous argument. We hence deduce (C.47). ∎

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