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The high type quadratic Siegel disks are Jordan domains

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Abstract. Let α be an irrational number of sufficiently high type and suppose $P_\alpha(z) = e^{2\pi i\alpha}z + z^2$ has a Siegel disk Δ_α centered at the origin. We prove that the boundary of Δ_α is a Jordan curve, and that it contains the critical point $-e^{2\pi i\alpha}/2$ if and only if α is a Herman number.

Keywords: Siegel disks, Jordan domains, Herman numbers, high type, near-parabolic renormalization.

Contents

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1. Introduction

Let f be a non-linear holomorphic function with $f(0) = 0$ and $f'(0) = e^{2\pi \mathrm{i}\alpha}$, where $0 < \alpha < 1$ is an irrational number. We say that f is *locally linearizable* at the fixed point 0 if there exists a holomorphic function defined near 0 which conjugates f to the *rigid rotation* $R_\alpha(z) = e^{2\pi \mathrm{i}\alpha} z$. The maximal region in which f is conjugate to R_α is a simply connected domain called the *Siegel disk* of f centered at 0.

The existence of the Siegel disk of f is dependent on the arithmetic condition of $\alpha \in (0,1) \setminus \mathbb{Q}$. Let

$$[0; a_1, a_2, \ldots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

be the *continued fraction expansion* of α. The rational numbers $p_n/q_n := [0; a_1, \ldots, a_n]$, $n \geqslant 1$, are the convergents of α, where p_n and q_n are coprime positive integers. If α belongs to the *Brjuno class*

$$\mathcal{B} := \Big\{ \alpha = [0; a_1, a_2, \ldots] \in (0,1) \setminus \mathbb{Q} : \sum_{n=1}^{\infty} q_n^{-1} \log q_{n+1} < +\infty \Big\},$$

then any holomorphic germ f with $f(0) = 0$ and $f'(0) = e^{2\pi \mathrm{i}\alpha}$ is locally linearizable at 0 and hence f has a Siegel disk centered at the origin [5, 47]. Yoccoz proved that the Brjuno condition is also necessary for the local linearization of the quadratic polynomial

$$P_\alpha(z) := e^{2\pi \mathrm{i}\alpha} z + z^2 : \mathbb{C} \to \mathbb{C}$$

at the origin [53].

1.1. Topology and obstructions of Siegel disk boundaries

The dynamics in the Siegel disks is simple and one mainly considers the properties of the boundaries. In the 1980s, Douady and Sullivan asked the following question (see [20,41]):

Question. Is the boundary of a Siegel disk a Jordan curve?

This question is still open, even for quadratic polynomials. However, much progress has been made on this problem for various families of functions under certain conditions. An irrational number $\alpha = [0; a_1, a_2, \ldots]$ is said to be of *bounded type* if $\sup_{n \geqslant 1} a_n < +\infty$.

Douady–Herman, Zakeri, Yampolsky–Zakeri, Shishikura and Zhang, respectively, proved that the boundaries of bounded type Siegel disks of quadratic polynomials, cubic polynomials, some quadratic rational maps, all polynomials and all rational maps with degree at least 2 are quasi-circles (hence are Jordan curves) (see [21,29,46,49,55,58]). This is also true for some transcendental entire functions (see [15,18,23,31,50,56,57,61]).

An important breakthrough was made by Petersen and Zakeri [36] in 2004. They proved that for almost all irrational numbers α, the boundary of the Siegel disk of the quadratic polynomial P_α is a Jordan curve. We refer to these irrational numbers as *PZ type*, i.e., $\log a_n = \mathcal{O}(\sqrt{n})$ as $n \to \infty$, where a_n is the n-th digit of the continued fraction expansion of α. Recently, Zhang [60] generalized this result to all polynomials and obtained the same result for the sine family [59].

Suppose the closure of the Siegel disk of f is compactly contained in the domain of definition of f. One may wonder what phenomena near the boundary of a Siegel disk prevent f from having a larger linearization domain. Obviously, the presence of periodic cycles near the boundary is one of the reasons since no Siegel disk can contain periodic points except the center itself. It was proved by Avila and Cheraghi [2] that under some condition on α every neighborhood of the Siegel disk of P_α contains infinitely many cycles, which is similar to the small cycle property that prevents linearization (see [34,52]).

On the other hand, no Siegel disk can contain a critical point. Hence the second question on the Siegel disk boundary is: Does the boundary of a Siegel disk always contain a critical point? The answer is no; Ghys and Herman gave the first examples of polynomials having a Siegel disk whose boundary does not contain a critical point (see [21,24,28]).

The results on the regularity[1] of the boundaries of the Siegel disks mentioned above (for bounded type or PZ type rotation numbers) often also include the statement that the boundaries of those Siegel disks pass through at least one critical point. In particular, for the bounded type rotation numbers, Graczyk and Świątek [25] proved a very general result: if an analytic function has a Siegel disk properly contained in the domain of holomorphy and the rotation number is of bounded type, then the boundary of the corresponding Siegel disk contains a critical point.

Herman [26] was one of the pioneers who studied analytic diffeomorphisms of circles. He introduced the following subset of irrational numbers.

Definition (Herman numbers). Let $\mathcal{H}$ be the set of irrational numbers α such that every orientation-preserving analytic circle diffeomorphism of rotation number α is analytically conjugate to a rigid rotation.

Herman [26] proved that the set $\mathcal{H}$ is non-empty and contains a subset of Diophantine numbers. Yoccoz [54] proved that $\mathcal{H}$ contains all Diophantine numbers (and hence contains all bounded type and PZ type numbers), and also gave an arithmetic characterization of the numbers in $\mathcal{H}$.

[1] The word "regularity" here means the topological and geometric properties of the boundaries of the Siegel disks. See [6].

Suppose f is an analytic function which has a Siegel disk properly contained in the domain of holomorphy. Ghys [24] proved that if the rotation number belongs to $\mathscr{H}$ and the boundary of the Siegel disk is a Jordan curve, then f has a critical point in the boundary of the Siegel disk. Later, Herman [27] generalized this result by dropping the topological condition on the Siegel disk boundary but requiring that the restriction of f to the Siegel disk boundary is injective (see also [35]). In particular, he proved that if a unicritical polynomial has a Siegel disk whose rotation number is in $\mathscr{H}$, then the boundary of the Siegel disk contains a critical point. Recently, Chéritat and Roesch [19] and Benini and Fagella [4] generalized this result to (respectively) the polynomials with two critical values and a special class of transcendental entire functions with two singular values.

For polynomials, Rogers [43] proved that if the Siegel disk Δ is fixed and the rotation number is in $\mathscr{H}$, then $\partial\Delta$ either contains a critical point or is an indecomposable continuum. For the exponential map $E_\theta(z) = e^{2\pi \mathrm{i}\theta}(e^z - 1)$, it was proved by Herman [27] that if E_θ has a bounded Siegel disk Δ_θ, then E_θ is injective on $\partial\Delta_\theta$. Hence Δ_θ is unbounded when $\theta \in \mathscr{H}$ since E_θ has no critical points. Conversely, Herman, Baker and Rippon asked: if Δ_θ is unbounded, is necessarily the singular value $-e^{2\pi\mathrm{i}\theta}$ contained in $\partial\Delta_\theta$? Rippon [40] showed that this is true for almost all θ and the question was fully answered positively by Rempe [38] and independently by Buff and Fagella (unpublished). Moreover, Rempe [39] also studied the Herman type Siegel disks of some other transcendental entire functions.

1.2. The statement of the main result

The proofs of the regularity results for bounded type and PZ type Siegel disks stated previously are all based on surgery, either quasiconformal or trans-quasiconformal. In these proofs, some pre-models, and usually a single Blaschke product or a family of Blaschke products, are needed. By surgery, the regularity and the existence of critical points on the boundaries of Siegel disks were proved at the same time.

In this paper, without using surgeries we shall prove that the Siegel disks of some holomorphic maps are Jordan domains and that Herman type rotation number is also necessary for the existence of critical points on the Siegel disk boundaries. To this end, we have to restrict the rotation numbers to a special class since we use the near-parabolic renormalization scheme.

In [30], a renormalization operator $\mathscr{R}$ and a compact class $\mathscr{F}$ that is invariant under $\mathscr{R}$ were introduced. All the maps in $\mathscr{F}$ have a special covering structure. They have a neutral fixed point at the origin and a unique simple critical point in their domains of definition. The renormalization operator assigns to a given map of $\mathscr{F}$ a new map in $\mathscr{F}$ that is obtained by considering the return map to a sector landing at the origin. As a return map, one iterate of $\mathscr{R}f$ corresponds to many iterates of $f \in \mathscr{F}$. To study very large iterates of f near 0, one hopes to repeat this process infinitely many times. However, to iterate $\mathscr{R}$ infinitely many times, the scheme requires the rotation number α, where $f'(0) = e^{2\pi\mathrm{i}\alpha}$, to be of *high type*, that is, to belong to

$$\mathrm{HT}_N := \{\alpha = [0; a_1, a_2, \ldots] \in (0,1) \setminus \mathbb{Q} : a_n \geqslant N \text{ for all } n \geqslant 1\}$$

for some large[2] $N \in \mathbb{N}$. In this paper we prove the following main result.

Main Theorem. *Let α be an irrational number of sufficiently high type and suppose $P_\alpha(z) = e^{2\pi i\alpha} z + z^2$ has a Siegel disk Δ_α centered at the origin. Then the boundary of Δ_α is a Jordan curve. Moreover, the boundary contains the critical point $-e^{2\pi i\alpha}/2$ if and only if α is a Herman number.*

Note that HT_N has measure zero if $N \geqslant 2$. However, all the usual types of irrational numbers have non-empty intersections with HT_N: bounded type, PZ type, Herman type, Brjuno type etc. In particular, HT_N contains some irrational numbers such that the Siegel disk boundary of P_α has the regularity studied in [1, 6] and the self-similarity studied in [32]. Rogers [42] proved that the boundary of any bounded irreducible Siegel disk Δ is either tame: the conformal map from Δ to the unit disk has a continuous extension to $\partial\Delta$, or wild: $\partial\Delta$ is an indecomposable continuum. Recently, Chéritat [16] constructed a holomorphic germ such that the corresponding Siegel disk is compactly contained in the domain of definition but the boundary is not locally connected. Our main theorem indicates that the boundaries of quadratic Siegel disks should be tame.

As we have seen, in order to guarantee the existence of critical points on the boundaries of Siegel disks, the Herman condition (i.e., the rotation number is of Herman type) appears usually as a requirement of sufficiency in most of the literature. As far as we know, the necessity only appears in [8], where it is proved that the Herman condition is equivalent to the existence of a critical point on the boundary of the Siegel disks of a family of toy models.

In fact, besides the quadratic polynomials, the proof of the Main Theorem in this paper is also valid for all the maps in Inou–Shishikura's invariant class. Hence the Main Theorem is also true for some rational maps and transcendental entire functions. We point out that it was proved in [3, 48] that the bounded type Siegel disks of the maps in Inou–Shishikura's class are quasi-disks if the rotation number is of sufficiently high type.

By constructing topological models of the post-critical sets of the maps in Inou–Shishikura's class for all high type numbers, Cheraghi [11] gave an alternative proof of the Main Theorem independently. Our proofs are different: we analyze the dynamics and carry out the computations in the renormalization tower directly.

Recently, Dudko and Lyubich [22] made significant progress on the quadratic Siegel polynomials P_α. They proved that the restriction of P_α to the boundary of the Siegel disk Δ_α of P_α is injective, which implies that $\partial\Delta_\alpha$ is not the whole Julia set of P_α (actually they proved a more general result for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$).

[2] The precise value of N is not known, but it is likely to be at least 20. It is conjectured that a variation of the invariant class and renormalization may be defined for $N = 1$.

1.3. Strategy of the proof

Let f_0 be the normalized quadratic polynomial or a map in Inou–Shishikura's class (see Section 2.1) satisfying $f_0(0) = 0$ and $f_0'(0) = e^{2\pi i\alpha}$, where α is of Brjuno-type and of sufficiently high type. For $n \geqslant 0$, let $f_{n+1} = \mathcal{R} f_n$ be the sequence of maps generated by the near-parabolic renormalization operator $\mathcal{R}$. For each $n \geqslant 0$, we use $\mathcal{P}_n$ to denote the perturbed petal of f_n, and Φ_n for the corresponding perturbed Fatou coordinate (see definitions in Section 2.2).

In order to prove that the boundary of the Siegel disk of f_0 is a Jordan curve, we construct a sequence $(\gamma_0^n : [0,1] \to \mathbb{C})_{n\in\mathbb{N}}$ of continuous curves in the perturbed Fatou coordinate plane of f_0 by using a renormalization tower. Each γ_0^n is obtained from γ_n^0 (in the perturbed Fatou coordinate plane of f_n) by going up through the renormalization tower, i.e., by lifting and then spreading around. In Lemma 3.2 we show that the inner radius of the Siegel disk Δ_n of f_n is estimated by the Brjuno sum up to a multiplicative constant. Then we choose the suitable height of γ_n^0 such that $\Phi_n^{-1}(\gamma_n^0)$ is contained in the Siegel disk Δ_n of f_n. Consequently, $\Phi_0^{-1}(\gamma_0^n)$ with $n \in \mathbb{N}$ are curves in the Siegel disk of f_0.

The key ingredient is Proposition 4.5: the sequence $(\gamma_0^n : [0,1] \to \mathbb{C})_{n\in\mathbb{N}}$ of continuous curves converges uniformly to a limit $\gamma^\infty : [0,1] \to \mathbb{C}$, which is also a continuous curve. For the proof, we use a family of "straight" curves η_n^0 to encode the difference between γ_n^0 and γ_n^1 in the Fatou coordinate plane of f_n. The diameters of the η_n^0 are discussed in Step 2 of the proof. The diameters of the lifts of η_n^0 are estimated by two kinds of contraction: one is uniform contraction with respect to the hyperbolic metrics in subdomains of the renormalization tower (see Lemma 4.7) and the other is "Brjuno-type arithmetic" – estimates from Section 2.4 (see also Lemma 4.8). In conclusion, the oscillations of the curves $(\gamma_0^n : [0,1] \to \mathbb{C})_{n\in\mathbb{N}}$ are bounded in terms of the Brjuno sum, i.e., the curves form an equicontinuous family. Because of contraction by going up the renormalization tower, the sequence $\Phi_0^{-1}(\gamma_0^n)$ converges exponentially fast towards the boundary of Δ_0 (see Proposition 4.9).

For the second part of the Main Theorem which concerns the Herman condition, we construct a Jordan arc Γ_0 in the non-escaping set of f_0 which connects the unique critical value cv to the origin, where $\gamma_0 := \Phi_0(\Gamma_0)$ is contained in a half-infinite strip $\mho$ with finite width. The existence of Γ_0 is proved in Lemma 5.3 and the proof is also based on contraction via going up the renormalization tower. To apply the contraction property successfully, the shape of $\Phi_0^{-1}(\mho)$ has to be controlled; this is Lemma 5.1 whose proof is given in the Appendix. The construction of Γ_0 guarantees that $\Gamma_n = \mathbb{E}\mathrm{xp} \circ \Phi_{n-1}(\Gamma_{n-1})$ is also a Jordan arc connecting cv to the origin and $\gamma_n = \Phi_n(\Gamma_n)$ is contained in $\mho$ for all $n \geqslant 1$.

We study the homeomorphism $s_{\alpha_n} := \Phi_n \circ \mathbb{E}\mathrm{xp} : \gamma_{n-1} \to \gamma_n$ from the simple curve in one level of the renormalization to another. Lemmas 5.4 and 5.5 estimate the dynamics of the s_{α_n} in terms of the Brjuno sum. Based on the sequence $(s_{\alpha_n})_{n\in\mathbb{N}}$, we define a new class $\widetilde{\mathcal{H}}_N$ of irrational numbers which is a subset of Brjuno numbers, where N is a large number. After comparing the properties of s_{α_n} and Yoccoz's arithmetic characterization of $\mathscr{H}$, we prove that $\widetilde{\mathscr{H}}_N$ is exactly equal to the set of high type Herman numbers (see Lemmas 6.4 and 6.6). On the other hand, we prove that the boundary of the Siegel disk of f_0 contains the critical value cv if and only if $\alpha \in \widetilde{\mathscr{H}}_N$ (see Proposition 5.7). This implies the second part of the Main Theorem.

1.4. Some observations

There are several applications of Inou–Shishikura's invariant class. The first remarkable application is that of Buff and Chéritat [7] who used it as one of the main tools to prove the existence of Julia sets of quadratic polynomials with positive area. Recently, Cheraghi and his collaborators have found several other important applications. In [9, 10], Cheraghi developed several elaborate analytic techniques based on Inou–Shishikura's results. The tools in [9, 10] have led to part of the recent major progress on the dynamics of quadratic polynomials. For example, Feigenbaum Julia sets with positive area (different from the examples in [7]) were found in [3], the Marmi–Moussa–Yoccoz conjecture for rotation numbers of high type was proved in [12], the local connectivity of the Mandelbrot set at some infinitely satellite renormalizable points was proved in [14], some statistical properties of the quadratic polynomials were depicted in [2], and the topological structure and the Hausdorff dimension of high type irrationally indifferent attractors were characterized in [11] and [13] respectively.

Recently, Chéritat [17] generalized the near-parabolic renormalization theory to the unicritical families of any finite degrees. See also [51] for the corresponding theory of local degree 3. Hence there is a hope to generalize the Main Theorem in this paper to all unicritical polynomials.

Notations. We use $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^+ = \{1, 2, \dots\}$. The Riemann sphere and the unit disk are denoted by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ respectively. A round disk in $\mathbb{C}$ is denoted by $\mathbb{D}(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$ and $\overline{\mathbb{D}}(a, r)$ is its closure. If $x \in \mathbb{R}$ is non-negative, we use $\lfloor x \rfloor$ to denote its integer part.

For $X \subset \mathbb{C}$ and $\delta > 0$, let $B_\delta(X) := \bigcup_{z \in X} \mathbb{D}(z, \delta)$ be the δ-neighborhood of X. For $a \in \mathbb{C}$ and $X \subset \mathbb{C}$, we denote $aX := \{az : z \in X\}$ and $X \pm a := \{z \pm a : z \in X\}$. Let $A, B \subset \mathbb{C}$. We say that A is *compactly contained* in B, and write $A \Subset B$, if the closure of A is compact and contained in the interior $\operatorname{int}(B)$ of B. We use $\operatorname{diam}(X)$ to denote the Euclidean diameter of a set $X \subset \mathbb{C}$ and $\operatorname{len}(\gamma)$ for the Euclidean length of a rectifiable curve $\gamma \subset \mathbb{C}$.

2. Near-parabolic renormalization scheme

In this section, we summarize some results in [2, 7, 10, 30] which will be used in this paper. Parts of the relevant theories can also be found in [44, 45].

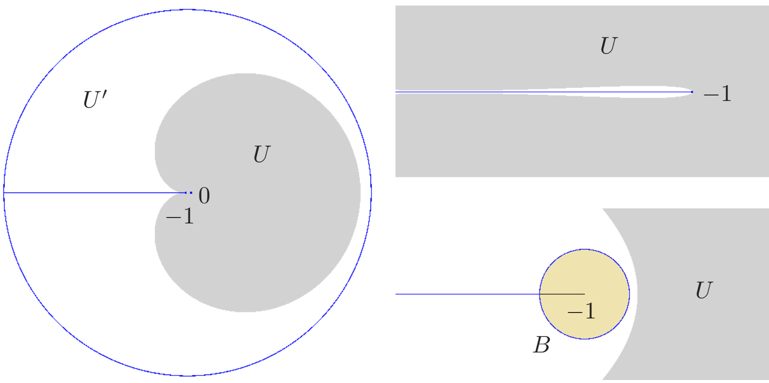

Fig. 1. The domains U (the gray part), U' (the white region bounded by the blue curves; see (2.5) for the definition) and their successive zooms near -1. The outer boundary of U' looks like a circle with radius about 35 and the rightmost point of U is about 32.2. The widths of these pictures are 72, 0.6 and 0.0075 respectively. It can be seen clearly that $\overline{U} \cap (-\infty, -1] = \emptyset$ and $U \Subset U'$.

2.1. *Inou–Shishikura's class*

Let $P(z) := z(1+z)^2$ be a cubic polynomial with a parabolic fixed point at 0 with multiplier 1. Then P has a critical point $\mathrm{cp}_P := -1/3$ which is mapped to the critical value $\mathrm{cv}_P := -4/27$. It also has another critical point -1 which is mapped to 0. Consider the ellipse

$$E := \left\{ x + yi \in \mathbb{C} : \left(\frac{x+0.18}{1.24}\right)^2 + \left(\frac{y}{1.04}\right)^2 \leqslant 1 \right\} \tag{2.1}$$

and define[3]

$$U := \psi_1(\widehat{\mathbb{C}} \setminus E), \quad \text{where} \quad \psi_1(z) := -\frac{4z}{(1+z)^2}. \tag{2.2}$$

The domain U is symmetric about the real axis, contains the parabolic fixed point 0 and the critical point cp_P, but $\overline{U} \cap (-\infty, -1] = \emptyset$ (see [30, Section 5.A] and Figure 1).

For a given function f, we denote its domain of definition by U_f. Following [30, Section 4], we define a class of maps[4]

$$\mathcal{IS}_0 := \left\{ f = P \circ \varphi^{-1} : U_f \to \mathbb{C} \;\middle|\; \begin{array}{l} 0 \in U_f \text{ open in } \mathbb{C},\ \varphi : U \to U_f \\ \text{conformal, } \varphi(0) = 0 \text{ and } \varphi'(0) = 1 \end{array} \right\}.$$

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[3]The domain U is denoted by V in [30].

[4]The definition of $\mathcal{IS}_0$ is based on the class $\mathcal{F}_1$ of [30]. There the conformal map φ in the definition of $\mathcal{IS}_0$ is required to have a quasiconformal extension to $\mathbb{C}$. This condition is used by Inou and Shishikura to prove the uniform contraction of the near-parabolic renormalization operator under the Teichmüller metric. We modify the definition here since we will not use this property in this paper.

Each map in this class has a parabolic fixed point at 0, a unique critical point at $\mathrm{cp}_f := \varphi(-1/3) \in U_f$ and a unique critical value at $\mathrm{cv} := -4/27$, which is independent of f.

For $\alpha \in \mathbb{R}$, we define

$$\mathcal{IS}_\alpha := \{f(z) = f_0(e^{2\pi \mathrm{i}\alpha} z) : e^{-2\pi \mathrm{i}\alpha}\, U_{f_0} \to \mathbb{C} \mid f_0 \in \mathcal{IS}_0\}.$$

For convenience, we normalize the quadratic polynomials to

$$Q_\alpha(z) = e^{2\pi \mathrm{i}\alpha} z + \frac{27}{16} e^{4\pi \mathrm{i}\alpha} z^2$$

so that all Q_α have the same critical value $-4/27$ as maps in $\mathcal{IS}_\alpha$. In particular, $Q_\alpha = Q_0 \circ R_\alpha$, where $R_\alpha(z) = e^{2\pi \mathrm{i}\alpha} z$. We mention that Q_α is not in the class $\mathcal{IS}_\alpha$.

Theorem 2.1 (Leau–Fatou [33, Section 10] and Inou–Shishikura [30]). *For all $f \in \mathcal{IS}_0 \cup \{Q_0\}$, there exist simply connected domains $\mathcal{P}_{\mathrm{attr},f}$, $\mathcal{P}_{\mathrm{rep},f} \subset U_f$ and univalent maps $\Phi_{\mathrm{attr},f} : \mathcal{P}_{\mathrm{attr},f} \to \mathbb{C}$, $\Phi_{\mathrm{rep},f} : \mathcal{P}_{\mathrm{rep},f} \to \mathbb{C}$ such that*

(a) *$\mathcal{P}_{\mathrm{attr},f}$ and $\mathcal{P}_{\mathrm{rep},f}$ are bounded by piecewise analytic curves and are compactly contained in U_f, $\mathrm{cp}_f \in \partial\mathcal{P}_{\mathrm{attr},f}$ and $\partial\mathcal{P}_{\mathrm{attr},f} \cap \partial\mathcal{P}_{\mathrm{rep},f} = \{0\}$;*

(b) *the image $\Phi_{\mathrm{attr},f}(\mathcal{P}_{\mathrm{attr},f})$ is a right half-plane and $\Phi_{\mathrm{rep},f}(\mathcal{P}_{\mathrm{rep},f})$ is a left half-plane;*

(c) *$\Phi_{\mathrm{attr},f}(f(z)) = \Phi_{\mathrm{attr},f}(z) + 1$ for $z \in \mathcal{P}_{\mathrm{attr},f}$ and $\Phi^{-1}_{\mathrm{rep},f}(\zeta) = f(\Phi^{-1}_{\mathrm{rep},f}(\zeta - 1))$ for $\zeta \in \Phi_{\mathrm{rep},f}(\mathcal{P}_{\mathrm{rep},f})$.*

Normalization of $\Phi_{\mathrm{attr},f}$ and $\Phi_{\mathrm{rep},f}$. The univalent map $\Phi_{\mathrm{attr},f}$ (resp. $\Phi_{\mathrm{rep},f}$) in Theorem 2.1 is called an *attracting* (resp. *repelling*) *Fatou coordinate* of f and $\mathcal{P}_{\mathrm{attr},f}$ (resp. $\mathcal{P}_{\mathrm{rep},f}$) is called an *attracting* (resp. *repelling*) *petal*. The attracting Fatou coordinate $\Phi_{\mathrm{attr},f}$ can be naturally extended to the immediate attracting basin $\mathcal{A}_{\mathrm{attr},f}$ of 0. Specifically, for $z \in \mathcal{A}_{\mathrm{attr},f}$ such that $f^{\circ k}(z) \in \mathcal{P}_{\mathrm{attr},f}$ with $k \geqslant 0$, one can define

$$\Phi_{\mathrm{attr},f}(z) := \Phi_{\mathrm{attr},f}(f^{\circ k}(z)) - k.$$

Since $\Phi_{\mathrm{attr},f}$ is unique up to an additive constant, we *normalize* it by $\Phi_{\mathrm{attr},f}(\mathrm{cp}_f) = 0$. Therefore, $\Phi_{\mathrm{attr},f}(\mathcal{P}_{\mathrm{attr},f}) = \{\zeta \in \mathbb{C} : \mathrm{Re}\,\zeta > 0\}$.

Every $f \in \mathcal{IS}_0 \cup \{Q_0\}$ can be written as $f(z) = z + a_2 z^2 + a_3 z^3 + \mathcal{O}(z^4)$ in a neighborhood of 0, where $a_2 \neq 0$. For z in a component Ω_f of $\mathcal{A}_{\mathrm{attr},f} \cap \mathcal{P}_{\mathrm{rep},f}$ such that $\mathrm{Im}\,\Phi_{\mathrm{rep},f}(z) \to +\infty$ as $z \to 0$, we have (see [45, Proposition 2.2.1])

$$\Phi_{\mathrm{attr},f}(z) = -\frac{1}{a_2 z} - \gamma \log\Big(-\frac{1}{a_2 z}\Big) + C_{\mathrm{attr}} + o(1),$$
$$\Phi_{\mathrm{rep},f}(z) = -\frac{1}{a_2 z} - \gamma \log\Big(-\frac{1}{a_2 z}\Big) + C_{\mathrm{rep}} + o(1),$$

where $\gamma = 1 - a_3/a_2^2$ is the *iterative residue* of f and C_{attr}, C_{rep} are constants. Since $\Phi_{\mathrm{rep},f}$ is also unique up to an additive constant, we *normalize* it by setting $C_{\mathrm{rep}} := C_{\mathrm{attr}}$, i.e., $\Phi_{\mathrm{attr},f}(z) - \Phi_{\mathrm{rep},f}(z) \to 0$ as $z \to 0$ in Ω_f.

2.2. Near-parabolic renormalization

We will consider the case that a sequence of functions converges to a limiting function and the neighborhoods of a function have to be defined.

Definition (Neighborhoods of a function). Let $f : U_f \to \mathbb{C}$ be a given function. A *neighborhood* of f is

$$\mathcal{N} = \mathcal{N}(f; K, \varepsilon) = \Big\{ g : U_g \to \widehat{\mathbb{C}} \;\Big|\; K \subset U_g \text{ and } \sup_{z \in K} d_{\widehat{\mathbb{C}}}(g(z), f(z)) < \varepsilon \Big\},$$

where $d_{\widehat{\mathbb{C}}}$ denotes the spherical distance, K is a compact subset contained in U_f and $\varepsilon > 0$. A sequence (f_n) is said to *converge* to f *uniformly on compact sets* if for any neighborhood $\mathcal{N}$ of f, there exists $n_0 > 0$ such that $f_n \in \mathcal{N}$ for all $n \geqslant n_0$.

If $f \in \bigcup_{\alpha \in [0,1)} \mathcal{IS}_\alpha \cup \{Q_\alpha\}$, we denote by α_f the rotation number of f at the origin, i.e., the real number $\alpha_f \in [0, 1)$ such that $f'(0) = e^{2\pi i \alpha_f}$. If $\alpha_f > 0$ is small, then besides the origin, the map f has another fixed point $\sigma_f \neq 0$ near 0 in U_f, which depends continuously on f (see [45, Section 3.2] or [7, Lemma 9, p. 707]).

Proposition 2.2 ([7, Proposition 12, p. 707]; see Figure 2). *There exist $\boldsymbol{k} \in \mathbb{N}^+$ and $\varepsilon_1 > 0$ satisfying $\lfloor 1/\varepsilon_1 \rfloor - \boldsymbol{k} > 1$ such that for all $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_1]$, there exist a Jordan domain $\mathcal{P}_f \subset U_f$ and a univalent map $\Phi_f : \mathcal{P}_f \to \mathbb{C}$ such that*

(a) *$\mathcal{P}_f$ contains* cv *and it is bounded by two arcs joining 0 and σ_f;*

(b) *$\Phi_f(\mathrm{cv}) = 1$, $\Phi_f(\mathcal{P}_f) = \{\zeta \in \mathbb{C} : 0 < \mathrm{Re}\,\zeta < \lfloor 1/\alpha_f \rfloor - \boldsymbol{k}\}$ with $\mathrm{Im}\,\Phi_f(z) \to +\infty$ as $z \to 0$ and $\mathrm{Im}\,\Phi_f(z) \to -\infty$ as $z \to \sigma_f$ in $\mathcal{P}_f$;*

(c) *if $z \in \mathcal{P}_f$ and $\mathrm{Re}\,\Phi_f(z) < \lfloor 1/\alpha_f \rfloor - \boldsymbol{k} - 1$, then $f(z) \in \mathcal{P}_f$ and $\Phi_f(f(z)) = \Phi_f(z) + 1$;*

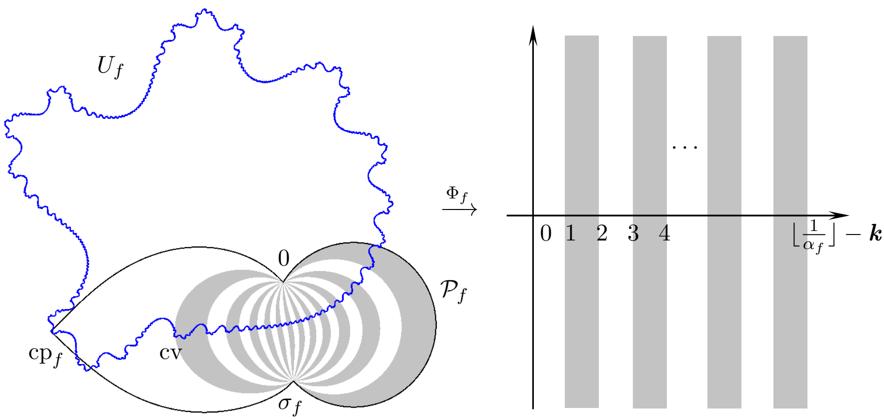

Fig. 2. The perturbed Fatou coordinate Φ_f and its domain of definition $\mathcal{P}_f$. The image of $\mathcal{P}_f$ under Φ_f has been colored by the same colors as on the right. The blue set on the left depicts the forward orbit of the critical point cp_f.

(d) *if* (f_n) *is a sequence of maps in* $\bigcup_{\alpha\in(0,\varepsilon_1]} \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ *converging to a map* $f_0 \in \mathcal{IS}_0 \cup \{Q_0\}$*, then any compact set* $K \subset \mathcal{P}_{\mathrm{attr},f_0}$ *is contained in* $\mathcal{P}_{f_n}$ *for* n *large enough and the sequence* (Φ_{f_n}) *converges to* Φ_{attr,f_0} *uniformly on* K*; moreover, any compact set* $K \subset \mathcal{P}_{\mathrm{rep},f_0}$ *is contained in* $\mathcal{P}_{f_n}$ *for* n *large enough and the sequence* $(\Phi_{f_n} - 1/\alpha_{f_n})$ *converges to* Φ_{rep,f_0} *uniformly on* K.

Proposition 2.2 was proved in [7] only for Inou–Shishikura's class. However, when $f = Q_\alpha$ with sufficiently small $\alpha > 0$, the existence of the domain $\mathcal{P}_f$ and the coordinate $\Phi_f : \mathcal{P}_f \to \mathbb{C}$ satisfying the properties in the above proposition is classical (see [45]). The map Φ_f in Proposition 2.2 is called the (*perturbed*) *Fatou coordinate* of f and $\mathcal{P}_f$ is called a (*perturbed*) *petal*.

Definition (see Figure 3). Let $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_1]$, where $\varepsilon_1 > 0$ is the constant of Proposition 2.2. Define

$$\mathcal{C}_f := \{z \in \mathcal{P}_f : 1/2 \leqslant \mathrm{Re}\,\Phi_f(z) \leqslant 3/2 \text{ and } -2 < \mathrm{Im}\,\Phi_f(z) \leqslant 2\},$$
$$\mathcal{C}_f^\sharp := \{z \in \mathcal{P}_f : 1/2 \leqslant \mathrm{Re}\,\Phi_f(z) \leqslant 3/2 \text{ and } \mathrm{Im}\,\Phi_f(z) \geqslant 2\}. \tag{2.3}$$

Note that $\mathrm{cv} = -4/27 \in \mathrm{int}\,\mathcal{C}_f$ and $0 \in \partial\mathcal{C}_f^\sharp$.

Proposition 2.3 ([10, Proposition 2.7]; see Figure 3). *There exist constants* $\varepsilon_1' \in (0, \varepsilon_1]$ *and* $\boldsymbol{k}_0 \in \mathbb{N}^+$ *such that for all* $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ *with* $\alpha \in (0, \varepsilon_1']$, *there exists a positive integer* $k_f \in [1, \boldsymbol{k}_0]$ *such that*

(a) *for all* $1 \leqslant k \leqslant k_f$*, the unique connected component* $(\mathcal{C}_f^\sharp)^{-k}$ *of* $f^{-k}(\mathcal{C}_f^\sharp)$ *that contains* 0 *in its closure is relatively compact in* U_f *and* $f^{\circ k} : (\mathcal{C}_f^\sharp)^{-k} \to \mathcal{C}_f^\sharp$ *is an isomorphism, and the unique connected component* $\mathcal{C}_f^{-k}$ *of* $f^{-k}(\mathcal{C}_f)$ *that intersects*

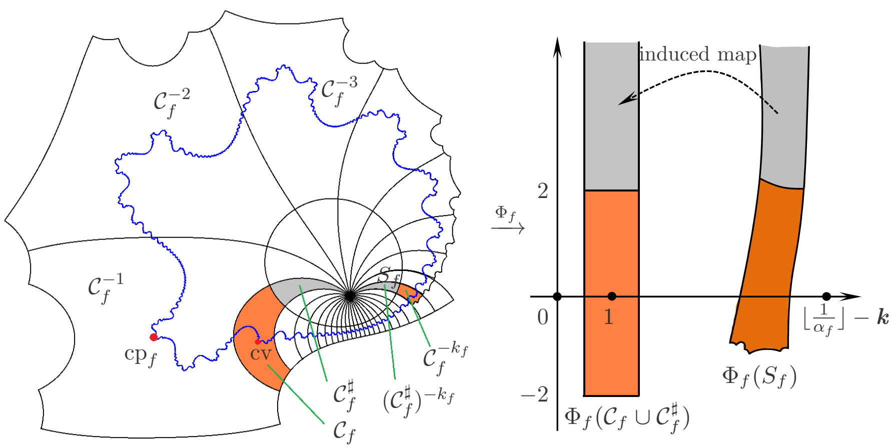

Fig. 3. Left: The sets $\mathcal{C}_f$, $\mathcal{C}_f^\sharp$ and some of their preimages. The blue set depicts the forward orbit of the critical point cp$_f$. Right: The images of $\mathcal{C}_f \cup \mathcal{C}_f^\sharp$ and S_f under the perturbed Fatou coordinate Φ_f. This shows how the near-parabolic renormalization map is induced.

$(\mathcal{C}_f^\sharp)^{-k}$ *is relatively compact in* U_f *and* $f^{\circ k}:\mathcal{C}_f^{-k}\to\mathcal{C}_f$ *is a covering of degree* 2 *ramified above* cv*;*

(b) k_f *is the* smallest *positive integer such that* $\mathcal{C}_f^{-k_f}\cup(\mathcal{C}_f^\sharp)^{-k_f}\subset\{z\in\mathcal{P}_f:0<\operatorname{Re}\Phi_f(z)<\lfloor 1/\alpha_f\rfloor-\boldsymbol{k}-1/2\}$.

The same statement as Proposition 2.3 without the uniform bound of k_f is proved in [7, Proposition 13, p. 713]. For the statements corresponding to Propositions 2.2 and 2.3 with $\alpha\in\mathbb{C}$ (specifically, when $|\arg\alpha|<\pi/4$ and $|\alpha|$ is small), we refer to [14, Section 2].

Definition (Near-parabolic renormalization; see Figure 3). For $f\in\mathcal{IS}_\alpha\cup\{Q_\alpha\}$ with $\alpha\in(0,\varepsilon_1']$, define

$$S_f:=\mathcal{C}_f^{-k_f}\cup(\mathcal{C}_f^\sharp)^{-k_f},$$

and consider the map

$$\Phi_f\circ f^{\circ k_f}\circ\Phi_f^{-1}:\Phi_f(S_f)\to\mathbb{C}.$$

This map commutes with translation by 1. Hence it projects by the *modified* exponential map[5]

$$\mathbb{E}\mathrm{xp}(\zeta):=-\frac{4}{27}s(e^{2\pi\mathrm{i}\zeta})\tag{2.4}$$

to a well-defined map $\mathcal{R}f$ which is defined on a set punctured at zero, where $s:z\mapsto\overline{z}$ is the complex conjugacy. One can check that $\mathcal{R}f$ extends across zero and satisfies $(\mathcal{R}f)(0)=0$ and $(\mathcal{R}f)'(0)=e^{2\pi\mathrm{i}/\alpha_f}$. The map $\mathcal{R}f$ is called the *near-parabolic renormalization*[6] of f.

Let $P(z)=z(1+z)^2$ be the cubic polynomial introduced at the beginning of Section 2.1. Define

$$U':=P^{-1}\big(\mathbb{D}\big(0,\tfrac{4}{27}e^{4\pi}\big)\big)\setminus((-\infty,-1]\cup\overline{B}),\tag{2.5}$$

where B is the connected component of $P^{-1}(\mathbb{D}(0,\frac{4}{27}e^{-4\pi}))$ containing -1. By an explicit calculation, one can prove that $\overline{U}\subset U'$ (see [30, Proposition 5.2] and Figure 1).

Theorem 2.4 ([30, Main Theorem 3]). *For every* $f=P\circ\varphi^{-1}\in\mathcal{IS}_\alpha$ *or* $f=Q_\alpha$ *with* $\alpha\in(0,\varepsilon_1']$*, the near-parabolic renormalization* $\mathcal{R}f$ *is well-defined and the restriction of* $\mathcal{R}f$ *to a domain containing* 0 *can be written as* $P\circ\psi^{-1}\in\mathcal{IS}_{1/\alpha}$*. Moreover,* ψ *extends to a univalent function on* $e^{-2\pi\mathrm{i}/\alpha}U'$*.*

From Theorem 2.4 we know that the near-parabolic renormalization $\mathcal{R}f$ can also be defined if the fractional part of $1/\alpha$ is contained in $(0,\varepsilon_1']$. This implies that the near-parabolic renormalization operator $\mathcal{R}$ can be applied infinitely many times to f if α is of sufficiently high type.

[5]Note that $\mathbb{E}\mathrm{xp}(0)=-4/27$ is a critical value of $\mathcal{R}f$ and $\mathbb{E}\mathrm{xp}(+\infty\mathrm{i})=0$. In some literature, the modified exponential map is defined as $\zeta\mapsto-\frac{4}{27}e^{2\pi\mathrm{i}\zeta}$ so that $(\mathcal{R}f)'(0)=e^{-2\pi\mathrm{i}/\alpha_f}$. In order to apply the classical continued fraction expansion conveniently, in this paper we put a complex conjugacy s in the definition of $\mathbb{E}\mathrm{xp}$.

[6]This is the *top* near-parabolic renormalization; the *bottom* near-parabolic renormalization around the fixed point σ_f can be defined similarly. See [30, Section 3].

2.3. *Some sets in the Fatou coordinate planes*

Let $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_1']$. In the Fatou coordinate plane of f, we define

$$\widetilde{\mathcal{D}}_f := \operatorname{int}\Big(\Phi_f(\mathcal{P}_f) \cup \bigcup_{j=0}^{b_f} (\Phi_f(S_f) + j)\Big), \tag{2.6}$$

where $b_f := k_f + \lfloor 1/\alpha \rfloor - \boldsymbol{k} - 2$ is the largest integer[7] such that one can extend $\Phi_f^{-1} : \Phi_f(\mathcal{P}_f) \to \mathcal{P}_f$ holomorphically to a domain like $\widetilde{\mathcal{D}}_f$. See Figure 4.

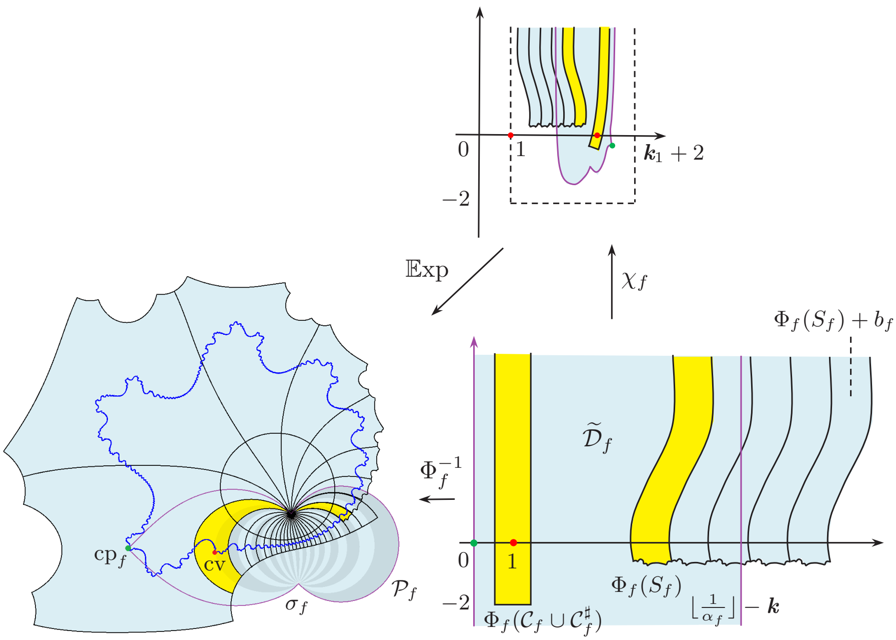

Fig. 4. The inverse Φ_f^{-1} of the perturbed Fatou coordinate can be extended holomorphically to $\widetilde{\mathcal{D}}_f$ (colored cyan). It can be seen that the image $\Phi_f^{-1}(\widetilde{\mathcal{D}}_f)$ wraps around 0. The holomorphic map Φ_f^{-1} has an anti-holomorphic lift χ_f such that $\mathbb{E}\mathrm{xp} \circ \chi_f = \Phi_f^{-1}$ (note that $\mathbb{E}\mathrm{xp}$ is anti-holomorphic). Some special points are also marked.

Lemma 2.5. *The map $\Phi_f^{-1} : \Phi_f(\mathcal{P}_f) \to \mathcal{P}_f$ can be extended to a holomorphic map*

$$\Phi_f^{-1} : \widetilde{\mathcal{D}}_f \to \mathcal{P}_f \cup \bigcup_{j=0}^{k_f} f^{\circ j}(S_f)$$

such that $\Phi_f^{-1}(\zeta + 1) = f \circ \Phi_f^{-1}(\zeta)$ for all $\zeta \in \mathbb{C}$ with $\zeta, \zeta + 1 \in \widetilde{\mathcal{D}}_f$.

[7] In particular, from the proof one can see that Lemma 2.5 will not be true if b_f is chosen as $k_f + \lfloor 1/\alpha \rfloor - \boldsymbol{k} - 1$.

This lemma has been proved in [2, Lemma 1.8]. For completeness and to clarify some ideas we include a sketch of the construction of Φ_f^{-1} here.

Proof of Lemma 2.5. By (2.3), the definition of S_f, and Propositions 2.2 (b) and 2.3 (a), we have $f^{\circ k_f}(S_f) = \mathcal{C}_f \cup \mathcal{C}_f^\sharp$ and $f^{\circ j}(S_f)$ is well-defined for all $0 \leqslant j \leqslant b_f$. If $\zeta \in \widetilde{\mathcal{D}}_f \setminus \Phi_f(\mathcal{P}_f)$, then there exists an integer $j \in [1, b_f]$ such that $\zeta \in \Phi_f(S_f) + j$. For such ζ we define

$$\Phi_f^{-1}(\zeta) := f^{\circ j}(\Phi_f^{-1}(\zeta - j)).$$

Note that there may exist two choices[8] of j for some point ζ. Assume that $\zeta \in \Phi_f(S_f) + j'$ for some $j' \in [1, b_f]$ and $j' \neq j$. Then $|j' - j| = 1$. Without loss of generality, we assume that $j' = j + 1$. By Proposition 2.2 (c), we have $\Phi_f^{-1}(\zeta + 1) = f \circ \Phi_f^{-1}(\zeta)$ for all $\zeta \in \mathbb{C}$ with $\zeta, \zeta + 1 \in \Phi_f(\mathcal{P}_f)$. Thus we have

$$f^{\circ j'}(\Phi_f^{-1}(\zeta - j')) = f^{\circ (j'-1)}(\Phi_f^{-1}(\zeta - j' + 1)) = f^{\circ j}(\Phi_f^{-1}(\zeta - j)).$$

This implies that Φ_f^{-1} is well-defined in $\widetilde{\mathcal{D}}_f$ and it is straightforward to check that Φ_f^{-1} is holomorphic. Finally, a completely similar calculation shows that $\Phi_f^{-1}(\zeta + 1) = f \circ \Phi_f^{-1}(\zeta)$ for all $\zeta \in \mathbb{C}$ with $\zeta, \zeta + 1 \in \widetilde{\mathcal{D}}_f$. ∎

Note that S_f is contained in $\{z \in \mathcal{P}_f : 0 < \operatorname{Re} \Phi_f(z) < \lfloor 1/\alpha \rfloor - \boldsymbol{k} - 1/2\}$ and $f^{\circ b_f}(S_f) = \{z \in \mathcal{P}_f : \lfloor 1/\alpha \rfloor - \boldsymbol{k} - 3/2 \leqslant \operatorname{Re} \Phi_f(z) \leqslant \lfloor 1/\alpha \rfloor - \boldsymbol{k} - 1/2 \text{ and } \operatorname{Im} \Phi_f(z) > -2\}$. According to Proposition 2.3 (b), if we consider the local rotation of f near the origin, this implies that

$$b_f = k_f + \lfloor 1/\alpha \rfloor - \boldsymbol{k} - 2 \geqslant \lfloor 1/\alpha \rfloor + 1, \quad \text{i.e.,} \quad k_f \geqslant \boldsymbol{k} + 3. \tag{2.7}$$

The modified exponential map $\mathbb{E}\mathrm{xp} : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ defined in (2.4) is a covering map which is anti-holomorphic. The map $\Phi_f^{-1} : \widetilde{\mathcal{D}}_f \to \mathbb{C} \setminus \{0\}$ can be lifted to an anti-holomorphic map

$$\chi_f : \widetilde{\mathcal{D}}_f \to \mathbb{C}$$

such that

$$\mathbb{E}\mathrm{xp} \circ \chi_f(\zeta) = \Phi_f^{-1}(\zeta) \quad \text{for all } \zeta \in \widetilde{\mathcal{D}}_f.$$

See Figure 4. There are infinitely many choices of $\chi_f : \widetilde{\mathcal{D}}_f \to \mathbb{C}$. But the following result holds.

Proposition 2.6 ([2, Proposition 1.9]). *There exists $\boldsymbol{k}_1 \in \mathbb{N}^+$ such that for all $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_1']$ and any choice of the lift χ_f, we have*

$$\sup \{|\operatorname{Re}(\zeta - \zeta')| : \zeta, \zeta' \in \chi_f(\widetilde{\mathcal{D}}_f)\} \leqslant \boldsymbol{k}_1.$$

[8] For example, this happens when ζ lies on $(\Phi_f(S_f) + j) \cap (\Phi_f(S_f) + j + 1)$ for $1 \leqslant j \leqslant b_f - 1$.

Proposition 2.6 was proved by applying Proposition 2.3, the pre-compactness of the class $\mathcal{IS}_\alpha$ and a uniform bound on the total spiral of the set $\mathcal{P}_f$ about the origin (see [7, Proposition 12] or [10, Proposition 2.4]).

From [30, Section 5.A] or [14, Propositions 2.6, 2.7] (the top and bottom near-parabolic renormalizations can be defined for all $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_1'])$, $\mathcal{P}_f$ is contained in the image of f. By Lemma 2.5, we have $\Phi_f^{-1}(\widetilde{\mathcal{D}}_f) \subset f(U_f)$. Since $f(U_f) \subset P(U') = \mathbb{D}(0, \frac{4}{27}e^{4\pi})$, it follows that $\operatorname{Im}\zeta > -2$ for every $\zeta \in \chi_f(\widetilde{\mathcal{D}}_f)$, where $P(z) = z(1+z)^2$ and U' is defined in (2.5). Therefore, by Proposition 2.6, there exists a choice of χ_f, denoted by $\chi_{f,0}$, such that

$$\chi_{f,0}(\widetilde{\mathcal{D}}_f) \subset \{\zeta \in \mathbb{C} : 1 \leqslant \operatorname{Re}\zeta < \boldsymbol{k}_1 + 2 \text{ and } \operatorname{Im}\zeta > -2\}. \tag{2.8}$$

We define

$$\mathcal{D}_f := \operatorname{int}\Big(\Phi_f(\mathcal{P}_f) \cup \bigcup_{j=0}^{k_f+\boldsymbol{k}_0+\boldsymbol{k}_1+2} (\Phi_f(S_f) + j)\Big), \tag{2.9}$$

where $\boldsymbol{k}_0, \boldsymbol{k}_1 \in \mathbb{N}^+$ are as in Propositions 2.3 and 2.6 respectively. Let $\boldsymbol{k} \in \mathbb{N}^+$ be as in Proposition 2.2.

Lemma 2.7. *For all $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $0 < \alpha \leqslant \widetilde{\varepsilon}_1 := \min\{\varepsilon_1', 1/(\boldsymbol{k} + \boldsymbol{k}_0 + \boldsymbol{k}_1 + 4)\}$, we have $\mathcal{D}_f \subset \widetilde{\mathcal{D}}_f$. Moreover,*

$$\begin{aligned} \mathcal{D}_f &\subset \Phi_f(\mathcal{P}_f) \cup \{\zeta \in \mathbb{C} : 0 \leqslant \operatorname{Re}\zeta - (\lfloor 1/\alpha \rfloor - \boldsymbol{k}) < 2\boldsymbol{k}_0 + \boldsymbol{k}_1 + 3/2\}, \\ \mathcal{D}_f &\supset \Phi_f(\mathcal{P}_f) \cup \{\zeta \in \mathbb{C} : 0 \leqslant \operatorname{Re}\zeta - (\lfloor 1/\alpha \rfloor - \boldsymbol{k}) \leqslant \boldsymbol{k}_0 + \boldsymbol{k}_1 + 3 \textit{ and } \operatorname{Im}\zeta \geqslant 0\}. \end{aligned} \tag{2.10}$$

Proof. The condition on α implies that $k_f + \boldsymbol{k}_0 + \boldsymbol{k}_1 + 2 \leqslant k_f + \lfloor 1/\alpha \rfloor - \boldsymbol{k} - 2$. Then we have $\mathcal{D}_f \subset \widetilde{\mathcal{D}}_f$ by definition.

Since $\Phi_f(S_f) \subset \{\zeta \in \mathbb{C} : 0 < \operatorname{Re}\zeta < \lfloor 1/\alpha \rfloor - \boldsymbol{k} - 1/2\}$ by Proposition 2.3 (b), for $\zeta \in \mathcal{D}_f$ we have $\operatorname{Re}\zeta < \lfloor 1/\alpha \rfloor + k_f + \boldsymbol{k}_0 + \boldsymbol{k}_1 - \boldsymbol{k} + 3/2 \leqslant \lfloor 1/\alpha \rfloor + 2\boldsymbol{k}_0 + \boldsymbol{k}_1 - \boldsymbol{k} + 3/2$. Hence (2.10) holds.

By (2.1) and (2.2), we have $U \supset \mathbb{D}(0, 8/9)$ (see also [10, Lemma 6.1]). For any $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$, by Koebe's $\frac{1}{4}$-theorem we have $U_f \supset \mathbb{D}(0, 2/9)$. Since $\operatorname{Exp}(\Phi_f(S_f)) \supset U_{\mathcal{R}f} \setminus \{0\}$ and $\mathcal{R}f \in \mathcal{IS}_{1/\alpha}$, we have $\mathbb{D}(0, 2/9) \subset \operatorname{Exp}(\Phi_f(S_f))$. Since $f^{\circ k_f}(S_f) = \mathcal{C}_f \cup \mathcal{C}_f^\sharp \subset \mathcal{P}_f$, we have $\operatorname{Re}\zeta > \lfloor 1/\alpha \rfloor - \boldsymbol{k}$ for all $\zeta \in \Phi_f(S_f) + k_f$. This implies that $\{\zeta \in \mathbb{C} : -3/2 \leqslant \operatorname{Re}\zeta - (\lfloor 1/\alpha \rfloor - \boldsymbol{k}) \leqslant 1 \text{ and } \operatorname{Im}\zeta > -\frac{1}{2\pi}\log\frac{3}{2}\}$ is contained in the interior of $\bigcup_{j=0}^{k_f}(\Phi_f(S_f) + j)$. Therefore, $\mathcal{D}_f \setminus \Phi_f(\mathcal{P}_f)$ contains $\{\zeta \in \mathbb{C} : 0 \leqslant \operatorname{Re}\zeta - (\lfloor 1/\alpha \rfloor - \boldsymbol{k}) \leqslant \boldsymbol{k}_0 + \boldsymbol{k}_1 + 3 \text{ and } \operatorname{Im}\zeta \geqslant 0\}$. ■

2.4. *Some quantitative estimates*

Let $\sigma_f \neq 0$ be another fixed point of $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ near 0 which is contained in $\partial\mathcal{P}_f$ for small $\alpha > 0$ (see Figure 2). It depends continuously on f and has asymptotic expansion

$$\sigma_f = -4\pi\alpha\mathrm{i}/f_0''(0) + o(\alpha) \tag{2.11}$$

as $f \to f_0 \in \mathcal{IS}_0 \cup \{Q_0\}$ in a fixed neighborhood of 0 (see [45, Section 3.2.1]). By [30, Main Theorem 1 (a)], $|f_0''(0)| \in [3,7]$ for all $f_0 \in \mathcal{IS}_0$. By the pre-compactness of $\mathcal{IS}_0$, there exists a constant $D_0' > 1$ such that for all $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_1]$, one has

$$\alpha/D_0' \leqslant |\sigma_f| \leqslant D_0'\alpha. \tag{2.12}$$

For a general statement of (2.12) (i.e., for $\alpha \in \mathbb{C}$), see [14, Lemma 3.25 (1)].

Let

$$\tau_f(w) := \frac{\sigma_f}{1 - e^{-2\pi \mathrm{i}\alpha w}} \tag{2.13}$$

be a universal covering from $\mathbb{C}$ to $\widehat{\mathbb{C}} \setminus \{0, \sigma_f\}$ with period $1/\alpha$. Then $\tau_f(w) \to 0$ as $\operatorname{Im} w \to +\infty$ and $\tau_f(w) \to \sigma_f$ as $\operatorname{Im} w \to -\infty$. There exists a unique lift F_f of f under τ_f such that

$$f \circ \tau_f(w) = \tau_f \circ F_f(w) \quad \text{with} \quad \lim_{\operatorname{Im} w \to +\infty} (F_f(w) - w) = 1.$$

The set $\tau_f^{-1}(\mathcal{P}_f)$ consists of countably many simply connected components. Each of them is bounded by piecewise analytic curves going from $-\infty\mathrm{i}$ to $+\infty\mathrm{i}$. Let $\widetilde{\mathcal{P}}_f$ be the unique component separating 0 from $1/\alpha$. Define

$$L_f := \Phi_f \circ \tau_f : \widetilde{\mathcal{P}}_f \to \mathbb{C}. \tag{2.14}$$

Then L_f is univalent and it is the Fatou coordinate of F_f since $L_f(F_f(w)) = L_f(w) + 1$ if both w and $F_f(w)$ are contained in $\widetilde{\mathcal{P}}_f$.

For $\alpha \in (0, \widetilde{\varepsilon}_1]$ and $R \in (0, +\infty)$, we define

$$\Theta_\alpha(R) := \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} \mathbb{D}(n/\alpha, R).$$

For $C > 0$, we denote $a_C := Ce^{5\pi\mathrm{i}/12}$ and define a piecewise analytic curve

$$\begin{aligned}\ell_C := &\left\{w \in \mathbb{C} : \arg(w - a_C) = \tfrac{11}{12}\pi\right\} \cup \left\{w \in \mathbb{C} : \arg(w - \overline{a}_C) = -\tfrac{11}{12}\pi\right\} \\ &\cup \left\{Ce^{\mathrm{i}\theta} : \theta \in [-\tfrac{5\pi}{12}, \tfrac{5\pi}{12}]\right\}.\end{aligned}$$

Then $\ell_C \cup (-\ell_C + 1/\alpha)$ divides $\mathbb{C}$ into three connected components.[9] Let $A_1(C)$ be the component of $\mathbb{C} \setminus (\ell_C \cup (-\ell_C + 1/\alpha))$ containing $1/(2\alpha)$. The following result is a summary of [10, Lemmas 6.4, 6.7 (2), 6.6, 6.11].

Lemma 2.8. *There are constants $\varepsilon_2 \in (0, \widetilde{\varepsilon}_1]$, C_0, $C_0' > 0$ and $C_0'' \geqslant 6$ such that for all $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_2]$,*

(a) *F_f is defined and univalent in $\Theta_\alpha(C_0')$, and for all $r \in (0, 1/2]$ and all $w \in \Theta_\alpha(r/\alpha) \cap \Theta_\alpha(C_0')$,*

$$|F_f(w) - (w+1)|,\ |F_f'(w) - 1| < \min\left\{\frac{1}{4}, C_0 \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im} w}\right\};$$

[9] We always assume that α is so small that $\Theta_\alpha(C)$ is connected and hence $1/(2\alpha) \in \Theta_\alpha(C)$.

(b) *for all*[10] $R \in [C_0'', 2/\alpha]$ *and all* w *with* $\mathbb{D}(w, R) \subset A_1 := A_1(C_0')$ *and* $\operatorname{Im} w \geqslant -1/\alpha$,

$$\frac{1}{|L_f'(w)|} \leqslant 1 + \frac{C_0}{R};$$

(c) $L_f : \widetilde{\mathcal{P}}_f \to \mathbb{C}$ *has a unique univalent extension onto* $\widetilde{\mathcal{P}}_f \cup A_1$ *such that* $L_f(F_f(w)) = L_f(w) + 1$ *if both* w *and* $F_f(w)$ *belong to* $\widetilde{\mathcal{P}}_f \cup A_1$;

(d) *for any* $r > 0$ *there is* $K_r \geqslant 1$ *depending only on* r *such that*[11]

$$K_r^{-1} \leqslant |(L_f^{-1})'(\zeta)| \leqslant K_r \quad \textit{for all } \zeta \in \Phi_f(\mathcal{P}_f) \setminus \mathbb{D}(0, r).$$

Lemma 2.9 and Proposition 2.10 below are useful in the estimates of the locations of the points under Φ_f^{-1} and χ_f.

Lemma 2.9. *There exists a constant* $D_0 > 0$ *such that for any* $D_1' > 0$, *there exists* $D_1 > 0$ *such that for all* $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ *with* $\alpha \in (0, \varepsilon_2]$,

(a) $D_0 \leqslant |L_f^{-1}(\zeta)| \leqslant D_1$ *for* $\zeta \in \Phi_f(\mathcal{P}_f) \cap \overline{\mathbb{D}}(0, D_1')$;

(b) $D_0 \leqslant |L_f^{-1}(\zeta) - 1/\alpha| \leqslant D_1$ *for* $\zeta \in \Phi_f(\mathcal{P}_f) \cap \overline{\mathbb{D}}(1/\alpha, D_1')$.

Proof. By the continuous dependence of the Fatou coordinates of the maps in $\mathcal{IS}_0$, the pre-compactness of $\mathcal{IS}_0$ and the fact that $\mathcal{P}_f$ is compactly contained in the domain of definition of f, there exists a constant $R_1 > 0$ such that

$$\mathcal{P}_f \subset \mathbb{D}(0, R_1) \quad \text{for all } f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\} \text{ with } \alpha \in (0, \varepsilon_2].$$

By (2.12) and the formula for τ_f in (2.13), a direct calculation shows that there exists a constant $D_0 > 0$ such that the Euclidean distance satisfies $\operatorname{dist}(L_f^{-1}(\zeta), \mathbb{Z}/\alpha) \geqslant D_0$ for all $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_2]$ and all $\zeta \in \Phi_f(\mathcal{P}_f)$.

By Lemma 2.8 (d), there exists a constant $K_1 > 1$ such that

$$K_1^{-1} \leqslant |(L_f^{-1})'(\zeta)| \leqslant K_1 \tag{2.15}$$

for all $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_2]$ and all $\zeta \in \Phi_f(\mathcal{P}_f) \setminus \mathbb{D}$. From [10, Proposition 6.17], there exists a constant $C_1 > 0$ such that for all $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_2]$ we have

$$|L_f^{-1}(3/2)| < C_1. \tag{2.16}$$

Without loss of generality we assume that $D_1' > 1$. Combining (2.15) and (2.16), there exists a constant $C_2 > 0$ depending only on K_1, C_1 and D_1' such that $|L_f^{-1}(\zeta)| < C_2$ for all $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_2]$ and all $\zeta \in (\Phi_f(\mathcal{P}_f) \cap \overline{\mathbb{D}}(0, D_1')) \setminus \mathbb{D}$. On the other

[10] In [10, Lemma 6.7(2)], R is contained in $[3.25, 1/(2\alpha)]$. In fact the estimate of $|L_f'(w)|$ there still holds if $R \in [3.25, C/\alpha]$ for every $C \geqslant 1/2$ (only the constants in the estimate have to be modified).

[11] By Lemma 2.8 (c), the number x_f defined in [10, equation (50)] satisfies $x_f \geqslant \lfloor 1/\alpha \rfloor - \boldsymbol{k}$. Hence by [10, Lemma 6.11] this part holds for all $\zeta \in \Phi_f(\mathcal{P}_f) \setminus \mathbb{D}(0, r)$.

hand, by Lemma 2.8 (a) and applying

$$L_f^{-1}(\zeta) = F_f^{-1} \circ L_f^{-1}(\zeta + 1),$$

there exists a constant $C_3 > 0$ such that $|L_f^{-1}(\zeta)| < C_3$ for all $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_2]$ and all $\zeta \in \Phi_f(\mathcal{P}_f) \cap \mathbb{D}$.

By Lemma 2.8 (d) and [10, Proposition 6.16], there exists a constant $C_4 > 0$ depending on D_1' such that $|L_f^{-1}(\zeta) - 1/\alpha| \leqslant C_4$ for all $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_2]$ and all $\zeta \in \Phi_f(\mathcal{P}_f) \cap \overline{\mathbb{D}}(1/\alpha, D_1')$. Then the proof is complete if we set $D_1 := \max\{C_2, C_3, C_4\}$. ■

Proposition 2.10 ([10, Propositions 6.19 and 6.17]). *There are constants* $\varepsilon_2' \in (0, \varepsilon_2]$ *and* $D_2 > 0$ *such that for all* $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ *with* $\alpha \in (0, \varepsilon_2']$,

(a) *if* $\zeta \in [0, \lfloor 1/\alpha \rfloor - \boldsymbol{k}] + \mathrm{i}\,[-3, +\infty)$, *then*

$$|L_f^{-1}(\zeta) - \zeta| \leqslant D_2 \log(1 + 1/\alpha);$$

(b) *if* $\zeta \in [0, \lfloor 1/\alpha \rfloor - \boldsymbol{k}] + \mathrm{i}\,[-3, 1/\alpha]$, *then*

$$|L_f^{-1}(\zeta) - \zeta| \leqslant D_2 \min\{\log(2 + |\zeta|),\ \log(2 + |\zeta - 1/\alpha|)\}.$$

Proposition 2.10 (a) was proved in [10, Proposition 6.19] (see also [10, Proposition 6.15]). Statement (b) was proved in [10, Proposition 6.17] for $\zeta \in [0, \lfloor 1/\alpha \rfloor - \boldsymbol{k}]$ (i.e., $\zeta \in \mathbb{R}$). However, the arguments there can be applied to $\zeta \in [0, \lfloor 1/\alpha \rfloor - \boldsymbol{k}] + \mathrm{i}\,[-3, 1/\alpha]$ completely similarly by using [10, Lemma 6.7] and Lemma 2.9. For more details on the study of L_f and L_f^{-1}, see [10, Sections 6.3–6.6] and [14, Section 3.5].

Let $X, Y \geqslant 0$. We use $X \asymp Y$ to denote that X and Y are of the same order, i.e., there exist universal positive constants C_1 and C_2 such that $C_1 Y \leqslant X \leqslant C_2 Y$. Let $\mathcal{D}_f$ be the set defined in (2.9).

Lemma 2.11. *There exist constants* $\varepsilon_3 \in (0, \varepsilon_2']$ *and* $D_3 > 0$ *such that for all* $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ *with* $\alpha \in (0, \varepsilon_3]$,

(a) *if* $\zeta \in \mathcal{D}_f$ *with* $\operatorname{Im}\zeta \geqslant 1/\alpha$, *then*

$$|\Phi_f^{-1}(\zeta)| \asymp \frac{\alpha}{e^{2\pi\alpha \operatorname{Im}\zeta}} \quad \text{and} \quad \left| \operatorname{Im}\chi_f(\zeta) - \left(\alpha \operatorname{Im}\zeta + \frac{1}{2\pi} \log \frac{1}{\alpha} \right) \right| \leqslant D_3;$$

(b) *if* $\zeta \in \mathcal{D}_f$ *with* $\operatorname{Im}\zeta \in [-3, 1/\alpha]$, *then*

$$|\Phi_f^{-1}(\zeta)| \asymp \max\left\{ \frac{1}{1 + |\zeta|}, \frac{1}{1 + |\zeta - 1/\alpha|} \right\},$$
$$\left| \operatorname{Im}\chi_f(\zeta) - \frac{1}{2\pi} \min\{\log(1 + |\zeta|), \log(1 + |\zeta - 1/\alpha|)\} \right| \leqslant D_3.$$

Proof. By the definition of Φ_f^{-1} in Lemma 2.5, if $\zeta \in \mathcal{D}_f \setminus \Phi_f(\mathcal{P}_f)$, then there exists a positive integer $j \in [1, k_f + \boldsymbol{k}_0 + \boldsymbol{k}_1 + 2]$ such that $\zeta - j \in \Phi_f(\mathcal{P}_f)$ and $\Phi_f^{-1}(\zeta) = f^{\circ j}(\Phi_f^{-1}(\zeta - j))$. By the pre-compactness of $\mathcal{IS}_\alpha$, it is sufficient to prove the statements in this lemma for $\zeta \in \Phi_f(\mathcal{P}_f)$.

(a) By Proposition 2.10 (a), we have

$$\operatorname{Im}\zeta - D_2\log(1+1/\alpha) \leqslant \operatorname{Im} L_f^{-1}(\zeta) \leqslant \operatorname{Im}\zeta + D_2\log(1+1/\alpha).$$

If α is small, then so is $\alpha\log(1+1/\alpha)$. Suppose $\zeta\in\mathcal{D}_f$ with $\operatorname{Im}\zeta\geqslant 1/\alpha$. Decreasing α if necessary, we assume that $\operatorname{Im}\zeta - D_2\log(1+1/\alpha) > 1/(2\alpha)$. Denote $w := L_f^{-1}(\zeta)$. Then $|e^{-2\pi \mathrm{i}\alpha w}| = |e^{2\pi\alpha \operatorname{Im} w}\cdot e^{-2\pi\mathrm{i}\alpha \operatorname{Re} w}| > e^{\pi}$. Note that $\alpha\log(1+1/\alpha)$ is uniformly bounded above. Since $\operatorname{Im}\zeta\geqslant 1/\alpha$, we have

$$|1-e^{-2\pi\mathrm{i}\alpha w}| \asymp e^{2\pi\alpha\operatorname{Im} w} \asymp e^{2\pi\alpha\operatorname{Im}\zeta}.$$

By (2.12)–(2.14), we have

$$|\Phi_f^{-1}(\zeta)| = |\tau_f\circ L_f^{-1}(\zeta)| = \left|\frac{\sigma_f}{1-e^{-2\pi\mathrm{i}\alpha w}}\right| \asymp \frac{\alpha}{e^{2\pi\alpha\operatorname{Im}\zeta}}.$$

Denote $y := \operatorname{Im}\mathbb{E}\mathrm{xp}^{-1}\circ\Phi_f^{-1}(\zeta)$. By definition we have $\frac{4}{27}e^{-2\pi y}\asymp \alpha/e^{2\pi\alpha\operatorname{Im}\zeta}$. A direct calculation shows that $y=\alpha\ \operatorname{Im}\zeta + \frac{1}{2\pi}\log\frac{1}{\alpha} + \mathcal{O}(1)$, where $\mathcal{O}(1)$ is a number whose absolute value is less than a universal constant.

(b) We divide the argument into two cases. Firstly we assume that $\zeta\in\mathcal{D}_f$ with $\operatorname{Re}\zeta\in[0,1/(2\alpha)]$. By Proposition 2.10 (b), we have

$$|L_f^{-1}(\zeta)-\zeta| \leqslant D_2\log(2+|\zeta|). \tag{2.17}$$

Let $D_1'>0$ be the smallest constant depending only on D_2 such that if $|\zeta|\geqslant D_1'$, then $|\zeta|\geqslant D_2\log(2+|\zeta|)+1$. If $|\zeta|\geqslant D_1'$, and $\operatorname{Re}\zeta\in[0,1/(2\alpha)]$ and $\operatorname{Im}\zeta\in[-3,1/\alpha]$, by (2.17) we have

$$|L_f^{-1}(\zeta)| \asymp |\zeta|+1. \tag{2.18}$$

If $|\zeta|\leqslant D_1'$, and $\operatorname{Re}\zeta\in[0,1/(2\alpha)]$ and $\operatorname{Im}\zeta\in[-3,1/\alpha]$, by Lemma 2.9 (a) there exists a constant $D_1>1$ depending only on D_1' such that $D_0\leqslant|L_f^{-1}(\zeta)|\leqslant D_1$. Therefore, we still have (2.18).

Next we assume that $\operatorname{Re}\zeta\in[1/(2\alpha),\lfloor 1/\alpha\rfloor-\boldsymbol{k}]$. By Proposition 2.10 (b), we have

$$|L_f^{-1}(\zeta)-\zeta| \leqslant D_2\log(2+|\zeta-1/\alpha|). \tag{2.19}$$

If $|\zeta-1/\alpha|\geqslant D_1'$, then $|\zeta-1/\alpha|\geqslant D_2\log(2+|\zeta-1/\alpha|)+1$. If $|\zeta-1/\alpha|\geqslant D_1'$, and $\operatorname{Re}\zeta\in[1/(2\alpha),\lfloor 1/\alpha\rfloor-\boldsymbol{k}]$ and $\operatorname{Im}\zeta\in[-3,1/\alpha]$, by (2.19) we have

$$|L_f^{-1}(\zeta)-1/\alpha| = |(L_f^{-1}(\zeta)-\zeta)+(\zeta-1/\alpha)| \asymp |\zeta-1/\alpha|+1. \tag{2.20}$$

If $|\zeta-1/\alpha|\leqslant D_1'$, and $\operatorname{Re}\zeta\in[1/(2\alpha),\lfloor 1/\alpha\rfloor-\boldsymbol{k}]$ and $\operatorname{Im}\zeta\in[-3,1/\alpha]$, by Lemma 2.9 (b) we have $D_0\leqslant|L_f^{-1}(\zeta)-1/\alpha|\leqslant D_1$. Therefore, in this case we still have (2.20).

Denote $w:=L_f^{-1}(\zeta)$. By (2.17) and (2.19), if α is small enough, then $-1/4\leqslant\operatorname{Re}(\alpha w)\leqslant 5/4$ and $|\alpha w|\leqslant 3/2$. By (2.12), (2.14), (2.18) and (2.20), we have

$$\begin{aligned}|\Phi_f^{-1}(\zeta)| = \left|\frac{\sigma_f}{1-e^{-2\pi\mathrm{i}\alpha w}}\right| &\asymp \max\left\{\frac{1}{|w|},\frac{1}{|w-1/\alpha|}\right\}\\ &\asymp \max\left\{\frac{1}{1+|\zeta|},\frac{1}{1+|\zeta-1/\alpha|}\right\}.\end{aligned}$$

Then the estimate of $\operatorname{Im}\mathbb{E}\mathrm{xp}^{-1}\circ\Phi_f^{-1}(\zeta)$ follows by a direct calculation. ■

Remark. (1) There exist some overlaps between the estimates in Lemma 2.11 (a, b). Indeed, if $\zeta \in \mathcal{D}_f$ and $\operatorname{Im}\zeta \asymp 1/\alpha$, then

$$|\Phi_f^{-1}(\zeta)| \asymp \alpha \quad \text{and} \quad \operatorname{Im}\mathbb{E}\mathrm{xp}^{-1} \circ \Phi_f^{-1}(\zeta) = \frac{1}{2\pi}\log\frac{1}{\alpha} + \mathcal{O}(1).$$

(2) Lemma 2.11 illustrates how the renormalization microscope χ_f reshapes the geometry of the Siegel disk at deeper scales. Specifically, part (a) is for the points deep in the Siegel disk, while (b) is for the points close to the Siegel boundary.

The following lemma can be seen as an inverse version of Lemma 2.11.

Lemma 2.12. *There exist constants $D_4, D_5 > 1$ and $\varepsilon_3' \in (0, \varepsilon_3]$ such that for all $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_3']$,*

(a) *if $\zeta' \in \mathbb{C}$ satisfies $\operatorname{Im}\zeta' \geqslant \frac{1}{2\pi}\log\frac{1}{\alpha} + D_4$, and $\mathbb{E}\mathrm{xp}(\zeta') \in \mathcal{P}_f$ and $\Phi_f \circ \mathbb{E}\mathrm{xp}(\zeta') \in (0, 2] + \mathrm{i}\,[-2, +\infty)$, then*

$$\left|\operatorname{Im}\Phi_f \circ \mathbb{E}\mathrm{xp}(\zeta') - \frac{1}{\alpha}\left(\operatorname{Im}\zeta' - \frac{1}{2\pi}\log\frac{1}{\alpha}\right)\right| \leqslant \frac{D_5}{\alpha};$$

(b) *if $\zeta' \in \mathbb{C}$ satisfies $\operatorname{Im}\zeta' < \frac{1}{2\pi}\log\frac{1}{\alpha} + D_4$, and $\mathbb{E}\mathrm{xp}(\zeta') \in \mathcal{P}_f$ and $\Phi_f \circ \mathbb{E}\mathrm{xp}(\zeta') \in (0, 2] + \mathrm{i}\,[-2, +\infty)$, then*

$$\left|\log\big(3 + \operatorname{Im}\Phi_f \circ \mathbb{E}\mathrm{xp}(\zeta')\big) - 2\pi\operatorname{Im}\zeta'\right| \leqslant D_5.$$

Proof. (a) Denote $\zeta = \Phi_f \circ \mathbb{E}\mathrm{xp}(\zeta') \in \Phi_f(\mathcal{P}_f)$. By Lemma 2.11 (a), if $\operatorname{Im}\zeta \geqslant 1/\alpha$ we have

$$\left|\operatorname{Im}\zeta - \frac{1}{\alpha}\left(\operatorname{Im}\zeta' - \frac{1}{2\pi}\log\frac{1}{\alpha}\right)\right| \leqslant \frac{D_3}{\alpha}. \tag{2.21}$$

Suppose $\operatorname{Re}\zeta \in (0, 2]$ and $\operatorname{Im}\zeta \in [-2, 1/\alpha)$. By Lemma 2.11 (b), we have

$$\operatorname{Im}\zeta' \leqslant \frac{1}{2\pi}\log(1 + |\zeta|) + D_3 < \frac{1}{2\pi}\log\left(\frac{1}{\alpha} + 3\right) + D_3 < \frac{1}{2\pi}\log\frac{1}{\alpha} + D_3 + 1.$$

Therefore, if $\operatorname{Im}\zeta' \geqslant \frac{1}{2\pi}\log\frac{1}{\alpha} + D_3 + 1$, then $\operatorname{Im}\zeta \geqslant 1/\alpha$ or $\operatorname{Im}\zeta < -2$. By the assumption of the lemma we have $\operatorname{Im}\zeta \geqslant 1/\alpha$ and (2.21) holds. Then part (a) follows if we set $D_4 := D_3 + 1$ and $D_5 := D_3$.

(b) Denote $\zeta = \Phi_f \circ \mathbb{E}\mathrm{xp}(\zeta') \in (0, 2] + \mathrm{i}\,[-2, +\infty)$. By (2.21), if $\operatorname{Im}\zeta \in [1/\alpha, (1 + 2D_3)/\alpha]$, we have $|\log\frac{1}{\alpha} + 2\pi\alpha\operatorname{Im}\zeta - 2\pi\operatorname{Im}\zeta'| \leqslant 2\pi D_3$ and hence

$$\begin{aligned}|\log(3 + \operatorname{Im}\zeta) - 2\pi\operatorname{Im}\zeta'| &\leqslant |\log(3\alpha + \alpha\operatorname{Im}\zeta) - 2\pi\alpha\operatorname{Im}\zeta| + 2\pi D_3\\ &\leqslant \log(4 + 2D_3) + 6\pi D_3 + 2\pi.\end{aligned}$$

By Lemma 2.11 (b), if $\operatorname{Re}\zeta \in (0, 2]$ and $\operatorname{Im}\zeta \in (-2, 1/\alpha)$ we have $|\log(1 + |\zeta|) - 2\pi\operatorname{Im}\zeta'| \leqslant 2\pi D_3$ and hence

$$\begin{aligned}|\log(3 + \operatorname{Im}\zeta) - 2\pi\operatorname{Im}\zeta'| &\leqslant |\log(3 + \operatorname{Im}\zeta) - \log(1 + |\zeta|)| + 2\pi D_3\\ &\leqslant \log 5 + 2\pi D_3.\end{aligned}$$

Set $D_5 = \log(4 + 2D_3) + 6\pi D_3 + 2\pi$. Then if $\operatorname{Im}\zeta < (1 + 2D_3)/\alpha$ we have

$$|\log(3 + \operatorname{Im}\zeta) - 2\pi \operatorname{Im}\zeta'| \leqslant D_5. \tag{2.22}$$

Suppose $\operatorname{Im}\zeta \geqslant (1 + 2D_3)/\alpha$. By Lemma 2.11 (a), we have

$$\operatorname{Im}\zeta' \geqslant \alpha \operatorname{Im}\zeta + \frac{1}{2\pi}\log\frac{1}{\alpha} - D_3 \geqslant \frac{1}{2\pi}\log\frac{1}{\alpha} + D_3 + 1.$$

Therefore, if $\operatorname{Im}\zeta' < \frac{1}{2\pi}\log\frac{1}{\alpha} + D_3 + 1$, then $\operatorname{Im}\zeta < (1 + 2D_3)/\alpha$ and we have (2.22).

Summing the constants in parts (a) and (b), the lemma follows if we set $D_4 := D_3 + 1$ and $D_5 := \log(4 + 2D_3) + 6\pi D_3 + 2\pi$. ∎

In the following, we use h' to denote $\partial h/\partial z$ if h is holomorphic and denote $\partial\overline{h}/\partial z$ if h is anti-holomorphic. The following result is useful in the estimate of the Euclidean length of curves in Fatou coordinate planes.

Proposition 2.13. *There exist positive constants $\varepsilon_4 \in (0, \varepsilon_3']$ and $D_2', D_6', D_6 > 1$ such that for all $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \varepsilon_4]$,*

(a) *if $\zeta \in \mathcal{D}_f$ with $\operatorname{Im}\zeta \geqslant 1/(4\alpha)$, then*

$$|\chi_f'(\zeta) - \alpha| \leqslant D_6 \alpha e^{-2\pi\alpha \operatorname{Im}\zeta};$$

(b) *if $\zeta \in \mathcal{D}_f$ with $\operatorname{Im}\zeta \in [-2, 1/(4\alpha)]$ and $r = \min\{|\zeta|, |\zeta - 1/\alpha|\} \geqslant D_6'$, then*

$$|\chi_f'(\zeta)| \leqslant \frac{\alpha}{1 - e^{-2\pi\alpha(r - D_2'\log(2+r))}}\left(1 + \frac{D_6}{r}\right),$$

where D_2' and D_6' are so chosen that $r - 2D_2'\log(2 + r) \geqslant 4$ if $r \geqslant D_6'$.

Proof. Part (a) is proved in [9, Proposition 3.3]. We only prove part (b). For the continuous function

$$\varphi(z) := |1 - e^{2\pi \mathrm{i} z}|,$$

where $z \in \Xi_\varrho := \{\varrho e^{\mathrm{i}\theta} : \theta \in [-\pi/4, 5\pi/4]\}$ with $0 < \varrho \leqslant 2/3$, by a direct calculation[12] we have

$$\min_{z\in\Xi_\varrho} \varphi(z) = \varphi(\varrho e^{\mathrm{i}\pi/2}) = \varphi(\varrho \mathrm{i}) = 1 - e^{-2\pi\varrho}. \tag{2.23}$$

Case 1. We first consider $\zeta \in \Lambda_1 := \mathcal{D}_f \cap \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta \in (0, 1/(2\alpha)]$ and $\operatorname{Im}\zeta \in [-2, 1/(4\alpha)]\}$ and denote $w := L_f^{-1}(\zeta) \in \widetilde{\mathcal{P}}_f$. By (2.4), (2.13), (2.14) and a straightforward calculation we have

$$\begin{aligned}\chi_f'(\zeta) &= (\mathbb{E}\mathrm{xp}^{-1} \circ \Phi_f^{-1})'(\zeta) = (\mathbb{E}\mathrm{xp}^{-1} \circ \tau_f \circ L_f^{-1})'(\zeta)\\ &= -\frac{\alpha}{1 - e^{2\pi \mathrm{i}\alpha w}} \cdot \frac{1}{L_f'(w)}.\end{aligned} \tag{2.24}$$

[12] By setting $r := 2\pi\varrho$, $\beta := \theta - \pi/2$ and considering the derivative of $\beta \mapsto (\varphi(\frac{r}{2\pi}e^{\mathrm{i}(\beta+\pi/2)}))^2$, it suffices to verify that $e^{-r\cos\beta}\sin\beta - \sin(\beta - r\sin\beta) > 0$ for any $r \in (0, 4\pi/3]$ and $\beta \in (0, 3\pi/4]$. This can be done by considering three cases: (1) $\beta - r\sin\beta \in [-\pi, 0]$; (2) $\beta - r\sin\beta \in (0, \pi/2]$ and $\beta \in (0, \pi/2]$; and (3) $\beta - r\sin\beta \in (0, 3\pi/4]$ and $\beta \in (\pi/2, 3\pi/4]$.

By Proposition 2.10 (b), we have

$$w \in \overline{\mathbb{D}}(\zeta, D_2 \log(2+|\zeta|)). \tag{2.25}$$

Let $C_0'' \geqslant 6$ be the constant and $A_1 = A_1(C_0')$ be the domain introduced in Lemma 2.8 (b). Let $C_1 \geqslant 1$ be a constant depending only on C_0'' and D_2 such that if $|\zeta| \geqslant C_1$, then

$$|\zeta| - 2D_2 \log(2+|\zeta|) \geqslant 4 \quad \text{and} \quad \overline{\mathbb{D}}(w, C_0'') \subset A_1. \tag{2.26}$$

We assume that $\widehat{\varepsilon}_1 > 0$ is so small that if $\alpha \in (0, \widehat{\varepsilon}_1]$, then $\alpha|\zeta| < 3/5$ and $D_2\alpha \log(2+|\zeta|) < 1/15$ for all $\zeta \in \Lambda_1$. Hence

$$\alpha|\zeta| + D_2\alpha \log(2+|\zeta|) < 2/3 \quad \text{for all } \zeta \in \Lambda_1. \tag{2.27}$$

By (2.25)–(2.27), for $\zeta \in \Lambda_1' := \Lambda_1 \cap \{\zeta \in \mathbb{C} : |\zeta| \geqslant C_1\}$ we have $\alpha w \in \{\varrho e^{i\theta} : 0 < \varrho \leqslant 2/3$ and $-\pi/4 < \theta < 3\pi/4\}$. According to (2.23), we have

$$|1 - e^{2\pi i \alpha w}| \geqslant 1 - e^{-2\pi\alpha(|\zeta| - D_2 \log(2+|\zeta|))}. \tag{2.28}$$

On the other hand, by (2.26), Lemma 2.8 (b, d) and Proposition 2.10 (b), there exists a constant $C_2 \geqslant 1$ depending only on C_1 and D_2 such that if $\zeta \in \Lambda_1'$ then

$$\frac{1}{|L_f'(w)|} \leqslant 1 + \frac{C_2}{|\zeta|}. \tag{2.29}$$

Combining (2.24), (2.28) and (2.29), if $\zeta \in \Lambda_1'$ we have

$$|\chi_f'(\zeta)| \leqslant \frac{\alpha}{1 - e^{-2\pi\alpha(|\zeta| - D_2 \log(2+|\zeta|))}} \left(1 + \frac{C_2}{|\zeta|}\right).$$

Case 2. Suppose $\zeta \in \Lambda_2 := \mathscr{D}_f \cap \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > 1/(2\alpha)$ and $\operatorname{Im}\zeta \in [-2, 1/(4\alpha)]\}$. By the definition of $\mathscr{D}_f$ in (2.9), there exist an integer $J \geqslant 1$ which is independent of f and $j_0 \in \mathbb{N}$ with $j_0 \leqslant J$ such that $\zeta - j_0 \in \Phi_f(\mathscr{P}_f) \cap \{\zeta : \operatorname{Re}\zeta > 1/(2\alpha)\}$. We denote $w := L_f^{-1}(\zeta - j_0) \in \widetilde{\mathscr{P}}_f$ and $\widetilde{w} := F_f^{\circ j_0}(w)$. Then

$$\begin{aligned}\chi_f'(\zeta) &= (\mathbb{E}\mathrm{xp}^{-1} \circ f^{\circ j_0} \circ \Phi_f^{-1})'(\zeta - j_0) \\ &= (\mathbb{E}\mathrm{xp}^{-1} \circ \tau_f \circ F_f^{\circ j_0} \circ L_f^{-1})'(\zeta - j_0) = -\frac{\alpha}{1 - e^{2\pi i \alpha \widetilde{w}}} \cdot \frac{(F_f^{\circ j_0})'(w)}{L_f'(w)}.\end{aligned} \tag{2.30}$$

By Proposition 2.10 (b), we have

$$w \in \overline{\mathbb{D}}\big(\zeta - j_0, D_2 \log\big(2 + \big|\zeta - j_0 - \tfrac{1}{\alpha}\big|\big)\big).$$

Let $C_0'' \geqslant 6$ and $A_1 = A_1(C_0')$ be as in Lemma 2.8 (b). By Lemma 2.8 (a), there exist positive constants C_1' and C_1'' depending only on C_0'', D_2 and J such that if $|\zeta - 1/\alpha| \geqslant C_1'$, then

$$\overline{\mathbb{D}}(w, C_0'') \subset A_1 \quad \text{and} \quad |F_f^{\circ j}(w) - 1/\alpha| \geqslant C_1''|\zeta - 1/\alpha| \tag{2.31}$$

for all $j = 0, 1, \ldots, j_0$. Also by Lemma 2.8 (a), there exists a constant $D_2' \geqslant D_2$ depending only on C_0'', C_1'', D_2 and J such that

$$\widetilde{w} = F_f^{\circ j_0}(w) \in \mathbb{D}\big(\zeta, D_2' \log(2 + |\zeta - 1/\alpha|)\big)$$

and

$$|(F_f^{\circ j_0})'(w)| \leqslant 1 + \frac{D_2'}{|\zeta - 1/\alpha|}. \tag{2.32}$$

Let $C_2' \geqslant C_1'$ be a constant depending only on C_1' and D_2' such that if $|\zeta - 1/\alpha| \geqslant C_2'$, then

$$|\zeta - 1/\alpha| - 2D_2' \log(2 + |\zeta - 1/\alpha|) \geqslant 4.$$

Moreover, we assume that $\widehat{\varepsilon}_2 > 0$ is so small that if $\alpha \in (0, \widehat{\varepsilon}_2]$, then

$$\alpha|\zeta - 1/\alpha| + D_2'\alpha \log(2 + |\zeta - 1/\alpha|) < 2/3 \quad \text{for all } \zeta \in \Lambda_2.$$

For $\zeta \in \Lambda_2' := \Lambda_2 \cap \{\zeta \in \mathbb{C} : |\zeta - 1/\alpha| \geqslant C_2'\}$, we have $\alpha\widetilde{w} - 1 \in \{\varrho e^{\mathrm{i}\theta} : 0 < \varrho \leqslant 2/3$ and $\pi/4 < \theta < 5\pi/4\}$. By (2.23) and $|1 - e^{2\pi \mathrm{i} z}| = |1 - e^{2\pi \mathrm{i}(z-1)}|$, we have

$$|1 - e^{2\pi \mathrm{i}\alpha\widetilde{w}}| \geqslant 1 - e^{-2\pi\alpha(|\zeta - 1/\alpha| - D_2' \log(2 + |\zeta - 1/\alpha|))}. \tag{2.33}$$

Similarly, by (2.31), Lemma 2.8 (b, d) and Proposition 2.10 (b), there exists a constant $C_3 \geqslant 1$ depending only on C_1'', C_2' and D_2' such that if $\zeta \in \Lambda_2'$ then

$$\frac{1}{|L_f'(w)|} \leqslant 1 + \frac{C_3}{|\zeta - 1/\alpha|}. \tag{2.34}$$

Combining (2.30), (2.32), (2.33) and (2.34), if $\zeta \in \Lambda_2'$ we have

$$|\chi_f'(\zeta)| \leqslant \frac{\alpha}{1 - e^{-2\pi\alpha(|\zeta - 1/\alpha| - D_2' \log(2 + |\zeta - 1/\alpha|))}} \left(1 + \frac{C_3'}{|\zeta - 1/\alpha|}\right)$$

for a constant $C_3' > 0$ depending only on C_3 and D_2'. The proof is complete if we set $\varepsilon_4 := \min\{\varepsilon_3', \widehat{\varepsilon}_1, \widehat{\varepsilon}_2\}$, $D_6' := \max\{C_1, C_2'\}$ and $D_6 := \max\{C_2, C_3'\}$. ∎

Remark. Proposition 2.13 will be used in the proof of Lemma 4.8. In [10, Proposition 6.18], an estimate of $|\chi_f'(\zeta)|$ has been obtained for $\zeta \in [1, 1/(2\alpha)]$ in another form.

2.5. *Renormalization tower and orbit relations*

In the rest of this paper, we always assume that the integer N is so large that $N \geqslant 1/\varepsilon_4$, where $\varepsilon_4 > 0$ is the constant of Proposition 2.13. Let $[0; a_1, a_2, \ldots]$ be the continued fraction expansion of $\alpha \in \mathrm{HT}_N$. Define $\alpha_0 := \alpha$, and inductively for $n \geqslant 1$, define the sequence of real numbers $\alpha_n \in (0, 1)$ as

$$\alpha_n = \frac{1}{\alpha_{n-1}} - \left\lfloor \frac{1}{\alpha_{n-1}} \right\rfloor \quad \text{for } n \geqslant 1. \tag{2.35}$$

Then each α_n has the continued fraction expansion $[0; a_{n+1}, a_{n+2}, \ldots]$. By definition, we have $\alpha_n \in (0, \varepsilon_4]$ for all $n \in \mathbb{N}$.

Let $\alpha \in \mathrm{HT}_N$ and $f_0 \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$. By Theorem 2.4, the following sequence of maps is well-defined for all $n \geqslant 0$:

$$f_{n+1} := \mathcal{R} f_n : U_{f_{n+1}} \to \mathbb{C}.$$

Let $U_n := U_{f_n}$ be the domain of definition of f_n for $n \geqslant 0$. Then for all n, we have

$$f_n : U_n \to \mathbb{C}, \quad f_n(0) = 0, \quad f_n'(0) = e^{2\pi i \alpha_n} \quad \text{and} \quad \mathrm{cv} = \mathrm{cv}_{f_n} = -4/27.$$

For $n \geqslant 0$, let $\Phi_n := \Phi_{f_n}$ be the Fatou coordinate of $f_n : U_n \to \mathbb{C}$ defined in the perturbed petal $\mathcal{P}_n := \mathcal{P}_{f_n}$ and let $\mathcal{C}_n := \mathcal{C}_{f_n}$ and $\mathcal{C}_n^\sharp := \mathcal{C}_{f_n}^\sharp$ be the corresponding sets for f_n defined in (2.3). Let $k_n := k_{f_n}$ be the positive integer in Proposition 2.3 such that

$$S_n^0 := S_{f_n} = \mathcal{C}_n^{-k_n} \cup (\mathcal{C}_n^\sharp)^{-k_n} \subset \{z \in \mathcal{P}_n : 0 < \operatorname{Re} \Phi_n(z) < \lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 1/2\}.$$

For $n \geqslant 0$, let $\widetilde{\mathcal{D}}_n := \widetilde{\mathcal{D}}_{f_n}$ and $\mathcal{D}_n := \mathcal{D}_{f_n}$ be the sets defined in (2.6) and (2.9) respectively. Note that $\mathcal{D}_n \subset \widetilde{\mathcal{D}}_n$ by Lemma 2.7. According to Lemma 2.5, we have a holomorphic map

$$\Phi_n^{-1} : \widetilde{\mathcal{D}}_n \to U_n \setminus \{0\}$$

such that $\Phi_n^{-1}(\zeta + 1) = f_n \circ \Phi_n^{-1}(\zeta)$ if $\zeta, \zeta + 1 \in \widetilde{\mathcal{D}}_n$. We denote the lift $\chi_{f_n,0}$ in (2.8) by $\chi_{n,0}$. Then for $n \geqslant 1$ we have

$$\chi_{n,0}(\widetilde{\mathcal{D}}_n) \subset \{\zeta \in \mathbb{C} : 1 \leqslant \operatorname{Re} \zeta < \boldsymbol{k}_1 + 2 \text{ and } \operatorname{Im} \zeta > -2\} \subset \Phi_{n-1}(\mathcal{P}_{n-1}). \tag{2.36}$$

Each $\chi_{n,0}$ is anti-holomorphic. For $j \in \mathbb{Z}$ we define

$$\chi_{n,j} := \chi_{n,0} + j. \tag{2.37}$$

In the following we are mainly interested in $\chi_{n,j}$ with $0 \leqslant j \leqslant a_n = \lfloor 1/\alpha_{n-1} \rfloor$.

Recall that for $\delta > 0$, $B_\delta(X)$ is the δ-neighborhood of a set $X \subset \mathbb{C}$ with respect to the Euclidean metric. The following lemma will be used to prove uniform contraction with respect to the hyperbolic metrics in the domains of adjacent renormalization levels (see Lemma 4.7).

Lemma 2.14 ([2, Lemma 2.1]). *There exists a constant $\delta_0 > 0$ depending only on the class $\mathcal{IS}_0$ such that for all $n \geqslant 1$ and $0 \leqslant j \leqslant a_n$,*

$$B_{\delta_0}(\chi_{n,j}(\mathcal{D}_n)) \subset \mathcal{D}_{n-1}.$$

For $n \geqslant 0$, recall that $\mathcal{P}_n$ is the perturbed petal of f_n. For $n \geqslant 1$, we define an anti-holomorphic map ψ_n by

$$\psi_n := \Phi_{n-1}^{-1} \circ \chi_{n,0} \circ \Phi_n : \mathcal{P}_n \to \mathcal{P}_{n-1}. \tag{2.38}$$

Hence we have the following diagrams:

$$
\begin{array}{ccc}
\mathcal{P}_{n-1} & \xleftarrow{\ \Phi_{n-1}^{-1}\ } & \Phi_{n-1}(\mathcal{P}_{n-1}) \\
{\scriptstyle \psi_n}\big\uparrow & & \big\uparrow{\scriptstyle \chi_{n,0}} \\
\mathcal{P}_n & \xrightarrow{\ \Phi_n\ } & \Phi_n(\mathcal{P}_n)
\end{array}
\qquad\qquad
\begin{array}{ccc}
U_{n-1} & \xleftarrow{\ \Phi_{n-1}^{-1}\ } & \mathcal{D}_{n-1} \\
 & & \big\uparrow{\scriptstyle \chi_{n,j}} \\
U_n & \xleftarrow{\ \Phi_n^{-1}\ } & \mathcal{D}_n
\end{array}
$$

Each ψ_n extends continuously to $0 \in \partial\mathcal{P}_n$ by mapping it to 0. For $n \geqslant 1$, we define

$$\Psi_n := \psi_1 \circ \cdots \circ \psi_n : \mathcal{P}_n \to \mathcal{P}_0 \subset U_0.$$

For $n \geqslant 0$ and $i \geqslant 1$, define the sector

$$S_n^i := \psi_{n+1} \circ \cdots \circ \psi_{n+i}(S_{n+i}^0) \subset \mathcal{P}_n.$$

In particular, $S_0^n \subset \mathcal{P}_0$ for all $n \geqslant 0$. Define

$$\mathcal{P}_n' := \{z \in \mathcal{P}_n : 0 < \operatorname{Re}\Phi_n(z) < \lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 1\}.$$

Let q_n be the denominator of the convergents $[0; a_1, \ldots, a_n]$ of the continued fraction expansion of α. Recall that $k_n = k_{f_n}$ is the positive integer as in Proposition 2.3. The following lemma was proved in [10, Section 3] and parts of the results can also be found in [7, Section 1.5.5]. The proof is based on the definition of near-parabolic renormalization.

Lemma 2.15 ([10, Lemmas 3.3, 3.4]). *For every $n \geqslant 1$,*

(a) *for every $z \in \mathcal{P}_n'$, $f_{n-1}^{\circ a_n} \circ \psi_n(z) = \psi_n \circ f_n(z)$ and $f_0^{\circ q_n} \circ \Psi_n(z) = \Psi_n \circ f_n(z)$;*

(b) *for every $z \in S_n^0$, $f_{n-1}^{\circ(k_n a_n+1)} \circ \psi_n(z) = \psi_n \circ f_n^{\circ k_n}(z)$ and $f_0^{\circ(k_n q_n + q_{n-1})} \circ \Psi_n(z) = \Psi_n \circ f_n^{\circ k_n}(z)$;*

(c) *for every $m < n$, $f_n : \mathcal{P}_n' \to \mathcal{P}_n$ and $f_n^{\circ k_n} : S_n^0 \to \mathcal{C}_n \cup \mathcal{C}_n^\sharp$ are conjugate to some iterates of f_m on the set $\psi_{m+1} \circ \cdots \circ \psi_n(\mathcal{P}_n)$.*

In particular, the dynamics of f_n is conjugate to the dynamics of f_0. Specifically, the first k_n iterates of f_n on S_n^0 correspond to $k_n q_n + q_{n-1}$ iterates of f_0, and the next $\lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 2$ iterates correspond to $q_n(\lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 2)$ iterates of f_0.

For each $n \in \mathbb{N}$, by (2.7) we have

$$b_n := k_n + \lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 2 \geqslant a_{n+1} + 1.$$

From the definition of $\widetilde{\mathcal{D}}_n$ in (2.6) and by Lemma 2.15, the following sets are well-defined for each $n \geqslant 0$:

$$\Omega_n^0 := \bigcup_{j=0}^{b_n} f_n^{\circ j}(S_n^0) \cup \{0\} \quad \text{and} \quad \Omega_0^n := \bigcup_{j=0}^{b_n q_n + q_{n-1}} f_0^{\circ j}(S_0^n) \cup \{0\}.$$

Definition (High type Brjunos). Let N be the integer fixed before. Define

$$\mathcal{B}_N := \left\{ \alpha = [0; a_1, a_2, \ldots] \in (0,1) \setminus \mathbb{Q} \;\middle|\; \begin{array}{l} \alpha \text{ is Brjuno and} \\ a_n \geqslant N, \ \forall n \geqslant 1 \end{array} \right\}. \tag{2.39}$$

Then $\mathcal{B}_N$ is strictly contained in HT_N.

Proposition 2.16 ([10, Propositions 3.5, 5.10 (2)]). *Let $f_0 \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in \mathrm{HT}_N$. Then for all $n \geqslant 0$,*

(a) *Ω_0^{n+1} is compactly contained in the interior of Ω_0^n and $f_0(\Omega_0^{n+1}) \subset \Omega_0^n$;*

(b) *if $\alpha \in \mathcal{B}_N$, then $\mathrm{int}(\bigcap_{n=0}^\infty \Omega_0^n) = \Delta_0$, where Δ_0 is the Siegel disk of f_0.*

In the rest of this paper, unless otherwise stated, for a given map $f_0 \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in \mathrm{HT}_N$, we use f_n to denote the map after n-th near-parabolic renormalization. We also use U_n, $\mathcal{P}_n$ and Φ_n etc. to denote respectively the domain of definition, the perturbed petal and the Fatou coordinate etc. of f_n.

3. The suitable heights

3.1. Radii of Siegel disks

The following classical distortion theorem can be found in [37, Theorem 1.6, p. 21].

Theorem 3.1 (Koebe's distortion theorem). *Suppose $f : \mathbb{D} \to \mathbb{C}$ is a univalent map with $f(0) = 0$ and $f'(0) = 1$. Then for each $z \in \mathbb{D}$ we have*

(a) $\dfrac{1 - |z|}{(1 + |z|)^3} \leqslant |f'(z)| \leqslant \dfrac{1 + |z|}{(1 - |z|)^3}$;

(b) $\dfrac{|z|}{(1 + |z|)^2} \leqslant |f(z)| \leqslant \dfrac{|z|}{(1 - |z|)^2}$;

(c) $|\arg f'(z)| \leqslant 2 \log \dfrac{1 + |z|}{1 - |z|}$.

Let $\alpha_0 := \alpha \in \mathcal{B}_N$ and $\alpha_n \in (0,1)$ be the number defined inductively as in (2.35) for $n \geqslant 1$. Denote $\beta_{-1} = 1$ and $\beta_n := \prod_{i=0}^n \alpha_i$ for $n \geqslant 0$. The *Brjuno sum* $\mathcal{B}(\alpha)$ of α in the sense of Yoccoz is defined as

$$\mathcal{B}(\alpha) := \sum_{n=0}^\infty \beta_{n-1} \log \frac{1}{\alpha_n} = \log \frac{1}{\alpha_0} + \alpha_0 \log \frac{1}{\alpha_1} + \alpha_0 \alpha_1 \log \frac{1}{\alpha_2} + \cdots. \tag{3.1}$$

It is proved in [53, Section 1.5] that $|\mathcal{B}(\alpha) - \sum_{n=0}^\infty q_n^{-1} \log q_{n+1}| \leqslant C'$ for a universal constant $C' > 0$.

Suppose a holomorphic map f has a Siegel disk Δ_f centered at the origin which is compactly contained in the domain of definition of f. The *inner radius* of Δ_f is the radius of the largest open disk centered at the origin that is contained in Δ_f.

Lemma 3.2. *There exists a universal constant $D_7 > 1$ such that for all $f_0 \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in \mathcal{B}_N$, the inner radius of the Siegel disk of f_n is $c_n e^{-\mathcal{B}(\alpha_n)}$ with $1/D_7 \leqslant c_n \leqslant D_7$ for every $n \in \mathbb{N}$.*

Proof. By the definition of near-parabolic renormalization, $f_n \in \mathcal{IS}_{\alpha_n}$ with $\alpha_n \in \mathcal{B}_N$ for all $n \geqslant 1$. Then according to [5], each f_n with $n \geqslant 0$ has a Siegel disk centered at the origin. By the definition of Inou–Shishikura's class and Koebe's distortion theorem (Theorem 3.1 (b)), f_n is univalent in $\mathbb{D}(0, \widetilde{c})$ for a universal constant $\widetilde{c} > 0$. According to Yoccoz [53, p. 21], the Siegel disk of f_n contains a round disk $\mathbb{D}(0, C_1 e^{-\mathcal{B}(\alpha_n)})$ for a universal constant $C_1 > 0$, where

$$\mathcal{B}(\alpha_n) := \log \frac{1}{\alpha_n} + \sum_{k=1}^{+\infty} \alpha_n \cdots \alpha_{n+k-1} \log \frac{1}{\alpha_{n+k}} \tag{3.2}$$

is the Brjuno sum of α_n defined in (3.1). On the other hand, by [10, Theorem G], there is a universal constant $C_2 > 1$ such that the inner radius of the Siegel disk of f_n is bounded above by $C_2 e^{-\mathcal{B}(\alpha_n)}$ for all $n \in \mathbb{N}$. The lemma follows with $D_7 := \max\{C_2, 1/C_1\}$. ∎

3.2. *Definition of the heights*

In the following, we use Δ_n to denote the Siegel disk of f_n for all $n \geqslant 0$, where $f_0 \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in \mathcal{B}_N$ and f_n is obtained by applying the near-parabolic renormalization operator.

Definition (The heights). Let $M \geqslant 1$. For $n \geqslant 0$, we define

$$h_n := \frac{\mathcal{B}(\alpha_{n+1})}{2\pi} + \frac{M}{\alpha_n}. \tag{3.3}$$

There are many choices of the height h_n. One of the candidates is $\frac{\mathcal{B}(\alpha_{n+1})}{2\pi} + M$. In order to apply Lemma 2.11 (a) directly, we choose h_n above so that $h_n > 1/\alpha_n$. Similar to (2.3) (see Figure 3), we define

$$\widetilde{\mathcal{C}}_n^\sharp := \{z \in \mathcal{P}_n : 1/2 \leqslant \operatorname{Re} \Phi_n(z) \leqslant 3/2 \text{ and } \operatorname{Im} \Phi_n(z) \geqslant h_n\}.$$

Let $(\widetilde{\mathcal{C}}_n^\sharp)^{-k_n}$ be the component of $f_n^{-k_n}(\widetilde{\mathcal{C}}_n^\sharp)$ contained in $(\mathcal{C}_n^\sharp)^{-k_n}$. Recall that ψ_n is defined in (2.38). For $n \geqslant 0$ and $i \geqslant 1$, we denote

$$V_n^0 := (\widetilde{\mathcal{C}}_n^\sharp)^{-k_n} \subset S_n^0 \quad \text{and} \quad V_n^i := \psi_{n+1} \circ \cdots \circ \psi_{n+i}(V_{n+i}^0) \subset S_n^i.$$

Lemma 3.3. *There exists a universal constant $M_1 \geqslant 1$ such that if $M \geqslant M_1$, then for all $n \geqslant 0$ and $i \geqslant 0$, V_n^i is compactly contained in Δ_n.*

Proof. We first prove that V_n^0 is compactly contained in Δ_n for all $n \geqslant 0$ if $M \geqslant 1$ is large enough. By a straightforward calculation, the image of $\Phi_n(\widetilde{\mathcal{C}}_n^\sharp)$ under $\mathbb{E}\mathrm{xp}$ is a punctured rounded disk centered at the origin with radius

$$\iota_n := \frac{4}{27} e^{-2\pi h_n} = \frac{4}{27} e^{-\frac{2\pi M}{\alpha_n}} \cdot e^{-\mathcal{B}(\alpha_{n+1})} < \frac{1}{D_7} e^{-\mathcal{B}(\alpha_{n+1})}$$

if $M \geqslant M_1 := \frac{1}{2\pi}\log D_7 + 1$, where $D_7 > 1$ is the constant of Lemma 3.2. This implies that $\mathbb{E}\mathrm{xp}\circ\Phi_n(\widetilde{\mathcal{C}}_n^\sharp)$ is compactly contained in the Siegel disk of f_{n+1} if $M \geqslant M_1$. Hence there exists a small open neighborhood D of $\widetilde{\mathcal{C}}_n^\sharp$ in $\mathcal{P}_n$ such that $\mathbb{E}\mathrm{xp}\circ\Phi_n(D)$ is compactly contained in the Siegel disk Δ_{n+1}. By Lemma 2.15 (c), f_n can be iterated infinitely many times in D and the orbit is compactly contained in the domain of definition of f_n. Note that $0 \in \overline{D}$. Therefore, D is contained in the Siegel disk of f_n and $\widetilde{\mathcal{C}}_n^\sharp \Subset \Delta_n$. Since $f_n^{\circ k_n}(V_n^0) = \widetilde{\mathcal{C}}_n^\sharp$ and $0 \in \partial V_n^0$, we have $V_n^0 \Subset \Delta_n$.

For each $z \in V_n^0$, there exists a small open neighborhood of z on which f_n can be iterated infinitely many times. By Lemma 2.15 (b), there exists a small open neighborhood of $\Psi_n(z) \in V_0^n$ on which f_0 can also be iterated infinitely many times. Since each $z \in V_n^0$ has this property and $0 \in \partial V_0^n$, it follows that $V_0^n \Subset \Delta_0$. By a completely similar argument, we have $V_n^i \Subset \Delta_n$ for any $i, n > 0$. ∎

Note that the forward orbit of V_n^i is compactly contained in Δ_n for any $n, i \geqslant 0$. Moreover, the backward orbit of V_n^i is also compactly contained in Δ_n if the preimage under f_n is chosen in Δ_n. In the following, we always assume that $M \geqslant M_1$ unless otherwise stated.

3.3. *The location of the neighborhoods*

For $n \geqslant 0$, each $V_n^0 \cup \{0\}$ is a closed topological triangle[13] whose boundary consists of three analytic curves. We use $\partial^l V_n^0$, $\partial^r V_n^0$ and $\partial^b V_n^0$ to denote the three smooth edges of V_n^0, where $f_n(\partial^l V_n^0) = \partial^r V_n^0$ and $\partial^l V_n^0 \cap \partial^r V_n^0 = \{0\}$. The superscripts '$l$', '$r$' and '$b$' denote 'left', 'right' and 'bottom', respectively. See Figure 5.

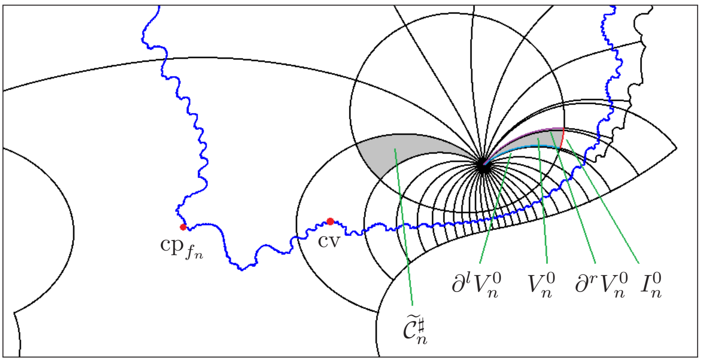

Fig. 5. In the dynamical plane of f_n, the sets $\partial^l V_n^0$, $\partial^r V_n^0$ and I_n^0 are colored cyan, purple and red respectively. The blue set is the (partial) forward orbit of the critical point cp_{f_n}. The sets V_n^0 and $\widetilde{\mathcal{C}}_n^\sharp = f_n^{\circ k_n}(V_n^0)$ are colored gray.

[13] Here we use the fact that for any $x \in (0, \lfloor 1/\alpha_n \rfloor - \boldsymbol{k})$, $\lim_{y\to+\infty}\Phi_n^{-1}(x + y\mathrm{i}) = 0$ (see [14, Proposition 2.4 (a)] or [10, Lemma 6.9]).

A similar naming convention is adopted for V_n^i and their forward images for all $n, i \geqslant 0$. For example, $\partial^l V_n^i := \psi_{n+1} \circ \cdots \circ \psi_{n+i}(\partial^l V_{n+i}^0)$ if i is even, while $\partial^l V_n^i := \psi_{n+1} \circ \cdots \circ \psi_{n+i}(\partial^r V_{n+i}^0)$ if i is odd (note that each ψ_j is anti-holomorphic). For simplicity, we denote

$$I_n^0 := \partial^b V_n^0 \subset \Delta_n.$$

The 'left' and the 'right' end points of I_n^0 are denoted by $\partial^l I_n^0$ and $\partial^r I_n^0$ respectively so that $f_n(\partial^l I_n^0) = \partial^r I_n^0$. A similar naming convention is adopted for I_n^i and their forward images for all $n, i \geqslant 0$. In particular, by Lemma 2.15 (a) we have $f_0^{\circ q_n}(\partial^l I_0^n) = \partial^r I_0^n$ if n is even and $f_0^{\circ q_n}(\partial^r I_0^n) = \partial^l I_0^n$ if n is odd. Moreover, let $\partial^l S_n^i$ and $\partial^r S_n^i$ be the smooth edges of S_n^i containing $\partial^l V_n^i$ and $\partial^r V_n^i$ respectively.

Let $k_n = k_{f_n} \geqslant 1$ be the integer of Proposition 2.3, $D_3 > 0$ be the constant of Lemma 2.11 and $\mathcal{D}_n = \mathcal{D}_{f_n}$ be the set defined in (2.9).

Lemma 3.4 (see Figure 6). *There exists a constant $M_2 \geqslant 1$ such that if $M \geqslant M_2$, then for all $n \in \mathbb{N}$,*

(a) $\operatorname{diam}(\Phi_n(I_n^0)) \leqslant 2$ *and* $|\operatorname{Im}\zeta - h_n| \leqslant 1$ *for all* $\zeta \in \Phi_n(I_n^0)$*;*

(b) $u_n(y) := \{\zeta \in \mathbb{C} : \operatorname{Im}\zeta = y\} \cap \Phi_n(\partial^l S_n^0)$ *is a singleton for all* $y \geqslant h_n - 1$*;*

(c) $\operatorname{diam}(\beta_n') \leqslant 1$*, where* β_n' *is the arc in* $\Phi_n(\partial^l S_n^0)$ *connecting* $u_n(h_n)$ *to* $\Phi_n(\partial^l I_n^0)$*.*

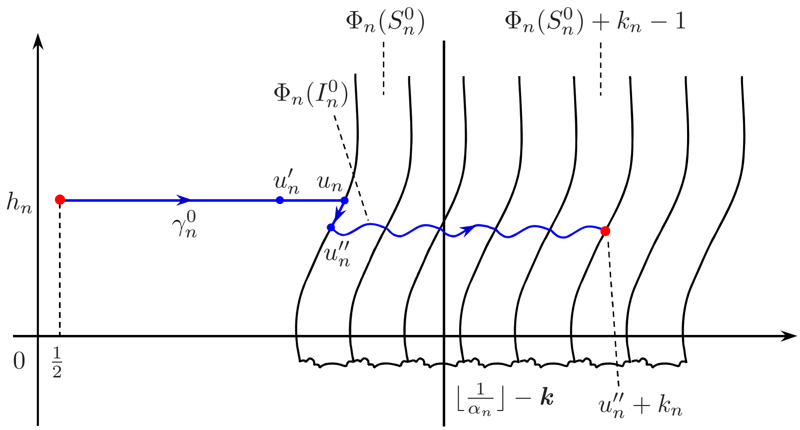

Fig. 6. The sketch of the construction of the continuous curve γ_n^0 (in blue) in the Fatou coordinate plane of f_n. The two red dots denote the initial and terminal points of γ_n^0 and they have the same image under the map Φ_n^{-1}. In particular, $\Phi_n^{-1}(\gamma_n^0)$ is a continuous closed curve in the Siegel disk of f_n.

Proof. The proof is mainly based on applying Koebe's distortion theorem and the definition of near-parabolic renormalization.

(a) By the definition of near-parabolic renormalization, we have

$$f_{n+1}(\mathbb{E}\mathrm{xp} \circ \Phi_n(V_n^0)) = \mathbb{E}\mathrm{xp} \circ \Phi_n(\widetilde{\mathcal{C}}_n^\sharp).$$

Note that $\mathbb{E}\mathrm{xp} \circ \Phi_n(\widetilde{\mathcal{C}}_n^\sharp) \cup \{0\}$ is a closed round disk with radius

$$\iota_n = \tfrac{4}{27} e^{-\frac{2\pi M}{\alpha_n}} \cdot e^{-\mathcal{B}(\alpha_{n+1})}.$$

By Lemma 3.2, Δ_{n+1} contains the disk $\mathbb{D}(0, \varsigma_n)$, where

$$\varsigma_n := D_7^{-1} e^{-\mathcal{B}(\alpha_{n+1})}.$$

Therefore,

$$g := f_{n+1}^{-1} : \mathbb{D}(0, \varsigma_n) \to \Delta_{n+1} \tag{3.4}$$

is a well-defined univalent map with $|g'(0)| = 1$. If M is large enough such that ι_n is much smaller than ς_n, then by Theorem 3.1 the distortion of the circle $g(\partial\mathbb{D}(0, \iota_n))$ relative to $\partial\mathbb{D}(0, \iota_n)$ can be arbitrarily small. Part (a) is proved if we notice that $\Phi_n(I_n^0)$ is the closure of a connected component of $\mathbb{E}\mathrm{xp}^{-1} \circ g(\partial\mathbb{D}(0, \iota_n) \setminus \{\iota_n\})$.

(b) Still by the definition of near-parabolic renormalization, we have

$$f_{n+1}(\mathbb{E}\mathrm{xp} \circ \Phi_n(\partial^l S_n^0)) = \big(0, \tfrac{4}{27} e^{4\pi}\big].$$

Since $\mathbb{D}(0, \varsigma_n) \subset \Delta_{n+1}$, we have $f_{n+1}^{-1}([0, \frac{4}{27} e^{4\pi}]) \cap g(\mathbb{D}(0, \varsigma_n)) = g([0, \varsigma_n))$, where g is defined in (3.4). On the other hand, by (3.4) and Theorem 3.1 (b), we assume that M is so large that ι_n is small and $g(\mathbb{D}(0, \varsigma_n)) \supset \overline{\mathbb{D}}(0, e^{2\pi}\iota_n)$. According to Theorem 3.1 (c), we assume further that M is large so that $g([0, \varsigma_n)) \cap \partial\mathbb{D}(0, r)$ is a singleton for any $0 < r \leqslant e^{2\pi}\iota_n$. Therefore,

$$\mathbb{E}\mathrm{xp} \circ \Phi_n(\partial^l S_n^0) \cap \{z \in \mathbb{C} : |z| = r\}$$

is a singleton, where $0 < r \leqslant e^{2\pi}\iota_n = \frac{4}{27} e^{-2\pi(h_n - 1)}$. This proves (b).

(c) By the definition of near-parabolic renormalization, we have

$$\mathbb{E}\mathrm{xp}(u_n(h_n)) = g([0, \varsigma_n)) \cap \partial\mathbb{D}(0, \iota_n) \quad \text{and} \quad \mathbb{E}\mathrm{xp} \circ \Phi_n(\partial^l I_n^0) = g(\iota_n).$$

Moreover, by the definition of β_n' we have $\mathbb{E}\mathrm{xp}(\beta_n') \subset g([0, \varsigma_n))$. By Theorem 3.1, the Euclidean length of the arc $\mathbb{E}\mathrm{xp}(\beta_n')$ with end points $g([0, \varsigma_n)) \cap \partial\mathbb{D}(0, \iota_n)$ and $g(\iota_n)$ can be arbitrarily small if M is large enough. This proves (c). ■

Let $D_3 > 0$ be as in Lemma 2.11. In the following we always assume that $M \geqslant \max\{M_2, D_3 + \frac{1}{2\pi}\log\frac{4D_7}{27} + 2\}$ unless otherwise stated. Then

$$y_n := \frac{\mathcal{B}(\alpha_{n+1})}{2\pi} + M - D_3 - \frac{3}{2} > \frac{\mathcal{B}(\alpha_{n+1})}{2\pi} - \frac{1}{2\pi}\log\frac{27 c_{n+1}}{4}. \tag{3.5}$$

This implies that if $\operatorname{Im}\zeta \geqslant y_n$, then $\zeta \in \mathbb{E}\mathrm{xp}^{-1}(\Delta_{n+1})$.

4. The sequence of the curves is convergent

In this section, we define a sequence $(\gamma_n^i)_{n\in\mathbb{N}}$ of continuous curves in the Fatou coordinate planes with $i \in \mathbb{N}$. The image of each γ_n^i under Φ_n^{-1} is a continuous closed curve contained in the Siegel disk Δ_n of f_n. We shall prove that $(\gamma_0^n)_{n\in\mathbb{N}}$ converges uniformly to the boundary of Δ_0.

4.1. Definition of the curves and their parametrization

Note that $a_{n+1} = \lfloor 1/\alpha_n \rfloor$ for each $n \in \mathbb{N}$. Recall that

$$u_n := u_n(h_n) = \{\zeta \in \mathbb{C} : \operatorname{Im}\zeta = h_n\} \cap \Phi_n(\partial^l S_n^0)$$

is introduced in Lemma 3.4 (b). Since $f_n^{\circ k_n}(S_n^0) = \mathcal{C}_n \cup \mathcal{C}_n^\sharp$, we have $\operatorname{Re}\zeta > a_{n+1} - \boldsymbol{k}$ for all $\zeta \in \Phi_n(S_n^0) + k_n$. Therefore,

$$a_{n+1} - \boldsymbol{k} - k_n < \operatorname{Re} u_n < a_{n+1} - \boldsymbol{k} - 3/2. \tag{4.1}$$

We denote

$$u_n' := a_{n+1} - \boldsymbol{k} - k_n - 1/2 + h_n \mathrm{i}.$$

According to (4.1), we have $\operatorname{Re} u_n' < \operatorname{Re} u_n$. Denote

$$u_n'' := \Phi_n(\partial^l I_n^0).$$

Let β_n' be the arc in $\Phi_n(\partial^l S_n^0)$ connecting u_n to u_n''; see Figure 6. We first give the definitions of $\gamma_n^0(t)$ and $\gamma_n^1(t)$ for $t \in [0,1]$, and then define the curves $(\gamma_n^i)_{n\in\mathbb{N}}$ inductively.

Definition of γ_n^0: The curve $\gamma_n^0 : [0,1] \to \mathbb{C}$ is defined piecewise as follows:

(a$_0$) for $t \in [0, 1 - \frac{\boldsymbol{k}+k_n+1}{a_{n+1}}]$, define $\gamma_n^0(t) := a_{n+1}t + \frac{1}{2} + h_n\mathrm{i}$;

(b$_0$) let $\gamma_n^0 : [1 - \frac{\boldsymbol{k}+k_n+1}{a_{n+1}}, 1 - \frac{k_n}{a_{n+1}}] \to [u_n', u_n] \cup \beta_n'$ be a homeomorphism such that

$$\gamma_n^0\left(1 - \frac{\boldsymbol{k}+k_n+1}{a_{n+1}}\right) = u_n' \quad \text{and} \quad \gamma_n^0\left(1 - \frac{k_n}{a_{n+1}}\right) = u_n'';$$

(c$_0$) let $\gamma_n^0 : [1 - \frac{k_n}{a_{n+1}}, 1 - \frac{k_n-1}{a_{n+1}}] \to \Phi_n(I_n^0)$ be a homeomorphism such that

$$\gamma_n^0\left(1 - \frac{k_n}{a_{n+1}}\right) = u_n'' \quad \text{and} \quad \gamma_n^0\left(1 - \frac{k_n-1}{a_{n+1}}\right) = u_n'' + 1;$$

(d$_0$) for $t \in [1 - \frac{k_n-j}{a_{n+1}}, 1 - \frac{k_n-j-1}{a_{n+1}}]$ with $1 \leqslant j \leqslant k_n - 1$, define

$$\gamma_n^0(t) := \gamma_n^0\left(t - \frac{j}{a_{n+1}}\right) + j.$$

Lemma 4.1 (see Figure 6). *The map $\gamma_n^0 : [0,1] \to \mathbb{C}$ has the following properties:*

(a) *γ_n^0 and $\gamma_n^0 + 1$ are simple arcs in $\mathcal{D}_n$;*

(b) *$\gamma_n^0(0) = \frac{1}{2} + h_n\mathrm{i}$ and $\gamma_n^0(1) = u_n'' + k_n$;*

(c) *$\Phi_n^{-1}(\gamma_n^0)$ is a continuous closed curve in Δ_n;*

(d) *$|\operatorname{Im}\gamma_n^0(t) - h_n| \leqslant 1$ for all $t \in [0,1]$.*

Proof. (a) and (b) follow from the definition of γ_n^0. For (c), since $f_n^{\circ k_n}(\Phi_n^{-1}(u_n'')) = \Phi_n^{-1}(1/2 + h_n\mathrm{i})$, we have $\Phi_n^{-1}(1/2 + h_n\mathrm{i}) = \Phi_n^{-1}(u_n'' + k_n)$ by Lemma 2.5. This implies that $\Phi_n^{-1}(\gamma_n^0)$ is a continuous closed curve in Δ_n. Part (d) is an immediate consequence of Lemma 3.4 (a, c). ∎

Before introducing γ_n^1, we define a thickened curve $\widetilde{\gamma}_n^0 : [0,1] \to \mathbb{C}$ of γ_n^0:

$$\widetilde{\gamma}_n^0(t) := \begin{cases} \gamma_n^0\big(\frac{a_{n+1}}{a_{n+1}-1}t\big) & \text{if } t \in \big[0, 1-\frac{1}{a_{n+1}}\big], \\ \gamma_n^0(t)+1 & \text{if } t \in \big(1-\frac{1}{a_{n+1}}, 1\big]. \end{cases}$$

One can see that $\widetilde{\gamma}_n^0 = \gamma_n^0 \cup (\gamma_n^0([1-\frac{1}{a_{n+1}},1])+1) = \gamma_n^0 \cup (\Phi_n(I_n^0)+k_n)$ and $\widetilde{\gamma}_n^0 : [0,1] \to \mathbb{C}$ is a continuous curve in $\mathcal{D}_n$. Let $\chi_{n,0} := \chi_{f_n,0}$ be the anti-holomorphic map defined in (2.8).

Definition of γ_n^1: The curve $\gamma_n^1 : [0,1] \to \mathbb{C}$ is defined piecewise as follows:

(a_1) For $t \in [0, \frac{1}{a_{n+1}}]$, define $\gamma_n^1(t) := \chi_{n+1,0} \circ \widetilde{\gamma}_{n+1}^0(1-a_{n+1}t)$;

(b_1) For $t \in (\frac{j}{a_{n+1}}, \frac{j+1}{a_{n+1}}]$, where $1 \leqslant j \leqslant a_{n+1}-1$, define

$$\gamma_n^1(t) := \chi_{n+1,j} \circ \gamma_{n+1}^0(j+1-a_{n+1}t),$$

where $\chi_{n+1,j} = \chi_{n+1,0} + j$ is defined in (2.37).

Let $D_3 > 0$ be the constant of Lemma 2.11.

Lemma 4.2. *The map* $\gamma_n^1 : [0,1] \to \mathbb{C}$ *has the following properties:*

(a) γ_n^1 *and* $\gamma_n^1 + 1$ *are continuous curves in* $\mathcal{D}_n$*;*

(b) $\gamma_n^1(0) = \chi_{n+1,0}(\gamma_{n+1}^0(1)+1)$ *and* $\gamma_n^1(1) = \chi_{n+1,0}(\gamma_{n+1}^0(1)) + a_{n+1}$*;*

(c) $\Phi_n^{-1}(\gamma_n^1(0)) = \Phi_n^{-1}(\gamma_n^1(1))$ *and* $\Phi_n^{-1}(\gamma_n^1)$ *is a continuous closed curve in* Δ_n*;*

(d) *there exists a constant* $D_8 > 0$ *independent of* n *such that for all* $t \in [0,1]$,

$$|\mathrm{Re}\,\gamma_n^0(t) - \mathrm{Re}\,\gamma_n^1(t)| \leqslant D_8 \quad \textit{and} \quad \left|\mathrm{Im}\,\gamma_n^1(t) - \frac{\mathcal{B}(\alpha_{n+1})}{2\pi} - M\right| \leqslant D_3 + \tfrac{1}{2}.$$

Proof. (a) Since $\chi_{n+1,j}$ is anti-holomorphic for all $j \in \mathbb{Z}$, we have

$$\chi_{n+1,j}(\gamma_{n+1}^0(0)) = \chi_{n+1,j}(\gamma_{n+1}^0(1)) + 1 = \chi_{n+1,j+1}(\gamma_{n+1}^0(1))$$

for $0 \leqslant j \leqslant a_{n+1}-2$. Therefore, $\gamma_n^1 : [0,1] \to \mathbb{C}$ is a continuous curve. By Lemma 2.14, γ_n^1 and γ_n^1+1 are continuous curves in $\mathcal{D}_n$.

(b) By the definition of γ_n^1, we have

$$\gamma_n^1(0) = \chi_{n+1,0} \circ \widetilde{\gamma}_{n+1}^0(1) = \chi_{n+1,0}(\gamma_{n+1}^0(1)+1)$$

and

$$\begin{aligned}\gamma_n^1(1) &= \chi_{n+1,a_{n+1}-1}(\gamma_{n+1}^0(0)) \\ &= \chi_{n+1,a_{n+1}-1}(\gamma_{n+1}^0(1)) + 1 = \chi_{n+1,0}(\gamma_{n+1}^0(1)) + a_{n+1}.\end{aligned}$$

(c) By Lemma 2.15 (a), we have

$$\Phi_n^{-1} \circ \chi_{n+1,0}(\gamma_{n+1}^0(1) + 1) = f_n^{\circ a_{n+1}}(\Phi_n^{-1} \circ \chi_{n+1,0}(\gamma_{n+1}^0(1))).$$

This implies that $\Phi_n^{-1}(\gamma_n^1(0)) = \Phi_n^{-1}(\gamma_n^1(1))$ by (b). Therefore, $\Phi_n^{-1}(\gamma_n^1)$ is a continuous closed curve in Δ_n.

(d) By (2.36) we have

$$\operatorname{Re} \chi_{n+1,j}(\widetilde{\gamma}_{n+1}^0) \subset [1 + j, \boldsymbol{k}_1 + 2 + j], \quad j \in \mathbb{Z}. \tag{4.2}$$

Hence for $t \in [0, 1 - \frac{\boldsymbol{k}+k_n+1}{a_{n+1}}]$, we have

$$|\operatorname{Re} \gamma_n^0(t) - \operatorname{Re} \gamma_n^1(t)| \leqslant \boldsymbol{k}_1 + 3/2.$$

For $t \in [1 - \frac{\boldsymbol{k}+k_n+1}{a_{n+1}}, 1 - \frac{k_n}{a_{n+1}}]$, by (4.1) and Lemma 3.4 (c) we have

$$\operatorname{Re} \gamma_n^0(t) \in [\operatorname{Re} u_n' - 1/2, \operatorname{Re} u_n + 1] \subset [a_{n+1} - \boldsymbol{k} - k_n - 1, a_{n+1} - \boldsymbol{k} - 1/2].$$

If $t \in [1 - \frac{\boldsymbol{k}+k_n+1}{a_{n+1}}, 1 - \frac{k_n}{a_{n+1}}]$, then $\gamma_n^1(t) \in \bigcup_{i=0}^{\boldsymbol{k}} \chi_{n+1,a_{n+1}-\boldsymbol{k}-k_n-1+i}(\gamma_{n+1}^0)$. By (4.2)

$$\operatorname{Re} \gamma_n^1(t) \in [a_{n+1} - k_n - \boldsymbol{k},\ a_{n+1} - k_n + \boldsymbol{k}_1 + 1].$$

Therefore, for $t \in [1 - \frac{\boldsymbol{k}+k_n+1}{a_{n+1}}, 1 - \frac{k_n}{a_{n+1}}]$ we have

$$|\operatorname{Re} \gamma_n^0(t) - \operatorname{Re} \gamma_n^1(t)| \leqslant \max\{k_n - 1/2, \boldsymbol{k} + \boldsymbol{k}_1 + 2\}.$$

By Lemma 3.4 (a, c), we have

$$u_n'' \in \overline{\mathbb{D}}(u_n, 1) \quad \text{and} \quad \Phi_n(I_n^0) \subset \overline{\mathbb{D}}(u_n'', 2). \tag{4.3}$$

For $t \in [1 - \frac{k_n}{a_{n+1}}, 1 - \frac{k_n-1}{a_{n+1}}]$, by (4.1) and (4.3) we have

$$\operatorname{Re} \gamma_n^0(t) \in [a_{n+1} - \boldsymbol{k} - k_n - 3, a_{n+1} - \boldsymbol{k} + 3/2].$$

On the other hand,

$$\operatorname{Re} \gamma_n^1(t) \in [a_{n+1} - k_n + 1, a_{n+1} - k_n + \boldsymbol{k}_1 + 2].$$

Since $\gamma_n^i(t + \frac{1}{a_{n+1}}) = \gamma_n^i(t) + 1$ for $t \in [1 - \frac{k_n}{a_{n+1}}, 1 - \frac{1}{a_{n+1}}]$, where $i = 0, 1$, this implies that for all $t \in [1 - \frac{k_n}{a_{n+1}}, 1]$, we have

$$|\operatorname{Re} \gamma_n^0(t) - \operatorname{Re} \gamma_n^1(t)| \leqslant \max\{k_n - \boldsymbol{k} + 1/2, \boldsymbol{k} + \boldsymbol{k}_1 + 5\}.$$

Since $k_n \leqslant \boldsymbol{k}_0$ by Proposition 2.3, this implies that $|\operatorname{Re} \gamma_n^0(t) - \operatorname{Re} \gamma_n^1(t)| \leqslant D_8 := \max\{\boldsymbol{k}_0 - 1/2, \boldsymbol{k} + \boldsymbol{k}_1 + 5\}$ for all $t \in [0, 1]$. Finally, the statement on $\operatorname{Im} \gamma_n^1(t)$ follows immediately from Lemmas 2.11 (a) and 4.1 (d). ∎

By (3.5) and Lemma 4.2 (d), for any $t \in [0,1]$ and $\zeta \in \mathbb{E}\mathrm{xp}^{-1}(\partial\Delta_{n+1})$, we have

$$\operatorname{Im}\gamma_n^1(t) \geqslant \frac{\mathcal{B}(\alpha_{n+1})}{2\pi} + M - D_3 - \frac{1}{2} > 1 + \operatorname{Im}\zeta. \tag{4.4}$$

For $\ell = 1$, we define a thickened curve $\widetilde{\gamma}_n^\ell : [0,1] \to \mathbb{C}$ of γ_n^ℓ by

$$\widetilde{\gamma}_n^\ell(t) := \begin{cases} \gamma_n^\ell\big(\frac{a_{n+1}}{a_{n+1}-1}t\big) & \text{if } t \in \big[0, 1 - \frac{1}{a_{n+1}}\big], \\ \gamma_n^\ell(t) + 1 & \text{if } t \in \big(1 - \frac{1}{a_{n+1}}, 1\big]. \end{cases} \tag{4.5}$$

One can see that $\widetilde{\gamma}_n^\ell = \gamma_n^\ell \cup (\gamma_n^\ell([1 - \frac{1}{a_{n+1}}, 1]) + 1) = \gamma_n^\ell \cup \chi_{n+1,a_{n+1}}(\gamma_{n+1}^{\ell-1})$, and $\widetilde{\gamma}_n^\ell : [0,1] \to \mathbb{C}$ is a continuous curve in $\mathcal{D}_n$.

Inductive definition of γ_n^i: For all $n \in \mathbb{N}$ and $1 \leqslant \ell \leqslant i$, we assume that the curves $\gamma_n^\ell : [0,1] \to \mathbb{C}$ and $\widetilde{\gamma}_n^\ell : [0,1] \to \mathbb{C}$ are defined and satisfy

(a_ℓ) $\widetilde{\gamma}_n^\ell$ is defined as in (4.5);

(b_ℓ) $$\gamma_n^\ell(t) := \begin{cases} \chi_{n+1,0} \circ \widetilde{\gamma}_{n+1}^{\ell-1}(1 - a_{n+1}t) & \text{for } t \in \big[0, \frac{1}{a_{n+1}}\big), \\ \chi_{n+1,j} \circ \gamma_{n+1}^{\ell-1}(j + 1 - a_{n+1}t) & \text{for } t \in \big(\frac{j}{a_{n+1}}, \frac{j+1}{a_{n+1}}\big], \end{cases}$$

where $1 \leqslant j \leqslant a_{n+1} - 1$;

(c_ℓ) γ_n^ℓ and $\gamma_n^\ell + 1$ are continuous curves in $\mathcal{D}_n$;

(d_ℓ) $\gamma_n^\ell(0) = \chi_{n+1,0}(\gamma_{n+1}^{\ell-1}(1) + 1)$ and $\gamma_n^\ell(1) = \chi_{n+1,0}(\gamma_{n+1}^{\ell-1}(1)) + a_{n+1}$;

(e_ℓ) $\Phi_n^{-1}(\gamma_n^\ell(0)) = \Phi_n^{-1}(\gamma_n^\ell(1))$ and $\Phi_n^{-1}(\gamma_n^\ell)$ is a continuous closed curve in Δ_n.

Similar to the construction of γ_n^i, the curve $\gamma_n^{i+1} : [0,1] \to \mathbb{C}$ is defined as follows:

(a_{i+1}) for $t \in [0, \frac{1}{a_{n+1}}]$, define $\gamma_n^{i+1}(t) := \chi_{n+1,0} \circ \widetilde{\gamma}_{n+1}^i(1 - a_{n+1}t)$;

(b_{i+1}) for $t \in (\frac{j}{a_{n+1}}, \frac{j+1}{a_{n+1}}]$, where $1 \leqslant j \leqslant a_{n+1} - 1$, define

$$\gamma_n^{i+1}(t) := \chi_{n+1,j} \circ \gamma_{n+1}^i(j + 1 - a_{n+1}t).$$

Lemma 4.3. *The map* $\gamma_n^{i+1} : [0,1] \to \mathbb{C}$ *has the following properties:*

(a) γ_n^{i+1} *and* $\gamma_n^{i+1} + 1$ *are continuous curves in* $\mathcal{D}_n$*;*

(b) $\gamma_n^{i+1}(0) = \chi_{n+1,0}(\gamma_{n+1}^i(1) + 1)$ *and* $\gamma_n^{i+1}(1) = \chi_{n+1,0}(\gamma_{n+1}^i(1)) + a_{n+1}$*;*

(c) $\Phi_n^{-1}(\gamma_n^{i+1}(0)) = \Phi_n^{-1}(\gamma_n^{i+1}(1))$ *and* $\Phi_n^{-1}(\gamma_n^{i+1})$ *is a continuous closed curve in* Δ_n*.*

The proof of Lemma 4.3 is completely similar to that of Lemma 4.2. Moreover, one can define the thickened curve $\widetilde{\gamma}_n^\ell$ of γ_n^ℓ with $\ell = i + 1$ as in (4.5) similarly.

By the definition of $\widetilde{\gamma}_n^i$, we have the following result.

Lemma 4.4. *For each* $t_0 \in [0,1]$*, there exist sequences* $(t_n)_{n\in\mathbb{N}}$ *with* $t_n \in [0,1]$ *and* $(j_n)_{n\geqslant 1}$ *with* $0 \leqslant j_n \leqslant a_n$ *such that for all* $n \geqslant 1$ *and all* $i \in \mathbb{N}$*,*

$$\widetilde{\gamma}_{n-1}^{i+1}(t_{n-1}) = \chi_{n,j_n}(\widetilde{\gamma}_n^i(t_n)).$$

4.2. The curves are convergent

Our main goal in this subsection is to prove the following.

Proposition 4.5. *There exists a constant $K > 0$ such that for all $n \in \mathbb{N}$,*

$$\sum_{i=0}^{n} \sup_{t\in[0,1]} |\gamma_0^i(t) - \gamma_0^{i+1}(t)| \leqslant K. \tag{4.6}$$

In particular, the sequence $(\gamma_0^n : [0,1] \to \mathbb{C})_{n\in\mathbb{N}}$ of continuous curves converges uniformly as $n \to \infty$.

In order to estimate the distance between $\gamma_0^i(t)$ and $\gamma_0^{i+1}(t)$ for $t \in [0,1]$, we will combine uniform contraction with respect to the hyperbolic metrics and some quantitative estimates (with respect to the Euclidean metric) obtained in Section 2.4. For any hyperbolic domain $X \subset \mathbb{C}$, we use $\rho_X(z)|\mathrm{d}z|$ to denote the hyperbolic metric of X. The following lemma appears in [10, Lemma 5.5] in another form. For completeness we include a proof.

Lemma 4.6. *Let X, Y be hyperbolic domains in $\mathbb{C}$ with $\operatorname{diam}(\operatorname{Re}(X)) \leqslant A'$ and $B_\delta(X) \subset Y$, where A' and δ are positive constants. Then there exists a number $0 < \lambda < 1$ depending only on A' and δ such that for any $z \in X$,*

$$\rho_Y(z) \leqslant \lambda\, \rho_X(z).$$

Proof. For any fixed $z_0 \in X$, we consider the holomorphic function

$$F(z) := z + \frac{\delta\,(z - z_0)}{z - z_0 + 2A' + \delta} : X \to \mathbb{C}.$$

Since $\operatorname{diam}(\operatorname{Re}(X)) \leqslant A'$, it follows that $|z - z_0| < |z - z_0 + 2A' + \delta|$ if $z \in X$. Thus we have $|F(z) - z| < \delta$ and $F(X) \subset Y$ by the assumption. Applying Schwarz–Pick's lemma to $F : X \to Y$ at $F(z_0) = z_0$, we have

$$\rho_Y(F(z_0))|F'(z_0)| = \rho_Y(z_0)\left(1 + \frac{\delta}{2A' + \delta}\right) \leqslant \rho_X(z_0).$$

The proof is finished by setting $\lambda := (2A' + \delta)/(2A' + 2\delta)$. ■

Let X be a set in $\mathbb{C}$ and $z_0 \in X$. We use $\operatorname{Comp}_{z_0} X$ to denote the connected component of X containing z_0. Let $\mathcal{D}_n$ be the set defined in (2.9). For $n \in \mathbb{N}$, we define

$$\mathcal{D}_n' := \operatorname{Comp}_1(\mathcal{D}_n \cap \{\zeta \in \mathbb{C} : -3 < \operatorname{Im}\zeta < h_n + 2\}),$$

where h_n is the height defined in (3.3). Note that each $\mathcal{D}_n'$ is a hyperbolic domain. Let $\rho_n(z)|\mathrm{d}z|$ be the hyperbolic metric of $\mathcal{D}_n'$. We use $\operatorname{len}(\cdot)$ and $\operatorname{len}_{\rho_n}(\cdot)$ to denote the length of curves with respect to the Euclidean and the hyperbolic metric $\rho_n(z)|\mathrm{d}z|$ respectively.

Lemma 4.7. *Let $A', \delta > 0$ be constants. Then there exist $A > 0$ and $0 < \nu < 1$ depending only on A' and δ such that for any piecewise continuous curve ϑ_n in $\mathcal{D}_n'$ with $\operatorname{len}(\vartheta_n) \leqslant A'$ and $B_\delta(\vartheta_n) \subset \mathcal{D}_n'$, we have*

$$\operatorname{len}(\chi_{1,j_1} \circ \cdots \circ \chi_{n,j_n}(\vartheta_n)) \leqslant A \cdot \nu^n$$

for all $0 \leqslant j_i \leqslant a_i$ and $1 \leqslant i \leqslant n$.

Proof. Let $1 \leqslant i \leqslant n$ and $0 \leqslant j_i \leqslant a_i$. Note that we have assumed that $M > D_3$ in (3.5). By Lemma 2.11, for $\zeta \in \mathcal{D}_i'$ we have

$$\operatorname{Im} \chi_{i,j_i}(\zeta) \leqslant \frac{\mathcal{B}(\alpha_i)}{2\pi} + M + D_3 + 1 < \frac{\mathcal{B}(\alpha_i)}{2\pi} + \frac{M}{\alpha_{i-1}} + 1 = h_{i-1} + 1. \tag{4.7}$$

Since $\Phi_i^{-1}(\mathcal{D}_i)$ is contained in the image of f_i, by the definition of near-parabolic renormalization (see also (2.8)) we have

$$\operatorname{Im} \chi_{i,j_i}(\zeta) > -2 \quad \text{for all } \zeta \in \mathcal{D}_i. \tag{4.8}$$

By Lemma 2.14, we have $B_{\delta_0}(\chi_{i,j_i}(\mathcal{D}_i)) \subset \mathcal{D}_{i-1}$ for a constant δ_0 depending only on the class $\mathcal{IS}_0$. Without loss of generality, we assume that $\delta_0 < 1$. Combining (4.7) and (4.8), we have

$$B_{\delta_0}(\chi_{i,j_i}(\mathcal{D}_i')) \subset \mathcal{D}_{i-1}'. \tag{4.9}$$

Note that $\chi_{i,j_i} : (\mathcal{D}_i', \rho_i) \to (\mathcal{D}_{i-1}', \rho_{i-1})$ can be decomposed as

$$(\mathcal{D}_i', \rho_i) \xrightarrow{\chi_{i,j_i}} (\chi_{i,j_i}(\mathcal{D}_i'), \tilde{\rho}_i) \overset{\text{inc.}}{\hookrightarrow} (\mathcal{D}_{i-1}', \rho_{i-1}),$$

where $\tilde{\rho}_i(z)|\mathrm{d}z|$ is the hyperbolic metric of $\chi_{i,j_i}(\mathcal{D}_i')$. According to Proposition 2.6, we have $\operatorname{diam}(\operatorname{Re} \chi_{i,j_i}(\mathcal{D}_i')) \leqslant \boldsymbol{k}_1$. By Lemma 4.6, the inclusion map

$$(\chi_{i,j_i}(\mathcal{D}_i'), \tilde{\rho}_i) \overset{\text{inc.}}{\hookrightarrow} (\mathcal{D}_{i-1}', \rho_{i-1})$$

is uniformly contracting with respect to the hyperbolic metrics (and the contracting factor depends only on $\boldsymbol{k}_1$ and δ_0). Since $\chi_{i,j_i} : \mathcal{D}_i' \to \chi_{i,j_i}(\mathcal{D}_i')$ do not expand the hyperbolic metric, it follows that $\chi_{i,j_i} : (\mathcal{D}_i', \rho_i) \to (\mathcal{D}_{i-1}', \rho_{i-1})$ is also uniformly contracting.

Since ϑ_n is a piecewise continuous curve satisfying $\operatorname{len}(\vartheta_n) \leqslant A'$ and $B_\delta(\vartheta_n) \subset \mathcal{D}_n'$, it follows that there exists a constant $A'' > 0$ depending only on A' and δ (not on n) such that $\operatorname{len}_{\rho_n}(\vartheta_n) \leqslant A''$. Define

$$G_n := \chi_{1,j_1} \circ \cdots \circ \chi_{n,j_n} : \mathcal{D}_n' \to \mathcal{D}_0'.$$

Since χ_{i,j_i} for $1 \leqslant i \leqslant n$ is a uniform contraction with respect to the hyperbolic metrics, there exists a constant $0 < \nu < 1$ depending only on $\boldsymbol{k}_1$ and δ_0 such that

$$\operatorname{len}_{\rho_0}(G_n(\vartheta_n)) \leqslant A'' \cdot \nu^n.$$

Since $B_{\delta_0}(G_n(\mathcal{D}_n')) \subset \mathcal{D}_0'$, the Euclidean metric and the hyperbolic metric ρ_0 of $\mathcal{D}_0'$ are comparable in $G_n(\mathcal{D}_n')$. Since $G_n(\vartheta_n) \subset G_n(\mathcal{D}_n') \subset \mathcal{D}_0'$, there exists a constant $A > 0$ depending only on A' and δ such that $\operatorname{len}(G_n(\vartheta_n)) \leqslant A \cdot \nu^n$. ∎

Let $D_6' > 1$ be the constant of Proposition 2.13.

Lemma 4.8. *There exists $K_1 > 0$ such that for any $n \geqslant 1$ and any continuous curve $\eta_n : [0,1] \to \mathscr{D}_n$ with $\eta_n(0) \in \widetilde{\gamma}_n^0$ and* $\mathrm{len}(\eta_n) \leqslant h_n - D_6' - 1$,

$$\mathrm{len}(\chi_{n,0}(\eta_n)) \leqslant \frac{1}{2\pi}\mathscr{B}(\alpha_n) + K_1.$$

Proof. By Proposition 2.13, we define

$$\phi_1(r) := (1 + D_6 e^{-2\pi\alpha_n r})\alpha_n \qquad \text{if } r \in \left[\tfrac{1}{4\alpha_n}, +\infty\right),$$
$$\phi_2(r) := \frac{\alpha_n}{1 - e^{-2\pi\alpha_n(r - D_2'\log(2+r))}}\left(1 + \frac{D_6}{r}\right) \qquad \text{if } r \in \left[D_6', \tfrac{1}{4\alpha_n}\right].$$

A direct calculation shows that

$$J' := \int_{1/(4\alpha_n)}^{h_n - 1} \phi_1(r)\,\mathrm{d}r < \frac{1}{2\pi}\alpha_n\mathscr{B}(\alpha_{n+1}) + M + D_6. \tag{4.10}$$

We claim that there exists $K_1' > 0$ independent of α_n such that

$$J'' := \int_{D_6'}^{1/(4\alpha_n)} \phi_2(r)\,\mathrm{d}r < \frac{1}{2\pi}\log\frac{1}{\alpha_n} + K_1'. \tag{4.11}$$

In fact, a direct calculation shows that $J'' = J_1 + D_2' J_2 + D_6 J_3$, where

$$J_1 = \frac{1}{2\pi}\int_{D_6'}^{1/(4\alpha_n)} \frac{2\pi\alpha_n e^{2\pi\alpha_n r} - 2\pi\alpha_n D_2'(r+2)^{2\pi\alpha_n D_2' - 1}}{e^{2\pi\alpha_n r} - (r+2)^{2\pi\alpha_n D_2'}}\,\mathrm{d}r,$$
$$J_2 = \int_{D_6'}^{1/(4\alpha_n)} \frac{\alpha_n (r+2)^{2\pi\alpha_n D_2' - 1}}{e^{2\pi\alpha_n r} - (r+2)^{2\pi\alpha_n D_2'}}\,\mathrm{d}r,$$
$$J_3 = \int_{D_6'}^{1/(4\alpha_n)} \frac{\alpha_n e^{2\pi\alpha_n r}}{e^{2\pi\alpha_n r} - (r+2)^{2\pi\alpha_n D_2'}}\cdot\frac{1}{r}\,\mathrm{d}r.$$

We assume that α_n is small such that $2\pi\alpha_n D_2' \leqslant 1/2$ and $2\pi\alpha_n D_2'\log(2 + \frac{1}{4\alpha_n}) \leqslant 1/2$. Since $1 + t \leqslant e^t \leqslant 1 + 2t$ for $t \in [0,1]$, if $D_6' \leqslant r \leqslant \frac{1}{4\alpha_n}$ we have

$$\begin{aligned} e^{2\pi\alpha_n r} - (r+2)^{2\pi\alpha_n D_2'} &\geqslant 1 + 2\pi\alpha_n r - (1 + 4\pi\alpha_n D_2'\log(r+2)) \\ &= 2\pi\alpha_n(r - 2D_2'\log(r+2)), \end{aligned} \tag{4.12}$$

where $r - 2D_2'\log(2+r) \geqslant 4$ if $r \geqslant D_6'$ (see Proposition 2.13 (b)).

By (4.12), there exist $C_1, C_1' > 0$ independent of α_n such that

$$J_1 \leqslant C_1 - \frac{1}{2\pi}\log(e^{2\pi\alpha_n D_6'} - (D_6' + 2)^{2\pi\alpha_n D_2'}) \leqslant \frac{1}{2\pi}\log\frac{1}{\alpha_n} + C_1'.$$

For J_2, since the integral

$$\int_{D_6'}^{+\infty} \frac{1}{r - 2D_2'\log(2+r)}\cdot\frac{1}{(r+2)^{1/2}}\,\mathrm{d}r$$

is convergent, there exists a constant $C_2 > 0$ independent of α_n such that $J_2 \leqslant C_2$. Similarly, there exists a constant $C_3 > 1$ independent of α_n such that $J_3 \leqslant C_3$. Hence (4.11) follows if we set $K_1' := C_1' + C_2 D_2' + C_3 D_6$.

Without loss of generality we assume that $r \mapsto r - D_2' \log(2+r)$ is increasing on $[D_6', +\infty)$. Then $\phi_1(r)$ and $\phi_2(r)$ are decreasing on $[\frac{1}{4\alpha_n}, +\infty)$ and $[D_6', \frac{1}{4\alpha_n}]$ respectively. Denote

$$\phi(r) := \begin{cases} \phi_1(r) & \text{if } r \in \big[\frac{1}{4\alpha_n}, +\infty\big), \\ \max\{\phi_2(r), \phi_1\big(\frac{1}{4\alpha_n}\big)\} & \text{if } r \in \big[D_6', \frac{1}{4\alpha_n}\big). \end{cases} \tag{4.13}$$

Then $\phi(r)$ is (maybe not strictly) decreasing on $[D_6', +\infty)$. By Lemma 4.1 (d), we have $|\operatorname{Im} \eta_n(0) - h_n| \leqslant 1$. Since $\operatorname{len}(\eta_n) \leqslant h_n - D_6' - 1$, we have $\eta_n \cap (\mathbb{D}(0, D_6') \cup \mathbb{D}(1/\alpha_n, D_6')) = \emptyset$. By (4.10) and (4.11) we have

$$\begin{aligned} \operatorname{len}(\chi_{n,0}(\eta_n)) &\leqslant \int_{D_6'}^{h_n-1} \phi(r)\, \mathrm{d}r \leqslant J' + \left(J'' + \left(\frac{1}{4\alpha_n} - D_6'\right)\phi_1\left(\frac{1}{4\alpha_n}\right)\right) \\ &< J' + J'' + \frac{1}{4}(D_6 + 1) < \frac{1}{2\pi}\mathcal{B}(\alpha_n) + K_1, \end{aligned} \tag{4.14}$$

where $K_1 := M + \frac{3}{2}D_6 + K_1'$. The proof is complete. ∎

Proof of Proposition 4.5. Note that $\gamma_0^n(t) = \widetilde{\gamma}_0^n(\frac{a_1-1}{a_1}t)$ for all $t \in [0,1]$ and all $n \in \mathbb{N}$. In order to prove (4.6), it suffices to prove that there exist $K > 0$ and a sequence $(y_i)_{i\geqslant 0}$ of non-negative numbers such that for any $n \in \mathbb{N}$, any $0 \leqslant i \leqslant n$ and any $t_0 \in [0,1]$,

$$|\widetilde{\gamma}_0^i(t_0) - \widetilde{\gamma}_0^{i+1}(t_0)| \leqslant y_i \quad \text{and} \quad \sum_{i=0}^{n} y_i \leqslant K. \tag{4.15}$$

We divide the argument into several steps.

Step 1. Basic settings. For any $t_0 \in [0,1]$, by Lemma 4.4 there exist sequences $(t_n)_{n\in\mathbb{N}}$ with $t_n \in [0,1]$ and $(j_n)_{n\geqslant 1}$ with $0 \leqslant j_n \leqslant a_n$ such that for all $n \geqslant 1$ and all $i \in \mathbb{N}$,

$$\widetilde{\gamma}_{n-1}^{i+1}(t_{n-1}) = \chi_{n,j_n}(\widetilde{\gamma}_n^i(t_n)). \tag{4.16}$$

For $n \in \mathbb{N}$, let

$$\xi_n^0 : [0,1] \to [\widetilde{\gamma}_n^0(t_n), \widetilde{\gamma}_n^1(t_n)]$$

be the segment with $\xi_n^0(0) = \widetilde{\gamma}_n^0(t_n)$ and $\xi_n^0(1) = \widetilde{\gamma}_n^1(t_n)$ (we assume that the parametrization of ξ_n^0 on $[0,1]$ is linear).

By the definition of $\mathcal{D}_n'$ and Lemma 2.7, the set $\mathcal{D}_n'$ contains

$$\{\zeta \in \mathbb{C} : 0 < \operatorname{Re}\zeta \leqslant \lfloor 1/\alpha_n \rfloor - \boldsymbol{k} + \boldsymbol{k}_0 + \boldsymbol{k}_1 + 3 \text{ and } 0 \leqslant \operatorname{Im}\zeta < h_n + 2\}.$$

By Lemma 3.4 (a, c), (4.1) and Lemma 4.1 (d), we have

$$\widetilde{\gamma}_n^0 \subset \{\zeta \in \mathbb{C} : 1/2 \leqslant \operatorname{Re}\zeta \leqslant \lfloor 1/\alpha_n \rfloor - \boldsymbol{k} + k_n + 3/2 \text{ and } -1 \leqslant \operatorname{Im}\zeta - h_n \leqslant 1\}.$$

In (3.5) we assume that $M > D_3 + \frac{1}{2\pi}\log\frac{4D_7}{27} + 2 > D_3 + \frac{3}{2}$ (since $D_7 > 1$). Hence by (4.2) and Lemma 4.2 (d), we have

$$\widetilde{\gamma}_n^1 \subset \{\zeta \in \mathbb{C} : 1 \leqslant \operatorname{Re}\zeta \leqslant \lfloor 1/\alpha_n \rfloor + \boldsymbol{k}_1 + 3 \text{ and } 1 \leqslant \operatorname{Im}\zeta \leqslant h_n + 1\}. \tag{4.17}$$

Note that $k_n \leqslant \boldsymbol{k}_0$ (see Proposition 2.3). Hence $B_{1/2}(\xi_n^0) \subset \mathcal{D}_n'$ for all $n \in \mathbb{N}$. For $\ell \geqslant 1$, we define the Jordan arc $\xi_n^\ell : [0,1] \to \mathbb{C}$ as

$$\xi_n^\ell(s) := \chi_{n+1,j_{n+1}} \circ \cdots \circ \chi_{n+\ell,j_{n+\ell}}(\xi_{n+\ell}^0(s)) \quad \text{for } s \in [0,1]. \tag{4.18}$$

By (4.16) and (4.18), the following curve is continuous:

$$\begin{aligned}\eta_n^\ell := \xi_n^0 \cup \xi_n^1 \cup \cdots \cup \xi_n^\ell &= \xi_n^0 \cup \chi_{n+1,j_{n+1}}(\eta_{n+1}^{\ell-1}) \\ &= \xi_n^0 \cup \chi_{n+1,j_{n+1}}(\xi_{n+1}^0 \cup \cdots \cup \xi_{n+1}^{\ell-1}).\end{aligned}$$

Denote $\eta_n^0 := \xi_n^0$. According to (4.9), for any $n \geqslant 0$ and $\ell \geqslant 0$ we have

$$B_\delta(\eta_n^\ell) \subset \mathcal{D}_n', \quad \text{where} \quad \delta := \min\{\delta_0, 1/4\}.$$

A parametrization of $\eta_n^\ell : [0,1] \to \mathbb{C}$ is given by

$$\eta_n^\ell(s) := \xi_n^j((\ell+1)s - j)$$

for $s \in [\frac{j}{\ell+1}, \frac{j+1}{\ell+1}]$ and $0 \leqslant j \leqslant \ell$ (note that $\xi_n^j(1) = \xi_n^{j+1}(0)$ for every $0 \leqslant j \leqslant \ell - 1$). By definition, $|\widetilde{\gamma}_0^i(t_0) - \widetilde{\gamma}_0^{i+1}(t_0)| \leqslant \operatorname{len}(\xi_0^i)$ for all $i \in \mathbb{N}$. Therefore, to obtain (4.15), it suffices to prove that there exist $K > 0$ and non-negative numbers $(y_i)_{i \geqslant 0}$ such that for any $n \in \mathbb{N}$ and any $0 \leqslant i \leqslant n$,

$$\operatorname{len}(\xi_0^i) \leqslant y_i \quad \text{and} \quad \sum_{i=0}^n y_i \leqslant K. \tag{4.19}$$

Step 2. Decompositions of the curves. Note that we have assumed that $M > D_3 + 3/2$ (see (3.5)). By (4.5), it follows that Lemma 4.2 (d) holds also for $\widetilde{\gamma}_n^0$ and $\widetilde{\gamma}_n^1$. By Lemma 4.1 (d) and a direct calculation, we have

$$\begin{aligned}\operatorname{len}(\eta_n^0) &= \operatorname{len}(\xi_n^0) = |\widetilde{\gamma}_n^0(t_n) - \widetilde{\gamma}_n^1(t_n)| \\ &\leqslant h_n + 1 - \frac{\mathcal{B}(\alpha_{n+1})}{2\pi} - M + D_3 + \frac{1}{2} + D_8 < h_n - \frac{\mathcal{B}(\alpha_{n+1})}{2\pi} + D_8.\end{aligned} \tag{4.20}$$

Hence $\eta_n^0 = \xi_n^0 : [0,1] \to \mathbb{C}$ can be written as the union of two continuous curves $\eta_{n,(0)}^0 := \eta_n^0([0,s_n])$ and $\eta_{n,(1)}^0 := \eta_n^0([s_n,1])$ for some $s_n \in (0,1)$ (the choice of s_n is not unique), such that

$$\operatorname{len}(\eta_{n,(0)}^0) \leqslant h_n - D_6' - 1 \quad \text{and} \quad \operatorname{len}(\eta_{n,(1)}^0) \leqslant D_6' + D_8 + 1. \tag{4.21}$$

Since $B_\delta(\eta_n^0) \subset \mathcal{D}_n'$, there exists a constant $K_2' > 0$ depending only on δ and $D_6' + D_8 + 1$ such that

$$\operatorname{len}_{\rho_n}(\eta_{n,(1)}^0) \leqslant K_2', \tag{4.22}$$

where $\rho_n(z)|\mathrm{d}z|$ is the hyperbolic metric of $\mathcal{D}_n'$.

Let $K_1>0$ be the constant of Lemma 4.8. There exists a constant $K_2>K_2'$ depending only on $A':=K_1+D_6'+D_8+1$ and δ such that for any $n\in\mathbb{N}$ and any piecewise continuous curve ξ' in $\mathcal{D}_n'$ with $B_\delta(\xi')\subset\mathcal{D}_n'$ and $\operatorname{len}(\xi')\leqslant K_1+D_6'+D_8+1$,

$$\operatorname{len}_{\rho_n}(\xi')\leqslant K_2. \tag{4.23}$$

Let $\nu\in(0,1)$ be as in Lemma 4.7 depending only on A' and δ.

Suppose $n\geqslant 1$. By Lemma 4.8 and (4.21), Lemma 4.7 and (4.22), ξ_{n-1}^1 is the union of two continuous curves $\chi_{n,j_n}(\eta_{n,(0)}^0)$ and $\chi_{n,j_n}(\eta_{n,(1)}^0)$, where

$$\begin{aligned}\operatorname{len}(\chi_{n,j_n}(\eta_{n,(0)}^0))&\leqslant\tfrac{1}{2\pi}\mathcal{B}(\alpha_n)+K_1,\\ \operatorname{len}_{\rho_{n-1}}(\chi_{n,j_n}(\eta_{n,(1)}^0))&\leqslant K_2'\nu<K_2\nu.\end{aligned} \tag{4.24}$$

Therefore, by (4.20) we have

$$\begin{aligned}&\operatorname{len}(\xi_{n-1}^0\cup\chi_{n,j_n}(\eta_{n,(0)}^0))\\ &\quad\leqslant\left(h_{n-1}-\frac{\mathcal{B}(\alpha_n)}{2\pi}+D_8\right)+\left(\frac{\mathcal{B}(\alpha_n)}{2\pi}+K_1\right)=h_{n-1}+D_8+K_1.\end{aligned} \tag{4.25}$$

This implies that $\xi_{n-1}^0\cup\chi_{n,j_n}(\eta_{n,(0)}^0)=\xi_{n-1}^0\cup\xi_{n-1}^1([0,s_n])=\eta_{n-1}^1([0,\frac{1+s_n}{2}])$ can be written as the union of two continuous curves $\eta_{n-1,(0)}^1:=\eta_{n-1}^1([0,s_{n-1}])$ and $\eta_{n-1,(1)}^1:=\eta_{n-1}^1([s_{n-1},\frac{1+s_n}{2}])$ for some $s_{n-1}\in(0,\frac{1+s_n}{2})$, where

$$\begin{aligned}\operatorname{len}(\eta_{n-1,(0)}^1)&\leqslant h_{n-1}-D_6'-1,\\ \operatorname{len}(\eta_{n-1,(1)}^1)&\leqslant A'=K_1+D_6'+D_8+1.\end{aligned} \tag{4.26}$$

Since $B_\delta(\eta_{n-1}^1)\subset\mathcal{D}_{n-1}'$, by (4.23) we have $\operatorname{len}_{\rho_{n-1}}(\eta_{n-1,(1)}^1)\leqslant K_2$.

Denote $\eta_{n-1,(2)}^1:=\eta_{n-1}^1([\frac{1+s_n}{2},1])=\chi_{n,j_n}(\eta_{n,(1)}^0)$, $s_{n-1}^{(1)}:=s_{n-1}$ and $s_{n-1}^{(2)}:=\frac{1+s_n}{2}$. Then the continuous curve

$$\eta_{n-1}^1=\xi_{n-1}^0\cup\xi_{n-1}^1=\eta_{n-1,(0)}^1\cup\eta_{n-1,(1)}^1\cup\eta_{n-1,(2)}^1$$

satisfies

- $\eta_{n-1,(0)}^1=\eta_{n-1}^1([0,s_{n-1}^{(1)}])$, $\eta_{n-1,(1)}^1=\eta_{n-1}^1([s_{n-1}^{(1)},s_{n-1}^{(2)}])$, $\eta_{n-1,(2)}^1=\eta_{n-1}^1([s_{n-1}^{(2)},1])$;
- $\operatorname{len}(\eta_{n-1,(0)}^1)\leqslant h_{n-1}-D_6'-1$, $\operatorname{len}_{\rho_{n-1}}(\eta_{n-1,(1)}^1)\leqslant K_2$, $\operatorname{len}_{\rho_{n-1}}(\eta_{n-1,(2)}^1)\leqslant K_2\nu$.

Step 3. Inductive procedure. Suppose there exists $1\leqslant i\leqslant n-1$ such that $\eta_{n-i}^i=\bigcup_{\ell=0}^{i}\xi_{n-i}^\ell=\bigcup_{k=0}^{i+1}\eta_{n-i,(k)}^i$ with $B_\delta(\eta_{n-i}^i)\subset\mathcal{D}_{n-i}'$ has the following properties:

- $\eta_{n-i,(k)}^i=\eta_{n-i}^i([s_{n-i}^{(k)},s_{n-i}^{(k+1)}])$ for all $0\leqslant k\leqslant i+1$ with some $0=s_{n-i}^{(0)}<s_{n-i}^{(1)}<\cdots<s_{n-i}^{(i+1)}<s_{n-i}^{(i+2)}=1$;
- $\operatorname{len}(\eta_{n-i,(0)}^i)\leqslant h_{n-i}-D_6'-1$ and $\operatorname{len}_{\rho_{n-i}}(\eta_{n-i,(k)}^i)\leqslant K_2\nu^{k-1}$ for every $1\leqslant k\leqslant i+1$.

By a similar argument to (4.24)–(4.26), there exist $0 = s_{n-i-1}^{(0)} < s_{n-i-1}^{(1)} < \cdots < s_{n-i-1}^{(i+2)} < s_{n-i-1}^{(i+3)} = 1$ such that the continuous curve

$$\eta_{n-i-1}^{i+1} = \bigcup_{\ell=0}^{i+1} \xi_{n-i-1}^{\ell} = \bigcup_{k=0}^{i+2} \eta_{n-i-1,(k)}^{i+1}$$

with $B_\delta(\eta_{n-i-1}^{i+1}) \subset \mathcal{D}'_{n-i-1}$ has the following properties:

- $\eta_{n-i-1,(k)}^{i+1} = \eta_{n-i-1}^{i+1}([s_{n-i-1}^{(k)}, s_{n-i-1}^{(k+1)}])$ for every $0 \leqslant k \leqslant i+2$;
- $\mathrm{len}(\eta_{n-i-1,(0)}^{i+1}) \leqslant h_{n-i-1} - D'_6 - 1$ and $\mathrm{len}_{\rho_{n-i-1}}(\eta_{n-i-1,(k)}^{i+1}) \leqslant K_2 \nu^{k-1}$ for every $1 \leqslant k \leqslant i+2$.

Inductively (as i increases), there exist $0 = s_0^{(0)} < s_0^{(1)} < \cdots < s_0^{(n+1)} < s_0^{(n+2)} = 1$ such that the continuous curve $\eta_0^n = \bigcup_{\ell=0}^n \xi_0^\ell = \bigcup_{k=0}^{n+1} \eta_{0,(k)}^n$ with $B_\delta(\eta_0^n) \subset \mathcal{D}'_0$ has the following properties:

- $\eta_{0,(k)}^n = \eta_0^n([s_0^{(k)}, s_0^{(k+1)}])$ for every $0 \leqslant k \leqslant n+1$;
- $\mathrm{len}(\eta_{0,(0)}^n) \leqslant h_0 - D'_6 - 1$ and $\mathrm{len}_{\rho_0}(\eta_{0,(k)}^n) \leqslant K_2 \nu^{k-1}$ for every $1 \leqslant k \leqslant n+1$.

Step 4. The conclusion. Since $B_\delta(\eta_0^n) \subset \mathcal{D}'_0$, the Euclidean metric and the hyperbolic metric ρ_0 of $\mathcal{D}'_0$ are comparable in a small neighborhood of η_0^n. Hence there exists a constant $C > 0$ depending only on δ such that

$$\sum_{k=1}^{n+1} \mathrm{len}(\eta_{0,(k)}^n) \leqslant C \sum_{k=1}^{n+1} \mathrm{len}_{\rho_0}(\eta_{0,(k)}^n) \leqslant \frac{CK_2}{1-\nu}.$$

Therefore, for all $n \geqslant 0$ we have

$$\mathrm{len}(\eta_0^n) = \sum_{i=0}^{n} \mathrm{len}(\xi_0^i) = \sum_{k=0}^{n+1} \mathrm{len}(\eta_{0,(k)}^n) \leqslant K := h_0 - D'_6 - 1 + \frac{CK_2}{1-\nu}.$$

By (4.13), (4.14) and the estimates similar to (4.24) and (4.25) in the above inductive procedure, it follows that for any $n \geqslant 0$, there exists a sequence $(y_i^{(n)})_{i=0}^n$ of non-negative numbers independent of $(t_n)_{n\in\mathbb{N}}$ such that for any $0 \leqslant i \leqslant n$, we have

$$\mathrm{len}(\xi_0^i) \leqslant y_i^{(n)} \quad \text{and} \quad \sum_{i=0}^{n} y_i^{(n)} \leqslant K.$$

Then (4.19) holds if we set $y_i := \inf_{n\in\mathbb{N}} y_i^{(n)}$.

The estimate (4.6) implies that the sequence $(\widetilde{\gamma}_n)_{n\in\mathbb{N}}$ of continuous curves converges uniformly on $[0, 1]$. Since $\gamma_0^n(t) = \widetilde{\gamma}_0^n(\frac{a_1-1}{a_1}t)$ for all $t \in [0, 1]$ and $n \in \mathbb{N}$, this implies that $(\gamma_0^n)_{n\in\mathbb{N}}$ converges uniformly on $[0, 1]$. ■

Remark. If α is of bounded type, or if there exists a universal constant $C > 0$ such that $\mathcal{B}(\alpha_{n+1}) \geqslant C/\alpha_n$ for all $n \in \mathbb{N}$, then $(\gamma_0^n)_{n\in\mathbb{N}}$ converges exponentially fast as $n \to \infty$.

4.3. The Siegel disks are Jordan domains

By Proposition 4.5, the sequence $(\gamma_0^n)_{n\geqslant 0}$ of continuous curves has a limit:

$$\gamma_0^\infty(t) := \lim_{n\to\infty} \gamma_0^n(t) \quad \text{for } t \in [0,1].$$

Proposition 4.9. *The limit $\Phi_0^{-1}(\gamma_0^\infty)$ is the boundary of the Siegel disk of f_0.*

Proof. For $\zeta_0 \in \gamma_0^{n+1}$, there exists $\zeta_n \in \widetilde{\gamma}_n^1 \subset \bigcup_{j_{n+1}=0}^{a_{n+1}} \chi_{n+1,j_{n+1}}(\widetilde{\gamma}_{n+1}^0)$ such that

$$\zeta_0 = \chi_{1,j_1} \circ \cdots \circ \chi_{n,j_n}(\zeta_n)$$

for some sequence $(j_1,\ldots,j_n)$, where $0 \leqslant j_i \leqslant a_i$ and $1 \leqslant i \leqslant n$. By Lemma 4.2 (d) and (2.8), we have

$$\left|\operatorname{Im}\zeta_n - \frac{1}{2\pi}\mathcal{B}(\alpha_{n+1}) - M\right| \leqslant D_3 + \frac{1}{2} \quad \text{and} \quad 1 \leqslant \operatorname{Re}\zeta_n \leqslant a_{n+1} + \boldsymbol{k}_1 + 2. \tag{4.27}$$

By Proposition 2.16 (b), each Siegel disk Δ_n is compactly contained in the domain of definition of f_n. For each $n \in \mathbb{N}$, Φ_n^{-1} is defined in $\mathcal{D}_n$ (see Lemma 2.5). We denote

$$\Delta_n' := \{\zeta \in \mathcal{D}_n : \Phi_n^{-1}(\zeta) \in \Delta_n\}.$$

By the definition of $\mathcal{D}_n$, we have $\Phi_n^{-1}(\Delta_n') = \Delta_n$ and $\mathbb{E}\mathrm{xp}(\Delta_n') = \Delta_{n+1}$. By Lemma 3.2, the inner radius of the Siegel disk of f_{n+1} is $c_{n+1}e^{-\mathcal{B}(\alpha_{n+1})}$, where $c_{n+1} \in [D_7^{-1}, D_7]$ and $D_7 > 1$ is a universal constant. According to the definition of the near-parabolic renormalization $f_{n+1} = \mathcal{R} f_n$, there exists a point $\zeta_n' \in \partial\Delta_n' \cap \mathbb{E}\mathrm{xp}^{-1}(\partial\Delta_{n+1})$ such that

$$|\operatorname{Re}(\zeta_n - \zeta_n')| \leqslant \frac{1}{2} \quad \text{and} \quad \operatorname{Im}\zeta_n' = \frac{1}{2\pi}\mathcal{B}(\alpha_{n+1}) - \frac{1}{2\pi}\log\frac{27c_{n+1}}{4}. \tag{4.28}$$

Let $[\zeta_n, \zeta_n']$ be the closed segment connecting ζ_n to ζ_n'. By (4.4), we have $[\zeta_n, \zeta_n') \subset \Delta_n'$. By Lemmas 2.7, 2.14 and (4.17), we have $B_\delta([\zeta_n, \zeta_n']) \subset \mathcal{D}_n'$ for $\delta = \min\{\delta_0, 1/4\}$. Combining (4.27) and (4.28), there exists a constant $A' > 0$ independent of n such that $|\zeta_n' - \zeta_n| \leqslant A'$. According to Lemma 4.7, there exist constants $A > 0$ and $0 < \nu < 1$ independent of n such that

$$\operatorname{len}(\chi_{1,j_1} \circ \cdots \circ \chi_{n,j_n}([\zeta_n, \zeta_n'])) \leqslant A \cdot \nu^n$$

whenever $0 \leqslant j_i \leqslant a_i$ and $1 \leqslant i \leqslant n$. Denote $\zeta_0' := \chi_{1,j_1} \circ \cdots \circ \chi_{n,j_n}(\zeta_n')$. Then $|\zeta_0 - \zeta_0'| \leqslant A \cdot \nu^n$. Since $\zeta_0' \in \partial\Delta_0'$, this implies that

$$\operatorname{dist}(\zeta_0, \partial\Delta_0') \leqslant A \cdot \nu^n. \tag{4.29}$$

For any $t_0 \in [0,1]$ and $n \geqslant 1$, we choose $\zeta_0 = \zeta_0^{(n)} := \gamma_0^{n+1}(t_0)$. By (4.29) we have $\gamma_0^\infty(t_0) \in \partial\Delta_0'$. By the arbitrariness of $t_0 \in [0,1]$, it follows that $\gamma_0^\infty \subset \partial\Delta_0'$. Therefore we have $\Phi_0^{-1}(\gamma_0^\infty) \subset \partial\Delta_0$.

By Lemma 4.3 (c), $\Phi_0^{-1}(\gamma_0^n)$ is a continuous closed curve for all $n \geqslant 0$. Since γ_0^n converges uniformly to the limit γ_0^∞ on $[0,1]$ as $n \to \infty$, it follows that $\Phi_0^{-1}(\gamma_0^\infty)$ is a continuous closed curve which separates Δ_0 from each component of $U_0 \setminus \overline{\Delta_0}$, where U_0 is the domain of definition of f_0. In particular, $\Phi_0^{-1}(\gamma_0^\infty) = \partial\Delta_0$. ■

Proof of the first part of the Main Theorem. Suppose $f_0 \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$, where $\alpha \in \mathcal{B}_N$ with N sufficiently large. By Proposition 4.9, the boundary $\partial\Delta_0 = \Phi_0^{-1}(\gamma_0^\infty)$ of the Siegel disk of f_0 is connected and locally connected. On the other hand, Δ_0 is compactly contained in the domain of definition of f_0 by Proposition 2.16 (b). By the definition of Δ_0, there exists a conformal map $\phi : \mathbb{D} \to \Delta_0$ such that $f_0 \circ \phi(w) = \phi(e^{2\pi\mathrm{i}\alpha} w)$. According to the Carathéodory theorem, the map ϕ can be extended continuously to $\phi : \overline{\mathbb{D}} \to \overline{\Delta}_0$.

For each $\theta \in [0, 2\pi)$, let $\gamma_\theta := \{\phi(re^{\mathrm{i}\theta}) : 0 \leqslant r \leqslant 1\}$ be the internal ray of Δ_0. Suppose there are two different rays γ_{θ_1} and γ_{θ_2} landing at a common point on $\partial\Delta_0$, i.e., $\phi(e^{\mathrm{i}\theta_1}) = \phi(e^{\mathrm{i}\theta_2})$. Then $\gamma_{\theta_1} \cup \gamma_{\theta_2}$ is a Jordan curve contained in $\overline{\Delta}_0$. By the maximum modulus principle, $\{f_0^{\circ n}\}_{n\in\mathbb{N}}$ forms a normal family in the bounded domain D_{θ_1,θ_2} which is bounded by $\gamma_{\theta_1} \cup \gamma_{\theta_2}$. This implies that D_{θ_1,θ_2} is contained in the Fatou set and hence contained in Δ_0. However, by Riesz brothers' theorem, ϕ must be a constant. This is a contradiction and so each point in $\partial\Delta_0$ is the landing point of exactly one internal ray. Hence $\partial\Delta_0$ is a Jordan curve. ∎

5. A Jordan arc and a new class of irrationals

In this section, we first define a Jordan arc Γ connecting the origin to the critical value $\mathrm{cv} = -4/27$ in the domain of definition of $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in \mathrm{HT}_N$. In particular, this arc is contained in $\mathcal{P}_f$. Then we define a new class of irrational numbers based on the mapping relations between different levels of renormalization.

5.1. A Jordan arc corresponding to $\alpha \in \mathrm{HT}_N$

Let $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in \mathrm{HT}_N$, where $N \geqslant 1/\varepsilon_4$ as in Section 2.5. We define a half-infinite strip

$$\mho := \{\zeta \in \mathbb{C} : 1/4 < \operatorname{Re}\zeta < 7/4 \text{ and } \operatorname{Im}\zeta > -2\} \tag{5.1}$$

and a topological triangle

$$\mathcal{Q}_f := \{z \in \mathcal{P}_f : \Phi_f(z) \in \mho\}.$$

Lemma 5.1. *There exists $\varepsilon_4' \in (0, \varepsilon_4]$ such that for all $f \in \mathcal{IS}_\alpha$ with $\alpha \in (0, \varepsilon_4']$,*

$$\overline{\mathcal{Q}}_f \setminus \{0\} \subset \mathbb{D}\big(0, \tfrac{4}{27}e^{3\pi}\big) \setminus \big[0, \tfrac{4}{27}e^{3\pi}\big). \tag{5.2}$$

We postpone the proof of Lemma 5.1 to Appendix A. The inclusion relation (5.2) is proved for the maps in $\mathcal{IS}_0$ first and then a continuity argument is used.

For $f_0 \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in \mathrm{HT}_N$, let $f_n := \mathcal{R} f_{n-1}$ be the maps defined by the renormalization operator inductively, where $n \geqslant 1$. In the following, we always assume that $N \geqslant 1/\varepsilon_4'$ and denote $\mathcal{Q}_n := \mathcal{Q}_{f_n}$. Recall that for $X \subset \mathbb{C}$ and $\delta > 0$, we denote $B_\delta(X) := \bigcup_{z\in X} \mathbb{D}(z, \delta)$.

Corollary 5.2. *For each $n \geqslant 1$, there exists a unique anti-holomorphic inverse branch of the modified exponential map* $\mathbb{E}\mathrm{xp}$*:*

$$\mathbb{L}\mathrm{og} : \mathcal{Q}_n \to \Phi_{n-1}(\mathcal{Q}_{n-1}) = \mho,$$

such that $\mathbb{L}\mathrm{og}(-\frac{4}{27}) = 1$. *Moreover,* $B_{1/4}(\mathbb{L}\mathrm{og}(\overline{\mathcal{Q}}_n \setminus \{0\})) \subset \mho$ *and* $\Phi_{n-1}^{-1} \circ \mathbb{L}\mathrm{og} : \overline{\mathcal{Q}}_n \setminus \{0\} \to \mathcal{Q}_{n-1}$ *is well-defined.*

Proof. Since $\mathbb{E}\mathrm{xp}$ takes the value $-4/27$ at each integer, it follows that $\mathbb{E}\mathrm{xp}$ has an inverse branch $\mathbb{L}\mathrm{og}$ defined on $\overline{\mathcal{Q}}_n \setminus \{0\}$ such that $\mathbb{L}\mathrm{og}(-4/27) = 1$ since $\overline{\mathcal{Q}}_n \setminus \{0\}$ is simply connected and avoids the origin. By Lemma 5.1, we have $\mathrm{Re}\,\mathbb{L}\mathrm{og}(\overline{\mathcal{Q}}_n \setminus \{0\}) \subset (1/2, 3/2)$ and $\mathrm{Im}\,\mathbb{L}\mathrm{og}(\overline{\mathcal{Q}}_n \setminus \{0\}) > -3/2$. Therefore, $B_{1/4}(\mathbb{L}\mathrm{og}(\overline{\mathcal{Q}}_n \setminus \{0\}))$ is contained in $\mho$ and $\Phi_{n-1}^{-1} \circ \mathbb{L}\mathrm{og} : \overline{\mathcal{Q}}_n \setminus \{0\} \to \mathcal{Q}_{n-1}$ is well-defined. ∎

Define a half-infinite strip

$$\mho' := \{\zeta \in \mathbb{C} : 1/2 < \mathrm{Re}\,\zeta < 3/2 \text{ and } \mathrm{Im}\,\zeta > -7/4\} \subset \mho \tag{5.3}$$

and a topological triangle, for every $n \geqslant 0$,

$$\mathcal{Q}_n' := \{z \in \mathcal{P}_n : \Phi_n(z) \in \mho'\}.$$

Definition (see Figure 7). Let $K_0 := \mathcal{Q}_0'$. For each $n \geqslant 1$, define

$$K_n := \Phi_0^{-1} \circ \mathbb{L}\mathrm{og} \circ \cdots \circ \Phi_{n-1}^{-1} \circ \mathbb{L}\mathrm{og}(\mathcal{Q}_n').$$

By Corollary 5.2, $K_{n+1} \subset K_n$ for all $n \geqslant 0$, the critical value $\mathrm{cv} = -4/27$ is in the interior of K_n and $0 \in \partial K_n$. Define

$$\Gamma := \bigcap_{n \geqslant 0} K_n. \tag{5.4}$$

Lemma 5.3. *The set* $\Gamma \cup \{0\}$ *is a Jordan arc connecting* $\mathrm{cv} = -4/27$ *to* 0.

Proof. The general idea is to use uniform contraction with respect to the hyperbolic metrics to prove that $\Gamma \cup \{0\}$ is locally connected and then prove that it must be a Jordan arc. Let us give the details.

Step 1: We first define two continuous curves $\gamma_{0,\pm}^0 : [0, +\infty) \to \mho$ by

$$\gamma_{0,\pm}^0(t) := \begin{cases} 1 \pm \frac{1}{2} + \left(t - \frac{11}{4}\right)\mathrm{i} & \text{if } t \in [1, +\infty), \\ 1 \pm \frac{t}{2} - \frac{7}{4}\mathrm{i} & \text{if } t \in [0, 1). \end{cases}$$

Then $\gamma_{0,+}^0$ and $\gamma_{0,-}^0$ have the same initial point $\gamma_{0,\pm}^0(0) = 1 - \frac{7}{4}\mathrm{i}$ and $\gamma_{0,+}^0 \cup \gamma_{0,-}^0 = \partial\mho'$, where $\mho'$ is defined in (5.3). For $\alpha \in (0, 1)$, we define

$$\varphi_\alpha(t) := \begin{cases} \dfrac{1}{\alpha}\left(t - \dfrac{1}{2\pi}\log\dfrac{1}{\alpha} + 1\right) & \text{if } t \geqslant \dfrac{1}{2\pi}\log\dfrac{1}{\alpha}, \\ e^{2\pi t} & \text{if } t < \dfrac{1}{2\pi}\log\dfrac{1}{\alpha}. \end{cases} \tag{5.5}$$

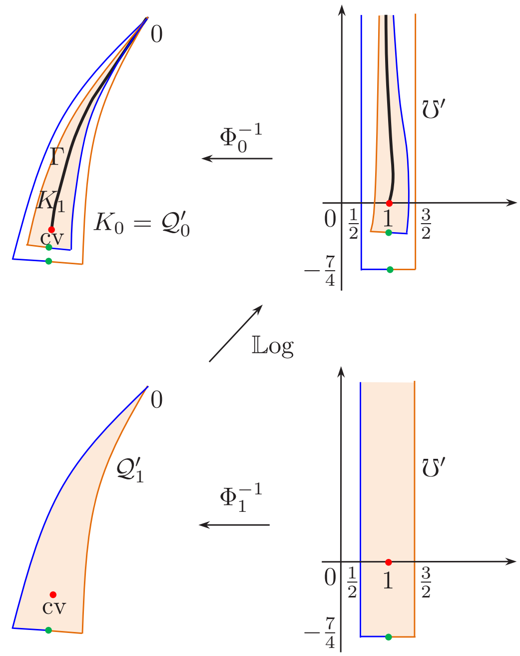

Fig. 7. A sketch of the renormalization microscope between levels 0 and 1. The sets Γ, $\mho'$, $\mathcal{Q}'_n$, K_n with $n = 0, 1$ and some special points are marked.

It is easy to see that φ_α is continuous on $\mathbb{R}$ and strictly increasing. For $n \geqslant 1$, we define $\varphi_n := \varphi_{\alpha_n}$. Then $\varphi_n \circ \cdots \circ \varphi_1(t) \to +\infty$ as $n \to \infty$ for all $t \in \mathbb{R}$.

In the following, we define two sequences $(\gamma^0_{n,\pm})_{n\geqslant 0}$ of continuous curves inductively. For $n \geqslant 1$, suppose $\gamma^0_{n-1,\pm} : [0, +\infty) \to \partial\mho'$ has been defined. We define $\gamma^0_{n,\pm} : [0, +\infty) \to \partial\mho'$ as

$$\gamma^0_{n,\pm}(t) := \begin{cases} 1 \pm \frac{1}{2} + \big(\varphi_n(\operatorname{Im} \gamma^0_{n-1,\pm}(t)) - e^{-7\pi/2} - \frac{7}{4}\big)\,\mathrm{i} & \text{if } t \in \big[1, +\infty\big), \\ 1 \pm \frac{t}{2} - \frac{7}{4}\,\mathrm{i} & \text{if } t \in [0, 1). \end{cases} \tag{5.6}$$

Note that $\gamma^0_{n,+}(1) = \frac{3}{2} - \frac{7}{4}\,\mathrm{i}$ and $\gamma^0_{n,-}(1) = \frac{1}{2} - \frac{7}{4}\,\mathrm{i}$. Then both $\gamma^0_{n,+} : [0, +\infty) \to \partial\mho'$ and $\gamma^0_{n,-} : [0, +\infty) \to \partial\mho'$ are continuous injections with the same initial point $\gamma^0_{n,\pm}(0) = 1 - \frac{7}{4}\mathrm{i}$. Moreover, $\gamma^0_{n,+} \cup \gamma^0_{n,-} = \partial\mho'$.

For all $t \in [0, +\infty)$, $n \geqslant 1$ and $1 \leqslant i \leqslant n$, by Corollary 5.2 the following curves are well-defined:

$$\gamma^i_{n-i,\pm}(t) := \begin{cases} \mathbb{L}\mathrm{og} \circ \Phi^{-1}_{n-i+1} \circ \cdots \circ \mathbb{L}\mathrm{og} \circ \Phi^{-1}_n(\gamma^0_{n,\pm}(t)) & \text{if } i \text{ is even}, \\ \mathbb{L}\mathrm{og} \circ \Phi^{-1}_{n-i+1} \circ \cdots \circ \mathbb{L}\mathrm{og} \circ \Phi^{-1}_n(\gamma^0_{n,\mp}(t)) & \text{if } i \text{ is odd}. \end{cases}$$

In particular, $\gamma^i_{n-i,\pm} \subset \overline{\mho'}$ for every $0 \leqslant i \leqslant n$. Define

$$\Gamma^i_{n-i,\pm}(t) := \Phi^{-1}_{n-i}(\gamma^i_{n-i,\pm}(t)) \quad \text{for } t \in [0, +\infty).$$

Then $\Gamma^i_{n-i,+}\cup\{0\}$ and $\Gamma^i_{n-i,-}\cup\{0\}$ are Jordan arcs, and $\Gamma^i_{n-i,+}\cup\Gamma^i_{n-i,-}\cup\{0\}$ is a Jordan curve.[14] In particular, we have $\Gamma^n_{0,+}\cup\Gamma^n_{0,-}\cup\{0\}=\partial K_n$ and two sequences of continuous curves $\gamma^n_{0,\pm}:[0,+\infty)\to\overline{\mho'}$, where $n\in\mathbb{N}$. In the following we prove that $\gamma^n_{0,\pm}$ and $\Gamma^n_{0,\pm}$ converge uniformly on $[0,+\infty)$ as $n\to\infty$.

Step 2: We first estimate the distance between $\gamma^0_{n-1,\pm}(t)$ and $\gamma^1_{n-1,\pm}(t)$ for all $n\geqslant 1$ and $t\in[0,+\infty)$. Let $t_n\in(1,+\infty)$ be the unique parameter such that

$$\operatorname{Im}\gamma^0_{n,\pm}(t_n)=\varphi_n(\operatorname{Im}\gamma^0_{n-1,\pm}(t_n))-e^{-7\pi/2}-7/4=1/\alpha_n.$$

Then $\frac{1}{2\pi}\log\frac{1}{\alpha_n}<\operatorname{Im}\gamma^0_{n-1,\pm}(t_n)<\frac{1}{2\pi}\log\frac{1}{\alpha_n}+2\alpha_n$. By definition, we have

$$\begin{aligned}|\gamma^0_{n-1,\pm}(t)-\gamma^1_{n-1,\pm}(t)|&=|\gamma^0_{n-1,\pm}(t)-\mathbb{L}\mathrm{og}\circ\Phi_n^{-1}(\gamma^0_{n,\mp}(t))|\\&\leqslant 1+|\operatorname{Im}\gamma^0_{n-1,\pm}(t)-\operatorname{Im}\mathbb{L}\mathrm{og}\circ\Phi_n^{-1}(\gamma^0_{n,\mp}(t))|.\end{aligned}$$

If $t\geqslant t_n$, then $\operatorname{Im}\gamma^0_{n,\pm}(t)\geqslant 1/\alpha_n$. By (5.5), (5.6) and Lemma 2.11 (a),

$$\begin{aligned}&|\operatorname{Im}\gamma^0_{n-1,\pm}(t)-\operatorname{Im}\mathbb{L}\mathrm{og}\circ\Phi_n^{-1}(\gamma^0_{n,\mp}(t))|\\&\qquad\leqslant D_3+\left|\operatorname{Im}\gamma^0_{n-1,\pm}(t)-\alpha_n\operatorname{Im}\gamma^0_{n,\mp}(t)-\frac{1}{2\pi}\log\frac{1}{\alpha_n}\right|\\&\qquad\leqslant D_3+1+\alpha_n(e^{-7\pi/2}+7/4)<D_3+2.\end{aligned}$$

If $t<t_n$, then $\operatorname{Im}\gamma^0_{n,\pm}(t)<1/\alpha_n$. By (5.5), (5.6) and Lemma 2.11 (b), there exist universal constants $C_1,C_2\geqslant 1$ such that

$$\begin{aligned}&|\operatorname{Im}\gamma^0_{n-1,\pm}(t)-\operatorname{Im}\mathbb{L}\mathrm{og}\circ\Phi_n^{-1}(\gamma^0_{n,\mp}(t))|\\&\quad\leqslant D_3+\left|\operatorname{Im}\gamma^0_{n-1,\pm}(t)-\frac{1}{2\pi}\log(1+|\gamma^0_{n,\mp}(t)|)\right|\\&\quad\leqslant D_3+C_1+\left|\operatorname{Im}\gamma^0_{n-1,\pm}(t)-\frac{1}{2\pi}\log(1+|\operatorname{Im}\gamma^0_{n,\mp}(t)|)\right|\leqslant D_3+C_1+C_2.\end{aligned}$$

Therefore, for all $n\geqslant 1$ and $t\in[0,+\infty)$,

$$|\gamma^0_{n-1,\pm}(t)-\gamma^1_{n-1,\pm}(t)|\leqslant D_3+C_1+C_2+1. \tag{5.7}$$

Step 3: Let $\rho_\mho(\zeta)|d\zeta|$ and $\rho_n(z)|dz|$ be the hyperbolic metrics of $\mho$ and $\mathcal{Q}_n$ respectively. Note that $\gamma^0_{n-1,\pm},\gamma^1_{n-1,\pm}\subset\overline{\mho'}$ and $B_{1/4}(\overline{\mho'})\subset\mho$. By (5.7), there exists $C_3>0$ such that the hyperbolic distance between $\gamma^0_{n-1,\pm}$ and $\gamma^1_{n-1,\pm}$ satisfies

$$\mathrm{dist}_{\rho_\mho}(\gamma^0_{n-1,\pm}(t),\gamma^1_{n-1,\pm}(t))\leqslant C_3\quad\text{for any } n\geqslant 1 \text{ and } t\in[0,+\infty).$$

[14] As before, we use the fact that $\lim_{\operatorname{Im}\zeta\to+\infty}\Phi_{n-i}^{-1}(\zeta)=0$, where $\zeta\in\Phi_{n-i}(\mathcal{P}_{n-i})$.

According to Corollary 5.2, each map $\mathbb{L}\mathrm{og} \circ \Phi_i^{-1} : (\mho, \rho_\mho) \to (\mho, \rho_\mho)$ for $1 \leqslant i \leqslant n$ can be decomposed as

$$\mathbb{L}\mathrm{og} \circ \Phi_i^{-1} : (\mho, \rho_\mho) \xrightarrow{\Phi_i^{-1}} (\mathcal{Q}_i, \rho_i) \xrightarrow{\mathbb{L}\mathrm{og}} (\mathbb{L}\mathrm{og}(\mathcal{Q}_i), \tilde{\rho}_i) \xhookrightarrow{\text{inc.}} (B_{1/4}(\mathbb{L}\mathrm{og}(\mathcal{Q}_i)), \hat{\rho}_i) \xhookrightarrow{\text{inc.}} (\mho, \rho_\mho),$$

where $\tilde{\rho}_i$ and $\hat{\rho}_i$ are hyperbolic metrics of $\mathbb{L}\mathrm{og}(\mathcal{Q}_i)$ and $B_{1/4}(\mathbb{L}\mathrm{og}(\mathcal{Q}_i))$ respectively. Since $\mathrm{diam}(\mathrm{Re}(\mathbb{L}\mathrm{og}(\mathcal{Q}_i))) \leqslant 1$, by Lemma 4.6 the inclusion map

$$(\mathbb{L}\mathrm{og}(\mathcal{Q}_i), \tilde{\rho}_i) \xhookrightarrow{\text{inc.}} (B_{1/4}(\mathbb{L}\mathrm{og}(\mathcal{Q}_i)), \hat{\rho}_i)$$

is uniformly contracting with respect to their hyperbolic metrics. Since Φ_i^{-1}, $\mathbb{L}\mathrm{og}$ and the second inclusion map do not expand the hyperbolic metrics, it follows that $\mathbb{L}\mathrm{og} \circ \Phi_i^{-1} : (\mho, \rho_\mho) \to (\mho, \rho_\mho)$ is uniformly contracting.

By the definition of $\gamma_{0,\pm}^n$, there exists a constant $0 < \nu < 1$ such that

$$\mathrm{dist}_{\rho_\mho}(\gamma_{0,\pm}^{n-1}(t), \gamma_{0,\pm}^n(t)) \leqslant C_3 \cdot \nu^{n-1} \quad \text{for } n \geqslant 1 \text{ and } t \in [0, +\infty).$$

This implies that the hyperbolic distance between $\Gamma_{0,\pm}^{n-1}(t)$ and $\Gamma_{0,\pm}^n(t)$ in $\mathcal{Q}_0 = \Phi_0^{-1}(\mho)$ satisfies

$$\mathrm{dist}_{\rho_0}(\Gamma_{0,\pm}^{n-1}(t), \Gamma_{0,\pm}^n(t)) \leqslant C_3 \cdot \nu^{n-1} \quad \text{for } n \geqslant 1 \text{ and } t \in [0, +\infty).$$

Let $\check{\mathcal{Q}}_0 := B_1(\mathcal{Q}_0)$ and $\check{\rho}_0(z)|\mathrm{d}z|$ be the hyperbolic metric of $\check{\mathcal{Q}}_0$. Then the Euclidean and hyperbolic metrics (with respect to $\check{\rho}_0$) are comparable on $\mathcal{Q}_0$. According to Schwarz–Pick's lemma, we have $\check{\rho}_0(z) < \rho_0(z)$ for all $z \in \mathcal{Q}_0$. Therefore, there exists a constant $C_4 > 0$ such that the distance in the Euclidean metric satisfies

$$|\Gamma_{0,\pm}^{n-1}(t) - \Gamma_{0,\pm}^n(t)| \leqslant C_4 \cdot \nu^{n-1} \quad \text{for } n \geqslant 1 \text{ and } t \in [0, +\infty).$$

Therefore, the following convergence is uniform for $t \in [0, +\infty)$:

$$\Gamma_{0,\pm}^\infty(t) := \lim_{n\to\infty} \Gamma_{0,\pm}^n(t).$$

Note that $1 \in \mho$ and $\mathbb{L}\mathrm{og} \circ \Phi_n^{-1}(1) = 1$. Since $\mathbb{L}\mathrm{og} \circ \Phi_i^{-1} : (\mho, \rho_\mho) \to (\mho, \rho_\mho)$ is uniformly contracting for all $1 \leqslant i \leqslant n$, we have

$$\begin{aligned} \lim_{n\to\infty} \Gamma_{0,\pm}^n(0) &= \lim_{n\to\infty} \Phi_0^{-1} \circ \mathbb{L}\mathrm{og} \circ \Phi_1^{-1} \circ \cdots \circ \mathbb{L}\mathrm{og} \circ \Phi_n^{-1}\big(1 - \tfrac{7}{4}\mathrm{i}\big) \\ &= \Phi_0^{-1}(1) = -\tfrac{4}{27}. \end{aligned}$$

Since $\gamma_{n-1,\pm}^0 \subset \overline{\mho'}$ and $B_{1/4}(\overline{\mho'}) \subset \mho$, there exists a constant $C_3' > 0$ such that

$$\mathrm{dist}_{\rho_\mho}(\gamma_{n-1,+}^0(t), \gamma_{n-1,-}^0(t)) \leqslant C_3' \quad \text{for any } n \geqslant 1 \text{ and } t \in [0, +\infty).$$

By a similar argument, we have

$$\Gamma_{0,+}^\infty(t) = \Gamma_{0,-}^\infty(t) \quad \text{for } t \in [0, +\infty).$$

Note that Γ is the intersection of the nested sequence $(K_n)_{n\geqslant 0}$, where K_n is the bounded component of $\mathbb{C} \setminus (\Gamma_{0,+}^n \cup \Gamma_{0,-}^n \cup \{0\})$ for all $n \geqslant 0$. Therefore, $\Gamma = \Gamma_{0,+}^\infty = \Gamma_{0,-}^\infty$ and $\Gamma \cup \{0\}$ is a Jordan arc connecting $-4/27$ to 0. ∎

5.2. Dynamical behavior of the points on the arcs

Let $\phi_0 := \mathrm{id}$. For each $n \geqslant 1$, we denote

$$\phi_n := \mathbb{E}\mathrm{xp} \circ \Phi_{n-1} \circ \cdots \circ \mathbb{E}\mathrm{xp} \circ \Phi_0.$$

Let Γ be the Jordan arc defined in (5.4). By the proof of Lemma 5.3, ϕ_n can be defined on $\Gamma_0 := \Gamma$ since

$$\Gamma_n := \phi_n(\Gamma_0) \subset \mathcal{Q}_n' = \Phi_n^{-1}(\mho') \quad \text{for } n \geqslant 1.$$

Since the restriction of $\mathbb{E}\mathrm{xp} \circ \Phi_{n-1}$ to Γ_{n-1} is a homeomorphism, each $\Gamma_n \cup \{0\}$ is also a Jordan arc connecting $-4/27$ to 0 in the dynamical plane of f_n. For each $n \geqslant 1$, the map $\phi_n : \Gamma_0 \to \Gamma_n$ can be extended homeomorphically to $\phi_n : \Gamma_0 \cup \{0\} \to \Gamma_n \cup \{0\}$ such that $\phi_n(-\frac{4}{27}) = -\frac{4}{27}$ and $\phi_n(0) = 0$. Moreover,

$$\gamma_n := \Phi_n(\Gamma_n) \tag{5.8}$$

is an unbounded arc in $\mho'$ with initial point 1.

Definition. For $n \geqslant 1$, we define

$$s_{\alpha_n} := \Phi_n \circ \mathbb{E}\mathrm{xp} : \gamma_{n-1} \to \gamma_n. \tag{5.9}$$

Then s_{α_n} is a homeomorphism with $s_{\alpha_n}(1) = 1$.

In the following, we assume that $\alpha = \alpha_0 \in \mathcal{B}_N$, where $\mathcal{B}_N$ is the set of high type Brjuno numbers defined in (2.39). Let $\mathcal{B}(\alpha_n)$ be the Brjuno sum defined in (3.2).

Definition. For $n \geqslant 0$, we define

$$\widetilde{\mathcal{B}}(\alpha_n) := \frac{\mathcal{B}(\alpha_n)}{2\pi} + M,$$

where $M \geqslant 1$ is a constant which will be determined in a moment.

Lemma 5.4. *There exists a constant $M_0 > 1$ such that if $M \geqslant M_0$, then for all $n \geqslant 1$ and $\zeta \in \gamma_{n-1}$ with $\mathrm{Im}\,\zeta \geqslant \widetilde{\mathcal{B}}(\alpha_n)$, we have $\mathrm{Im}\, s_{\alpha_n}(\zeta) \geqslant \widetilde{\mathcal{B}}(\alpha_{n+1})$.*

Proof. Let $D_4 > 0$ be the constant of Lemma 2.12. If $M \geqslant D_4$, then

$$\widetilde{\mathcal{B}}(\alpha_n) = \frac{\mathcal{B}(\alpha_n)}{2\pi} + M > \frac{1}{2\pi} \log \frac{1}{\alpha_n} + D_4.$$

By Lemma 2.12 (a), if $M \geqslant 2D_5$ and $\mathrm{Im}\,\zeta \geqslant \widetilde{\mathcal{B}}(\alpha_n)$, then

$$\begin{aligned}\mathrm{Im}\, s_{\alpha_n}(\zeta) &\geqslant \frac{1}{\alpha_n}\left(\mathrm{Im}\,\zeta - \frac{1}{2\pi}\log\frac{1}{\alpha_n} - D_5\right) \geqslant \frac{1}{\alpha_n}\left(\widetilde{\mathcal{B}}(\alpha_n) - \frac{1}{2\pi}\log\frac{1}{\alpha_n} - D_5\right)\\ &= \widetilde{\mathcal{B}}(\alpha_{n+1}) + \frac{1}{\alpha_n}((1-\alpha_n)M - D_5) \geqslant \widetilde{\mathcal{B}}(\alpha_{n+1}).\end{aligned}$$

Then the lemma follows by setting $M_0 := \max\{D_4, 2D_5\}$. ∎

Since $\alpha \in \mathcal{B}_N$, every $f_0 \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ has a Siegel disk Δ_0 centered at the origin. Let $D_7 > 1$ be the universal constant of Lemma 3.2. In the following we fix

$$M \geqslant \max\left\{M_0, \frac{1}{2\pi}\log\frac{27D_7}{4}\right\}. \tag{5.10}$$

Let $\Gamma_0 \cup \{0\}$ be the Jordan arc connecting the critical value $\mathrm{cv} = -\frac{4}{27}$ to 0 corresponding to f_0 (see Lemma 5.3). For a given point $z_0 \in \Gamma_0$, let $(\zeta_n)_{n\geqslant 0}$ be the sequence defined by

$$\zeta_0 := \Phi_0(z_0) \in \gamma_0 \quad \text{and} \quad \zeta_n := s_{\alpha_n}(\zeta_{n-1}) \in \gamma_n \quad \text{for } n \geqslant 1.$$

Lemma 5.5. *If $z_0 \in \Gamma_0 \cap \Delta_0$, then there exists $n_0 \geqslant 0$ such that $\operatorname{Im}\zeta_n \geqslant \widetilde{\mathcal{B}}(\alpha_{n+1})$ for all $n \geqslant n_0$.*

Proof. Let $z_0 \in \Gamma_0 \cap \Delta_0$. By Lemma 3.2, for every $n \in \mathbb{N}$, the inner radius of the Siegel disk of f_n is $c_n e^{-\mathcal{B}(\alpha_n)}$, where $c_n \in [1/D_7, D_7]$. Let $\mho$ be the half-infinite strip defined in (5.1). By the definition of the near-parabolic renormalization $f_{n+1} = \mathcal{R} f_n$, there exists $\widetilde{\zeta}_n \in \overline{\mho'}$ such that $\mathbb{E}\mathrm{xp}(\widetilde{\zeta}_n) \in \partial\Delta_{n+1}$ and (see (4.28))

$$\operatorname{Im}\widetilde{\zeta}_n = \frac{1}{2\pi}\mathcal{B}(\alpha_{n+1}) - \frac{1}{2\pi}\log\frac{27c_{n+1}}{4}. \tag{5.11}$$

Assume there exists a subsequence $(n_j)_{j\geqslant 1}$ such that

$$\operatorname{Im}\zeta_{n_j} < \widetilde{\mathcal{B}}(\alpha_{n_j+1}) = \frac{1}{2\pi}\mathcal{B}(\alpha_{n_j+1}) + M.$$

If $\operatorname{Im}\zeta_{n_j} \leqslant \operatorname{Im}\widetilde{\zeta}_{n_j}$, there exists $\zeta'_{n_j} \in \Phi_{n_j}(\partial\Delta_{n_j} \cap \mathcal{P}_{n_j}) \cap \overline{\mho'}$ with $\operatorname{Im}\zeta'_{n_j} = \operatorname{Im}\zeta_{n_j}$ such that

$$|\zeta_{n_j} - \zeta'_{n_j}| \leqslant 1. \tag{5.12}$$

If $\operatorname{Im}\zeta_{n_j} > \operatorname{Im}\widetilde{\zeta}_{n_j}$, we have

$$\frac{1}{2\pi}\mathcal{B}(\alpha_{n_j+1}) - \frac{1}{2\pi}\log\frac{27c_{n_j+1}}{4} < \operatorname{Im}\zeta_{n_j} < \frac{1}{2\pi}\mathcal{B}(\alpha_{n_j+1}) + M$$

and hence

$$|\zeta_{n_j} - \widetilde{\zeta}_{n_j}|^2 \leqslant 1 + \left(M + \frac{1}{2\pi}\log\frac{27D_7}{4}\right)^2. \tag{5.13}$$

By (5.12) and (5.13), for each ζ_{n_j} with $j \geqslant 1$, one can find a point (ζ'_{n_j} or $\widetilde{\zeta}_{n_j}$) in $\Phi_{n_j}(\partial\Delta_{n_j} \cap \mathcal{P}_{n_j}) \cap \overline{\mho'}$ such that the hyperbolic distance with respect to $\rho_\mho$ between them are uniformly bounded above. By a similar argument to Proposition 4.9 based on Lemma 4.6, we conclude that $\zeta_0 \in \Phi_0(\partial\Delta_0 \cap \mathcal{P}_0) \cap \overline{\mho'}$ and $z_0 \in \partial\Delta_0$, which violates our assumption that $z_0 \in \Delta_0$. Therefore, there exists $n_0 \geqslant 0$ such that $\operatorname{Im}\zeta_n \geqslant \widetilde{\mathcal{B}}(\alpha_{n+1})$ for all $n \geqslant n_0$. ∎

Lemma 5.6. $\Gamma_0 \cap \partial\Delta_0$ *is a singleton. In particular,* $\Gamma_0 \setminus \{\mathrm{cv}\} \subset \Delta_0$ *if and only if* $\mathrm{cv} \in \partial\Delta_0$.

Proof. Since $\Gamma_0 \cup \{0\}$ is a Jordan arc connecting $\mathrm{cv} = -\frac{4}{27}$ to 0, there exists a homeomorphism $\beta : [0,1] \to \Gamma_0 \cup \{0\}$ such that $\beta(0) = \mathrm{cv}$ and $\beta(1) = 0$. Assume that $\Gamma_0 \cap \partial\Delta_0$ is not a singleton. Then there exist $0 \leqslant t_1 < t_2 < 1$ such that

- $\beta(t_i) \in \partial\Delta_0$ for $i = 1, 2$;
- $\beta([0,t_1]) \cap \Delta_0 = \emptyset$ and $\beta((t_2,1]) \subset \Delta_0$.

Let $\Gamma_0' := \beta([t_1,t_2])$ be a subarc of Γ_0. Then we have the following two cases.

(1) Assume $\Gamma_0' \subset \partial\Delta_0$. There exists $z_0 \in \Gamma_0'$ such that $f_0^{\circ q_n}(z_0) \in \Gamma_0'$ for some large integer n since the restriction of f_0 to $\partial\Delta_0$ is conjugate to the rigid rotation. Denote $\Gamma_n' := \mathbb{E}\mathrm{xp} \circ \Phi_{n-1} \circ \cdots \circ \mathbb{E}\mathrm{xp} \circ \Phi_0(\Gamma_0')$. Then Γ_n' is a Jordan arc contained in $\Gamma_n \subset \mathcal{Q}_n'$. By Lemma 2.15 (a), Γ_n' and hence Γ_n contains a point z_n and $f_n(z_n)$, which is impossible.

(2) Assume $\Gamma_0' \not\subset \partial\Delta_0$. Since $\Phi_n(\Gamma_n) \subset \mho'$, it follows that $f_n(\Gamma_n)$ is well-defined and contained in $\mathcal{P}_n$. Thus by Lemma 2.15, Γ_0 (and hence Γ_0') can be iterated infinitely many times by f_0. Let $W \neq \Delta_0$ be any bounded component of $\mathbb{C} \setminus (\partial\Delta_0 \cup \Gamma_0')$. Since $\partial\Delta_0 \cup \Gamma_0'$ and W can be iterated infinitely many times by f_0, it follows from the maximum modulus principle that W is contained in the Fatou set of f_0. Since $\partial W \cap \partial\Delta_0$ contains a subarc of $\partial\Delta_0$, it follows that W is contained in Δ_0, which is a contradiction. This finishes the proof that $\Gamma_0 \cap \partial\Delta_0$ is a singleton.

From $\Gamma_0 \setminus \{\mathrm{cv}\} \subset \Delta_0$ we obtain $\mathrm{cv} \in \partial\Delta_0$ immediately. If $\mathrm{cv} \in \partial\Delta_0$, since Γ_0 is a Jordan arc and $\Gamma_0 \cap \partial\Delta_0$ is a singleton, we conclude that $\Gamma_0 \setminus \{\mathrm{cv}\} \subset \Delta_0$. ∎

5.3. *A new class of irrational numbers*

For $n \geqslant 1$, let $s_{\alpha_n} : \gamma_{n-1} \to \gamma_n$ be the homeomorphism defined in (5.9). In the following, we use Γ_α (resp. γ_α) to denote Γ_0 (resp. $\gamma_0 = \Phi_0(\Gamma_0)$) when we want to emphasize the dependence on $\alpha = \alpha_0 \in \mathrm{HT}_N$.

Definition. Let $\widetilde{\mathcal{H}}_N$ be a subset of $\mathcal{B}_N$ defined as

$$\widetilde{\mathcal{H}}_N := \left\{ \alpha \in \mathcal{B}_N \;\middle|\; \begin{array}{l} \forall \zeta \in \gamma_\alpha \setminus \{1\},\ \exists n \geqslant 1 \text{ such that} \\ \operatorname{Im} s_{\alpha_n} \circ \cdots \circ s_{\alpha_1}(\zeta) \geqslant \widetilde{\mathcal{B}}(\alpha_{n+1}) \end{array} \right\}.$$

In the next section we show that $\widetilde{\mathcal{H}}_N$ is independent of the choice of $f_0 \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ by proving that $\widetilde{\mathcal{H}}_N$ coincides with the set of high type Herman numbers.

Proposition 5.7. *The critical value* $\mathrm{cv} = -\frac{4}{27}$ *is in* $\partial\Delta_0$ *if and only if* $\alpha \in \widetilde{\mathcal{H}}_N$.

Proof. For each $\zeta \in \gamma_\alpha \setminus \{1\}$ and $n \geqslant 1$, we denote

$$\zeta_n := s_{\alpha_n} \circ \cdots \circ s_{\alpha_1}(\zeta).$$

Suppose $\alpha \in \widetilde{\mathcal{H}}_N$. Then there exists $n \geqslant 1$ such that $\operatorname{Im} \zeta_n \geqslant \widetilde{\mathcal{B}}(\alpha_{n+1})$. By (5.11) and the choice of M in (5.10), we have $\Phi_n^{-1}(\zeta_n) \in \Delta_n$ and hence $\Phi_0^{-1}(\zeta) \in \Delta_0$. Therefore, $\Gamma_\alpha \setminus \{\mathrm{cv}\} = \Phi_0^{-1}(\gamma_\alpha \setminus \{1\})$ is contained in Δ_0 and $\mathrm{cv} \in \partial\Delta_0$.

Conversely, suppose $\alpha \in \mathcal{B}_N$ and $\mathrm{cv} \in \partial\Delta_0$. By Lemma 5.6, we have $\Phi_0^{-1}(\zeta) \in \Delta_0 \cap \Gamma_\alpha$. According to Lemma 5.5, there exists an integer $n \geqslant 1$ such that $\operatorname{Im} \zeta_n \geqslant \widetilde{\mathcal{B}}(\alpha_{n+1})$. This implies that $\alpha \in \widetilde{\mathcal{H}}_N$. ∎

6. Optimality of the Herman condition

The Herman condition is not easy to verify in general. Yoccoz gave this condition an arithmetic characterization so that one can check easily whether an irrational number is of Herman type. In this section, we first recall Yoccoz's characterization and then prove that under the high type condition, an irrational number is of Herman type if and only if it belongs to the set $\widetilde{\mathcal{H}}_N$ defined in Section 5.3.

6.1. *Yoccoz's characterization of* $\mathcal{H}$

For $\alpha \in (0,1)$ and $x \in \mathbb{R}$, define

$$r_\alpha(x) := \begin{cases} \dfrac{1}{\alpha}\left(x - \log\dfrac{1}{\alpha} + 1\right) & \text{if } x \geqslant \log\dfrac{1}{\alpha}, \\ e^x & \text{if } x < \log\dfrac{1}{\alpha}. \end{cases}$$

The map r_α is of class C^1 on $\mathbb{R}$ and satisfies $r_\alpha(\log\frac{1}{\alpha}) = r'_\alpha(\log\frac{1}{\alpha}) = \frac{1}{\alpha}$ and $x+1 \leqslant r_\alpha(x) \leqslant e^x$ for all $x \in \mathbb{R}$, and $r'_\alpha(x) \geqslant 1$ for all $x \geqslant 0$.

For an irrational number $\alpha \in (0,1)$, we use $(\alpha_n)_{n\geqslant 0}$ to denote the sequence of irrationals defined as in (2.35). Let $\mathcal{B}(\alpha)$ be the Brjuno sum of α (see (3.1)). A Brjuno number α is a *Herman number* (or belongs to *Herman type*) if every orientation-preserving analytic circle diffeomorphism of rotation number α is analytically conjugate to a rigid rotation. Let $\mathcal{H}$ be the set of all Herman numbers.

Theorem 6.1 ([54, Section 2.5]). *The Herman condition has the following arithmetic characterization:*

$$\mathcal{H} = \{\alpha \in \mathcal{B} : \forall m \geqslant 0,\ \exists n > m \text{ such that } r_{\alpha_{n-1}} \circ \cdots \circ r_{\alpha_m}(0) \geqslant \mathcal{B}(\alpha_n)\}.$$

6.2. *Two conditions are equivalent*

In this subsection, we prove that the set of Herman numbers is equal to $\widetilde{\mathcal{H}}_N$ defined in Section 5.3 under the high type condition.

Lemma 6.2 ([54, Lemma 4.9]). *Let α be irrational and $x \geqslant 0$. Then $\alpha \notin \mathcal{H}$ if and only if there exist m and an infinite set $I = I(m,x,\alpha) \subset \mathbb{N}$ such that, for all $k \in I$,*

$$r_{\alpha_{m+k-1}} \circ \cdots \circ r_{\alpha_m}(x) < \log\tfrac{1}{\alpha_{m+k}}.$$

Let D_4 and $D_5 > 1$ be the constants of Lemma 2.12.

Definition. For $\alpha \in (0,1)$ and $y \in \mathbb{R}$, we define

$$\overline{s}_\alpha(y) := \begin{cases} \dfrac{1}{\alpha}\left(y - \dfrac{1}{2\pi}\log\dfrac{1}{\alpha} + D_5\right) & \text{if } y \geqslant \dfrac{1}{2\pi}\log\dfrac{1}{\alpha} + D_4, \\ e^{D_5}\, e^{2\pi y} & \text{if } y < \dfrac{1}{2\pi}\log\dfrac{1}{\alpha} + D_4. \end{cases} \tag{6.1}$$

Let $\gamma_\alpha = \gamma_{\alpha_0}$ be the unbounded arc defined in (5.8) and $s_{\alpha_n} := \Phi_n \circ \mathbb{E}\mathrm{xp} : \gamma_{n-1} \to \gamma_n$ the map defined in (5.9). By Lemma 2.12 and the definition of $\overline{s}_\alpha$, we have the following immediate result.

Lemma 6.3. *For each $\alpha \in \mathcal{B}_N$ and $\zeta \in \gamma_\alpha$, we have*

$$\operatorname{Im} s_\alpha(\zeta) \leqslant \overline{s}_\alpha(\operatorname{Im} \zeta).$$

Define $\mathcal{H}_N := \mathcal{H} \cap \mathcal{B}_N$.

Lemma 6.4. *We have $\widetilde{\mathcal{H}}_N \subset \mathcal{H}_N$.*

Proof. Assume by contradiction that $\alpha \in \widetilde{\mathcal{H}}_N \setminus \mathcal{H}_N$. Define

$$C_0 := 8\pi e^{D_5 + 2\pi D_4}. \tag{6.2}$$

By Lemma 6.2, for the number $2C_0$, there exist $m \geqslant 1$ and an infinite subset $I = I(m, 2C_0, \alpha)$ of $\mathbb{N}$ such that for all $k \in I$, we have

$$r_{\alpha_{m+k-1}} \circ \cdots \circ r_{\alpha_m}(2C_0) < \log \frac{1}{\alpha_{m+k}}. \tag{6.3}$$

Denote $x_{m-1} := 2C_0$ and $y_{m-1} := 1$. For $k \geqslant 1$, we define

$$x_{m+k-1} := r_{\alpha_{m+k-1}} \circ \cdots \circ r_{\alpha_m}(2C_0) \quad \text{and} \quad y_{m+k-1} := \overline{s}_{\alpha_{m+k-1}} \circ \cdots \circ \overline{s}_{\alpha_m}(1),$$

where $\overline{s}_{\alpha_n}$ is the map defined in (6.1). We claim that

$$x_{m+k-1} \geqslant 2\pi y_{m+k-1} + C_0 \quad \text{for all } k \geqslant 0. \tag{6.4}$$

Assume temporarily that (6.4) holds. Since γ_α is an arc starting at 1 and finally going up to infinity, there exists $\zeta \in \gamma_{\alpha_{m-1}}$ such that $\operatorname{Im} \zeta = 1$. For $k \geqslant 1$, we denote

$$\zeta_{m+k-1} := s_{\alpha_{m+k-1}} \circ \cdots \circ s_{\alpha_m}(\zeta),$$

where each s_{α_n} is defined in (5.9). By Lemma 6.3, we have $y_{m+k-1} \geqslant \operatorname{Im} \zeta_{m+k-1}$ for all $k \geqslant 1$.

Since $\alpha \in \widetilde{\mathcal{H}}_N$, by the definition of $\widetilde{\mathcal{H}}_N$ and Lemma 5.4 there exists an integer $k_0 \geqslant 1$ such that for all $k \geqslant k_0$,

$$y_{m+k-1} \geqslant \operatorname{Im} \zeta_{m+k-1} \geqslant \widetilde{\mathcal{B}}(\alpha_{m+k}) = \frac{\mathcal{B}(\alpha_{m+k})}{2\pi} + M > \frac{1}{2\pi} \log \frac{1}{\alpha_{m+k}} + M.$$

On the other hand, since $\alpha \notin \mathcal{H}_N$, by (6.3) there exists $k \in I$ with $k \geqslant k_0$ such that $x_{m+k-1} < \log \frac{1}{\alpha_{m+k}}$. This is a contradiction since by (6.4) we have $x_{m+k-1} \geqslant 2\pi y_{m+k-1} + C_0 > \log \frac{1}{\alpha_{m+k}}$. Hence it suffices to prove (6.4).

Obviously, (6.4) is true when $k = 0$ since $C_0 \geqslant 2\pi$. Suppose $x_{m+k-1} \geqslant 2\pi y_{m+k-1} + C_0$ for some $k \geqslant 0$. It suffices to obtain $x_{m+k} \geqslant 2\pi y_{m+k} + C_0$. The argument is divided into three cases.

Case I: Suppose $x_{m+k-1} < \log\frac{1}{\alpha_{m+k}}$ and $y_{m+k-1} < \frac{1}{2\pi}\log\frac{1}{\alpha_{m+k}} + D_4$. By (6.2), we have $C_0 > 2(D_5 + \log(2\pi))$ and hence $e^{y+C_0} > e^{y+D_5+\log(2\pi)} + C_0$ for any $y \geqslant 1$. Therefore,

$$\begin{aligned} x_{m+k} &= e^{x_{m+k-1}} \geqslant e^{2\pi y_{m+k-1}+C_0} > e^{2\pi y_{m+k-1}+D_5+\log(2\pi)} + C_0 \\ &= 2\pi\,\overline{s}_{\alpha_{m+k}}(y_{m+k-1}) + C_0 = 2\pi y_{m+k} + C_0. \end{aligned}$$

Case II: Suppose $x_{m+k-1} \geqslant \log\frac{1}{\alpha_{m+k}}$ and $y_{m+k-1} \geqslant \frac{1}{2\pi}\log\frac{1}{\alpha_{m+k}} + D_4$. Then

$$\begin{aligned} x_{m+k} &= \frac{1}{\alpha_{m+k}}\left(x_{m+k-1} - \log\frac{1}{\alpha_{m+k}} + 1\right) \\ &\geqslant \frac{2\pi}{\alpha_{m+k}}\left(y_{m+k-1} - \frac{1}{2\pi}\log\frac{1}{\alpha_{m+k}} + D_5\right) + \frac{1}{\alpha_{m+k}}(C_0 + 1 - 2\pi D_5) \\ &\geqslant 2\pi y_{m+k} + 2(C_0 + 1 - 2\pi D_5) > 2\pi y_{m+k} + C_0. \end{aligned}$$

Case III: Suppose $x_{m+k-1} \geqslant \log\frac{1}{\alpha_{m+k}}$ and $y_{m+k-1} < \frac{1}{2\pi}\log\frac{1}{\alpha_{m+k}} + D_4$. We consider the following two subcases:

Subcase (i): Suppose $2\pi y_{m+k-1} < \log\frac{1}{\alpha_{m+k}} - \frac{C_0}{4}$. Note that

$$x_{m+k} = \frac{1}{\alpha_{m+k}}\left(x_{m+k-1} - \log\frac{1}{\alpha_{m+k}} + 1\right) \geqslant \frac{1}{\alpha_{m+k}}.$$

Since $x_{m-1} = 2C_0$, we have $x_{m+k} \geqslant \max\left\{2C_0, \frac{1}{\alpha_{m+k}}\right\}$. By (6.2), we have $C_0 > 4D_5 + 4\log(4\pi)$ and hence $2\pi e^{D_5 - C_0/4} < 1/2$. Then

$$\begin{aligned} x_{m+k} \geqslant \max\left\{2C_0, \frac{1}{\alpha_{m+k}}\right\} &\geqslant \frac{2\pi e^{D_5 - C_0/4}}{\alpha_{m+k}} + C_0 \\ &\geqslant 2\pi e^{D_5} e^{2\pi y_{m+k-1}} + C_0 = 2\pi y_{m+k} + C_0. \end{aligned}$$

Subcase (ii): Suppose $\log\frac{1}{\alpha_{m+k}} - \frac{C_0}{4} \leqslant 2\pi y_{m+k-1} < \log\frac{1}{\alpha_{m+k}} + 2\pi D_4$. Then

$$\begin{aligned} &\alpha_{m+k}(x_{m+k} - (2\pi y_{m+k} + C_0)) \\ &\quad = x_{m+k-1} - \log\tfrac{1}{\alpha_{m+k}} + 1 - \alpha_{m+k}(2\pi e^{D_5} e^{2\pi y_{m+k-1}} + C_0) \\ &\quad \geqslant 2\pi y_{m+k-1} + C_0 + 1 - \log\tfrac{1}{\alpha_{m+k}} - 2\pi\alpha_{m+k} e^{D_5} e^{2\pi y_{m+k-1}} - C_0\alpha_{m+k}. \end{aligned} \tag{6.5}$$

For $\alpha \in (0, 1/2]$, we consider the following continuous function:

$$h(t) := t + C_0 + 1 - \log\tfrac{1}{\alpha} - 2\pi\alpha e^{D_5} e^t - C_0\alpha \quad \text{for } t \in \mathbb{R}.$$

Then $h'(t) = 1 - 2\pi\alpha e^{D_5} e^t$. Hence h is increasing on $(-\infty, \log\frac{1}{\alpha} - D_5 - \log(2\pi)]$ and decreasing on $[\log\frac{1}{\alpha} - D_5 - \log(2\pi), +\infty)$. By (6.2) and a direct calculation, we have

$$\begin{aligned} h\big(\log\tfrac{1}{\alpha} - C_0/4\big) &= (3/4 - \alpha)C_0 + 1 - 2\pi e^{D_5 - C_0/4} > 0, \\ h\big(\log\tfrac{1}{\alpha} + 2\pi D_4\big) &= (1 - \alpha)C_0 + 2\pi D_4 + 1 - 2\pi e^{D_5 + 2\pi D_4} > 0. \end{aligned} \tag{6.6}$$

By (6.5) and (6.6), we have $x_{m+k} > 2\pi y_{m+k} + C_0$. This finishes the proof of (6.4), and so the lemma holds. ∎

Let $D_3 > 0$ be the constant of Lemma 2.11.

Definition. For $\alpha \in (0, 1)$ and $y \in \mathbb{R}$, we define

$$\underline{s}_\alpha(y) := \begin{cases} \dfrac{1}{\alpha}\left(y - \dfrac{1}{2\pi}\log\dfrac{1}{\alpha} - D_3\right) & \text{if } y \geqslant \dfrac{1}{2\pi}\log\dfrac{1}{\alpha} + D_3 + 1, \\ e^{-D_5}\, e^{2\pi y} - 3 & \text{if } y < \dfrac{1}{2\pi}\log\dfrac{1}{\alpha} + D_3 + 1. \end{cases} \tag{6.7}$$

Lemma 6.5. *For each $\alpha \in \mathcal{B}_N$ and $\zeta \in \gamma_\alpha$, we have*

$$\underline{s}_\alpha(\operatorname{Im}\zeta) \leqslant \operatorname{Im} s_\alpha(\zeta).$$

Proof. It follows from the proof of Lemma 2.12 that $D_4 = D_3 + 1$. Moreover, we choose $D_5 = D_3$ in the proof of Lemma 2.12 (a). Then the assertion follows immediately from Lemma 2.12 and the definition of $\underline{s}_\alpha$. ∎

Lemma 6.6. *We have $\mathcal{H}_N \subset \widetilde{\mathcal{H}}_N$.*

Proof. The proof is similar to that of Lemma 6.4. Suppose $\alpha \in \mathcal{H}_N \setminus \widetilde{\mathcal{H}}_N$ by contradiction. Since $\alpha \notin \widetilde{\mathcal{H}}_N$, by the definition of $\widetilde{\mathcal{H}}_N$ there exist a point $\zeta \in \gamma_\alpha \setminus \{1\}$ and an infinite sequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\operatorname{Im}\zeta_{n_k} < \widetilde{\mathcal{B}}(\alpha_{n_k+1}), \tag{6.8}$$

where

$$\zeta_n := s_{\alpha_n} \circ \cdots \circ s_{\alpha_1}(\zeta) \quad \text{for all } n \in \mathbb{N}.$$

By uniform contraction with respect to the hyperbolic metric as in the proof of Proposition 4.9 and Lemma 5.3 there exists an integer $m \geqslant 1$ such that

$$\zeta_{m-1} \in \gamma_{m-1} \quad \text{and} \quad \operatorname{Im}\zeta_{m-1} \geqslant 2C_0,$$

where $C_0 > 2M$ is a large number and $M \geqslant 1$ is as in the definition of $\widetilde{\mathcal{B}}(\alpha_n)$. Then by (6.8) there exists an infinite subset $I' = I'(m, \zeta, \alpha)$ of $\mathbb{N}$ such that for all $k \in I'$,

$$\operatorname{Im}\zeta_{m+k-1} < \widetilde{\mathcal{B}}(\alpha_{m+k}). \tag{6.9}$$

Since $\alpha \in \mathcal{H}_N$, by Theorem 6.1 there exists $k_0 = k_0(m) \geqslant 1$ such that $r_{\alpha_{m+k_0-1}} \circ \cdots \circ r_{\alpha_m}(0) \geqslant \mathcal{B}(\alpha_{m+k_0})$. A direct calculation shows that for all $k \geqslant k_0$,

$$r_{\alpha_{m+k-1}} \circ \cdots \circ r_{\alpha_m}(0) \geqslant \mathcal{B}(\alpha_{m+k}). \tag{6.10}$$

Denote $x_{m-1} := 0$ and $y_{m-1} := 2C_0$. For $k \geqslant 1$, we define

$$x_{m+k-1} := r_{\alpha_{m+k-1}} \circ \cdots \circ r_{\alpha_m}(0) \quad \text{and} \quad y_{m+k-1} := \underline{s}_{\alpha_{m+k-1}} \circ \cdots \circ \underline{s}_{\alpha_m}(2C_0),$$

where $\underline{s}_{\alpha_n}$ is the map defined in (6.7). We claim that if C_0 is large enough, then

$$2\pi y_{m+k-1} \geqslant x_{m+k-1} + C_0 \quad \text{for all } k \geqslant 0. \tag{6.11}$$

Assume temporarily that (6.11) holds. By Lemma 6.5, we have $y_{m+k-1} \leqslant \operatorname{Im} \zeta_{m+k-1}$ for all $k \geqslant 1$. By (6.9), there exists an integer $k \in I'$ with $k \geqslant k_0$ such that

$$y_{m+k-1} \leqslant \operatorname{Im} \zeta_{m+k-1} < \widetilde{\mathcal{B}}(\alpha_{m+k}) = \frac{\mathcal{B}(\alpha_{m+k})}{2\pi} + M.$$

On the other hand, by (6.10), we have $x_{m+k-1} \geqslant \mathcal{B}(\alpha_{m+k})$. However, by (6.11) we have $x_{m+k-1} \leqslant 2\pi y_{m+k-1} - C_0 < \mathcal{B}(\alpha_{m+k})$, which is a contradiction. Hence it suffices to prove (6.11).

Obviously, (6.11) is true when $k = 0$. Suppose $2\pi y_{m+k-1} \geqslant x_{m+k-1} + C_0$ for some $k \geqslant 0$. Then one can divide the argument into three cases as in Lemma 6.4 to obtain $2\pi y_{m+k} \geqslant x_{m+k} + C_0$. We omit the details since the rest of the proof is completely the same. ∎

Remark. In fact, if $\alpha \in \mathcal{H}_N$, then according to [24,27], the boundary of the Siegel disk of each $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ contains the unique critical value $-\frac{4}{27}$. This implies that $\alpha \in \widetilde{\mathcal{H}}_N$ by Proposition 5.7. Therefore in this way we also obtain $\mathcal{H}_N \subset \widetilde{\mathcal{H}}_N$.

Proof of the second part of the Main Theorem. Let $\alpha \in \mathrm{HT}_N$ be an irrational number of sufficiently high type. By Lemmas 6.4 and 6.6, $\alpha \in \mathcal{H}_N$ if and only if $\alpha \in \widetilde{\mathcal{H}}_N$. By Proposition 5.7, $\alpha \in \widetilde{\mathcal{H}}_N$ if and only if $\mathrm{cv} = f(\mathrm{cp}_f) \in \partial\Delta_f$, where Δ_f is the Siegel disk of $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ and cp_f is the unique critical point of f. Therefore, $\alpha \in \mathcal{H}_N$ if and only if $\mathrm{cp}_f \in \partial\Delta_f$. ∎

Appendix A. Some calculations in Fatou coordinate planes

In this appendix we give the proof of Lemma 5.1 based on some estimates of [30]. Let $0 < \alpha < 1/2$. Define

$$Y := \left\{ w = x + yi \in \mathbb{C} : -\frac{1}{2\pi\alpha}\left(\arccos\frac{\sqrt{3}}{2e^{2\pi\alpha y}} - \frac{\pi}{6}\right) < x < \frac{2}{3\alpha} \quad \text{and} \quad y > 1 \right\}$$

and $R := \frac{4}{27}e^{3\pi}$ (see Figure 8).

Lemma A.1. *There exists $\varepsilon' > 0$ such that for all $f \in \mathcal{IS}_\alpha$ with $\alpha \in (0, \varepsilon']$,*

$$\tau_f(Y) \subset \mathbb{D}(0, R) \setminus [0, R),$$

where $\tau_f : \mathbb{C} \to \widehat{\mathbb{C}} \setminus \{0, \sigma_f\}$ is the universal covering defined in (2.13).

Proof. By a direct calculation, we have

$$\begin{aligned}\{e^{-2\pi i\alpha w} : w \in Y\} &= \left\{ \xi \in \mathbb{C} : |\xi| > e^{2\pi\alpha} \text{ and } -\frac{4\pi}{3} < \arg\xi < \arccos\frac{\sqrt{3}}{2|\xi|} - \frac{\pi}{6} \right\} \\ &= \mathbb{C} \setminus \left(\overline{\mathbb{D}}(0, e^{2\pi\alpha}) \cup \left\{ \xi \in \mathbb{C} : \frac{\pi}{3} \leqslant \arg\left(\xi - \frac{1-\sqrt{3}i}{2}\right) \leqslant \frac{2\pi}{3} \right\} \right).\end{aligned}$$

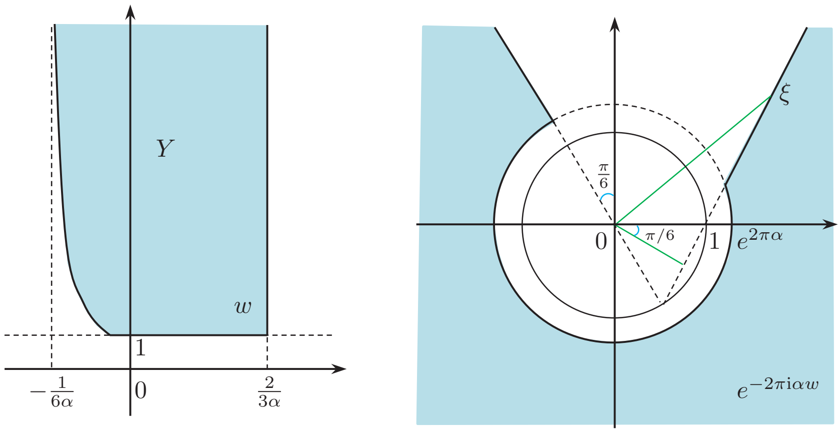

Fig. 8. The domain Y and its image under $w \mapsto e^{-2\pi i\alpha w}$.

Since $4\pi\alpha/(3R) < e^{2\pi\alpha} - 1$, we have (see Figure 8)

$$e^{-2\pi i\alpha w} \in \mathbb{C} \setminus \left(\overline{\mathbb{D}}\left(1, \frac{4\pi\alpha}{3R}\right) \cup \left\{\xi \in \mathbb{C} : \frac{\pi}{3} \leqslant \arg(\xi - 1) \leqslant \frac{2\pi}{3}\right\}\right).$$

This implies that

$$\frac{1}{1 - e^{-2\pi i\alpha w}} \in \mathbb{D}\left(0, \frac{3R}{4\pi\alpha}\right) \setminus \left\{\xi \in \mathbb{C} : \frac{\pi}{3} \leqslant \arg \xi \leqslant \frac{2\pi}{3}\right\}. \tag{A.1}$$

Note that $\arcsin x \leqslant \frac{\pi}{3} x$ for $0 \leqslant x \leqslant 1/2$. By [30, Main Theorem 1 (a)], we have $|f_0''(0) - 4.91| \leqslant 1.14$ for all $f_0 \in \mathcal{IS}_0$. Hence $|\arg f_0''(0)| < \arcsin \frac{1}{3} \leqslant \frac{\pi}{9}$ and

$$-\frac{4\pi i\alpha}{f_0''(0)} \in \left\{z \in \mathbb{C} : \frac{4\pi\alpha}{7} < |z| < \frac{8\pi\alpha}{7} \text{ and } \frac{25\pi}{18} < \arg z < \frac{29\pi}{18}\right\}.$$

By (2.11) and the pre-compactness of $\mathcal{IS}_\alpha$, there exists a small $\varepsilon' > 0$ such that for all $f \in \mathcal{IS}_\alpha$ with $\alpha \in (0, \varepsilon']$,

$$\sigma_f \in \left\{z \in \mathbb{C} : \frac{\pi\alpha}{2} < |z| < \frac{4\pi\alpha}{3} \text{ and } \frac{4\pi}{3} < \arg z < \frac{5\pi}{3}\right\}. \tag{A.2}$$

By (A.1) and (A.2) we have

$$\tau_f(w) = \frac{\sigma_f}{1 - e^{-2\pi i\alpha w}} \in \mathbb{D}(0, R) \setminus [0, R).$$

The proof is complete. ■

For each $C \geqslant 1$, we define a subset of $\mho$ (see (5.1)) by

$$\mho_1(C) := \{\zeta \in \mathbb{C} : 1/4 < \operatorname{Re} \zeta < 7/4 \text{ and } \operatorname{Im} \zeta \geqslant C\}. \tag{A.3}$$

Lemma A.2. *There exist $C > 1$ and $\varepsilon'' > 0$ such that for all $f \in \mathcal{IS}_\alpha$ with $\alpha \in (0, \varepsilon'']$, we have*

$$L_f^{-1}(\overline{\mho_1(C)}) \subset Y,$$

where $L_f : \widetilde{\mathcal{P}}_f \to \mathbb{C}$ is the univalent map defined in (2.14).

Proof. Let $D_2 > 0$ be as in Proposition 2.10. For $y > 0$, we define

$$\varphi_1(y) := \log\big(2 + \sqrt{y^2 + (7/4)^2}\big).$$

There exists a constant $C > 0$ depending only on D_2 such that if $y \geqslant C$, then

$$y - 2D_2\varphi_1(y) > 1 \tag{A.4}$$

and

$$\frac{y}{2\pi}\left(\arccos\frac{\sqrt{3}}{2e^{2\pi}} - \frac{\pi}{6}\right) - D_2\varphi_1(y) > 0. \tag{A.5}$$

Let $0 < \alpha \leqslant 1/C$. By Proposition 2.10, we have $L_f^{-1}(\overline{\mho_1(C)}) \subset X_1 \cup X_2 \cup X_3$, where

$$X_1 = \{x + y\mathrm{i} : -D_2 \log(1 + 1/\alpha) \leqslant x \leqslant D_2 \log(1 + 1/\alpha) + 7/4 \text{ and } y \geqslant 1/\alpha\},$$
$$X_2 = \{x + y\mathrm{i} : -D_2\varphi_1(y) \leqslant x \leqslant D_2\varphi_1(y) + 7/4 \text{ and } y \in [C, 1/\alpha]\},$$
$$X_3 = \{x + y\mathrm{i} : -D_2\varphi_1(C) \leqslant x \leqslant D_2\varphi_1(C) + 7/4 \text{ and } y \in [C - D_2\varphi_1(C), C]\}.$$

For $y > 0$, we define a continuous function

$$\phi(y) := \frac{1}{2\pi\alpha}\left(\arccos\frac{\sqrt{3}}{2e^{2\pi\alpha y}} - \frac{\pi}{6}\right).$$

Note that $\alpha \log(1 + 1/\alpha)$ is uniformly bounded above for $0 < \alpha < 1$. There exists a constant $\kappa_1 > 0$ depending only on D_2 such that if $\alpha \in (0, \kappa_1]$, then for $y \geqslant 1/\alpha$,

$$\phi(y) - D_2 \log\left(1 + \frac{1}{\alpha}\right) \geqslant \frac{1}{2\pi\alpha}\left(\arccos\frac{\sqrt{3}}{2e^{2\pi}} - \frac{\pi}{6}\right) - D_2 \log\left(1 + \frac{1}{\alpha}\right) > 0.$$

For $y \in [C, 1/\alpha]$, we denote $t = 2\pi\alpha y \in [2\pi\alpha C, 2\pi]$. Then

$$\phi(y) - D_2\varphi_1(y) = y\psi(t) - D_2\varphi_1(y),$$

where

$$\psi(t) := \frac{1}{t}\left(\arccos\frac{\sqrt{3}}{2e^{t}} - \frac{\pi}{6}\right). \tag{A.6}$$

A direct calculation[15] shows that $\psi(t)$ is decreasing on $(0, 2\pi]$. By (A.5) we have

$$\phi(y) - D_2\varphi_1(y) \geqslant \frac{y}{2\pi}\left(\arccos\frac{\sqrt{3}}{2e^{2\pi}} - \frac{\pi}{6}\right) - D_2\varphi_1(y) > 0.$$

[15]Note that $\psi(t) = \frac{1}{t}\int_0^t (\frac{4}{3}e^{2s} - 1)^{-1/2}\,\mathrm{d}s$ can be seen as the average of the integral of $\widetilde{\psi}(s) = (\frac{4}{3}e^{2s} - 1)^{-1/2}$ in the interval $(0, t)$. Since $s \mapsto \widetilde{\psi}(s)$ is strictly decreasing in $(0, +\infty)$, so is $t \mapsto \psi(t)$.

Finally, let $y \in [C - D_2\varphi_1(C), C]$ and still denote $t = 2\pi\alpha y$. A direct calculation shows that $\lim_{t\to 0^+} \psi(t) = \sqrt{3}$, where ψ is defined in (A.6). By (A.4), there exists a constant $\kappa_2 > 0$ depending only on D_2 such that if $\alpha \in (0, \kappa_2]$, then for $y \in [C - D_2\varphi_1(C), C]$ we have

$$\phi(y) - D_2\varphi_1(C) \geqslant y - D_2\varphi_1(C) \geqslant (C - D_2\varphi_1(C)) - D_2\varphi_1(C) > 1.$$

Let $\kappa_3 > 0$ be a constant depending only on D_2 such that $D_2\varphi_1(\frac{1}{\alpha}) + \frac{7}{4} < \frac{2}{3\alpha}$ for all $\alpha \in (0, \kappa_3]$. The proof is finished by setting $\varepsilon'' := \min\{1/C, \kappa_1, \kappa_2, \kappa_3\}$. ∎

Proof of Lemma 5.1. For $f_0 \in \mathcal{IS}_0$, one can define $\mathcal{C}_{f_0}$ and $\mathcal{C}_{f_0}^\sharp$ as in (2.3) similarly (replacing $\mathcal{P}_f$ and Φ_f there by $\mathcal{P}_{\mathrm{attr},f_0}$ and Φ_{attr,f_0}). We first show that (5.2) holds for $f_0 \in \mathcal{IS}_0$ and then use an argument of continuity.

The Main Theorem 1 in [30] was proved by transferring the parabolic fixed point 0 of $f_0 \in \mathcal{IS}_0$ to ∞, and a class corresponding to $\mathcal{IS}_0$ was defined (see [30, Section 5.A]):

$$\mathcal{IS}_0^Q := \left\{ F = Q \circ \varphi^{-1} \;\middle|\; \begin{array}{l} \varphi : \widehat{\mathbb{C}} \setminus E \to \widehat{\mathbb{C}} \setminus \{0\} \text{ is univalent,} \\ \varphi(\infty) = \infty \text{ and } \varphi'(\infty) = 1 \end{array} \right\},$$

where E is the ellipse defined in (2.1) and $Q(z) = z(1 + 1/z)^6/(1 - 1/z)^4$ is a parabolic map. Each map in this class has a parabolic fixed point at ∞, a critical point at $\mathrm{cp}_F := \varphi(5 + 2\sqrt{6})$ and a critical value at $\mathrm{cv}_Q = 27$ which is independent of F.

By [30, Lemma 5.14 (a)], P and Q are related by $Q = \psi_0^{-1} \circ P \circ \psi_1$, where $\psi_1(z) = -4z/(1 + z)^2$ is defined in (2.2) and $\psi_0(z) = -4/z$. By [30, Proposition 5.3 (c)], there exists a one-to-one correspondence between $\mathcal{IS}_0$ and $\mathcal{IS}_0^Q$. For $F \in \mathcal{IS}_0^Q$, one has natural definitions of the attracting petal $\mathcal{P}_{\mathrm{attr},F}$, repelling petal $\mathcal{P}_{\mathrm{rep},F}$, attracting Fatou coordinate $\Phi_{\mathrm{attr},F}$ and repelling Fatou coordinate $\Phi_{\mathrm{rep},F}$ etc. based on the definitions relating to $f_0 \in \mathcal{IS}_0$ in Section 2.1. For example, the attracting Fatou coordinate of F is defined as $\Phi_{\mathrm{attr},F}(z) = \Phi_{\mathrm{attr},f_0} \circ \psi_0(z)$.

For $f_0 \in \mathcal{IS}_0$, we define a topological triangle

$$\mathcal{Q}_{f_0} := \{z \in \mathcal{P}_{\mathrm{attr},f_0} : \Phi_{\mathrm{attr},f_0}(z) \in \mho\}.$$

In order to prove (5.2), it is convenient to work in the corresponding dynamical plane of $F = \psi_0^{-1} \circ f_0 \circ \psi_0 \in \mathcal{IS}_0^Q$. Define

$$D_{0,F} := \{z \in \mathcal{P}_{\mathrm{attr},F} : 0 < \operatorname{Re} \Phi_{\mathrm{attr},F}(z) < 1 \text{ and } \operatorname{Im} \Phi_{\mathrm{attr},F}(z) > -2\}$$

and $D_{1,F} := F(D_{0,F})$. By [30, Proposition 5.7 (e)], for $z \in \overline{D}_{0,F}$ we have

$$|z| \geqslant 0.05 > 27e^{-3\pi} \quad \text{and} \quad z \notin \mathbb{R}_-.$$

By [30, Proposition 5.6 (b)], for $z \in \overline{D}_{1,F}$ we have

$$|z| \geqslant \tfrac{25}{\sqrt{3}} \sin \tfrac{\pi}{3} = \tfrac{25}{2} > 27e^{-3\pi} \quad \text{and} \quad z \notin \mathbb{R}_-.$$

Let $R = \frac{4}{27}e^{3\pi}$. We have

$$\overline{D}_{0,F} \cup \overline{D}_{1,F} \subset \psi_0^{-1}(\mathbb{D}(0, R) \setminus [0, R)) = \mathbb{C} \setminus \big(\overline{\mathbb{D}}(0, 27e^{-3\pi}) \cup \mathbb{R}^-\big). \tag{A.7}$$

By the definition of $\mathcal{Q}_{f_0}$, we have

$$\psi_0^{-1}(\mathcal{Q}_{f_0}) = \{z \in \mathcal{P}_{\mathrm{attr},F} : 1/4 < \operatorname{Re}\Phi_{\mathrm{attr},F}(z) < 7/4 \text{ and } \operatorname{Im}\Phi_{\mathrm{attr},F}(z) > -2\}.$$

Therefore, by (A.7) we have $\psi_0^{-1}(\overline{\mathcal{Q}}_{f_0} \setminus \{0\}) \subset \overline{D}_{0,F} \cup \overline{D}_{1,F}$. This implies that

$$\overline{\mathcal{Q}}_{f_0} \setminus \{0\} \subset \mathbb{D}(0,R) \setminus [0,R) \quad \text{for all } f_0 \in \mathcal{IS}_0. \tag{A.8}$$

Let $C > 1$ be the constant of Lemma A.2 and $\mho_1 = \mho_1(C)$ be defined in (A.3). By Lemmas A.1 and A.2, for every $f \in \mathcal{IS}_\alpha$ with $0 < \alpha \leqslant \min\{\varepsilon', \varepsilon''\}$ we have

$$\Phi_f^{-1}(\overline{\mho}_1) = \tau_f \circ L_f^{-1}(\overline{\mho}_1) \subset \mathbb{D}(0,R) \setminus [0,R).$$

Define

$$\mho_2 := \overline{\mho \setminus \mho_1} = \{\zeta \in \mathbb{C} : 1/4 \leqslant \operatorname{Re}\zeta \leqslant 7/4 \text{ and } -2 \leqslant \operatorname{Im}\zeta \leqslant C\}.$$

By (A.8), the continuity of the Fatou coordinates in Proposition 2.2 (d) (see also [45, Proposition 3.2.2]) and the pre-compactness of $\mathcal{IS}_0$, there exists a constant $0 < \varepsilon_4' \leqslant \min\{\varepsilon', \varepsilon''\}$ such that for all $f \in \mathcal{IS}_\alpha$ with $\alpha \in (0, \varepsilon_4']$, we have $\Phi_f^{-1}(\mho_2) \subset \mathbb{D}(0,R) \setminus [0,R)$ and hence $\overline{\mathcal{Q}}_f \setminus \{0\} = \Phi_f^{-1}(\overline{\mho}) \subset \mathbb{D}(0,R) \setminus [0,R)$. ∎

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