J. Eur. Math. Soc. 27, 4705–4712 (2025) DOI 10.4171/JEMS/1458

Tom Bachmann

The very effective covers of KO and KGL over Dedekind schemes

Received 17 February 2022; revised 6 May 2022

Abstract. We answer a question of Hoyois–Jelisiejew–Nardin–Yakerson regarding framed models of motivic connective K-theory spectra over Dedekind schemes. That is, we show that the framed suspension spectrum of the presheaf of groupoids of vector bundles (resp. non-degenerate symmetric bilinear bundles) is the effective cover of KGL (resp. very effective cover of KO). One consequence is that, over any scheme, we obtain a spectral sequence from Spitzweck's motivic cohomology to homotopy algebraic K-theory; it is strongly convergent under mild assumptions.

Keywords: motivic cohomology, algebraic K-theory, framed correspondence.

1. Statement of results

Let S be a scheme. The category $\mathcal{P}_\Sigma(\mathrm{Cor}^{\mathrm{fr}}(S))$ of presheaves with framed transfers [5, §2.3] is a motivic analog of the classical category of $\mathcal{E}_\infty$-monoids. We have the *framed suspension spectrum* functor

$$\Sigma^\infty_{\mathrm{fr}}\colon \mathcal{P}_\Sigma(\mathrm{Cor}^{\mathrm{fr}}(S)) \to \mathcal{SH}(S)$$

which was constructed in [6, Theorem 18]. By analogy with the classical situation, one might expect that many interesting motivic spectra can be obtained as framed suspension spectra. This is indeed the case; see [8, §1.1] for a summary.

This note concerns the following examples of the above idea. One has framed presheaves $\mathrm{Vect}, \mathrm{Bil} \in \mathcal{P}_\Sigma(\mathrm{Cor}^{\mathrm{fr}}(S))$ [8, §6], where $\mathrm{Vect}(X)$ is the groupoid of vector bundles on X and $\mathrm{Bil}(X)$ is the groupoid of vector bundles with a non-degenerate symmetric bilinear form. There exist Bott elements

$$\beta \in \pi_{2,1}\Sigma^\infty_{\mathrm{fr}}\mathrm{Vect} \quad \text{and} \quad \tilde{\beta} \in \pi_{8,4}\Sigma^\infty_{\mathrm{fr}}\mathrm{Bil}$$

Tom Bachmann: Mathematisches Institut, Ludwig-Maximilians-Universität München, Theresienstr. 39, 80333 München, Germany; Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, 0316 Oslo, Norway; Institut für Mathematik, FB 08 – Physik, Mathematik und Informatik, Johannes Gutenberg-Universität Mainz, Staudingerweg 9, 55128 Mainz, Germany; tom.bachmann@zoho.com

Mathematics Subject Classification 2020: 14F42 (primary); 19E20, 19G38 (secondary).

and canonical equivalences [7, Proposition 5.1], [8, Proposition 6.7]

$$(\Sigma^\infty_{\mathrm{fr}}\mathrm{Vect})[\beta^{-1}] \simeq \mathrm{KGL} \quad \text{and} \quad (\Sigma^\infty_{\mathrm{fr}}\mathrm{Bil})[\widetilde{\beta}^{-1}] \simeq \mathrm{KO}.$$

Here KGL is the motivic spectrum representing homotopy algebraic K-theory, and KO is the motivic spectrum representing homotopy hermitian K-theory.[1] Again by comparison with the classical situation, this suggests that $\Sigma^\infty_{\mathrm{fr}}\mathrm{Vect}$ and $\Sigma^\infty_{\mathrm{fr}}\mathrm{Bil}$ should be motivic analogs of *connective* K-theory spectra. Another way of producing "connective" versions is by passing to (very) effective covers [11, 12]. It was proved in [7, 8] that these two notions of connective motivic K-theory spectra coincide, provided that S is regular over a field.

Our main result is to extend this comparison to more general base schemes. We denote by $H\mathbb{Z}$ Spitzweck's motivic cohomology spectrum [11] and by $H\mathrm{W}$ the periodic Witt cohomology spectrum [3, Definition 4.6].

Theorem 1.1. *Let S be a scheme.*

(1) *Suppose that $f_1(H\mathbb{Z}) = 0 \in \mathcal{SH}(S)$. The canonical map*

$$\Sigma^\infty_{\mathrm{fr}}\mathrm{Vect} \to f_0\mathrm{KGL} \in \mathcal{SH}(S)$$

is an equivalence.

(2) *Suppose in addition that $1/2 \in S$ and $H\mathrm{W}_{\geq 2} = 0 \in \mathcal{SH}(S)$. The canonical map*

$$\Sigma^\infty_{\mathrm{fr}}\mathrm{Bil} \to \widetilde{f}_0\mathrm{KO} \in \mathcal{SH}(S)$$

is an equivalence.

These assumptions are satisfied if S is essentially smooth over a Dedekind scheme (containing $1/2$ in case (2)).

Remark 1.2. That the assumptions are satisfied for Dedekind schemes is proved in [4, Proposition B.4] for (1) and in [3, Lemma 3.8] for (2). They in fact hold for all schemes; this will be recorded elsewhere.

Example 1.3. Bott periodicity implies formally that

$$f_n\mathrm{KGL} \simeq \Sigma^{2n,n} f_0\mathrm{KGL} \quad \text{and} \quad s_n(\mathrm{KGL}) \simeq \Sigma^{2n,n} f_0(\mathrm{KGL})/\beta.$$

Theorem 1.1 (1) implies that $f_0(\mathrm{KGL})/\beta \simeq H\mathbb{Z}$ (see Lemma 2.1). Hence in this situation, the slice filtration for KGL yields a convergent spectral sequence, with E_2-page given by (Spitzweck's) motivic cohomology.

Notation. We use notation for standard motivic categories and spectra as in [3, 8].

[1] As a notational convention for this introduction, whenever we mention KO we shall assume that $1/2 \in S$.

2. Proofs

As a warm-up, we treat the case of KGL. Recall that the functor $\Sigma^\infty_{\mathrm{fr}}$ inverts group-completion. The Bott element lifts to $\beta\colon (\mathbb{P}^1, \infty) \to \mathrm{Vect}^{\mathrm{gp}}$ [7, §5]. We also have the rank map $\mathrm{Vect}^{\mathrm{gp}} \to \mathbb{Z} \in \mathcal{P}_\Sigma(\mathrm{Cor}^{\mathrm{fr}}(S))$. The composite

$$(\mathbb{P}^1, \infty) \wedge \mathrm{Vect}^{\mathrm{gp}} \xrightarrow{\beta} \mathrm{Vect}^{\mathrm{gp}} \wedge \mathrm{Vect}^{\mathrm{gp}} \xrightarrow{m} \mathrm{Vect}^{\mathrm{gp}} \to \mathbb{Z}$$

is null-homotopic after motivic localization since $\mathbb{Z}$ is motivically local and truncated and $(\mathbb{P}^1, \infty) \overset{\mathrm{mot}}{\simeq} S^1 \wedge \mathbb{G}_m$.

Lemma 2.1. *The induced map*

$$(\Sigma^\infty_{\mathrm{fr}}\mathrm{Vect})/\beta \to \Sigma^\infty_{\mathrm{fr}}\mathbb{Z} \simeq H\mathbb{Z}$$

is an equivalence.

Proof. The equivalence $\Sigma^\infty_{\mathrm{fr}}\mathbb{Z} \simeq H\mathbb{Z}$ is proved in [6, Theorem 21]. Since all terms are stable under base change [8, proof of Lemma 7.5], [6, Lemma 16], we may assume that $S = \mathrm{Spec}(\mathbb{Z})$. Using [4, Proposition B.3], we further reduce to the case where S is the spectrum of a perfect field. In this case, $\Sigma^\infty_{\mathrm{fr}}\mathrm{Vect} \simeq f_0\mathrm{KGL}$ and so $(\Sigma^\infty_{\mathrm{fr}}\mathrm{Vect})/\beta \simeq s_0\mathrm{KGL} \simeq H\mathbb{Z}$ (see, e.g., [1, Proposition 2.7]). ■

Proof of Theorem 1.1 (1). Note first that if $U \subset S$ is an open subscheme, and any of the assumptions of Theorem 1.1 holds for S, it also holds for U. On the other hand, if one of the conclusions holds for all U in an open cover, it holds for S. It follows that we may assume that S is qcqs (quasicompact quasiseparated), e.g., affine.

Since $f_1(H\mathbb{Z}) = 0$, we find (using Lemma 2.1) that

$$\beta\colon \Sigma^\infty_{\mathrm{fr}}\mathrm{Vect} \to \Sigma^{-2,-1}\Sigma^\infty_{\mathrm{fr}}\mathrm{Vect}$$

induces an equivalence on f_i for $i \geq 0$. It follows that in the directed system

$$\Sigma^\infty_{\mathrm{fr}}\mathrm{Vect} \xrightarrow{\beta} \Sigma^{-2,-1}\Sigma^\infty_{\mathrm{fr}}\mathrm{Vect} \xrightarrow{\beta} \Sigma^{-4,-2}\Sigma^\infty_{\mathrm{fr}}\mathrm{Vect} \xrightarrow{\beta} \cdots,$$

all maps induce an equivalence on f_0. Since the colimit is KGL, f_0 commutes with colimits (here we use that X is qcqs, via [4, Proposition A.3 (2)]) and $\Sigma^\infty_{\mathrm{fr}}\mathrm{Vect}$ is effective (like any framed suspension spectrum), the result follows. ■

The proof for KO is an elaboration on these ideas. From now on, we assume that $1/2 \in S$. Recall from [3, Definition 2.6 and Lemma 2.7] the motivic spectrum

$$\underline{k}^M \simeq (H\mathbb{Z}/2)/\tau \in \mathcal{SH}(S).$$

For the time being, assume S is a Dedekind scheme. Taking framed loops, we obtain

$$\underline{k}^M_1 := \Omega^\infty_{\mathrm{fr}}\Sigma^{1,1}\underline{k}^M \in \mathcal{P}_\Sigma(\mathrm{Cor}^{\mathrm{fr}}(S)).$$

Lemma 2.2. *Let S be a Dedekind scheme, $1/2 \in S$.*

(1) *We have $\underline{k}_1^M \simeq a_{\mathrm{Nis}}\tau_{\le 0}\mathbb{G}_m/2$, where $\mathbb{G}_m \in \mathcal{P}_\Sigma(\mathrm{Cor}^{\mathrm{fr}}(S))$ denotes the sheaf $\mathcal{O}^\times$ with its usual structure of transfers* [9, Example 2.4].

(2) *If $f: S' \to S$ is a morphism of Dedekind schemes, then $f^*\underline{k}_1^M \overset{\mathrm{mot}}{\simeq} \underline{k}_1^M \in \mathcal{P}_\Sigma(\mathrm{Cor}^{\mathrm{fr}}(S'))$.*

(3) *The canonical map $\Sigma^\infty_{\mathrm{fr}}\underline{k}_1^M \to \Sigma^{1,1}\underline{k}^M \in \mathcal{SH}(S)$ is an equivalence.*

For this and some of the following arguments, it will be helpful to recall that we have an embedding of $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)^{\mathrm{gp}}$ into the stable category of spectral presheaves on $\mathrm{Cor}^{\mathrm{fr}}(S)$. In particular, many fiber sequences in $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)$ are cofiber sequences.

Proof. (1) It is clear by construction since $H^1_{\mathrm{et}}(X, \mu_2) \simeq \mathcal{O}^\times(X)/2$ for affine X.

(2) By (1), we have a cofiber sequence $\Sigma\mu_2 \to a_{\mathrm{Nis}}\mathbb{G}_m/2 \to \underline{k}_1^M \in \mathcal{P}_\Sigma(\mathrm{Cor}^{\mathrm{fr}}(S))$. Since pullback of framed presheaves preserves cofiber sequences and commutes with forgetting transfers up to motivic equivalence [6, Lemma 16], we reduce to the same assertion about $\mathbb{G}_m$, μ_2, viewed as presheaves without transfers. Since they are representable, the assertion is clear.

(3) Using [4, Proposition B.3], (2) and [3, Theorem 4.4], we may assume that S is the spectrum of a perfect field. In this case, $\Sigma^\infty_{\mathrm{fr}}\Omega^\infty_{\mathrm{fr}} \simeq \widetilde{f_0}$ [5, Theorem 3.5.14 (i)], so we need only prove that $\Sigma^{1,1}\underline{k}_1^M$ is very effective. But this is clear since we have the cofiber sequence $\Sigma^{1,0}H\mathbb{Z}/2 \xrightarrow{\tau} \Sigma^{1,1}H\mathbb{Z}/2 \to \Sigma^{1,1}\underline{k}_1^M$ and $H\mathbb{Z}/2$ is very effective. ∎

Construction 2.3. The assignment $V \mapsto (V \oplus V^*, \varphi_V)$ sending a vector bundle to its associated (hyperbolic) symmetric bilinear bundle upgrades to a morphism

$$\mathrm{Vect} \to \mathrm{Bil} \in \mathcal{P}_\Sigma(\mathrm{Cor}^{\mathrm{fr}}(S))^{BC_2},$$

where Vect carries the C_2-action coming from passing to dual bundles, and Bil carries the trivial C_2-action.

Proof. Since the presheaves are 1-truncated, all the required coherence data can be written down by hand. ∎

Lemma 2.4. *Let S be a Dedekind scheme containing $1/2$.*

(1) *The map $(\mathrm{Vect}^{\mathrm{gp}})_{hC_2} \to \mathrm{Bil}^{\mathrm{gp}}$ induces an isomorphism on $a_{\mathrm{Nis}}\pi_i$ for $i = 1, 2$.*

(2) *The homotopy orbits spectral sequence yields $a_{\mathrm{Nis}}\pi_0(\mathrm{Vect}^{\mathrm{gp}})_{hC_2} \simeq \mathbb{Z}$, an exact sequence $0 \to \underline{k}_1^M \to a_{\mathrm{Nis}}\pi_1(\mathrm{Vect}^{\mathrm{gp}})_{hC_2} \to \mathbb{Z}/2 \to 0$ and a map $a_{\mathrm{Nis}}\pi_2(\mathrm{Vect}^{\mathrm{gp}})_{hC_2} \to \mathbb{Z}/2$, all as presheaves with framed transfers.*

Proof. (1) This follows from the cofiber sequence $K_{hC_2} \to \mathrm{GW} \to L$ [10, Theorem 7.6] using that $a_{\mathrm{Nis}}\pi_i L = 0$ unless $i \equiv 0 \pmod 4$.

(2) The homotopy orbit spectral sequence just arises from the Postnikov filtration of $\mathrm{Vect}^{\mathrm{gp}}$ and the formation of homotopy orbits and hence is compatible with transfers. Its E_2 page takes the form

$$H_i(C_2, a_{\mathrm{Nis}}\pi_j \mathrm{Vect}^{\mathrm{gp}}) \Rightarrow a_{\mathrm{Nis}}\pi_{i+j}(\mathrm{Vect}^{\mathrm{gp}})_{hC_2}.$$

The form of the differentials of the spectral sequence implies that $H_i(C_2, a_{\mathrm{Nis}}\pi_j \mathrm{Vect}^{\mathrm{gp}})$ consists of permanent cycles for $i \leq 1$, and survive to E_∞ for $(i, j) = (0, 0)$ and $(i, j) = (1, 1)$. One has $a_{\mathrm{Nis}}\pi_0\mathrm{Vect}^{\mathrm{gp}} = \mathbb{Z}$ with the trivial action and $a_{\mathrm{Nis}}\pi_1\mathrm{Vect}^{\mathrm{gp}} = \mathbb{G}_m$ [13, Lemma III.1.4] with the inversion action. This already yields the first assertion. A straightforward computation shows that

$$H_*(C_2, \mathbb{Z}) = \mathbb{Z},\ \mathbb{Z}/2,\ 0,\ \mathbb{Z}/2,\ \ldots$$

and

$$H_*(C_2, \mathbb{G}_m) = \underline{k}_1^M,\ \mu_2,\ \underline{k}_1^M,\ \ldots$$

Since $H_2(C_2, \mathbb{Z}) = 0$, no differential can hit the $(i, j) = (0, 1)$ spot either, yielding the second assertion. Moreover, this implies that $H_1(C_2, \mathbb{G}_m) = \mu_2$ is the bottom of the filtration of π_2. It follows that there is a map $a_{\mathrm{Nis}}\pi_2(\mathrm{Vect}^{\mathrm{gp}})_{hC_2} \to A$, where A is a quotient of μ_2. To prove that $A = \mu_2$, it suffices to check this on sections over a field, in which case we can use the hermitian motivic spectral sequence of [2]. ■

We have $a_{\mathrm{Nis}}\pi_0\mathrm{Bil}^{\mathrm{gp}} \simeq \underline{GW}$. Thus we can form the following filtration of $\mathrm{Bil}^{\mathrm{gp}}$ refining the Postnikov filtration

$$\mathrm{Bil}^{\mathrm{gp}} \leftarrow F_1\mathrm{Bil}^{\mathrm{gp}} \leftarrow F_2\mathrm{Bil}^{\mathrm{gp}} \leftarrow F_3\mathrm{Bil}^{\mathrm{gp}} \leftarrow F_4\mathrm{Bil}^{\mathrm{gp}} \in \mathcal{P}_\Sigma(\mathrm{Cor}^{\mathrm{fr}}(S))$$

with subquotients given Nisnevich-locally by

$$\underline{GW},\ \Sigma\mathbb{Z}/2,\ \Sigma\underline{k}_1^M,\ \Sigma^2\mathbb{Z}/2. \tag{2.1}$$

Recall also the framed presheaf $\mathrm{Alt} \in \mathcal{P}_\Sigma(\mathrm{Cor}^{\mathrm{fr}}(S))$ sending a scheme to the groupoid of vector bundles with a non-degenerate alternating form. Tensoring with the canonical alternating (virtual) form $H(1) - h$ on $H\mathbb{P}^1$ (where $H(1)$ is the tautological rank 2 alternating form on $H\mathbb{P}^1$, and h is the standard alternating form on a trivial vector bundle of rank 2) yields maps

$$\sigma_1\colon H\mathbb{P}^1 \wedge \mathrm{Alt}^{\mathrm{gp}} \to \mathrm{Bil}^{\mathrm{gp}} \quad \text{and} \quad \sigma_2\colon H\mathbb{P}^1 \wedge \mathrm{Bil}^{\mathrm{gp}} \to \mathrm{Alt}^{\mathrm{gp}}.$$

By construction, we have $\widetilde{\beta} = \sigma_1\sigma_2$ (recall that $H\mathbb{P}^1 \overset{\mathrm{mot}}{\simeq} S^{4,2}$).

Lemma 2.5. *Let S be a Dedekind scheme, $1/2 \in S$.*

(1) *The composite $H\mathbb{P}^1 \wedge \mathrm{Alt}^{\mathrm{gp}} \xrightarrow{\sigma_1} \mathrm{Bil}^{\mathrm{gp}} \to \mathrm{Bil}^{\mathrm{gp}}/F_4\mathrm{Bil}^{\mathrm{gp}}$ is motivically null. The induced map $\Sigma^\infty_{\mathrm{fr}}\mathrm{cof}(\sigma_1) \to \Sigma^\infty_{\mathrm{fr}}\mathrm{Bil}^{\mathrm{gp}}/F_4\mathrm{Bil}^{\mathrm{gp}}$ is an equivalence.*

(2) *The composite $H\mathbb{P}^1 \wedge \mathrm{Bil}^{\mathrm{gp}} \xrightarrow{\sigma_2} \mathrm{Alt}^{\mathrm{gp}} \xrightarrow{rk/2} \mathbb{Z}$ is motivically null. The induced map $\Sigma^\infty_{\mathrm{fr}}\mathrm{cof}(\sigma_2) \to \Sigma^\infty_{\mathrm{fr}}\mathbb{Z}$ is an equivalence.*

Proof. (1) Write C for the cofiber *computed in the category of spectral presheaves on* $\mathrm{Cor}^{\mathrm{fr}}(S)$. Then C admits a finite filtration, with subquotients corresponding to those in (2.1). Since each of those is the infinite loop space of a motivic spectrum, it follows that C is in fact motivically local. Consequently, C corresponds to $\mathrm{Bil}^{\mathrm{gp}}/F_4\mathrm{Bil}^{\mathrm{gp}}$

under the embedding into spectral presheaves. These contortions tell us that there are *fiber* sequences

$$F_{i+1}\mathrm{Bil}^{\mathrm{gp}}/F_4\mathrm{Bil}^{\mathrm{gp}} \to F_i\mathrm{Bil}^{\mathrm{gp}}/F_4\mathrm{Bil}^{\mathrm{gp}} \to F_i\mathrm{Bil}^{\mathrm{gp}}/F_{i+1}\mathrm{Bil}^{\mathrm{gp}}$$

for $i < 4$. Hence to prove that the composite is null, it suffices to prove that there are no maps from $\Sigma^{4,2}\mathrm{Alt}^{\mathrm{gp}}$ into the motivic localizations of the subquotients of the filtration given in (2.1). These motivic localizations are $\underline{GW}$, $L_{\mathrm{Nis}}K(\mathbb{Z}/2,1)$, $L_{\mathrm{Nis}}K(\underline{k}_1^M,1)$ and $L_{\mathrm{Nis}}K(\mathbb{Z}/2,2)$ (since they are motivically equivalent to the subquotients, and motivically local because they are infinite loop spaces of the motivic spectra $H\widetilde{\mathbb{Z}}$, $\Sigma\underline{k}^M$, $\Sigma^{2,1}\underline{k}^M$, $\Sigma^2\underline{k}^M$). It suffices to prove that $\Omega^{4,2}$ of these subquotients vanishes, which is clear. Next we claim that $\Sigma^\infty_{\mathrm{fr}}\mathrm{Bil}^{\mathrm{gp}}/F_4\mathrm{Bil}^{\mathrm{gp}}$ is stable under base change (among Dedekind schemes containing $1/2$). Indeed, the defining fiber sequences of $F_4\mathrm{Bil}^{\mathrm{gp}}$ are also cofiber sequences, and so $\Sigma^\infty_{\mathrm{fr}}\mathrm{Bil}^{\mathrm{gp}}/F_4\mathrm{Bil}^{\mathrm{gp}}$ is obtained by iterated extension from spectra stable under base change (see Lemma 2.2 (2) for $\underline{k}_1^M$, [8, proof of Lemma 7.5] for Bil and Alt, and [6, Lemma 16] for $\mathbb{Z}/2$). To prove that the induced map is an equivalence, we thus reduce as before to $S = \mathrm{Spec}(k)$, where k is a perfect field of characteristic $\neq 2$. In this case, the result is a straightforward consequence of the hermitian motivic filtration of [2].

(2) The proof is essentially the same as for (1), but easier. ∎

We now arrive at the main result.

Theorem 2.6. *Let S be a scheme containing $1/2$ such that*

$$f_1(H\mathbb{Z}) = 0 = H\mathrm{W}_{\geq 2} \in \mathcal{SH}(S).$$

The canonical maps

$$\Sigma^\infty_{\mathrm{fr}}\mathrm{Bil} \to \tilde{f}_0\mathrm{KO} \quad \textit{and} \quad \Sigma^\infty_{\mathrm{fr}}\mathrm{Alt} \to \tilde{f}_0\Sigma^{4,2}\mathrm{KO}$$

are equivalences.

Proof. As before, we may assume that S is qcqs.

We know that KO is the colimit of

$$\Sigma^\infty_{\mathrm{fr}}\mathrm{Bil} \xrightarrow{\sigma_2} \Sigma^{-4,-2}\Sigma^\infty_{\mathrm{fr}}\mathrm{Alt} \xrightarrow{\sigma_1} \Sigma^{-8,-4}\mathrm{Bil} \xrightarrow{\sigma_2} \cdots .$$

It is hence enough to prove that

$$\sigma_1\colon \Sigma^{-8n,-4n}\Sigma^\infty_{\mathrm{fr}}\mathrm{Bil} \to \Sigma^{-8n-4,-4n-2}\Sigma^\infty_{\mathrm{fr}}\mathrm{Alt}$$

induces an equivalence on $\tilde{f}_0$ for every $n \geq 0$, and similarly for σ_2. (Here we use that S is qcqs, so that $\tilde{f}_0$ preserves filtered colimits.) Given a cofiber sequence $A \to B \to C$, in order to prove that $\tilde{f}_0 A \simeq \tilde{f}_0 B$, it suffices to show that $\mathrm{Map}(X, C) = *$ for every $X \in \mathcal{SH}(S)^{\mathrm{veff}}$, i.e., that $C \in \mathcal{SH}(S)^{\mathrm{veff}\perp}$.

Over $\mathbb{Z}[1/2]$, the cofiber of σ_1 has a finite filtration, with subquotients

$$\Sigma^{-4,-2}\Sigma^\infty_{\mathrm{fr}}\underline{GW},\ \Sigma^{-3,-2}\Sigma^\infty_{\mathrm{fr}}\mathbb{Z}/2,\ \Sigma^{-3,-2}\Sigma^\infty_{\mathrm{fr}}\underline{k}_1^M,\ \Sigma^{-2,-2}\Sigma^\infty_{\mathrm{fr}}\mathbb{Z}/2,$$

and the cofiber of σ_2 is $\Sigma^{-4,-2}\Sigma^\infty_{\mathrm{fr}}\mathbb{Z}$. Using [6, Corollary 22], [8, Theorem 7.3] and Lemma 2.2 (3), we can identify the list of cofibers as

$$\Sigma^{-4,-2}H\widetilde{\mathbb{Z}},\ \Sigma^{-3,-2}H\mathbb{Z}/2,\ \Sigma^{-2,-1}\underline{k}^M,\ \Sigma^{-2,-2}H\mathbb{Z}/2,\ \Sigma^{-4,-2}H\mathbb{Z}.$$

These spectra are stable under arbitrary base change (essentially by definition), and hence for arbitrary S the cofibers of σ_1, σ_2 are obtained as finite extensions, with cofibers in the above list. To conclude the proof, it will thus suffice to show that all spectra in the above list are in $\mathcal{SH}(S)^{\mathrm{veff}\perp}$.

Note that if $E \in \mathcal{SH}(S)$, then $E \in \mathcal{SH}(S)^{\mathrm{veff}\perp}$ if and only if $\Omega^\infty E \simeq *$. In particular, this holds if $f_0 E = 0$. This holds for $\Sigma^{m,n} H\mathbb{Z}$ as soon as $n < 0$, by assumption. Hence it also holds for $\Sigma^{m,n} H\mathbb{Z}/2$ in the same case (f_0 being a stable functor) and for

$$\Sigma^{m,n}\underline{k}^M \simeq \mathrm{cof}(\Sigma^{m,n-1}H\mathbb{Z}/2 \xrightarrow{\tau} \Sigma^{m,n}H\mathbb{Z}/2).$$

The only spectrum left in our list is $\Sigma^{-4,-2}H\widetilde{\mathbb{Z}}$. Using [3, Definition 4.1], we see now that $\Omega^\infty\Sigma^{-4,-2}H\widetilde{\mathbb{Z}} \simeq \Omega^\infty\Sigma^{-4,-2}\underline{K}^W$, so we may treat the latter spectrum. We have $\underline{K}^W/\eta \simeq \underline{k}^M$ [3, Lemma 3.9], whence $\eta\colon \Sigma^{-4-n,-2-n}\underline{K}^W \to \Sigma^{-5-n,-3-n}\underline{K}^W$ induces an equivalence on Ω^∞. Since Ω^∞ commutes with filtered colimits, we see that $\Sigma^{-4,-2}\underline{K}^W \in \mathcal{SH}(S)^{\mathrm{veff}\perp}$ if and only if $\Sigma^{-4,-2}\underline{K}^W[\eta^{-1}] \in \mathcal{SH}(S)^{\mathrm{veff}\perp}$. This latter spectrum is the same as $\Sigma^{-2}H\mathrm{W}$ [3, Lemma 3.9], and

$$\tilde{f}_0(\Sigma^{-2}H\mathrm{W}) \simeq \tilde{f}_0((\Sigma^{-2}H\mathrm{W})_{\geq 0}) \simeq \tilde{f}_0(\Sigma^{-2}(H\mathrm{W}_{\geq 2})) = 0$$

by assumption. ∎

References

[1] Ananyevskiy, A., Röndigs, O., Østvær, P. A.: On very effective hermitian K-theory. Math. Z. **294**, 1021–1034 (2020) Zbl 1453.14064 MR 4074031

[2] Bachmann, T.: The generalized slices of Hermitian K-theory. J. Topol. **10**, 1124–1144 (2017) Zbl 1453.14065 MR 3743071

[3] Bachmann, T.: η-periodic motivic stable homotopy theory over Dedekind domains. J. Topol. **15**, 950–971 (2022) Zbl 1520.14043 MR 4441609

[4] Bachmann, T., Hoyois, M.: Norms in motivic homotopy theory. Astérisque **425**, ix+207 pp. (2021) Zbl 1522.14028 MR 4288071

[5] Elmanto, E., Hoyois, M., Khan, A. A., Sosnilo, V., Yakerson, M.: Motivic infinite loop spaces. Camb. J. Math. **9**, 431–549 (2021) Zbl 07422194 MR 4325285

[6] Hoyois, M.: The localization theorem for framed motivic spaces. Compos. Math. **157**, 1–11 (2021) Zbl 1455.14042 MR 4215649

[7] Hoyois, M., Jelisiejew, J., Nardin, D., Totaro, B., Yakerson, M.: The Hilbert scheme of infinite affine space and algebraic K-theory. J. Eur. Math. Soc. (JEMS) (2024)

[8] Hoyois, M., Jelisiejew, J., Nardin, D., Yakerson, M.: Hermitian K-theory via oriented Gorenstein algebras. J. Reine Angew. Math. **793**, 105–142 (2022) Zbl 1520.14046 MR 4513165

[9] Mazza, C., Voevodsky, V., Weibel, C.: Lecture notes on motivic cohomology. Clay Math. Monogr. 2, American Mathematical Society, Providence, RI (2006) Zbl 1115.14010 MR 2242284

[10] Schlichting, M.: Hermitian K-theory, derived equivalences and Karoubi's fundamental theorem. J. Pure Appl. Algebra **221**, 1729–1844 (2017) Zbl 1360.19008 MR 3614976

[11] Spitzweck, M., Østvær, P. A.: Motivic twisted K-theory. Algebr. Geom. Topol. **12**, 565–599 (2012) Zbl 1282.14040 MR 2916287

[12] Voevodsky, V.: Open problems in the motivic stable homotopy theory. I. In: Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998), Int. Press Lect. Ser. 3, International Press, Somerville, MA, 3–34 (2002) Zbl 1047.14012 MR 1977582

[13] Weibel, C. A.: The K-book: An introduction to algebraic K-theory. Grad. Stud. Math. 145, American Mathematical Society, Providence, RI (2013) Zbl 1273.19001 MR 3076731