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Qualitative analysis on the critical points of the Robin function

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Abstract. Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain with $N \geq 2$ and $\Omega_\varepsilon = \Omega \setminus B(P, \varepsilon)$, where $B(P, \varepsilon)$ is the ball centered at $P \in \Omega$ with radius ε. In this paper, we establish the number, location and non-degeneracy of critical points of the Robin function in Ω_ε for ε small enough. We will show that the location of P plays a crucial role in the existence and multiplicity of the critical points. The proof of our result is a consequence of delicate estimates on the Green's function near $\partial B(P, \varepsilon)$. Some applications to computing the exact number of solutions of related well-studied nonlinear elliptic problems are shown.

Keywords: Robin function, Green's function, critical points, non-degeneracy.

1. Introduction and main results

Let $D \subset \mathbb{R}^N$, $N \geq 2$, be a smooth domain. For $(x, y) \in D \times D$, $x \neq y$, denote by $G_D(x, y)$ the Green's function in D. It satisfies

$$\begin{cases} -\Delta_x G_D(x, y) = \delta_x(y) & \text{in } D, \\ G_D(x, y) = 0 & \text{on } \partial D, \end{cases}$$

in the sense of distribution. We have the classical representation formula

$$G_D(x, y) = S(x, y) - H_D(x, y), \tag{1.1}$$

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where $S(x, y)$ is the *fundamental solution* given by

$$S(x, y) = \begin{cases} -\dfrac{1}{2\pi} \ln |x - y| & \text{if } N = 2, \\ \dfrac{C_N}{|x - y|^{N-2}} & \text{if } N \geq 3, \end{cases}$$

where $C_N := \frac{1}{N(N-2)\omega_N}$, with ω_N the volume of the unit ball in $\mathbb{R}^N$. The function $H_D(x, y)$ is the *regular part of the Green's function* which is harmonic in both variables x and y. The *Robin* function is defined as

$$\mathcal{R}_D(x) := H_D(x, x) \quad \text{in } D.$$

Note that our definition differs, up to a multiplicative constant, from that of some other authors (see, for example, [2,6] where $\mathcal{R}_D(x) = 2\pi H_D(x, x)$ for $N = 2$). However, since we are interested in the critical points of the Robin function, this difference plays no role in our results.

The *Robin* function plays a fundamental role in a great number of problems (see [2,12] and the references therein). It also plays a role in the theory of conformal mappings and is closely related to the inner radius function (see [19]) and to some geometric quantities such as the capacity and transfinite diameter of sets (see [2, 12]). For elliptic problems involving critical Sobolev exponent [16,23], the number of solutions is linked to the number of non-degenerate critical points of the Robin function. Despite the great interest in the Robin function, many questions are still unanswered and we are far from a complete understanding of its properties.

The only smooth bounded domain where the Robin function is explicitly known is the ball centered at a point $Q \in \mathbb{R}^N$; in this case, the Robin function is *radial* and Q is the only critical point (it turns out to be non-degenerate). The computation of the number of critical points as well as other geometric properties (for example, the shape of level sets) in general domains of $\mathbb{R}^N$ interested a great number of experts in PDEs, but the results on this subject are very few. One additional difficulty is that it is not known if the Robin function satisfies some differential equation (see [2] for more information). This is known only for planar simply-connected domains [2] where it solves the Liouville equation.

To our knowledge, one of the first result in general domains is [6], in which Caffarelli and Friedman proved that the Robin function admits only one non-degenerate critical point in convex and bounded domain in $\mathbb{R}^2$. Note that here a crucial role is played by the Liouville equation. Later, the existence and uniqueness of critical points of the Robin function for a convex and bounded domain in higher dimension was proved in [10]. However, the question of non-degeneracy of this critical point is still open. Also some results on the non-degeneracy of the critical points of the Robin function for some symmetric domains can be found in [17]. For non-convex domains, for example, domains with "rich" topology, we cannot expect the uniqueness of the critical point of the Robin function. However, how the topology of the domain impacts the number of critical points of the Robin function is still unclear and seems to be a very difficult issue.

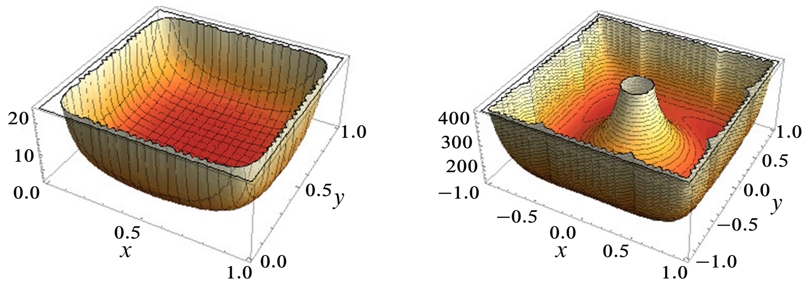

Fig. 1. Left: Robin function in Ω; right: Robin function in Ω_ε.

In this paper, we study what happens when we remove a small hole in the domain $\Omega \subset \mathbb{R}^N$. More precisely, denoting by $B(x_0, r)$ the ball centered at x_0 of radius r, set

$$\Omega_\varepsilon = \Omega \setminus B(P, \varepsilon) \quad \text{with } P \in \Omega. \tag{1.2}$$

Note that the Robin function $\mathcal{R}_{\Omega_\varepsilon}$ blows up at $\partial B(P, \varepsilon)$ (see Figure 1). So, for ε small enough, $\mathcal{R}_\Omega$ and $\mathcal{R}_{\Omega_\varepsilon}$ look very differently near P.

Our aim is to study the number of critical points of the Robin function $\mathcal{R}_{\Omega_\varepsilon}$ as well as their non-degeneracy. Observe that the regular part $H_{\Omega_\varepsilon}(x, y)$ satisfies

$$\begin{cases} \Delta H_{\Omega_\varepsilon}(x, y) = 0 & \text{in } \Omega_\varepsilon, \\ H_{\Omega_\varepsilon}(x, y) = S(x, y) & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Hence, by the standard regularity theory, we have that

$$H_{\Omega_\varepsilon}(x, y) \to H_\Omega(x, y) \tag{1.3}$$

in any compact set $K \subset \overline{\Omega} \setminus B(P, r)$ as $\varepsilon \to 0$ for some small *fixed* $r > 0$. Setting $x = y$ in (1.3), we get that $\mathcal{R}_{\Omega_\varepsilon}(x) \to \mathcal{R}_\Omega(x)$ for any $x \in K$, which is a good information about the behavior of $\mathcal{R}_{\Omega_\varepsilon}$ *far away* from P. The behavior of $\mathcal{R}_{\Omega_\varepsilon}$ close to ∂B_ε is much more complicated and is the most delicate problem to be addressed in this paper. Here a careful use of the maximum principle for harmonic functions will be crucial. Actually, sharp estimates up to $\partial B(P, \varepsilon)$ will allow us to prove the existence of critical points for $\mathcal{R}_{\Omega_\varepsilon}$ which converge to P as $\varepsilon \to 0$.

Our first result emphasizes the role of the center of the ball $B(P, \varepsilon)$. Indeed, the scenarios are very different depending on whether P is a critical point of $\mathcal{R}_\Omega$ or not.

In all the paper, we denote, for $x \in \Omega_\varepsilon$, by $O(f(\varepsilon, x))$ a quantity such that

$$O(f(\varepsilon, x)) \leq C|f(\varepsilon, x)|,$$

where C is a constant independent of ε and x.

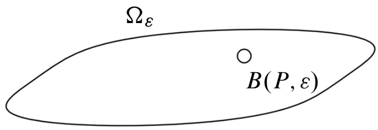

Fig. 2. An example of a domain Ω_ε.

Theorem 1.1. *Suppose Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, with $P \in \Omega$ (see Figure 2). If $\nabla \mathcal{R}_\Omega(P) \neq 0$, then for ε small enough,*

$$\sharp\{\text{critical points of } \mathcal{R}_{\Omega_\varepsilon} \text{ in } B(P,r) \setminus B(P,\varepsilon)\} = 1,$$

where $B(P,r) \subset \Omega$ is chosen not containing any critical point of $\mathcal{R}_\Omega$. Moreover, the critical point $x_\varepsilon \in B(P,r)$ of $\mathcal{R}_{\Omega_\varepsilon}$ satisfies, for ε small enough, the following:

(P_1) *The asymptotic behavior of x_ε is*

$$x_\varepsilon = P + \begin{cases} \varepsilon^{\frac{N-2}{2N-3}}\Big(\Big(\dfrac{2}{N\omega_N|\nabla \mathcal{R}_\Omega(P)|^{2N-2}}\Big)^{\frac{1}{2N-3}} \nabla \mathcal{R}_\Omega(P) + o(1)\Big) & \text{for } N \geq 3, \\ r_\varepsilon\Big(\dfrac{\nabla \mathcal{R}_\Omega(P)}{\pi^2|\nabla \mathcal{R}_\Omega(P)|^2} + o(1)\Big) & \text{for } N = 2, \end{cases} \tag{1.4}$$

where r_ε is the unique solution of the equation

$$r - \frac{\ln r}{\ln \varepsilon} = 0 \quad \text{in } (0,\infty). \tag{1.5}$$

(P_2) *x_ε is a non-degenerate critical point with* $\text{index}_{x_\varepsilon}(\nabla \mathcal{R}_{\Omega_\varepsilon}) = (-1)^{N+1}$.

(P_3) *$\mathcal{R}_{\Omega_\varepsilon}(x_\varepsilon) \to \mathcal{R}_\Omega(P)$.*

Remark 1.2. The condition $\nabla \mathcal{R}_\Omega(P) \neq 0$ cannot be removed. Indeed, if $\Omega = B(0,R)$, we know that 0 is the unique critical point of $\mathcal{R}_\Omega$, and in the shrinking annulus $\Omega_\varepsilon = B(0,R) \setminus B(0,\varepsilon)$ we have that $\mathcal{R}_{\Omega_\varepsilon}$ is radial with respect to the origin. Since $\mathcal{R}_{\Omega_\varepsilon}|_{\partial\Omega_\varepsilon} = +\infty$, we have that the set of minima of $\mathcal{R}_{\Omega_\varepsilon}$ is a sphere, and then $\mathcal{R}_{\Omega_\varepsilon}$ admits infinitely many minima.

Remark 1.3. Although $\mathcal{R}_{\Omega_\varepsilon}|_{\partial B_\varepsilon} = +\infty$, we have by (P_3) that $\mathcal{R}_{\Omega_\varepsilon}(x_\varepsilon) \leq C$. Roughly speaking, this means that x_ε is not "so close" to $\partial B(P,\varepsilon)$ (actually, by (P_1) we have that $\frac{|x_\varepsilon - P|}{\varepsilon} \to +\infty$).

Remark 1.4. Let us give the idea of the proof of the theorem. Our starting point is the following basic representation formula for the gradient of the Robin function (see [13,24] for $N = 2,3$ and [4] for any $N \geq 2$),

$$\begin{aligned} \nabla \mathcal{R}_{\Omega_\varepsilon}(x) &= \int_{\partial\Omega_\varepsilon} \nu_\varepsilon(y)\Big(\frac{\partial G_{\Omega_\varepsilon}(x,y)}{\partial \nu_y}\Big)^2 d\sigma_y \qquad (1.6)\\ &= \int_{\partial\Omega} \nu(y)\Big(\frac{\partial G_{\Omega_\varepsilon}(x,y)}{\partial \nu_y}\Big)^2 d\sigma_y - \int_{\partial B(P,\varepsilon)} \frac{y-P}{\varepsilon}\Big(\frac{\partial G_{\Omega_\varepsilon}(x,y)}{\partial \nu_y}\Big)^2 d\sigma_y, \end{aligned}$$

where $\nu_\varepsilon(y)$ and $\nu(y)$ are the outer unit normal to $\partial\Omega_\varepsilon$ and $\partial\Omega$, respectively. By (1.6), we will derive some C^1-estimates which are crucial to prove our results and, in our opinion, have an independent interest. Let us start to discuss the case $N \geq 3$, where we get, *uniformly* for $x \in \Omega_\varepsilon$,

$$\nabla\mathcal{R}_{\Omega_\varepsilon}(x) = \nabla\mathcal{R}_\Omega(x) + \underbrace{\nabla\mathcal{R}_{\mathbb{R}^N\setminus B(P,\varepsilon)}(x)}_{=-\frac{2\varepsilon^{N-2}}{N\omega_N}\frac{x-P}{(|x-P|^2-\varepsilon^2)^{N-1}}} + O\Big(\frac{\varepsilon^{N-2}}{|x-P|^{N-1}}\Big) + O(\varepsilon). \quad (1.7)$$

The previous estimate is a *second-order* expansion of the Robin function in Ω_ε.

It turns out that $\nabla\mathcal{R}_\Omega(x)$ and $\nabla\mathcal{R}_{\mathbb{R}^N\setminus B(P,\varepsilon)}(x)$ are the leading terms of the expansion of $\nabla\mathcal{R}_{\Omega_\varepsilon}(x)$.

Note that from (1.7) we get that

$$\nabla\mathcal{R}_{\Omega_\varepsilon}(x) \to \nabla\mathcal{R}_\Omega(x)$$

uniformly on the compact sets of Ω not containing P. So, under some non-degeneracy assumptions on the critical points of $\mathcal{R}_\Omega(x)$, we get that the number of critical points of $\mathcal{R}_{\Omega_\varepsilon}(x)$ far away from P is the same of $\mathcal{R}_\Omega(x)$. Next let us study what happens when $x \to P$. Here the analysis is very delicate but formally, using the uniform convergence in (1.7), we get

$$\begin{aligned}\nabla\mathcal{R}_{\Omega_\varepsilon}(P + \varepsilon^{\frac{N-2}{2N-3}} y) &\sim \nabla\mathcal{R}_\Omega(P) - \frac{2}{N\omega_N}\frac{y}{|y|^{2N-2}} \\ &= \nabla\Big(\sum_{j=1}^N \frac{\partial\mathcal{R}_\Omega(P)}{\partial x_j} y_j - \frac{2}{N\omega_N(4-2N)}\frac{1}{|y|^{2N-4}}\Big).\end{aligned}$$

Since the function

$$F(y) = \sum_{j=1}^N \frac{\partial\mathcal{R}_\Omega(P)}{\partial x_j} y_j - \frac{2}{N\omega_N(4-2N)}\frac{1}{|y|^{2N-4}}$$

admits a unique non-degenerate critical point with index $(-1)^{N+1}$, we get that the same holds for $\mathcal{R}_{\Omega_\varepsilon}$, which proves (P_1) and (P_2) for $N \geq 3$.

The case $N = 2$ is a little bit more complicated because $\mathcal{R}_{\mathbb{R}^2\setminus B(P,\varepsilon)}(x)$ does not goes to 0 as $\varepsilon \to 0$. Actually, an additional term appears in the expansion,

$$\nabla\mathcal{R}_{\Omega_\varepsilon}(x) = \nabla\mathcal{R}_\Omega(x) + \underbrace{\nabla\mathcal{R}_{\mathbb{R}^2\setminus B(P,\varepsilon)}(x)}_{=-\frac{1}{\pi}\frac{x-P}{|x-P|^2-\varepsilon^2}} + \frac{1}{\pi}\Big(1 - \frac{\ln|x-P|}{\ln\varepsilon}\Big)\frac{x-P}{|x-P|^2} + O\Big(\frac{1}{|x-P||\ln\varepsilon|}\Big). \quad (1.8)$$

However, up to some technicalities, the proof follows the same line as in the case $N \geq 3$.

Finally, let us point out that the maximum principle plays a crucial role in getting uniform estimates up to $\partial\Omega$ in (1.7) and (1.8).

Remark 1.5. An interesting asymptotic formula for the Robin function in Ω_ε is the following (see [2, pp. 198–199]): for every $x \neq P$ and $N \geq 3$,

$$\mathcal{R}_{\Omega_\varepsilon}(x) = \mathcal{R}_\Omega(x) + \frac{G_\Omega^2(x,P)}{N(N-2)\omega_N(\varepsilon^{2-N} - \mathcal{R}_\Omega(P))} + O(\varepsilon^{N-1}), \tag{1.9}$$

(an analogous formula holds for $N = 2$). Formula (1.9) is a consequence of the Schiffer–Spencer formula (see [25] for $N = 2$ and [22] for $N \geq 3$) but the remainder term $O(\varepsilon^{N-1})$ is not uniform with respect to x (as stated in [22, p. 771]).

In Proposition 3.1, we prove the following:

$$\mathcal{R}_{\Omega_\varepsilon}(x) = \mathcal{R}_\Omega(x) + \mathcal{R}_{B_\varepsilon^c}(x) + \begin{cases} O\Big(\dfrac{\varepsilon^{N-2}}{|x-P|^{N-2}}\Big) + O(\varepsilon) & \text{for } N \geq 3, \\ -\dfrac{1}{2\pi}\dfrac{(\ln\frac{|x-P|}{\varepsilon})^2}{\ln\varepsilon} + O\Big(\Big|\dfrac{\ln|x-P|}{\ln\varepsilon}\Big|\Big) & \text{for } N = 2, \end{cases}$$

where the remainder terms are uniform with respect to $x \in \Omega_\varepsilon$. This can be seen as an extension of (1.9).

Theorem 1.1 states the *uniqueness* and *non-degeneracy* of the critical points of $\mathcal{R}_{\Omega_\varepsilon}$ near the hole $B(P,\varepsilon)$. Under a non-degeneracy condition on the critical points of $\mathcal{R}_\Omega$, we can compute the exact number of the critical points of $\mathcal{R}_{\Omega_\varepsilon}$ in Ω_ε.

Corollary 1.6. *Suppose that Ω and Ω_ε are the domains as in Theorem 1.1. If*

$$\nabla\mathcal{R}_\Omega(P) \neq 0$$

and all critical points of $\mathcal{R}_\Omega$ in Ω are non-degenerate, then for ε small enough, all critical points of $\mathcal{R}_{\Omega_\varepsilon}$ are non-degenerate and

$$\sharp\{\text{critical points of } \mathcal{R}_{\Omega_\varepsilon} \text{ in } \Omega_\varepsilon\} = \sharp\{\text{critical points of } \mathcal{R}_\Omega \text{ in } \Omega\} + 1.$$

As previously mentioned, the Robin function of the ball $B(0,R)$ has a unique non-degenerate critical point. So Corollary 1.6 applies and then if $P \neq 0$, the Robin function $\mathcal{R}_{\Omega_\varepsilon}$ has *two* non-degenerate critical points. On the other hand, if $P = 0$, we are in the situation described in Remark 1.2. Hence a nice consequence of the previous corollary is the following one.

Corollary 1.7. *Assume that $\Omega = B(0,R) \subset \mathbb{R}^N$, $N \geq 2$, and $\Omega_\varepsilon = B(0,R) \setminus B(P,\varepsilon)$. Then, for ε small enough,*

$$\sharp\{\text{critical points of } \mathcal{R}_{\Omega_\varepsilon} \text{ in } \Omega_\varepsilon\} = \begin{cases} 2 & \text{if } P \neq 0, \\ \infty & \text{if } P = 0, \end{cases}$$

and if $P \neq 0$, the two critical points are non-degenerate (see Figure 3).

In next theorem, we study what happens when $\nabla\mathcal{R}_\Omega(P) = 0$. This case is more delicate and it seems very hard to give a complete answer. However, in some cases it is possible to compute the number of the critical points, as stated in the following.

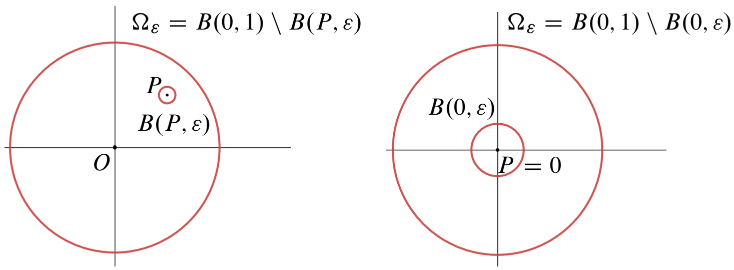

Fig. 3. Left: $P \neq 0$ (two critical points); right: $P = 0$ (infinitely many critical points).

Theorem 1.8. *Suppose Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, and $P \in \Omega$. Suppose*

- $\nabla \mathcal{R}_\Omega(P) = 0$.
- *The critical point P is non-degenerate, i.e.,* $\det(\mathrm{Hess}(\mathcal{R}_\Omega(P))) \neq 0$.
- *The Hessian matrix* $\mathrm{Hess}(\mathcal{R}_\Omega(P))$ *has $m \leq N$ positive eigenvalues*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$$

with associated eigenvectors $v_1, \ldots, v_m$ with $|v_i| = 1$ for $i = 1, \ldots, m$.

Assume that $B(P,r) \subset \Omega$ admits P as the only critical point of $\mathcal{R}_\Omega$. If the eigenvalue λ_l is simple for some $l \in \{1, \ldots, m\}$, then we have two non-degenerate critical points $x^\pm_{l,\varepsilon}$ of $\mathcal{R}_{\Omega_\varepsilon}(x)$ in $B(P,r) \setminus B(P,\varepsilon)$ which satisfy

$$x^\pm_{l,\varepsilon} = P \pm \begin{cases} \left(\dfrac{2 + o(1)}{N\omega_N \lambda_l}\right)^{\frac{1}{2N-2}} \varepsilon^{\frac{N-2}{2N-2}} v_l & \textit{if } N \geq 3, \\ \hat{r}_{\varepsilon,l}(1 + o(1)) v_l & \textit{if } N = 2, \end{cases} \tag{1.10}$$

where v_l is the l-th eigenvector associated to λ_l with $|v_l| = 1$, $\hat{r}_{\varepsilon,l}$ is the unique solution of

$$r^2 - \frac{\ln r}{\lambda_l \pi \ln \varepsilon} = 0 \quad \textit{in } (0, \infty).$$

Finally, it holds

$$\mathcal{R}_{\Omega_\varepsilon}(x^\pm_{l,\varepsilon}) \to \mathcal{R}_\Omega(P). \tag{1.11}$$

Moreover, if all the positive eigenvalues of $\mathcal{R}_\Omega(P)$ are simple, we have that

$$\sharp\{\text{critical points of } \mathcal{R}_{\Omega_\varepsilon}(x) \text{ in } B(P,r) \setminus B(P,\varepsilon)\} = 2m \tag{1.12}$$

and all critical points satisfy (1.10) *for $l = 1, \ldots, m$.*

Remark 1.9. The condition $\det(\mathrm{Hess}\, \mathcal{R}_\Omega(P)) \neq 0$ is satisfied by many examples. In [21], it was proved that it holds up small perturbations of the domain Ω. Note that only positive eigenvalue of the Hessian matrix of $\mathcal{R}_\Omega(P)$ “generates” critical points for $\mathcal{R}_\Omega$ (see Proposition 5.1). Hence saddle points of $\mathcal{R}_\Omega$ give less contribution to the number of critical points of $\mathcal{R}_{\Omega_\varepsilon}$.

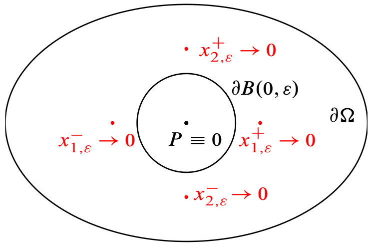

Fig. 4. An example of a two-dimensional domain Ω_ε and the new critical points $x^{\pm}_{1,\varepsilon}$ and $x^{\pm}_{2,\varepsilon}$.

Corollary 1.10. *Let Ω be a convex and symmetric domain (see Gidas, Ni and Nirenberg* [14]*) with respect to the origin (see Figure* 4*). We have that*

(i) *If $P \neq 0$, then $\mathcal{R}_{\Omega_\varepsilon}(x)$ admits exactly two non-degenerate critical points in Ω_ε.*

(ii) *If $P = 0$ and all the eigenvalues of $\mathrm{Hess}(\mathcal{R}_\Omega(0))$ are simple, then $\mathcal{R}_{\Omega_\varepsilon}(x)$ admits exactly $2N$ non-degenerate critical points in Ω_ε.*

Remark 1.11. An example of a domain Ω which satisfies the conditions of Theorem 1.8 for $N \geq 2$ is the following (see Section 6):

$$\Omega_\delta = \left\{x \in \mathbb{R}^N, \sum_{i=1}^N x_i^2(1+\alpha_i\delta)^2 < 1 \text{ with } \delta > 0 \text{ and } 0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_N\right\}.$$

We will give a precise description of the Robin function $\mathcal{R}_{\Omega_\delta}$ for small δ in Theorem 6.1.

Remark 1.12. The proof of Theorem 1.8 uses again estimates (1.7) and (1.8). In this case, we get that

$$\frac{\partial \mathcal{R}_{\Omega_\varepsilon}(P + \varepsilon^{\frac{N-2}{2N-2}} y)}{\partial x_i} \sim \sum_{j=1}^N \frac{\partial^2 \mathcal{R}_\Omega(P)}{\partial x_i \partial x_j} y_j - \frac{2}{N\omega_N} \frac{y_i}{|y|^{2N-2}} \quad \text{for } N \geq 2.$$

In other words, after diagonalization we get that

$$\nabla \mathcal{R}_{\Omega_\varepsilon}(P + \varepsilon^{\frac{N-2}{2N-2}} y) \sim \nabla\left(\sum_{j=1}^N \lambda_j y_j^2 - \frac{2}{N\omega_N(4-2N)} \frac{1}{|y|^{2N-4}}\right) \quad \text{for } N \geq 3.$$

Note that if $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m$ and $\lambda_j < 0$ for $j = m+1, \ldots, N$, then the function

$$F(y) = \sum_{j=1}^N \lambda_j y_j^2 - \frac{2}{N\omega_N(4-2N)} \frac{1}{|y|^{2N-4}}$$

admits $2m$ non-degenerate critical points and the claim follows as in Remark 1.4. On the other hand, if some positive eigenvalue is multiple, say $\lambda_1 = \lambda_2 = \cdots = \lambda_k > 0$, then the function F is spherically symmetric in $(y_1, y_2, \ldots, y_k)$. This implies that there is a set of critical points of F given by a sphere S^k. Since we have a manifold of critical points, the number of the critical points depends on the approximation function and it leads to a *third-order* expansion of $\mathcal{R}_{\Omega_\varepsilon}(x)$. Further considerations on multiple eigenvalues deserve to be studied apart. However, when Ω is a symmetric domain, we obtain partial results on the critical points of $\mathcal{R}_{\Omega_\varepsilon}(x)$ (see Theorem 5.7).

Remark 1.13. Our results can be iterated to handle the case in which k ($k \geq 2$) small holes are removed from Ω. Moreover, using similar ideas, our main theorems are true if we replace $B(P, \varepsilon)$ by a small convex set.

As in the case $\nabla \mathcal{R}_\Omega(P) = 0$, we have the following corollary.

Corollary 1.14. *Suppose Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, and $P \in \Omega$. If $\nabla \mathcal{R}_\Omega(P) = 0$, all critical points of $\mathcal{R}_\Omega$ are non-degenerate and* $\mathrm{Hess}(\mathcal{R}_\Omega(P))$ *has $m \leq N$ positive eigenvalues which are all simple, and for small ε it holds*

$$\sharp\{\text{critical points of } \mathcal{R}_{\Omega_\varepsilon}(x) \text{ in } \Omega_\varepsilon\} = \sharp\{\text{critical points of } \mathcal{R}_\Omega(x) \text{ in } \Omega\} + 2m - 1.$$

Finally, all critical points of $\mathcal{R}_{\Omega_\varepsilon}(x)$ are non-degenerate.

As said before, the non-degeneracy of critical points of the Robin function plays an important role in PDEs. Now we would like to give applications of our results to some elliptic problems. For example, let us consider the following:

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega_\varepsilon, \\ \quad u > 0 & \text{in } \Omega_\varepsilon, \\ \quad u = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{1.13}$$

where the solution $u_{\varepsilon,p}$ satisfies either

$$\lim_{p \to +\infty} p \int_{\Omega_\varepsilon} |\nabla u_{\varepsilon,p}|^2 dx = 8\pi e \quad \text{for } N = 2 \tag{1.14}$$

or

$$\lim_{p \to \frac{N+2}{N-2}^-} \frac{\int_{\Omega_\varepsilon} |\nabla u_{\varepsilon,p}|^2 dx}{(\int_{\Omega_\varepsilon} u_{\varepsilon,p}^{p+1} dx)^{\frac{2}{p+1}}} = S \quad \text{for } N \geq 3, \tag{1.15}$$

with S the best constant in Sobolev inequality. We have the following results.

Theorem 1.15. *Suppose $N = 2$ or $N \geq 4$, $\Omega \subset \mathbb{R}^N$ is a convex domain and the critical point of $\mathcal{R}_\Omega(x)$ is non-degenerate (the uniqueness was proved in* [5, 10]*, the non-degeneracy in* [5] *for $N = 2$). Then there exists some $\varepsilon_0 > 0$ such that for any fixed $\varepsilon \in (0, \varepsilon_0]$, p large for $N = 2$ or $\frac{N+2}{N-2} - p > 0$ small for $N \geq 4$,*

(1) *if $\nabla \mathcal{R}_\Omega(P) \neq 0$, we have exactly two solutions of* (1.13) *satisfying* (1.14) *for $N = 2$ or* (1.15) *for $N \geq 4$ (see Figure 5);*

(2) *if $\nabla \mathcal{R}_\Omega(P) = 0$ and all the eigenvalues of $\nabla^2 \mathcal{R}_\Omega(P)$ are simple, then we have exactly $2N$ solutions of* (1.13) *satisfying* (1.14) *for $N = 2$ or* (1.15) *for $N \geq 4$.*

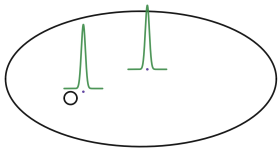

Fig. 5. Picture of two solutions concentrating at critical points of $\mathcal{R}_{\Omega_\varepsilon}$.

Proof of Theorem 1.15. Firstly, let us fix $\varepsilon \in (0, \varepsilon_0]$ such that Theorems 1.1 and 1.8 apply. Then if $\nabla \mathcal{R}_\Omega(P) \neq 0$, from Corollary 1.6 we get that $\mathcal{R}_{\Omega_\varepsilon}$ admits exactly two critical points in Ω_ε which are non-degenerate. If $\nabla \mathcal{R}_\Omega(P) = 0$ and all the eigenvalues of $\nabla^2 \mathcal{R}_\Omega(P)$ are simple, then Theorem 1.8 gives us that $\mathcal{R}_{\Omega_\varepsilon}$ admits exactly $2N$ critical points in Ω_ε which are non-degenerate. Next it is known by [11, 20, 23] that the solutions of (1.13) with (1.14) or (1.15) concentrate at critical points of $\mathcal{R}_{\Omega_\varepsilon}$ when p is large for $N = 2$ or $\frac{N+2}{N-2} - p > 0$ is small for $N \geq 4$. Moreover, from [1, 18], using the non-degeneracy assumption of the critical points of $\mathcal{R}_{\Omega_\varepsilon}$, we have the *local* uniqueness of these solutions. ■

Remark 1.16. Observe that in the above corollary, the assumption $N \geq 4$ instead of the natural one $N \geq 3$ is due to technical reason in proving the uniqueness result in [1]. If the uniqueness result in [1] is extended to $N = 3$, we will get the claim also in this case.

Similar applications can also be given on Brezis–Nirenberg problem [3, 16], planar vortex patch in incompressible steady flow [7, 9] and plasma problem [5, 8] for some non-convex domains.

The paper is organized as follows. In Section 2, we prove some lemmas which estimate the regular part H_{Ω_ε} in terms of H_Ω and $H_{\mathbb{R}^N \setminus B(P,\varepsilon)}$. Here the maximum principle for harmonic functions allows us to get *uniform* estimates for $\mathcal{R}_{\Omega_\varepsilon}$ up to $\partial B(P, \varepsilon)$. These will be the basic tools to give the expansion of $\mathcal{R}_{\Omega_\varepsilon}$ and its derivatives which will be proved in Section 3. In Section 4, we consider the case $\nabla \mathcal{R}_\Omega(P) \neq 0$ and prove Theorem 1.1 and Corollary 1.6. In Section 5, we consider the case $\nabla \mathcal{R}_\Omega(P) = 0$ and prove Theorem 1.8 and Corollaries 1.10 and 1.14. In Section 6, we will give an example of domains which satisfy the assumptions of Theorem 1.8. Finally, in the appendix we recall some known properties of the Robin function in the exterior of the ball as well as some useful identities involving the Green's function.

2. Uniform estimates on the regular part of the Green's function

Set $\Omega_\varepsilon = \Omega \setminus B_\varepsilon$, where $B_\varepsilon = B(P, \varepsilon)$ with $P \in \Omega$. Observe that $B_\varepsilon \subset \Omega$ for ε sufficiently small, and we will always assume this is the case. We denote by

$$B_\varepsilon^c := \mathbb{R}^N \setminus B_\varepsilon$$

and without loss of generality, we take $P = 0 \in \Omega$.

In this section, we will prove two crucial lemmas that will be repeatedly used in the proof of our expansion of the Robin function and its derivatives. In order to clarify their role, let us write down the following representation formula for the gradient of the Robin function proved in [4, p. 170] for $x \in \Omega_\varepsilon$ and letting ν_y be the outer unit normal to the boundary of the domain,

$$\begin{aligned}\nabla \mathcal{R}_{\Omega_\varepsilon}(x) &= \int_{\partial\Omega_\varepsilon} \nu(y)\Big(\frac{\partial G_{\Omega_\varepsilon}(y,x)}{\partial \nu_x}\Big)^2 d\sigma_y = \int_{\partial\Omega_\varepsilon} \nu(y)\Big(\frac{\partial G_{\Omega_\varepsilon}(x,y)}{\partial \nu_y}\Big)^2 d\sigma_y \\ &= \int_{\partial\Omega} \nu(y)\Big(\frac{\partial G_{\Omega_\varepsilon}(x,y)}{\partial \nu_y}\Big)^2 d\sigma - \int_{\partial B_\varepsilon} \frac{y}{\varepsilon}\Big(\frac{\partial G_{\Omega_\varepsilon}(x,y)}{\partial \nu_y}\Big)^2 d\sigma_y. \end{aligned} \tag{2.1}$$

Using the identities $G_{\Omega_\varepsilon} = G_\Omega + H_\Omega - H_{\Omega_\varepsilon}$ and $G_{\Omega_\varepsilon} = G_{B_\varepsilon^c} + H_{B_\varepsilon^c} - H_{\Omega_\varepsilon}$,

$$\begin{aligned}\text{r.h.s.} = &\underbrace{\int_{\partial\Omega} \nu(y)\Big(\frac{\partial G_\Omega(x,y)}{\partial \nu_y}\Big)^2 d\sigma_y}_{=\nabla\mathcal{R}_\Omega(x)} + \int_{\partial\Omega} \nu(y)\Big(\frac{\partial (H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial \nu_y}\Big)^2 d\sigma_y \\ &+ 2\int_{\partial\Omega} \nu(y)\frac{\partial G_\Omega(x,y)}{\partial \nu_y}\frac{\partial (H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial \nu_y} d\sigma_y \\ &- \underbrace{\int_{\partial B_\varepsilon} \frac{y}{\varepsilon}\Big(\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y}\Big)^2 d\sigma_y}_{=-\nabla\mathcal{R}_{B_\varepsilon^c}(x) \text{ by Lemma A.2}} - \int_{\partial B_\varepsilon} \frac{y}{\varepsilon}\Big(\frac{\partial (H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial \nu_y}\Big)^2 d\sigma_y \\ &- 2\int_{\partial B_\varepsilon} \frac{y}{\varepsilon}\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y}\frac{\partial (H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial \nu_y} d\sigma_y. \end{aligned}$$

Hence in order to estimate the previous integrals, we need to known the behavior of $H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y)$ on $\partial\Omega$ and $H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y)$ on ∂B_ε, respectively. It will be done in the next lemmas.

Lemma 2.1. *For any $x \in \Omega_\varepsilon$ and $y \in \overline{\Omega}_\varepsilon$ with $|y| \geq C_0 > 0$ and $i = 1, \ldots, N$, it holds*

$$\frac{\partial (H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial y_i} = \begin{cases} O\Big(\dfrac{\varepsilon^{N-2}}{|x|^{N-2}}\Big) + O(\varepsilon) & \text{for } N \geq 3, \\ O\Big(\Big|\dfrac{\ln|x|}{\ln\varepsilon}\Big|\Big) + O\Big(\dfrac{1}{|\ln\varepsilon|}\Big) & \text{for } N = 2. \end{cases} \tag{2.2}$$

Proof. Let us point out that the functions $H_\Omega(x,y)$ and $H_{\Omega_\varepsilon}(x,y)$ are well defined if $x \in \Omega_\varepsilon$.

For any $y \in \overline{\Omega}$ with $|y| \geq C_0 > 0$, we have that

$$\begin{cases} \Delta_x(H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y)) = 0 & \text{for } x \in \Omega\setminus B_\varepsilon, \\ H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y) = 0 & \text{for } x \in \partial\Omega, \\ H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y) = -G_\Omega(x,y) & \text{for } x \in \partial B_\varepsilon. \end{cases}$$

By the representation formula for harmonic function, we obtain

$$H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y) = \int_{\partial B_\varepsilon} \frac{\partial G_{\Omega_\varepsilon}(x,t)}{\partial \nu_t} G_\Omega(y,t) d\sigma_t.$$

Since $G_\Omega(y,t)$ has no singularity if $t \in \partial B_\varepsilon$ and $|y| \geq C_0 > 0$, we see that $\nabla_y(H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))$ is well defined for $y \in \overline{\Omega}$ with $|y| \geq C_0 > 0$.

Now fix $y \in \Omega$ with $|y| \geq C_0 > 0$ and observe that

$$\begin{cases} \Delta_x \dfrac{\partial(H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial y_i} = 0 & \text{for } x \in \Omega \setminus B_\varepsilon, \\ \dfrac{\partial(H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial y_i} = 0 & \text{for } x \in \partial\Omega, \\ \dfrac{\partial(H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial y_i} = -\dfrac{\partial G_\Omega(x,y)}{\partial y_i} & \text{for } x \in \partial B_\varepsilon. \end{cases}$$

For $N \geq 3$, we consider, in Ω_ε, the function

$$b_\varepsilon(x,y) := \frac{\partial(H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial y_i} + \frac{\varepsilon^{N-2}}{|x|^{N-2}} \frac{\partial G_\Omega(0,y)}{\partial y_i}.$$

We have

$$\begin{cases} \Delta_x b_\varepsilon(x,y) = 0 & \text{for } x \in \Omega \setminus B_\varepsilon, \\ b_\varepsilon(x,y) = \dfrac{\varepsilon^{N-2}}{|x|^{N-2}} \dfrac{\partial G_\Omega(0,y)}{\partial y_i} = O(\varepsilon^{N-2}) & \text{for } x \in \partial\Omega, \\ b_\varepsilon(x,y) = \underbrace{-\dfrac{\partial G_\Omega(x,y)}{\partial y_i} + \dfrac{\partial G_\Omega(0,y)}{\partial y_i}}_{=O(\varepsilon) \text{ uniformly for } |y| \geq C_0 > 0} & \text{for } x \in \partial B_\varepsilon. \end{cases}$$

So, by the maximum principle, we get that

$$\frac{\partial(H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial y_i} + \frac{\varepsilon^{N-2}}{|x|^{N-2}} \frac{\partial G_\Omega(0,y)}{\partial y_i} = O(\varepsilon)$$

for any $x \in \Omega_\varepsilon$ which implies (2.2) for $N \geq 3$.

For $N = 2$, we consider

$$b_\varepsilon(x,y) := \frac{\partial(H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial y_i} + \frac{\ln|x|}{\ln\varepsilon} \frac{\partial G_\Omega(0,y)}{\partial y_i},$$

so that, for any $|y| \geq C_0 > 0$,

$$\begin{cases} \Delta_x b_\varepsilon(x,y) = 0 & \text{for } x \in \Omega \setminus B_\varepsilon, \\ b_\varepsilon(x,y) = \dfrac{\ln|x|}{\ln\varepsilon} \dfrac{\partial G_\Omega(0,y)}{\partial y_i} = O\Big(\dfrac{1}{|\ln\varepsilon|}\Big) & \text{for } x \in \partial\Omega, \\ b_\varepsilon(x,y) = -\dfrac{\partial G_\Omega(x,y)}{\partial y_i} + \dfrac{\partial G_\Omega(0,y)}{\partial y_i} = O(\varepsilon) & \text{for } x \in \partial B_\varepsilon, \end{cases}$$

and exactly as for $N \geq 3$, we get $b_\varepsilon(x,y) = O(\frac{1}{|\ln\varepsilon|})$ for any $x \in \Omega_\varepsilon$, $|y| \geq C_0 > 0$. So the claim follows. ■

The next lemma concerns the estimate for $H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y)$ as $y \in \partial B_\varepsilon$. Since the corresponding integrals of (2.1) are harder to estimate, we will need to write the leading term of the expansion as $\varepsilon \to 0$.

Lemma 2.2. *For any $x \in \Omega_\varepsilon$, $y \in \partial B_\varepsilon$, we have*

$$\nabla_y\big(H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y)\big) = \phi_\varepsilon(x,y) + \begin{cases} O(1) & \text{for } N \geq 3,\\ O\Big(\dfrac{1}{|\ln \varepsilon|}\Big) & \text{for } N = 2, \end{cases} \tag{2.3}$$

where the function $\phi_\varepsilon(x,y)$ is given by

$$\phi_\varepsilon(x,y) = \begin{cases} (2-N)\dfrac{y}{\varepsilon^2}\Big(H_\Omega(x,0) - H_\Omega(0,0)\dfrac{\varepsilon^{N-2}}{|x|^{N-2}}\Big) & \text{for } N \geq 3,\\ \dfrac{y}{\varepsilon^2}\Big[\dfrac{1}{2\pi}\Big(\dfrac{G_\Omega(x,0)}{-\frac{1}{2\pi}\ln\varepsilon - H_\Omega(0,0)} - 1\Big) & \\ \qquad - 2\Big(\nabla_y H_\Omega(x,0)\cdot y - \nabla_y H_\Omega(0,0)\cdot y\dfrac{\ln|x|}{\ln\varepsilon}\Big)\Big] & \text{for } N = 2. \end{cases}$$

Remark 2.3. It is possible to improve estimate (2.3) for $N \geq 3$ in order to have a lower-order term like $o(1)$. This can be achieved by adding other suitable terms to ϕ_ε like in the 2-dimensional case. However, the remainder term $O(1)$ will be enough for our aims.

Proof of Lemma 2.2. As in the proof of Lemma 2.1, we can prove that $\nabla_y(H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y))$ is well defined for $x \in \Omega_\varepsilon$ and $y \in \partial B_\varepsilon$. Then for $N \geq 2$ and $i = 1,\dots,N$,

$$\begin{cases} \Delta_x \dfrac{\partial(H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial y_i} = 0 & \text{for } x \in \Omega \setminus B_\varepsilon,\\ \dfrac{\partial(H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial y_i} = -\dfrac{\partial G_{B_\varepsilon^c}(x,y)}{\partial y_i} & \text{for } x \in \partial\Omega,\\ \dfrac{\partial(H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial y_i} = 0 & \text{for } x \in \partial B_\varepsilon. \end{cases}$$

Next we consider in Ω_ε the functions

$$b_{\varepsilon,i}(x,y) := \frac{\partial(H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial y_i} - \phi_{\varepsilon,i}(x,y)$$

for $i = 1,\dots,N$ and $\phi_{\varepsilon,i}(x,y)$ being the i-th component of $\phi_\varepsilon(x,y)$. Then denoting by

$$\widehat{b}_\varepsilon(x,y) = (b_{\varepsilon,1}(x,y),\dots,b_{\varepsilon,N}(x,y)),$$

we have

$$\begin{cases} \Delta_x b_{\varepsilon,i}(x,y) = 0 & \text{for } x \in \Omega \setminus B_\varepsilon,\\ b_{\varepsilon,i}(x,y) = -\dfrac{\partial G_{B_\varepsilon^c}(x,y)}{\partial y_i} - \phi_{\varepsilon,i}(x,y) & \text{for } x \in \partial\Omega,\\ b_{\varepsilon,i}(x,y) = -\phi_{\varepsilon,i}(x,y) & \text{for } x \in \partial B_\varepsilon. \end{cases}$$

Case 1: $N=2$. In this case, we have, for $x\in\partial B_\varepsilon$ (recall that $y\in\partial B_\varepsilon$),

$$\begin{aligned}|\widehat{b}_\varepsilon(x,y)| &= |\phi_\varepsilon(x,y)|\\ &= \frac{1}{\varepsilon}\Big[-\frac{1}{2\pi}\underbrace{\Big(\frac{-\frac{1}{2\pi}\ln\varepsilon - H_\Omega(x,0)}{-\frac{1}{2\pi}\ln\varepsilon - H_\Omega(0,0)}-1\Big)}_{=O(\frac{|x|}{|\ln\varepsilon|})=O(\frac{\varepsilon}{|\ln\varepsilon|})}-2\underbrace{(\nabla_y H_\Omega(x,0)-\nabla_y H_\Omega(0,0))\cdot y}_{=O(|x|\cdot|y|)=O(\varepsilon^2)}\Big]\\ &= O\Big(\frac{1}{|\ln\varepsilon|}\Big).\end{aligned}$$

On the other hand, for $x\in\partial\Omega$, using (A.1), we have

$$\begin{aligned}b_{\varepsilon,i}(x,y) &= \frac{1}{2\pi}\Big[\frac{y_i-x_i}{|x|^2+\varepsilon^2-2x\cdot y}-\frac{|x|^2y_i-\varepsilon^2x_i}{\varepsilon^2(|x|^2+\varepsilon^2-2x\cdot y)}+\frac{y_i}{\varepsilon^2}\\ &\quad +2\frac{y_i}{\varepsilon^2}\frac{x\cdot y}{|x|^2}+\frac{y_i}{\varepsilon^2}O\Big(\frac{|y|}{|\ln\varepsilon|}\Big)\Big]\\ &= \underbrace{\frac{1}{2\pi}\Big[\frac{2\varepsilon^2|x|^2y_i+2\varepsilon^2(x\cdot y)y_i-4(x\cdot y)^2y_i}{\varepsilon^2|x|^2(|x|^2+\varepsilon^2-2x\cdot y)}\Big]}_{=O(\varepsilon)}+O\Big(\frac{1}{|\ln\varepsilon|}\Big)=O\Big(\frac{1}{|\ln\varepsilon|}\Big).\end{aligned}$$

So by the maximum principle, we get

$$\frac{\partial(H_{B_\varepsilon^c}(x,y)-H_{\Omega_\varepsilon}(x,y))}{\partial y_i}-\phi_{\varepsilon,i}(x,y)=O\Big(\frac{1}{|\ln\varepsilon|}\Big),$$

which gives the claim.

Case 2: $N\geq 3$. In this case, we have, for $x\in\partial B_\varepsilon$,

$$|\widehat{b}_\varepsilon(x,y)|=|\phi_\varepsilon(x,y)|=(N-2)\frac{1}{\varepsilon}\underbrace{|H_\Omega(x,0)-H_\Omega(0,0)|}_{=O(|x|)=O(\varepsilon)}=O(1),$$

while, when $x\in\partial\Omega$ using (A.1) with $|y|=\varepsilon$,

$$\begin{aligned}b_{\varepsilon,i}(x,y) &= -C_N(N-2)\Big[\underbrace{\frac{x_i-y_i}{|x-y|^N}}_{=O(1)}+\varepsilon^{-2}\underbrace{\frac{|x|^2y_i-\varepsilon^2x_i}{(|x|^2+O(\varepsilon))^{\frac{N}{2}}}}_{=|x|^{2-N}y_i+O(\varepsilon^2)}-\frac{y_i}{\varepsilon^2}\frac{1}{|x|^{N-2}}+O(\varepsilon^{N-3})\Big]\\ &= O(1).\end{aligned}$$

As in the previous case, the maximum principle gives the claim. ■

Lemmas 2.1 and 2.2 will allow us to give sharp estimates for $\mathcal{R}_{\Omega_\varepsilon}$ and $\nabla\mathcal{R}_{\Omega_\varepsilon}$ in Ω_ε. For what concerns $H_{B_\varepsilon^c}(x,y)-H_{\Omega_\varepsilon}(x,y)$, we need additional information on its second derivative. For this it will be useful to use the following known result for harmonic function.

Lemma 2.4. *Let $u(x)$ be a harmonic function in a domain $D \subset \mathbb{R}^N$, $N \geq 2$, and let $B(x,r) \subset D$, then*

$$|\nabla u(x)| \leq \frac{N}{r} \sup_{\partial B(x,r)} |u(x)|.$$

Proof. See [15, p. 22]. ∎

By the previous lemma, we deduce the following corollary.

Corollary 2.5. *For any $x \in \Omega_\varepsilon$, we have that*

- *if $|y| \geq C_0$, then*

$$\frac{\partial^2 (H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial x_j \partial y_i} = \begin{cases} O\Big(\frac{\varepsilon^{N-2}}{\mathrm{dist}(x,\partial\Omega_\varepsilon)|x|^{N-2}}\Big) + O\Big(\frac{\varepsilon}{\mathrm{dist}(x,\partial\Omega_\varepsilon)}\Big) & \text{for } N \geq 3, \\ O\Big(\Big|\frac{\ln|x|}{\mathrm{dist}(x,\partial\Omega_\varepsilon)\ln\varepsilon}\Big|\Big) + O\Big(\frac{1}{\mathrm{dist}(x,\partial\Omega_\varepsilon)|\ln\varepsilon|}\Big) & \text{for } N = 2, \end{cases} \tag{2.4}$$

- *if $y \in \partial B_\varepsilon$, then*

$$\frac{\partial^2 (H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial x_j \partial y_i} = \frac{\partial \phi_{\varepsilon,i}(x,y)}{\partial x_j} + \begin{cases} O\Big(\frac{1}{\mathrm{dist}(x,\partial\Omega_\varepsilon)}\Big) & \text{for } N \geq 3, \\ O\Big(\frac{1}{\mathrm{dist}(x,\partial\Omega_\varepsilon)|\ln\varepsilon|}\Big) & \text{for } N = 2, \end{cases} \tag{2.5}$$

where ϕ_ε is the function introduced in Lemma 2.2.

Proof. Let $N = 2$ and $|y| \geq C_0$. From (2.2) and by Lemma 2.4 with $r = \mathrm{dist}(x,\partial\Omega_\varepsilon)$, we get

$$\begin{aligned} &\Big|\frac{\partial^2 (H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial x_j \partial y_i}\Big| \\ &\quad \leq \frac{C}{\mathrm{dist}(x,\partial\Omega_\varepsilon)} \sup_{\partial B(x,\mathrm{dist}(x,\partial\Omega_\varepsilon))} \Big|\frac{\partial (H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial y_i}\Big| \\ &\quad = O\Big(\Big|\frac{\ln|x|}{\mathrm{dist}(x,\partial\Omega_\varepsilon)\ln\varepsilon}\Big|\Big) + O\Big(\frac{1}{\mathrm{dist}(x,\partial\Omega_\varepsilon)|\ln\varepsilon|}\Big), \end{aligned}$$

which gives the claim. In the same way, we get, for $N \geq 3$,

$$\Big|\frac{\partial}{\partial x_j}\Big(\frac{\partial (H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial y_i}\Big)\Big| = O\Big(\frac{\varepsilon^{N-2}}{\mathrm{dist}(x,\partial\Omega_\varepsilon)|x|^{N-2}} + \frac{\varepsilon}{\mathrm{dist}(x,\partial\Omega_\varepsilon)}\Big),$$

which proves (2.4) for $N \geq 3$. In the same way, applying Lemma 2.4 to the function $\nabla_y (H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y)) - \phi_\varepsilon(x,y)$ and using (2.3), we have (2.5). ∎

3. Estimates on the Robin function $\mathcal{R}_{\Omega_\varepsilon}$ and its derivatives

In this section, we prove an asymptotic estimate for $\mathcal{R}_{\Omega_\varepsilon}$ and its derivatives in the domain $\Omega_\varepsilon = \Omega \setminus B_\varepsilon$. It is worth to remark that we get uniform estimates up to ∂B_ε for small ε. These allow us to find the additional critical point for $\mathcal{R}_{\Omega_\varepsilon}$ (which will be actually close to ∂B_ε), but we believe that these estimates are interesting themselves. There is a common strategy in the proof of the estimates both for $\mathcal{R}_{\Omega_\varepsilon}$ and its derivatives. We start using some representation formula and after some manipulations (as in (2.1)) we reduce our estimate to some boundary integrals. Lastly, we use the lemmas of Section 2 to conclude.

3.1. *Estimate of* $\mathcal{R}_{\Omega_\varepsilon}$

The main result of this section is the following.

Proposition 3.1. *We have that, for any* $x \in \Omega_\varepsilon$,

$$\mathcal{R}_{\Omega_\varepsilon}(x) = \mathcal{R}_\Omega(x) + \mathcal{R}_{B_\varepsilon^c}(x) + \begin{cases} O\Big(\dfrac{\varepsilon^{N-2}}{|x|^{N-2}}\Big) + O(\varepsilon) & \text{for } N \geq 3, \\ -\dfrac{1}{2\pi}\dfrac{(\ln\frac{|x|}{\varepsilon})^2}{\ln\varepsilon} + O\Big(\Big|\dfrac{\ln|x|}{\ln\varepsilon}\Big|\Big) & \text{for } N = 2. \end{cases} \tag{3.1}$$

Remark 3.2. If $N = 2$, taking in account the explicit expression of $\mathcal{R}_{B_\varepsilon^c}$ (see (A.2)), we have that (3.1) can be written in this way,

$$\mathcal{R}_{\Omega_\varepsilon}(x) = \mathcal{R}_\Omega(x) - \frac{1}{2\pi}\frac{(\ln\frac{|x|}{\varepsilon})^2}{\ln\varepsilon} + \frac{1}{2\pi}\ln\frac{\varepsilon}{|x|^2-\varepsilon^2} + O\Big(\Big|\frac{\ln|x|}{\ln\varepsilon}\Big|\Big).$$

Proof of Proposition 3.1. Starting by (1.1) and the representation formula for harmonic function, we have

$$H_{\Omega_\varepsilon}(x,t) = -\int_{\partial\Omega_\varepsilon}\frac{\partial G_{\Omega_\varepsilon}(x,y)}{\partial\nu_y}S(y,t)d\sigma_y,$$

and so

$$\begin{aligned}
\mathcal{R}_{\Omega_\varepsilon}(x) &= -\int_{\partial\Omega_\varepsilon}\frac{\partial G_{\Omega_\varepsilon}(x,y)}{\partial\nu_y}S(x,y)d\sigma_y \\
&= -\int_{\partial\Omega}\frac{\partial G_{\Omega_\varepsilon}(x,y)}{\partial\nu_y}S(x,y)d\sigma_y - \int_{\partial B_\varepsilon}\frac{\partial G_{\Omega_\varepsilon}(x,y)}{\partial\nu_y}S(x,y)d\sigma_y \\
&= \underbrace{-\int_{\partial\Omega}\frac{\partial G_{\Omega}(x,y)}{\partial\nu_y}S(x,y)d\sigma_y}_{=\mathcal{R}_\Omega(x)} \underbrace{-\int_{\partial\Omega}\frac{\partial(H_\Omega(x,y)-H_{\Omega_\varepsilon}(x,y))}{\partial\nu_y}S(x,y)d\sigma_y}_{K_{1,\varepsilon}} \\
&\quad \underbrace{-\int_{\partial B_\varepsilon}\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial\nu_y}S(x,y)d\sigma_y}_{=\text{ see (A.10)}} \underbrace{-\int_{\partial B_\varepsilon}\frac{\partial(H_{B_\varepsilon^c}(x,y)-H_{\Omega_\varepsilon}(x,y))}{\partial\nu_y}S(x,y)d\sigma_y}_{K_{2,\varepsilon}} \\
&= \mathcal{R}_\Omega(x) - K_{1,\varepsilon} - K_{2,\varepsilon} + \begin{cases} \mathcal{R}_{B_\varepsilon^c}(x) & \text{if } N \geq 3, \\ \mathcal{R}_{B_\varepsilon^c}(x) + \dfrac{1}{2\pi}\ln\dfrac{|x|}{\varepsilon} & \text{if } N = 2. \end{cases}
\end{aligned} \tag{3.2}$$

Computation of $K_{1,\varepsilon}$. By (2.2), we get that

$$\begin{aligned} K_{1,\varepsilon} &= \int_{\partial\Omega} \frac{\partial(H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial \nu_y} S(x,y)d\sigma_y \\ &= O(|\nabla(H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))|) \underbrace{\int_{\partial\Omega} S(x,y)d\sigma_y}_{=O(1)} \\ &= \begin{cases} O\Big(\frac{\varepsilon^{N-2}}{|x|^{N-2}}\Big) + O(\varepsilon) & \text{for } N \geq 3, \\ O\Big(\Big|\frac{\ln|x|}{\ln\varepsilon}\Big|\Big) + O\Big(\frac{1}{|\ln\varepsilon|}\Big) & \text{for } N = 2. \end{cases} \end{aligned}$$

Computation of $K_{2,\varepsilon}$. First we observe that

$$\begin{aligned} &\frac{\partial(H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial \nu_y}\Big|_{\partial B_\varepsilon} \\ &= \phi_\varepsilon(x,y)\cdot\frac{-y}{\varepsilon} + \begin{cases} O(1) & \text{if } N \geq 3, \\ O\Big(\frac{1}{|\ln\varepsilon|}\Big) & \text{if } N = 2 \end{cases} \\ &= \begin{cases} \frac{N-2}{\varepsilon}\Big(H_\Omega(x,0) - H_\Omega(0,0)\frac{\varepsilon^{N-2}}{|x|^{N-2}}\Big) + O(1) & \text{if } N \geq 3, \\ -\frac{1}{\varepsilon}\Big[\frac{1}{2\pi}\Big(\frac{G_\Omega(x,0)}{-\frac{1}{2\pi}\ln\varepsilon - H_\Omega(0,0)} - 1\Big) \\ \quad - 2\Big(\nabla_y H_\Omega(x,0)\cdot y - \nabla_y H_\Omega(0,0)\cdot y\frac{\ln|x|}{\ln\varepsilon}\Big)\Big] \\ \quad + O\Big(\frac{1}{|\ln\varepsilon|}\Big) & \text{if } N = 2. \end{cases} \end{aligned} \tag{3.3}$$

Case $N \geq 3$. By (3.3), we get that

$$\begin{aligned} K_{2,\varepsilon} &= \int_{\partial B_\varepsilon} \frac{\partial(H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial \nu_y} \frac{C_N}{|x-y|^{N-2}} d\sigma_y \\ &= \Big[\frac{N-2}{\varepsilon}\Big(H_\Omega(x,0) - H_\Omega(0,0)\frac{\varepsilon^{N-2}}{|x|^{N-2}}\Big) + O(1)\Big] \underbrace{\int_{\partial B_\varepsilon} \frac{C_N}{|x-y|^{N-2}} d\sigma_y}_{\text{by (A.12)} = \frac{1}{N-2}\frac{\varepsilon^{N-1}}{|x|^{N-2}}} \\ &= O\Big(\frac{\varepsilon^{N-2}}{|x|^{N-2}}\Big), \end{aligned}$$

and so the claim follows for $N \geq 3$.

Case $N = 2$. In the same way, we get by (3.3),

$$K_{2,\varepsilon} = -\frac{1}{2\pi}\int_{\partial B_\varepsilon} \frac{\partial(H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial \nu_y} \ln|x-y| d\sigma_y$$

$$= \frac{1}{4\pi^2\varepsilon}\underbrace{\Big(\frac{G_\Omega(x,0)}{-\frac{1}{2\pi}\ln\varepsilon - H_\Omega(0,0)}-1\Big)}_{=-1+\frac{\ln|x|}{\ln\varepsilon}+O(\frac{|x|}{|\ln\varepsilon|})}\underbrace{\int_{\partial B_\varepsilon}\ln|x-y|d\sigma_y}_{=2\pi\varepsilon\ln|x| \text{ by (A.13)}}$$

$$-\frac{1}{\pi\varepsilon}\Big(\nabla_y H_\Omega(x,0)-\nabla_y H_\Omega(0,0)\frac{\ln|x|}{\ln\varepsilon}\Big)\underbrace{\int_{\partial B_\varepsilon} y\ln|x-y|d\sigma_y}_{=O(\varepsilon^2|\ln|x||) \text{ by (A.14)}}$$

$$+O\Big(\frac{1}{|\ln\varepsilon|}\Big)\underbrace{\int_{\partial B_\varepsilon}|\ln|x-y||d\sigma_y}_{=O(\varepsilon|\ln|x||)\text{ by (A.14)}}$$

$$=\frac{1}{2\pi}\ln\frac{|x|}{\varepsilon}+\frac{1}{2\pi}\frac{(\ln\frac{|x|}{\varepsilon})^2}{\ln\varepsilon}+O\Big(\Big|\frac{\ln|x|}{\ln\varepsilon}\Big|\Big).$$

By (3.2) and the estimates for $K_{1,\varepsilon}$ and $K_{2,\varepsilon}$, the claim follows. ■

3.2. Estimate of $\nabla\mathcal{R}_{\Omega_\varepsilon}$

The main result of this section is the following.

Proposition 3.3. *We have that for any $x\in\Omega_\varepsilon$,*

$$\nabla\mathcal{R}_{\Omega_\varepsilon}(x)=\nabla\mathcal{R}_\Omega(x)+\nabla\mathcal{R}_{B_\varepsilon^c}(x)$$
$$+\begin{cases} O\Big(\frac{\varepsilon^{N-2}}{|x|^{N-1}}\Big)+O(\varepsilon) & \text{for } N\ge 3,\\ \frac{1}{\pi}\Big(1-\frac{\ln|x|}{\ln\varepsilon}\Big)\frac{x}{|x|^2}+O\Big(\frac{1}{|x||\ln\varepsilon|}\Big) & \text{for } N=2.\end{cases} \tag{3.4}$$

Proof. Recalling (2.1), we have that

$$\nabla\mathcal{R}_{\Omega_\varepsilon}(x)=\nabla\mathcal{R}_\Omega(x)+\nabla\mathcal{R}_{B_\varepsilon^c}(x)$$
$$+\underbrace{\int_{\partial\Omega}\nu(y)\Big(\frac{\partial(H_\Omega(x,y)-H_{\Omega_\varepsilon}(x,y))}{\partial\nu_y}\Big)^2 d\sigma_y}_{I_{1,\varepsilon}}$$
$$+\underbrace{2\int_{\partial\Omega}\nu(y)\frac{\partial G_\Omega(x,y)}{\partial\nu_y}\frac{\partial(H_\Omega(x,y)-H_{\Omega_\varepsilon}(x,y))}{\partial\nu_y}d\sigma_y}_{I_{2,\varepsilon}}$$
$$-\underbrace{\int_{\partial B_\varepsilon}\frac{y}{\varepsilon}\Big(\frac{\partial(H_{B_\varepsilon^c}(x,y)-H_{\Omega_\varepsilon}(x,y))}{\partial\nu_y}\Big)^2 d\sigma_y}_{I_{3,\varepsilon}}$$
$$-\underbrace{2\int_{\partial B_\varepsilon}\frac{y}{\varepsilon}\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial\nu_y}\frac{\partial(H_{B_\varepsilon^c}(x,y)-H_{\Omega_\varepsilon}(x,y))}{\partial\nu_y}d\sigma_y}_{I_{4,\varepsilon}}. \tag{3.5}$$

We will show that, for $N \geq 3$, the integrals $I_{1,\varepsilon}, \ldots, I_{4,\varepsilon}$ are lower-order terms with respect to $\nabla \mathcal{R}_\Omega$ and $\nabla \mathcal{R}_{B_\varepsilon^c}$. If $N = 2$, the situation is more complicated because both integrals $I_{3,\varepsilon}$ and $I_{4,\varepsilon}$ give a contribution.

Computation of $I_{1,\varepsilon}$. By (2.2), we get that

$$I_{1,\varepsilon} = \begin{cases} O\Big(\dfrac{\varepsilon^{2(N-2)}}{|x|^{2(N-2)}}\Big) + O(\varepsilon^2) & \text{for } N \geq 3, \\ O\Big(\Big|\dfrac{\ln^2|x|}{\ln^2\varepsilon}\Big|\Big) + O\Big(\dfrac{1}{|\ln\varepsilon|^2}\Big) & \text{for } N = 2. \end{cases}$$

Computation of $I_{2,\varepsilon}$. By (2.2), we get that

$$|I_{2,\varepsilon}| = O(|\nabla(H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))|) \underbrace{\int_{\partial\Omega} \Big|\frac{\partial G_\Omega(x,y)}{\partial \nu_y}\Big| d\sigma_y}_{=1}$$
$$= \begin{cases} O\Big(\dfrac{\varepsilon^{N-2}}{|x|^{N-2}}\Big) + O(\varepsilon) & \text{for } N \geq 3, \\ O\Big(\Big|\dfrac{\ln|x|}{\ln\varepsilon}\Big|\Big) + O\Big(\dfrac{1}{|\ln\varepsilon|}\Big) & \text{for } N = 2. \end{cases}$$

Computation of $I_{3,\varepsilon}$. Then we look at the cases $N \geq 3$ and $N = 2$ separately.

Case $N \geq 3$. By (3.3), we get

$$\begin{aligned} I_{3,\varepsilon} &= \int_{\partial B_\varepsilon} \frac{y}{\varepsilon}\Big(\frac{\partial(H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial\nu_y}\Big)^2 d\sigma_y \\ &= (N-2)^2 \int_{\partial B_\varepsilon} \frac{y}{\varepsilon}\Big[\frac{1}{\varepsilon}\Big(H_\Omega(x,0) - H_\Omega(0,0)\frac{\varepsilon^{N-2}}{|x|^{N-2}}\Big) + O(1)\Big]^2 d\sigma_y \\ &= \frac{(N-2)^2}{\varepsilon^2}\Big(H_\Omega(x,0) - H_\Omega(0,0)\frac{\varepsilon^{N-2}}{|x|^{N-2}}\Big)^2 \underbrace{\int_{\partial B_\varepsilon} \frac{y}{\varepsilon} d\sigma_y}_{=0} + O\Big(\frac{1}{\varepsilon}\Big) \underbrace{\int_{\partial B_\varepsilon} \Big|\frac{y}{\varepsilon}\Big| d\sigma_y}_{=O(\varepsilon^{N-1})} \\ &\quad + O(1) \int_{\partial B_\varepsilon} \Big|\frac{y}{\varepsilon}\Big| d\sigma_y = O(\varepsilon^{N-2}). \end{aligned}$$

Case $N = 2$. Again by (3.3), we have that

$$\begin{aligned} I_{3,\varepsilon} &= \int_{\partial B_\varepsilon} \frac{y}{\varepsilon}\Big(\frac{\partial(H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial\nu_y}\Big)^2 d\sigma_y \\ &= \int_{\partial B_\varepsilon} \frac{y}{\varepsilon}\Big(\phi_\varepsilon(x,y)\cdot\nu_y + O\Big(\frac{1}{|\ln\varepsilon|}\Big)\Big)^2 d\sigma_y \\ &= \int_{\partial B_\varepsilon} \frac{y}{\varepsilon}(\phi_\varepsilon(x,y)\cdot\nu_y)^2 d\sigma_y + O\Big(\frac{1}{|\ln\varepsilon|}\Big) \underbrace{\int_{\partial B_\varepsilon} |\phi_\varepsilon(x,y)| d\sigma_y}_{=O(1)} + O\Big(\frac{\varepsilon}{(\ln\varepsilon)^2}\Big) \end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial B_\varepsilon} \frac{y}{\varepsilon}(\phi_\varepsilon(x,y)\cdot\nu_y)^2 d\sigma_y + O\Big(\frac{1}{|\ln\varepsilon|}\Big) \\
&= \frac{1}{4\pi^2\varepsilon^2}\Big(\frac{G_\Omega(x,0)}{-\frac{1}{2\pi}\ln\varepsilon - H_\Omega(0,0)} - 1\Big)^2 \underbrace{\int_{\partial B_\varepsilon}\frac{y}{\varepsilon}d\sigma_y}_{=0} + \frac{2}{\pi\varepsilon^2}\Big(1-\frac{\ln|x|}{\ln\varepsilon}+O\Big(\frac{1}{|\ln\varepsilon|}\Big)\Big) \\
&\quad\times\int_{\partial B_\varepsilon}\frac{y}{\varepsilon}\Big[\nabla_y H_\Omega(x,0)\cdot y - \nabla_y H_\Omega(0,0)\cdot y\frac{\ln|x|}{\ln\varepsilon}\Big]d\sigma_y \\
&\quad+\underbrace{\int_{\partial B_\varepsilon}\frac{|y|}{\varepsilon}O(1)d\sigma_y}_{=O(\varepsilon)} + O\Big(\frac{1}{|\ln\varepsilon|}\Big) = \frac{2}{\pi\varepsilon^3}\Big(1-\frac{\ln|x|}{\ln\varepsilon}+O\Big(\frac{1}{|\ln\varepsilon|}\Big)\Big) \\
&\quad\times\sum_{j=1}^{2}\Big(\frac{\partial H_\Omega(x,0)}{\partial y_j} - \frac{\partial H_\Omega(0,0)}{\partial y_j}\frac{\ln|x|}{\ln\varepsilon}\Big)\underbrace{\int_{\partial B_\varepsilon} y y_j d\sigma_y}_{=\pi\varepsilon^3\delta_j^i} + O\Big(\frac{1}{|\ln\varepsilon|}\Big) \\
&= 2\Big(1-\frac{\ln|x|}{\ln\varepsilon}\Big)\Big(\nabla_y H_\Omega(x,0) - \nabla_y H_\Omega(0,0)\frac{\ln|x|}{\ln\varepsilon}\Big) + O\Big(\frac{1}{|\ln\varepsilon|}\Big) \\
&= 2\nabla_y H_\Omega(x,0) + O\Big(\Big|\frac{\ln|x|}{\ln\varepsilon}\Big|\Big) + O\Big(\frac{1}{|\ln\varepsilon|}\Big).
\end{aligned}
$$

Computation of $I_{4,\varepsilon}$. As in the previous step, let us consider the case $N \geq 3$ firstly.

Case $N \geq 3$. Recalling (3.3) and using (A.4) and (A.5), we have

$$
\begin{aligned}
I_{4,\varepsilon} &= 2\int_{\partial B_\varepsilon}\frac{y}{\varepsilon}\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial\nu_y}\frac{\partial(H_{B_\varepsilon^c}(x,y)-H_{\Omega_\varepsilon}(x,y))}{\partial\nu_y}d\sigma_y \\
&= 2\frac{N-2}{\varepsilon}\underbrace{\Big(H_\Omega(x,0)-H_\Omega(0,0)\frac{\varepsilon^{N-2}}{|x|^{N-2}}\Big)}_{=O(1)}\underbrace{\frac{1}{\varepsilon}\int_{\partial B_\varepsilon} y\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial\nu_y}d\sigma_y}_{=O(\frac{\varepsilon^{N-1}}{|x|^{N-1}})\text{ by (A.5)}} \\
&\quad + O(1)\underbrace{\int_{\partial B_\varepsilon}\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial\nu_y}d\sigma_y}_{=O(\frac{\varepsilon^{N-2}}{|x|^{N-2}})\text{ by (A.4)}} = O\Big(\frac{\varepsilon^{N-2}}{|x|^{N-1}}\Big).
\end{aligned}
$$

Case $N = 2$. Using (3.3), we have

$$
\begin{aligned}
I_{4,\varepsilon} &= 2\int_{\partial B_\varepsilon}\frac{y}{\varepsilon}\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial\nu_y}\frac{\partial(H_{B_\varepsilon^c}(x,y)-H_{\Omega_\varepsilon}(x,y))}{\partial\nu_y}d\sigma_y \\
&= 2\int_{\partial B_\varepsilon}\frac{y}{\varepsilon}\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial\nu_y}\Big(\phi_\varepsilon(x,y)\cdot\nu_y + O\Big(\frac{1}{|\ln\varepsilon|}\Big)\Big)d\sigma_y \\
&= \frac{1}{\pi\varepsilon}\Big(1-\frac{\ln|x|}{\ln\varepsilon}+O\Big(\frac{1}{|\ln\varepsilon|}\Big)\Big)\underbrace{\int_{\partial B_\varepsilon}\frac{y}{\varepsilon}\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial\nu_y}d\sigma_y}_{=-\frac{x}{|x|^2}\varepsilon\text{ by (A.5)}}
\end{aligned}
$$

$$+\underbrace{\frac{4}{\varepsilon}\int_{\partial B_\varepsilon}\frac{y}{\varepsilon}\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y}\Big(\nabla_y H_\Omega(x,0)\cdot y-\nabla_y H_\Omega(0,0)\cdot y\frac{\ln|x|}{\ln\varepsilon}\Big)d\sigma_y}_{:=\widetilde{K}_{4,\varepsilon}}$$

$$+\,O\Big(\frac{1}{|\ln\varepsilon|}\Big)\underbrace{\int_{\partial B_\varepsilon}\frac{|y|}{\varepsilon}\Big|\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y}\Big|d\sigma_y}_{=O(1)\text{ by (A.4)}}\,. \tag{3.6}$$

Also, by (A.6), we have $\int_{\partial B_\varepsilon} yy_j\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial\nu_y}d\sigma_y=-\varepsilon^4\frac{xx_j}{|x|^4}-\delta_j^i\frac{\varepsilon^2}{2}(1-\frac{\varepsilon^2}{|x|^2})$ and then

$$\begin{aligned}\widetilde{K}_{4,\varepsilon}=&-2\Big(1-\frac{\varepsilon^2}{|x|^2}\Big)\Big(\nabla_y H_\Omega(x,0)-\nabla_y H_\Omega(0,0)\frac{\ln|x|}{\ln\varepsilon}\Big)\\&-4\varepsilon^2\frac{x}{|x|^4}\sum_{j=1}^{2}\Big(\frac{\partial H_\Omega(x,0)}{\partial y_j}-\frac{\partial H_\Omega(0,0)}{\partial y_j}\frac{\ln|x|}{\ln\varepsilon}\Big)x_j\\=&-2\Big(\nabla_y H_\Omega(x,0)-\nabla_y H_\Omega(0,0)\frac{\ln|x|}{\ln\varepsilon}\Big)+O\Big(\frac{\varepsilon^2}{|x|^2}\Big).\end{aligned}\tag{3.7}$$

Hence from (3.6) and (3.7), we find

$$I_{4,\varepsilon}=-\frac{1}{\pi}\Big(1-\frac{\ln|x|}{\ln\varepsilon}\Big)\frac{x}{|x|^2}-2\nabla_y H_\Omega(x,0)+O\Big(\frac{1}{|x|\cdot|\ln\varepsilon|}\Big).$$

Collecting the estimates for $I_{1,\varepsilon},\dots,I_{4,\varepsilon}$, we have the expansion for $\nabla\mathcal{R}_{\Omega_\varepsilon}$. ∎

3.3. *Estimate of $\nabla^2\mathcal{R}_{\Omega_\varepsilon}$*

These estimates will be crucial to prove the uniqueness of the critical point of $\mathcal{R}_{\Omega_\varepsilon}$ close to $\partial B(0,\varepsilon)$. The basic result of this section is the following.

Proposition 3.4. *For any $i,j=1,\dots,N$ and for $x\in\Omega_\varepsilon$ such that* $\mathrm{dist}(x,\partial\Omega)\ge C>0$, *we have*

$$\frac{\partial^2\mathcal{R}_{\Omega_\varepsilon}(x)}{\partial x_i\partial x_j}=\frac{\partial^2\mathcal{R}_{\Omega}(x)}{\partial x_i\partial x_j}+\frac{\partial^2\mathcal{R}_{B_\varepsilon^c}(x)}{\partial x_i\partial x_j}$$
$$+\begin{cases}O\Big(\frac{\varepsilon^{N-2}}{\mathrm{dist}(x,\partial B_\varepsilon)|x|^{N-2}}+\frac{\varepsilon}{\mathrm{dist}(x,\partial B_\varepsilon)}+\frac{\varepsilon^{N-2}}{|x|^N}\\ \quad+\frac{\varepsilon^{N-2}|x|}{\mathrm{dist}(x,\partial B_\varepsilon)^N}+\frac{\varepsilon^{N-2}|x|^2}{\mathrm{dist}(x,\partial B_\varepsilon)^{N+1}}\Big) & \text{for } N\ge 3,\\ \frac{1}{\pi}\Big(1-\frac{\ln|x|}{\ln\varepsilon}\Big)\frac{\partial}{\partial x_j}\Big(\frac{x_i}{|x|^2}\Big)\\ \quad+O\Big(\frac{1}{|x|^2|\ln\varepsilon|}+\Big|\frac{\ln|x|}{\mathrm{dist}(x,\partial B_\varepsilon)\ln\varepsilon}\Big|\Big)\\ \quad+O\Big(\frac{|x|}{\mathrm{dist}(x,\partial B_\varepsilon)^2|\ln\varepsilon|}+\frac{|x|^2}{\mathrm{dist}(x,\partial B_\varepsilon)^3|\ln\varepsilon|}\Big) & \text{for } N=2.\end{cases}\tag{3.8}$$

Proof. Differentiating formula (3.5) for $i, j = 1, \dots, N$, we have

$$
\begin{aligned}
\frac{\partial^2 \mathcal{R}_{\Omega_\varepsilon}(x)}{\partial x_i \partial x_j}
&= \frac{\partial^2 \mathcal{R}_{\Omega}(x)}{\partial x_i \partial x_j} + \frac{\partial^2 \mathcal{R}_{B_\varepsilon^c}(x)}{\partial x_i \partial x_j} \\
&\quad + \underbrace{2\int_{\partial\Omega} \nu_i(y)\Big(\frac{\partial^2 (H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial x_j \partial \nu_y}\Big)\Big(\frac{\partial (H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial \nu_y}\Big) d\sigma_y}_{J_{1,\varepsilon}} \\
&\quad + \underbrace{2\int_{\partial\Omega} \nu_i(y)\frac{\partial^2 G_\Omega(x,y)}{\partial x_j \partial \nu_y}\frac{\partial (H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial \nu_y} d\sigma_y}_{J_{2,\varepsilon}} \\
&\quad + \underbrace{2\int_{\partial\Omega} \nu_i(y)\frac{\partial G_\Omega(x,y)}{\partial \nu_y}\frac{\partial^2 (H_\Omega(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial x_j \partial \nu_y} d\sigma_y}_{J_{3,\varepsilon}} \\
&\quad \underbrace{- 2\int_{\partial B_\varepsilon} \frac{y_i}{\varepsilon}\Big(\frac{\partial^2 (H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial x_j \partial \nu_y}\Big)\Big(\frac{\partial (H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial \nu_y}\Big) d\sigma_y}_{J_{4,\varepsilon}} \\
&\quad \underbrace{- 2\int_{\partial B_\varepsilon} \frac{y_i}{\varepsilon}\frac{\partial^2 G_{B_\varepsilon^c}(x,y)}{\partial x_j \partial \nu_y}\frac{\partial (H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial \nu_y} d\sigma_y}_{J_{5,\varepsilon}} \\
&\quad \underbrace{- 2\int_{\partial B_\varepsilon} \frac{y_i}{\varepsilon}\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y}\frac{\partial^2 (H_{B_\varepsilon^c}(x,y) - H_{\Omega_\varepsilon}(x,y))}{\partial x_j \partial \nu_y} d\sigma_y}_{J_{6,\varepsilon}}.
\end{aligned}
$$

We have to estimate the integrals $J_{1,\varepsilon}, \dots, J_{6,\varepsilon}$. The computations are very similar to those of Proposition 3.3.

Computation of $J_{1,\varepsilon}$. By Lemma 2.1 and (2.4), we get

$$
J_{1,\varepsilon} = \begin{cases} O\Big(\dfrac{\varepsilon^{2N-4}}{\operatorname{dist}(x,\partial B_\varepsilon)|x|^{2N-4}}\Big) + O\Big(\dfrac{\varepsilon^2}{\operatorname{dist}(x,\partial B_\varepsilon)}\Big) & \text{for } N \geq 3, \\ O\Big(\dfrac{\ln^2|x|}{\operatorname{dist}(x,\partial B_\varepsilon)\ln^2\varepsilon}\Big) & \text{for } N = 2. \end{cases}
$$

Computation of $J_{2,\varepsilon}$. Here we use the assumption that x satisfies

$$\operatorname{dist}(x,\partial\Omega) \geq C > 0.$$

We need it to have

$$\frac{\partial^2 G_\Omega(x,y)}{\partial x_j \partial \nu_y} = O(1) \quad \text{for } y \in \partial\Omega.$$

Using Lemma 2.1, we immediately have

$$
\begin{aligned}
|J_{2,\varepsilon}| &\le \int_{\partial\Omega} \Big|\frac{\partial^2 G_\Omega(x,y)}{\partial x_j \partial \nu_y}\Big| \Big|\frac{\partial(H_\Omega(x,y)-H_{\Omega_\varepsilon}(x,y))}{\partial \nu_y}\Big| d\sigma_y \\
&= \begin{cases} O\Big(\dfrac{\varepsilon^{N-2}}{|x|^{N-2}}\Big) + (\varepsilon) & \text{for } N \ge 3, \\ O\Big(\Big|\dfrac{\ln|x|}{\ln\varepsilon}\Big|\Big) & \text{for } N = 2. \end{cases}
\end{aligned}
$$

Computation of $J_{3,\varepsilon}$. Recalling that

$$
\int_{\partial\Omega} \Big|\frac{\partial G_\Omega(x,y)}{\partial \nu_y}\Big| d\sigma_y = 1,
$$

we have

$$
\begin{aligned}
J_{3,\varepsilon} &= \int_{\partial\Omega} \Big|\frac{\partial G_\Omega(x,y)}{\partial \nu_y}\Big| \Big|\frac{\partial^2(H_\Omega(x,y)-H_{\Omega_\varepsilon}(x,y))}{\partial x_j \partial \nu_y}\Big| d\sigma_y \\
&= \begin{cases} O\Big(\dfrac{\varepsilon^{N-2}}{\mathrm{dist}(x,\partial B_\varepsilon)|x|^{N-2}}\Big) + O\Big(\dfrac{\varepsilon}{\mathrm{dist}(x,\partial B_\varepsilon)}\Big) & \text{for } N \ge 3, \\ O\Big(\Big|\dfrac{\ln|x|}{\mathrm{dist}(x,\partial B_\varepsilon)\ln\varepsilon}\Big|\Big) & \text{for } N = 2. \end{cases}
\end{aligned}
$$

Computation of $J_{4,\varepsilon}$. We will show that

$$
J_{4,\varepsilon} = \begin{cases} O\Big(\dfrac{\varepsilon^{N-2}}{\mathrm{dist}(x,\partial B_\varepsilon)}\Big) + O\Big(\dfrac{\varepsilon^{2N-4}}{|x|^{N-1}}\Big) & \text{for } N \ge 3, \\ \dfrac{\partial^2 H_\Omega(x,0)}{\partial x_j \partial y_i} + O\Big(\dfrac{1}{\mathrm{dist}(x,\partial B_\varepsilon)|\ln\varepsilon|}\Big) + O\Big(\Big|\dfrac{\ln|x|}{\ln\varepsilon}\Big|\Big) & \text{for } N = 2. \end{cases} \tag{3.9}
$$

By (2.5), we derive that

$$
\begin{aligned}
&\frac{\partial^2(H_{\Omega_\varepsilon}(x,y)-H_{B_\varepsilon^c}(x,y))}{\partial x_j \partial \nu_y}\Big|_{y\in\partial B_\varepsilon} \\
&= \begin{cases} \dfrac{N-2}{\varepsilon}\Big(\dfrac{\partial H(x,0)}{\partial x_j} - (2-N)H_\Omega(0,0)\dfrac{\varepsilon^{N-2}x_j}{|x|^N}\Big) \\ \quad + O\Big(\dfrac{1}{\mathrm{dist}(x,\partial B_\varepsilon)}\Big) & \text{for } N \ge 3, \\ -\dfrac{1}{\varepsilon}\dfrac{1}{-\ln\varepsilon - 2\pi H_\Omega(0,0)}\dfrac{\partial G_\Omega(x,0)}{\partial x_j} \\ \quad + \dfrac{2}{\varepsilon}\displaystyle\sum_{l=1}^{2}\Big(\dfrac{\partial^2 H_\Omega(x,0)}{\partial x_j \partial y_l} - \dfrac{\partial H_\Omega(0,0)}{\partial y_l}\dfrac{x_j}{|x|^2\ln\varepsilon}\Big)y_l \\ \quad + O\Big(\dfrac{1}{\mathrm{dist}(x,\partial B_\varepsilon)|\ln\varepsilon|}\Big) & \text{for } N = 2. \end{cases}
\end{aligned} \tag{3.10}
$$

Hence for $N \geq 3$, we have (see (3.3))

$$
\begin{aligned}
J_{4,\varepsilon} &= \frac{(N-2)^2}{\varepsilon^2} \int_{\partial B_\varepsilon} \frac{y_i}{\varepsilon}\Big(H_\Omega(x,0) - H_\Omega(0,0)\frac{\varepsilon^{N-2}}{|x|^{N-2}} + O(\varepsilon)\Big) \\
&\quad \times \Big(\frac{\partial H(x,0)}{\partial x_j} - (2-N)H_\Omega(0,0)\frac{\varepsilon^{N-2}x_j}{|x|^N} + O\Big(\frac{\varepsilon}{\mathrm{dist}(x,\partial B_\varepsilon)}\Big)\Big) d\sigma_y \\
&= \frac{(N-2)^2}{\varepsilon^2}\Big(H_\Omega(x,0) - H_\Omega(0,0)\frac{\varepsilon^{N-2}}{|x|^{N-2}}\Big) \\
&\quad \times \Big(\frac{\partial H(x,0)}{\partial x_j} - (2-N)H_\Omega(0,0)\frac{\varepsilon^{N-2}x_j}{|x|^N}\Big) \underbrace{\int_{\partial B_\varepsilon} \frac{y_i}{\varepsilon} d\sigma_y}_{=0} \\
&\quad + \frac{1}{\varepsilon^2}\int_{\partial B_\varepsilon} \Big(O\Big(\frac{\varepsilon}{\mathrm{dist}(x,\partial B_\varepsilon)}\Big) + O\Big(\frac{\varepsilon^{N-1}}{|x|^{N-1}}\Big)\Big) d\sigma_y \\
&= O\Big(\frac{\varepsilon^{N-2}}{\mathrm{dist}(x,\partial B_\varepsilon)}\Big) + O\Big(\frac{\varepsilon^{2N-4}}{|x|^{N-1}}\Big),
\end{aligned}
$$

which gives (3.9) for $N \geq 3$.

If $N = 2$, we have, again using (3.3),

$$
\begin{aligned}
J_{4,\varepsilon} = \int_{\partial B_\varepsilon} \frac{y_i}{\varepsilon^3}\Bigg[&\frac{1}{-\ln\varepsilon - 2\pi H_\Omega(0,0)}\frac{\partial G_\Omega(x,0)}{\partial x_j} \\
&- 2\Big(\sum_{l=1}^{2} \frac{\partial^2 H_\Omega(x,0)}{\partial x_j \partial y_l} y_l - \nabla_y H_\Omega(0,0)\cdot y \frac{x_j}{|x|^2 \ln\varepsilon}\Big) \\
&+ O\Big(\frac{\varepsilon}{\mathrm{dist}(x,\partial B_\varepsilon)|\ln\varepsilon|}\Big)\Bigg]\Bigg[\frac{1}{2\pi}\Big(\frac{G_\Omega(x,0)}{-\frac{1}{2\pi}\ln\varepsilon - H_\Omega(0,0)} - 1\Big) + O(\varepsilon)\Bigg] d\sigma_y \\
= \frac{1}{2\pi\varepsilon^3}&\Big(\frac{1}{-\ln\varepsilon - 2\pi H_\Omega(0,0)}\frac{\partial G_\Omega(x,0)}{\partial x_j}\Big)\Big(\frac{G_\Omega(x,0)}{-\frac{1}{2\pi}\ln\varepsilon - H_\Omega(0,0)} - 1\Big) \underbrace{\int_{\partial B_\varepsilon} y_i d\sigma_y}_{=0} \\
- \frac{1}{\pi\varepsilon^3}&\Big(\frac{G_\Omega(x,0)}{-\frac{1}{2\pi}\ln\varepsilon - H_\Omega(0,0)} - 1\Big) \\
\times &\sum_{l=1}^{2}\Big(\frac{\partial^2 H_\Omega(x,0)}{\partial x_j \partial y_l} - \frac{\partial H_\Omega(0,0)}{\partial y_l}\frac{x_j}{|x|^2\ln\varepsilon}\Big) \underbrace{\int_{\partial B_\varepsilon} y_i y_l d\sigma_y}_{\pi\varepsilon^3\delta_i^l} \\
- \frac{1}{2\pi\varepsilon^3}&\Big(\frac{G_\Omega(x,0)}{-\frac{1}{2\pi}\ln\varepsilon - H_\Omega(0,0)} - 1\Big) O\Big(\frac{\varepsilon}{\mathrm{dist}(x,\partial B_\varepsilon)|\ln\varepsilon|}\Big) \underbrace{\int_{\partial B_\varepsilon} |y_i| d\sigma_y}_{=O(\varepsilon^2)} \\
+ O(\varepsilon)&\Big(\frac{1}{-\ln\varepsilon - 2\pi H_\Omega(0,0)}\frac{\partial G_\Omega(x,0)}{\partial x_j}\Big) \underbrace{\int_{\partial B_\varepsilon} \frac{|y_i|}{\varepsilon^3} d\sigma_y}_{=O(\frac{1}{\varepsilon})}
\end{aligned}
$$

$$= \frac{1}{\pi\varepsilon^3}\Big(1 - \frac{\ln|x|}{\ln\varepsilon} + O\Big(\frac{1}{|x|\cdot|\ln\varepsilon|}\Big)\Big)\frac{\partial^2 H_\Omega(x,0)}{\partial x_j \partial y_i}\underbrace{\int_{\partial B_\varepsilon} y_i^2 d\sigma_y}_{=\pi\varepsilon^3}$$
$$+ O\Big(\frac{1}{\mathrm{dist}(x,\partial B_\varepsilon)|\ln\varepsilon|} + \Big|\frac{\ln|x|}{\ln\varepsilon}\Big|\Big)$$
$$= \frac{\partial^2 H_\Omega(x,0)}{\partial x_j \partial y_i} + O\Big(\frac{1}{\mathrm{dist}(x,\partial B_\varepsilon)|\ln\varepsilon|}\Big) + O\Big(\Big|\frac{\ln|x|}{\ln\varepsilon}\Big|\Big),$$

which proves (3.9).

Computation of $J_{5,\varepsilon}$. We will show that

$$J_{5,\varepsilon} = \begin{cases} O\Big(\frac{\varepsilon^{N-2}}{|x|^N}\Big) + O\Big(\frac{\varepsilon^{N-2}|x|}{\mathrm{dist}(x,\partial B_\varepsilon)^N}\Big) + O\Big(\frac{\varepsilon^{N-2}|x|^2}{\mathrm{dist}(x,\partial B_\varepsilon)^{N+1}}\Big) & \text{for } N \ge 3, \\ \frac{1}{2\pi}\Big(-1 + \frac{\ln|x|}{\ln\varepsilon} + O\Big(\frac{1}{|\ln\varepsilon|}\Big)\Big)\frac{\partial}{\partial x_j}\Big(\frac{x_i}{|x|^2}\Big) & \\ \quad + O\Big(\frac{|x|}{|\ln\varepsilon|\,\mathrm{dist}(x,\partial B_\varepsilon)^2}\Big) + O\Big(\frac{|x|^2}{|\ln\varepsilon|\,\mathrm{dist}(x,\partial B_\varepsilon)^3}\Big) + O\Big(\frac{\varepsilon^2}{|x|^3}\Big) & \text{for } N = 2. \end{cases}$$

Here we will use that, for $N \ge 2$,

$$N\omega_N \frac{\partial^2 G_{B_\varepsilon^c}(x,y)}{\partial x_j \partial \nu_y}\Big|_{|y|=\varepsilon} = -\frac{2x_j}{\varepsilon|x-y|^N} + N\frac{|x|^2-\varepsilon^2}{\varepsilon|x-y|^{N+2}}(x_j - y_j)$$
$$= O\Big(\frac{|x|}{\varepsilon|x-y|^N}\Big) + O\Big(\frac{|x|^2}{\varepsilon|x-y|^{N+1}}\Big). \tag{3.11}$$

We have that, by (3.3) and for $N \ge 3$,

$$J_{5,\varepsilon} = \int_{\partial B_\varepsilon} \frac{y_i}{\varepsilon}\frac{\partial^2 G_{B_\varepsilon^c}(x,y)}{\partial x_j \partial \nu_y}\Big(\frac{N-2}{\varepsilon}\Big(H_\Omega(x,0) - H_\Omega(0,0)\frac{\varepsilon^{N-2}}{|x|^{N-2}}\Big) + O(1)\Big)d\sigma_y$$
$$= \frac{N-2}{\varepsilon^2}\Big(H_\Omega(x,0) - H_\Omega(0,0)\frac{\varepsilon^{N-2}}{|x|^{N-2}}\Big)\underbrace{\int_{\partial B_\varepsilon} y_i \frac{\partial^2 G_{B_\varepsilon^c}(x,y)}{\partial x_j \partial \nu_y} d\sigma_y}_{=\varepsilon^N\left(\frac{\delta_j^i|x|^2 - N x_i x_j}{|x|^{N+2}}\right)}$$
$$+ \int_{\partial B_\varepsilon} O\Big(\Big|\frac{\partial^2 G_{B_\varepsilon^c}(x,y)}{\partial x_j \partial y_i}\Big|\Big)d\sigma_y.$$

Differentiating (A.5), and using (3.11),

$$\text{r.h.s.} = O\Big(\frac{\varepsilon^{N-2}}{|x|^N}\Big) + O\Big(\frac{|x|}{\varepsilon}\Big)\int_{\partial B_\varepsilon}\frac{1}{|x-y|^N}d\sigma_y + O\Big(\frac{|x|^2}{\varepsilon}\Big)\int_{\partial B_\varepsilon}\frac{1}{|x-y|^{N+1}}d\sigma_y,$$

and observing that for $y \in \partial B_\varepsilon$, we get $\mathrm{dist}(x,\partial B_\varepsilon) \le |x-y|$,

$$\text{r.h.s.} = O\Big(\frac{\varepsilon^{N-2}}{|x|^N}\Big) + O\Big(\frac{\varepsilon^{N-2}|x|}{\mathrm{dist}(x,\partial B_\varepsilon)^N}\Big) + O\Big(\frac{\varepsilon^{N-2}|x|^2}{\mathrm{dist}(x,\partial B_\varepsilon)^{N+1}}\Big), \tag{3.12}$$

which gives the claim. If $N=2$, we have, using (2.5),

$$J_{5,\varepsilon}=-\int_{\partial B_\varepsilon}\frac{y_i}{\varepsilon^2}\frac{\partial^2 G_{B_\varepsilon^c}(x,y)}{\partial x_j\partial\nu_y}\Big[\frac{1}{2\pi}\Big(\frac{G_\Omega(x,0)}{-\frac{1}{2\pi}\ln\varepsilon-H_\Omega(0,0)}-1\Big)$$

$$-2\Big(\nabla_y H_\Omega(x,0)\cdot y-\nabla_y H_\Omega(0,0)\cdot y\frac{\ln|x|}{\ln\varepsilon}\Big)+O\Big(\frac{\varepsilon}{|\ln\varepsilon|}\Big)\Big]d\sigma_y$$

$$=-\frac{1}{2\pi\varepsilon^2}\underbrace{\Big(\frac{G_\Omega(x,0)}{-\frac{1}{2\pi}\ln\varepsilon-H_\Omega(0,0)}-1\Big)}_{=-1+\frac{\ln|x|}{\ln\varepsilon}+O(\frac{1}{|\ln\varepsilon|})}\underbrace{\int_{\partial B_\varepsilon}y_i\frac{\partial^2 G_{B_\varepsilon^c}(x,y)}{\partial x_j\partial\nu_y}d\sigma_y}_{=-\frac{\partial}{\partial x_j}(\frac{x_i}{|x|^2}\varepsilon^2)}$$

$$+\frac{2}{\varepsilon^2}\sum_{l=1}^2\frac{\partial H_\Omega(x,0)}{\partial x_l}\underbrace{\int_{\partial B_\varepsilon}y_iy_l\frac{\partial^2 G_{B_\varepsilon^c}(x,y)}{\partial x_j\partial\nu_y}d\sigma_y}_{=O(\frac{\varepsilon^4}{|x|^3})}$$

$$-\frac{2}{\varepsilon^2}\sum_{l=1}^2\frac{\partial H_\Omega(0,0)}{\partial x_l}\frac{\ln|x|}{\ln\varepsilon}\underbrace{\int_{\partial B_\varepsilon}y_iy_l\frac{\partial^2 G_{B_\varepsilon^c}(x,y)}{\partial x_j\partial\nu_y}d\sigma_y}_{=-\frac{\partial}{\partial x_j}(\frac{\varepsilon^2}{|x|^2}(\varepsilon^2\frac{x_ix_l}{|x|^2}+\frac{\delta_l^i}{2}(|x|^2-\varepsilon^2)))=O(\frac{\varepsilon^4}{|x|^3})}$$

$$+O\Big(\frac{1}{|\ln\varepsilon|}\Big)\int_{\partial B_\varepsilon}\Big|\frac{\partial^2 G_{B_\varepsilon^c}(x,y)}{\partial x_j\partial\nu_y}\Big|d\sigma_y.$$

Differentiating (A.6) and using (3.11) as in (3.12),

$$\text{r.h.s.}=\frac{1}{2\pi}\Big(-1+\frac{\ln|x|}{\ln\varepsilon}+O\Big(\frac{1}{|\ln\varepsilon|}\Big)\Big)\frac{\partial}{\partial x_j}\Big(\frac{x_i}{|x|^2}\Big)+O\Big(\frac{|x|}{|\ln\varepsilon|\,\mathrm{dist}(x,\partial B_\varepsilon)^2}\Big)$$

$$+O\Big(\frac{|x|^2}{|\ln\varepsilon|\,\mathrm{dist}(x,\partial B_\varepsilon)^3}\Big)+O\Big(\frac{\varepsilon^2}{|x|^3}\Big).$$

Computation of $J_{6,\varepsilon}$. We will show that

$$J_{6,\varepsilon}=\begin{cases}O\Big(\frac{\varepsilon^{N-2}}{|x|^{N-1}}\Big)+O\Big(\frac{\varepsilon^{N-2}}{\mathrm{dist}(x,\partial B_\varepsilon)|x|^{N-2}}\Big) & \text{for } N\ge 3,\\ -\frac{\partial^2 H_\Omega(x,0)}{\partial x_j\partial y_i}+O\Big(\frac{1}{|x|^2\cdot|\ln\varepsilon|}\Big)+O\Big(\frac{1}{\mathrm{dist}(x,\partial B_\varepsilon)|\ln\varepsilon|}\Big) & \text{for } N=2.\end{cases}$$

We have, by (3.10) and $N\ge 3$,

$$J_{6,\varepsilon}=\int_{\partial B_\varepsilon}\frac{y_i}{\varepsilon}\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial\nu_y}$$

$$\times\Big(\frac{N-2}{\varepsilon}\Big(\frac{\partial H_\Omega(x,0)}{\partial x_j}-(2-N)H_\Omega(0,0)\frac{\varepsilon^{N-2}x_j}{|x|^N}\Big)+O\Big(\frac{1}{\mathrm{dist}(x,\partial\Omega_\varepsilon)}\Big)\Big)d\sigma_y$$

$$=\frac{N-2}{\varepsilon^2}\Big(\frac{\partial H(x,0)}{\partial x_j}-(2-N)H_\Omega(0,0)\frac{\varepsilon^{N-2}x_j}{|x|^N}\Big)\underbrace{\int_{\partial B_\varepsilon}y_i\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial\nu_y}d\sigma_y}_{=O(\frac{\varepsilon^N}{|x|^{N-1}})\text{ by (A.5)}}$$

$$+ O\Big(\frac{1}{\mathrm{dist}(x,\partial\Omega_\varepsilon)}\Big)\underbrace{\int_{\partial B_\varepsilon}\Big|\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial\nu_y}\Big|d\sigma_y}_{=O(\frac{\varepsilon^{N-2}}{|x|^{N-2}})\text{ by (A.4)}}$$

$$= O\Big(\frac{\varepsilon^{N-2}}{|x|^{N-1}}\Big) + O\Big(\frac{\varepsilon^{N-2}}{\mathrm{dist}(x,\partial B_\varepsilon)|x|^{N-2}}\Big),$$

and for $N = 2$,

$$J_{6,\varepsilon} = -\int_{\partial B_\varepsilon}\frac{y_i}{\varepsilon^2}\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial\nu_y}\Big[\frac{1}{-\ln\varepsilon - 2\pi H_\Omega(0,0)}\frac{\partial G_\Omega(x,0)}{\partial x_j}$$
$$-2\Big(\sum_{l=1}^{2}\frac{\partial^2 H_\Omega(x,0)}{\partial x_j\partial y_l}y_l - \nabla_y H_\Omega(0,0)\cdot y\frac{x_j}{|x|^2\ln\varepsilon}\Big) + O\Big(\frac{\varepsilon}{\mathrm{dist}(x,\partial B_\varepsilon)|\ln\varepsilon|}\Big)\Big]d\sigma_y$$
$$= \frac{1}{\varepsilon^2}\frac{1}{\ln\varepsilon + 2\pi H_\Omega(0,0)}\underbrace{\frac{\partial G_\Omega(x,0)}{\partial x_j}}_{=O(\frac{1}{|x|})}\underbrace{\int_{\partial B_\varepsilon}y_i\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial\nu_y}d\sigma_y}_{=O(\frac{\varepsilon^2}{|x|})\text{ by (A.5)}}$$
$$+ \frac{2}{\varepsilon^2}\frac{\partial^2 H_\Omega(x,0)}{\partial x_j\partial y_l}\underbrace{\int_{\partial B_\varepsilon}y_i y_l\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial\nu_y}d\sigma_y}_{=-\frac{\delta_i^l}{2}\varepsilon^2 + O(\frac{\varepsilon^4}{|x|^2})\text{ by (A.6)}} + \frac{2}{\varepsilon^2}\nabla_y H_\Omega(0,0)\frac{x_j}{|x|^2\ln\varepsilon}$$
$$\times\underbrace{\int_{\partial B_\varepsilon}y_i y_j\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial\nu_y}d\sigma_y}_{=-\frac{\delta_i^j}{2}\varepsilon^2 + O(\frac{\varepsilon^4}{|x|^2})\text{ by (A.6)}} + O\Big(\frac{1}{\mathrm{dist}(x,\partial B_\varepsilon)|\ln\varepsilon|}\Big)\underbrace{\int_{\partial B_\varepsilon}\Big|\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial\nu_y}\Big|d\sigma_y}_{=1\text{ by (A.4)}}$$
$$= -\frac{\partial^2 H_\Omega(x,0)}{\partial x_j\partial y_i} + O\Big(\frac{1}{|x|^2\cdot|\ln\varepsilon|}\Big) + O\Big(\frac{1}{\mathrm{dist}(x,\partial B_\varepsilon)|\ln\varepsilon|}\Big),$$

which proves the claim.

Now we look at the cases $N = 2$ and $N \ge 3$ separately. If $N \ge 3$, collecting the previous estimates, we have that

$$2J_{1,\varepsilon} + 2J_{2,\varepsilon} + 2J_{3,\varepsilon} - 2J_{4,\varepsilon} - 2J_{5,\varepsilon} - 2J_{6,\varepsilon}$$
$$= O\Big(\frac{\varepsilon^{2N-4}}{\mathrm{dist}(x,\partial B_\varepsilon)|x|^{2N-4}}\Big) + O\Big(\frac{\varepsilon^2}{\mathrm{dist}(x,\partial B_\varepsilon)}\Big) + O\Big(\frac{\varepsilon^{N-2}}{|x|^{N-2}}\Big) + O(\varepsilon)$$
$$+ O\Big(\frac{\varepsilon^{N-2}}{\mathrm{dist}(x,\partial B_\varepsilon)|x|^{N-2}}\Big) + O\Big(\frac{\varepsilon}{\mathrm{dist}(x,\partial B_\varepsilon)}\Big) + O\Big(\frac{\varepsilon^{N-2}}{\mathrm{dist}(x,\partial B_\varepsilon)}\Big)$$
$$+ O\Big(\frac{\varepsilon^{2N-4}}{|x|^{N-1}}\Big) + O\Big(\frac{\varepsilon^{N-2}}{|x|^N}\Big) + O\Big(\frac{\varepsilon^{N-2}|x|}{\mathrm{dist}(x,\partial B_\varepsilon)^N}\Big) + O\Big(\frac{\varepsilon^{N-2}|x|^2}{\mathrm{dist}(x,\partial B_\varepsilon)^{N+1}}\Big)$$
$$+ O\Big(\frac{\varepsilon^{N-2}}{|x|^{N-1}}\Big) + O\Big(\frac{\varepsilon^{N-2}}{\mathrm{dist}(x,\partial B_\varepsilon)|x|^{N-2}}\Big) = O\Big(\frac{\varepsilon^{N-2}}{\mathrm{dist}(x,\partial B_\varepsilon)|x|^{N-2}}\Big)$$
$$+ O\Big(\frac{\varepsilon}{\mathrm{dist}(x,\partial B_\varepsilon)}\Big) + O\Big(\frac{\varepsilon^{N-2}}{|x|^N}\Big) + O\Big(\frac{\varepsilon^{N-2}|x|}{\mathrm{dist}(x,\partial B_\varepsilon)^N}\Big) + O\Big(\frac{\varepsilon^{N-2}|x|^2}{\mathrm{dist}(x,\partial B_\varepsilon)^{N+1}}\Big).$$

If $N=2$, collecting the previous estimates, we have that

$$
\begin{aligned}
&2J_{1,\varepsilon}+2J_{2,\varepsilon}+2J_{3,\varepsilon}-2J_{4,\varepsilon}-2J_{5,\varepsilon}-2J_{6,\varepsilon}\\
&\quad= O\Big(\frac{\ln^2|x|}{\operatorname{dist}(x,\partial B_\varepsilon)\ln^2\varepsilon}\Big)+O\Big(\frac{1}{\operatorname{dist}(x,\partial B_\varepsilon)|\ln\varepsilon|^2}\Big)+O\Big(\Big|\frac{\ln|x|}{\ln\varepsilon}\Big|\Big)\\
&\qquad+O\Big(\frac{1}{|\ln\varepsilon|}\Big)+O\Big(\frac{1+|\ln|x||}{\operatorname{dist}(x,\partial B_\varepsilon)|\ln\varepsilon|}\Big)-2\frac{\partial^2 H_\Omega(x,0)}{\partial x_j\partial y_i}\\
&\qquad+\frac{1}{\pi}\Big(1-\frac{\ln|x|}{\ln\varepsilon}+O\Big(\frac{1}{|\ln\varepsilon|}\Big)\Big)\frac{\partial}{\partial x_j}\Big(\frac{x_i}{|x|^2}\Big)+O\Big(\frac{|x|}{|\ln\varepsilon|\operatorname{dist}(x,\partial B_\varepsilon)^2}\Big)\\
&\qquad+O\Big(\frac{|x|^2}{|\ln\varepsilon|\operatorname{dist}(x,\partial B_\varepsilon)^3}\Big)+O\Big(\frac{\varepsilon^2}{|x|^3}\Big)+2\frac{\partial^2 H_\Omega(x,0)}{\partial x_j\partial y_i}+O\Big(\frac{1}{|x|^2\cdot|\ln\varepsilon|}\Big)\\
&\quad=\frac{1}{\pi}\Big(1-\frac{\ln|x|}{\ln\varepsilon}\Big)\frac{\partial}{\partial x_j}\Big(\frac{x_i}{|x|^2}\Big)\\
&\qquad+O\Big(\frac{1}{|x|^2|\ln\varepsilon|}+\frac{1}{\operatorname{dist}(x,\partial B_\varepsilon)|\ln\varepsilon|}+\frac{\ln|x|}{\operatorname{dist}(x,\partial B_\varepsilon)|\ln\varepsilon|}\Big)\\
&\qquad+O\Big(\frac{|x|}{\operatorname{dist}(x,\partial B_\varepsilon)^2|\ln\varepsilon|}+\frac{|x|^2}{\operatorname{dist}(x,\partial B_\varepsilon)^3|\ln\varepsilon|}\Big),
\end{aligned}
$$

which ends the proof. ∎

We will apply the C^2-estimates of $\mathcal{R}_{\Omega_\varepsilon}$ at the *critical points* of $\mathcal{R}_{\Omega_\varepsilon}$. This leads to the following corollary.

Corollary 3.5. *Set, for every $c>0$ and $0<q\le 1$ and for small $\varepsilon>0$,*

$$
\mathcal{D}_{\varepsilon,q,c}=\left\{x\in\Omega_\varepsilon \text{ such that } \operatorname{dist}(x,\partial\Omega)\ge C>0,\ \begin{cases}|x|\ge c\varepsilon^q & \text{if } N\ge 3,\\ |x|\ge c r_\varepsilon^q & \text{if } N=2\end{cases}\right\}.
$$

Then for any $i,j=1,\dots,N$ and $x\in\mathcal{D}_{\varepsilon,q,c}$, we have that

$$
\frac{\partial^2\mathcal{R}_{\Omega_\varepsilon}(x)}{\partial x_i\partial x_j}=\frac{\partial^2\mathcal{R}_\Omega(x)}{\partial x_i\partial x_j}
+\begin{cases}
-\Big(\frac{2}{N\omega_N}+o(1)\Big)\frac{2(1-N)x_ix_j+\delta_i^j|x|^2}{|x|^{2N}}\varepsilon^{N-2}\\
\quad+O(\varepsilon^{1-q}+\varepsilon^{N-2-Nq}) & \text{for } N\ge 3,\\
-\frac{1}{\pi}\frac{\ln|x|}{\ln\varepsilon}\frac{-2x_ix_j+\delta_i^j|x|^2}{|x|^4}\\
\quad+O\Big(\frac{1}{r_\varepsilon^{2q}|\ln\varepsilon|}+r_\varepsilon^{1-q}\Big)+o(1) & \text{for } N=2.
\end{cases}
\tag{3.13}
$$

Remark 3.6. We will apply this corollary with $q\le\frac{N-2}{2N-3}$ for $N\ge 3$ and any $q<1$ for $N=2$.

Proof of Corollary 3.5. First of all, we have that, for $N \geq 2$ and since $|x| \geq c\varepsilon^q$,

$$\frac{\partial^2 \mathcal{R}_{B_\varepsilon^c}(x)}{\partial x_i \partial x_j} = -\frac{2}{N\omega_N} \frac{2(1-N)x_i x_j + \delta_i^j (|x|^2 - \varepsilon^2)}{(|x|^2-\varepsilon^2)^N} \varepsilon^{N-2}$$
$$= -\Big(\frac{2}{N\omega_N} + O\Big(\frac{\varepsilon^2}{|x|^{2N}}\Big)\Big) \frac{2(1-N)x_i x_j + \delta_i^j |x|^2}{|x|^{2N}} \varepsilon^{N-2}.$$

Next we observe that $\mathrm{dist}(x, \partial B_\varepsilon) = |x| - \varepsilon \sim |x|$ since $x \in \mathcal{D}_{\varepsilon,q,c}$ and by Remark 4.2. So we have that (3.8) becomes

$$\frac{\partial^2 \mathcal{R}_{\Omega_\varepsilon}(x)}{\partial x_i \partial x_j} = \frac{\partial^2 \mathcal{R}_{\Omega}(x)}{\partial x_i \partial x_j} + \begin{cases} -\Big(\frac{2}{N\omega_N} + o(1)\Big) \frac{2(1-N)x_i x_j + \delta_i^j |x|^2}{|x|^{2N}} \varepsilon^{N-2} \\ \quad + O(\varepsilon^{1-q} + \varepsilon^{N-2-Nq}) & \text{for } N \geq 3, \\ -\frac{1}{\pi} \frac{\ln|x|}{\ln \varepsilon} \frac{\partial}{\partial x_j}\Big(\frac{x_i}{|x|^2}\Big) \\ \quad + O\Big(\frac{1}{r_\varepsilon^{2q} |\ln \varepsilon|}\Big) + O(r_\varepsilon^{1-q}) + \underbrace{O\Big(\frac{\varepsilon^2}{|x|^6}\Big)}_{=o(1)} & \text{for } N = 2, \end{cases}$$

which gives the claim. ∎

4. The case $\nabla \mathcal{R}_\Omega(0) \neq 0$. Proof of Theorem 1.1 and Corollary 1.6

We start this section with a *necessary condition* on the location of the critical points of $\mathcal{R}_{\Omega_\varepsilon}$. Basically, it is a consequence of Proposition 3.3.

Proposition 4.1. *Set*

$$y_0 = \frac{2^{\frac{1}{2N-3}}}{(N\omega_N)^{\frac{1}{2N-3}} |\nabla \mathcal{R}_\Omega(0)|^{\frac{2N-2}{2N-3}}} \nabla \mathcal{R}_\Omega(0). \tag{4.1}$$

If x_ε is a critical point of $\mathcal{R}_{\Omega_\varepsilon}(x)$, then for $\varepsilon \to 0$ we have that either

$$x_\varepsilon \to x_0 \neq 0 \quad \text{with } \nabla \mathcal{R}_\Omega(x_0) = 0 \tag{4.2}$$

or

$$\begin{cases} x_\varepsilon = (y_0 + o(1)) \varepsilon^{\frac{N-2}{2N-3}} & \text{if } N \geq 3, \\ x_\varepsilon = [y_0 + o(1)] r_\varepsilon & \text{if } N = 2, \end{cases} \tag{4.3}$$

where r_ε is defined in (1.5).

Proof. Let x_ε be a critical point of $\mathcal{R}_{\Omega_\varepsilon}$ and first consider $N \geq 3$. By (3.4), we get

$$0 = \nabla \mathcal{R}_\Omega(x_\varepsilon) - \frac{2\varepsilon^{N-2}}{N\omega_N} \frac{x_\varepsilon}{(|x_\varepsilon|^2 - \varepsilon^2)^{N-1}} + O\Big(\frac{\varepsilon^{N-2}}{|x_\varepsilon|^{N-1}}\Big) + O(\varepsilon). \tag{4.4}$$

If $x_\varepsilon \to x_0 \neq 0$, we have that (4.2) holds. So let us suppose that

$$x_\varepsilon \to 0.$$

By (3.4), we have

$$\nabla\mathcal{R}_\Omega(0) + o(1) = \frac{2\varepsilon^{N-2}}{N\omega_N|x_\varepsilon|^{N-2}}\Big[\frac{x_\varepsilon}{|x_\varepsilon|^N(1-\frac{\varepsilon^2}{|x_\varepsilon|^2})^{N-1}} + O\Big(\frac{1}{|x_\varepsilon|}\Big)\Big] \tag{4.5}$$

and, since

$$\Big|\frac{x_\varepsilon}{|x_\varepsilon|^N(1-\frac{\varepsilon^2}{|x_\varepsilon|^2})^{N-1}} + O\Big(\frac{1}{|x_\varepsilon|}\Big)\Big| \geq \frac{1}{|x_\varepsilon|^{N-1}} - \frac{C}{|x_\varepsilon|} \to +\infty \quad \text{if } N \geq 3,$$

then $\lim_{\varepsilon\to 0}\frac{\varepsilon}{|x_\varepsilon|} = 0$ and (4.5) becomes

$$\nabla\mathcal{R}_\Omega(0) + o(1) = \frac{2\varepsilon^{N-2}}{N\omega_N|x_\varepsilon|^{2N-3}}\Big[\frac{x_\varepsilon}{|x_\varepsilon|} + o(1)\Big].$$

Therefore,

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{N-2}}{|x_\varepsilon|^{2N-3}} = \frac{N\omega_N}{2}|\nabla\mathcal{R}_\Omega(0)|,$$

which gives the claim.

Next we consider the case $N = 2$. As for $N \geq 3$, we get that if $x_\varepsilon \to x_0 \neq 0$, by (3.4) we derive that (4.2) holds. So assume that $x_\varepsilon \to 0$. By (3.4), we have

$$\nabla\mathcal{R}_\Omega(0) + o(1) = \frac{x_\varepsilon}{\pi|x_\varepsilon|^2}\Big(\frac{\ln|x_\varepsilon|}{\ln\varepsilon} + \frac{\varepsilon^2}{|x_\varepsilon|^2-\varepsilon^2} + O\Big(\frac{1}{|\ln\varepsilon|}\Big)\Big). \tag{4.6}$$

Since $\frac{1}{|x_\varepsilon|} \to +\infty$ and at least one among $\frac{(x_\varepsilon)_i}{|x_\varepsilon|} \neq 0$, using that the two terms $\frac{\ln|x_\varepsilon|}{\ln\varepsilon}$, $\frac{\varepsilon^2}{|x_\varepsilon|^2-\varepsilon^2}$ have the same sign, we get that

$$\frac{\ln|x_\varepsilon|}{\ln\varepsilon} = o(1) \quad \text{and} \quad \frac{\varepsilon^2}{|x_\varepsilon|^2-\varepsilon^2} = o(1), \tag{4.7}$$

which implies also that

$$\frac{\varepsilon^2}{|x_\varepsilon|^3} = o(1) \quad \text{and then} \quad \frac{\varepsilon^2}{|x_\varepsilon|(|x_\varepsilon|^2-\varepsilon^2)} = o(1), \tag{4.8}$$

(indeed, if by contradiction $\frac{\varepsilon^2}{|x_\varepsilon|^3} \geq k > 0$, then $2 - 3\frac{\ln|x_\varepsilon|}{\ln\varepsilon} \leq \frac{\ln k}{\ln\varepsilon}$, a contradiction as $\varepsilon \to 0$.) So that (4.6) becomes

$$\begin{aligned}\pi|\nabla\mathcal{R}_\Omega(0)|(1+o(1)) &= \frac{1}{|x_\varepsilon|}\frac{\ln|x_\varepsilon|}{\ln\varepsilon}\Big(1 + O\Big(\frac{1}{|\ln|x_\varepsilon||}\Big)\Big)\\ &= \frac{r_\varepsilon}{|x_\varepsilon|}\frac{\ln|x_\varepsilon|}{\ln r_\varepsilon}(1+o(1))\\ &= \Big[\frac{\ln\frac{|x_\varepsilon|}{r_\varepsilon}}{\frac{|x_\varepsilon|}{r_\varepsilon}\ln r_\varepsilon} + \frac{1}{\frac{|x_\varepsilon|}{r_\varepsilon}}\Big](1+o(1)).\end{aligned} \tag{4.9}$$

Since the both terms $\frac{\ln(|x_\varepsilon|/r_\varepsilon)}{(|x_\varepsilon|/r_\varepsilon)\ln r_\varepsilon}$ and $\frac{|x_\varepsilon|}{r_\varepsilon}$ are positive and $r_\varepsilon \to 0$, it cannot happen that $\frac{|x_\varepsilon|}{r_\varepsilon} \to 0$. Moreover, since $\nabla\mathcal{R}_\Omega(0) \neq 0$ it is not possible that $\frac{|x_\varepsilon|}{r_\varepsilon} \to +\infty$. Then $\frac{|x_\varepsilon|}{r_\varepsilon} \to A \in (0,+\infty)$ and by (4.9) we have that $A = \frac{1}{\pi|\nabla\mathcal{R}_\Omega(0)|}$ which proves the claim. ∎

Remark 4.2. Let us point out that

$$r_\varepsilon \in \Big(\frac{1}{|\ln\varepsilon|}, \frac{1}{\sqrt{|\ln\varepsilon|}}\Big). \tag{4.10}$$

In fact, taking $g(r) = r - \frac{\ln r}{\ln\varepsilon}$, then

$$g'(r) = 1 - \frac{1}{r\ln\varepsilon} > 0 \quad \text{for any } r \in (0,\infty),$$

and

$$g\Big(\frac{1}{|\ln\varepsilon|}\Big) = \frac{1-\ln|\ln\varepsilon|}{|\ln\varepsilon|} < 0, \quad g\Big(\frac{1}{\sqrt{|\ln\varepsilon|}}\Big) = \frac{1}{\sqrt{|\ln\varepsilon|}} - \frac{\ln|\ln\varepsilon|}{2|\ln\varepsilon|} > 0.$$

Remark 4.3. Proposition 4.1 implies that the critical points of $\mathcal{R}_{\Omega_\varepsilon}$ that converge to 0 belong to the ball

$$\mathcal{C}_\varepsilon = \begin{cases} B(y_0\varepsilon^{\frac{N-2}{2N-3}}, \delta\varepsilon^{\frac{N-2}{2N-3}}) & \text{if } N \geq 3, \\ B(y_0 r_\varepsilon, \delta r_\varepsilon) & \text{if } N = 2, \end{cases}$$

where $\delta > 0$ is a small fixed constant. So, if x_ε is a critical point of $\mathcal{R}_{\Omega_\varepsilon}$, then either $x_\varepsilon \in \Omega_\varepsilon \setminus \mathcal{C}_\varepsilon$, and then x_ε converges to a critical point of $\mathcal{R}_\Omega$ or $x_\varepsilon \to 0$, and then $x_\varepsilon \in \mathcal{C}_\varepsilon$.

In the next lemma, we introduce a function F which plays a crucial role in the proof of the main results.

Lemma 4.4. *Let us consider the function $F : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ as*

$$F(y) = \begin{cases} \displaystyle\sum_{j=1}^{N} \frac{\partial\mathcal{R}_\Omega(0)}{\partial x_j} y_j - \frac{D_N}{4-2N}\frac{1}{|y|^{2N-4}} & \text{if } N \geq 3, \\ \displaystyle\sum_{j=1}^{2} \frac{\partial\mathcal{R}_\Omega(0)}{\partial x_j} y_j - \frac{1}{\pi}\log|y| & \text{if } N = 2 \end{cases}$$

with

$$D_N = \frac{2}{N\omega_N} \quad \text{for } N \geq 2. \tag{4.11}$$

Then, the function F has a unique critical point y_0 (see (4.1)) which is non-degenerate and satisfies

$$\det(\mathrm{Hess}(F(y_0))) = (-1)^N \frac{|\nabla\mathcal{R}_\Omega(0)|^{\frac{N(2N-2)}{2N-3}}}{D_N^{\frac{N}{2N-3}}}(3-2N) \neq 0. \tag{4.12}$$

Remark 4.5. The non-degeneracy of y_0, together with (4.12), implies that there exists $\overline{\delta} > 0$ such that $B(y_0, \overline{\delta}) \subset \mathbb{R}^N \setminus \{0\}$ and

$$\det(\mathrm{Hess}(F(y))) \neq 0$$

in $B(y_0, \overline{\delta})$. If necessary, choosing a smaller $\overline{\delta}$ we can assume that $|y| \geq c > 0$ for every $y \in B(y_0, \overline{\delta})$. From now, we fix $\delta = \overline{\delta}$ in the definition of $\mathcal{C}_\varepsilon$.

Proof of Lemma 4.4. By a straightforward computation, we have that y_0 is the unique critical point of $F(y)$. Next we have that for $N \geq 2$, the Hessian matrix of $F(y)$ computed at $y_0 = (y_{1,0}, \dots, y_{N,0})$ is given by

$$\mathrm{Hess}(F(y_0)) = -D_N \frac{1}{|y_0|^{2N-2}} \underbrace{\left(\delta_j^i + (2-2N) \frac{y_{i,0} y_{j,0}}{|y_0|^2}\right)_{1 \leq i,j \leq N}}_{A_{ij}}.$$

Note that $\lambda_1 = 3 - 2N$ is the first eigenvalue of the matrix A_{ij} with associated eigenvector $v_1 = y_0$. Next we observe that any vector of the space

$$X := \{x \in \mathbb{R}^N,\ x \perp y_0\}$$

is an eigenvector of the matrix A_{ij} corresponding to the eigenvalue $\lambda = 1$, so that $\det(A_{ij}) = 3 - 2N$. Hence we have that

$$\begin{aligned}
\det \mathrm{Hess}(F(y_0)) &= \det\Big[-\frac{|\nabla \mathcal{R}_\Omega(0)|^{\frac{2N-2}{2N-3}}}{D_N^{\frac{1}{2N-3}}} \left(\delta_j^i - (2N-2) \frac{y_{i,0} y_{j,0}}{|y_0|^2}\right)_{1 \leq i,j \leq N}\Big] \\
&= (-1)^N \frac{|\nabla \mathcal{R}_\Omega(0)|^{\frac{N(2N-2)}{2N-3}}}{D_N^{\frac{N}{2N-3}}} \prod_{i=1}^N \lambda_i \\
&= (-1)^N \frac{|\nabla \mathcal{R}_\Omega(0)|^{\frac{N(2N-2)}{2N-3}}}{D_N^{\frac{N}{2N-3}}} (3-2N) \neq 0,
\end{aligned}$$

which proves (4.12). ∎

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Recall that we assume that $P = 0 \in \Omega$. If $x_\varepsilon \in B(0,r) \setminus B(0,\varepsilon)$ is a critical point of $\mathcal{R}_{\Omega_\varepsilon}$, then, since $B(0,r) \subset \Omega$ is chosen not containing any critical point of $\mathcal{R}_\Omega$, by Proposition 4.1, we necessarily have that $x_\varepsilon \to 0$ and the expansion in (4.3) holds. Then by Proposition 4.1 and Remark 4.3, it is enough to prove the existence and the uniqueness of the critical point x_ε in the ball $\mathcal{C}_\varepsilon$.

Let us introduce the function $F_\varepsilon \colon B(y_0, \delta) \to \mathbb{R}$ as

$$F_\varepsilon(y) = \begin{cases} \dfrac{1}{\varepsilon^{\frac{N-2}{2N-3}}} \mathcal{R}_{\Omega_\varepsilon}(\varepsilon^{\frac{N-2}{2N-3}} y) & \text{if } N \geq 3, \\ \dfrac{1}{r_\varepsilon} \mathcal{R}_{\Omega_\varepsilon}(r_\varepsilon y) & \text{if } N = 2, \end{cases}$$

where y_0 is defined in (4.1) and δ is chosen as in Remark 4.5. Let us show that

$$\nabla F_\varepsilon \to \nabla F \quad \text{in } C^1(\overline{B(y_0,\delta)}),$$

where F is the function defined in Lemma 4.4. Indeed, using Proposition 3.3 we have that

$$\nabla F_\varepsilon(y) = \nabla \mathcal{R}_{\Omega_\varepsilon}(\varepsilon^{\frac{N-2}{2N-3}} y) = \nabla \mathcal{R}_\Omega(0) - \frac{2}{N\omega_N}\frac{y}{(|y|^2+o(1))^{N-1}} + o(1),$$

which gives the uniform convergence of ∇F_ε to ∇F in $B(y_0,\delta)$.

Concerning $N=2$, we have that, again by Proposition 3.3, (1.5) and since $\frac{\varepsilon^k}{r_\varepsilon} = o(1)$ for any $k>0$ by Remark 4.2,

$$\begin{aligned}
\nabla F_\varepsilon(y) &= \nabla \mathcal{R}_{\Omega_\varepsilon}(r_\varepsilon y)\\
&= \nabla \mathcal{R}_\Omega(0) + o(1) - \frac{y}{\pi|y|^2}\frac{\ln|y| + \ln r_\varepsilon}{r_\varepsilon \ln \varepsilon} - \underbrace{\frac{\varepsilon^2 y}{\pi r_\varepsilon^3(|y|^2+o(1))|y|^2}}_{=o(1)}\\
&= \nabla \mathcal{R}_\Omega(0) - \frac{y}{\pi|y|^2}\left(1+\frac{\ln|y|}{\ln r_\varepsilon}\right) + o(1),
\end{aligned}$$

which gives the uniform convergence of ∇F_ε to ∇F in $B(y_0,\delta)$ for $N=2$.

Let us show the C^1-convergence. By Remark 4.5, we can apply Corollary 3.5 with $q = \frac{N-2}{2N-3}$ and a suitable $c>0$ such that for $N \geq 3$,

$$\begin{aligned}
\frac{\partial^2 F_\varepsilon(y)}{\partial x_i \partial x_j} &= \varepsilon^{\frac{N-2}{2N-3}} \frac{\partial^2 \mathcal{R}_{\Omega_\varepsilon}(\varepsilon^{\frac{N-2}{2N-3}} y)}{\partial x_i \partial x_j}\\
&= o(1) + \varepsilon^{\frac{N-2}{2N-3}} \frac{\partial^2 \mathcal{R}_{B_\varepsilon^c}(\varepsilon^{\frac{N-2}{2N-3}} y)}{\partial x_i \partial x_j} + \underbrace{O(\varepsilon) + O(\varepsilon^{\frac{(N-2)^2}{2N-3}})}_{=o(1)}\\
&= \underbrace{2C_N(2-N)}_{=-\frac{2}{N\omega_N}}\left(\frac{\delta_j^i}{|y|^{2N-2}} + (2-2N)\frac{y_i y_j}{|y^{2N}|}\right) + o(1),
\end{aligned}$$

which gives the claim. In the same way, if $N=2$, using (1.5) and again the Corollary 3.5 with any $q<1$ and a suitable $c>0$,

$$\begin{aligned}
\frac{\partial^2 F_\varepsilon(y)}{\partial x_i \partial x_j} &= r_\varepsilon \frac{\partial^2 \mathcal{R}_{\Omega_\varepsilon}(r_\varepsilon y)}{\partial x_i \partial x_j} = -\frac{1}{\pi}\frac{\ln|y| + \ln r_\varepsilon}{r_\varepsilon \ln\varepsilon}\frac{-2y_i y_j + \delta_i^j|y|^2}{|y|^4} + O\left(\frac{r_\varepsilon^{2-2q}}{r_\varepsilon|\ln\varepsilon|}\right)\\
&= -\frac{1}{\pi}\left(1+\frac{\ln|y|}{\ln r_\varepsilon}\right)\frac{-2y_i y_j + \delta_i^j|y|^2}{|y|^4} + O\left(\frac{r_\varepsilon^{2-2q}}{|\ln r_\varepsilon|}\right)
\end{aligned}$$

since $r_\varepsilon \to 0$,

$$\text{r.h.s.} = -\frac{1}{\pi}\frac{-2y_i y_j + \delta_i^j|y|^2}{|y|^4} + o(1),$$

which gives the claim.

Finally, the C^1-convergence of ∇F_ε to ∇F and (4.12) gives that

$$\deg(\nabla F_\varepsilon, B(y_0,\delta),0)=\deg(\nabla F, B(y_0,\delta),0)\neq 0,$$

which, jointly with the non-degeneracy of y_0, implies the existence and uniqueness of a critical point $y_\varepsilon \in B(y_0,\delta)$ of F_ε. By the definition of F_ε, this implies the existence of a unique critical point x_ε for $\mathcal{R}_{\Omega_\varepsilon}$ in $\mathcal{C}_\varepsilon$. Finally, by the definition of $\mathcal{C}_\varepsilon$, $x_\varepsilon \to 0$ and by (4.3) of Proposition 4.1 we get (1.4). Moreover,

$$\begin{aligned}\mathrm{index}_{x_\varepsilon}(\mathcal{R}_{\Omega_\varepsilon}(x)) &= \mathrm{index}_{y_\varepsilon}(F_\varepsilon(y)) = \mathrm{sgn}(\det \mathrm{Jac}(F_\varepsilon(y_\varepsilon)))\\ &= \mathrm{sgn}(\det \mathrm{Jac}(F(y_0))) = (-1)^{N+1}.\end{aligned}$$

We end the proof showing that $\mathcal{R}_{\Omega_\varepsilon}(x_\varepsilon)\to\mathcal{R}_\Omega(0)$. By Proposition 3.1, we have that for $N\geq 3$,

$$\begin{aligned}\mathcal{R}_{\Omega_\varepsilon}(x_\varepsilon) &= \underbrace{\mathcal{R}_\Omega(x_\varepsilon)}_{=\mathcal{R}_\Omega(0)+o(1)} + \mathcal{R}_{B_\varepsilon^c}(x_\varepsilon) + O\underbrace{\left(\frac{\varepsilon^{N-2}}{|x_\varepsilon|^{N-2}}\right)}_{=O(\varepsilon^{\frac{(N-2)(N-1)}{2N-3}})} + O(\varepsilon)\\ &= \mathcal{R}_\Omega(0) + C_N\underbrace{\frac{\varepsilon^{N-2}}{(|x_\varepsilon|^2-\varepsilon^2)^{N-2}}}_{=O(\varepsilon^{\frac{N-2}{2N-3}})} + o(1) = \mathcal{R}_\Omega(0)+o(1),\end{aligned} \tag{4.13}$$

which gives the claim. For $N=2$, we have by Remark 3.2

$$\begin{aligned}\mathcal{R}_{\Omega_\varepsilon}(x_\varepsilon) &= \underbrace{\mathcal{R}_\Omega(x_\varepsilon)}_{=\mathcal{R}_\Omega(0)+o(1)} + \underbrace{\frac{1}{2\pi}\ln\left(1-\frac{\varepsilon^2}{|x_\varepsilon|^2}\right)}_{=\frac{1}{2\pi}\ln(1+o(1)) \text{ by (4.7)}} + \underbrace{O\left(\left|\frac{\ln|x_\varepsilon|}{\ln\varepsilon}\right|\right)}_{=o(1) \text{ by (4.7)}}\\ &= \mathcal{R}_\Omega(0)+o(1),\end{aligned} \tag{4.14}$$

which gives the claim. ■

Next we prove Corollary 1.6.

Proof of Corollary 1.6. Set $\mathcal{B}=\{x\in\Omega \text{ such that } \nabla\mathcal{R}_\Omega(x)=0\}$. Since $\nabla\mathcal{R}_\Omega(P)\neq 0$, we get that $\mathcal{B}\cap B(P,r)=\emptyset$ for any small fixed $r>0$. Now we write

$$\Omega_\varepsilon = \mathcal{C}_1\cup\mathcal{C}_2\cup(B(P,r)\setminus B(P,\varepsilon))$$

with

$$\mathcal{C}_1=\{x, \mathrm{dist}(x,\mathcal{B})\leq r\} \quad \text{and} \quad \mathcal{C}_2 := \Omega_\varepsilon\setminus(\mathcal{C}_1\cup B(P,r)),$$

where r is such that $\det(\mathrm{Hess}(\mathcal{R}_\Omega(x)))\neq 0$ in $\mathcal{C}_1$.

By Proposition 3.3 and Corollary 3.5, we have that $\mathcal{R}_{\Omega_\varepsilon}\to\mathcal{R}_\Omega$ in $\mathcal{C}_1$ and so by the choice of r, the non-degeneracy of the critical points of $\mathcal{R}_\Omega$ and the C^2-convergence of $\mathcal{R}_{\Omega_\varepsilon}$ to $\mathcal{R}_\Omega$ in $\mathcal{C}_1$ we have

$$\sharp\{\text{critical points of } \mathcal{R}_{\Omega_\varepsilon}(x) \text{ in } \mathcal{C}_1\} = \sharp\{\mathcal{B}\},$$

while the C^1-convergence of $\mathcal{R}_{\Omega_\varepsilon}$ to $\mathcal{R}_\Omega$ in $\overline{\Omega_\varepsilon \setminus B(P,r)}$ gives

$$\sharp\{\text{critical points of } \mathcal{R}_{\Omega_\varepsilon}(x) \text{ in } \mathcal{C}_2\} = 0.$$

Finally, from Theorem 1.1, we get that

$$\sharp\{\text{critical points of } \mathcal{R}_{\Omega_\varepsilon}(x) \text{ in } B(P,r) \setminus B(P,\varepsilon)\} = 1,$$

which proves the claim. ■

5. The case $\nabla \mathcal{R}_\Omega(P) = 0$, proof of Theorem 1.8

As in the previous sections, we assume that $P = 0$. We will follow the line of the proof of Theorem 1.1. So we start with a necessary condition satisfied by the critical points of $\mathcal{R}_{\Omega_\varepsilon}(x)$.

Proposition 5.1. *If 0 is a non-degenerate critical point for $\mathcal{R}_\Omega(x)$ and x_ε is a critical point of $\mathcal{R}_{\Omega_\varepsilon}(x)$, then for $\varepsilon \to 0$ we have that either*

$$x_\varepsilon \to x_0 \neq 0 \quad \text{with } \nabla \mathcal{R}_\Omega(x_0) = 0$$

or

$$x_\varepsilon = \begin{cases} \left(\dfrac{(2+o(1))}{N\omega_N \lambda}\right)^{\frac{1}{2N-2}} \varepsilon^{\frac{N-2}{2N-2}} v & \text{if } N \geq 3, \\ \hat{r}_\varepsilon (1+o(1)) v_l & \text{if } N = 2, \end{cases} \tag{5.1}$$

where λ is a positive eigenvalue of the Hessian matrix $\mathrm{Hess}(\mathcal{R}_\Omega(0))$, *$v$ is an associated eigenvector with $|v| = 1$ and $\hat{r}_\varepsilon$ is the unique solution of*

$$r^2 - \frac{\ln r}{\lambda\pi \ln \varepsilon} = 0 \quad \text{in } (0,\infty). \tag{5.2}$$

Remark 5.2. It is immediate to check that (5.2) admits only one solution $\hat{r}_\varepsilon$ satisfying

$$\hat{r}_\varepsilon \in \left(\frac{1}{(\lambda\pi|\ln\varepsilon|)^{\frac12}}, \frac{1}{(\lambda\pi|\ln\varepsilon|)^{\frac14}}\right).$$

This can be seen by observing that $r_{\varepsilon,1} = \frac{1}{(\lambda\pi|\ln\varepsilon|)^{\frac12}}$ and $r_{\varepsilon,2} = \frac{1}{(\lambda\pi|\ln\varepsilon|)^{\frac14}}$ satisfy

$$h(r_{\varepsilon,1}) = \frac{1 - \ln\sqrt{\lambda\pi|\ln\varepsilon|}}{\lambda\pi|\ln\varepsilon|} < 0, \quad h(r_{\varepsilon,2}) = \frac{1}{\sqrt{\lambda\pi|\ln\varepsilon|}} - \frac{\ln(\lambda\pi|\ln\varepsilon|)}{4\lambda\pi|\ln\varepsilon|} > 0.$$

Remark 5.3. Unlike when $\nabla\mathcal{R}_\Omega(P) \neq 0$, here in general we do not have the uniqueness of the limit point of x_ε. It depends on the simplicity of the eigenvalue λ. This will play a crucial role in the next results.

Proof of Proposition 5.1. The first assertion follows arguing as in Proposition 4.1 (see (4.4) or (3.4)). On the other hand, if $x_\varepsilon \to 0$, using (4.5) and (4.6) for $N \ge 3$ and (4.7) and (4.8) for $N = 2$, we get again $\frac{\varepsilon}{|x_\varepsilon|} \to 0$. Furthermore, from Proposition 3.3 and $\nabla \mathcal{R}_\Omega(0) = 0$, we get

$$0 = \sum_{j=1}^{N} \Big(\frac{\partial^2 \mathcal{R}_\Omega(0)}{\partial x_i \partial x_j} + o(1) \Big) x_{j,\varepsilon} + \begin{cases} -\dfrac{2 x_{i,\varepsilon} \varepsilon^{N-2}}{N \omega_N |x_\varepsilon|^{2N-2}} (1 + o(1)) & \text{for } N \ge 3, \\ \dfrac{x_{i,\varepsilon} \ln |x_\varepsilon|}{\pi |x_\varepsilon|^2 |\ln \varepsilon|} (1 + o(1)) & \text{for } N = 2. \end{cases} \tag{5.3}$$

From (5.3), we immediately get that, as $\varepsilon \to 0$,

$$\begin{cases} \dfrac{2 \varepsilon^{N-2}}{N \omega_N |x_\varepsilon|^{2N-2}} \to \lambda & \text{for } N \ge 3, \\ -\dfrac{\ln |x_\varepsilon|}{\pi |x_\varepsilon|^2 |\ln \varepsilon|} \to \lambda & \text{for } N = 2, \end{cases}$$

where λ is a positive eigenvalue of the Hessian matrix $\mathrm{Hess}(\mathcal{R}_\Omega(0))$. Hence (5.1) follows by dividing (5.3) by $|x_\varepsilon|$ and passing to the limit. ∎

Analogously to the previous section, let us introduce "the limit function" of a suitable rescaling of $\mathcal{R}_{\Omega_\varepsilon}$.

Lemma 5.4. *Let us consider the function* $\widehat{F} : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ *as*

$$\widehat{F}(y) = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{i,j=1}^{N} \frac{\partial^2 \mathcal{R}_\Omega(0)}{\partial x_i \partial x_j} y_i y_j - \frac{D_N}{(4 - 2N)|y|^{2N-4}} & \text{for } N \ge 3, \\ \dfrac{1}{2} \displaystyle\sum_{i,j=1}^{2} \frac{\partial^2 \mathcal{R}_\Omega(0)}{\partial x_i \partial x_j} y_i y_j - D_2 \ln |y| & \text{for } N = 2, \end{cases} \tag{5.4}$$

where D_N *($N \ge 2$) is the same as in* (4.11). *Suppose that* $\mathrm{Hess}(\mathcal{R}_\Omega(0))$ *has* $m \le N$ *positive eigenvalues* $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m$. *Then we have that*

$$\text{if } \nabla \widehat{F}(\overline{y}) = 0, \text{ then } \overline{y} = \overline{y}^{(l)} = \Big(\frac{D_N}{\lambda_l} \Big)^{\frac{1}{2N-2}} v^{(l)} \text{ for some } l \in \{1, \dots, m\}, \tag{5.5}$$

where $v^{(l)}$ *is an eigenvector of the matrix* $(\frac{\partial^2 \mathcal{R}_\Omega(0)}{\partial x_i \partial x_j})_{i,j=1,\dots,N}$ *associated to* λ_l *such that* $|v^{(l)}| = 1$.

Moreover, it holds

$$\det(\mathrm{Hess}(\widehat{F}(\overline{y}^{(l)}))) = (2N - 2) \lambda_l \prod_{\substack{s=1 \\ s \ne l}}^{N} (\lambda_s - \lambda_l). \tag{5.6}$$

Remark 5.5. If the eigenvalue λ_l is *simple*, we get two corresponding eigenvectors $v_\pm^{(l)}$ with $|v_\pm^{(l)}| = 1$ and then two critical points $\overline{y}_\pm^{(l)}$ in (5.5). Moreover, by (5.6) $\overline{y}_\pm^{(l)}$ are non-degenerate critical points and then it is possible to select $\delta^{(l)} > 0$ such that in $B(\overline{y}_\pm^{(l)}, \delta^{(l)})$ we have $\det(\mathrm{Hess}(\widehat{F}(y))) \ne 0$.

Proof of Lemma 5.4. Observe that $\nabla \widehat{F}(y)$ is given by

$$\frac{\partial \widehat{F}(y)}{\partial y_i} = \sum_{j=1}^{N} \frac{\partial^2 \mathcal{R}_\Omega(0)}{\partial x_i \partial x_j} y_j - \frac{D_N y_i}{|y|^{2N-2}} \quad \text{for } N \geq 2, \tag{5.7}$$

and then if $\nabla \widehat{F}(y) = 0$, we immediately get that $\frac{D_N}{|y|^{2N-2}} = \lambda_l$ for some $l \in \{1, \ldots, m\}$ proving (5.5).

Claim (5.6) will be proved by diagonalizing the matrix $\mathrm{Hess}(\mathcal{R}_\Omega(0))$. Here we consider the case $N \geq 3$ ($N = 2$ can be handled in the same way). Let $\mathbf{P}$ be the orthogonal matrix such that

$$\mathbf{P}^\mathsf{T}\, \mathrm{Hess}(\mathcal{R}_\Omega(0))\mathbf{P} = \mathrm{diag}(\lambda_1, \ldots, \lambda_N).$$

Taking $Z = \mathbf{P}^\mathsf{T}(y)$, we get that the system $\nabla \widehat{F}(y) = 0$ becomes

$$\lambda_i Z_i - D_N \frac{Z_i}{|Z|^{2N-2}} = 0 \quad \text{for } N \geq 2 \text{ and } i = 1, \ldots, N.$$

Note that these zeros are critical points of the function

$$T(y) = \frac{1}{2} \sum_{j=1}^{N} \lambda_j y_j^2 - \frac{D_N}{(4-2N)|y|^{2N-4}} \quad \text{for } N \geq 3. \tag{5.8}$$

Next step is the computation of the determinant of the Hessian matrix of $\widehat{F}$. We know that

$$\begin{aligned}
&\det \mathrm{Hess}(\widehat{F}(y^{(l)})) \\
&\quad = \det(\mathbf{P}^\mathsf{T}\, \mathrm{Hess}(T(y^{(l)}))\mathbf{P}) = \det(\mathrm{Hess}(T(y^{(l)}))) \\
&\quad = \det\Big(\mathrm{diag}(\lambda_1, \ldots, \lambda_N) + \Big(-\frac{D_N}{|y^{(l)}|^{2N-2}}\delta_{ik} + (2N-2)\frac{D_N y_i^{(l)} y_k^{(l)}}{|y^{(l)}|^{2N}}\Big)_{1 \leq i,k \leq N}\Big) \\
&\quad = \det\underbrace{\Big(\mathrm{diag}(\lambda_1 - \lambda_l, \ldots, \lambda_N - \lambda_l) + (2N-2)\lambda_l\Big(\frac{y_i^{(l)} y_k^{(l)}}{|y^{(l)}|^2}\Big)_{1 \leq i,k \leq N}\Big)}_{:=\mathbf{M_0}}.
\end{aligned}$$

By the basic theory of linear algebra, we can find that the eigenvalues of $\mathbf{M_0}$ are $\lambda_s - \lambda_l$ for $s = 1, \ldots, l-1, l+1, \ldots, N$ and $(2N-2)\lambda_l$. Hence we have

$$\det \mathrm{Hess}(\widehat{F}(y^{(l)})) = (2N-2)\lambda_l \prod_{\substack{s=1 \\ s \neq l}}^{N} (\lambda_s - \lambda_l). \qquad \blacksquare$$

Assume that λ_l is a *simple* eigenvalue. Then, following the notations of Remark 5.5, analogously to the previous section, we define

$$\mathcal{C}_{\varepsilon,\pm}^{(l)} = \begin{cases} B(y_\pm^{(l)} \varepsilon^{\frac{N-2}{2N-2}}, \delta^{(l)} \varepsilon^{\frac{N-2}{2N-2}}) & \text{if } N \geq 3, \\ B(y_\pm^{(l)} \widehat{r}_\varepsilon, \delta^{(l)} \widehat{r}_\varepsilon) & \text{if } N = 2, \end{cases} \quad \text{and} \quad \widehat{\mathcal{C}}_\varepsilon = \bigcup_l \mathcal{C}_{\varepsilon,\pm}^{(l)}.$$

Proposition 5.1 implies that, under the assumption that all the eigenvalues of the matrix $(\frac{\partial^2 \mathcal{R}_\Omega(0)}{\partial x_i \partial x_j})_{i,j=1,\ldots,N}$ are positive, the critical points x_ε satisfy either $x_\varepsilon \in \Omega_\varepsilon \setminus \widehat{\mathcal{C}}_\varepsilon$, and then x_ε converge to a critical point of $\mathcal{R}_\Omega$ or $x_\varepsilon \to 0$, and then $x_\varepsilon \in \widehat{\mathcal{C}}_\varepsilon$.

Proof of Theorem 1.8. Recall that we assume that $P = 0 \in \Omega$. As in the proof of Theorem 1.1, if $x_\varepsilon \in B(0,r) \setminus B(0,\varepsilon)$ is a critical point of $\mathcal{R}_{\Omega_\varepsilon}$, then, since $B(0,r) \subset \Omega$ is chosen to not contain any critical point of $\mathcal{R}_\Omega$ different from 0, by Proposition 5.1 we necessarily have that $x_\varepsilon \to 0$. Following the notation of Remark 5.5, denote by $\overline{y}_+^{(l)}$ the critical point of $\widehat{F}$ associated to $\overline{v}_+^{(l)}$ and let us show the existence of *one* critical point of $\mathcal{R}_{\Omega_\varepsilon}$ in $B(0,r)$ which by (5.1) satisfies (1.10). The same holds for $\overline{y}_-^{(l)}$, giving the proof of the first claim of Theorem 1.8.

Let us introduce the function $\widehat{F}_\varepsilon : B(\overline{y}_+^{(l)}, \delta^{(l)}) \to \mathbb{R}$ as

$$\widehat{F}_\varepsilon(y) = \begin{cases} \dfrac{1}{\varepsilon^{\frac{N-2}{2N-2}}} \mathcal{R}_{\Omega_\varepsilon}(\varepsilon^{\frac{N-2}{2N-2}} y) & \text{if } N \geq 3, \\ \dfrac{1}{\widehat{r}_\varepsilon} \mathcal{R}_{\Omega_\varepsilon}(\widehat{r}_\varepsilon y) & \text{if } N = 2. \end{cases}$$

Furthermore, using Corollary 3.5 with

$$q = \frac{N-2}{2N-2} \text{ for } N \geq 3 \quad \text{and} \quad q < 1 \text{ for } N = 2$$

and arguing as in the proof of Theorem 1.1, we get that

$$\nabla \widehat{F}_\varepsilon \to \nabla \widehat{F} \quad \text{in } C^1(B(\overline{y}_+^{(l)}, \delta^{(l)})).$$

Since $\overline{y}_+^{(l)}$ is a non-degenerate critical point of $\widehat{F}$, then $\nabla \widehat{F}_\varepsilon(y)$ admits a unique critical point $y_{\varepsilon,+}^{(l)} \to y_+^{(l)}$ in $B(\overline{y}_+^{(l)}, \delta^{(l)})$ and also $y_{\varepsilon,+}^{(l)}$ is a non-degenerate critical point of $\mathcal{R}_{\Omega_\varepsilon}(0)$.

Finally, (1.10) follows by (5.1) and (1.11) can be proved repeating step by step the proof of (4.13)–(4.14) in Theorem 1.1.

Next let us assume that all the eigenvalues of the Hessian matrix $\mathrm{Hess}(\mathcal{R}_\Omega(0))$ are simple. Again by Proposition 5.1, Remark 5.5 and the discussion above, we have that if x_ε is a critical point belonging to $B(0,r) \setminus B(0,\varepsilon)$, then $x_\varepsilon \in \widehat{\mathcal{C}}_\varepsilon$. The simplicity of the eigenvalue and the previous claim prove (1.12). ■

Proof of Corollary 1.10. In this case, the Robin function $\mathcal{R}_\Omega(x)$ has only one critical point $P = 0$ which is a non-degenerate minimum point (see [10, 17]). This means that $\mathrm{Hess}(\mathcal{R}_\Omega(0))$ has N positive eigenvalues and, if they are simple, the Robin function $\mathcal{R}_{\Omega \setminus B(0,\varepsilon)}(x)$ has exactly $2N$ critical points for ε small enough. So the claim follows by Theorems 1.1 and 1.8. ■

Remark 5.6. What about multiple eigenvalues of $\mathrm{Hess}(\mathcal{R}_\Omega(0))$?

In this case, we are not able to give a complete description of the critical points of $\mathcal{R}_{\Omega_\varepsilon}$. Suppose that we have a multiple eigenvalue satisfying

$$\lambda_j = \lambda_{j+1} = \cdots = \lambda_{j+k}, \quad j,k \geq 1.$$

In this case, the function $T(y)$ defined in formula (5.8) admits a manifold of critical points given by

$$S^k = \Big\{x_j^2 + \cdots + x_{j+k}^2 = \Big(\frac{D_N}{\lambda_j}\Big)^{\frac{1}{2N-2}}\Big\}.$$

By (5.6), we have that the Hessian matrix is non-degenerate in the directions different from $x_j, \ldots, x_{j+k}$ and so it is a *non-degenerate* manifold of critical points for N in the sense of Morse–Bott theory. However, even in this explicit case, it seems hard to get the existence results of critical points for $T(y)$ under non-radial small perturbations. Of course, no possible information can be deduced about the non-degeneracy.

Without additional assumptions, as pointed out in the introduction, it is even possible to have *infinitely many* solutions (the radial case). For these reasons, the case of multiple eigenvalues is unclear.

Before the close of this section, we give a partial result when Ω is a symmetric domain.

Theorem 5.7. *Let $\Omega \subset \mathbb{R}^N$ be convex and symmetric with respect to $x_1, \ldots, x_N$ for $N \geq 2$. Then we have that $\mathcal{R}_{\Omega_\varepsilon}(x)$ has at least $2N$ critical points which are located on the coordinate axis, i.e.,*

$$x_{i,\varepsilon}^{\pm} = \Big(\underbrace{0, \ldots, 0}_{i-1}, \pm\Big(\frac{2}{N\omega_N\lambda_i} + o(1)\Big)^{\frac{1}{N-2}} \varepsilon^{\frac{N-2}{2N-2}}, \underbrace{0, \ldots, 0}_{N-i}\Big) \quad \textit{for } i = 1, \ldots, N.$$

Moreover, we have $\frac{\partial^2 \mathcal{R}_{\Omega_\varepsilon}(x_{i,\varepsilon}^{\pm})}{\partial x_i^2} \neq 0$ *for* $i = 1, \ldots, N$.

Proof. We prove the claim by constructing $2N$ zeros for $\nabla \mathcal{R}_{\Omega_\varepsilon}(x)$. For the case $N \geq 3$, as in the previous theorem, we study the equation

$$0 = (\lambda_i + o(1))Z_i - \Big(\frac{2}{N\omega_N} + o(1)\Big)\frac{Z_i}{|Z|^{2N-2}}\varepsilon^{N-2} \quad \text{for } N \geq 3.$$

Here we introduce the points

$$Y_{1,\varepsilon} = ((1-a)\alpha_1 \varepsilon^{\frac{N-2}{2N-2}}, 0, \ldots, 0) \quad \text{and} \quad Y_{2,\varepsilon} = ((1+a)\alpha_1 \varepsilon^{\frac{N-2}{2N-2}}, 0, \ldots, 0)$$

for any $a \in (0, 1)$ and $\alpha_1 = (\frac{D_N}{\lambda_1})^{\frac{1}{2N-2}}$. We have that $\Omega_\varepsilon = \Omega \setminus B(0, \varepsilon)$ is symmetric, and then

$$\frac{\partial \mathcal{R}_{\Omega_\varepsilon}(Y_{1,\varepsilon})}{\partial y_i} = \frac{\partial \mathcal{R}_{\Omega_\varepsilon}(Y_{2,\varepsilon})}{\partial y_i} = 0 \quad \text{for any } i = 2, \ldots, N.$$

Also we have

$$\begin{aligned}\frac{\partial \mathcal{R}_{\Omega_\varepsilon}(Y_{1,\varepsilon})}{\partial y_1} &= \mathbf{P}^{\mathsf{T}}\frac{\partial \mathcal{R}_{\Omega_\varepsilon}(Y_{1,\varepsilon})}{\partial x_1} = (\lambda_1 + o(1))Y_{1,\varepsilon} - \Big(\frac{2}{N\omega_N} + o(1)\Big)\frac{Y_{1,\varepsilon}}{|Y_{1,\varepsilon}|^{2N-2}}\varepsilon^{N-2} \\ &= (1-a)\alpha_1\lambda_1\varepsilon^{\frac{N-2}{2N-2}}\Big[1 - \frac{1}{(1-a)^{2N-2}} + o(1)\Big] > 0,\end{aligned}$$

and similarly,

$$\frac{\partial \mathcal{R}_{\Omega_\varepsilon}(Y_{2,\varepsilon})}{\partial y_1} = (1+a)\alpha_1\lambda_1\varepsilon^{\frac{N-2}{2N-2}}\Big[1-\frac{1}{(1+a)^{2N-2}}+o(1)\Big]<0.$$

This implies that there exists $Z_{1,\varepsilon}=(b\varepsilon^{\frac{N-2}{2N-2}},0,\dots,0)$ with $b\in((1-a)\alpha_1,(1+a)\alpha_1)$ satisfying $\nabla\mathcal{R}_{\Omega_\varepsilon}(Z_{1,\varepsilon})=0$. Taking a small, we can write $b=\alpha_1+o(1)$. Hence there exists $Z_{1,\varepsilon}=((\alpha_1+o(1))\varepsilon^{\frac{N-2}{2N-2}},0,\dots,0)$ satisfying $\nabla\mathcal{R}_{\Omega_\varepsilon}(Z_{1,\varepsilon})=0$.

The same computation holds if we replace $Y_{1,\varepsilon}$, $Y_{2,\varepsilon}$ by $-Y_{1,\varepsilon}$, $-Y_{2,\varepsilon}$ getting the existence of a *second* critical point $-Z_{1,\varepsilon}$. Since $\mathrm{Hess}(\mathcal{R}_\Omega(0))$ has N positive eigenvalues $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_N$, then repeating the argument above for any positive eigenvalue, we have that $\mathcal{R}_{\Omega_\varepsilon}(x)$ has at least $2N$ critical points $x^\pm_{i,\varepsilon}$ $(i=1,\dots,N)$ in $B(0,r)\setminus B(0,\varepsilon)$, where $B(0,r)\setminus\{0\}\subset\Omega$ does not contain any critical point of $\mathcal{R}_\Omega(x)$. Moreover,

$$x^\pm_{i,\varepsilon}=\pm Z_{i,\varepsilon}=\Big(\underbrace{0,\dots,0}_{i-1},\pm\Big(\frac{2}{N\omega_N\lambda_i}+o(1)\Big)^{\frac{1}{N-2}}\varepsilon^{\frac{N-2}{2N-2}},\underbrace{0,\dots,0}_{N-i}\Big)\quad\text{for } i=1,\dots,N.$$

Also using (3.13) for $N\ge 3$, we compute

$$\begin{aligned}\frac{\partial^2\mathcal{R}_{\Omega_\varepsilon}(x^\pm_{i,\varepsilon})}{\partial x_i^2}&=\lambda_i+\Big(\frac{2}{N\omega_N}+o(1)\Big)(2N-3)\frac{1}{|x^\pm_{i,\varepsilon}|^{2N-2}}\varepsilon^{N-2}+o(1)\\&=(2N-2)\lambda_i+o(1)\ne 0\quad\text{for } i=1,\dots,N.\end{aligned}$$

For the case $N=2$, as in the previous theorem, we study the equation

$$0=(\lambda_i+o(1))Z_i-\Big(\frac{1}{\pi}+o(1)\Big)\frac{Z_i\ln|Z|}{|Z|^2\ln\varepsilon}.$$

Then, similarly to the idea in proving the case $N\ge 3$, we can write

$$x^\pm_{1,\varepsilon}=((1+o(1))\widehat{r}_{\varepsilon,1},0)\quad\text{and}\quad x^\pm_{2,\varepsilon}=(0,(1+o(1))\widehat{r}_{\varepsilon,2}).$$

Finally, using (3.13) for $N=2$, we compute

$$\frac{\partial^2\mathcal{R}_{\Omega_\varepsilon}(x^\pm_{i,\varepsilon})}{\partial x_i^2}=\lambda_i+\frac{1}{\pi|x^\pm_{i,\varepsilon}|^{2N-2}}\frac{\ln\widehat{r}_{\varepsilon,i}}{\ln\varepsilon}+o(1)=2\lambda_i+o(1)\ne 0\quad\text{for } i=1,2.\ \blacksquare$$

Remark 5.8. Here we point out that the positive eigenvalues may be multiple in Theorem 5.7.

6. Examples on which the conditions of Theorem 1.8 hold

In Theorem 1.8, we proved that if all positive eigenvalues of the Hessian matrix of $\mathcal{R}_\Omega$ are simple, then we can give the precise number and non-degeneracy of the critical points of the Robin function $\mathcal{R}_{\Omega_\varepsilon}(x)$. In this section, we exhibit some domains on which this assumption is satisfied.

We recall that the regular part of the Green's function in $B_1 = B_1(0)$ is

$$H_{B_1}(x,y) = \begin{cases} \dfrac{1}{2\pi} \ln \dfrac{1}{|y| \cdot |x - \frac{y}{|y|^2}|} & \text{for } y \in B_1(0) \text{ and } N = 2, \\ \dfrac{1}{N(N-2)\omega_N} \dfrac{1}{|y|^{N-2} \cdot |x - \frac{y}{|y|^2}|^{N-2}} & \text{for } y \in B_1(0) \text{ and } N \geq 3. \end{cases}$$

Then we have the following result.

Theorem 6.1. *Let $N \geq 2$ and*

$$\Omega_\delta = \left\{ x \in \mathbb{R}^N, \sum_{i=1}^{N} x_i^2 (1 + \alpha_i \delta)^2 < 1 \text{ with } \delta > 0 \text{ and } 0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_N \right\}.$$

Then the Robin function $\mathcal{R}_{\Omega_\delta}$ has a unique critical point $P = 0$ and is even with respect to $x_1, \ldots, x_N$.

Moreover, if $\alpha_1 \leq \alpha_2 \leq \cdots < \alpha_{l_0+1} = \alpha_{l_0+2} = \cdots = \alpha_{l_0+k} < \alpha_{l_0+k+1} \leq \cdots \leq \alpha_N$, then $\mathcal{R}_{\Omega_\delta}(x)$ is radial function with respect to $x_{l_0+1} \cdots x_{l_0+k}$, i.e.,

$$\mathcal{R}_{\Omega_\delta}(x) = \mathcal{R}_{\Omega_\delta}\Big(x_1, \ldots, x_{l_0}, \sqrt{x_{l_0+1}^2 + \cdots + x_{l_0+k}^2}, x_{l_0+k+1}, \ldots, x_N\Big),$$

and we have that $\lambda_{l_0+1} = \lambda_{l_0+2} = \cdots = \lambda_{l_0+k}$ for any $\delta > 0$.

Moreover, if there exists some k such that $\alpha_k \neq \alpha_j$ for any $j \in \{1, \ldots, N\}$ with $j \neq k$, then the k-th eigenvalue of $\mathrm{Hess}(\mathcal{R}_{\Omega_\delta}(0))$ *is simple for small $\delta > 0$.*

Finally, if $\alpha_i \neq \alpha_j$ for any $i, j \in \{1, \ldots, N\}$ with $i \neq j$, then all the eigenvalues of $\mathrm{Hess}(\mathcal{R}_{\Omega_\delta}(0))$ *are simple.*

Proof. Firstly, suppose that A is a reflection or a rotation such that $A\Omega = \Omega$. We have

(i) $H(y,x)$ and $H(Ay, Ax)$ are harmonic in y.

(ii) For any $y \in \partial\Omega$ and $x \in \Omega$ it holds that $H(y,x) = H(Ay, Ax)$.

Hence we have $H(y,x) = H(Ay, Ax)$ for any $x, y \in \Omega$, which gives

$$\mathcal{R}(x) = \mathcal{R}(Ax).$$

Now since Ω_δ is symmetric with respect to $x_1, \ldots, x_N$, then the Robin function $\mathcal{R}_{\Omega_\delta}$ is even function with respect to $x_1, \ldots, x_N$. Also from the fact that Ω_δ is convex and [6, 10], we have that $\mathcal{R}_{\Omega_\delta}$ has a unique critical point $P = 0$. Moreover, by [17] $\mathrm{Hess}(\mathcal{R}_{\Omega_\delta}(0))$ is a diagonal matrix with i-th eigenvalue

$$\lambda_i = \frac{\partial^2 \mathcal{R}_{\Omega_\delta}(0)}{\partial x_i^2} \quad \text{for } i = 1, \ldots, N.$$

If $\alpha_1 \leq \alpha_2 < \cdots < \alpha_{l_0+1} = \alpha_{l_0+2} = \cdots = \alpha_{l_0+k} < \alpha_{l_0+k+1} \leq \cdots \leq \alpha_N$, then $\mathcal{R}_{\Omega_\delta}(x)$ is a radial function with respect to $x_{l_0+1} \cdots x_{l_0+k}$ and $\lambda_{l_0+1} = \lambda_{l_0+2} = \cdots = \lambda_{l_0+k}$ for any $\delta > 0$.

Set $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_N$ and consider the general case for small $\delta > 0$. Let $g_\delta(x,y) = H_\delta(x,y) - H_{B(0,1)}(x,y)$, then we have

$$\begin{cases} \Delta_x g_\delta(x,y) = 0, & \text{in } \Omega_\delta, \\ g_\delta(x,y) = \begin{cases} \dfrac{1}{N(N-2)\omega_N}\Big(\dfrac{1}{|x-y|^{N-2}} \\ \quad -\dfrac{1}{|y|^{N-2}\cdot|x-\frac{y}{|y|^2}|^{N-2}}\Big) & \text{for } N \ge 3, \\ \dfrac{1}{2\pi}\ln\dfrac{1}{|x-y|} - \dfrac{1}{2\pi}\ln\dfrac{1}{|y|\cdot|x-\frac{y}{|y|^2}|} & \text{for } N = 2 \end{cases} & \text{on } \partial\Omega_\delta. \end{cases} \tag{6.1}$$

Next, on $\partial\Omega_\delta$, we have

$$\begin{aligned} &\frac{1}{|x-y|^{N-2}} - \frac{1}{|y|^{N-2}\cdot|x-\frac{y}{|y|^2}|^{N-2}} \\ &= \frac{1}{|x-y|^{N-2}}\Big(1-\frac{|x-y|^{N-2}}{|y|^{N-2}\cdot|x-\frac{y}{|y|^2}|^{N-2}}\Big) = \frac{1}{|x-y|^{N-2}}\Big(1-\frac{1}{(1-\alpha(x,y))^{\frac{N-2}{2}}}\Big) \\ &= \frac{(N-2)\alpha(x,y)}{2|x-y|^{N-2}} + O(\alpha^2(x,y)) \quad \text{for } N \ge 3, \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} &\ln\frac{1}{|x-y|} - \ln\frac{1}{|y|\cdot|x-\frac{y}{|y|^2}|} \\ &= \frac{1}{2}\ln\frac{|y|^2\cdot|x-\frac{y}{|y|^2}|^2}{|x-y|^2} = \frac{1}{2}\ln(1-\alpha(x,y)) = -\frac{1}{2}\alpha(x,y) + O(\alpha(x,y)^2), \end{aligned} \tag{6.3}$$

where

$$\alpha(x,y) := \frac{|x-y|^2 - |y|^2\cdot|x-\frac{y}{|y|^2}|^2}{|x-y|^2}.$$

Also for $x \in \partial\Omega_\delta$, it holds

$$\begin{aligned} |y|^2\cdot|x-\frac{y}{|y|^2}|^2 &= |y|^2\cdot|x|^2 - 2\langle x,y\rangle + 1 \\ &= -2\delta(|y|^2-1)\sum_{i=1}^N \alpha_i x_i^2 + |x-y|^2 + O(\delta^2). \end{aligned} \tag{6.4}$$

Hence for $x \in \partial\Omega_\delta$, from (6.2), (6.3) and (6.4), we have

$$g_\delta(x,y) = -\frac{\delta(|y|^2-1)}{N\omega_N|x-y|^N}\sum_{i=1}^N \alpha_i x_i^2 + O(\delta^2) \quad \text{for } N \ge 2. \tag{6.5}$$

Now using (6.1) and (6.5), we deduce

$$H_\delta(x,y) = H_{B(0,1)}(x,y) - \frac{\delta(|y|^2-1)}{N\omega_N}v_\delta(x,y) + O(\delta^2) \quad \text{for } N \ge 2,$$

where $v_\delta(x,y)$ is the solution of

$$\begin{cases} \Delta_x v_\delta(x,y)=0 & \text{in } \Omega_\delta,\\ v_\delta(x,y)=\dfrac{1}{|x-y|^N}\displaystyle\sum_{i=1}^N \alpha_i x_i^2 & \text{on } \partial\Omega_\delta. \end{cases} \tag{6.6}$$

Then it follows

$$\mathcal{R}_\delta(x)=\mathcal{R}(x)-\frac{\delta(|x|^2-1)}{N\omega_N}v_\delta(x,x)+O(\delta^2)\quad \text{for } N\ge 2. \tag{6.7}$$

Also for $x\in\partial\Omega_\delta$ and small $|y|$, by Taylor's expansion, it holds

$$\begin{aligned}\frac{1}{|x-y|^N}&=\Big(\frac{1}{|x|^2+|y|^2-2\langle x,y\rangle}\Big)^{\frac N2}\\ &=\frac{1}{|x|^N}\Big(1+\frac{2\langle x,y\rangle}{|x|^2}-\frac{|y|^2}{|x|^2}+\frac{4\langle x,y\rangle^2}{|x|^4}+O(|y|^3)\Big)^{\frac N2}\\ &=\frac{1}{|x|^N}\Big(1+\frac{N\langle x,y\rangle}{|x|^2}-\frac N2\frac{|y|^2}{|x|^2}+\frac{N(N+2)}{2}\frac{\langle x,y\rangle^2}{|x|^4}+O(|y|^3)\Big). \end{aligned}\tag{6.8}$$

From (6.6) and (6.8), we get

$$v_\delta(x,y)=v_\delta^{(1)}(x,y)+Nv_\delta^{(2)}(x,y)-\frac N2 v_\delta^{(3)}(x,y)+\frac{N(N+2)}{2}v_\delta^{(4)}(x,y)+v_\delta^{(5)}(x,y),$$

where

$$\begin{cases} \Delta_x v_\delta^{(1)}(x,y)=0 & \text{in } \Omega_\delta,\\ v_\delta^{(1)}(x,y)=\dfrac{1}{|x|^N}\displaystyle\sum_{i=1}^N \alpha_i x_i^2 & \text{on } \partial\Omega_\delta, \end{cases}$$

$$\begin{cases} \Delta_x v_\delta^{(2)}(x,y)=0 & \text{in } \Omega_\delta,\\ v_\delta^{(2)}(x,y)=\dfrac{\langle x,y\rangle}{|x|^{N+2}}\displaystyle\sum_{i=1}^N \alpha_i x_i^2 & \text{on } \partial\Omega_\delta, \end{cases}$$

$$\begin{cases} \Delta_x v_\delta^{(3)}(x,y)=0 & \text{in } \Omega_\delta,\\ v_\delta^{(3)}(x,y)=\dfrac{|y|^2}{|x|^{N+2}}\displaystyle\sum_{i=1}^N \alpha_i x_i^2 & \text{on } \partial\Omega_\delta, \end{cases}$$

$$\begin{cases} \Delta_x v_\delta^{(4)}(x,y)=0 & \text{in } \Omega_\delta,\\ v_\delta^{(4)}(x,y)=\dfrac{\langle x,y\rangle^2}{|x|^{N+4}}\displaystyle\sum_{i=1}^N \alpha_i x_i^2 & \text{on } \partial\Omega_\delta, \end{cases}$$

$$\begin{cases} \Delta_x v_\delta^{(5)}(x,y)=0 & \text{in } \Omega_\delta,\\ v_\delta^{(5)}(x,y)=O(|y|^3) & \text{on } \partial\Omega_\delta. \end{cases}$$

We know that as $\delta \to 0$, $v_\delta^{(1)}(x,y) \to v^{(1)}(x,y)$, where

$$\Delta v^{(1)}(x,y)=0 \text{ in } B_1(0), \quad v^{(1)}(x,y)=\sum_{i=1}^N \alpha_i x_i^2 \text{ on } \partial B_1(0). \tag{6.9}$$

Solving (6.9), we can get

$$v^{(1)}(x,y)=-\frac{(N-1)}{2}(|x|^2-1)+\sum_{i=1}^N \alpha_i x_i^2.$$

Then it holds

$$(|x|^2-1)v^{(1)}(x,x)=-\frac{(N-1)}{2}(|x|^2-1)^2+(|x|^2-1)\sum_{i=1}^N \alpha_i x_i^2,$$

$$\begin{aligned}\nabla^2((|x|^2-1)v^{(1)}(x,x))\big|_{x=0}&=\nabla^2\Big((N-1)|x|^2-\sum_{i=1}^N \alpha_i x_i^2\Big)\Big|_{x=0}\\&=2(N-1)\mathbf{E}_N-2\,\mathrm{diag}(\alpha_1,\alpha_2,\dots,\alpha_N),\end{aligned}$$

where $\mathbf{E}_N$ is the unit matrix. In the same way, we have that for $\delta \to 0$, $v_\delta^{(i)}(x,y) \to v^{(i)}(x,y)$ for $i=2,3,4$, where

$$\begin{aligned}v^{(2)}(x,y)&=\langle x,y\rangle\sum_{i=1}^N \alpha_i x_i^2-\frac{N(N-1)}{2(N+2)}(|x|^2-1)\langle x,y\rangle-\frac{2(|x|^2-1)}{N+2}\sum_{i=1}^N \alpha_i x_i y_i,\\
v^{(3)}(x,y)&=|y|^2v^{(1)}(x,y)=|y|^2\Big(-\frac{(N-1)}{2}(|x|^2-1)+\sum_{i=1}^N \alpha_i x_i^2\Big),\\
v^{(4)}(x,y)&=-2|y|^2\Big(\sum_{i=1}^N \alpha_i g_i(x)\Big)-\sum_{i=1}^N(N(N-1)+8\alpha_i)g_i(x)y_i^2\\
&\quad-\sum_{i=1}^N\sum_{j\neq i}(N(N-1)+8\alpha_i)f_{ij}(x)y_iy_j+\Big(\sum_{i=1}^N \alpha_i x_i^2\Big)\langle x,y\rangle^2,\end{aligned}$$

with

$$f_{ij}(x)=\frac{x_ix_j(|x|^2-1)}{12}$$

and

$$g_i(x)=\frac{(|x|^2-1)x_i^2}{2(N+4)}-\frac{(|x|^4-1)}{4(N+2)(N+4)}+\frac{(|x|^2-1)}{2N(N+4)}.$$

For future aims, it will be useful to remark that

$$v^{(4)}(x,x)=\frac{N(N-1)|x|^2+4\sum_{i=1}^N \alpha_i x_i^2}{2N(N+2)}+O(|x|^4).$$

Finally, since $v_\delta^{(5)}(x,y) = O(|y|^3)$ as $\delta \to 0$, we get

$$\nabla^2((|x|^2-1)v_\delta^{(5)}(x,x))\big|_{x=0} \to \mathbf{O}_N,$$

where $\mathbf{O}_N$ is the zero matrix in $\mathbb{R}^N$. Hence by the explicit form of $v_\delta^{(1)},\dots,v_\delta^{(4)}$, a straightforward (and tedious) computation gives that

$$\begin{aligned}&\nabla^2((|x|^2-1)v_\delta(x,x))\big|_{x=0}\\ &\quad\to -\frac{(N-1)(N^2-2N-4)}{N+2}\mathbf{E}_N - \frac{8(N+1)}{N+2}\operatorname{diag}(\alpha_1,\dots,\alpha_N).\end{aligned} \tag{6.10}$$

Observe that

$$\frac{\partial^2 \mathcal{R}_{B_1}(x)}{\partial x_i \partial x_j} = \frac{2}{N\omega_N}\delta_j^i \quad \text{for } 1 \le i,j \le N. \tag{6.11}$$

Finally, let λ_i for $i=1,\dots,N$ be the eigenvalues of $\operatorname{Hess}(\mathcal{R}_{\Omega_\delta}(0))$; for $N \ge 2$, from (6.7), (6.10) and (6.11), we find

$$\lambda_i = \frac{2}{N\omega_N} + \frac{(N-1)(N^2-2N-4)+8(N+1)\alpha_i}{N(N+2)\omega_N}\delta + o(\delta) \quad \text{for } i=1,\dots,N.$$

Hence we deduce that if there exists some k such that $\alpha_k \ne \alpha_j$ for any $j \in \{1,\dots,N\}$ with $j \ne k$, then the corresponding k-th eigenvalue of $\operatorname{Hess}(\mathcal{R}_{\Omega_\delta}(0))$ is simple for small $\delta > 0$.

Furthermore, if $\alpha_i \ne \alpha_j$ for any $i,j \in \{1,\dots,N\}$ with $i \ne j$, then all the eigenvalues of $\operatorname{Hess}(\mathcal{R}_{\Omega_\delta}(0))$ are simple. ■

Appendix A. Some useful computations on the Green's function in the exterior of the ball

We recall that the Green's function $G_{B_\varepsilon^c}(x,y)$ of $\mathbb{R}^N \setminus B_\varepsilon$ is given by (see [2], for example)

$$G_{B_\varepsilon^c}(x,y) = \begin{cases} -\dfrac{1}{2\pi}\Big(\ln|x-y| - \ln\sqrt{\dfrac{|x|^2|y|^2}{\varepsilon^2} + \varepsilon^2 - 2x\cdot y}\Big) & \text{for } N=2,\\[2ex] C_N\Big(\dfrac{1}{|x-y|^{N-2}} - \dfrac{\varepsilon^{N-2}}{\big||x|y-\varepsilon^2\frac{x}{|x|}\big|^{N-2}}\Big) & \text{for } N \ge 3,\end{cases}$$

and, for $i=1,\dots,N$,

$$\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial y_i} = \frac{1}{N\omega_N}\Big(\frac{x_i-y_i}{|x-y|^N} + \varepsilon^{N-2}\frac{|x|^2 y_i - \varepsilon^2 x_i}{(|x|^2|y|^2 - 2\varepsilon^2 x\cdot y + \varepsilon^4)^{\frac{N}{2}}}\Big). \tag{A.1}$$

The *Robin function* of $\mathbb{R}^N \setminus B_\varepsilon$ is given by (see [2])

$$\mathcal{R}_{B_\varepsilon^c}(x) = \begin{cases} \dfrac{1}{2\pi}\ln\dfrac{\varepsilon}{|x|^2-\varepsilon^2} & \text{for } N=2,\\[2ex] C_N\dfrac{\varepsilon^{N-2}}{(|x|^2-\varepsilon^2)^{N-2}} & \text{for } N \ge 3.\end{cases} \tag{A.2}$$

It will be also useful the well-known representation formula for harmonic function in the ball B_ε: if u is harmonic in B_ε we have that

$$u(x) = -\int_{\partial B_\varepsilon} \frac{\partial G_{B_\varepsilon}(x,y)}{\partial \nu_y} u(y) ds_y = \frac{1}{N\omega_N} \frac{\varepsilon^2 - |x|^2}{\varepsilon} \int_{\partial B_\varepsilon} \frac{u(y)}{|x-y|^N} ds_y. \tag{A.3}$$

In the next lemma, we prove some identities which we use in our computations.

Lemma A.1. *We have that for any $x \in B_\varepsilon^c$ and $N \geq 2$, the following equalities hold:*

$$\int_{\partial B_\varepsilon} \frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y} d\sigma_y = -\frac{\varepsilon^{N-2}}{|x|^{N-2}}, \tag{A.4}$$

$$\int_{\partial B_\varepsilon} y_j \frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y} d\sigma_y = -\frac{x_j}{|x|^N} \varepsilon^N, \tag{A.5}$$

$$\int_{\partial B_\varepsilon} y_i y_j \frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y} d\sigma_y = -\frac{\varepsilon^N}{|x|^N} \Big[\varepsilon^2 \frac{x_i x_j}{|x|^2} + \frac{\delta_j^i}{N} (|x|^2 - \varepsilon^2) \Big]. \tag{A.6}$$

Proof. In order to prove (A.4) and (A.5), we will use the well-known facts that for $x \in B_\varepsilon$,

$$\int_{\partial B_\varepsilon} \frac{\partial G_{B_\varepsilon}(x,y)}{\partial \nu_y} d\sigma_y = -1 \quad \text{and} \quad \int_{\partial B_\varepsilon} y_j \frac{\partial G_{B_\varepsilon}(x,y)}{\partial \nu_y} d\sigma_y = -x_j, \tag{A.7}$$

(these can be deduced by (A.3)). Let us prove (A.4) for $N \geq 3$. We have that, for $x \in B_\varepsilon^c$,

$$\int_{\partial B_\varepsilon} \frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y} d\sigma_y = \frac{\varepsilon^2 - |x|^2}{N\omega_N \varepsilon} \int_{\partial B_\varepsilon} \frac{1}{|x-y|^N} d\sigma_y.$$

Setting $x = \varepsilon^2 \frac{t}{|t|^2}$, and then $|t| \leq \varepsilon$,

$$\text{r.h.s.} = \varepsilon \frac{|t|^2 - \varepsilon^2}{N\omega_N |t|^2} \int_{\partial B_\varepsilon} \frac{|t|^{2N}}{|\varepsilon^2 t - |t|^2 y|^N} d\sigma_y = \frac{(|t|^2 - \varepsilon^2)|t|^{N-2}}{N\omega_N \varepsilon^{N-1}} \int_{\partial B_\varepsilon} \frac{1}{|t-y|^N} d\sigma_y,$$

and using (A.7) for $t \in B_\varepsilon$,

$$\text{r.h.s.} = \frac{|t|^{N-2}}{\varepsilon^{N-2}} \int_{\partial B_\varepsilon} \frac{\partial G_{B_\varepsilon}(t,y)}{\partial \nu_y} d\sigma_y = -\frac{\varepsilon^{N-2}}{|x|^{N-2}}.$$

In the same way, we get (A.5),

$$\int_{\partial B_\varepsilon} y_j \frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y} d\sigma_y = \frac{\varepsilon^2 - |x|^2}{N\omega_N \varepsilon} \int_{\partial B_\varepsilon} \frac{y_j}{|x-y|^N} d\sigma_y.$$

Setting $x = \varepsilon^2 \frac{t}{|t|^2}$, and then $|t| \leq \varepsilon$,

$$\text{r.h.s.} = \varepsilon \frac{|t|^2 - \varepsilon^2}{N\omega_N |t|^2} \int_{\partial B_\varepsilon} \frac{|t|^{2N} y_j}{|\varepsilon^2 t - |t|^2 y|^N} d\sigma_y = \frac{(|t|^2 - \varepsilon^2)|t|^{N-2}}{N\omega_N \varepsilon^{N-1}} \int_{\partial B_\varepsilon} \frac{y_j}{|t-y|^N} d\sigma_y,$$

and using (A.7) for $t \in B_\varepsilon$,

$$\text{r.h.s.} = \frac{|t|^{N-2}}{\varepsilon^{N-2}} t_j \int_{\partial B_\varepsilon} \frac{\partial G_{B_\varepsilon}(x,y)}{\partial \nu_y} d\sigma_y = -\frac{x_j}{|x|^N}\varepsilon^N.$$

Let us prove (A.6). We have that the function

$$a(x) = x_i x_j + \frac{\delta_j^i}{N}(\varepsilon^2 - |x|^2)$$

is harmonic and $a|_{\partial B_\varepsilon} = x_i x_j$. So

$$\int_{\partial B_\varepsilon} y_j y_j \frac{\partial G_{B_\varepsilon}(x,y)}{\partial \nu_y} d\sigma_y = -\Big(x_i x_j + \frac{\delta_j^i}{N}(\varepsilon^2 - |x|^2)\Big), \tag{A.8}$$

and arguing as before, we obtain that

$$\int_{\partial B_\varepsilon} y_i y_j \frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y} d\sigma_y = \frac{\varepsilon^2 - |x|^2}{N\omega_N \varepsilon} \int_{\partial B_\varepsilon} \frac{y_i y_j}{|x-y|^N} d\sigma_y.$$

Setting $x = \varepsilon^2 \frac{t}{|t|^2}$, and then $|t| \le \varepsilon$,

$$\text{r.h.s.} = \varepsilon \frac{|t|^2 - \varepsilon^2}{N\omega_N |t|^2} \int_{\partial B_\varepsilon} \frac{|t|^{2N} y_i y_j}{|\varepsilon^2 t - |t|^2 y|^N} d\sigma_y = \frac{(|t|^2 - \varepsilon^2)|t|^{N-2}}{N\omega_N \varepsilon^{N-1}} \int_{\partial B_\varepsilon} \frac{y_i y_j}{|t-y|^N} d\sigma_y,$$

and using (A.8) for $t \in B_\varepsilon$,

$$\begin{aligned}\text{r.h.s.} &= -\frac{|t|^{N-2}}{\varepsilon^{N-2}}\Big(t_i t_j + \frac{\delta_j^i}{N}(\varepsilon^2 - |t|^2)\Big) = -\frac{\varepsilon^{N-2}}{|x|^{N-2}}\Big[\varepsilon^4 \frac{x_i x_j}{|x|^4} + \frac{\delta_j^i}{N}\varepsilon^2\Big(1 - \frac{\varepsilon^2}{|x|^2}\Big)\Big] \\ &= -\frac{\varepsilon^N}{|x|^N}\Big[\varepsilon^2 \frac{x_i x_j}{|x|^2} + \frac{\delta_j^i}{N}(|x|^2 - \varepsilon^2)\Big],\end{aligned}$$

which proves the claim. ■

The next lemma concerns some identities on the Robin function.

Lemma A.2. *We have that for any $x \in B_\varepsilon^c$ and $N \ge 2$, the following identities hold:*

$$\int_{\partial B_\varepsilon} \frac{y}{\varepsilon}\Big(\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y}\Big)^2 d\sigma_y = -\nabla \mathcal{R}_{B_\varepsilon^c}(x) = \frac{2}{N\omega_N}\varepsilon^{N-2}\frac{x}{(|x|^2 - \varepsilon^2)^{N-1}} \tag{A.9}$$

and

$$\begin{aligned} &-\int_{\partial B_\varepsilon} \frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y} S(x,y) d\sigma_y \\ &= \begin{cases} C_N \dfrac{\varepsilon^{N-2}}{(|x|^2 - \varepsilon^2)^{N-2}} = \mathcal{R}_{B_\varepsilon^c}(x) & \text{for } N \ge 3, \\ \dfrac{1}{2\pi}\Big(-\ln \dfrac{|x|^2}{|x|^2 - \varepsilon^2} + \ln |x|\Big) = \mathcal{R}_{B_\varepsilon^c}(x) + \dfrac{1}{2\pi} \ln \dfrac{|x|}{\varepsilon} & \text{for } N = 2. \end{cases} \end{aligned} \tag{A.10}$$

Proof. Let us prove (A.9). For $t \in B_\varepsilon$, using the expressions of the Green's and the Robin functions in B_ε and recalling [4, p. 170], we have

$$\int_{\partial B_\varepsilon} \nu(y)\Big(\frac{\partial G_{B_\varepsilon}(t,y)}{\partial \nu_y}\Big)^2 d\sigma_y = \nabla \mathcal{R}_{B_\varepsilon}(t) = \frac{2}{N\omega_N}\varepsilon^{N-2}\frac{t}{(\varepsilon^2-|t|^2)^{N-1}},$$

and we deduce the following formula:

$$\frac{1}{(N\omega_N)^2}\int_{\partial B_\varepsilon}\frac{y}{\varepsilon}\frac{1}{|t-y|^{2N}}d\sigma_y = \frac{2}{N\omega_N}\varepsilon^{N-2}\frac{t}{(\varepsilon^2-|t|^2)^{N-1}} \quad \text{for any } t \in B_\varepsilon. \tag{A.11}$$

Now, let $x \in B_\varepsilon^c$, then using (A.1)

$$\int_{\partial B_\varepsilon}\frac{y}{\varepsilon}\Big(\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y}\Big)^2 d\sigma_y = \frac{1}{(N\omega_N)^2}\frac{(\varepsilon^2-|x|^2)^2}{\varepsilon^2}\int_{\partial B_\varepsilon}\frac{y}{\varepsilon}\frac{1}{|x-y|^{2N}}d\sigma_y.$$

Setting $x = \varepsilon^2\frac{t}{|t|^2}$ and so $|t| \le \varepsilon$,

$$\begin{aligned}\text{r.h.s.} &= \frac{1}{(N\omega_N)^2}\frac{\varepsilon^2}{|t|^4}(|t|^2-\varepsilon^2)^2\int_{|y|=\varepsilon}\frac{y}{\varepsilon}\frac{1}{|\frac{\varepsilon^2 t-|t|^2 y}{|t|^2}|^{2N}}d\sigma_y\\ &= \frac{1}{(N\omega_N)^2}(|t|^2-\varepsilon^2)^2\frac{|t|^{2N-4}}{\varepsilon^{2N-2}}\int_{|y|=\varepsilon}\frac{y}{\varepsilon}\frac{1}{|t-y|^{2N}}d\sigma_y,\end{aligned}$$

and using (A.11) for $|t| \le \varepsilon$,

$$\begin{aligned}\text{r.h.s.} &= \frac{2}{N\omega_N}\frac{|t|^{2N-4}}{\varepsilon^{N-2}}\frac{t}{(\varepsilon^2-|t|^2)^{N-1}}\\ &= \frac{2}{N\omega_N}\varepsilon^{N-2}\frac{x}{(|x|^2-\varepsilon^2)^{N-1}} = -\nabla\mathcal{R}_{B_\varepsilon^c}(x),\end{aligned}$$

which gives the claim.

Next let us prove (A.10). If $N \ge 3$, we have

$$\int_{\partial B_\varepsilon}\frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y}S(x,y)d\sigma_y = \frac{\varepsilon^2-|x|^2}{N\omega_N\varepsilon}C_N\int_{\partial B_\varepsilon}\frac{1}{|x-y|^{2N-2}}d\sigma_y.$$

Setting $x = \varepsilon^2\frac{t}{|t|^2}$,

$$\begin{aligned}\text{r.h.s.} &= \frac{|t|^2-\varepsilon^2}{N\omega_N\varepsilon}C_N\frac{|t|^{2N-4}}{\varepsilon^{2N-4}}\int_{\partial B_\varepsilon}\frac{1}{|t-y|^{2N-2}}d\sigma_y\\ &= \frac{|t|^{2N-4}}{\varepsilon^{2N-4}}\int_{\partial B_\varepsilon}\frac{\partial G_{B_\varepsilon}(t,y)}{\partial \nu_y}S(t,y)d\sigma_y\\ &= -C_N\frac{|t|^{2N-4}}{\varepsilon^{2N-4}}\frac{\varepsilon^{N-2}}{(\varepsilon^2-|t|^2)^{N-2}} = -C_N\frac{\varepsilon^{N-2}}{(|x|^2-\varepsilon^2)^{N-2}}.\end{aligned}$$

If $N = 2$, we have, arguing as above,

$$\begin{aligned}
&\int_{\partial B_\varepsilon} \frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y} S(x,y) d\sigma_y \\
&= -\frac{\varepsilon^2 - |x|^2}{4\pi^2 \varepsilon} \int_{\partial B_\varepsilon} \frac{\ln|x-y|}{|x-y|^2} d\sigma_y = -\varepsilon^2 \frac{|t|^2 - \varepsilon^2}{4\pi^2 \varepsilon |t|^2} \int_{\partial B_\varepsilon} \frac{\ln|\varepsilon^2 \frac{t}{|t|^2} - y|}{|\varepsilon^2 \frac{t}{|t|^2} - y|^2} d\sigma_y \\
&= -\frac{|t|^2 - \varepsilon^2}{4\pi^2 \varepsilon} \int_{\partial B_\varepsilon} \frac{\ln|t-y| + \ln\varepsilon - \ln|t|}{|t-y|^2} d\sigma_y \\
&= \int_{\partial B_\varepsilon} \frac{\partial G_{B_\varepsilon}(t,y)}{\partial \nu_y} S(t,y) d\sigma_y - \frac{|t|^2 - \varepsilon^2}{4\pi^2 \varepsilon} \ln\frac{\varepsilon}{|t|} \int_{\partial B_\varepsilon} \frac{1}{|t-y|^2} d\sigma_y \\
&= -\mathcal{R}_{B_\varepsilon}(t) + \frac{1}{2\pi} \ln\frac{\varepsilon}{|t|} = -\frac{1}{2\pi} \ln\frac{\varepsilon}{\varepsilon^2 - |t|^2} + \frac{1}{2\pi} \ln\frac{\varepsilon}{|t|} \\
&= -\mathcal{R}_{B_\varepsilon^c}(x) - \frac{1}{2\pi} \ln\frac{|x|}{\varepsilon},
\end{aligned}$$

which ends the proof. ∎

The final lemma computes some useful integrals.

Lemma A.3. *We have that for any $x \in B_\varepsilon^c$ and $N \geq 2$, the following identities hold:*

$$\int_{\partial B_\varepsilon} \frac{C_N}{|x-y|^{N-2}} d\sigma_y = \frac{1}{N-2} \frac{\varepsilon^{N-1}}{|x|^{N-2}}, \tag{A.12}$$

$$\int_{\partial B_\varepsilon} \ln|x-y| d\sigma_y = 2\pi\varepsilon \ln|x|, \tag{A.13}$$

$$\int_{\partial B_\varepsilon} |\ln|x-y|| d\sigma_y = O(\varepsilon |\ln|x||). \tag{A.14}$$

Proof. For $x \in B_\varepsilon^c$, using (A.4) and (A.5), we have

$$\begin{aligned}
\int_{\partial B_\varepsilon} \frac{C_N}{|x-y|^{N-2}} d\sigma_y &= C_N \int_{\partial B_\varepsilon} \frac{|x|^2 + \varepsilon^2 - 2x \cdot y}{|x-y|^N} d\sigma_y \\
&= \frac{\varepsilon}{N-2} \frac{|x|^2 + \varepsilon^2}{\varepsilon^2 - |x|^2} \int_{\partial B_\varepsilon} \frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y} d\sigma_y \\
&\quad - \frac{2\varepsilon}{N-2} \frac{x}{\varepsilon^2 - |x|^2} \int_{\partial B_\varepsilon} y \frac{\partial G_{B_\varepsilon^c}(x,y)}{\partial \nu_y} d\sigma_y \\
&= \frac{\varepsilon^{N-1}}{N-2} \frac{1}{|x|^{N-2}}.
\end{aligned}$$

Let us prove (A.13). Denoting by $F(x) = \int_{\partial B_\varepsilon} \ln|x-y| d\sigma_y$, by (A.3) we get, for any $x \in B_\varepsilon$,

$$\frac{\partial F(x)}{\partial x_i} = x_i \int_{\partial B_\varepsilon} \frac{1}{|x-y|^2} d\sigma_y - \int_{\partial B_\varepsilon} \frac{y_i}{|x-y|^2} d\sigma_y = 0,$$

and then $F(x) = F(0) = \int_{\partial B_\varepsilon} \ln|y| d\sigma_y = 2\pi\varepsilon \ln\varepsilon$. Next, if $x \in B_\varepsilon^c$, setting $x = \varepsilon^2 \frac{t}{|t|^2}$,

$$\int_{\partial B_\varepsilon} \ln|x-y| d\sigma_y = \int_{\partial B_\varepsilon} \Big(\ln|x-y| + \ln\frac{\varepsilon}{|t|}\Big) d\sigma_y$$
$$= 2\pi\varepsilon\ln\varepsilon + 2\pi\varepsilon \ln\frac{|x|}{\varepsilon} = 2\pi\varepsilon\ln|x|,$$

which proves (A.13).

Finally, to estimate $\int_{\partial B_\varepsilon} |\ln|x-y|| d\sigma_y$ we consider the alternative either $\varepsilon < |x| \le \frac{1}{2}$ or $|x| \ge \frac{1}{2}$.

If $|x| \ge \frac{1}{2}$, then

$$\int_{\partial B_\varepsilon} |\ln|x-y|| d\sigma_y \le \int_{\partial B_\varepsilon} \Big[|\ln|x|| + \frac{1}{2}\underbrace{\Big|\ln\Big(1 - 2\frac{x\cdot y}{|x|^2} + \frac{\varepsilon^2}{|x|^2}\Big)\Big|}_{=o(1)}\Big] d\sigma_y$$
$$= (2\pi|\ln|x|| + o(1))\varepsilon.$$

On the other hand, if $\varepsilon < |x| \le \frac{1}{2}$, for $y \in \partial B_\varepsilon$ we have $|x-y| < 1$. So

$$\int_{\partial B_\varepsilon} |\ln|x-y|| d\sigma_y = -\int_{\partial B_\varepsilon} \ln|x-y| d\sigma_y = -2\pi\varepsilon\ln|x|,$$

which ends the proof. ∎

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