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An inverse problem for Hankel operators and turbulent solutions of the cubic Szegő equation on the line

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Abstract. We construct inverse spectral theory for finite rank Hankel operators acting on the Hardy space of the upper half-plane. A particular feature of our theory is that we completely characterise the set of spectral data. As an application of this theory, we prove the genericity of turbulent solutions of the cubic Szegő equation on the real line.

Keywords: Hankel operators, inverse spectral problem, Hardy space, model spaces.

Contents

1. Introduction

1.1. The cubic Szegő equation and Hankel operators

Let $H^2(\mathbb{C}_+)$ be the standard Hardy class in the upper half-plane, with the inner product (linear in the first factor and anti-linear in the second) denoted by $\langle \cdot, \cdot \rangle$. We will routinely

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identify functions $f \in H^2(\mathbb{C}_+)$ with their boundary values on the real line, and with this identification one has $H^2(\mathbb{C}_+) \subset L^2(\mathbb{R})$ as a closed subspace, $\langle \cdot, \cdot \rangle$ being the restriction of the L^2 inner product. Let $\mathbb{P}_+$ be the orthogonal projection onto $H^2(\mathbb{C}_+)$ in $L^2(\mathbb{R})$.

In [24, 25] Pocovnicu, by analogy with the unit circle case [6, 7] (which will be discussed later in Section 1.5), introduced and studied the *cubic Szegő equation*

$$i\frac{\partial u}{\partial t} = \mathbb{P}_+(|u|^2 u), \quad u = u(x,t), \quad x, t \in \mathbb{R}. \tag{1.1}$$

Here, for every $t \in \mathbb{R}$, $u(\cdot, t)$ belongs to a suitable Sobolev class of functions in $H^2(\mathbb{C}_+)$. Following the strategy of the unit circle case [6, 7], it was proven in [24, 25] that this equation is completely integrable and possesses a Lax pair. The Lax pair involves the anti-linear Hankel operator H_u on $H^2(\mathbb{C}_+)$, defined by

$$H_u f = \mathbb{P}_+(u\overline{f}), \quad f \in H^2(\mathbb{C}_+); \tag{1.2}$$

notice that the conjugate of f is always taken on the real line. It is well known that the boundedness of H_u is equivalent to the inclusion $u \in \mathrm{BMOA}(\mathbb{R})$. By a version of Kronecker's theorem, the rationality of u is equivalent to H_u being finite rank. We refer to [23] for the background on Hankel operators. Observe that while H_u is anti-linear, the square H_u^2 is linear (and positive semi-definite). We will say that $\lambda > 0$ is a *singular value* of H_u if the corresponding *Schmidt subspace*

$$E_{H_u}(\lambda) = \mathrm{Ker}(H_u^2 - \lambda^2 I)$$

is non-trivial: $E_{H_u}(\lambda) \neq \{0\}$. The Lax pair formulation for (1.1) ensures, in particular, that all singular values of H_u are integrals of motion of the Szegő equation. In order to solve the Cauchy problem for the Szegő equation, one must develop a suitable version of direct and inverse spectral theory for the Hankel operator H_u. In [24], this programme was completely achieved when the symbol is rational and all singular values are simple (i.e., all Schmidt subspaces $E_{H_u}(\lambda)$ are one-dimensional).

The purpose of this paper is to extend the spectral analysis of [24] to the case of multiplicities, i.e., to the case when the symbol is rational but the dimensions of the subspaces $E_{H_u}(\lambda)$ may be > 1. This requires a more detailed analysis of the structure of these subspaces and of the action of H_u on them. Such analysis was performed in [11] and is recalled later on in this introduction.

As an application of our spectral analysis, we prove the genericity of turbulent solutions of the cubic Szegő equation on the line. In [24,25] it was proved that, for every $s \geqslant \frac{1}{2}$, the initial value problem for (1.1) is globally well posed on the intersection $W^{s,2}(\mathbb{C}_+)$ of $H^2(\mathbb{C}_+)$ with the Sobolev space $W^{s,2}(\mathbb{R})$ on the line. Though the trajectories are bounded in $W^{1/2,2}(\mathbb{C}_+)$ because of some conservation laws, they may not be bounded in $W^{s,2}(\mathbb{C}_+)$ if $s > \frac{1}{2}$. We shall call *turbulent* a solution of (1.1) with an unbounded trajectory in $W^{s,2}(\mathbb{C}_+)$ for some $s > \frac{1}{2}$. An example of a turbulent solution is provided in [24] as a rational solution with two poles such that the associated Hankel operator has a singular value of multiplicity 2. Using our spectral analysis, we are able to find many more such turbulent rational solutions, leading to the following result.

Theorem 1.1. *There exists a dense G_δ subset G of $W^{1,2}(\mathbb{C}_+)$ such that any solution u of the cubic Szegő equation with initial datum in G satisfies*

$$\int_1^{+\infty} \frac{\|\partial_x u(t)\|_{L^2}}{t^2}\, dt = +\infty.$$

In the above statement, the regularity exponent 1 in $W^{1,2}$ is probably not essential but it is technically easier to handle.

We close this paragraph by some comments about the phenomenon of turbulent solutions for Hamiltonian equations, which has been actively studied by mathematicians in the last two decades. Bourgain [1] asked whether there is a solution of the cubic defocusing nonlinear Schrödinger equation on the two-dimensional torus $\mathbb{T}^2$ with initial data

$$u_0 \in W^{s,2}(\mathbb{T}^2), \quad s > 1,$$

such that

$$\limsup_{t\to\infty} \|u(t)\|_{W^{s,2}} = \infty.$$

There is still no complete answer to this question, despite a first partial result in this direction by Colliander–Keel–Staffilani–Takaoka–Tao [2]. Using this approach, Hani [17] proved the existence of turbulent solutions for a totally resonant version of the cubic nonlinear Schrödinger equation on $\mathbb{T}^2$, and Hani–Pausader–Tzvetkov–Visciglia [18] established the first – and, still at this time, the unique – example of solution of the turbulent nonlinear Schrödinger equation, in the case of the cubic Schrödinger equation on the cylinder $\mathbb{T}^2 \times \mathbb{R}$. In the direction of the growth of Sobolev norms for the nonlinear Schrödinger equation, let us also mention the works of Guardia [12], Guardia–Kaloshin [15, 16], Haus–Procesi [19], Guardia–Haus–Procesi [14], and more recently the work of Guardia–Hani–Haus–Maspero–Procesi [13], which uses the integrable structure of the defocusing cubic nonlinear Schrödinger equation on the one-dimensional torus.

The phenomenon of growth of Sobolev norms also occurs for two-dimensional incompressible Euler equations: the sharp double exponentially growing vorticity gradient on the disc was constructed by Kiselev–Šverak [21], and the existence of exponentially growing vorticity gradient solutions on the torus was shown by Zlatoš [28]. It also has been recently observed by Schwinte and Thomann [27] for a system of two lowest Landau level equations.

At this stage, observe that the above results provide examples of turbulent solutions *without establishing their genericity*. In fact, as far as we know, the only equation where genericity of turbulent solutions has been proved before Theorem 1.1 is the cubic Szegő equation on the circle [5, 8].

1.2. Model spaces and isometric multipliers

Before recalling relevant results from [11], we need to talk about model spaces in $H^2(\mathbb{C}_+)$ and isometric multipliers on them. Let θ be an inner function in the upper half-plane (i.e., $\theta \in H^\infty(\mathbb{C}_+)$ and $|\theta(x)| = 1$ for a.e. $x \in \mathbb{R}$). We will only be concerned with the case when θ is a finite Blaschke product, i.e.,

$$\theta(x) = e^{i\alpha} \prod_{j=1}^{m} \frac{x - a_j}{x - \overline{a_j}}, \quad x \in \mathbb{C}_+, \tag{1.3}$$

where $e^{i\alpha}$ is a unimodular complex number and the parameters a_j satisfy $\operatorname{Im} a_j > 0$.

The model space $K_\theta \subset H^2(\mathbb{C}_+)$ is defined by

$$K_\theta = H^2(\mathbb{C}_+) \cap (\theta H^2(\mathbb{C}_+))^\perp;$$

here

$$\theta H^2(\mathbb{C}_+) = \{\theta f : f \in H^2(\mathbb{C}_+)\}$$

and the orthogonal complement is taken in $L^2(\mathbb{R})$. It is clear that $f \in H^2(\mathbb{C}_+)$ belongs to K_θ if and only if $\theta \overline{f} \in H^2(\mathbb{C}_+)$. When θ is a finite Blaschke product (1.3), the model space K_θ is finite-dimensional and can be explicitly described by

$$K_\theta = \left\{\frac{p(z)}{q(z)}, \deg p \leqslant m-1\right\}, \quad q(z) = \prod_{j=1}^{m}(x - \overline{a_j}),$$

where p represents any polynomial of degree $\leqslant m-1$ (see, e.g., [4, Corollary 5.18]). In particular, all elements of K_θ are rational functions.

Let $S(\tau)$ be the semigroup on $H^2(\mathbb{C}_+)$ defined by

$$(S(\tau)f)(x) = e^{i\tau x} f(x), \quad x \in \mathbb{C}_+, \quad \tau > 0. \tag{1.4}$$

Further, let P_θ be the orthogonal projection onto K_θ, and let $S_\theta(\tau)$ be Beurling–Lax semigroup on K_θ, defined by

$$S_\theta(\tau) f = P_\theta(S(\tau)f), \quad f \in K_\theta, \quad \tau > 0 \tag{1.5}$$

(see Section 2 for a more detailed discussion). This is a strongly continuous contractive semigroup and therefore, by the theory of [20], it can be written as

$$S_\theta(\tau) = e^{i\tau A_\theta}, \quad \tau > 0, \tag{1.6}$$

where A_θ is the infinitesimal generator of $S_\theta(\tau)$. For a general inner function θ, the operator A_θ may be unbounded, but under our assumption of rationality of θ the operator A_θ is easily seen to be bounded and of finite rank. Furthermore, it is dissipative, i.e., $\operatorname{Im}\langle A_\theta f, f\rangle \geqslant 0$ for every f in K_θ.

Let p be a holomorphic function in $\mathbb{C}_+$. It is called an *isometric multiplier* on K_θ if for every $f \in K_\theta$, we have $pf \in H^2(\mathbb{C}_+)$ and

$$\|pf\| = \|f\| \quad \forall f \in K_\theta.$$

In this case, we can consider the subspace

$$pK_\theta = \{pf : f \in K_\theta\}.$$

We note that the choice of the parameters p, θ in the representation pK_θ for this subspace is not unique. In fact, if p (resp. $\widetilde{p}$) is an isometric multiplier on K_θ (resp. on $K_{\widetilde{\theta}}$), then (see, e.g., [3, Theorem 10])

$$pK_\theta = \widetilde{p}K_{\widetilde{\theta}}$$

if and only if for some $|w| < 1$ and some unimodular constants $e^{i\alpha}$, $e^{i\beta}$ we have

$$\widetilde{\theta} = e^{i\alpha}\frac{w-\theta}{1-\overline{w}\theta}, \quad \widetilde{p} = e^{i\beta}\frac{1-\overline{w}\theta}{\sqrt{1-|w|^2}}p. \tag{1.7}$$

The transformation $\theta \mapsto \widetilde{\theta}$, $p \mapsto \widetilde{p}$ is called a *Frostman shift*.

1.3. The Schmidt subspaces of Hankel operators

The relevance of these objects to Hankel operators transpires from the following result.

Theorem 1.2 ([11]). *Let $u \in \mathrm{BMOA}(\mathbb{R})$ and let $\lambda > 0$ be a singular value of H_u. Then there exist an inner function ψ and an isometric multiplier p on K_ψ such that the Schmidt subspace $E_{H_u}(\lambda)$ is represented as*

$$E_{H_u}(\lambda) = pK_\psi.$$

Moreover, there exists a unimodular constant $e^{i\alpha}$ such that the action of H_u on $E_{H_u}(\lambda)$ is given by

$$H_u(ph) = \lambda e^{i\alpha} pH_\psi h, \quad h \in K_\psi. \tag{1.8}$$

In particular, by normalising p and ψ suitably (see (1.7)), one can always achieve $e^{i\alpha} = 1$.

We note that H_ψ acts on K_ψ in a simple explicit way,

$$H_\psi h = \psi\overline{h}, \quad h \in K_\psi,$$

because $\psi\overline{h} \in H^2(\mathbb{C}_+)$ and so $\mathbb{P}_+(\psi\overline{h}) = \psi\overline{h}$.

1.4. Direct and inverse spectral problems for Hankel operators with rational symbols

Let u be a rational symbol, analytic in the upper half-plane. We normalise u so that $u(\infty) = 0$ (subtracting a constant from u does not change the Hankel operator H_u). It follows that $u \in H^2(\mathbb{C}_+)$.

First note that we have the commutation relation

$$S(\tau)^* H_u = H_u S(\tau), \quad \tau > 0. \tag{1.9}$$

In fact, H_u is a Hankel operator if and only if it satisfies this relation. It follows from (1.9) that the kernel of H_u is an invariant subspace for $S(\tau)$ and therefore, by the Beurling–Lax theorem [22],

$$\text{either} \quad \overline{\operatorname{Ran} H_u} = H^2(\mathbb{C}_+) \quad \text{or} \quad \overline{\operatorname{Ran} H_u} = K_\theta$$

for some inner function θ. For rational symbols u, the range of H_u is finite-dimensional, and so we have the second possibility here with some rational inner function θ. Further, the range of H_u is closed, so finally we have

$$\operatorname{Ran} H_u = K_\theta$$

with some finite Blaschke product θ. It will be convenient to normalise θ so that

$$\theta(\infty) = 1.$$

Let A_θ be the corresponding infinitesimal generator as in (1.6).

Let us denote the singular values of H_u by $\lambda_1 > \lambda_2 > \cdots > \lambda_N > 0$; to each of these singular values there corresponds a Schmidt subspace $E_{H_u}(\lambda_n)$, which may have arbitrary finite dimension. According to Theorem 1.2, we have

$$E_{H_u}(\lambda_n) = p_n K_{\psi_n}, \quad n = 1, \dots, N, \tag{1.10}$$

where $\psi_1, \dots, \psi_N$ are finite Blaschke products in $\mathbb{C}_+$ and each p_n is an isometric multiplier on K_{ψ_n} (in fact, each p_n is a rational function, see Section 3.2). We will normalise each ψ_n so that $\psi_n(\infty) = 1$.

The "direct spectral problem" for H_u is given by the following theorem.

Theorem 1.3. *Let u be a rational function analytic in the closed upper half-plane and going to zero at infinity. Then*

- *For all n, the vector $u \in H^2(\mathbb{C}_+)$ is not orthogonal to $E_{H_u}(\lambda_n)$.*
- *Denoting by u_n the orthogonal projection of u onto $E_{H_u}(\lambda_n)$, we have*

$$H_u u_n = \lambda_n e^{i\varphi_n} u_n$$

  for some unimodular constants $e^{i\varphi_n}$.
- *The numbers*

$$\omega_n := \langle A_\theta u_n, u_n \rangle, \quad n = 1, \dots, N$$

  have strictly positive imaginary parts.

Next, we introduce the spectral data corresponding to a rational symbol u,

$$(\{\lambda_n\}_{n=1}^N, \{\psi_n\}_{n=1}^N, \{e^{i\varphi_n}\}_{n=1}^N, \{\omega_n\}_{n=1}^N). \tag{1.11}$$

Here λ_n, $e^{i\varphi_n}$ and ω_n are defined in the previous theorem and ψ_n are defined in (1.10). According to the discussion of Section 1.2, the inner functions ψ_n are defined only up to Frostman shifts. So it would be more precise to say that the spectral data contains the orbits of ψ_n under all Frostman shifts, but for notational convenience we will be talking about representatives ψ_n of these orbits. Moreover, it will turn out that our solution to the inverse problem is independent of the choice of a representative.

Now we can state, somewhat informally, one of our main results; the precise statements are Theorems 5.5 and 6.2 below.

Theorem 1.4. (i) *Uniqueness: A rational symbol* u *is uniquely determined by the spectral data* (1.11)*. In fact,* u *is given by an explicit formula* (5.17) *below.*

(ii) *Surjectivity: Let* $N \in \mathbb{N}$*, and let*

- $\lambda_1 > \cdots > \lambda_N > 0$ *be any positive real numbers,*
- $\psi_1, \ldots, \psi_N$ *be any finite Blaschke products,*
- $e^{i\varphi_1}, \ldots, e^{i\varphi_N}$ *be any unimodular complex numbers,*
- $\omega_1, \ldots, \omega_N$ *be any complex numbers with positive imaginary parts.*

Then there exists a rational symbol u *such that* H_u *corresponds to the spectral data* (1.11).

If all singular values λ_n are simple, we recover the result of Pocovnicu [24]. In this case, the spectral data do not contain inner functions ψ_n. Let us explain this. If λ_n is simple, then ψ_n is a single Blaschke factor

$$\psi_n(x) = \frac{x - a_n}{x - \overline{a_n}}, \quad a_n \in \mathbb{C}_+.$$

By using a Frostman shift, a_n can be changed to any given number in $\mathbb{C}_+$. In other words, in this case the orbit of ψ_n by the action of Frostman shifts consists of all single Blaschke factors, and so this orbit does not contain any information apart from the fact that the dimension of the corresponding Schmidt subspace is one. In Section 5.9, we give a more detailed comparison with the spectral data of [24].

Observe that the isometric multipliers p_n are not part of our spectral data; in fact, they can be explicitly determined from the spectral data, see Section 5.7.

1.5. The unit circle case

The Szegő equation (1.1) was originally introduced in [6, 7], where it was considered for functions u defined on the unit circle; u was assumed to be in a suitable Sobolev subspace of the Hardy space $H^2(\mathbb{D})$, where $\mathbb{D}$ is the unit disc. In this context, $\mathbb{P}_+$ becomes the orthogonal projection in $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$, often called the Szegő projection (hence the name for the equation). In fact, [6, 7] provided the blueprint for the study of the Lax pair structure of [24, 25]. More precisely, in the unit circle case the Szegő equation is completely integrable and possesses a Lax pair, which involves a Hankel operator H_u in $H^2(\mathbb{D})$. Solving the equation reduces to a solution of a direct and inverse spectral problem for H_u. Despite many similarities, we would like to stress some important differences between the unit circle and the real line cases:

- The choice of the spectral data in the unit circle case is very different. It involves introducing an auxiliary Hankel operator (denoted by K_u in [7]) and looking at its singular values and Schmidt subspaces.
- The property $u \not\perp E_{H_u}(\lambda_n)$ of Theorem 1.3 is *false* in the unit circle case! Roughly speaking, in the unit circle case for about half of the singular values (the ones termed *K-dominant* in [7]) we have $u \perp E_{H_u}(\lambda_n)$.

Our proof of the second part of Theorem 1.4 is strongly inspired by [10], where a new "algebraic" approach to the inverse spectral problem on the unit circle was developed.

1.6. The structure of the paper

In Section 2, we recall various well-known facts from the theory of model spaces, focussing on K_θ with rational θ. In Section 3, we consider the set-up

$$pK_\psi \subset K_\theta,$$

where p is an isometric multiplier on K_ψ and derive identities relating the infinitesimal generators A_θ and A_ψ. Although model spaces is a well-studied subject, some of our results here appear to be new.

In Section 4, we introduce Hankel operators and prove Theorem 1.3. The key ingredient here is the commutation relation (recall that $\theta(\infty) = 1$)

$$A_\theta H_u^2 - H_u^2 A_\theta = \frac{i}{2\pi}\langle\cdot, H_u u\rangle(1-\theta) - \frac{i}{2\pi}\langle\cdot, u\rangle u.$$

In Section 5, we prove the uniqueness part of Theorem 1.4 by giving an explicit expression for $u(x)$ in terms of the spectral data. Our starting point is the formula

$$u(x) = \langle (A_\theta^* - x)^{-1} u, 1-\theta\rangle, \quad \operatorname{Im} x > 0,$$

which follows directly from the fact that $1-\theta$ is the "reproducing kernel of K_θ at infinity" (see Lemma 2.5 below). We then consider the action of A_θ on the model space $K_\theta =$ Ran H_u, represented as the orthogonal sum

$$K_\theta = \bigoplus_{n=1}^{N} p_n K_{\psi_n}. \tag{1.12}$$

We use the results of Section 3 and the above commutation relation for A_θ to show that A_θ has a rather special block-matrix structure in this representation. Using this block-matrix structure, we express the resolvent $(A_\theta^* - x)^{-1}$ in terms of our spectral data.

In Sections 6 and 7, we prove the last part of Theorem 1.4. Here we use the "algebraic" approach of [10]. Namely, we take u given by the explicit expression established at the previous step of the proof and check directly that the corresponding operator H_u has the "correct" spectral data. An important step in the proof is checking that the functions p_n, given by certain explicit matrix formulas, are isometric multipliers on K_{ψ_n}. Here we are guided by intuition coming from Sarason's work [26], which gives a general representation formula for all isometric multipliers on a given model space. Lemma 7.2 provides a partial extension of this formula to the matrix case.

Section 8 is devoted to describing the evolution of the spectral data (1.11) of a solution of the cubic Szegő equation (1.1). Finally, in Section 9, we combine the previous results to prove Theorem 1.1.

2. Model spaces

Almost all of this section is either well known or folklore; see, e.g., the monograph [4]. Some of the facts that we need are easy to find in the literature but for the case of the Hardy spaces on the unit disc rather than the upper half-plane. In any case, for completeness we present all necessary statements with simple proofs.

2.1. Model spaces

In what follows, we denote by H^2 the standard Hardy class in the upper half-plane $\mathbb{C}_+$. Let v_ζ, $\zeta \in \mathbb{C}_+$, be the reproducing kernel in H^2,

$$v_\zeta(x) = \frac{1}{2\pi i}\frac{1}{\overline{\zeta} - x},$$

so that $f(\zeta) = \langle f, v_\zeta\rangle$ for every $f \in H^2$. Let θ be a finite Blaschke product in $\mathbb{C}_+$, and let K_θ be the corresponding model space

$$K_\theta = H^2 \cap (\theta H^2)^\perp.$$

Further, let P_θ be the orthogonal projection in H^2 onto K_θ; it is straightforward to obtain (see, e.g., [4, Proposition 5.14]) that P_θ is given by

$$P_\theta\colon f \mapsto f - \theta\mathbb{P}_+(\overline{\theta} f), \quad f \in H^2. \tag{2.1}$$

Let $S(\tau)$ and $S_\theta(\tau)$ be semigroups (1.4), (1.5), and let A_θ be the infinitesimal generator of $S_\theta(\tau)$ as in (1.6). For $\zeta \in \mathbb{C}_+$, the following resolvents are well defined and bounded on K_θ:

$$(A_\theta - \overline{\zeta})^{-1} = -i\int_0^\infty S_\theta(\lambda)e^{-i\lambda\overline{\zeta}}d\lambda,$$
$$(A_\theta^* - \zeta)^{-1} = i\int_0^\infty S_\theta(\lambda)^* e^{i\lambda\zeta}d\lambda,$$

see, e.g., [20]. Using the definition of S_θ and computing the integrals, we see that these resolvents can be expressed as

$$(A_\theta - \overline{\zeta})^{-1} f = -2\pi i P_\theta(f v_\zeta), \tag{2.2}$$
$$(A_\theta^* - \zeta)^{-1} f = 2\pi i\mathbb{P}_+(f\overline{v_\zeta}) = \frac{f(x) - f(\zeta)}{x - \zeta}. \tag{2.3}$$

Lemma 2.1. *The operator A_θ is completely non-self-adjoint, i.e., there is no non-trivial subspace $N \subset K_\theta$, invariant for A_θ, where A_θ is self-adjoint.*

Proof. From the definition of $S(\tau)$, it follows that

$$\lim_{\tau\to\infty} \|S(\tau)^* f\| = 0$$

for any $f \in H^2$. Since θH^2 is invariant under $S(\tau)$, the model space K_θ is invariant under $S(\tau)^*$, and therefore $S_\theta(\tau)^* f = S(\tau)^* f$ for any $f \in K_\theta$. Thus, we also have

$$\lim_{\tau\to\infty} \|S_\theta(\tau)^* f\| = 0$$

for any $f \in K_\theta$. It follows that $S_\theta(\tau)^*$ is completely non-unitary on K_θ, i.e., there is no non-trivial subspace $N \subset K_\theta$, invariant for $S_\theta(\tau)^*$ for all $\tau > 0$ and such that $S_\theta(\tau)^*$ is unitary on N for all $\tau > 0$. From here we get the claim. ∎

We write $\theta(\infty) = \lim_{|x|\to\infty} \theta(x)$; clearly, we have $|\theta(\infty)| = 1$. In what follows, we normalise θ so that $\theta(\infty) = 1$.

Lemma 2.2. *Let θ be a finite Blaschke product and $\theta(\infty) = 1$. Then $1 - \theta \in K_\theta$.*

Proof. For $h \in H^2$, we have

$$\langle \theta h, 1-\theta\rangle = \langle h, \overline{\theta}(1-\theta)\rangle = \langle h, \overline{\theta} - 1\rangle = 0.$$

It follows that $1 - \theta$ is orthogonal to θH^2, and so it belongs to K_θ. ∎

Lemma 2.3. *Let θ be a finite Blaschke product and $\theta(\infty) = 1$. Then for any $f \in K_\theta$ and for any $\zeta \in \mathbb{C}_+$, we have*

$$\begin{aligned} f(\zeta) &= \frac{1}{2\pi i}\langle f, (A_\theta - \overline{\zeta})^{-1}(1-\theta)\rangle, \\ (H_\theta f)(\zeta) &= -\frac{1}{2\pi i}\langle (A_\theta^* - \zeta)^{-1}(1-\theta), f\rangle. \end{aligned} \tag{2.4}$$

Proof. We have

$$f(\zeta) = \langle f, v_\zeta\rangle = \langle f, P_\theta v_\zeta\rangle = \langle f, P_\theta(v_\zeta(1-\theta))\rangle$$

because $P_\theta(\theta v_\zeta) = 0$. By (2.2),

$$\langle f, P_\theta(v_\zeta(1-\theta))\rangle = \frac{1}{2\pi i}\langle f, (A_\theta - \overline{\zeta})^{-1}(1-\theta)\rangle,$$

which yields (2.4). Similarly,

$$\begin{aligned} H_\theta f(\zeta) &= \langle \theta \overline{f}, v_\zeta\rangle = \langle \theta\overline{v_\zeta}, f\rangle = \langle \mathbb{P}_+(\theta\overline{v_\zeta}), f\rangle = -\langle \mathbb{P}_+((1-\theta)\overline{v_\zeta}), f\rangle \\ &= -\frac{1}{2\pi i}\langle (A_\theta^* - \zeta)^{-1}(1-\theta), f\rangle, \end{aligned}$$

where we have used (2.3) at the last step. ∎

Corollary 2.4. *Let θ be a finite Blaschke product and $\theta(\infty) = 1$. Then the linear span of each of the two sets*

$$\{(A_\theta - \overline{\zeta})^{-1}(1-\theta)\}_{\zeta\in\mathbb{C}_+}, \quad \{(A_\theta^* - \zeta)^{-1}(1-\theta)\}_{\zeta\in\mathbb{C}_+}$$

is dense in K_θ.

2.2. Behaviour at infinity

For a rational function f with the Laurent expansion

$$f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots, \quad |x| \to \infty,$$

at infinity, we will denote

$$\Lambda_1(f) = a_1, \quad \Lambda_2(f) = a_2.$$

Lemma 2.5. *Let θ be a finite Blaschke product and $\theta(\infty) = 1$. For any $f \in K_\theta$, we have*

$$\Lambda_1(f) = -\frac{1}{2\pi i}\langle f, 1-\theta\rangle, \quad \Lambda_2(f) = -\frac{1}{2\pi i}\langle A_\theta^* f, 1-\theta\rangle,$$

and, in particular,

$$\Lambda_1(\theta) = \frac{1}{2\pi i}\|1-\theta\|^2, \quad \Lambda_2(\theta) = \frac{1}{2\pi i}\langle A_\theta^*(1-\theta), 1-\theta\rangle. \tag{2.5}$$

Proof. Since A_θ is bounded, we can expand the resolvent in (2.4), which yields

$$f(x) = -\frac{1}{2\pi i}\frac{1}{x}\langle f, 1-\theta\rangle - \frac{1}{2\pi i}\frac{1}{x^2}\langle A_\theta^* f, 1-\theta\rangle + O\Big(\frac{1}{x^3}\Big)$$

as $|x| \to \infty$. This yields the required identities. ■

2.3. Formulas for A_θ and A_θ^*

Lemma 2.6. *Let θ be a finite Blaschke product and $\theta(\infty) = 1$. Then the operator A_θ on K_θ satisfies the identities*

$$A_\theta f(x) = xf(x) - \Lambda_1(f)\theta(x) = xf(x) + \frac{1}{2\pi i}\langle f, 1-\theta\rangle\theta(x), \tag{2.6}$$

$$A_\theta^* f(x) = xf(x) - \Lambda_1(f) = xf(x) + \frac{1}{2\pi i}\langle f, 1-\theta\rangle, \tag{2.7}$$

$$\operatorname{Im} A_\theta = \frac{1}{4\pi}\langle \cdot, 1-\theta\rangle(1-\theta). \tag{2.8}$$

Proof. Let A_θ be the operator given by the right-hand side of (2.6). We need to check that for all $f \in K_\theta$,

$$\left\| \frac{1}{i\tau}(S_\theta(\tau) - I)f - A_\theta f \right\| \to 0, \quad \tau \to 0_+. \tag{2.9}$$

First we observe that for all $\tau > 0$ and any $h \in K_\theta$,

$$\int_{-\infty}^{\infty} \frac{e^{i\tau x} - 1}{i\tau x}\theta(x)\overline{h(x)}\, dx = 0, \tag{2.10}$$

since

$$\frac{e^{i\tau x} - 1}{i\tau x} \in H^2 \quad \text{and} \quad \theta\overline{h} \in H^2.$$

Using this, we compute

$$\frac{1}{i\tau}\langle (S_\theta(\tau)-I)f,h\rangle = \frac{1}{i\tau}\langle (S(\tau)-I)f,h\rangle = \int_{-\infty}^{\infty}\frac{e^{i\tau x}-1}{i\tau x}xf(x)\overline{h(x)}\,dx$$
$$= \int_{-\infty}^{\infty}\frac{e^{i\tau x}-1}{i\tau x}(xf(x)-\Lambda_1(f)\theta(x))\overline{h(x)}\,dx$$
$$= \int_{-\infty}^{\infty}\frac{e^{i\tau x}-1}{i\tau x}A_\theta f(x)\overline{h(x)}\,dx,$$

and therefore

$$\Big\langle \frac{1}{i\tau}(S_\theta(\tau)-I)f - A_\theta f, h\Big\rangle = \int_{-\infty}^{\infty}\Big(\frac{e^{i\tau x}-1}{i\tau x}-1\Big)A_\theta f(x)\overline{h(x)}\,dx.$$

By the Cauchy–Schwarz inequality,

$$\Big|\Big\langle \frac{1}{i\tau}(S_\theta(\tau)-I)f - A_\theta f, h\Big\rangle\Big|^2 \leqslant \|g\|^2\int_{-\infty}^{\infty}\Big|\frac{e^{i\tau x}-1}{i\tau x}-1\Big|^2|A_\theta f(x)|^2dx,$$

and the integral in the right-hand side tends to zero as $\lambda\to 0_+$ by dominated convergence. This yields (2.9).

Similarly, let A_θ^* be the operator given by the right-hand side of (2.7); we need to check that

$$\Big\|\frac{1}{i\tau}(S(\tau)^*-I)f - A_\theta^* f\Big\| \to 0, \quad \tau\to 0_+,$$

for all $f\in K_\theta$. This is achieved by following the same line of reasoning, with the only difference that instead of (2.10) we use the identity

$$\int_{-\infty}^{\infty}\frac{e^{-i\tau x}-1}{i\tau x}\overline{h(x)}\,dx = 0, \quad \tau>0, \quad g\in K_\theta,$$

because both factors in the integral are complex conjugates of elements of H^2.

Finally, (2.8) follows by subtracting (2.7) from (2.6). ∎

2.4. *The action of Frostman shifts*

Let ψ be a finite Blaschke product. Here we compute the action of Frostman shifts (1.7) on various quantities relevant to the subsequent analysis of inverse problems. First note that if we require that $\psi(\infty)=\widetilde{\psi}(\infty)=1$, this fixes the unimodular constant $e^{i\alpha}$ in (1.7), so we obtain

$$\widetilde{\psi} = \frac{1-\overline{w}}{w-1}\frac{w-\psi}{1-\overline{w}\psi}, \quad |w|<1. \tag{2.11}$$

Next, observe that the function $i\frac{1+\psi}{1-\psi}$ is a Herglotz function, i.e., it maps the upper half-plane into itself. As a rational Herglotz function, it admits the representation

$$i\frac{1+\psi(x)}{1-\psi(x)} = Ax+B+\sum_j\frac{c_j}{\alpha_j-x}, \quad \operatorname{Im}x>0,$$

where $A \geqslant 0$, $B \in \mathbb{R}$, $c_j \geqslant 0$, $\alpha_j \in \mathbb{R}$, and the sum is finite; the points α_j are the solutions to the equation $\psi(x) = 1$. Recalling that

$$\|1-\psi\|^2 = 2\pi i \Lambda_1(\psi) \tag{2.12}$$

and renormalising, we obtain the representation

$$i\frac{\|1-\psi\|^2}{4\pi}\frac{1+\psi(x)}{1-\psi(x)} = x + B + \sum_j \frac{c_j}{\alpha_j - x}, \quad \operatorname{Im} x > 0, \tag{2.13}$$

with the same conditions on the parameters B, c_j, α_j.

Lemma 2.7. *Let ψ be a finite Blaschke product with $\psi(\infty) = 1$, and let $\widetilde{\psi}$ be as in (2.11). Then*

$$\|1-\widetilde{\psi}\|^2 = \frac{1-|w|^2}{|1-w|^2}\|1-\psi\|^2, \tag{2.14}$$

$$i\frac{\|1-\widetilde{\psi}\|^2}{4\pi}\frac{1+\widetilde{\psi}}{1-\widetilde{\psi}} = i\frac{\|1-\psi\|^2}{4\pi}\frac{1+\psi}{1-\psi} + \frac{1}{2\pi}\frac{\operatorname{Im} w}{|1-w|^2}\|1-\psi\|^2. \tag{2.15}$$

Proof. Computing the asymptotics of $\widetilde{\psi}$ at infinity, comparing with the asymptotics of ψ and using (2.12), we obtain (2.14). After this, the proof of (2.15) is direct algebra. ∎

This lemma shows that under the Frostman shift, only the constant B in representation (2.13) changes. The next lemma gives more precise information about this constant.

Lemma 2.8. *Let ψ be a finite Blaschke product with $\psi(\infty) = 1$. Then*

$$i\frac{\|1-\psi\|^2}{4\pi}\frac{1+\psi(x)}{1-\psi(x)} = x - \frac{\operatorname{Re}\langle A_\psi(1-\psi), 1-\psi\rangle}{\|1-\psi\|^2} + O\Big(\frac{1}{x}\Big) \tag{2.16}$$

as $|x| \to \infty$. As a consequence, the function

$$i\frac{\|1-\psi\|^2}{4\pi}\frac{1+\psi(x)}{1-\psi(x)} + \frac{\operatorname{Re}\langle A_\psi(1-\psi), 1-\psi\rangle}{\|1-\psi\|^2} \tag{2.17}$$

is invariant under the Frostman shifts $\psi \mapsto \widetilde{\psi}$ as in (2.11).

Proof. Using (2.12) and expanding the function at infinity, we get

$$\begin{aligned}
i\frac{\|1-\psi\|^2}{4\pi}\frac{1+\psi(x)}{1-\psi(x)} &= -\frac{1}{2}\Lambda_1(\psi)\frac{2 + \frac{\Lambda_1(\psi)}{x} + O(\frac{1}{x^2})}{-\frac{\Lambda_1(\psi)}{x} - \frac{\Lambda_2(\psi)}{x^2} + O(\frac{1}{x^3})} \\
&= x\frac{1 + \frac{\Lambda_1(\psi)}{2x} + O(\frac{1}{x^2})}{1 + \frac{\Lambda_2(\psi)}{x\Lambda_1(\psi)} + O(\frac{1}{x^2})} \\
&= x - \left(\frac{\Lambda_2(\psi)}{\Lambda_1(\psi)} - \frac{\Lambda_1(\psi)}{2}\right) + O\Big(\frac{1}{x}\Big)
\end{aligned}$$

as $|x| \to \infty$. By (2.5), (2.12) and (2.8), the constant term in the above formula transforms as

$$\frac{\Lambda_2(\psi)}{\Lambda_1(\psi)} - \frac{\Lambda_1(\psi)}{2} = \frac{\langle A_\psi^*(1-\psi), 1-\psi\rangle}{\|1-\psi\|^2} - \frac{1}{4\pi i}\|1-\psi\|^2 = \frac{\operatorname{Re}\langle A_\psi(1-\psi), 1-\psi\rangle}{\|1-\psi\|^2},$$

which yields (2.16). By Lemma 2.7, function (2.17) changes by a constant under the Frostman shift. By (2.16), this constant is zero. ∎

3. Isometric multipliers on model spaces

In this section, we consider the following scenario: θ is a finite Blaschke product, ψ is an inner function in $\mathbb{C}_+$ and p is an isometric multiplier on K_ψ. We assume that $pK_\psi \subset K_\theta$ and derive some identities relating the infinitesimal generators A_θ and A_ψ. Although the results of this section are relatively straightforward, the set-up is rather special and hard to locate in the literature.

3.1. Formula for the projection onto pK_ψ

Let ψ be a non-constant inner function in $\mathbb{C}_+$, and let p be an isometric multiplier on K_ψ. Denote $M = pK_\psi$ and let P_M be the orthogonal projection onto M. First we need a formula for P_M.

Lemma 3.1. *Let $f \in H^2$ be such that $\overline{p} f \in L^2(\mathbb{R})$. Then*

$$P_M f = p\mathbb{P}_+(\overline{p} f) - p\psi\mathbb{P}_+(\overline{p}\overline{\psi} f). \tag{3.1}$$

Proof. Using formula (2.1) for the orthogonal projection P_ψ onto K_ψ, we see that the right-hand side of (3.1) can be written as

$$p\mathbb{P}_+(\overline{p} f) - p\psi\mathbb{P}_+(\overline{p}\overline{\psi} f) = pP_\psi\mathbb{P}_+(\overline{p} f).$$

It is clear that the right-hand side here is in pK_ψ. It remains to check that its difference with f is orthogonal to pK_ψ. For $h \in K_\psi$, we have

$$\langle f - pP_\psi\mathbb{P}_+(\overline{p} f), ph\rangle = \langle f, ph\rangle - \langle P_\psi\mathbb{P}_+(\overline{p} f), h\rangle = \langle f, ph\rangle - \langle \overline{p} f, h\rangle = 0,$$

as required. ∎

3.2. Projecting onto $M = pK_\psi$ in K_θ

Here we work out formulas for projections of various functions onto $M = pK_\psi$ in K_θ.

Lemma 3.2. *Let ψ and θ be non-constant inner functions in $\mathbb{C}_+$, and let p be an isometric multiplier on K_ψ. Assume that $pK_\psi \subset K_\theta$ and that θ is a finite Blaschke product. Then both ψ and p are rational (in particular, ψ is also a finite Blaschke product). Furthermore,*

$$p - p(\infty) \in K_\theta. \tag{3.2}$$

Proof. It is easy to compute the reproducing kernel of K_ψ (cf., e.g., [4, Section 5.5]):

$$F_\zeta(x) := P_\psi v_\zeta(x) = \frac{1}{2\pi i}\frac{1-\overline{\psi(\zeta)}\psi(x)}{\overline{\zeta}-x}.$$

Since $F_\zeta \in K_\psi$ and $pK_\psi \subset K_\theta$, we see that pF_ζ is rational with $(pF_\zeta)(\infty) = 0$. Multiplying by $(\overline{\zeta}-x)$, we find that the function

$$(1-\overline{\psi(\zeta)}\psi(x))p(x)$$

is rational. Since this is true for any $\zeta \in \mathbb{C}_+$ and ψ is non-constant, we conclude that both p and ψp are rational. Since p is not identically zero, we finally conclude that ψ is also rational.

Let us prove (3.2). Take some $h \in K_\psi$ with $\Lambda_1(h) \neq 0$, for example, $h = 1-\psi$ (if ψ is normalised by $\psi(\infty) = 1$), see (2.5). By (2.7), we have

$$\begin{aligned} A_\theta^*(ph) - pA_\psi^* h &= xp(x)h(x) - \Lambda_1(ph) - p(x)(xh(x) - \Lambda_1(h)) \\ &= p(x)\Lambda_1(h) - \Lambda_1(ph) = (p(x) - p(\infty))\Lambda_1(h). \end{aligned}$$

Here the left-hand side is in K_θ, and so the right-hand side is also in K_θ. ■

Lemma 3.3. *Let ψ and θ be finite Blaschke products with $\psi(\infty) = \theta(\infty) = 1$, and let p be an isometric multiplier on K_ψ. Assume that $M = pK_\psi \subset K_\theta$. Then*

$$P_M(1-\theta) = \overline{p(\infty)}p(1-\psi)$$

and

$$\|P_M(1-\theta)\| = |p(\infty)|\|1-\psi\|. \tag{3.3}$$

Proof. By formula (3.1), we have

$$P_M(1-\theta) = p\mathbb{P}_+(\overline{p}(1-\theta)) - p\psi\mathbb{P}_+(\overline{p}\overline{\psi}(1-\theta)).$$

Let us compute the two terms in the right-hand side. We have

$$\begin{aligned} \overline{p}(1-\theta) &= (\overline{p} - \overline{p(\infty)})(1-\theta) + \overline{p(\infty)}(1-\theta) \\ &= (\overline{p} - \overline{p(\infty)}) - \theta(\overline{p} - \overline{p(\infty)}) + \overline{p(\infty)}(1-\theta). \end{aligned}$$

By (3.2), the first term in the right-hand side is anti-analytic and the second term is in H^2, so we get

$$\mathbb{P}_+(\overline{p}(1-\theta)) = -\theta(\overline{p} - \overline{p(\infty)}) + \overline{p(\infty)}(1-\theta) = \overline{p(\infty)} - \theta\overline{p}. \tag{3.4}$$

Next, we have

$$\overline{p}\overline{\psi}(1-\theta) = \overline{p}(\overline{\psi}-1) - \theta\overline{p}(\overline{\psi}-1) + \overline{p}(1-\theta).$$

Since $p(\psi - 1) \in pK_\psi \subset K_\theta$, we have $\theta\,\overline{p}(\overline{\psi} - 1) \in H^2$. Thus, using (3.4),

$$\mathbb{P}_+(\overline{p}\overline{\psi}(1-\theta)) = -\theta\,\overline{p}(\overline{\psi} - 1) + \mathbb{P}_+(\overline{p}(1-\theta)) = -\theta\,\overline{p}(\overline{\psi} - 1) + \overline{p(\infty)} - \theta\,\overline{p}.$$

Combining this, we obtain

$$\begin{aligned} P_M(1-\theta) &= p(\overline{p(\infty)} - \theta\,\overline{p}) - p\psi(\overline{p(\infty)} - \theta\,\overline{p} - \theta\,\overline{p}(\overline{\psi} - 1)) \\ &= (1-\psi)p(\overline{p(\infty)} - \theta\,\overline{p}) + \theta\psi|p|^2(\overline{\psi} - 1) = \overline{p(\infty)}p(1-\psi), \end{aligned}$$

as claimed. Computing the norms and using the isometricity of p, we obtain (3.3). ■

Lemma 3.4. *Assume the hypothesis of the previous lemma. Then*

$$P_M(p - p(\infty)) = \varkappa p(1-\psi)$$

with some $\varkappa \in \mathbb{C}$.

Unlike in the previous lemma, we do not have a direct argument for it, nor an expression for the constant $\varkappa$. Our proof requires two intermediate steps.

Lemma 3.5. *Assume the hypothesis of Lemma 3.3 and let* $\zeta \in \mathbb{C}_+$. *Then*

$$P_M(A_\theta^* - \zeta)^{-1}(p(1-\psi)) = cp(A_\psi^* - \zeta)^{-1}(1-\psi)$$

with some $c \in \mathbb{C}$.

Proof. The statement of the lemma is equivalent to the following one. Let $f \in pK_\psi$, $f \perp p(A_\psi^* - \zeta)^{-1}(1-\psi)$; then

$$f \perp (A_\theta^* - \zeta)^{-1}(p(1-\psi)). \tag{3.5}$$

Write $f = ph$, $h \in K_\psi$; then condition $f \perp p(A_\psi^* - \zeta)^{-1}(1-\psi)$ is equivalent to $h \perp (A_\psi^* - \zeta)^{-1}(1-\psi)$. We have

$$\langle f, (A_\theta^* - \zeta)^{-1}(p(1-\psi))\rangle = \langle (A_\theta - \overline{\zeta})^{-1}(ph), p(1-\psi)\rangle,$$

and, by (2.2),

$$(A_\theta - \overline{\zeta})^{-1}(ph) = -2\pi i P_\theta(phv_\zeta) = -2\pi i phv_\zeta + w, \quad w \in \theta H^2.$$

It follows that

$$\langle (A_\theta - \overline{\zeta})^{-1}(ph), p(1-\psi)\rangle = -2\pi i\langle phv_\zeta, p(1-\psi)\rangle. \tag{3.6}$$

We would like to use the isometricity of p in the right-hand side of (3.6). In order to be able to do so, we must check that hv_ζ is in K_ψ. Let $g \in H^2$; consider

$$\langle hv_\zeta, \psi g\rangle = \langle h, \overline{v_\zeta}\psi g\rangle = \langle h, \mathbb{P}_+(\overline{v_\zeta}\psi g)\rangle.$$

Further, we have

$$\begin{aligned}\mathbb{P}_+(\overline{v_\zeta}\psi g)(x) &= \frac{1}{2\pi i}\frac{\psi(x)g(x)-\psi(\zeta)g(\zeta)}{x-\zeta}\\ &= \frac{1}{2\pi i}\psi(x)\frac{g(x)-g(\zeta)}{x-\zeta}+\frac{1}{2\pi i}g(\zeta)\frac{(\psi(x)-1)-(\psi(\zeta)-1)}{x-\zeta}\\ &= \psi\widetilde{w}+\frac{1}{2\pi i}g(\zeta)(A_\psi^*-\zeta)^{-1}(\psi-1),\end{aligned}$$

where $\widetilde{w}\in H^2$. Now

$$\langle h,\mathbb{P}_+(\overline{v_\zeta}\psi g)\rangle=\langle h,\psi\widetilde{w}\rangle-\frac{1}{2\pi i}\overline{g(\zeta)}\langle h,(A_\psi^*-\zeta)^{-1}(1-\psi)\rangle=0,$$

where the first term in the right-hand side vanishes because $h\in K_\psi$ and the second term vanishes because of our condition on h. Thus, we have checked that $hv_\zeta\in K_\psi$.

We come back to (3.6),

$$\langle phv_\zeta,p(1-\psi)\rangle=\langle hv_\zeta,1-\psi\rangle=-(2\pi i)\Lambda_1(hv_\zeta),$$

where we have used Lemma 2.5 at the last step. Finally, we have $h(x)=O(\frac{1}{x})$ and $v_\zeta(x)=O(\frac{1}{x})$ at infinity, and therefore $\Lambda_1(hv_\zeta)=0$. We have checked (3.5); the proof is complete. ■

Lemma 3.6. *Assume the hypothesis of Lemma 3.3 and let $\zeta\in\mathbb{C}_+$. Then*

$$P_M(A_\theta^*-\zeta)^{-1}(p-p(\infty))=c'p(A_\psi^*-\zeta)^{-1}(1-\psi) \tag{3.7}$$

with some $c'\in\mathbb{C}$.

Proof. We have, from (2.3),

$$\begin{aligned}(A_\theta^*-\zeta)^{-1}(p(1-\psi)) &= \frac{p(x)(1-\psi(x))-p(\zeta)(1-\psi(\zeta))}{x-\zeta}\\ &= p(x)\frac{(1-\psi(x))-(1-\psi(\zeta))}{x-\zeta}\\ &\quad+(1-\psi(\zeta))\frac{(p(x)-p(\infty))-(p(\zeta)-p(\infty))}{x-\zeta}\\ &= p(A_\psi^*-\zeta)^{-1}(1-\psi)+(1-\psi(\zeta))(A_\theta^*-\zeta)^{-1}(p-p(\infty)).\end{aligned}$$

Let us apply P_M to both sides of this identity. Observing that $p(A_\psi^*-\zeta)^{-1}(1-\psi)\in pK_\psi$ and using the previous lemma, we obtain

$$cp(A_\psi^*-\zeta)^{-1}(1-\psi)=p(A_\psi^*-\zeta)^{-1}(1-\psi)+(1-\psi(\zeta))P_M(A_\theta^*-\zeta)^{-1}(p-p(\infty)).$$

Rearranging, we obtain the required identity with $c'=\frac{c-1}{1-\psi(\zeta)}$. ■

Proof of Lemma 3.4. We have the strong convergence

$$-\zeta(A_\theta^*-\zeta)^{-1}\to I,\quad |\zeta|\to\infty, \tag{3.8}$$

and the same is true for the resolvent of A_ψ^*. Applying this to (3.7), we obtain the required result. ■

3.3. Relation between A_θ and A_ψ

Theorem 3.7. *Let ψ and θ be finite Blaschke products normalised so that $\psi(\infty) = \theta(\infty) = 1$. Let p be an isometric multiplier on K_ψ; assume that $M = pK_\psi \subset K_\theta$. Then there exists a constant $c \in \mathbb{C}$ such that for any $h_1, h_2 \in K_\psi$, we have*

$$\langle A_\theta(ph_1), ph_2\rangle = \langle A_\psi h_1, h_2\rangle + c\langle h_1, 1-\psi\rangle\langle 1-\psi, h_2\rangle. \tag{3.9}$$

Moreover,

$$\frac{\langle A_\theta P_M(1-\theta), P_M(1-\theta)\rangle}{\|P_M(1-\theta)\|^2} = \frac{\langle A_\psi(1-\psi), 1-\psi\rangle}{\|1-\psi\|^2} + c\|1-\psi\|^2, \tag{3.10}$$

and

$$\frac{1}{4\pi}\|P_M(1-\theta)\|^2 = \frac{1}{4\pi}\|1-\psi\|^2 + \operatorname{Im} c\|1-\psi\|^2. \tag{3.11}$$

Proof. By complex conjugation, (3.9) is equivalent to

$$\langle A_\theta^*(ph_2), ph_1\rangle = \langle A_\psi^* h_2, h_1\rangle + \overline{c}\langle h_2, 1-\psi\rangle\langle 1-\psi, h_1\rangle. \tag{3.12}$$

Let us prove the last identity. By (2.7), we have

$$\begin{aligned} A_\theta^*(ph_2)(x) - p(x)A_\psi^* h(x) &= xp(x)h_2(x) - \Lambda_1(ph_2) - p(x)(xh_2(x) - \Lambda_1(h_2)) \\ &= -\Lambda_1(ph_2) + p(x)\Lambda_1(h_2) = (p(x) - p(\infty))\Lambda_1(h_2). \end{aligned}$$

Let us apply P_M to both sides here and use Lemmas 3.4 and 2.5:

$$P_M A_\theta^*(ph_2) - pA_\psi^* h_2 = \varkappa p(1-\psi)\Lambda_1(h_2) = -\frac{\varkappa}{2\pi i} p(1-\psi)\langle h_2, 1-\psi\rangle.$$

Denote

$$\overline{c} = -\frac{\varkappa}{2\pi i};$$

taking the inner product with ph_1 and using the isometricity of p on K_ψ, we arrive at (3.12).

In order to prove (3.10), it suffices to take $h_1 = h_2 = \overline{p(\infty)}(1-\psi)$ in (3.9) and divide by $\|P_M(1-\theta)\|^2$, using Lemma 3.3. Taking imaginary part and using (2.8), we arrive at (3.11). ∎

4. Direct spectral problem: Proof of Theorem 1.3

Throughout this section, u is a bounded rational symbol with no poles in the closed upper half-plane, normalised so that $u(\infty) = 0$ and H_u is the Hankel operator (1.2). Furthermore, θ is the finite Blaschke product such that

$$\operatorname{Ran} H_u = K_\theta$$

and normalised by $\theta(\infty) = 1$.

4.1. Commutation relations for H_u and A_θ

Lemma 4.1. *Let u, θ be as above. Then $H_u(1-\theta) = u$. Furthermore,*

$$A_\theta^* H_u = H_u A_\theta, \tag{4.1}$$

$$A_\theta H_u^2 - H_u^2 A_\theta = \frac{i}{2\pi}\langle\cdot, H_u u\rangle(1-\theta) - \frac{i}{2\pi}\langle\cdot, u\rangle u. \tag{4.2}$$

Proof. Let v_ζ be the reproducing kernel of H^2, then

$$H_u v_\zeta(x) = \frac{1}{2\pi i}\frac{u(x)-u(\zeta)}{x-\zeta}. \tag{4.3}$$

Comparing (4.3) with (2.3), we find that

$$H_u v_\zeta = \frac{1}{2\pi i}(A_\theta^* - \zeta)^{-1} u.$$

By (3.8), it follows that

$$\|(2\pi i\zeta) H_u v_\zeta + u\| \to 0, \quad |\zeta| \to \infty,$$

and so $u \in \overline{\operatorname{Ran} H_u} = K_\theta$. Using this, we obtain

$$H_u(1-\theta) = \mathbb{P}_+(u - u\overline{\theta}) = \mathbb{P}_+ u = u.$$

Next, the kernel of H_u is θH^2 and therefore $H_u(I - P_\theta) = 0$. It follows that for any $f \in K_\theta$,

$$H_u S(\tau) f = H_u P_\theta S(\tau) f = H_u S_\theta(\tau) f.$$

Thus, restricting the commutation relation (1.9) onto K_θ, we obtain

$$S_\theta(\tau)^* H_u = H_u S_\theta(\tau)$$

or equivalently

$$e^{-i\tau A_\theta^*} H_u = H_u e^{i\tau A_\theta}.$$

Differentiating this with respect to τ at $\tau = 0$ and taking into account the anti-linearity of H_u, we arrive at (4.1).

The proof of (4.2) is a twice repeated application of (4.1) and (2.8). We have

$$\begin{aligned} A_\theta H_u^2 &= (A_\theta - A_\theta^* + A_\theta^*) H_u^2 = \frac{i}{2\pi}\langle\cdot, H_u^2(1-\theta)\rangle(1-\theta) + A_\theta^* H_u^2 \\ &= \frac{i}{2\pi}\langle\cdot, H_u u\rangle(1-\theta) + H_u A_\theta H_u. \end{aligned}$$

Further, using the anti-linearity of H_u,

$$\begin{aligned} H_u A_\theta H_u &= H_u(A_\theta - A_\theta^* + A_\theta^*) H_u = -\frac{i}{2\pi}\langle 1-\theta, H_u\cdot\rangle H_u(1-\theta) + H_u A_\theta^* H_u \\ &= -\frac{i}{2\pi}\langle\cdot, u\rangle u + H_u^2 A_\theta. \end{aligned}$$

Combining these identities, we obtain (4.2). ∎

4.2. The action of H_u on the cyclic subspace generated by $1-\theta$

As in the introduction, we denote by $\lambda_1 > \cdots > \lambda_N > 0$ the singular values of H_u and by $E_{H_u}(\lambda_j)$ the corresponding Schmidt subspaces. We set for brevity $E_j := E_{H_u}(\lambda_j)$ and denote by P_j the orthogonal projection onto E_j. Observe that H_u commutes with H_u^2 and therefore it commutes with P_j. We also set $g = 1-\theta$ and

$$g_j := P_j g, \quad u_j := P_j u.$$

By the previous lemma, we have $H_u g = u$ and therefore $H_u g_j = u_j$.

Lemma 4.2. *For any j, none of the elements g, u, $H_u u$ are orthogonal to E_j. Furthermore, we have*

$$H_u u_j = \lambda_j e^{i\varphi_j} u_j, \quad H_u g_j = \lambda_j e^{-i\varphi_j} g_j \tag{4.4}$$

for some unimodular constants $e^{i\varphi_j}$.

Proof. Since $H_u E_j = E_j$ (see Theorem 1.2), the three conditions $g \perp E_j$, $u \perp E_j$ and $H_u u \perp E_j$ are equivalent to each other.

Suppose, to get a contradiction, that for some j we have $P_j g = 0$, i.e., the three orthogonality conditions above hold. For $f \in E_j$, applying the commutation relation (4.2), we get

$$A_\theta H_u^2 f = H_u^2 A_\theta f,$$

which can be rewritten as

$$H_u^2 A_\theta f = \lambda A_\theta f.$$

Thus, $A_\theta f \in E_j$, and so we obtain that E_j is an invariant subspace for A_θ. Further, by (2.8), we get $A_\theta^* f = A_\theta f$ for any $f \in E_j$. This contradicts the complete non-self-adjointness of A_θ (see Lemma 2.1).

Let us prove that $H_u u_j$ and u_j are collinear. Let $f \in E_j \cap u_j^\perp$; then

$$\langle A_\theta H_u^2 f, u_j\rangle - \langle H_u^2 A_\theta f, u_j\rangle = \lambda_j^2 \langle A_\theta f, u_j\rangle - \lambda_j^2 \langle A_\theta f, u_j\rangle = 0.$$

By the commutation relation (4.2), this yields $f \perp H_u u$, hence $f \perp H_u u_j$. Thus, we get

$$E_j \cap u_j^\perp \subset E_j \cap (H_u u_j)^\perp.$$

Both of these subspaces are non-trivial and have codimension one in E_j. We conclude that these two subspaces must coincide, which means that u_j and $H_u u_j$ are collinear.

Since

$$\|H_u u_j\|^2 = \langle H_u^2 u_j, u_j\rangle = \lambda_j^2 \|u_j\|^2,$$

from the previous step we get the first one of relations (4.4) with some unimodular constant $e^{i\varphi_j}$. Substituting $u_j = H_u g_j$, we obtain the second relation in (4.4). ■

4.3. *Proof of Theorem 1.3*

The first two statements of the theorem are already proved in Lemmas 4.1 and 4.2. It remains to prove the third one. Using (4.1), we obtain

$$\begin{aligned}\omega_j &= \langle A_\theta H_u g_j, H_u g_j\rangle = \langle g_j, H_u A_\theta H_u g_j\rangle = \langle g_j, A_\theta^* H_u^2 g_j\rangle \\ &= \lambda_j^2 \langle A_\theta g_j, g_j\rangle. \end{aligned} \tag{4.5}$$

Further, using (2.8),

$$\operatorname{Im}\omega_j = \lambda_j^2 \operatorname{Im}\langle A_\theta g_j, g_j\rangle = \frac{\lambda_j{}^2}{4\pi}|\langle g_j, g\rangle|^2 = \frac{\lambda_j^2\|g_j\|^4}{4\pi} > 0. \tag{4.6}$$

4.4. *The matrix structure of A_θ*

In preparation for our discussion of the inverse problem in the next section, here we discuss the matrix structure of A_θ with respect to the orthogonal decomposition (1.12). It turns out that the off-diagonal entries of the matrix of A_θ are rank one. We compute these entries here. Recall that P_j be the orthogonal projection in K_θ onto E_j.

Lemma 4.3. *For all $j \neq k$, we have*

$$P_k A_\theta P_j = \frac{i}{2\pi}\frac{\lambda_j^2 - \lambda_j\lambda_k e^{i(\varphi_j - \varphi_k)}}{\lambda_j^2 - \lambda_k^2}\langle \cdot, g_j\rangle g_k. \tag{4.7}$$

Proof. Let $f_j \in E_j$ and $f_k \in E_k$; then, taking the bilinear form of the second commutation relation in Lemma 4.1, we obtain

$$(\lambda_j^2 - \lambda_k^2)\langle A_\theta f_j, f_k\rangle = \frac{i}{2\pi}\langle f_j, H_u u_j\rangle\langle g_k, f_k\rangle - \frac{i}{2\pi}\langle f_j, u_j\rangle\langle u_k, f_k\rangle.$$

Recall that

$$u_j = H_u g_j = \lambda_j e^{-i\varphi_j} g_j, \quad H_u u_j = H_u^2 g_j = \lambda_j^2 g_j.$$

Substituting this into the above formula and dividing by $\lambda_j^2 - \lambda_k^2$, we obtain

$$\langle A_\theta f_j, f_k\rangle = \frac{i}{2\pi}\frac{\lambda_j^2 - \lambda_j\lambda_k e^{i(\varphi_j - \varphi_k)}}{\lambda_j^2 - \lambda_k^2}\langle f_j, g_j\rangle\langle g_k, f_k\rangle,$$

as required. ∎

We note here that the diagonal entries $P_j A_\theta P_j$ have more complicated structure, to be discussed in the next section.

5. Inverse spectral problem: The spectral data and uniqueness

5.1. Preliminaries and notation

The aim of this section is to prove the uniqueness part of Theorem 1.4 and to give an explicit formula for the symbol u (and other objects) in terms of the spectral data (1.11). In order to motivate these formulas, we make some preliminary remarks and calculations. We follow the notation of the previous subsection; in particular, u, θ, λ_j, E_j, $e^{i\varphi_j}$, ω_j, g_j, u_j are as above. We also denote for brevity

$$\nu_j = \|g_j\|.$$

Then formula (4.6) becomes

$$\operatorname{Im}\omega_j = \frac{\lambda_j^2 \nu_j^4}{4\pi}; \tag{5.1}$$

in particular, ν_j is determined by the spectral data (1.11).

First observe that by (2.4), we have

$$w(x) = \frac{1}{2\pi i}\langle (A_\theta^* - x)^{-1} w, g\rangle, \quad \operatorname{Im} x > 0, \tag{5.2}$$

for any $w \in K_\theta$. Applying this to u and recalling that by Lemma 4.2,

$$u = \sum_{k=1}^N u_k = \sum_{k=1}^N \lambda_k e^{-i\varphi_k} g_k,$$

we find

$$\begin{aligned} u(x) &= \frac{1}{2\pi i}\sum_{k=1}^N \lambda_k e^{-i\varphi_k}\langle (A_\theta^* - x)^{-1} g_k, g\rangle \\ &= \frac{1}{2\pi i}\sum_{k,j=1}^N \lambda_k e^{-i\varphi_k}\langle (A_\theta^* - x)^{-1} g_k, g_j\rangle. \end{aligned} \tag{5.3}$$

Thus, $u(x)$ will be determined if we compute all matrix entries

$$\langle (A_\theta^* - x)^{-1} g_k, g_j\rangle. \tag{5.4}$$

This leads us to the consideration of the matrix structure of A_θ^* in the orthogonal decomposition (1.12). We have computed the off-diagonal entries of A_θ^* in this decomposition in the previous section. Here we start by discussing the diagonal entries.

In this section, we will use some matrix notation which we explain here. Below $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{C}^N$. We denote by $\mathbb{1}_1, \dots, \mathbb{1}_N$ the standard basis in $\mathbb{C}^N$, and $\mathbb{1} = (1, \dots, 1)^\top \in \mathbb{C}^N$. If $(\alpha_1, \dots, \alpha_N)$ are complex numbers, we will denote by $D(\alpha)$ the diagonal $N \times N$ matrix with $\alpha_1, \dots, \alpha_N$ on the diagonal.

5.2. *Diagonal elements of A_θ^**

Here we consider the operator $P_k A_\theta^* P_k$ acting in E_k. Recall that by Theorem 3.7, we have

$$\langle A_\theta(p_j h), p_j w\rangle = \langle A_{\psi_j} h, w\rangle + c_j\langle h, 1-\psi_j\rangle\langle 1-\psi_j, w\rangle, \quad h, w \in K_{\psi_j}, \tag{5.5}$$

with some constants c_j. Further, by Lemma 3.3, we have

$$g_k = \overline{p_{k,\infty}}\, p_k(1-\psi_k), \tag{5.6}$$

where $p_{k,\infty} = p_k(\infty)$.

Lemma 5.1. *For* $\operatorname{Im} x > 0$, *we have*

$$R_k(x) := \langle (P_k A_\theta^* P_k - xI)^{-1} g_k, g_k\rangle = \frac{\nu_k^2}{\|1-\psi_k\|^2}\frac{1-\psi_k(x)}{\frac{1}{2\pi i} + \overline{c_k}(1-\psi_k(x))}. \tag{5.7}$$

Proof. Let us compute the vector

$$f = (P_k A_\theta^* P_k - xI)^{-1} g_k$$

by solving the equation

$$P_k A_\theta^* f - xf = g_k, \quad f \in E_k.$$

Using (5.6) and writing $f = p_k h$, our equation becomes

$$P_k A_\theta^*(p_k h) - x p_k h = \overline{p_{k,\infty}}\, p_k(1-\psi_k).$$

Let us take an inner product of this with an arbitrary element $p_k w \in E_k$, $w \in K_{\psi_k}$:

$$\langle A_\theta^*(p_k h), p_k w\rangle - x\langle h, w\rangle = \overline{p_{k,\infty}}\langle 1-\psi_k, w\rangle.$$

Using (5.5), we obtain

$$\langle (A_{\psi_k}^* - xI)h, w\rangle + \overline{c_k}\langle h, 1-\psi_k\rangle\langle 1-\psi_k, w\rangle = \overline{p_{k,\infty}}\langle 1-\psi_k, w\rangle.$$

Since $w \in K_{\psi_k}$ is arbitrary, this implies

$$(A_{\psi_k}^* - x)h + \overline{c_k}\langle h, 1-\psi_k\rangle(1-\psi_k) = \overline{p_{k,\infty}}(1-\psi_k).$$

Let us apply $(A_{\psi_k}^* - x)^{-1}$ and take the inner product with $1-\psi_k$:

$$\begin{aligned}\langle h, 1-\psi_k\rangle &+ \overline{c_k}\langle h, 1-\psi_k\rangle\langle (A_{\psi_k}^* - x)^{-1}(1-\psi_k), 1-\psi_k\rangle\\ &= \overline{p_{k,\infty}}\langle (A_{\psi_k}^* - x)^{-1}(1-\psi_k), 1-\psi_k\rangle.\end{aligned}$$

By (2.4), we have

$$1-\psi_k(x) = \frac{1}{2\pi i}\langle (A_{\psi_k}^* - x)^{-1}(1-\psi_k), 1-\psi_k\rangle,$$

and therefore our equation becomes

$$\langle h, 1-\psi_k\rangle\Big(\frac{1}{2\pi i}+\overline{c_k}(1-\psi_k(x))\Big)=\overline{p_{k,\infty}}(1-\psi_k(x)).$$

Finally,

$$R_k(x)=\langle p_k h, g_k\rangle = p_{k,\infty}\langle h, 1-\psi_k\rangle = |p_{k,\infty}|^2\frac{1-\psi_k(x)}{\frac{1}{2\pi i}+\overline{c_k}(1-\psi_k(x))}.$$

Computing the norm in (5.6), we obtain

$$\nu_k = \|g_k\| = |p_{k,\infty}|\|p_k(1-\psi_k)\| = |p_{k,\infty}|\|1-\psi_k\|.$$

Putting this together, we arrive at the required formula. ■

5.3. *Matrix elements of the resolvent of A_θ^**

Here we compute the matrix entries (5.4). First let us introduce notation for the (normalised) matrix elements of A_θ with respect to the vectors g_j,

$$\mathcal{A}_{kj} := \frac{1}{\nu_j^2\nu_k^2}\langle A_\theta g_j, g_k\rangle.$$

The off-diagonal entries have already appeared in the right-hand side of (4.7):

$$\mathcal{A}_{kj} = \frac{i}{2\pi}\frac{\lambda_j^2-\lambda_j\lambda_k e^{i(\varphi_j-\varphi_k)}}{\lambda_j^2-\lambda_k^2}, \quad j\neq k. \tag{5.8}$$

The diagonal entries are given by (4.5), viz.

$$\mathcal{A}_{jj} = \frac{\omega_j}{\lambda_j^2\nu_j^4} = \frac{\omega_j}{4\pi\,\mathrm{Im}\,\omega_j}. \tag{5.9}$$

Next, for any x in the open upper half-plane we define an $N\times N$ matrix $Q(x)$ as follows:

$$Q_{kj}(x) = (\mathcal{A}^*)_{kj}, \quad k\neq j, \tag{5.10}$$

$$Q_{kk}(x) = \frac{1}{R_k(x)}, \tag{5.11}$$

where $R_k(x)$ is defined in the previous lemma. The following lemma is nothing but some linear algebra.

Lemma 5.2. *For any n, m, we have*

$$\langle (A_\theta^*-x)^{-1}g_m, g_n\rangle = \langle Q(x)^{-1}\mathbb{1}_m, \mathbb{1}_n\rangle. \tag{5.12}$$

Proof. Fix m and denote $f=(A_\theta^*-x)^{-1}g_m$. Our aim is to compute $\langle f, g_n\rangle$; the element f satisfies the equation

$$(A_\theta^*-x)f = g_m. \tag{5.13}$$

Let us write $f = \sum_{j=1}^{N} f_j$ with $f_j \in E_j$. For every j, we have

$$A_\theta^* f_j = \sum_{k \neq j} (\mathcal{A}^*)_{kj} \langle f_j, g_j \rangle g_k + P_j A_\theta^* P_j f_j.$$

Thus, our equation (5.13) becomes a system

$$(P_k A_\theta^* P_k - x) f_k + \sum_{j \neq k} (\mathcal{A}^*)_{kj} \langle f_j, g_j \rangle g_k = \delta_{km} g_k, \quad k = 1, \dots, N.$$

Inverting $P_k A_\theta^* P_k - x$, we get

$$f_k + \sum_{j \neq k} (\mathcal{A}^*)_{kj} \langle f_j, g_j \rangle (P_k A_\theta^* P_k - x)^{-1} g_k = \delta_{km} (P_k A_\theta^* P_k - x)^{-1} g_k, \quad k = 1, \dots, N.$$

Let us take the inner product with g_k and use the notation $R_k(x)$:

$$\langle f_k, g_k \rangle + \sum_{j \neq k} (\mathcal{A}^*)_{kj} \langle f, g_j \rangle R_k(x) = \delta_{km} R_k(x), \quad k = 1, \dots, N.$$

Denote $\xi = (\xi_1, \dots, \xi_N)^\top \in \mathbb{C}^N$, $\xi_k = \langle f_k, g_k \rangle$; then we obtain

$$\frac{1}{R_k(x)} \xi_k + \sum_{j \neq k} (\mathcal{A}^*)_{kj} \xi_j = \delta_{km}, \quad k = 1, \dots, N.$$

By the definition of $Q(x)$, this rewrites as

$$Q(x) \xi = \mathbb{1}_m.$$

Thus, $\xi = Q(x)^{-1} \mathbb{1}_m$ and

$$\langle (A_\theta^* - x)^{-1} g_m, g_n \rangle = \langle f, g_n \rangle = \xi_n = \langle \xi, \mathbb{1}_n \rangle = \langle Q(x)^{-1} \mathbb{1}_m, \mathbb{1}_n \rangle,$$

as required. ∎

5.4. *Expression for $Q(x)$ in terms of the spectral data*

We have defined the diagonal entries of $Q(x)$ in terms of $R_k(x)$, and expression (5.7) for $R_k(x)$ involves the constants c_k. It is not obvious that $Q(x)$ can be expressed entirely in terms of the spectral data (1.11). Let us show that this can be done.

First we need some notation. For every j and $\operatorname{Im} x > 0$, we define

$$b_j(x) = \frac{1}{\nu_j^2} \Big(i \frac{\|1 - \psi_j\|^2}{4\pi} \frac{1 + \psi_j(x)}{1 - \psi_j(x)} + \frac{\operatorname{Re} \langle A_{\psi_j}(1 - \psi_j), 1 - \psi_j \rangle}{\|1 - \psi_j\|^2} - x \Big). \tag{5.14}$$

Clearly, b_j is determined by the spectral data; we recall that $\nu_j^2 = \frac{\sqrt{4\pi \operatorname{Im} \omega_j}}{\lambda_j}$. By (2.13) and Lemma 2.8, this is a rational Herglotz function with the representation

$$b_j(x) = \sum_k \frac{c_{jk}}{\alpha_{jk} - x} \tag{5.15}$$

with some $c_{jk} \geqslant 0$ and $\alpha_{jk} \in \mathbb{R}$. Again by Lemma 2.8, this function is Frostman-invariant (i.e., invariant with respect to the action of Frostman shifts on ψ_j).

Let $Q(x)$ be as defined by (5.10), (5.11). Recall that $D(\alpha) := \operatorname{diag}\{\alpha_1, \dots, \alpha_N\}$.

Lemma 5.3. *The matrix $Q(x)$ can be expressed in terms of the spectral data by*

$$Q(x) = \mathcal{A}^* - xD(\nu^2)^{-1} - D(b(x)), \quad \operatorname{Im} x > 0. \tag{5.16}$$

Proof. For the off-diagonal entries, (5.16) evidently agrees with (5.10). The issue is only to check that the diagonal entries agree. First we compute, using (4.5), (5.6) and (5.5):

$$\begin{aligned}
&\frac{\operatorname{Re}\omega_k}{\lambda_k^2\nu_k^2} - \frac{\operatorname{Re}\langle A_{\psi_k}(1-\psi_k), 1-\psi_k\rangle}{\|1-\psi_k\|^2} \\
&\quad = \frac{\operatorname{Re}\langle A_\theta g_k, g_k\rangle}{\|g_k\|^2} - \frac{\operatorname{Re}\langle A_{\psi_k}(1-\psi_k), 1-\psi_k\rangle}{\|1-\psi_k\|^2} \\
&\quad = \frac{\operatorname{Re}\langle A_\theta(p_k(1-\psi_k)), p_k(1-\psi_k)\rangle}{\|p_k(1-\psi_k)\|^2} - \frac{\operatorname{Re}\langle A_{\psi_k}(1-\psi_k), 1-\psi_k\rangle}{\|1-\psi_k\|^2} \\
&\quad = \frac{\operatorname{Re}\langle A_\theta(p_k(1-\psi_k)), p_k(1-\psi_k)\rangle}{\|1-\psi_k\|^2} - \frac{\operatorname{Re}\langle A_{\psi_k}(1-\psi_k), 1-\psi_k\rangle}{\|1-\psi_k\|^2} \\
&\quad = \frac{\operatorname{Re} c_k \|1-\psi_k\|^4}{\|1-\psi_k\|^2} = \|1-\psi_k\|^2 \operatorname{Re} c_k.
\end{aligned}$$

Writing the diagonal entry of the right-hand side of (5.16) and using the last formula, we get

$$\begin{aligned}
\overline{\mathcal{A}_{kk}} - \frac{x}{\nu_k^2} - b_k(x) &= \operatorname{Re}\mathcal{A}_{kk} - i\operatorname{Im}\mathcal{A}_{kk} - \frac{x}{\nu_k^2} - b_k(x) \\
&= \frac{\operatorname{Re}\omega_k}{\lambda_k^2\nu_k^4} - \frac{i}{4\pi} + \frac{1}{4\pi i}\frac{\|1-\psi_k\|^2}{\nu_k^2}\frac{1+\psi_k(x)}{1-\psi_k(x)} \\
&\quad - \frac{\operatorname{Re}\langle A_{\psi_k}(1-\psi_k), 1-\psi_k\rangle}{\nu_k^2\|1-\psi_k\|^2} \\
&= \frac{\|1-\psi_k\|^2}{\nu_k^2}\operatorname{Re} c_k - \frac{i}{4\pi} + \frac{1}{4\pi i}\frac{\|1-\psi_k\|^2}{\nu_k^2}\frac{1+\psi_k(x)}{1-\psi_k(x)}.
\end{aligned}$$

On the other hand, let us compute $\frac{1}{R_k(x)}$ and use (3.11),

$$\begin{aligned}
\frac{1}{R_k(x)} &= \frac{\|1-\psi_k\|^2}{\nu_k^2}\Big(\frac{1}{2\pi i}\frac{1}{1-\psi_k(x)} + \overline{c_k}\Big) \\
&= \frac{\|1-\psi_k\|^2}{\nu_k^2}\Big(\operatorname{Re} c_k + \frac{1}{4\pi i}\frac{\nu_k^2}{\|1-\psi_k\|^2} + \frac{i}{4\pi} + \frac{1}{2\pi i}\frac{1}{1-\psi_k(x)}\Big) \\
&= -\frac{i}{4\pi} + \frac{\|1-\psi_k\|^2}{\nu_k^2}\Big(\operatorname{Re} c_k + \frac{1}{4\pi i}\frac{1+\psi_k(x)}{1-\psi_k(x)}\Big).
\end{aligned}$$

Putting this together, we obtain the required identity. ∎

Remark 5.4. Note that in the simple spectrum case, by performing a Frostman shift in each subspace $p_j K_{\psi_j}$, we can choose

$$\psi_j(x) = \frac{x-i}{x+i}$$

for all j. Then a calculation shows that $\|1-\psi_j\|^2 = 4\pi$ and the formula for $Q(x)$ becomes

$$Q(x) = \mathcal{A}^* - xD(\nu^2)^{-1}.$$

5.5. *Uniqueness and formula for u*

Now we can put it all together and give a formula for u in terms of the spectral data. The theorem below is more precise form of Theorem 1.4 (i).

Theorem 5.5. *Let u be a bounded rational function without poles in the closed upper half-plane, normalised so that $u(\infty) = 0$. Then u is uniquely determined by the spectral data* (1.11) *according to the following formula:*

$$u(x) = \frac{1}{2\pi i}\langle Q(x)^{-1} D(\lambda) D(e^{-i\varphi})\mathbb{1}, \mathbb{1}\rangle, \quad \operatorname{Im} x > 0. \tag{5.17}$$

Here

$$Q(x) = \mathcal{A}^* - xD(\nu^2)^{-1} - D(b(x)),$$

where the matrix $\mathcal{A}$ is defined by

$$\mathcal{A}_{kj} = \frac{i}{2\pi}\frac{\lambda_j^2 - \lambda_j\lambda_k e^{i(\varphi_j-\varphi_k)}}{\lambda_j^2-\lambda_k^2}, \quad j \neq k,$$
$$\mathcal{A}_{jj} = \frac{\omega_j}{\lambda_j^2\nu_j^4} = \frac{\omega_j}{4\pi \operatorname{Im}\omega_j},$$

and b_j are the functions defined by (5.14).

Proof. Combining (5.3) with (5.12) gives

$$u(x) = \frac{1}{2\pi i}\sum_{k,j=1}^{N} \lambda_k e^{-i\varphi_k}\langle Q(x)^{-1}\mathbb{1}_k, \mathbb{1}_j\rangle = \frac{1}{2\pi i}\langle Q(x)^{-1}D(\lambda)D(e^{-i\varphi})\mathbb{1}, \mathbb{1}\rangle,$$

as required. ∎

5.6. *Formula for g_j*

For our proof of surjectivity in the following sections, we will need a formula for g_j, which appeared in the proof of the above theorem. It will be convenient to have it in the vector form.

Lemma 5.6. *The vector*

$$\mathbf{g}(x) = (g_1(x), \dots, g_N(x))^\top$$

can be expressed in terms of the spectral data by

$$\mathbf{g}(x) = \frac{1}{2\pi i}(Q(x)^\top)^{-1}\mathbb{1}. \tag{5.18}$$

Proof. As in the proof of Theorem 5.5, taking $w = g_k$ in (5.2) and using (5.12), we obtain

$$\begin{aligned} g_k(x) &= \frac{1}{2\pi i}\langle (A_\theta^* - x)^{-1} g_k, g\rangle = \frac{1}{2\pi i}\langle Q(x)^{-1}\mathbb{1}_k, \mathbb{1}\rangle \\ &= \frac{1}{2\pi i}\langle (Q(x)^\top)^{-1}\mathbb{1}, \mathbb{1}_k\rangle. \end{aligned}$$

Writing this in the vector form, we obtain (5.18). ∎

5.7. *Formula for p_j*

For our proof of surjectivity in the following sections, it will be convenient to have a formula for the isometric multiplier p_j in the representation $E_j = p_j K_{\psi_j}$. First let us fix the unimodular multiplicative constant in the definition of p_j. By (1.8), this can be done so that

$$H_u(p_j h) = \lambda_j p_j \psi_j \overline{h}, \quad h \in K_{\psi_j}. \tag{5.19}$$

This does not fix p_j uniquely but up to a factor of ± 1 (observe that (1.8) is invariant under the change $p \mapsto -p$).

Take $h = 1 - \psi_j$ in (5.19),

$$H_u(p_j(1 - \psi_j)) = -\lambda_j p_j (1 - \psi_j).$$

On the other hand, (4.4) gives the equation $H_u g_j = \lambda_j e^{-i\varphi_j} g_j$ and (5.6) gives a relation between g_j and $p_j(1 - \psi_j)$. Putting this together, after a little algebra we obtain

$$e^{i\varphi_j} = -\frac{\overline{p_{j,\infty}}}{p_{j,\infty}}. \tag{5.20}$$

Next, by (5.6) again we find

$$p_j = \frac{1}{\overline{p_{j,\infty}}}\frac{g_j}{1 - \psi_j}.$$

Taking into account (5.20), this becomes

$$p_j = \pm i e^{-i\varphi_j/2}\frac{1}{|p_{j,\infty}|}\frac{g_j}{1 - \psi_j} = \pm i e^{-i\varphi_j/2}\frac{\|1 - \psi_j\|}{\nu_j}\frac{g_j}{1 - \psi_j}, \tag{5.21}$$

where the sign $\pm$ remains undetermined.

5.8. *Formula for* $\theta(x)$

Below, we give a formula for $\theta(x)$, $\operatorname{Im} x > 0$, in terms of the spectral data. We do not need this formula in the rest of the paper, and so we give it without proof. Along with $Q(x)$, consider the matrix

$$Q^{\#}(x) = \mathcal{A} - xD(\nu^2)^{-1} - D(b(x)), \quad \operatorname{Im} x > 0;$$

the difference with $Q(x)$ is that here we take $\mathcal{A}$ instead of $\mathcal{A}^*$. The matrix $Q^{\#}(x)$ is no longer necessarily invertible in $\mathbb{C}_+$. The inner function θ can be recovered from the spectral data by the formula

$$\theta(x) = \frac{\det Q^{\#}(x)}{\det Q(x)}, \quad \operatorname{Im} x > 0.$$

The idea of the proof is to start from formula

$$1 - \theta = \sum_{j=1}^{N} g_j = \sum_{j=1}^{N} \overline{p_{j,\infty}} p_j (1 - \psi_j),$$

express p_j according to (5.21) and rearrange the result using some matrix algebra similar to the one of Section 7.1 below.

5.9. *Comparison with* [24]

For the reader's convenience, we compare the spectral data of Pocovnicu's paper [24] (where the case of singular values of multiplicity one was considered) with the one of this paper. In [24], notation γ_j is used for $\operatorname{Re} \omega_j$ and ϕ_j is used for φ_j. Notation λ_j and ν_j in [24] have the same meaning as here. The spectral data in [24] is

$$(\{2\lambda_j^2 \nu_j^2\}_{j=1}^N, \{4\pi\lambda_j^2\}_{j=1}^N, \{2\varphi_j\}_{j=1}^N, \{\operatorname{Re} \omega_j\}_{j=1}^N);$$

these are the generalised action-angle variables for the Szegő equation. Taking into account (5.1), we see that this spectral data is in a one-to-one correspondence with our spectral data (1.11).

6. Inverse spectral problem: The surjectivity of the spectral map

6.1. *The set-up*

Suppose we are given $N \in \mathbb{N}$, and the spectral data (1.11), where

- $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N$ are real numbers;
- $\{\psi_j\}_{j=1}^N$ are finite Blaschke products with the normalisation condition $\psi_j(\infty) = 1$;
- $\{e^{i\varphi_j}\}_{j=1}^N$ are unimodular complex numbers;
- $\{\omega_j\}_{j=1}^N$ are complex numbers with positive imaginary parts.

We define the numbers $\nu_j > 0$ so that (4.6) holds, i.e.,

$$4\pi \operatorname{Im} \omega_j = \lambda_j^2 \nu_j^4.$$

With these parameters, let us define the $N \times N$ matrices $\mathcal{A}$, $Q(x)$ as in Section 5.5. Since

$$Q(x) = \mathcal{A}^* - xD(\nu^2)^{-1} - D(b(x))$$

and, by Lemma 2.8,

$$b(x) = O\Big(\frac{1}{x}\Big), \quad |x| \to \infty,$$

we find that

$$(Q(x))^{-1} = -\frac{1}{x} D(\nu^2) + O\Big(\frac{1}{|x|^2}\Big), \quad |x| \to \infty. \tag{6.1}$$

Theorem 6.1. *For every $x \in \mathbb{C}_+$, the matrix $Q(x)$ is invertible. Furthermore,*

$$\sup_{x \in \mathbb{C}_+} \|Q(x)^{-1}\| < \infty.$$

Observe that as a consequence, the radial limits

$$\lim_{\varepsilon \to 0+} Q(x + i\varepsilon)^{-1}$$

exist for a.e. $x \in \mathbb{R}$. We now define $u(x)$ by formula (5.17). It is evident that u is a rational function; by (6.1), we have $u(x) \to 0$ as $|x| \to \infty$. Furthermore, by Theorem 6.1, $u(x)$ has no poles in the closed upper half-plane. The main result of this section is the following.

Theorem 6.2. *The Hankel operator H_u corresponds to the spectral data* (1.11) *according to Theorem* 5.5.

The main step of the proof is as follows. Let us define the functions $g_j(x)$ by (5.18). By definition, these are rational functions going to zero at infinity and by Theorem 6.1 they do not have poles in the closed upper half-plane. Thus, $g_j \in H^2$ for all j.

Theorem 6.3. *The eigenvalue equations*

$$H_u g_j = \lambda_j e^{-i\varphi_j} g_j, \quad j = 1, \dots, N,$$

or in the vector form,

$$\mathbb{P}_+(u\overline{\mathbf{g}}) = D(\lambda)D(e^{-i\varphi})\mathbf{g}, \tag{6.2}$$

hold true.

In this section, we prove Theorems 6.1 and 6.3. In the following section, we prove Theorem 6.2.

6.2. *Algebraic properties of the matrix $\mathcal{A}$*

For convenience of notation and also to make the connection with Hankel operators more transparent, let us define the anti-linear operator $\mathcal{H}$ in $\mathbb{C}^N$ by

$$\mathcal{H} f = D(\lambda)D(e^{-i\varphi})\overline{f}, \quad \overline{f} = (\overline{f_1}, \dots, \overline{f_N})^\top. \tag{6.3}$$

Lemma 6.4. *The matrix $\mathcal{A}$ satisfies*

$$\operatorname{Im} \mathcal{A} = \frac{1}{4\pi}\langle \cdot, \mathbb{1}\rangle \mathbb{1}, \tag{6.4}$$

$$\mathcal{A}^* \mathcal{H} = \mathcal{H} \mathcal{A}. \tag{6.5}$$

Proof. First let us check (6.4). For $j \neq k$, we have

$$\begin{aligned} \frac{1}{2i}(\mathcal{A}_{kj} - \overline{\mathcal{A}_{jk}}) &= \frac{1}{2i}\frac{i}{2\pi}\frac{1}{\lambda_j^2 - \lambda_k^2}(\lambda_j^2 - \lambda_j\lambda_k e^{i\varphi_j}e^{-i\varphi_k} - \lambda_k^2 + \lambda_j\lambda_k e^{-i\varphi_k}e^{i\varphi_j}) \\ &= \frac{1}{4\pi}, \end{aligned}$$

which agrees with the right-hand side of (6.4). For $j = k$, we have

$$\operatorname{Im} \mathcal{A}_{jj} = \frac{1}{4\pi},$$

which again agrees with (6.4). The second identity (6.5) can be written as

$$\overline{\mathcal{A}_{jk}}\lambda_j e^{-i\varphi_j} = \overline{\mathcal{A}_{kj}}\lambda_k e^{-i\varphi_k}$$

in terms of the matrix entries. For $j \neq k$, the matrix $\mathcal{A}$ satisfies this relation by an inspection of the definition of $\mathcal{A}_{jk}$; for $j = k$ this relation is trivially true. ∎

Lemma 6.5. *The eigenvalues of $\mathcal{A}$ lie in the open upper half-plane.*

Proof. Since by (6.4) we have $\operatorname{Im} \mathcal{A} \geqslant 0$, the question reduces to proving that $\mathcal{A}$ has no real eigenvalues. Assume, to get a contradiction, that $\operatorname{Ker}(\mathcal{A} - \lambda I) \neq \{0\}$ for some $\lambda \in \mathbb{R}$. Since $\operatorname{Im} \mathcal{A} \geqslant 0$, from here we easily check that

$$\operatorname{Ker}(\mathcal{A} - \lambda I) = \operatorname{Ker}(\mathcal{A}^* - \lambda I).$$

Now let $f \in \operatorname{Ker}(\mathcal{A} - \lambda I)$; by (6.5), we have

$$\mathcal{A}^* \mathcal{H} f = \lambda \mathcal{H} f,$$

i.e., $(\mathcal{A}^* - \lambda I)\mathcal{H} f = 0$, and therefore $(\mathcal{A} - \lambda I)\mathcal{H} f = 0$. Thus, we see that $\operatorname{Ker}(\mathcal{A} - \lambda I)$ is an invariant subspace of $\mathcal{H}$. It follows that it is also an invariant subspace of the linear Hermitian operator $\mathcal{H}^2 = D(\lambda)^2$. It is also clear from (6.4) that $\operatorname{Ker}(\mathcal{A} - \lambda I)$ is orthogonal to the vector $\mathbb{1}$. But this vector is clearly cyclic for $D(\lambda)^2$. This contradiction completes the proof. ∎

6.3. *Proof of Theorem* 6.1

Let $\sigma \subset \{1, \dots, N\}$; we shall denote by P_σ the orthogonal projection from $\mathbb{C}^N$ onto the $|\sigma|$-dimensional subspace

$$\{x \in \mathbb{C}^N : x_s = 0 \text{ for } s \notin \sigma\}.$$

We shall denote by $\overline{\mathbb{C}_+}$ the closed upper half-plane $\operatorname{Im} z \geqslant 0$.

Lemma 6.6. *Let $\mathcal{A}$ be an $N \times N$ matrix such that for any subset $L \subset \{1, \dots, N\}$, and for all $\beta_\ell \in \overline{\mathbb{C}_+}$, $\ell \in L$, the $|L| \times |L|$ matrix*

$$D(\beta) + P_L \mathcal{A} P_L^*$$

is invertible. Then

$$\sup_{\alpha_1, \dots, \alpha_N \in \overline{\mathbb{C}_+}} \|(D(\alpha) + \mathcal{A})^{-1}\| < \infty.$$

Proof. Assume, to get a contradiction, that there exist sequences $\alpha^{(n)} \in (\overline{\mathbb{C}_+})^N$ and $X^{(n)} \in \mathbb{C}^N$ such that $\|X^{(n)}\| = 1$ for all n and

$$\|(D(\alpha^{(n)}) + \mathcal{A})X^{(n)}\| \to 0, \quad n \to \infty.$$

After extracting a subsequence, we can achieve

$$X^{(n)} \to X, \quad \|X\| = 1.$$

Furthermore, again extracting subsequences, we can split the index set $\{1, \dots, N\}$ into disjoint subsets $J \cup L$ as follows:

$$J = \{j \in \{1, \dots, N\} : |\alpha_j^{(n)}| \to \infty, n \to \infty\};$$
$$L = \{\ell \in \{1, \dots, N\} : \alpha_\ell^{(n)} \to \alpha_\ell \in \overline{\mathbb{C}_+}\}.$$

Now for every $j \in J$, we have

$$\alpha_j^{(n)} X_j^{(n)} + (\mathcal{A} X^{(n)})_j \to 0,$$

as $n \to \infty$, and therefore

$$\alpha_j^{(n)} X_j^{(n)} \to -(\mathcal{A} X)_j.$$

It follows that $X_j^{(n)} \to 0$ as $n \to \infty$, and so $X_j = 0$.

Next, for every $\ell \in L$, we have

$$\alpha_\ell^{(n)} X_\ell^{(n)} + (\mathcal{A} X^{(n)})_\ell \to 0,$$

which yields

$$\alpha_\ell X_\ell + (\mathcal{A} X)_\ell = 0.$$

Denoting $\beta_\ell = \alpha_\ell$, $\ell \in L$, this can be written as

$$D(\beta)P_L X + P_L \mathcal{A} X = 0.$$

But by the previous step, $X = P_L^* P_L X$, and so we obtain

$$D(\beta)P_L X + P_L \mathcal{A} P_L^* P_L X = 0.$$

By the assumption of the invertibility, we conclude $P_L X = 0$, and so $X = 0$ – contradiction! ■

Proof of Theorem 6.1. It suffices to prove the corresponding statement for $Q(x)^*$ in place of $Q(x)$. Further, since

$$Q(x)^* = \mathcal{A} - \overline{x} D(\nu^2)^{-1} - D(b(x))^*,$$

and $\operatorname{Im}(x\nu_j^{-2} + b_j(x)) > 0$ for all $x \in \mathbb{C}_+$, it suffices to prove that for any $\zeta_1, \dots, \zeta_N \in \mathbb{C}_+$ the matrix $\mathcal{A} + D(\zeta)$ is invertible and

$$\sup_{\zeta_1,\dots,\zeta_N \in \mathbb{C}_+} \|(\mathcal{A} + D(\zeta))^{-1}\| < \infty.$$

Let us show that this follows from the previous lemma. For $\zeta_j \in \mathbb{C}_+$, write $\zeta_j = \alpha_j + i\beta_j$ with $\alpha_j \in \mathbb{R}$, $\beta_j > 0$. First, notice that Lemma 6.5 remains valid if we replace $\mathcal{A}$ by $\mathcal{A} + D(\alpha)$. Next, since

$$\operatorname{Im}(\mathcal{A} + D(\zeta)) = \operatorname{Im} \mathcal{A} + D(\beta) \geqslant 0,$$

it is clear that all eigenvalues of $\mathcal{A} + D(\zeta)$ lie in $\overline{\mathbb{C}_+}$. If $(\mathcal{A} + D(\zeta))f = \lambda f$ for some $\lambda \in \mathbb{R}$, then taking the imaginary part of the quadratic form, we obtain $\operatorname{Im}\langle \mathcal{A} f, f\rangle = 0$, and so λ is an eigenvalue of $\mathcal{A} + D(\alpha)$, which is impossible. Thus, all eigenvalues of $\mathcal{A} + D(\zeta)$ lie in $\mathbb{C}_+$.

Finally, considering any square submatrix of $\mathcal{A}$, we observe that it has the same structure as $\mathcal{A}$ itself, and so the above argument applies to this submatrix. It follows that the hypothesis of Lemma 6.6 is satisfied, and we arrive at the required result. ■

6.4. *Orthogonality of g_j*

Our aim here is to prove that g_j form an orthogonal set in H^2, normalised by

$$\|g_j\| = \nu_j.$$

In view of (5.18), this is a consequence of the following.

Lemma 6.7. *For any $X, Y \in \mathbb{C}^N$, we have*

$$\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \langle Q(x)^{-1} X, \mathbb{1}\rangle \overline{\langle Q(x)^{-1} Y, \mathbb{1}\rangle}\, dx = \langle D(\nu^2) X, Y\rangle. \tag{6.6}$$

Proof. As a first step, let us prove the identity

$$\operatorname{Im} Q(x)^{-1} = \frac{1}{4\pi} Q^*(x)^{-1}(\langle\cdot, \mathbb{1}\rangle\mathbb{1})Q(x)^{-1} \quad \text{for a.e. } x \in \mathbb{R}. \tag{6.7}$$

Recall that

$$Q(x) = \mathcal{A}^* - xD(\nu^2)^{-1} - D(b(x)),$$

and $\operatorname{Im} b_j(x) = 0$ for a.e. $x \in \mathbb{R}$. It follows that

$$\operatorname{Im} Q(x) = \operatorname{Im} \mathcal{A}^* = -\operatorname{Im} \mathcal{A} = -\frac{1}{4\pi}\langle\cdot, \mathbb{1}\rangle\mathbb{1} \quad \text{for a.e. } x \in \mathbb{R}.$$

From here we get the required identity (6.7).

Next, observe that by (6.7), the integrand in the left-hand side of (6.6) rewrites as

$$\begin{aligned}\frac{1}{4\pi^2}\langle Q(x)^{-1}X, \mathbb{1}\rangle\overline{\langle Q(x)^{-1}Y, \mathbb{1}\rangle} &= \frac{1}{\pi}\langle \operatorname{Im} Q(x)^{-1}X, Y\rangle \\ &= \frac{1}{2\pi i}(\langle Q(x)^{-1}X, Y\rangle - \overline{\langle Q(x)^{-1}Y, X\rangle})\end{aligned}$$

for a.e. $x \in \mathbb{R}$. Therefore, the left-hand side in (6.6) rewrites as

$$\frac{1}{2\pi i}\lim_{R\to\infty}\int_{-R}^{R}\langle Q(x)^{-1}X, Y\rangle\, dx - \frac{1}{2\pi i}\lim_{R\to\infty}\int_{-R}^{R}\overline{\langle Q(x)^{-1}Y, X\rangle}\, dx.$$

Deforming the integration contour from $[-R, R]$ to the upper semi-circle of radius R centered at the origin and using (6.1), we obtain

$$\frac{1}{2\pi i}\lim_{R\to\infty}\int_{-R}^{R}\langle Q(x)^{-1}X, Y\rangle\, dx = \frac{1}{2}\langle D(\nu^2)X, Y\rangle,$$

and, by complex conjugation,

$$\frac{1}{2\pi i}\lim_{R\to\infty}\int_{-R}^{R}\overline{\langle Q(x)^{-1}Y, X\rangle}\, dx = -\frac{1}{2}\langle D(\nu^2)X, Y\rangle.$$

Putting this together, we obtain (6.6). ∎

6.5. *The action of H_u on g_j*

Our aim here is to prove Theorem 6.3. We recall that by our definitions,

$$u(x) = \sum_{j=1}^{N}\lambda_j e^{-i\varphi_j} g_j(x) = \langle D(\lambda)D(\nu)\mathbf{g}(x), \mathbb{1}\rangle. \tag{6.8}$$

Lemma 6.8. *Let $f \in L^2(\mathbb{R})$ be such that $xf \in L^2(\mathbb{R})$. Then*

$$\mathbb{P}_+(xf) = x\mathbb{P}_+(f) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} f(x)\, dx.$$

Proof. We have

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x - z}\, dx,$$

and therefore

$$\begin{aligned} \mathbb{P}_+(xf)(z) - z\mathbb{P}_+(f)(z) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{xf(x)}{x - z}\, dx - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{zf(x)}{x - z}\, dx \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(x)\, dx, \end{aligned}$$

as required. ∎

Lemma 6.9. *We have the identity*

$$(\mathcal{A} - xD(\nu^2)^{-1})\mathbb{P}_+(u\overline{\mathbf{g}}) - \mathbb{P}_+(uD(b)\overline{\mathbf{g}}) = \frac{1}{2\pi i} D(\lambda)D(e^{-i\varphi})\mathbb{1} - \frac{u(x)}{2\pi i}\mathbb{1}. \tag{6.9}$$

Proof. By the definition of $\mathbf{g}$ (cf. (5.18)), we have

$$Q^\top(x)\mathbf{g}(x) = \frac{1}{2\pi i}\mathbb{1}. \tag{6.10}$$

Let us take the complex conjugate of this equation, multiply by $u(x)$ and apply $\mathbb{P}_+$:

$$\mathbb{P}_+(uQ^*\overline{\mathbf{g}}) = -\frac{1}{2\pi i} u\mathbb{1}.$$

By the definition of $Q(x)$ and by using Lemma 6.8, we rewrite the left-hand side as

$$\begin{aligned} \mathbb{P}_+(uQ^*\overline{\mathbf{g}}) &= \mathcal{A}\mathbb{P}_+(u\overline{\mathbf{g}}) - \mathbb{P}_+(xuD(\nu^2)^{-1}\overline{\mathbf{g}}) - \mathbb{P}_+(D(b)u\overline{\mathbf{g}}) \\ &= (\mathcal{A} - xD(\nu^2)^{-1})\mathbb{P}_+(u\overline{\mathbf{g}}) - \frac{1}{2\pi i} D(\nu^2)^{-1} \int_{-\infty}^{\infty} u(x)\overline{\mathbf{g}(x)}\, dx \\ &\quad - \mathbb{P}_+(D(b)u\overline{\mathbf{g}}). \end{aligned}$$

Observe that by (6.10), we have $Q^\top(x)\mathbf{g}(x) \in H^\infty$, and also by the definition of $\mathbf{g}(x)$, we have $x\mathbf{g}(x) \in H^\infty$; thus, all the expressions above are well defined.

By (6.8) and the orthogonality of g_j, we get

$$\int_{-\infty}^{\infty} u(x)\overline{\mathbf{g}(x)}\, dx = D(\nu^2)D(\lambda)D(e^{-i\varphi})\mathbb{1}.$$

Putting this together, we obtain the required identity. ∎

Lemma 6.10. *We have the identity*

$$Q^*(x)D(\lambda)D(e^{-i\varphi})\mathbf{g}(x) = \frac{1}{2\pi i} D(\lambda)D(e^{-i\varphi})\mathbb{1} - \frac{u(x)}{2\pi i}\mathbb{1} \tag{6.11}$$

for a.e. $x \in \mathbb{R}$.

Proof. Since $\operatorname{Im} \mathcal{A} = \frac{1}{4\pi}\langle \cdot, \mathbb{1}\rangle \mathbb{1}$ and $D(b)$ is real on $\mathbb{R}$, we have

$$Q^*(x)D(\lambda)D(e^{-i\varphi})\mathbf{g}(x) = (\mathcal{A} - \mathcal{A}^* + \mathcal{A}^* - xD(\nu^2)^{-1} - D(b))D(\lambda)D(e^{-i\varphi})\mathbf{g}(x)$$
$$= \frac{2i}{4\pi}\langle D(\lambda)D(e^{-i\varphi})\mathbf{g}(x), \mathbb{1}\rangle \mathbb{1} + Q(x)D(\lambda)D(e^{-i\varphi})\mathbf{g}(x)$$

for a.e. $x \in \mathbb{R}$. By (6.8),

$$\frac{2i}{4\pi}\langle D(\lambda)D(e^{-i\varphi})\mathbf{g}(x), \mathbb{1}\rangle \mathbb{1} = -\frac{u(x)}{2\pi i}\mathbb{1}.$$

Further, by the commutation relation (6.5), we see that $Q(x)$ satisfies

$$D(\lambda)D(e^{-i\varphi})Q(x)^\top = Q(x)D(\lambda)D(e^{-i\varphi}).$$

Using this formula, we obtain

$$Q(x)D(\lambda)D(e^{-i\varphi})\mathbf{g}(x) = D(\lambda)D(e^{-i\varphi})Q^\top(x)\mathbf{g}(x) = \frac{1}{2\pi i}D(\lambda)D(e^{-i\varphi})\mathbb{1},$$

where we have used (6.10) at the last step. Putting this together, we obtain the required identity. ■

The next lemma involves multiplication by $b_j(x)$ on the real axis. Here we need to proceed with caution because $b_j(x)$ has poles on the real axis, see (5.15). We claim that

$$D(b(x))Q(x)^{-1} \in H^\infty(\mathbb{C}_+) \quad \text{and} \quad D(b(x))u(x) \in H^\infty(\mathbb{C}_+). \tag{6.12}$$

Indeed, from $Q(x) = \mathcal{A}^* - xD(\nu^2)^{-1} - D(b(x))$ it is clear that $Q(x)$ has singularities at the same points as $D(b(x))$, and so these singularities cancel out. As an alternative argument, one can write

$$I = (\mathcal{A}^* - xD(\nu^2)^{-1})Q(x)^{-1} - D(b(x))Q(x)^{-1}, \quad x \in \mathbb{C}_+;$$

by Theorem 6.1, one finds that the product $D(b(x))Q(x)^{-1}$ is bounded outside a neighbourhood of infinity. On the other hand, this product is bounded in the neighbourhood of infinity because both factors are bounded there. This gives the first inclusion in (6.12); the second one follows by recalling that definition (5.17) of $u(x)$ involves $Q(x)^{-1}$.

Lemma 6.11. *We have the identity*

$$\mathbb{P}_+(Q^*(x)^{-1}\mathbb{P}_+(uD(b)\overline{\mathbf{g}})) = \mathbb{P}_+(Q^*(x)^{-1}D(b)\mathbb{P}_+(u\overline{\mathbf{g}})).$$

Note that by (6.12), both sides here are well defined.

Proof. Let us take the inner product of the left-hand side of the required identity with an arbitrary element $\mathbf{f} \in H^2$:

$$\langle \mathbb{P}_+ Q^*(x)^{-1}\mathbb{P}_+(uD(b)\overline{\mathbf{g}}), \mathbf{f}\rangle = \langle \mathbb{P}_+(uD(b)\overline{\mathbf{g}}), Q(x)^{-1}\mathbf{f}\rangle$$
$$= \langle uD(b)\overline{\mathbf{g}}, Q(x)^{-1}\mathbf{f}\rangle = \langle u\overline{\mathbf{g}}, D(b)Q(x)^{-1}\mathbf{f}\rangle.$$

Since $D(b)Q(x)^{-1} \in H^\infty$, we have

$$\begin{aligned}\langle u\overline{\mathbf{g}}, D(b)Q(x)^{-1}\mathbf{f}\rangle &= \langle \mathbb{P}_+(u\overline{\mathbf{g}}), D(b)Q(x)^{-1}\mathbf{f}\rangle \\ &= \langle Q^*(x)^{-1}D(b)\mathbb{P}_+(u\overline{\mathbf{g}}), \mathbf{f}\rangle = \langle \mathbb{P}_+(Q^*(x)^{-1}D(b)\mathbb{P}_+(u\overline{\mathbf{g}})), \mathbf{f}\rangle.\end{aligned}$$

This proves the required identity. ∎

Proof of Theorem 6.3. Let us apply $Q^*(x)^{-1}$ to both sides of (6.11) and then use (6.9):

$$\begin{aligned}D(\lambda)D(e^{-i\varphi})\mathbf{g}(x) &= Q^*(x)^{-1}\Big(\frac{1}{2\pi i}D(\lambda)D(e^{-i\varphi})\mathbb{1} - \frac{u(x)}{2\pi i}\mathbb{1}\Big) \\ &= Q^*(x)^{-1}(\mathcal{A} - xD(\nu^2)^{-1})\mathbb{P}_+(u\overline{\mathbf{g}}) - Q^*(x)^{-1}\mathbb{P}_+(uD(b)\overline{\mathbf{g}}).\end{aligned}$$

Next, let us apply $\mathbb{P}_+$ to both sides and use Lemma 6.11:

$$\begin{aligned}D(\lambda)D(e^{-i\varphi})\mathbf{g}(x) &= \mathbb{P}_+Q^*(x)^{-1}(\mathcal{A} - xD(\nu^2)^{-1})\mathbb{P}_+(u\overline{\mathbf{g}}) - \mathbb{P}_+Q^*(x)^{-1}\mathbb{P}_+(uD(b)\overline{\mathbf{g}}) \\ &= \mathbb{P}_+Q^*(x)^{-1}(\mathcal{A} - xD(\nu^2)^{-1})\mathbb{P}_+(u\overline{\mathbf{g}}) - \mathbb{P}_+Q^*(x)^{-1}D(b)\mathbb{P}_+(u\overline{\mathbf{g}}) \\ &= \mathbb{P}_+Q^*(x)^{-1}Q(x)\mathbb{P}_+(u\overline{\mathbf{g}}) = \mathbb{P}_+(u\overline{\mathbf{g}}) = H_u\mathbf{g}.\end{aligned}$$

This is exactly the required eigenvalue equation in the vector form (6.2). ∎

7. Proof of Theorem 6.2

In this section, we complete the proof of the surjectivity of the spectral map. An important step consists in establishing that the rational function p_j is indeed an isometric multiplier on the model space K_{ψ_j}. An ingredient of this proof is a representation of the functions p_j in terms of a completely non-unitary contraction on $\mathbb{C}^N$, which is inspired by Sarason's work [26].

7.1. *Rational function p_j is an isometric multiplier on K_{ψ_j}*

Let us define p_j as in (5.21), selecting for definiteness the sign "+":

$$p_j = ie^{-i\varphi_j/2}\frac{\|1-\psi_j\|}{\nu_j}\frac{g_j}{1-\psi_j}.$$

In the vector form, denoting

$$\mathbf{p} = (p_1, \dots, p_N)^\top,$$

recalling formula (5.18) for $\mathbf{g}$ and setting $\gamma_j = \frac{1}{2\sqrt{\pi}}\|1-\psi_j\|$, we obtain

$$\mathbf{p} = \frac{1}{\sqrt{\pi}}D(e^{-i\varphi/2})D(\gamma)D(\nu)^{-1}D(1-\psi)^{-1}(Q(x)^\top)^{-1}\mathbb{1}.$$

We need to rearrange this expression as follows.

Lemma 7.1. *We have*

$$\mathbf{p} = D(e^{-i\varphi/2})(I - BD(\psi))^{-1}\beta, \tag{7.1}$$

where $\beta \in \mathbb{C}^N$ and B is a completely non-unitary contraction in $\mathbb{C}^N$ (with respect to the usual Euclidean norm), satisfying

$$BB^* + \langle \cdot, \beta\rangle\beta = I. \tag{7.2}$$

Proof. First we rewrite formula (5.16) for $Q(x)$ as

$$Q(x) = \mathcal{A}_1^* - iD(\gamma^2)D(\nu^2)^{-1}D\Big(\frac{1+\psi}{1-\psi}\Big),$$

where

$$\mathcal{A}_1 := \mathcal{A} - \operatorname{diag}\Big\{\frac{\operatorname{Re}\langle A_{\psi_j}(1-\psi_j), 1-\psi_j\rangle}{\nu_j^2\|1-\psi_j\|^2}\Big\}.$$

Then we have

$$\begin{aligned}
&D(\gamma)D(\nu)^{-1}D(1-\psi)^{-1}(Q(x)^\top)^{-1}\\
&\quad= D(\gamma)D(\nu)^{-1}D(1-\psi)^{-1}(\overline{\mathcal{A}_1} - iD(\gamma^2)D(\nu^2)^{-1}D(1+\psi)D(1-\psi)^{-1})^{-1}\\
&\quad= (\overline{\mathcal{A}_1}D(\gamma)^{-1}D(\nu)D(1-\psi) - iD(\gamma)D(\nu)^{-1}D(1+\psi))^{-1}\\
&\quad= (D(\nu)D(\gamma)^{-1}\overline{\mathcal{A}_1}D(\gamma)^{-1}D(\nu)D(1-\psi) - iD(1+\psi))^{-1}D(\gamma)^{-1}D(\nu).
\end{aligned}$$

Denote for brevity

$$\mathcal{A}_2 = D(\nu)D(\gamma)^{-1}\mathcal{A}_1D(\gamma)^{-1}D(\nu),$$

then

$$\begin{aligned}
&(\overline{\mathcal{A}_2}D(1-\psi) - iD(1+\psi))^{-1}D(\gamma)^{-1}D(\nu)\\
&\quad= (\overline{\mathcal{A}_2} - i - (\overline{\mathcal{A}_2} + i)D(\psi))^{-1}D(\gamma)^{-1}D(\nu)\\
&\quad= (I - BD(\psi))^{-1}(\overline{\mathcal{A}_2} - i)^{-1}D(\gamma)^{-1}D(\nu),
\end{aligned}$$

where

$$B = (\overline{\mathcal{A}_2} - i)^{-1}(\overline{\mathcal{A}_2} + i).$$

This yields (7.1) with

$$\beta = \frac{1}{\sqrt{\pi}}(\overline{\mathcal{A}_2} - i)^{-1}D(\gamma)^{-1}D(\nu)\mathbb{1}.$$

Next, let us prove (7.2). We will see that this is a consequence of the rank one relation (6.4) for $\mathcal{A}$. Since

$$\operatorname{Im}\mathcal{A} = \operatorname{Im}\mathcal{A}_1 = \frac{1}{4\pi}\langle\cdot, \mathbb{1}\rangle\mathbb{1},$$

we have

$$\operatorname{Im}\mathcal{A}_2 = \frac{1}{4\pi}\langle\cdot, D(\gamma)^{-1}D(\nu)\mathbb{1}\rangle D(\gamma)^{-1}D(\nu)\mathbb{1},$$

and therefore

$$\begin{aligned} BB^* &= (\overline{\mathcal{A}_2} - i)^{-1}(\overline{\mathcal{A}_2} + i)(\overline{\mathcal{A}_2}^* - i)(\overline{\mathcal{A}_2}^* + i)^{-1} \\ &= (\overline{\mathcal{A}_2} - i)^{-1}(\overline{\mathcal{A}_2 \mathcal{A}_2}^* + I + 2\operatorname{Im}\overline{\mathcal{A}_2})(\overline{\mathcal{A}_2}^* + i)^{-1} \\ &= I - (\overline{\mathcal{A}_2} - i)^{-1}(4\operatorname{Im}\overline{\mathcal{A}_2})(\overline{\mathcal{A}_2}^* + i)^{-1} = I - \langle \cdot, \beta\rangle\beta. \end{aligned}$$

Finally, let us check that B is a completely non-unitary contraction in $\mathbb{C}^N$. The fact that B is a contraction is clear from (2.11). To check that B is completely non-unitary, let us first consider the matrix $\mathcal{A}_2$ and check that it is completely non-self-adjoint. Indeed, suppose $\mathcal{A}_2$ has a real eigenvalue λ with an eigenvector f; then

$$(\mathcal{A}_1 - \lambda D(\gamma)^2 D(\nu^2)^{-1})D(\gamma)^{-1}D(\nu) f = 0.$$

But this is impossible for $f \neq 0$, because $\mathcal{A}_1 - \lambda D(\gamma)^2 D(\nu^2)^{-1}$ is completely non-self-adjoint (see Lemma 6.5). Since $\mathcal{A}_2$ is completely non-self-adjoint, so is $\overline{\mathcal{A}_2}$; it follows that B is completely non-unitary. ■

The isometricity of p_j is a consequence of the following lemma.

Lemma 7.2. *Let B be a completely non-unitary contraction in $\mathbb{C}^N$, and let*

$$BB^* + \langle \cdot, \beta\rangle\beta = I$$

with some vector $\beta \in \mathbb{C}^N$. Let $\psi_1, \dots, \psi_N$ be inner functions in $\mathbb{C}_+$, and let the vector $\mathbf{p}$ *be defined by*

$$\mathbf{p}(z) = (I - BD(\psi(z)))^{-1}\beta, \quad z \in \mathbb{C}_+.$$

Then each p_j is an isometric multiplier on K_{ψ_j}.

Proof. Step 1: Let $\zeta_1, \dots, \zeta_N$ be complex numbers in the closed unit disc, $|\zeta_j| \leqslant 1$. As B is a completely non-unitary contraction, so is $BD(\zeta)$, and therefore, by a compactness argument, the norms

$$\|(I - BD(\zeta))^{-1}\|$$

are bounded uniformly for $|\zeta_j| \leqslant 1$, $j = 1, \dots, N$. It follows that the inverse

$$(I - BD(\psi(x)))^{-1}, \quad x \in \mathbb{C}_+,$$

is analytic and bounded in $\mathbb{C}_+$.

Step 2: Let $x \in \mathbb{R}$; denote for brevity $A = BD(\psi(x))$. Observe that we have $BB^* = AA^*$. Furthermore,

$$\begin{aligned} |((I - A)^{-1}\beta)_j|^2 &= [(1 - A)^{-1}(\langle \cdot, \beta\rangle\beta)(I - A^*)^{-1}]_{jj} \\ &= [(I - A)^{-1}(I - AA^*)(I - A^*)^{-1}]_{jj}, \end{aligned}$$

and

$$(I - A)^{-1}(I - AA^*)(I - A^*)^{-1} = I + (I - A)^{-1}A + A^*(I - A^*)^{-1}.$$

It follows that

$$|p_j(x)|^2 = 1 + [(I - BD(\psi(x)))^{-1}B]_{jj}\psi_j(x) + [B^*(I - D(\psi(x))^*B^*)^{-1}]_{jj}\overline{\psi_j(x)}.$$

Let us multiply this by $|h(x)|^2$, where $h \in K_{\psi_j}$. We obtain, for $x \in \mathbb{R}$,

$$\begin{aligned}|p_j(x)|^2|h(x)|^2 = |h(x)|^2 + [(I - BD(\psi(x)))^{-1}B]_{jj}(\psi_j(x)\overline{h(x)})h(x) \\ + \overline{[B^\top(I - D(\psi(x))B^\top)^{-1}]_{jj}(\psi_j(x)\overline{h(x)})h(x)}.\end{aligned}$$

Observe that the second term in the right-hand side is the boundary value of a function in $H^1(\mathbb{C}_+)$, while the third term is the complex conjugate of such boundary value. It follows that the integrals over $\mathbb{R}$ of both these terms vanish, and so integrating yields the required isometricity of p_j. ■

7.2. The action of H_u on $p_j K_{\psi_j}$

In Theorem 6.3 above, we have checked the eigenvalue equation

$$H_u g_j = \lambda_j e^{-i\varphi_j} g_j. \tag{7.3}$$

Here our aim is to compute the action of H_u on the whole subspace $p_j K_{\psi_j}$.

Lemma 7.3. *For every $j = 1, \dots, N$ and for every $h \in K_{\psi_j}$, we have*

$$H_u(p_j h) = \lambda_j p_j \psi_j \overline{h}. \tag{7.4}$$

Proof. Recall that p_j is defined by the formula

$$p_j = i\sqrt{4\pi}e^{-i\varphi_j/2}\frac{\gamma_j}{\nu_j}\frac{g_j}{1-\psi_j},$$

and we have already checked that p_j is an isometric multiplier on K_{ψ_j}. The desired equation (7.4) can be written on the real line as

$$\mathbb{P}_+(u\overline{p_j h}) = \lambda_j p_j \psi_j \overline{h},$$

i.e., we need to check that

$$F := u\overline{p_j h} - \lambda_j p_j \psi_j \overline{h} \in H^2(\mathbb{C}_-).$$

Let us check this inclusion. Observe that by construction, F is a rational function without poles on $\mathbb{R}$ and $F(x) = O(\frac{1}{x^2})$ as $|x| \to \infty$. Thus, we only need to check that F has no poles in the open lower half-plane.

First note that by the same logic the eigenvalue equation (7.3) can be transformed into the condition

$$G := u\overline{g_j} - \lambda_j e^{-i\varphi_j} g_j \in H^2(\mathbb{C}_-).$$

Observe that G is a rational function without poles in the closed lower half-plane.

Next, recalling the definition of p_j and using that $|\psi_j| = 1$ on the real line, we find

$$F = -i\sqrt{4\pi}\frac{\gamma_j}{\nu_j}e^{i\varphi_j/2}\Big(\frac{u\overline{g_j}\overline{h}}{1-\overline{\psi_j}} + \lambda_j e^{-i\varphi_j}\frac{g_j\psi_j\overline{h}}{1-\psi_j}\Big) = -i\sqrt{4\pi}\frac{\gamma_j}{\nu_j}e^{i\varphi_j/2}\frac{G\overline{h}}{1-\overline{\psi_j}}.$$

From this representation, we see that F has no poles in the open lower half-plane. The proof is complete. ■

7.3. Identification of θ

Lemma 7.4. *Let θ be the finite Blaschke product such that* $\operatorname{Ran} H_u = K_\theta$ *and* $\theta(\infty) = 1$. *Let g be defined by*

$$g = \sum_{j=1}^{N} g_j,$$

where g_j is given by (5.18)*; then $g = 1 - \theta$.*

Proof. By Theorem 6.3, we have

$$H_u g = \sum_{j=1}^{N} H_u g_j = \sum_{j=1}^{N} \lambda_j e^{-i\varphi_j} g_j = u.$$

On the other hand, we know from Lemma 4.1 that $H_u(1-\theta) = u$. It follows that $g - (1-\theta) \in \operatorname{Ker} H_u$.

Further, by the definition of g and by Theorem 6.3, we see that $g \in \operatorname{Ran} H_u$. Also, $1 - \theta \in K_\theta = \operatorname{Ran} H_u$. It follows that $g - (1-\theta) \in \operatorname{Ran} H_u$; we conclude that $g - (1-\theta) = 0$. ■

7.4. Identification of the range of H_u

Lemma 7.5. *The range of H_u is given by*

$$\operatorname{Ran} H_u = \bigoplus_{k=1}^{N} p_k K_{\psi_k}.$$

Proof. By Lemma 7.3, the subspaces $p_k K_{\psi_k}$ are mutually orthogonal and

$$\bigoplus_{k=1}^{N} p_k K_{\psi_k} \subset \operatorname{Ran} H_u.$$

It suffices to check that for some dense set D in $\operatorname{Ran} H_u$, we have

$$D \subset \bigoplus_{k=1}^{N} p_k K_{\psi_k}.$$

We use Corollary 2.4 and Lemma 7.4; let us prove that for any $\zeta \in \mathbb{C}_+$,

$$(A_\theta^* - \zeta)^{-1} g \in \bigoplus_{k=1}^{N} p_k K_{\psi_k}.$$

By the definition of g, it suffices to check that

$$(A_\theta^* - \zeta)^{-1} g_j \in \bigoplus_{k=1}^{N} p_k K_{\psi_k}$$

for all $j = 1, \dots, N$ and all $\zeta \in \mathbb{C}_+$. From formula (2.3) for the resolvent of A_θ^* and from formula (5.18) for g_j, we find

$$\begin{aligned}(A_\theta^* - \zeta)^{-1} g_j(x) &= \frac{g_j(x) - g_j(\zeta)}{x - \zeta} = \frac{1}{2\pi i}\langle Q(x)^{-1}(Q(\zeta) - Q(x))Q(\zeta)^{-1}\mathbb{1}_j, \mathbb{1}\rangle \\ &= \frac{1}{2\pi i}\langle (Q(\zeta) - Q(x))Q(\zeta)^{-1}\mathbb{1}_j, \overline{(Q^\top(x))^{-1}\mathbb{1}}\rangle.\end{aligned}$$

Recalling that

$$\mathbf{g}(x) = \frac{1}{2\pi i}(Q^\top(x))^{-1}\mathbb{1}$$

and on the other hand

$$\begin{aligned}Q(x) - Q(\zeta) &= -iD(\gamma^2)D(\nu^2)^{-1}D\Big(\frac{1+\psi(\zeta)}{1-\psi(\zeta)} - \frac{1+\psi(x)}{1-\psi(x)}\Big) \\ &= -iD(\gamma^2)D(\nu^2)^{-1}D\Big(\frac{-2(\psi(x) - \psi(\zeta))}{(1-\psi(\zeta))(1-\psi(x))}\Big),\end{aligned}$$

we obtain, for some constants $c_{jk}(\zeta)$,

$$\frac{g_j(x) - g_j(\zeta)}{x - \zeta} = \sum_{k=1}^{N} c_{jk}(\zeta)\frac{g_k(x)(\psi_k(x) - \psi_k(\zeta))}{(1 - \psi_k(x))(x - \zeta)}.$$

Since $g_k(x) = \overline{p_{k,\infty}}\, p_k(x)(1 - \psi_k(x))$, we are left with a linear combination of terms of the form

$$p_k(x)\frac{\psi_k(x) - \psi_k(\zeta)}{x - \zeta} = -p_k(x)(A_{\psi_k}^* - \zeta)^{-1}(1 - \psi_k),$$

which belong to $p_k K_{\psi_k}$. This completes the proof. ∎

7.5. Identification of ω_j

Lemma 7.6. *For any $X, Y \in \mathbb{C}^N$, we have*

$$\int_{-\infty}^{\infty} \langle Q(x)^{-1}D(b(x))X, \mathbb{1}\rangle\overline{\langle Q(x)^{-1}Y, \mathbb{1}\rangle}\, dx = 0. \tag{7.5}$$

Proof. Following the proof of Lemma 6.7, we obtain

$$\frac{1}{4\pi^2}\langle Q(x)^{-1}D(b(x))X,\mathbb{1}\rangle\overline{\langle Q(x)^{-1}Y,\mathbb{1}\rangle}$$
$$=\frac{1}{2\pi i}\big(\langle Q(x)^{-1}D(b(x))X,Y\rangle-\overline{\langle Q(x)^{-1}Y,D(b(x))X\rangle}\big).$$

Since b_j are real-valued on $\mathbb{R}$, we have

$$\overline{\langle Q(x)^{-1}Y,D(b(x))X\rangle}=\overline{\langle D(b(x))Q(x)^{-1}Y,X\rangle}$$

for a.e. $x\in\mathbb{R}$. Therefore, the left-hand side of (7.5) rewrites as

$$\frac{1}{2\pi i}\lim_{R\to\infty}\int_{-R}^{R}\langle Q(x)^{-1}D(b(x))X,Y\rangle\,dx-\frac{1}{2\pi i}\lim_{R\to\infty}\int_{-R}^{R}\overline{\langle D(b(x))Q(x)^{-1}Y,X\rangle}\,dx.$$

Deforming the integration contour as in the proof of Lemma 6.7 and using that $b(x)=O(\frac{1}{x})$ at infinity, we find that both limits are equal to zero. ∎

Lemma 7.7. *For the operator A_θ corresponding to H_u, we have*

$$\langle A_\theta g_j,g_k\rangle=\nu_j^2\nu_k^2\mathcal{A}_{kj}$$

and, in particular,

$$\langle A_\theta g_j,g_j\rangle=\frac{\omega_j}{\lambda_j^2}.$$

Proof. We shall prove the equivalent identity

$$\langle A_\theta^* g_j,g_k\rangle=\nu_j^2\nu_k^2(\mathcal{A}^*)_{kj}.$$

By (2.7), we have

$$A_\theta^* g_j(x)=xg_j(x)-\Lambda_1(g_j).$$

Recall that

$$g_j(x)=\frac{1}{2\pi i}\langle Q(x)^{-1}\mathbb{1}_j,\mathbb{1}\rangle.$$

By the asymptotic formula (6.1), we find

$$\Lambda_1(g_j)=-\frac{1}{2\pi i}\nu_j^2=-\frac{1}{2\pi i}\langle D(\nu^2)\mathbb{1}_j,\mathbb{1}\rangle,$$

and therefore

$$\begin{aligned}A_\theta^* g_j(x)&=\frac{1}{2\pi i}\langle (xQ(x)^{-1}+D(\nu^2))\mathbb{1}_j,\mathbb{1}\rangle\\&=\frac{1}{2\pi i}\langle Q(x)^{-1}(xI+Q(x)D(\nu^2))\mathbb{1}_j,\mathbb{1}\rangle\\&=\frac{1}{2\pi i}\langle Q(x)^{-1}(xD(\nu^2)^{-1}+Q(x))D(\nu^2)\mathbb{1}_j,\mathbb{1}\rangle\\&=\frac{1}{2\pi i}\langle Q(x)^{-1}\mathcal{A}^*D(\nu^2)\mathbb{1}_j,\mathbb{1}\rangle-\frac{1}{2\pi i}\langle Q(x)^{-1}D(b(x))D(\nu^2)\mathbb{1}_j,\mathbb{1}\rangle.\end{aligned}$$

It follows that

$$\langle A_\theta^* g_j, g_k\rangle = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \langle Q(x)^{-1}\mathcal{A}^* D(\nu^2)\mathbb{1}_j, \mathbb{1}\rangle \overline{\langle Q(x)^{-1}\mathbb{1}_k, \mathbb{1}\rangle}\, dx$$
$$- \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \langle Q(x)^{-1} D(b(x))D(\nu^2)\mathbb{1}_j, \mathbb{1}\rangle \overline{\langle Q(x)^{-1}\mathbb{1}_k, \mathbb{1}\rangle}\, dx.$$

Here the first term in the right-hand side equals

$$\langle D(\nu^2)\mathcal{A}^* D(\nu^2)\mathbb{1}_j, \mathbb{1}_k\rangle = \nu_j^2\nu_k^2(\mathcal{A}^*)_{kj}$$

by Lemma 6.7 and the second one equals zero by Lemma 7.6. ∎

7.6. *Proof of Theorem 6.2*

The theorem follows by putting together Theorem 6.3 and the lemmas of this section. Indeed, by Theorem 6.3 and by Lemma 7.5, the set of singular values of H_u is exactly $\{\lambda_n\}_{n=1}^N$. Again by Theorem 6.3 and by Lemma 7.3, the inner function ψ_j and the unimodular constant $e^{i\varphi_j}$ corresponds to the eigenvalue λ_j. Finally, by Lemma 7.7, the parameters ω_j correspond to A_θ and g_j.

8. The Szegő dynamics

8.1. *Formulas for the Szegő dynamics*

In this section, we express the Szegő dynamics for rational solutions in terms of the spectral data. The main result here is

Theorem 8.1. *Let u be a solution of the cubic Szegő equation*

$$i\frac{d}{dt}u = \mathbb{P}_+(|u|^2u) \tag{8.1}$$

with $u|_{t=0}$ *rational. Then the solution is rational for all* $t > 0$*, and the spectral data of u satisfy the following law:*

$$\frac{d}{dt}\lambda_j = \frac{d}{dt}\psi_j = 0; \tag{8.2}$$
$$\frac{d}{dt}\varphi_j = \lambda_j^2, \tag{8.3}$$
$$\frac{d}{dt}\omega_j = \frac{1}{2\pi}\lambda_j^4\nu_j^4. \tag{8.4}$$

Before coming to the proof of Theorem 8.1, observe the following. Since we know from [24] that the initial value problem for the cubic Szegő equation is well posed in every Sobolev space $W^{s,2}(\mathbb{C}_+)$ for every $s \geqslant \frac{1}{2}$, the uniqueness implies that it is enough to prove that the rational function corresponding to the spectral data defined by the evolution laws (8.2), (8.3), (8.4) is indeed a solution of the cubic Szegő equation.

We consider the spectral data evolving according to the evolution laws (8.2), (8.3) and (8.4). For each $t > 0$, we define the matrices $\mathcal{A}$ and $Q(x)$ as in Section 5, suppressing the dependence on t in our notation. We define the anti-linear operator $\mathcal{H}$ in $\mathbb{C}^N$ as in (6.3). According to Section 5, the functions u and u_j can be recovered from the spectral data by the formulas

$$u_j(x) = \frac{1}{2\pi i}\langle Q(x)^{-1}\mathcal{H}\mathbb{1}_j, \mathbb{1}\rangle,$$
$$u(x) = \frac{1}{2\pi i}\langle Q(x)^{-1}\mathcal{H}\mathbb{1}, \mathbb{1}\rangle. \tag{8.5}$$

Our aim is to differentiate (8.5) with respect to t and check that u satisfies (8.1).

8.2. *The time derivative of Q*

We first observe that equation (8.4) means, in particular, that $\operatorname{Im}\omega_j$ are fixed by the dynamics. By (5.1), it follows that ν_j are also fixed.

The time derivative of $\mathcal{H}\mathbb{1}$ is straightforward to compute

$$\frac{d}{dt}\mathcal{H}\mathbb{1} = D(\lambda)\frac{d}{dt}D(e^{-i\varphi})\mathbb{1} = -i\mathcal{H}^2\mathcal{H}\mathbb{1} = -i\mathcal{H}^3\mathbb{1}. \tag{8.6}$$

Let us compute the time derivative of the matrix $\mathcal{A}$. For $j \neq k$, the only time-dependent quantities of $\mathcal{A}_{kj}$ are φ_j and φ_k, and so we get

$$\frac{d}{dt}\mathcal{A}_{kj} = \frac{i}{2\pi}\frac{-\lambda_j\lambda_k(i\lambda_j^2 - i\lambda_k^2)e^{i\varphi_j}e^{-i\varphi_k}}{\lambda_j^2-\lambda_k^2} = \frac{1}{2\pi}\lambda_j\lambda_k e^{i\varphi_j}e^{-i\varphi_k}.$$

For the diagonal entries $\mathcal{A}_{jj}$, we have, by our definitions,

$$\frac{d}{dt}\mathcal{A}_{jj} = \frac{1}{\lambda_j^2\nu_j^4}\frac{d}{dt}\omega_j = \frac{1}{2\pi}\frac{\lambda_j^4\nu_j^4}{\lambda_j^2\nu_j^4} = \frac{1}{2\pi}\lambda_j^2.$$

Putting this together, we find

$$\frac{d}{dt}\mathcal{A} = \frac{1}{2\pi}\langle\cdot, \mathcal{H}\mathbb{1}\rangle\mathcal{H}\mathbb{1}.$$

Since b_j is independent of t, we obtain

$$\frac{d}{dt}Q(x) = \frac{1}{2\pi}\langle\cdot, \mathcal{H}\mathbb{1}\rangle\mathcal{H}\mathbb{1}, \quad \operatorname{Im} x > 0,$$

and finally, taking inverses,

$$\frac{d}{dt}Q(x)^{-1} = -\frac{1}{2\pi}\langle\cdot, (Q^*(x))^{-1}\mathcal{H}\mathbb{1}\rangle Q(x)^{-1}\mathcal{H}\mathbb{1}, \tag{8.7}$$

for $\operatorname{Im} x > 0$ and, by taking limits, everywhere on the real axis apart from finitely many points.

8.3. *Formulas for $H_u u$ and $H_u^2 u$*

One easily verifies the identity

$$uH_u u + H_u^2 u = \mathbb{P}_+(|u|^2 u).$$

Using this, one can rewrite the Szegő equation in the following equivalent form:

$$i\frac{d}{dt}u = uH_u u + H_u^2 u. \tag{8.8}$$

Now, in preparation for what comes next, let us express $H_u u$ and $H_u^2 u$ in terms of the spectral data. We have

$$H_u u = \sum_j \lambda_j e^{i\varphi_j} u_j = \frac{1}{2\pi i}\sum_j \lambda_j e^{i\varphi_j}\langle Q(x)^{-1}\mathcal{H}\mathbb{1}_j, \mathbb{1}\rangle = \frac{1}{2\pi i}\langle Q(x)^{-1}\mathcal{H}^2\mathbb{1}, \mathbb{1}\rangle,$$

$$H_u^2 u = \sum_j \lambda_j^2 u_j = \frac{1}{2\pi i}\sum_j \lambda_j^2 \langle Q(x)^{-1}\mathcal{H}\mathbb{1}_j, \mathbb{1}\rangle = \frac{1}{2\pi i}\langle Q(x)^{-1}\mathcal{H}^3\mathbb{1}, \mathbb{1}\rangle.$$

8.4. *Concluding the proof of Theorem 8.1*

First we need an identity relating Q and $\mathcal{H}$. From the matrix identity (6.5), we get

$$Q(x)\mathcal{H} = \mathcal{H}Q^*(x), \quad \operatorname{Im} x > 0.$$

Passing to the inverses,

$$\mathcal{H}(Q^*(x))^{-1} = Q(x)^{-1}\mathcal{H}, \tag{8.9}$$

for $\operatorname{Im} x > 0$ and, by taking limits, also everywhere on the real axis apart from finitely many points (the poles of b_j).

Using (8.6) and (8.7), we find (suppressing the dependence on x)

$$\begin{aligned}
i\frac{d}{dt}u &= \frac{i}{2\pi i}\frac{d}{dt}\langle Q^{-1}\mathcal{H}\mathbb{1}, \mathbb{1}\rangle \\
&= \frac{1}{2\pi}\Big\langle\Big(\frac{d}{dt}Q^{-1}\Big)\mathcal{H}\mathbb{1}, \mathbb{1}\Big\rangle + \frac{1}{2\pi}\Big\langle Q^{-1}\Big(\frac{d}{dt}\mathcal{H}\mathbb{1}\Big), \mathbb{1}\Big\rangle \\
&= \Big(\frac{1}{2\pi i}\Big)^2\langle\mathcal{H}\mathbb{1}, (Q^*)^{-1}\mathcal{H}\mathbb{1}\rangle\langle Q^{-1}\mathcal{H}\mathbb{1}, \mathbb{1}\rangle + \frac{1}{2\pi i}\langle Q^{-1}\mathcal{H}^3\mathbb{1}, \mathbb{1}\rangle \\
&= u(x)\frac{1}{2\pi i}\langle Q^{-1}\mathcal{H}\mathbb{1}, \mathcal{H}\mathbb{1}\rangle + H_u^2 u.
\end{aligned}$$

Using (8.9), we transform the inner product in the right-hand side as

$$\langle Q^{-1}\mathcal{H}\mathbb{1}, \mathcal{H}\mathbb{1}\rangle = \langle\mathcal{H}(Q^*)^{-1}\mathbb{1}, \mathcal{H}\mathbb{1}\rangle = \langle\mathcal{H}^2\mathbb{1}, (Q^*)^{-1}\mathbb{1}\rangle = \langle Q^{-1}\mathcal{H}^2\mathbb{1}, \mathbb{1}\rangle.$$

Putting this together, we obtain the Szegő equation in form (8.8).

9. The genericity of turbulent solutions: Proof of Theorem 1.1

In this last section, we prove Theorem 1.1. The key argument is to establish that, if u is a rational solution of the cubic Szegő equation (1.1) such that one of the singular values of the Hankel operator H_u is multiple while the other singular values are simple, then the L^2 norm of $\partial_x u$ tends to infinity as t tends to infinity. This can be achieved thanks to the representation of rational solutions obtained in previous sections.

9.1. *Upper bound for general rational solutions*

We start with a general a priori bound for rational solutions.

Proposition 9.1. *If u is a rational solution of the cubic Szegő equation, then*

$$\limsup_{t\to+\infty} \frac{\|\partial_x u(\cdot,t)\|_{L^2}}{t} < +\infty.$$

Before proceeding with the proof, we need a simple lemma. For every spectral data

$$(\lambda, \psi, e^{i\varphi}, \omega) := (\{\lambda_j\}_{j=1}^N, \{\psi_j\}_{j=1}^N, \{e^{i\varphi_j}\}_{j=1}^N, \{\omega_j\}_{j=1}^N),$$

denote by $\mathcal{A}(\lambda, \psi, \varphi, \omega)$ the matrix defined in Section 5.3. We also denote by $\mathbb{P}_j$ the projector matrix onto the j-th direction in $\mathbb{C}^N$.

Lemma 9.2. *Fix (λ, ψ, ω); there exists $C > 0$ such that for every $j = 1, \dots, N$, we have*

$$\forall \xi \in \mathbb{R}^N, \quad \sup_{\varphi\in\mathbb{T}^N} \|\mathbb{P}_j(\mathcal{A}(\lambda, \psi, \varphi, \omega)^* + D(\xi))^{-1}\| \leqslant \frac{C}{1+|\xi_j|}.$$

A similar result holds for $\mathcal{A}(\lambda, \psi, \varphi, \omega)$ in place of $\mathcal{A}(\lambda, \psi, \varphi, \omega)^*$.

Proof. Let $X \in \mathbb{C}^N$ of norm 1, and let

$$Z := (\mathcal{A}(\lambda, \psi, \varphi, \omega)^* + D(\xi))^{-1} X.$$

We want to prove that the components of Z satisfy

$$|Z_j| \leqslant \frac{C}{1+|\xi_j|}.$$

We already know, from Lemma 6.6 and from the proof of Theorem 6.1, that $\|Z\|$ is bounded. Furthermore, one can easily check from the proof of Lemma 6.6 that this estimate is uniform in $\varphi \in \mathbb{T}^N$. Then we come back to the equations in Z, which read

$$\xi_j Z_j + (\mathcal{A}(\lambda, \psi, \varphi, \omega)^* Z)_j = X_j$$

and this immediately leads to the required estimate. The proof for $\mathcal{A}$ is similar. ∎

Proof of Proposition 9.1. By Theorems 5.5 and 8.1, the rational solution of the cubic Szegő equation reads

$$u(x,t) = \frac{1}{2\pi i}\langle R_t(\xi(x,t))^{-1}X(t), \mathbb{1}\rangle$$

with, for every $\xi = (\xi_1, \dots, \xi_N) \in \overline{\mathbb{C}_-}^N$,

$$R_t(\xi) := \mathcal{A}(\lambda, \psi, \varphi(t), \omega(0))^* + D(\xi), \quad X(t) := D(\lambda)D(e^{-i\varphi(t)})\mathbb{1},$$

and

$$\xi_j(x,t) := \frac{\lambda_j^2}{2\pi}t - b_j(x) - \frac{x}{\nu_j^2}. \tag{9.1}$$

Notice that the time dependence of $\xi_j(x,t)$ is due to the time dependence of $\operatorname{Re}\omega_j(t)$ coming from (8.4), and that

$$\partial_x\xi_j(x,t) = -b_j'(x) - \frac{1}{\nu_j^2}$$

is independent of t. Also recall that b_j is a rational Herglotz function, representable as in (5.15). Consequently, denoting by $\mathbb{P}_j$ the projector matrix onto the j-th direction in $\mathbb{C}^N$, we have

$$\partial_x u(x,t) = \sum_{j=1}^{N}\frac{i\,\partial_x\xi_j(x,t)}{2\pi}\langle R_t(\xi(x,t))^{-1}\mathbb{P}_j R_t(\xi(x,t))^{-1}X(t), \mathbb{1}\rangle. \tag{9.2}$$

Write $\mathcal{A}(t) := \mathcal{A}(\lambda, \psi, \varphi(t), \omega(0))$ for brevity and observe that

$$\begin{aligned}&\langle R_t(\xi(x,t))^{-1}\mathbb{P}_j R_t(\xi(x,t))^{-1}X(t), \mathbb{1}\rangle\\ &\quad= \langle \mathbb{P}_j(\mathcal{A}(t)^* + D(\xi(x,t)))^{-1}X(t), \mathbb{P}_j(\mathcal{A}(t) + D(\xi(x,t)))^{-1}\mathbb{1}\rangle\end{aligned}$$

so that Lemma 9.2 and identity (9.2) lead to

$$|\partial_x u(x,t)| \lesssim \sum_{j=1}^{N}\frac{|\partial_x\xi_j(x,t)|}{1+\xi_j(x,t)^2}, \quad x \in \mathbb{R},$$

where $\lesssim$ denotes inequality up to a multiplicative constant. Let us fix j; observe that $b_j(x) + \frac{x}{\nu_j^2}$ is strictly increasing between the poles of b_j. We decompose the integral

$$\int_{\mathbb{R}}\frac{|\partial_x\xi_j(x,t)|^2}{(1+\xi_j(x,t)^2)^2}\,dx$$

into a finite sum of integrals over the open intervals between the adjacent poles of b_j (plus two semi-infinite intervals). Then on each interval the map $x \mapsto \xi_j(x,t)$ is strictly decreasing. We write in each of these integrals

$$|\partial_x\xi_j(t,x)|\,dx = d\xi_j.$$

We have

$$|b_j'(x)| \lesssim \Big(b_j(x) + \frac{x}{\nu_j^2}\Big)^2 + 1;$$

indeed, this follows by observing that both sides are rational functions with poles of second order located at the same points and by inspecting the behaviour at infinity. Next, we have, as $t \to +\infty$,

$$|\partial_x \xi_j(t,x)| = \Big|b_j'(x) + \frac{1}{\nu_j^2}\Big| \lesssim \Big(b_j(x) + \frac{x}{\nu_j^2}\Big)^2 + 1 = \Big(\xi_j(t,x) - \frac{\lambda_j^2}{2\pi}t\Big)^2 + 1$$
$$\lesssim \xi_j(t,x)^2 + t^2.$$

Plugging this estimate into each of our integrals and summing over j, we get the required bound $\|\partial_x u(\cdot,t)\|_{L^2}^2 \lesssim t^2$. ∎

9.2. *Lower bound in the case of one multiple eigenvalue*

Proposition 9.3. *Let u be a rational solution of the cubic Szegő equation on the line such that H_u has singular values $\lambda_1, \dots, \lambda_N$, with λ_1 being multiple and λ_j being simple for every $j \geq 2$. Then*

$$\liminf_{t\to+\infty} \frac{\|\partial_x u(t)\|_{L^2}}{t} > 0.$$

Proof. We decompose $\partial_x u(x,t)$ as in the proof of Proposition 9.1, starting from (9.2). Since λ_j is simple for $j \geqslant 2$, it is easy to check (see, e.g., Remark 5.4) that $\xi_j(x,t)$ is linear in x, and therefore that $\partial_x \xi_j(x,t)$ is uniformly bounded. Consequently, using again Lemma 9.2, the quantity

$$\sum_{j\geq 2} \int_{\mathbb{R}} |\partial_x \xi_j(x,t)|^2 |\langle R_t(\xi(x,t))^{-1} \mathbb{P}_j R_t(\xi(x,t))^{-1} X(t), \mathbb{1}\rangle|^2 \, dx$$

is bounded as $t \to \infty$. It remains to study the integral

$$\int_{\mathbb{R}} |\partial_x \xi_1(x,t)|^2 |\langle R_t(\xi(x,t))^{-1} \mathbb{P}_1 R_t(\xi(x,t))^{-1} X(t), \mathbb{1}\rangle|^2 \, dx,$$

which we minorize by the integral $I(t)$ of the same function on an interval J_t constructed as follows. Since λ_1 is a multiple eigenvalue, $1 - \psi_1$ has at least one zero on the real line. Denote by x_c such a zero. Since ψ_1 is a Blaschke product, we know that $i\psi_1'(x_c)$ is a non-zero real number. Consider the interval

$$J_t = \Big[x_c + \frac{\mu}{t} + \frac{\varkappa_1}{t^2}, x_c + \frac{\mu}{t} + \frac{\varkappa_2}{t^2}\Big],$$

where $\mu \neq 0$ and $\varkappa_1 < \varkappa_2$ are real numbers which we are going to choose. For $x \in J_t$, we can expand

$$\psi_1(x) = 1 + \psi_1'(x_c)\Big(\frac{\mu}{t} + \frac{\varkappa}{t^2}\Big) + \frac{1}{2}\psi_1''(x_c)\frac{\mu^2}{t^2} + O(t^{-3})$$
$$= 1 + \psi_1'(x_c)\frac{\mu}{t} + \Big(\psi_1'(x_c)\varkappa + \frac{1}{2}\psi_1''(x_c)\mu^2\Big)\frac{1}{t^2} + O(t^{-3})$$

with $\varkappa_1 \leqslant \varkappa \leqslant \varkappa_2$. It follows that

$$\begin{aligned}
\frac{1+\psi_1(x)}{1-\psi_1(x)} &= \frac{2+\psi_1'(x_c)\frac{\mu}{t}+O(t^{-2})}{-\psi_1'(x_c)\frac{\mu}{t}-(\psi_1'(x_c)\varkappa+\frac{1}{2}\psi_1''(x_c)\mu^2)\frac{1}{t^2}+O(t^{-3})} \\
&= -\frac{2t}{\psi_1'(x_c)\mu}\frac{1+\frac{1}{2}\psi_1'(x_c)\frac{\mu}{t}+O(t^{-2})}{1+(\frac{\varkappa}{\mu}+\frac{\mu}{2}\frac{\psi_1''(x_c)}{\psi_1'(x_c)})\frac{1}{t}+O(t^{-2})} \\
&= -\frac{2t}{\psi_1'(x_c)\mu}\Big(1+\Big(\frac{\mu}{2}\psi_1'(x_c)-\frac{\varkappa}{\mu}-\frac{\mu}{2}\frac{\psi_1''(x_c)}{\psi_1'(x_c)}\Big)\frac{1}{t}+O(t^{-2})\Big) \\
&= -\frac{2}{\psi_1'(x_c)\mu}t+\Big(-1+\frac{2\varkappa}{\mu^2\psi_1'(x_c)}+\frac{\psi_1''(x_c)}{(\psi_1'(x_c))^2}\Big)+O(t^{-1})
\end{aligned}$$

so that, in view of expression (5.14) of b_1, we obtain

$$\begin{aligned}
\xi_1(x,t) &= \frac{\lambda_1^2}{2\pi}t-b_1(x)-\frac{x}{\nu_1^2} \\
&= \frac{\lambda_1^2}{2\pi}t-\frac{i}{4\pi}\frac{\|1-\psi_1\|^2}{\nu_1^2}\frac{1+\psi_1(x)}{1-\psi_1(x)}+\frac{\operatorname{Re}\langle A_{\psi_1}(1-\psi_1),(1-\psi_1)\rangle}{\nu_1^2\|1-\psi_1\|^2} \\
&= \Big(\frac{\lambda_1^2}{2\pi}+\frac{i}{2\pi}\frac{\|1-\psi_1\|^2}{\mu\nu_1^2\psi_1'(x_c)}\Big)t-\frac{i}{4\pi}\frac{\|1-\psi_1\|^2}{\nu_1^2}\Big(-1+\frac{2\varkappa}{\mu^2\psi_1'(x_c)}+\frac{\psi_1''(x_c)}{(\psi_1'(x_c))^2}\Big) \\
&\quad -\frac{\operatorname{Re}\langle A_{\psi_1}(1-\psi_1),(1-\psi_1)\rangle}{\nu_1^2\|1-\psi_1\|^2}+O(t^{-1})
\end{aligned}$$

or equivalently

$$\begin{aligned}
\xi_1(x,t) &= \zeta_1 t+\eta_1(\varkappa)+O(t^{-1}), \quad x=x_c+\frac{\mu}{t}+\frac{\varkappa}{t^2}, \quad \varkappa\in[\varkappa_1,\varkappa_2], \\
\zeta_1 &:= \frac{\lambda_1^2}{2\pi}+\frac{i}{2\pi}\frac{\|1-\psi_1\|^2}{\mu\nu_1^2\psi_1'(x_c)}, \\
\eta_1(\varkappa) &:= -\frac{i}{4\pi}\frac{\|1-\psi_1\|^2}{\nu_1^2}\Big(-1+\frac{2\varkappa}{\mu^2\psi_1'(x_c)}+\frac{\psi_1''(x_c)}{(\psi_1'(x_c))^2}\Big) \\
&\quad -\frac{\operatorname{Re}\langle A_{\psi_1}(1-\psi_1),(1-\psi_1)\rangle}{\nu_1^2\|1-\psi_1\|^2}.
\end{aligned}$$

We define μ such that $\zeta_1=0$. Consequently, for x in the interval J_t,

$$\xi_1(x,t)=\eta_1(\varkappa)+O\Big(\frac{1}{t}\Big).$$

Let us write $Z(x,t):=R_t(\xi(x,t))^{-1}X(t)$, $\widetilde{Z}(x,t):=R_t^*(\xi(x,t))^{-1}\mathbb{1}$, so that

$$\langle R_t(\xi(x,t))^{-1}\mathbb{P}_1R_t(\xi(x,t))^{-1}X(t),\mathbb{1}\rangle=Z_1(x,t)\overline{\widetilde{Z}_1(x,t)}.$$

In order to estimate Z_1, $\widetilde{Z}_1$ for $x\in J_t$, we write, by the definition of Z and $\widetilde{Z}$,

$$\xi_1Z_1+[\mathcal{A}(t)^*Z]_1=\lambda_1e^{-i\varphi_1(t)}, \quad \xi_1\widetilde{Z}_1+[\mathcal{A}(t)\widetilde{Z}]_1=1.$$

Observe that (9.1) implies that, for $x \in J_t$ and for $j \geq 2$, $\xi_j(x,t) \sim c_j t$ for some $c_j > 0$. Therefore, from Lemma 9.2, we infer

$$|Z_j(x,t)| + |\widetilde{Z}_j(x,t)| \leqslant O\Big(\frac{1}{t}\Big), \quad j \geqslant 2,$$

and consequently,

$$Z_1(x,t) = z_1(\varkappa)e^{-i\varphi_1(t)} + O\Big(\frac{1}{t}\Big), \quad \widetilde{Z}_1(x,t) = \widetilde{z}_1(\varkappa) + O\Big(\frac{1}{t}\Big),$$

where

$$z_1(\varkappa) := \frac{\lambda_1}{\eta_1(\varkappa) + \frac{\overline{\omega}_1}{\lambda_1^2 \nu_1^4}}, \quad \widetilde{z}_1(\varkappa) := \frac{1}{\eta_1(\varkappa) + \frac{\omega_1}{\lambda_1^2 \nu_1^4}}.$$

Here the parameters $\varkappa_1 < \varkappa_2$ are chosen so that the denominators of $z_1(\varkappa)$ and $\widetilde{z}_1(\varkappa)$ do not cancel for $\varkappa \in [\varkappa_1, \varkappa_2]$. Consequently, for $x \in J_t$,

$$|\langle R_t(\xi(x,t))^{-1}\mathbb{P}_1 R_t(\xi(x,t))^{-1}X(t), \mathbb{1}\rangle|^2 = |z_1(\varkappa)\widetilde{z}_1(\varkappa)|^2 + O\Big(\frac{1}{t}\Big),$$

where $\varkappa := t^2(x - x_c - \frac{\mu}{t}) \in [\varkappa_1, \varkappa_2]$. On the other hand,

$$|\partial_x \xi_1(x,t)| = \frac{\|1-\psi_1\|^2}{2\pi\nu_1^2}\frac{|\psi_1'(x)|}{|1-\psi_1(x)|^2} \sim \frac{\|1-\psi_1\|^2}{2\pi\nu_1^2|\psi_1'(x_c)|\mu^2}t^2$$

for $x \in J_t$. Making the change of variable

$$\varkappa := t^2\Big(x - x_c - \frac{\mu}{t}\Big)$$

in the integral, we conclude that

$$\begin{aligned} I(t) &= \int_{J_t} |\partial_x \xi_1(x,t)|^2 |\langle R_t(\xi(x,t))^{-1}\mathbb{P}_1 R_t(\xi(x,t))^{-1}X(t), \mathbb{1}\rangle|^2\,dx \\ &\sim \frac{\|1-\psi_1\|^4}{4\pi^2\nu_1^4|\psi_1'(x_c)|^2\mu^4}\, t^2 \int_{\varkappa_1}^{\varkappa_2} |z_1(\varkappa)\widetilde{z}_2(\varkappa)|^2\,d\varkappa, \end{aligned}$$

which completes the proof. ∎

Remark. We note the following:

- Proposition 9.3 includes the case studied in [24], which corresponds to $N = 1$ and was revisited in [9, Appendix B] by solving explicitly the corresponding ODE system. Here our approach is more flexible so that we can deal with more general data, providing enough turbulent solutions to establish genericity in the next subsection.

- In [24], it is proved that, if u is a rational solution and if H_u has only simple singular values, then all the Sobolev norms of u stay bounded. Proposition 9.3 shows that the situation may be dramatically different if there exists a multiple singular value for H_u. In fact, we expect that the existence of such a multiple singular value always implies that the norms of the solution in $W^{s,2}$ with large s are unbounded.

9.3. Lower bound for generic data in $W^{1,2}(\mathbb{C}_+)$

Denote by $\Phi(t)$ the flow map of the cubic Szegő equation on $W^{1,2}(\mathbb{C}_+)$. The main step in the proof of Theorem 1.1 is the following approximation result.

Lemma 9.4. *For every $u_0 \in W^{1,2}(\mathbb{C}_+)$, there exists a family (u_0^ε) in $W^{1,2}(\mathbb{C}_+)$ such that, as $\varepsilon \to 0$,*

$$u_0^\varepsilon \to u_0 \quad \text{in } W^{1,2}(\mathbb{C}_+)$$

and, for every ε,

$$\int_1^{+\infty} \frac{\|\partial_x \Phi(t) u_0^\varepsilon\|_{L^2}}{t^2}\, dt = +\infty.$$

Proof. *Step* 1: *Reduction to rational* u_0. Recall that $W^{1,2}(\mathbb{C}_+)$ is a Hilbert space with the inner product

$$\langle u, v\rangle + \langle \partial_x u, \partial_x v\rangle.$$

Let $u_0 \in W^{1,2}(\mathbb{C}_+)$. We first claim that u_0 can be approximated in $W^{1,2}(\mathbb{C}_+)$ by a sequence of rational functions. Indeed, the Fourier transform of a rational function in $W^{1,2}(\mathbb{C}_+)$ is a linear combination of

$$\xi^k \mathrm{e}^{-\alpha\xi}, \quad \xi > 0,$$

where k is a nonnegative integer and α is a complex number of positive real part. By the Plancherel theorem, if $u \in W^{1,2}(\mathbb{C}_+)$ is orthogonal to all rational functions, then

$$\int_0^\infty (1+\xi^2)\hat{u}(\xi)\xi^k \mathrm{e}^{-\alpha\xi}\, d\xi = 0$$

for every complex number α with $\operatorname{Re}\alpha > 0$. Consequently, by making $k = 0$ and α tend to the imaginary axis, we infer that $u = 0$. Therefore, it is enough to prove the lemma if u_0 is a rational function in $W^{1,2}(\mathbb{C}_+)$, which means

$$u_0(x) = \frac{A(x)}{B(x)}$$

where B is a polynomial of degree $N \geq 1$ with zeros in the open lower half-plane only, and A is a polynomial of degree at most $N-1$, with no common factors with B.

Step 2: *Reduction to u_0 with simple eigenvalues.* For a given N, the set of rational functions u_0 as above is a complex manifold of dimension $2N$, on which the condition that $H_{u_0}^2$ has N simple positive eigenvalues defines a dense open subset, characterised by

$$\det(\langle H_{u_0}^{2(n+m)} u_0, u_0\rangle)_{0\leqslant n,m\leqslant N-1} \neq 0$$

(see [24] for more detail). So we are reduced to proving the statement for u_0 belonging to this dense open subset.

Step 3: *Defining* u_0^ε. Denote by $(\lambda_n, \psi_n, \mathrm{e}^{i\varphi_n}, \omega_n)_{1\leqslant n\leqslant N}$ the spectral data of u_0. Let $\varphi_{N+1} = 0$, $\omega_{N+1} = i$ and $\lambda_{N+1} = \varepsilon > 0$; we also set

$$\psi_{N+1}(x) = \left(\frac{x-i}{x+i}\right)^2$$

(although any Blaschke product of degree $\geqslant 2$ will do). We define u_0^ε to be the rational function with spectral data $(\lambda_n, \psi_n, \mathrm{e}^{i\varphi_n}, \omega_n)_{1\leqslant n\leqslant N+1}$. Our aim is to check that $u_0^\varepsilon \to u_0$ in $W^{1,2}(\mathbb{C}_+)$ as $\varepsilon \to 0$ by applying the inverse spectral formula (5.17) of Theorem 5.5. But first we need to go through some preliminaries.

Denote by $\mathcal{A}^\varepsilon$ the $(N+1)\times(N+1)$ matrix associated to u_0^ε, and by $\mathcal{A}$ the $N \times N$ matrix associated to u_0. In view of formulas (5.8) and (5.9), we have

$$\mathcal{A}^\varepsilon_{N+1,N+1} = \frac{\omega_{N+1}}{\lambda^2_{N+1}\nu^4_{N+1}} = \frac{\omega_{N+1}}{4\pi\,\mathrm{Im}\,\omega_{N+1}} = \frac{i}{4\pi},$$

and

$$\mathcal{A}^\varepsilon_{N+1,k} = \frac{i}{2\pi} + O(\varepsilon), \quad \mathcal{A}^\varepsilon_{k,N+1} = O(\varepsilon),$$

as $\varepsilon \to 0$ for any $k \leqslant N$. It follows that

$$\mathcal{A}^\varepsilon = \mathcal{A}_0 + X^\varepsilon, \quad \mathcal{A}_0 = \begin{pmatrix} \mathcal{A} & 0 \\ \frac{i}{2\pi}\langle\cdot, \mathbb{1}_N\rangle & \frac{i}{2\pi}\end{pmatrix}, \quad \|X^\varepsilon\| = O(\varepsilon)$$

as $\varepsilon \to 0$, where $\mathbb{1}_N = (1,\dots,1) \in \mathbb{C}^N$. Denote $\zeta \in \mathbb{C}^N_+$, $\zeta_{N+1} \in \mathbb{C}_+$ and $\widetilde{\zeta} = (\zeta, \zeta_{N+1}) \in \mathbb{C}^{N+1}_+$. By Theorem 6.1, the inverse of $\mathcal{A}^* - D(\zeta)$ exists and

$$\sup_{\zeta_1,\dots,\zeta_N\in\mathbb{C}_+} \|(\mathcal{A}^* - D(\zeta))^{-1}\| < \infty.$$

We express the inverse of $\mathcal{A}_0^* - D(\widetilde{\zeta})$ as

$$(\mathcal{A}_0^* - D(\widetilde{\zeta}))^{-1} = \begin{pmatrix} (\mathcal{A}^* - D(\zeta))^{-1} & \frac{i}{2\pi}(-\frac{i}{4\pi} - \zeta_{N+1})^{-1}(\mathcal{A}^* - D(\zeta))^{-1}\mathbb{1}_N \\ 0 & (-\frac{i}{4\pi} - \zeta_{N+1})^{-1}\end{pmatrix}. \quad (9.3)$$

From here we find that this inverse is uniformly bounded,

$$\sup_{\widetilde{\zeta}\in\mathbb{C}^{N+1}_+} \|(\mathcal{A}_0^* - D(\widetilde{\zeta}))^{-1}\| < \infty.$$

From the resolvent identity

$$((\mathcal{A}^\varepsilon)^* - D(\widetilde{\zeta}))^{-1} - (\mathcal{A}_0^* - D(\widetilde{\zeta}))^{-1} = -((\mathcal{A}^\varepsilon)^* - D(\widetilde{\zeta}))^{-1}X^\varepsilon(\mathcal{A}_0^* - D(\widetilde{\zeta}))^{-1}$$

and the estimate $\|X^\varepsilon\| = O(\varepsilon)$, we find that the inverse of $(\mathcal{A}^\varepsilon)^* - D(\widetilde{\zeta})$ is similarly uniformly bounded, and moreover

$$\sup_{\widetilde{\zeta}\in\mathbb{C}^{N+1}_+} \|(\mathcal{A}_0^* - D(\widetilde{\zeta}))^{-1} - ((\mathcal{A}^\varepsilon)^* - D(\widetilde{\zeta}))^{-1}\| = O(\varepsilon) \quad (9.4)$$

as $\varepsilon \to 0$. Now let $Q(x)$ be the $N \times N$ matrix associated with u_0, and

$$Q_\varepsilon(x) = (\mathcal{A}^\varepsilon)^* - xD(\nu^2)^{-1} - D(b(x)),$$
$$Q_{\varepsilon,0}(x) = \mathcal{A}_0^* - xD(\nu^2)^{-1} - D(b(x)).$$

Since the eigenvalues $\lambda_1, \dots, \lambda_N$ are simple, we have $b_k(x) = 0$ for $k = 1, \dots, N$ (see Remark 5.4), and from the explicit form of ψ_{N+1} one obtains

$$b_{N+1}(x) = -\frac{1}{\nu_{N+1}^2}\frac{1}{x} = -\frac{\varepsilon}{\sqrt{4\pi}}\frac{1}{x}.$$

We denote by $\mathbb{P}_{N+1}$ the projection in $\mathbb{C}^{N+1}$ onto the subspace spanned by the last vector $(0, \dots, 0, 1)$ of the canonical basis, and let $\mathbb{P}_{N+1}^\perp = I - \mathbb{P}_{N+1}$ be the projection onto the orthogonal subspace in $\mathbb{C}^{N+1}$. Denoting by $D(\lambda)$, $D(e^{i\varphi})$ the diagonal operators in $\mathbb{C}^{N+1}$, we have

$$u_0^\varepsilon(x) = \frac{1}{2\pi i}\langle Q_\varepsilon(x)^{-1} D(\lambda) D(e^{-i\varphi})\mathbb{1}, \mathbb{1}\rangle_{\mathbb{C}^{N+1}},$$
$$u_0(x) = \frac{1}{2\pi i}\langle Q(x)^{-1} \mathbb{P}_{N+1}^\perp D(\lambda) D(e^{-i\varphi})\mathbb{1}, \mathbb{1}\rangle_{\mathbb{C}^{N}}.$$

Step 4: *Proof that $u_0^\varepsilon \to u_0$ in $L^2(\mathbb{R})$.* We will check two facts:

$$u_0^\varepsilon(x) \to u_0(x) \quad \text{uniformly in } x \in \mathbb{R}; \tag{9.5}$$
$$|u_0^\varepsilon(x)| + |u_0(x)| \leqslant \frac{C}{|x|}, \quad |x| \geqslant 2, \tag{9.6}$$

where the constant C is independent of ε.

Let us check (9.5). Since $\lambda_{N+1} = \varepsilon$, we have

$$\begin{aligned} u_0^\varepsilon(x) = {} & \frac{1}{2\pi i}\langle Q_\varepsilon(x)^{-1} \mathbb{P}_{N+1}^\perp D(\lambda) D(e^{-i\varphi})\mathbb{1}, \mathbb{1}\rangle_{\mathbb{C}^{N+1}} \\ & + \frac{\varepsilon}{2\pi i}\langle Q_\varepsilon(x)^{-1} \mathbb{P}_{N+1}\mathbb{1}, \mathbb{1}\rangle_{\mathbb{C}^{N+1}}, \end{aligned} \tag{9.7}$$

where the second term in the right-hand side is $O(\varepsilon)$ uniformly in $x \in \mathbb{R}$. Using (9.4), we replace $Q_\varepsilon(x)^{-1}$ by $Q_{\varepsilon,0}(x)^{-1}$ in the first term in the right-hand side, accruing a uniform $O(\varepsilon)$ error. By the matrix structure in (9.3), we find

$$\begin{aligned} & \frac{1}{2\pi i}\langle Q_{\varepsilon,0}(x)^{-1} \mathbb{P}_{N+1}^\perp D(\lambda) D(e^{-i\varphi})\mathbb{1}, \mathbb{1}\rangle_{\mathbb{C}^{N+1}} \\ & \quad = \frac{1}{2\pi i}\langle Q(x)^{-1} \mathbb{P}_{N+1}^\perp D(\lambda) D(e^{-i\varphi})\mathbb{1}, \mathbb{1}\rangle_{\mathbb{C}^{N}} = u_0(x). \end{aligned}$$

This proves (9.5).

Let us check (9.6). For $u_0(x)$, the estimate is obvious. For $u_0^\varepsilon(x)$, we use decomposition (9.7) again. By the resolvent identity and the matrix structure in (9.3),

$$\|Q_\varepsilon(x)^{-1}\mathbb{P}_{N+1}^\perp\| \leqslant C\|Q_{\varepsilon,0}(x)^{-1}\mathbb{P}_{N+1}^\perp\| \leqslant C\|Q(x)^{-1}\| \leqslant \frac{C}{|x|},$$

which gives the required estimate for the first term in the right-hand side of (9.7). For the second term, for the same reasons,

$$\begin{aligned}\varepsilon\|Q_\varepsilon(x)^{-1}\mathbb{P}_{N+1}\| &\leqslant C\varepsilon\|Q_{\varepsilon,0}(x)^{-1}\mathbb{P}_{N+1}\|\\ &\leqslant C\varepsilon\Big|\Big(-\frac{i}{4\pi}-\frac{\varepsilon}{\sqrt{4\pi}}\Big(x-\frac{1}{x}\Big)\Big)^{-1}\Big| \leqslant \frac{C}{|x|}\end{aligned} \tag{9.8}$$

for $|x|\geqslant 2$. This concludes the proof of (9.6); we have checked that $u_0^\varepsilon\to u_0$ in $L^2(\mathbb{R})$.

Step 5: *Proof that* $\partial_x u_0^\varepsilon\to\partial_x u_0$ *in* $L^2(\mathbb{R})$. We have

$$\partial_x Q_\varepsilon(x) = -D(\nu^2)^{-1} - D(b'(x)),$$

and so

$$\partial_x Q_\varepsilon(x)^{-1} = Q_\varepsilon(x)^{-1}D(\nu^2)^{-1}Q_\varepsilon(x)^{-1} + \frac{\varepsilon}{\sqrt{4\pi}}\frac{1}{x^2}Q_\varepsilon(x)^{-1}\mathbb{P}_{N+1}Q_\varepsilon(x)^{-1},$$

because $b_1'(x)=\cdots=b_N'(x)=0$ and $b_{N+1}'(x)=\frac{\varepsilon}{\sqrt{4\pi}}\frac{1}{x^2}$. Thus,

$$\begin{aligned}\partial_x u_0^\varepsilon(x) &= \frac{1}{2\pi i}\langle Q_\varepsilon(x)^{-1}D(\nu^2)^{-1}Q_\varepsilon(x)^{-1}D(\lambda)D(e^{-i\varphi})\mathbb{1},\mathbb{1}\rangle\\ &\quad+\frac{1}{2\pi i}\frac{\varepsilon}{\sqrt{4\pi}}\frac{1}{x^2}\langle Q_\varepsilon(x)^{-1}\mathbb{P}_{N+1}Q_\varepsilon(x)^{-1}D(\lambda)D(e^{-i\varphi})\mathbb{1},\mathbb{1}\rangle.\end{aligned}$$

In the same way as on the previous step of the proof, one proves that the first term in the right-hand side here converges in $L^2(\mathbb{R})$ to $\partial_x u_0(x)$. It remains to check that the second term converges to zero in $L^2(\mathbb{R})$.

Along with (9.8), and for the same reasons, we have the estimate

$$\|\mathbb{P}_{N+1}Q_\varepsilon(x)^{-1}\|\leqslant C\Big|\Big(-\frac{i}{4\pi}-\frac{\varepsilon}{\sqrt{4\pi}}\Big(x-\frac{1}{x}\Big)\Big)^{-1}\Big|.$$

Using this, we find

$$\begin{aligned}&\frac{\varepsilon}{x^2}|\langle Q_\varepsilon(x)^{-1}\mathbb{P}_{N+1}Q_\varepsilon(x)^{-1}D(\lambda)D(e^{-i\varphi})\mathbb{1},\mathbb{1}\rangle|\\ &\quad\lesssim \frac{\varepsilon}{x^2}\|Q_\varepsilon(x)^{-1}\mathbb{P}_{N+1}Q_\varepsilon(x)^{-1}D(\lambda)D(e^{-i\varphi})\|\\ &\quad\lesssim \frac{\varepsilon}{x^2}\Big|\Big(-\frac{i}{4\pi}-\frac{\varepsilon}{\sqrt{4\pi}}\Big(x-\frac{1}{x}\Big)\Big)^{-1}\Big|\|\mathbb{P}_{N+1}Q_\varepsilon(x)^{-1}D(\lambda)D(e^{-i\varphi})\|\\ &\quad\lesssim \frac{\varepsilon}{x^2}\Big|\Big(-\frac{i}{4\pi}-\frac{\varepsilon}{\sqrt{4\pi}}\Big(x-\frac{1}{x}\Big)\Big)^{-1}\Big|\\ &\qquad\times(\|\mathbb{P}_{N+1}Q_\varepsilon(x)^{-1}\mathbb{P}_{N+1}^\perp\|+\varepsilon\|\mathbb{P}_{N+1}Q_\varepsilon(x)^{-1}\mathbb{P}_{N+1}\|).\end{aligned}$$

Further, by (9.3) we have

$$\mathbb{P}_{N+1}Q_{\varepsilon,0}(x)^{-1}\mathbb{P}_{N+1}^\perp = 0$$

and so, by the resolvent identity,

$$\begin{aligned}\|\mathbb{P}_{N+1}Q_\varepsilon(x)^{-1}\mathbb{P}_{N+1}^\perp\| &= \|\mathbb{P}_{N+1}Q_{\varepsilon,0}(x)^{-1}(X^\varepsilon)^*Q_\varepsilon(x)^{-1}\mathbb{P}_{N+1}^\perp\| \\ &\lesssim \|\mathbb{P}_{N+1}Q_{\varepsilon,0}(x)^{-1}\|\|X^\varepsilon\| \lesssim \varepsilon\Big|\Big(-\frac{i}{4\pi}-\frac{\varepsilon}{\sqrt{4\pi}}\Big(x-\frac{1}{x}\Big)\Big)^{-1}\Big|.\end{aligned}$$

Putting this together, we find

$$\frac{\varepsilon}{x^2}|\langle Q_\varepsilon(x)^{-1}\mathbb{P}_{N+1}Q_\varepsilon(x)^{-1}D(\lambda)D(e^{-i\varphi})\mathbb{1},\mathbb{1}\rangle| \lesssim \frac{\varepsilon^2}{x^2}\Big|\Big(-\frac{i}{4\pi}-\frac{\varepsilon}{\sqrt{4\pi}}\Big(x-\frac{1}{x}\Big)\Big)^{-2}\Big|.$$

Now it is a matter of elementary calculation to check that the function in the right-hand side converges to zero in $L^2(\mathbb{R})$. For example, it is easy to see that this function is bounded by $C\min\{1,\frac{\varepsilon^2}{x^4}\}$, which yields the required convergence. We have checked that $\partial_x u_0^\varepsilon \to \partial_x u_0$ in $L^2(\mathbb{R})$, and so $u_0^\varepsilon \to u_0$ in $W^{1,2}(\mathbb{C}_+)$.

Step 6: *Concluding the proof.* From Proposition 9.3, we know that

$$\liminf_{t\to+\infty}\frac{\|\partial_x\Phi(t)u_0^\varepsilon\|_{L^2}}{t} > 0$$

for every $\varepsilon > 0$, and the lemma is proved. ∎

Let us complete the proof of Theorem 1.1. For every positive integer n, we consider

$$\Omega_n := \Big\{u_0 \in W^{1,2}(\mathbb{C}_+) : \exists t_n, \int_1^{t_n}\frac{\|\partial_x\Phi(t)u_0^\varepsilon\|_{L^2}}{t^2}\,dt > n\Big\}.$$

By the well-posedness of the cubic Szegő equation on $W^{1,2}(\mathbb{C}_+)$ (see [24]), the map

$$u_0 \in W^{1,2}(\mathbb{C}_+) \mapsto \Phi(\cdot)u_0 \in C([0,T], W^{1,2}(\mathbb{C}_+))$$

is continuous for every $T > 0$, and therefore Ω_n is an open subset of $W^{1,2}(\mathbb{C}_+)$. Furthermore, by Lemma 9.4, Ω_n is dense in $W^{1,2}(\mathbb{C}_+)$. Hence, Baire's theorem implies that $G := \bigcap_{n\ge1}\Omega_n$ is a dense G_δ subset of $W^{1,2}(\mathbb{C}_+)$. This completes the proof of Theorem 1.1.

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