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Arthur's multiplicity formula for even orthogonal and unitary groups

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Abstract. Let F be a number field and G an even orthogonal or unitary group over a number field. Based on the method used by Gan and Ichino (2018), we prove Arthur's multiplicity formula for the generic part of the automorphic discrete spectrum of G by using the theta lift. Enhancing this method, we also obtain a description of the full automorphic discrete spectrum of G of F-rank ≤ 1.

Keywords: Arthur packet, local Langlands correspondence, theta lift.

1. Introduction

Let F be a number field, $\mathbb{A}$ the adele ring of F, and G a reductive group over F. A central question in representation theory is to determine completely the spectral decomposition of $L^2(G(F)\backslash G(\mathbb{A}))$ as a unitary representation of $G(\mathbb{A})$. By some results in number theory and functional analysis, we have a decomposition

$$L^2(G(F)\backslash G(\mathbb{A})) = L^2_{\mathrm{disc}}(G) \oplus L^2_{\mathrm{cts}}(G).$$

Here $L^2_{\mathrm{disc}}(G)$ is called the "discrete spectrum", because it decomposes discretely, and $L^2_{\mathrm{cts}}(G)$ is called the "continuous spectrum". Furthermore, by Langlands [44], the continuous spectrum of G can be described in terms of the discrete spectrum of Levi subgroups of G using Eisenstein series. Therefore, the question is reduced to studying the discrete spectrum $L^2_{\mathrm{disc}}(G)$. In his monumental book [4], Arthur obtained a description of the discrete spectrum $L^2_{\mathrm{disc}}(G)$ for quasi-split special orthogonal and symplectic groups G. Roughly speaking, the classification can be divided into two steps:

Step I: Decompose $L^2_{\mathrm{disc}}(G)$ into a direct sum of so-called "near equivalence classes" ("NEC" for short), and show that each NEC can be represented by an elliptic A-parameter

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(in the sense of weak transfer to a certain general linear group). For an elliptic A-parameter ψ of G, we denote by $L^2_\psi(G)$ the summand of $L^2_{\mathrm{disc}}(G)$ represented by ψ.

Step II: Establish Arthur's multiplicity formula ("AMF" for short), which gives a further decomposition of $L^2_\psi(G)$ for each ψ.

Following Arthur's work, many papers have appeared:

- Mok [61] established AMF for quasi-split unitary groups.
- Kaletha–Mínguez–Shin–White [39] studied the case of inner forms of unitary groups, and obtained AMF for the generic part of the automorphic discrete spectrum of these groups.
- Gee–Taïbi [28] proved AMF for GSp_4.
- Taïbi [73] also studied certain inner forms of classical groups. For such a group G, he proved AMF for automorphic representations of G with algebraic regular infinitesimal character at Archimedean places.
- B. Xu [78, 79] established AMF for the generic part of the automorphic discrete spectrum of quasi-split similitude symplectic and similitude even orthogonal groups.
- Ishimoto [33] established AMF for the generic part of the automorphic discrete spectrum of non-quasi-split odd special orthogonal groups.

All these works use the stable trace formula as the main tool.

On the other hand, theta correspondence provides a way to transfer AMF from one group to another. Let (G, H) be a reductive dual pair over F such that (G, H) is in the stable range and G is the smaller group. For a unitary representation π of $G(\mathbb{A})$, we denote by $m_{\mathrm{disc}}(\pi)$ the multiplicity of π in $\mathcal{A}^2(G)$. Likewise, for a unitary representation σ of $H(\mathbb{A})$, we can define $m_{\mathrm{disc}}(\sigma)$. J.-S. Li [49] proved that

$$m_{\mathrm{disc}}(\pi) \le m_{\mathrm{disc}}(\theta^{\mathrm{abs}}(\pi)),$$

where $\theta^{\mathrm{abs}}(\pi)$ is the (placewise) theta lift of π. In [22, Section 4], Gan–Ichino observed that when π belongs to a generic NEC, J.-S. Li's inequality is indeed an equality. Based on this observation, they established AMF for the generic part of $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$ by transferring it from $L^2_{\mathrm{disc}}(\mathrm{SO}_{2r+1})$ with r sufficiently large. Using the same idea, but combined with some other results, they were also able to describe in [24] the full automorphic discrete spectrum of Mp_4. We should also mention that W.-W. Li [50] established AMF for the whole $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$ by combining the stable trace formula and the theta correspondence.

In this paper we follow Gan–Ichino's method and try to deduce some results on AMF for even orthogonal or unitary groups. Our goal is two-fold. First, we prove AMF for the generic part of $L^2_{\mathrm{disc}}(G)$ when G is an even orthogonal or unitary group by transferring it from $L^2_{\mathrm{disc}}(H)$, where H is a large symplectic or quasi-split unitary group. This part is almost parallel to [22]. Second, we enhance this method using a new observation and deduce a description of the full automorphic discrete spectrum of even orthogonal or unitary groups of F-rank ≤ 1. A case worth noting is when G is a unitary group and

$G_v \simeq \mathrm{U}_{1,n-1}$ at one real place v. In this case, the description of $L^2_{\mathrm{disc}}(G)$ might have some arithmetic applications to Shimura varieties of type $\mathrm{U}_{1,n-1}$.

Since some of the results in this article have already been proved elsewhere and we also rely on part of these results, here we list the results we are using in this work and compare the results we obtain with others.

- Our work relies on Arthur [4] and Mok [61] for the description of the discrete spectrum $L^2_{\mathrm{disc}}(G)$ for quasi-split symplectic and unitary groups. However, certain key statements such as the twisted weighted fundamental lemma and the local intertwining relation ("LIR" for short) are not proved in [4, 61]. LIR is now established by the joint work of Atobe, Gan, Ichino, Kaletha, Mínguez and Shin [10].
- Our results for unitary groups are independent of [39]. We obtain the same results as [39] on AMF for the generic part of the automorphic discrete spectrum of pure inner forms of unitary groups. However, we also obtain AMF for the full automorphic discrete spectrum of unitary groups with F-rank ≤ 1, which is not covered by [39].
- Our results partially overlap with the work of Taïbi [73], but our proof is independent of his.

We now give a summary of each section. In the first part of this paper (Sections 2–6), we follow the structure of Gan–Ichino [22].

- In Section 2, we first recall some basic notions and results for even orthogonal and unitary groups, and then formulate two of our main theorems. The first (Theorem 2.1) concerns the existence of weak transfers to certain general linear groups; it is in full generality and has no restrictive conditions. The second (Theorem 2.6) is AMF for the generic part of the automorphic discrete spectrum of even orthogonal groups or unitary groups. For the even orthogonal group case, this result is new; for the unitary group, we provide an alternative approach to [39].
- In Section 3, after recalling some basic notions on theta correspondence, we review the endoscopic classification for quasi-split groups, which will serve as the input of our later proofs. We also highlight several works due to Howe and J.-S. Li. It is their works that suggested the possibility of transferring AMF.
- In Section 4, we prove Theorem 2.1. The proof is actually the same as that of [22, Theorem A]: combining the local unramified calculations and some results on partial L-functions, one can show that the (abstract) theta lift realizes certain functoriality. Then the desired theorem for our target group G follows from the same result for the auxiliary quasi-split group H.
- Section 5 is the core of this paper. In this section, we first recall Gan–Ichino's observation, and then illustrate how to use their key equality to transfer AMF from the auxiliary quasi-split group H to G. Although Gan–Ichino's observation is only for generic NEC, we shall work in a general setting. Let ψ be an elliptic A-parameter for G. Locally, we define certain packets $\Pi^{\theta}_{\psi_v}(G_v)$ of G at each place v of F using the theta lift between (G, H). Globally, we "glue" these local packets using the canonical sign character ϵ_ψ defined by Arthur (Section 5.3), and define the global packet

$\Pi_{\psi}^{\theta}(G, \epsilon_{\psi})$. These local and global packets are possible candidates for (conjectural) A-packets for our target group G, and we call them "theta packets" to distinguish various notions of packets. By the key equality of Gan–Ichino, when $\psi = \phi$ is generic, $L_{\phi}^{2}(G)$ will decompose according to the associated global theta packet $\Pi_{\phi}^{\theta}(G, \epsilon_{\phi})$ (Proposition 5.6).

- Section 6 is the most technical section in this paper. In this section, we study those theta packets where $\psi = \phi$ is generic. In this case, we prove that the local theta packets $\Pi_{\phi_v}^{\theta}(G_v)$ are equal to the local L-packets $\Pi_{\phi_v}^{L}(G_v)$. The proof of this local comparison result is also the main difference from [22]. Gan–Ichino proved similar local comparison results mainly using the global method. Since they assumed AMF for both quasi-split and non-quasi-split special odd orthogonal groups, they could use theta lifts to two Witt towers to obtain the desired information. In our case, we only assume AMF holds for symplectic groups and quasi-split unitary groups, so essentially we only have one Witt tower to do theta lifts and cannot simply apply Gan–Ichino's arguments. We overcome this difficulty by combining local and global arguments. Firstly we use Prasad's conjecture and the induction principle to prove a comparison result for a large class of parameters and representations. With these special cases at hand, we then appeal to the global method to prove the comparison result for the remaining cases.

At this point, we will have already completed the proofs of the two theorems stated in Section 2. The second part of this paper (Section 7) is devoted to generalizing some results to the non-generic case. This part is short but perhaps more novel than the first part.

- In Section 7 we try to enhance Gan–Ichino's method by using some other observations. As suggested above, as long as one has the multiplicity preservation

$$m_{\mathrm{disc}}(\pi) = m_{\mathrm{disc}}(\theta^{\mathrm{abs}}(\pi)),$$

one can play the same game as in Section 5.3. We venture to conjecture that all representations in global theta packets only have square-integrable automorphic realizations (Conjecture 7.1). A trivial but noteworthy observation is that when G is anisotropic, all automorphic forms on G are automatically cuspidal. This implies that our conjecture holds in this case. Using the square-integrability criterion, we also prove the conjecture when G is of F-rank 1. With this conjecture at hand, the desired multiplicity preservation is then granted. Thus, we get a description of the full automorphic discrete spectrum of G of F-rank ≤ 1 (Theorem 7.7).

In Appendix A, we define local Langlands correspondence for real full even orthogonal groups and discuss the so-called Prasad's conjecture in this case. In Appendix B, we prove some result on the irreducibility of a certain parabolic induction. In Appendices C and D, we prove some results on irreducible self-dual or conjugate self-dual Galois representations. Finally, in Appendix E, we discuss the existence of certain number fields. These appendices supplement the main results of the paper.

We end up this introduction with a remark on a companion work. A disadvantage of Gan–Ichino's method is that, a priori, AMF for G we get might depend on the auxiliary group H. One would also like to prove that AMF is actually independent of the choice of H. By the local-global structure of AMF, the problem can be reduced to studying the local theta packets $\Pi^{\theta}_{\psi_v}(G_v)$ of G at each place v. It is predicted by Adams' conjecture [1, Section 4] that the theta packets $\Pi^{\theta}_{\psi_v}(G_v)$ should be exactly the (conjectural) local A-packets $\Pi^{A}_{\psi_v}(G_v)$ of G_v. In [12], we prove that these packets $\Pi^{\theta}_{\psi_v}(G_v)$ are independent of the choice of H, and compare them with the local A-packets $\Pi^{A}_{\psi_v}(G_v)$ when G_v is quasi-split. In fact, a large part of this has been studied by Mœglin [53]; see Remark 7.9.

2. Statement of results in the generic case

In this section, we formulate two of our main results. Let F be a local or global field of characterisitic zero, and E either F itself or a quadratic field extension of F. Let

$$c = \begin{cases} \text{the identity of } F & \text{if } E = F, \\ \text{the non-trivial element in } \operatorname{Gal}(E/F) & \text{if } [E:F] = 2. \end{cases}$$

For convenience, we denote by C_F (resp. C_E) the multiplicative group $F^{\times}$ (resp. $E^{\times}$) or $F^{\times}\backslash\mathbb{A}^{\times}$ (resp. $E^{\times}\backslash\mathbb{A}_E^{\times}$), depending on whether F is local or global. When $[E:F] = 2$, we denote by $\omega_{E/F}$ the quadratic character of C_F by class field theory. Let $V = V_{(n)}$ be a finite-dimensional vector space over E equipped with a non-degenerate Hermitian c-sesquilinear form

$$\langle \cdot, \cdot \rangle_V : V \times V \to E.$$

We consider the following three cases:

$$\begin{cases} \text{Case O:} & E = F \text{ and } \dim V = 2n, \\ \text{Case U}_0\text{:} & [E:F] = 2 \text{ and } \dim V = 2n, \\ \text{Case U}_1\text{:} & [E:F] = 2 \text{ and } \dim V = 2n+1, \end{cases}$$

where $n \geq 0$ is an integer. Sometimes, when we want to deal with Cases U_0 and U_1 at the same time, we shall simply write "Case U". Let $G = G(V)$ be the group of elements g in $\mathrm{GL}(V)$ such that

$$\langle gv, gw \rangle_V = \langle v, w \rangle_V \quad \text{for } v, w \in V.$$

If $\dim V = 0$, we interpret $G = G(V)$ and its pure inner form as the trivial group. Now assume that $\dim V > 0$. Let $\operatorname{disc} V = (-1)^n \cdot \det(V)$ be the discriminant of V. In Case O, we let

$$\chi_V : C_F \to \mathbb{C}^{\times} \tag{2.1}$$

be the quadratic character associated to $\operatorname{disc} V$ by class field theory, and $\epsilon(V)$ be the (normalized) Hasse–Witt invariant of V [68, pp. 80–81]. In Case U, we define the sign $\epsilon(V) = \omega_{E/F}(\operatorname{disc} V)$.

Sometimes we also need to consider pure inner forms of $G = G(V)$. It is well known that all pure inner forms of G arise in the form $G' = G(V')$ for some space V'. We briefly describe the classification of V' in both local and global situations. If F is a local field, then all these spaces V' are classified by some invariants.

When F is non-Archimedean:

- In Case O, the V' are orthogonal spaces of the same dimension and discriminant as V. There are at most two such spaces, distinguished by their (normalized) Hasse–Witt invariant $\epsilon(V)$. We shall denote by V^+ the one with Hasse–Witt invariant $+1$ (which always exists), and by V^- the one with Hasse–Witt invariant -1 (which exists unless $n = 1$ and χ_V is trivial). Since V^+ has the maximal possible Witt index, V^+ must be isometric to
$$V^+ \simeq V_{(d,c)} + \mathcal{H}^{n-1}$$
for some $d, c \in F^\times$, where
$$V_{(d,c)} = F[X]/(X^2 - d)$$
is a two-dimensional vector space over F equipped with the quadratic form
$$a + bX \mapsto c \cdot (a^2 - b^2 d),$$
and $\mathcal{H}$ is the (orthogonal) hyperbolic plane. We fix such a tuple (d, c) and the isometry, and we shall say that V^+ is *of type* (d, c). Notice that the tuple (d, c) is not unique.
- In Case U, the V' are Hermitian spaces of the same dimension as V. There are exactly two such spaces, distinguished by their sign $\epsilon(V) = \omega_{E/F}(\mathrm{disc}\, V)$. We shall denote by V^+ the one of sign $+1$, and by V^- the one of sign -1.

When F is real:

- In Case O, the space V is determined by its signature (p, q). In this case, the spaces V' are classified by their signatures (p', q') such that
$$p' + q' = 2n \quad \text{and} \quad p' \equiv p \bmod 2.$$
We shall denote by V^+ the one with Hasse–Witt invariant $+1$ and maximal Witt index.
- In Case U, the V' are classified by their signatures (p', q') satisfying $p' + q' = \dim V$. We shall denote by V^+ the one of sign $+1$ and with maximal Witt index.

When F is complex:

- There is only one such space V' (up to isometry) with given dimension, and we shall denote it by V^+.

With these local classifications at hand, we can now describe the classification of V' when F is a global field.

- In Case O, the V' are orthogonal spaces of the same dimension and discriminant as V. Let $d = \mathrm{disc}(V)$. The local-global principle for orthogonal spaces [68, p. 225, Theorem 6.10] implies that, whenever we are given a collection $\{V'_v\}_v$ of local orthogonal spaces over F_v for all places v of F, such that $\dim V'_v = 2n$, $\mathrm{disc}(V'_v) = d_v$, $\epsilon(V'_v) = 1$ for almost all v, and

$$\prod_v \epsilon(V'_v) = 1,$$

there exists a global orthogonal space V' with these localizations. Moreover, these spaces V' are classified by such coherent data $\{V'_v\}_v$.

- In Case U, the V' are Hermitian spaces of the same dimension as V. The local-global principle for Hermitian spaces [68, p. 377, Theorem 6.9] implies that, whenever we are given a collection $\{V'_v\}_v$ of local Hermitian spaces over E_v for all places v of F, such that $\dim V'_v = \dim V$, $\epsilon(V'_v) = 1$ for almost all v, and

$$\prod_v \epsilon(V'_v) = 1,$$

there exists a global Hermitian space V' with these localizations. Moreover, these spaces V' are classified by such coherent data $\{V'_v\}_v$.

Given V and $G = G(V)$, we let V^+ be the space such that for each place v of F, V_v^+ is (isometric to) the space we have defined in local situations, i.e. $(V^+)_v \simeq (V_v)^+$. In all the cases above, $G^* = G(V^+)$ is quasi-split, and we refer to it as the *quasi-split pure inner form* of G.

2.1. Near equivalence classes and Arthur parameters

Let F be a number field. We first describe the decomposition of $L^2_{\mathrm{disc}}(G)$ into near equivalence classes of representations. We say two irreducible representations $\pi = \bigotimes_v \pi_v$ and $\pi' = \bigotimes_v \pi'_v$ of $G(\mathbb{A})$ are *nearly equivalent* if π_v and π'_v are equivalent for almost all places v of F. The decomposition into near equivalence classes will be expressed in terms of elliptic A-parameters. Recall that a (global) *A-parameter* for G is nothing but a formal finite sum

$$\psi = \sum_i \phi_i \boxtimes S_{d_i}, \tag{2.2}$$

where

- ϕ_i is an irreducible (conjugate) self-dual cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$;
- S_{d_i} is the d_i-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$;
- $\sum_i n_i d_i = \dim V$;
- if d_i is odd, then ϕ_i is

$$\begin{cases} \text{orthogonal} & \text{in Case O,} \\ \text{conjugate symplectic} & \text{in Case } \mathrm{U}_0, \\ \text{conjugate orthogonal} & \text{in Case } \mathrm{U}_1; \end{cases}$$

- if d_i is even, then ϕ_i is

$$\begin{cases} \text{symplectic} & \text{in Case O,} \\ \text{conjugate orthogonal} & \text{in Case U}_0, \\ \text{conjugate symplectic} & \text{in Case U}_1; \end{cases}$$

- in Case O, if we denote the central character of ϕ_i by ω_i, then

$$\prod_i \omega_i^{d_i} = \chi_V.$$

Following Arthur, we make the following definitions:

- if there is only one term in the summation (2.2), i.e. $\psi = \phi_1 \boxtimes S_{d_1}$, then we say that ψ is *simple*;
- if $(\phi_i, d_i) \neq (\phi_j, d_j)$ for all $i \neq j$, then ψ is *elliptic*;
- if $d_i = 1$ for all i, then ψ is *generic* (or *tempered*).

Note that an (elliptic) A-parameter for G is also an (elliptic) A-parameter for G^*, and vice versa. We denote the set of all elliptic A-parameters of G by $\Psi_{\text{ell}}(G^*)$. One can formally associate to ψ a free $\mathbb{Z}/2\mathbb{Z}$-module

$$\mathcal{S}_\psi = \prod_i (\mathbb{Z}/2\mathbb{Z}) e_i$$

with a canonical basis $\{e_i\}_i$, where each e_i corresponds to the summand $\phi_i \boxtimes S_{d_i}$. We shall call $\mathcal{S}_\psi$ the *global component group* of ψ.

For each place v of F, one can also define local A-parameters and local component groups for G_v. First suppose that we are either in Case O, or in Case U and v is not split in E. Then there is a unique place of E above v, which we shall still denote by v. Let

$$L_{E_v} = \begin{cases} \text{the Weil group of } E_v & \text{if } v \text{ is Archimedean,} \\ \text{the Weil–Deligne group of } E_v & \text{if } v \text{ is non-Archimedean.} \end{cases}$$

A local A-parameter for G_v is a representation

$$\psi_v = \sum_i m_i \cdot \phi_{i,v} \boxtimes S_{d_i} \tag{2.3}$$

of $L_{E_v} \times \mathrm{SL}_2(\mathbb{C})$, where

- $\phi_{i,v} \boxtimes S_{d_i}$ are pairwise inequivalent irreducible (conjugate) self-dual representations of $L_{E_v} \times \mathrm{SL}_2(\mathbb{C})$ with multiplicities m_i;
- as a representation of $L_{E_v} \times \mathrm{SL}_2(\mathbb{C})$, ψ_v is

$$\begin{cases} \text{orthogonal} & \text{in Case O,} \\ \text{conjugate symplectic} & \text{in Case U}_0, \\ \text{conjugate orthogonal} & \text{in Case U}_1; \end{cases}$$

- in Case O, $\det(\psi_v) = \chi_{V,v}$.

By [19, Section 8], the component group $\mathcal{S}_{\psi_v}$ has an explicit description of the form

$$\mathcal{S}_{\psi_v} = \prod_j (\mathbb{Z}/2\mathbb{Z})e_j$$

with a canonical basis $\{e_j\}$, where the product ranges over all j such that $\phi_{j,v} \boxtimes S_{d_j}$ is (conjugate) self-dual of the same parity as ψ_v. For $e = e_{j_1} + \cdots + e_{j_r} \in \mathcal{S}_\psi$, we put

$$\psi_v^e = \phi_{j_1,v} \boxtimes S_{d_{j_1}} + \cdots + \phi_{j_r,v} \boxtimes S_{d_{j_r}}.$$

Next suppose that we are in Case U, and v is split into two places $\{w, \overline{w}\}$ in E. In this case the local A-parameter for G_v can be similarly defined as a formal sum as in (2.3), but now each $\phi_{i,v}$ is an irreducible conjugate self-dual representation of some $\mathrm{GL}_{n_i}(E_v) \simeq \mathrm{GL}_{n_i}(E_w) \times \mathrm{GL}_{n_i}(E_{\overline{w}})$. Indeed, if we identify $F_v \simeq E_w \simeq E_{\overline{w}}$, then the conjugate self-duality of each $\phi_{i,v}$ will imply that

$$\phi_{i,v} \simeq \phi_{i,w} \boxtimes \phi_{i,w}^\vee$$

for some irreducible representation $\phi_{i,w}$ of $\mathrm{GL}_{n_i}(E_w)$. Regarding these $\phi_{i,w}$ as representations of L_{E_w} by using the local Langlands correspondence for general linear groups (see [29, 30, 46, 69]), we get a local A-parameter for $G_v \simeq \mathrm{GL}(V_w)$,

$$\psi_w = \sum_i m_i \cdot \phi_{i,w} \boxtimes S_{d_i},$$

in the usual sense. In this case the component group $\mathcal{S}_{\psi_v}$ is trivial. In both cases, if further $d_i = 1$ for all i, i.e. the restriction of ψ_v to $\mathrm{SL}_2(\mathbb{C})$ is trivial, then we say that ψ_v is an *L-parameter* for G_v. Again following Arthur, we shall use $\Psi(G_v)$ (resp. $\Phi(G_v)$) to denote the set of A-parameters (resp. L-parameters) of G_v with bounded image on the Weil group, and also $\Psi^+(G_v)$ (resp. $\Phi^+(G_v)$) for the set of A-parameters (resp. L-parameters) of G_v.

Now given an elliptic A-parameter $\psi = \sum_i \phi_i \boxtimes S_{d_i}$ of G, let $\psi_v = \sum_i \phi_{i,v} \boxtimes S_{d_i}$ be the localization of ψ at v. Here each $\phi_{i,v}$ is an irreducible representation of $\mathrm{GL}_{n_i}(E_v)$, and we shall also regard it as an L-parameter via LLC for general linear groups. Then by definitions, ψ_v gives rise to an A-parameter for G_v. We associate to ψ_v an L-parameter ϕ_{ψ_v} by the formula

$$\phi_{\psi_v}(w) = \psi_v\left(w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right). \tag{2.4}$$

Our first theorem shows that each NEC has a weak transfer to a certain general linear group.

Theorem 2.1. *There exists a decomposition*

$$L^2_{\mathrm{disc}}(G) = \bigoplus_{\psi \in \Psi_{\mathrm{ell}}(G^*)} L^2_\psi(G),$$

where $L^2_\psi(G)$ is a full near equivalence class of irreducible representations π in $L^2_{\mathrm{disc}}(G)$ such that the L-parameter of π_v is ϕ_{ψ_v} for almost all places v of F.

Note that at this point, $L^2_\psi(G)$ could be zero for some $\psi \in \Psi_{\mathrm{ell}}(G^*)$.

2.2. Local Langlands correspondence

Our next goal is to describe the near equivalence class $L^2_\psi(G)$ when ψ is generic. For this purpose, we need to make use of the (Vogan version) local Langlands correspondence ("LLC" for short), which provides a bijection between irreducible representations of G_v and enhanced L-parameters. Here are some existing results.

When v is Archimedean: LLC was proved in [46] for connected reductive groups. However, since we need to deal with the disconnected group O_{2n} in Case O, we need to extend LLC for special even orthogonal groups to full even orthogonal groups. When F is complex, there is only one orthogonal space of dimension $2n$, and the corresponding even orthogonal group is quasi-split. In this case LLC is already provided by Arthur's results; see the remark below. When F is real, we provide LLC for full even orthogonal groups in Appendix A using theta lifts.

When v is non-Archimedean: LLC was proved in [4, 8] when G is a quasi-split even orthogonal group; in [61] when G is a quasi-split unitary group; and in [39] when G is an inner form of a unitary group. Also, in our previous papers [13, 14], we established LLC for pure inner forms of even orthogonal groups and unitary groups over non-Archimedean fields using theta lifts.

Remark 2.2. In fact, Arthur [4] also established a weak version of LLC for quasi-split special even orthogonal groups over Archimedean fields. According to [8, Theorem 3.10], Arthur's results also implicitly imply LLC for quasi-split full even orthogonal groups, though he did not highlight it. We can show that our extension of LLC for real full even orthogonal groups coincides with Arthur's when the group is quasi-split, by appealing to the global method. We sketch the proof at the end of Appendix A.

Now we briefly recall the above results. Assume F is local for a moment. There is a canonical finite-to-one surjection

$$\mathcal{L} : \bigsqcup_{G'} \mathrm{Irr}(G') \to \Phi^+(G),$$

where the disjoint union is taken over all pure inner forms of G. For each L-parameter ϕ, we denote

$$\Pi^L_\phi(G') = \mathcal{L}^{-1}(\phi) \cap \mathrm{Irr}(G')$$

and we call it the *L-packet* of G' associated to ϕ. There is a canonical bijection (depending on the choice of a Whittaker datum $\mathscr{W}$ of G^*)

$$\mathcal{J}^L_{\mathscr{W}} : \bigsqcup_{G'} \Pi^L_\phi(G') \to \widehat{\mathcal{S}_\phi}, \tag{2.5}$$

where the disjoint union is again taken over all pure inner forms of G. Furthermore, the bijection $\mathcal{J}^L_{\mathscr{W}}$ is compatible with the Kottwitz isomorphism [40, Theorem 1.2], and this property characterizes the image of $\Pi^L_\phi(G')$ under $\mathcal{J}^L_{\mathscr{W}}$. We shall denote by $\pi(\phi, \eta)$ the irreducible representation of some G' with L-parameter ϕ and corresponding to η. We may also regard the L-packet $\Pi_\phi^L(G')$ as a representation of $\mathcal{S}_\phi \times G'$ by letting

$$\Pi_\phi^L(G') = \bigoplus_\pi \mathcal{J}_{\mathscr{W}}^L(\pi) \boxtimes \pi,$$

where the summation on the RHS is over all irreducible representations in $\Pi_\phi^L(G')$. Sometimes we will adopt this point of view.

Remark 2.3. Let κ_ϕ be the character of $\mathcal{S}_\phi$ defined by the formula

$$\kappa_\phi(e) = (-1)^{\dim \phi^e} \tag{2.6}$$

for $e \in \mathcal{S}_\phi$.

(1) In Case O, for any irreducible representation $\pi = \pi(\phi, \eta)$ of G, we have

$$\pi \otimes \det = \pi(\phi, \eta \cdot \kappa_\phi).$$

When F is non-Archimedean, this property is proved for example in [13, Theorem 4.4]. When F is real, see Remark A.2. When F is complex, this follows from the compatibility of LLC for full even orthogonal groups with LLC for special even orthogonal groups [8, Desideratum 3.9(8)].

(2) In Case U_1, if F is non-Archimedean, we can take $V^- = a \cdot V^+$ for some a in $F^\times \backslash \mathrm{Nm}_{E/F}(E^\times)$. Then $G(V^+)$ and $G(V^-)$ are physically equal as subgroups of $\mathrm{GL}(V^+)$. For any irreducible representation $\pi = \pi(\phi, \eta)$ of $G(V^+)$, if we consider it as a representation of $G(V^-)$, then we have

$$\pi = \pi(\phi, \eta \cdot \kappa_\phi).$$

The readers can consult [14, Theorem 2.5.5] for a more detailed discussion of this property.

Remark 2.4. In Case U, we simply use LLC for general linear groups at split places.

2.3. *Multiplicity formula*

We now assume that F is a number field, and $\psi = \phi$ a generic elliptic A-parameter of G, i.e. ϕ is a multiplicity-free sum

$$\phi = \sum_i \phi_i$$

of irreducible (conjugate) self-dual cuspidal automorphic representations ϕ_i of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$ with appropriate parity. Fix a global Whittaker datum $\mathscr{W}$ of G^*. At each place v of F, we have a localization map

$$\mathcal{S}_\phi \to \mathcal{S}_{\phi_v}.$$

We define a global packet

$$\Pi_\phi(G) = \bigotimes_v{}' \, \Pi^L_{\phi_v}(G_v)$$
$$= \left\{ \pi = \bigotimes_v{}' \pi_v \;\middle|\; \pi_v \in \Pi^L_{\phi_v}(G_v),\ \pi_v \text{ unramified with } L\text{-parameter } \phi_v \text{ for almost all } v \right\}.$$

We then have a map

$$\mathcal{J}_{\mathscr{W}} : \Pi_\phi(G) \to \widehat{\mathcal{S}_\phi}, \quad \pi \mapsto \mathcal{J}_{\mathscr{W}}(\pi), \quad \mathcal{J}_{\mathscr{W}}(\pi)(x) := \prod_v \mathcal{J}^L_{\mathscr{W}_v}(\pi_v)(x_v),$$

where $x \in \mathcal{S}_\phi$ and x_v is the localization of x at v.

Remark 2.5. According to the main local theorems of [4, 61], if G_v and π_v are both unramified, then $\mathcal{J}^L_{\mathscr{W}_v}(\pi_v)$ is the trivial character 1. Hence $\mathcal{J}_{\mathscr{W}}$ is well-defined.

Let $\epsilon_\phi = 1$ be the trivial character of $\mathcal{S}_\phi$. We put

$$\Pi_\phi(G, \epsilon_\phi) = \{\pi \in \Pi_\phi(G) \mid \mathcal{J}_{\mathscr{W}}(\pi) = \epsilon_\phi\}.$$

Our second theorem is the following.

Theorem 2.6. *Let ϕ be a generic elliptic A-parameter for G. Then we have the decomposition*

$$L^2_\phi(G) = \bigoplus_{\pi \in \Pi_\phi(G, \epsilon_\phi)} \pi.$$

In particular, $L^2_\phi(G)$ is multiplicity-free.

Remark 2.7. (1) Suppose that $V = V^+$. Then $G = G^*$ is quasi-split. Hence, by the results of [4, 8] (for Case O), and [61] (for Case U), Theorem 2.6 holds for G. Our results generalize these works to the case of (not necessarily quasi-split) pure inner forms.

(2) We should mention that in [4], Arthur only formulated and proved his results for quasi-split special even orthogonal groups $\mathrm{SO}(V^+)$. His results do not distinguish between a square-integrable automorphic representation π and its twist by the outer automorphism corresponding to an element of $\mathrm{O}(V) \setminus \mathrm{SO}(V)$. Therefore, in AMF for $\mathrm{SO}(V^+)$, some multiplicity 2 phenomenon occurs. In [8], Atobe–Gan formulated AMF for quasi-split even orthogonal groups $\mathrm{O}(V^+)$ precisely and explicated that Arthur's results in [4] already implied Theorem 2.6 for $\mathrm{O}(V^+)$.

2.4. A special case

We first deal with a special case of Theorem 2.6, which will be used in later proofs. In this subsection, suppose that we are in Case U_1, and F is a totally imaginary number field.

In this case, we take $a \in F^\times$, so that a is in the same $\mathrm{Nm}_{E/F}(E^\times)$-orbit as $\mathrm{disc}(V)$. Then by [68, Corollary 6.6], we have $V^+ \simeq a \cdot V$. This implies that $G \simeq G^*$ as abstract groups. Therefore

$$L^2_\psi(G) = L^2_\psi(G^*)$$

for any elliptic A-parameter ψ of G. However, to establish Theorem 2.6 for G, we need to consider G not only as an abstract group, but also as a pure inner form of G^*, i.e. the Hermitian form V should also be taken into consideration. When $\psi = \phi$ is generic, we need to distinguish $\Pi^L_{\phi_v}(G_v)$ and $\Pi^L_{\phi_v}(G^*_v)$ at some places. To be more precise, let v be a place of F. There are two cases:

- If $a_v \in \mathrm{Nm}_{E_v/F_v}(E_v^\times)$, then $V_v \simeq V_v^+$, and hence $\Pi^L_{\phi_v}(G_v) = \Pi^L_{\phi_v}(G^*_v)$ as representations of $\mathcal{S}_{\phi_v} \times G_v$. Note that all complex places satisfy this condition.
- If $a_v \notin \mathrm{Nm}_{E_v/F_v}(E_v^\times)$, then $V_v \not\simeq V_v^+$. According to Remark 2.3 (2), as representations of $\mathcal{S}_{\phi_v} \times G_v$, we have
$$\Pi^L_{\phi_v}(G_v) = \Pi^L_{\phi_v}(G^*_v) \otimes \kappa_{\phi_v}.$$
Note that by the local-global principle for Hermitian forms, the number of places v satisfying this condition is even.

Therefore, it is not hard to check that

$$\Pi_\phi(G, \epsilon_\phi) = \Pi_\phi(G^*, \epsilon_\phi)$$

as sets of representations of $G(\mathbb{A})$. We deduce the following.

Proposition 2.8. *Suppose we are in Case* U_1*, and F is a totally imaginary number field. Then the conclusion of Theorem* 2.6 *holds.*

This proposition will be used in the later proof of Theorem 2.6.

3. Preliminaries

In this section, we recall some preliminaries we will need in the proof of our main theorems.

3.1. Theta lifts

Fix a trace zero element $\delta \in E^\times$. Let $W = W_{(r)}$ be a vector space over E which is

$$\begin{cases} 2r\text{-dimensional} & \text{in Case O}, \\ (2r+1)\text{-dimensional} & \text{in Case } \mathrm{U}_0, \\ (2r+2)\text{-dimensional} & \text{in Case } \mathrm{U}_1, \end{cases} \tag{3.1}$$

and equipped with a non-degenerate skew-Hermitian c-sesquilinear form

$$\langle \cdot, \cdot \rangle_W : W \times W \to E,$$

such that W is split (in Case U_0 we require that the anisotropic kernel of W is the one-dimensional skew-Hermitian space represented by δ). Let $H = H(W)$ be the group of elements h in $\mathrm{GL}(W)$ which preserve the form:

$$\langle hv, hw \rangle_W = \langle v, w \rangle_W \quad \text{for } v, w \in W.$$

Note that H is quasi-split. The pair (G, H) is then an example of a reductive dual-pair. We fix a pair of characters (χ_V, χ_W) of C_E as follows:

$$\chi_V = \begin{cases} \text{the quadratic character associated to } V & \text{in Case O,} \\ \text{a character of } C_E \text{ such that } \chi_V|_{C_F} = \omega_{E/F}^{\dim V} & \text{in Case U,} \end{cases}$$

$$\chi_W = \begin{cases} \text{the trivial character of } F^\times & \text{in Case O,} \\ \text{a character of } C_E \text{ such that } \chi_W|_{C_F} = \omega_{E/F}^{\dim W} & \text{in Case U.} \end{cases}$$

Assume F is local for a moment. With respect to a non-trivial additive character ψ_F of F and the auxiliary data (χ_V, χ_W), one can define the Weil representation ω of $G \times H$. For any irreducible representation π of G, the maximal π-isotypic quotient of ω is of the form

$$\pi \boxtimes \Theta(\pi)$$

for some smooth representation $\Theta(\pi)$ of H of finite length. Then by Howe duality [26, 27, 76], the maximal semisimple quotient $\theta(\pi)$ of $\Theta(\pi)$ is either zero or irreducible. Similarly, for any irreducible representation σ of H, we can define $\Theta(\sigma)$ and $\theta(\sigma)$.

Suppose next that F is a number field. Fix a non-trivial additive character ψ_F of $F\backslash\mathbb{A}$, and also characters (χ_V, χ_W). We define an abstract irreducible representation of π of $G(\mathbb{A})$ as a tensor product of irreducible representations π_v of G_v, which is at almost all places unramified. We write $\pi = \bigotimes_v \pi_v$. At each place v of F, we can form the local theta lift $\theta(\pi_v)$ with respect to $(\psi_{F,v}, \chi_{V,v}, \chi_{W,v})$. Assume that they are all non-vanishing. Then $\theta(\pi_v)$ is irreducible for all v and is unramified for almost all v. Hence, we may define an abstract irreducible representation

$$\theta^{\mathrm{abs}}(\pi) = \bigotimes_v \theta(\pi_v)$$

of $H(\mathbb{A})$. We call $\theta^{\mathrm{abs}}(\pi)$ the *abstract theta lift* of π to $H(\mathbb{A})$. On the other hand, if π is an irreducible cuspidal automorphic representation of $G(\mathbb{A})$, then we can define its *global theta lift* $\Theta^{\mathrm{aut}}(\pi)$ as the subspace of $\mathcal{A}(H)$ spanned by all automorphic forms of the form

$$\theta(f, \varphi)(h) = \int_{G(F)\backslash G(\mathbb{A})} \theta(f)(g, h)\overline{\varphi(g)}\, dg$$

for $f \in \omega$ and $\varphi \in \pi$. Here ω is the Weil representation of $G(\mathbb{A}) \times H(\mathbb{A})$ and $\theta(f)$ is the theta function associated to f. According to [43], if $\Theta^{\mathrm{aut}}(\pi)$ is non-zero and contained in $\mathcal{A}^2(H)$, then $\Theta^{\mathrm{aut}}(\pi)$ is irreducible and

$$\Theta^{\mathrm{aut}}(\pi) \simeq \theta^{\mathrm{abs}}(\pi).$$

3.2. *Unitary representations of low rank*

The notion of rank for unitary representations was first introduced by Howe [31] for symplectic groups and was extended to other classical groups by J.-S. Li [47]. Following [47], we say that an irreducible unitary representation of $H = H(W_{(r)})$ is *of low rank* if its rank is less than r. Such representations can be described using theta lifts as follows.

Assume $\dim V < r$, so that the reductive dual pair (G, H) is in the stable range (see [48, Definition 5.1]). If F is local, then for any irreducible representation π of G, its theta lift $\theta(\pi)$ to H is non-vanishing. Moreover, if π is unitary, then by [48], so is $\theta(\pi)$. In [47], J.-S. Li showed that this theta lift provides a bijection

$$\bigsqcup_{V'} \mathrm{Irr}_{\mathrm{unit}}\, G(V') \times \{\text{Characters of } H\}$$
$$\updownarrow$$
$$\{\text{Irreducible unitary representations of } H \text{ of rank } \dim V\}.$$

where the disjoint union is taken over all isometry classes of vector spaces V' over E of the same dimension as V, and equipped with a non-degenerate Hermitian c-sesquilinear form. The map sends a pair (π, χ) in the first set to a representation $\theta(\pi) \otimes \chi$ of H. Note that in Case O, χ is always trivial since H is simple.

This result has a global analog. Let F be a number field and $\sigma = \bigotimes_v \sigma_v$ an irreducible unitary representation of $H(\mathbb{A})$ that occurs as a subrepresentation of $\mathcal{A}(H)$. Then, by [48, Lemma 3.2], the following are equivalent:

- σ is of rank $\dim V$;
- σ_v is of rank $\dim V$ for all v;
- σ_v is of rank $\dim V$ for some v.

Suppose that σ satisfies the above equivalent conditions. Then [49, Proposition 5.7] asserts that there exists some $G = G(V')$ with $\dim V' = \dim V$, together with an abstract representation $\pi = \bigotimes_v \pi_v$ of $G(\mathbb{A})$, and an automorphic character χ of $H(\mathbb{A})$, such that

$$\sigma \simeq \theta^{\mathrm{abs}}(\pi) \otimes \chi.$$

3.3. Some inequalities

Finally, we recall a result of J.-S. Li, which allows us to lift square-integrable automorphic representations of $G(\mathbb{A})$ to $H(\mathbb{A})$. For any irreducible representation π of $G(\mathbb{A})$, we define its multiplicities $m(\pi)$ and $m_{\mathrm{disc}}(\pi)$ by

$$m(\pi) = \dim \mathrm{Hom}_{G(\mathbb{A})}(\pi, \mathcal{A}(G)),$$
$$m_{\mathrm{disc}}(\pi) = \dim \mathrm{Hom}_{G(\mathbb{A})}(\pi, \mathcal{A}^2(G)).$$

Obviously, $m_{\mathrm{disc}}(\pi) \le m(\pi)$. Similarly, if σ is an irreducible representation of $H(\mathbb{A})$, we have its multiplicities $m(\sigma)$ and $m_{\mathrm{disc}}(\sigma)$. By [49, Theorem A], we have the following.

Theorem 3.1. *Assume that* $\dim V < r$. *Let* π *be an irreducible unitary representation of* $G(\mathbb{A})$ *and* $\theta^{\mathrm{abs}}(\pi)$ *its abstract theta lift to* $H(\mathbb{A})$. *Then*

$$m_{\mathrm{disc}}(\pi) \le m_{\mathrm{disc}}(\theta^{\mathrm{abs}}(\pi)) \le m(\theta^{\mathrm{abs}}(\pi)) \le m(\pi).$$

3.4. *Review of results for quasi-split groups*

We review some results in [4, 61].

First, let F be a local field. Recall that by our definition, $H = H(W_{(r)})$ is either a symplectic group or a quasi-split unitary group. Similarly to Section 2.1, one can also define the local A-parameter and the component group for H. Following Arthur, we use $\Psi(H)$ to denote the set of local A-parameters for H with bounded images on the Weil group. Let $\psi \in \Psi(H)$ be an A-parameter for H. If we write

$$\psi = \sum_i m_i \cdot \phi_i \boxtimes S_{d_i}$$

with pairwise inequivalent irreducible subrepresentations $\phi_i \boxtimes S_{d_i}$ of ψ, then by [19, Section 8], we have

$$\mathcal{S}_\psi = \prod_j (\mathbb{Z}/2\mathbb{Z})e_j \quad \text{and} \quad \overline{\mathcal{S}_\psi} = \mathcal{S}_\psi/\langle z_\psi \rangle,$$

where the product is taken over all j such that $\phi_j \boxtimes S_{d_j}$ is (conjugate) self-dual of the same parity as ψ, and $z_\psi = \sum_j m_j \cdot e_j$. We shall call $\mathcal{S}_\psi$ or $\overline{\mathcal{S}_\psi}$ the *component group* associated to ψ. To such a ψ, Arthur [4] and Mok [61] assigned a finite multi-set $\Pi_\psi(H)$ of irreducible unitary representations of H, together with a canonical map (after fixing a Whittaker datum $\mathscr{W}'$ of H)

$$\mathcal{J}_{\mathscr{W}'} : \Pi_\psi(H) \to \widehat{\mathcal{S}_\psi}.$$

They proved that $\Pi_\psi(H)$ and the assignment $\sigma \mapsto \mathcal{J}_{\mathscr{W}'}(\sigma)$ have the following two properties:

(1) If both H and σ are unramified, then $\mathcal{J}_{\mathscr{W}'}(\sigma) = 1$.

(2) Let ϕ_ψ be the L-parameter associated to ψ as in (2.4). Then the A-packet $\Pi_\psi(H)$ contains the L-packet $\Pi^L_{\phi_\psi}(H)$ as a subset. We have a commutative diagram

$$\begin{array}{ccc} \Pi^L_{\phi_\psi}(H) & \xrightarrow{\mathcal{J}^L_{\mathscr{W}'}} & \widehat{\mathcal{S}_{\phi_\psi}} \\ \big\downarrow & & \big\downarrow \\ \Pi_\psi(H) & \xrightarrow{\mathcal{J}_{\mathscr{W}'}} & \widehat{\mathcal{S}_\psi} \end{array}$$

where the left vertical arrow is the natural inclusion and the right one is induced by the surjection $\mathcal{S}_\psi \to \mathcal{S}_{\phi_\psi}$. Moreover, if $\psi = \phi$ is a tempered L-parameter, then $\Pi_\psi(H)$ is multiplicity-free and coincides with the L-packet $\Pi^L_\phi(H)$.

Besides these two properties, the packet $\Pi_\psi(H)$ also satisfies the so-called "endoscopic character identity", but we will not use this fact in this paper.

Let $\Psi^+(H)$ be the set of local A-parameters, whose elements do not necessarily have bounded images on the Weil group. Due to the potential failure of the generalized Ramanujan conjecture for general linear groups, for the purpose of the global classification, Arthur and Mok also defined the local A-packet $\Pi_\psi(H)$ and the canonical map $\mathcal{J}_{\mathcal{W}'}$ for $\psi \in \Psi^+(H)$ by using parabolic inductions. These local A-packets $\Pi_\psi(H)$ and maps $\mathcal{J}_{\mathcal{W}'}$ also have the two properties listed above. In Section 6.2, we will discuss this issue in more detail.

Now we turn to the global classification, so F is now a number field. Let ψ be an elliptic A-parameter for H and write

$$\psi = \sum_i \phi_i \boxtimes S_{d_i}$$

as in (2.2). Let

$$\mathcal{S}_\psi = \prod_i (\mathbb{Z}/2\mathbb{Z})e_i \quad \text{and} \quad \overline{\mathcal{S}_\psi} = \mathcal{S}_\psi/\langle z_\psi \rangle$$

be the global component groups of ϕ, where $z_\psi = \sum_i e_i$. For each place v of F, the localization ψ_v of ψ at v gives rise to a local A-parameter for H_v. We also have a localization map

$$\mathcal{S}_\psi \to \mathcal{S}_{\psi_v} \quad (\text{or} \quad \overline{\mathcal{S}_\psi} \to \overline{\mathcal{S}_{\psi_v}}).$$

Fix a global Whittaker datum $\mathcal{W}'$ of H. Given an elliptic A-parameter ψ, we define the global packet $\Pi_\psi(H)$ associated to ψ as the restricted tensor product of the local A-packets $\Pi_{\psi_v}(H_v)$:

$$\Pi_\psi(H) = {\bigotimes_v}' \Pi_{\psi_v}(H_v)$$
$$= \Big\{\sigma = {\bigotimes_v}' \sigma_v \;\Big|\; \sigma_v \in \Pi_{\psi_v}(H_v),\, \sigma_v \text{ unramified with } L\text{-parameter } \phi_{\psi_v} \text{ for almost all } v\Big\}.$$

We then have a map

$$\mathcal{J}_{\mathcal{W}'} : \Pi_\psi(H) \to \widehat{\overline{\mathcal{S}_\psi}}, \quad \sigma \mapsto \mathcal{J}_{\mathcal{W}'}(\sigma), \quad \mathcal{J}_{\mathcal{W}'}(\sigma)(x) := \prod_v \mathcal{J}_{\mathcal{W}'_v}(\sigma_v)(x_v),$$

where $x \in \mathcal{S}_\psi$ and x_v is the localization of x at v.

Remark 3.2. According to the main local theorems of [4, 61], if H_v and σ_v are both unramified, then $\mathcal{J}_{\mathcal{W}'_v}(\sigma_v)$ is the trivial character 1. Hence $\mathcal{J}_{\mathcal{W}'}$ is well-defined.

Let $\epsilon_\psi \in \widehat{\mathcal{S}_\psi}$ be the canonical sign character defined by Arthur [4, p. 47] and Mok [61, p. 29]. Put

$$\Pi_\psi(H, \epsilon_\psi) = \{\sigma \in \Pi_\psi(H) \mid \mathcal{J}_{\mathcal{W}'}(\sigma) = \epsilon_\psi\}.$$

Then the main global theorems in [4, 61] assert the following.

Theorem 3.3. *There exists a decomposition*

$$L^2_{\text{disc}}(H) = \bigoplus_{\psi \in \Psi_{\text{ell}}(H)} L^2_\psi(H),$$

where $L^2_\psi(H)$ is a full near equivalence class of irreducible representations σ in $L^2_{\text{disc}}(H)$ such that the L-parameter of σ_v is ϕ_{ψ_v} for almost all places v of F.

Theorem 3.4. *Let ψ be an elliptic A-parameter for H. Then we have the decomposition*

$$L^2_\psi(H) = \bigoplus_{\sigma \in \Pi_\psi(H,\epsilon_\psi)} \sigma.$$

Our Theorem 2.1 is an analog of Theorem 3.3 for the group G, and our Theorem 2.6 is an analog of Theorem 3.4 for generic elliptic A-parameters of G.

3.5. Remarks on Whittaker data

In the rest of this paper, we will prove Theorems 2.1 and 2.6 by using the theta lift between G and H. Since the local and global classifications of both G and H depend on the choices of Whittaker data, and the theta lift also depends on the choice of an additive character, we need to specify the data we are using and their relation. Here we are following [6, Conjecture 4.4, 4.6] and [21, Sections 4.4, 4.6], as we will use their results in later proofs (see Theorem 6.11). We now briefly describe the way we choose these data.

When F is a local field, we first fix a non-trivial additive character ψ_F of F. We also need some auxiliary data in different cases:

- in Case O, we fix an isometry

$$V^+ \simeq V_{(d,c)} + \mathcal{H}^{n-1}$$

  for some $d, c \in F^\times$, as described at the beginning of Section 2;
- in Case U, we fix a trace zero element $\delta \in E^\times$.

We define a Whittaker datum $\mathscr{W} = \mathscr{W}_{\psi_F}$ of $G^* = G(V^+)$ as follows.

Case O*:* In this case, recall that we have fixed an isometry

$$V^+ \simeq V_{(d,c)} + \mathcal{H}^{n-1},$$

where $\mathcal{H}$ is the (orthogonal) hyperbolic plane. We denote by e, e' the images of $1, X \in F[X]$ in $V_{(d,c)}$ respectively. For $1 \leq k \leq n-1$, we define the k-th hyperbolic plane $\mathcal{H} = Fv_k + Fv_k^*$ with

$$\langle v_k, v_k\rangle_V = \langle v_k^*, v_k^*\rangle_V = 0 \quad \text{and} \quad \langle v_k, v_k^*\rangle_V = 1,$$

and we set

$$X_k = Fv_1 + \cdots + Fv_k \quad \text{and} \quad X_k^* = Fv_1^* + \cdots + Fv_k^*.$$

Let $B = TU$ be the F-rational Borel subgroup of G^* stabilizing the complete flag

$$X_1 \subset \cdots \subset X_{n-1},$$

where T is the F-rational torus stabilizing the lines Fv_i for $1 \leq k \leq n-1$. We define a generic character μ_c of U by

$$\mu_c(u) = \psi_F\big(\langle uv_2, v_1^*\rangle_V + \cdots + \langle uv_{n-1}, v_{n-2}^*\rangle_V + \langle ue, v_{n-1}^*\rangle_V\big). \tag{3.2}$$

Let $\mathscr{W} = (V^+, B, T, \mu_c)$. Here the notion of Whittaker datum is slightly different from the usual one: the datum is not only associated to the group G^*, but rather the orthogonal space V^+ is part of the datum. Moreover, this datum $\mathscr{W}$ is indeed independent of the choice of the additive character ψ, and only depends on the choice of $c \in F^\times$. The readers may consult [8, Section 2.2] for a discussion of this.

Case U_0*:* In this case, the Witt index of V^+ is n. We choose a basis $\{v_i, v_i^* \mid i = 1, \ldots, n\}$ of V^+ such that

$$\langle v_i, v_j \rangle_V = \langle v_i^*, v_j^* \rangle_V = 0 \quad \text{and} \quad \langle v_i, v_j^* \rangle_V = \delta_{i,j}$$

for $1 \le i, j \le n$. We set

$$X_k = Ev_1 + \cdots + Ev_k \quad \text{and} \quad X_k^* = Ev_1^* + \cdots + Ev_k^*$$

for $1 \le i, j \le n$. We denote by $B = TU$ the F-rational Borel subgroup of G^* stabilizing the complete flag

$$X_1 \subset \cdots \subset X_n,$$

where T is the F-rational torus stabilizing the lines Ev_i for $1 \le k \le n$. We define a generic character μ of U by

$$\mu(u) = \psi_F\big(\tfrac{1}{2}\operatorname{Tr}_{E/F}\big(\delta \cdot (\langle uv_2, v_1^* \rangle_V + \cdots + \langle uv_n, v_{n-1}^* \rangle_V + \langle uv_n^*, v_n^* \rangle_V)\big)\big).$$

Let $\mathscr{W} = (U, \mu)$.

Case U_1*:* In this case, there is a unique Whittaker datum $\mathscr{W}$ of G^*.

Then we define a Whittaker datum $\mathscr{W}' = \mathscr{W}'_{\psi_F}$ of $H = H(W)$ as follows.

Case O*:* In this case W is the $2r$-dimensional symplectic space. We choose a basis $\{w_i, w_i^* \mid i = 1, \ldots, r\}$ of W such that

$$\langle w_i, w_j \rangle_W = \langle w_i^*, w_j^* \rangle_W = 0 \quad \text{and} \quad \langle w_i, w_j^* \rangle_W = \delta_{i,j}$$

for $1 \le i, j \le r$. We set

$$Y_k = Fw_1 + \cdots + Fw_k \quad \text{and} \quad Y_k^* = Fw_1^* + \cdots + Fw_k^*$$

for $1 \le i, j \le r$. We denote by $B' = T'U'$ the F-rational Borel subgroup of H stabilizing the complete flag

$$Y_1 \subset \cdots \subset Y_r,$$

where T' is the F-rational torus stabilizing the lines Fw_i for $1 \le k \le r$. We define a generic character μ' of U' by

$$\mu'(u) = \psi_F\big(\langle uw_2, w_1^* \rangle_W + \cdots + \langle uw_r, w_{r-1}^* \rangle_W + c\langle uw_r^*, w_r^* \rangle_W\big). \tag{3.3}$$

Let $\mathscr{W}' = (U', \mu')$.

Case U_0*:* In this case W is a split $(2r+1)$-dimensional skew-Hermitian space. There is a unique Whittaker datum $\mathscr{W}'$ of H.

Case U_1*:* In this case W is a split $2r$-dimensional skew-Hermitian space. We choose a basis $\{w_i, w_i^* \mid i = 1, \ldots, r\}$ of W such that

$$\langle w_i, w_j \rangle_W = \langle w_i^*, w_j^* \rangle_W = 0 \quad \text{and} \quad \langle w_i, w_j^* \rangle_W = \delta_{i,j}$$

for $1 \leq i, j \leq r$. We set

$$Y_k = Fw_1 + \cdots + Fw_k \quad \text{and} \quad Y_k^* = Fw_1^* + \cdots + Fw_k^*$$

for $1 \leq i, j \leq r$. We denote by $B' = T'U'$ the F-rational Borel subgroup of H stabilizing the complete flag $Y_1 \subset \cdots \subset Y_r$, where T' is the F-rational torus stabilizing the lines Fw_i for $1 \leq k \leq r$. We define a generic character μ' of U' by

$$\mu'(u) = \psi_F\big(\tfrac{1}{2}\,\mathrm{Tr}_{E/F}\big(\langle uw_2, w_1^* \rangle_W + \cdots + \langle uw_r, w_{r-1}^* \rangle_W + \langle uw_r^*, w_r^* \rangle_W\big)\big).$$

Let $\mathscr{W}' = (U', \mu')$.

When F is a number field, after fixing an additive character ψ_F of $\mathbb{A}/F$ (and also some auxiliary data as in the local case), we can define (global) Whittaker data $\mathscr{W} = \mathscr{W}_{\psi_F}$ and $\mathscr{W}' = \mathscr{W}'_{\psi_F}$ of G^* and H using the same formulas as above. By our definitions, these Whittaker data satisfy the local-global property, in the sense that

$$\mathscr{W}_v = \mathscr{W}_{\psi_{F,v}} \quad \text{and} \quad \mathscr{W}'_v = \mathscr{W}'_{\psi_{F,v}}$$

for all places v of F.

4. Weak transfer to the general linear group

From now on we assume that $\dim V < r$, hence the reductive dual pair (G, H) is in the stable range. In this section, we prove Theorem 2.1, which shows that each NEC in $L^2_{\mathrm{disc}}(G)$ has a weak transfer to the general linear group $\mathrm{GL}(\mathscr{V})$, where $\mathscr{V}$ is the standard representation of the dual group $\widehat{G}$ of G.

4.1. Attaching Arthur parameters

Let F be a number field. Let C be a near equivalence class in $L^2_{\mathrm{disc}}(G)$. Then C gives rise to a collection of L-parameters

$$\phi_v : L_{F_v} \to {}^L G$$

for almost all v, such that for any irreducible summand π of C, the L-parameter of π_v is ϕ_v for almost all v.

Proposition 4.1. *There exists a unique elliptic A-parameter ψ for G such that $\phi_{\psi_v} = \phi_v$ for almost all v.*

Proof. The strategy of the proof is the same as that of [22, Proposition 3.1]. For the convenience of the reader, we sketch the proof. Let π be any irreducible summand of C, and consider the theta lift between (G, H). Since $m_{\mathrm{disc}}(\pi) \geq 1$, we deduce from J.-S. Li's inequality (Theorem 3.1) that

$$m_{\mathrm{disc}}(\theta^{\mathrm{abs}}(\pi)) \geq 1.$$

Therefore, Theorem 3.3 attaches an elliptic A-parameter $\theta(\psi)$ to $\theta^{\mathrm{abs}}(\pi)$.

Next we show that $\theta(\psi)$ contains $\chi_V \boxtimes S_{2r-2n+1}$ as a direct summand. Consider the partial L-function $L^S(s, \theta^{\mathrm{abs}}(\pi) \times \chi_V^{-1})$ associated to $\theta^{\mathrm{abs}}(\pi)$ and χ_V^{-1}. Here S is a sufficiently large finite set of places of F such that for any place v outside S, the L-parameter of π_v is ϕ_v, and $G_v, H_v, \psi_{F,v}, \chi_{V,v}, \chi_{W,v}, \pi_v$ are all unramified. If we write $\theta(\psi) = \sum_i \phi_i \boxtimes S_{d_i}$ as in (2.2), then

$$L^S(s, \theta^{\mathrm{abs}}(\pi) \times \chi_V^{-1}) = \prod_i \prod_{j=1}^{d_i} L^S\left(s + \frac{d_i+1}{2} - j, \phi_i \chi_V^{-1}\right). \tag{4.1}$$

It follows from [36, Lemma 4.4, Theorem 5.3] that $L^S(s, \phi_i \chi_V^{-1})$ is holomorphic for $\Re(s) > 1$ for all i, and it has a pole at $s = 1$ if and only if $\phi_i = \chi_V$ is an automorphic character of $\mathrm{GL}_1(\mathbb{A}_E)$. On the other hand, by local theta correspondence for unramified representations [67, Theorem 6.1], for all $v \notin S$, the L-parameter of $\theta(\pi_v)$ is

$$\phi_v \chi_{W,v}^{-1} \chi_{V,v} + \left(\bigoplus_{j=n-r}^{r-n} |\cdot|^j\right)\chi_{V,v}. \tag{4.2}$$

Hence

$$L^S(s, \theta^{\mathrm{abs}}(\pi) \times \chi_V^{-1}) = L^S(s, \pi \times \chi_W^{-1}) \prod_{j=n-r}^{r-n} \zeta^S(s+j), \tag{4.3}$$

where $L^S(s, \pi \times \chi_W^{-1})$ is the partial L-function associated to π and χ_W^{-1}. By [81, Theorem 9.1], we know that the partial L-function $L^S(s, \pi \times \chi_W^{-1})$ is holomorphic when

$$\Re(s) > \begin{cases} n & \text{in Case O,} \\ n + \frac{1}{2} & \text{in Case } \mathrm{U}_0, \\ n+1 & \text{in Case } \mathrm{U}_1. \end{cases}$$

It follows from (4.3) that $L^S(s, \theta^{\mathrm{abs}}(\pi) \times \chi_V^{-1})$ is holomorphic for $\Re(s) > r - n + 1$ but has a simple pole at $s = r - n + 1$. This and (4.1) imply that $\theta(\psi)$ contains $\chi_V \boxtimes S_t$ as a direct summand for some t. Let t be the largest integer with this property. Then $L^S(s, \theta^{\mathrm{abs}}(\pi) \times \chi_V^{-1})$ has a largest pole at $s = \frac{t+1}{2}$. So we have $t = 2r - 2n + 1$. Thus we may write

$$\theta(\psi) = \psi \chi_W^{-1} \chi_V + \chi_V \boxtimes S_{2r-2n+1}$$

for some elliptic A-parameter $\psi \in \Psi_{\mathrm{ell}}(G^*)$. This and (4.2) imply that $\phi_{\psi_v} = \phi_v$ for almost all v. The uniqueness of ψ follows easily from the strong multiplicity one theorem for general linear groups [35]. ∎

Now we denote by $L^2_\psi(G)$ the near equivalence class C with the associated A-parameter ψ. Then we have a decomposition

$$L^2(G) = \bigoplus_{\psi \in \Psi_{\mathrm{ell}}(G^*)} L^2_\psi(G).$$

This completes the proof of Theorem 2.1.

5. A key equality: Generic case

Let F be a number field, and G an even orthogonal or unitary group over F as in the setting of Section 2. In this section, we study the structure of $L^2_\phi(G)$ for a generic elliptic A-parameter ϕ.

5.1. Gan–Ichino's observation

For any irreducible representation π of $G(\mathbb{A})$, we define the multiplicity $m_{\mathrm{cusp}}(\pi)$ by

$$m_{\mathrm{cusp}}(\pi) = \dim \mathrm{Hom}_{G(\mathbb{A})}(\pi, \mathcal{A}_{\mathrm{cusp}}(G)).$$

Obviously,

$$m_{\mathrm{cusp}}(\pi) \leq m_{\mathrm{disc}}(\pi) \leq m(\pi).$$

The following observation due to Gan–Ichino is the core of this work.

Proposition 5.1. *Let ϕ be a generic elliptic A-parameter for G. Let π be an irreducible representation of $G(\mathbb{A})$ such that the L-parameter of π_v is ϕ_v for almost all v. Then*

$$m_{\mathrm{cusp}}(\pi) = m_{\mathrm{disc}}(\pi) = m(\pi).$$

Proof. The proof is the same as that of [22, Proposition 4.1]. ■

5.2. Multiplicity preservation

Consider the theta lift between (G, H) with respect to a datum (ψ_F, χ_V, χ_W) as described in Section 3.1. Let σ be an irreducible representation of $H(\mathbb{A})$. We say σ is *relevant* to G if there is an irreducible unitary representation π of $G(\mathbb{A})$ and an automorphic character χ of $H(\mathbb{A})$ such that

$$\sigma \simeq \theta^{\mathrm{abs}}(\pi) \otimes \chi.$$

As a consequence of the previous proposition, we deduce the following corollary.

Corollary 5.2. *Let ϕ be a generic elliptic A-parameter for G. Suppose that*

$$L^2_\phi(G) = \bigoplus_\pi m_\pi \pi.$$

Let $\theta(\phi) = \phi\chi_W^{-1}\chi_V + \chi_V \boxtimes S_{2r-2n+1}$ *be an elliptic A-parameter for* H*. Then*

$$L^2_{\theta(\phi)}(H) = \Big(\bigoplus_{\pi} m_\pi \theta^{\mathrm{abs}}(\pi)\Big) \oplus \Big(\bigoplus_{\sigma} m_\sigma \sigma\Big),$$

where the second summation on the RHS is over all σ *with A-parameter* $\theta(\phi)$ *and not relevant to* G*.*

Proof. It follows from Theorem 3.1 and Howe duality that there is an injection

$$\bigoplus_{\pi} m_\pi \theta^{\mathrm{abs}}(\pi) \hookrightarrow L^2_{\theta(\phi)}(H).$$

It remains to show that for any irreducible summand σ of $L^2_{\theta(\phi)}(H)$ relevant to G, there is an irreducible summand π of $L^2_\phi(G)$ such that $\sigma \simeq \theta^{\mathrm{abs}}(\pi)$ and $m_{\mathrm{disc}}(\pi) = m_{\mathrm{disc}}(\theta^{\mathrm{abs}}(\pi))$.

So now suppose that σ is relevant to G. By definition there is a unique irreducible unitary representation π of $G(\mathbb{A})$ and a unique automorphic character χ of $H(\mathbb{A})$ such that

$$\sigma \simeq \theta^{\mathrm{abs}}(\pi) \otimes \chi.$$

It follows from J.-S. Li's results [49, Proposition 5.7] that π and χ satisfying this condition are unique. By Theorem 3.3, we know that the L-parameter of σ_v is

$$\phi_v \chi_{W,v}^{-1}\chi_{V,v} + \Big(\bigoplus_{j=n-r}^{r-n} |\cdot|^j\Big)\chi_{V,v} \tag{5.1}$$

for almost all v. Then it follows from the local theta lift for unramified representations that

$$\sigma_v \simeq \theta(\pi'_v)$$

for almost all v, where π'_v is the unramified representation of G_v with L-parameter ϕ_v. Therefore, by the uniqueness of π and χ, the L-parameter of π_v is ϕ_v and χ_v is trivial for almost all v. Since χ is automorphic, it must be trivial, so that $\sigma \simeq \theta^{\mathrm{abs}}(\pi)$. Moreover, since the L-parameter of π_v is ϕ_v for almost all v, we have

$$m_{\mathrm{disc}}(\pi) = m_{\mathrm{disc}}(\theta^{\mathrm{abs}}(\pi))$$

thanks to Theorem 3.1 and Proposition 5.1. This completes the proof. ∎

Remark 5.3. As explicated at the beginning of Section 2, all pure inner forms of G arise in the form $G' = G(V')$ for some space V'. Suppose that for G' we have the decomposition

$$L^2_\phi(G') = \bigoplus_{\pi'} m_{\pi'}\pi'.$$

Consider the theta lift between (G', H) for all such G' simultaneously. For any irreducible summand σ of $L^2_{\theta(\phi)}(H)$, at almost all places v of F the localization σ_v is unramified with L-parameter as in (5.1). Hence by using the same argument as in the proof of Corollary 5.2, one can show that σ is of the form $\theta^{\mathrm{abs}}(\pi')$ for a unique summand π' of $L^2_\phi(G')$, where G' is a pure inner form of G. This implies that

$$L^2_{\theta(\phi)}(H) = \bigoplus_{G'} \Big(\bigoplus_{\pi'} m_{\pi'} \theta^{\mathrm{abs}}(\pi') \Big),$$

where the first summation on the RHS is over all pure inner forms of G.

5.3. Transferring the multiplicity formula

In this subsection we define some notions and "transfer" AMF from H to G. We shall work in a general setting first, and then specialize to the generic case.

Let $\psi = \sum_i \phi_i \boxtimes S_{d_i}$ be an elliptic A-parameter for G, and

$$\theta(\psi) = \psi \chi_W^{-1} \chi_V + \chi_V \boxtimes S_{2r-2n+1}$$

be an elliptic A-parameter for H. Then

$$\mathcal{S}_\psi = \prod_i (\mathbb{Z}/2\mathbb{Z}) e_i \quad \text{and} \quad \mathcal{S}_{\theta(\psi)} = \Big(\prod_i (\mathbb{Z}/2\mathbb{Z}) e_i' \Big) \times (\mathbb{Z}/2\mathbb{Z}) a,$$

where e_i corresponds to $\phi_i \boxtimes S_{d_i} \subset \psi$, e_i' corresponds to $\phi_i \chi_W^{-1} \chi_V \boxtimes S_{d_i} \subset \theta(\psi)$, and a corresponds to $\chi_V \boxtimes S_{2r-2n+1} \subset \theta(\psi)$. The composition

$$\ell : \mathcal{S}_\psi \to \mathcal{S}_{\theta(\psi)} \to \overline{\mathcal{S}_{\theta(\psi)}}$$

is an isomorphism, where the first map sends e_i to e_i', and the second is the natural projection. Recall that Arthur [4, p. 47] and Mok [61, p. 29] have defined the so-called canonical sign characters ϵ_ψ and $\epsilon_{\theta(\psi)}$ of $\mathcal{S}_\psi$ and $\overline{\mathcal{S}_{\theta(\psi)}}$. We first give an explicit description of these characters.

Proposition 5.4. *We have*

$$\epsilon_\psi(e_i) = \prod_{j \neq i} \epsilon\big(\tfrac{1}{2}, \phi_i \times \phi_j^\vee\big)^{\min\{d_i, d_j\}}.$$

A similar formula holds for $\epsilon_{\theta(\psi)}$.

Proof. For Case O, this is proved in [15, Proposition-Definition 8.3.7]. So we will only consider Case U. To make the proof more streamlined, we shall make use of hypothetical global Langlands groups $\mathcal{L}_F$ and $\mathcal{L}_E$. However, we should mention that these two groups are not necessary: one can introduce the substitutes $\mathcal{L}_\psi$ and $\mathcal{L}_{\psi/E}$ as in [61, p. 19, (2.4.3)], and our proof still works after some slight modifications.

By the global Langlands conjecture, there is a one-to-one correspondence between irreducible cuspidal representations of $\mathrm{GL}_k(E)$ and irreducible k-dimensional representations of $\mathcal{L}_E$. Therefore each ϕ_i corresponds to an irreducible representation of $\mathcal{L}_E$, which we shall still denote by ϕ_i. For each i, let $\psi_i = \phi_i \boxtimes S_{d_i}$ be an irreducible representation of $\mathcal{L}_E \times \mathrm{SL}_2(\mathbb{C})$. Then the global A-parameter ψ can be regarded as the direct sum of these ψ_i, and the conjugate self-duality of ψ allows us to extend it to an L-homomorphism

$$\widetilde{\psi} : \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G.$$

Using this L-homomorphism, the global component group $\mathcal{S}_\psi$ can be identified with the centralizer of $\mathrm{Im}(\widetilde{\psi})$ in $\widehat{G}$. To define the character ϵ_ψ, we need to consider the adjoint representation of ${}^L G$ on the Lie algebra $\widehat{\mathfrak{g}}$ of the Langlands dual group $\widehat{G}$. Let

$$\tau_\psi : \mathcal{S}_\psi \times \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(\widehat{\mathfrak{g}})$$

be the representation given by the formula

$$\tau_\psi(x, g, h) = \mathrm{Ad}(x \cdot \widetilde{\psi}(g, h))$$

for $x \in \mathcal{S}_\psi$, $g \in \mathcal{L}_F$ and $h \in \mathrm{SL}_2(\mathbb{C})$. If this representation τ_ψ is decomposed as

$$\tau_\psi = \bigoplus_\alpha \lambda_\alpha \boxtimes \mu_\alpha \boxtimes \nu_\alpha$$

for some irreducible representations $\lambda_\alpha, \mu_\alpha, \nu_\alpha$ of $\mathcal{S}_\psi$, $\mathcal{L}_F$ and $\mathrm{SL}_2(\mathbb{C})$ respectively, then the character ϵ_ψ is defined by

$$\epsilon_\psi(e_i) = \prod_\alpha{}' \det(\lambda_\alpha(e_i)),$$

where $\prod'_\alpha$ denotes the product over the indices α such that μ_α is symplectic and $\epsilon(\frac{1}{2}, \mu_\alpha) = -1$. By some elementary computations, we know that the representation τ_ψ can be decomposed as

$$\tau_\psi \simeq \Big(\sum_i \varepsilon_i^2 \,\mathrm{As}^{\pm}(\psi_i)\Big) \oplus \Big(\sum_{i<j} \varepsilon_i \varepsilon_j \cdot \mathrm{Ind}_{\mathcal{L}_E \times \mathrm{SL}_2(\mathbb{C})}^{\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C})} (\psi_i \otimes \psi_j^\vee)\Big),$$

where ε_i is the character of $\mathcal{S}_\psi$ defined by

$$\varepsilon_i : \mathcal{S}_\psi \to \{\pm 1\}, \quad e_j \mapsto \begin{cases} +1 & \text{if } j = i, \\ -1 & \text{if } j \neq i, \end{cases}$$

and $\mathrm{As}^{\pm 1}$ are the Asai representations defined in [19, Section 7]. Also note that

$$\mathrm{Ind}_{\mathcal{L}_E \times \mathrm{SL}_2(\mathbb{C})}^{\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C})}(\psi_i \otimes \psi_j^\vee) = \sum_{k=1}^{\min\{d_i, d_j\}} \mathrm{Ind}_{\mathcal{L}_E}^{\mathcal{L}_F}(\phi_i \otimes \phi_j^\vee) \boxtimes S_{d_i + d_j - 2k + 1}.$$

Therefore by the definition we have

$$\epsilon_\psi(e_i) = \prod_{j \neq i} \epsilon\big(\tfrac{1}{2}, \mathrm{Ind}_{\mathcal{L}_E}^{\mathcal{L}_F}(\phi_i \otimes \phi_j^\vee)\big)^{\min\{d_i, d_j\}} = \prod_{j \neq i} \epsilon\big(\tfrac{1}{2}, \phi_i \times \phi_j^\vee\big)^{\min\{d_i, d_j\}}.$$

Here in the last equality we use the inductivity of the epsilon factors. ∎

Remark 5.5. When $\psi = \phi$ is generic, for any indices i and j, both ϕ_i and ϕ_j are (conjugate) self-dual of the same parity. By [4, Theorem 1.5.3] (for Case O) and [61, Theorem 2.5.4] (for Case U), we have $\epsilon(\frac{1}{2}, \phi_i \times \phi_j^\vee) = 1$. This implies that

$$\epsilon_\phi = 1. \tag{5.2}$$

Next we compare the character ϵ_ψ and the pull back of $\epsilon_{\theta(\psi)}$ along the map ℓ using this description.

Lemma 5.6. *We have*

$$\epsilon_\psi = \ell^*(\epsilon_{\theta(\psi)}).$$

Proof. By Proposition 5.4,

$$\begin{aligned}
\epsilon_{\theta(\psi)}(e_i') &= \Big(\prod_{j\neq i} \epsilon\big(\tfrac{1}{2}, \phi_i \chi_V \chi_W^{-1} \times \phi_j^\vee \chi_V^{-1} \chi_W\big)^{\min\{d_i, d_j\}}\Big) \times \epsilon\big(\tfrac{1}{2}, \phi_i \chi_V \chi_W^{-1} \times \chi_V^{-1}\big)^{d_i} \\
&= \Big(\prod_{j\neq i} \epsilon\big(\tfrac{1}{2}, \phi_i \times \phi_j\big)^{\min\{d_i, d_j\}}\Big) \times \epsilon\big(\tfrac{1}{2}, \phi_i \chi_W^{-1}\big)^{d_i} \\
&= \epsilon_\psi(e_i) \cdot \epsilon\big(\tfrac{1}{2}, \phi_i \chi_W^{-1}\big)^{d_i}.
\end{aligned} \tag{5.3}$$

If ϕ_i has the same parity as ψ, then it also has the same parity as χ_W. By [4, Theorem 1.5.3] (for Case O) and [61, Theorem 2.5.4] (for Case U), we have $\epsilon(\frac{1}{2}, \phi_i \chi_W^{-1}) = 1$. On the other hand, if ϕ_i and ψ have different parities, then d_i must be even. Therefore we always have $\epsilon(\frac{1}{2}, \phi_i \chi_W^{-1})^{d_i} = 1$. The desired conclusion then follows from (5.3). ∎

Locally, for each place v of F, we also have the natural map of component groups

$$\ell_v : \mathcal{S}_{\psi_v} \to \mathcal{S}_{\theta(\psi_v)} \to \overline{\mathcal{S}_{\theta(\psi_v)}}.$$

Again ℓ_v is an isomorphism. We regard the local A-packet $\Pi_{\theta(\psi_v)}(H_v)$ as a representation of $\overline{\mathcal{S}_{\theta(\psi_v)}} \times H_v$ by setting

$$\Pi_{\theta(\psi_v)}(H_v) = \bigoplus_\sigma \mathcal{J}_{\mathscr{W}'}(\sigma) \boxtimes \sigma,$$

where the summation on the RHS is over all irreducible unitary representations of H_v in $\Pi_{\theta(\psi_v)}(H_v)$. Let

$$\Pi_{\psi_v}^\theta(G_v) = \bigoplus_\sigma \ell_v^*(\mathcal{J}_{\mathscr{W}'}(\sigma)) \boxtimes \theta(\sigma).$$

Note that $\theta(\sigma)$ could be zero. We shall discard those σ being mapped to zero under the theta lift. We regard $\Pi_{\phi_v}^\theta(G_v)$ as a (multi-) set equipped with a map

$$\mathcal{J}_v^\theta : \Pi_{\psi_v}^\theta(G_v) \to \mathcal{S}_{\psi_v}, \quad \theta(\sigma) \mapsto \ell_v^*(\mathcal{J}_{\mathscr{W}'}(\sigma)).$$

Define a global packet

$$\begin{aligned}
\Pi_\psi^\theta(G) &= \sideset{}{'}\bigotimes_v \Pi_{\psi_v}^\theta(G_v) \\
&= \Big\{\pi = \sideset{}{'}\bigotimes_v \pi_v \;\Big|\; \pi_v \in \Pi_{\psi_v}^\theta(G_v),\ \pi_v \text{ unramified with } L\text{-parameter } \phi_{\psi_v} \text{ for almost all } v\Big\}.
\end{aligned}$$

We then have a map

$$\mathcal{J}^{\theta} : \Pi^{\theta}_{\psi}(G) \to \widehat{\mathcal{S}_{\psi}}, \quad \pi \mapsto \mathcal{J}^{\theta}(\pi), \quad \mathcal{J}^{\theta}(\pi)(x) := \prod_{v} \mathcal{J}^{\theta}_{v}(\pi_v)(x_v),$$

where $x \in \mathcal{S}_{\psi}$ and x_v is the localization of x at v. It is easy to check that

$$\mathcal{J}^{\theta}(\pi) = \ell^{*}(\mathcal{J}_{\mathscr{W}'}(\theta^{\mathrm{abs}}(\pi))).$$

We put

$$\Pi^{\theta}_{\psi}(G, \epsilon_{\psi}) = \{\pi \in \Pi^{\theta}_{\psi}(G) \mid \mathcal{J}^{\theta}(\pi) = \epsilon_{\psi}\}.$$

Next we specialize to the generic case. As a direct consequence of Corollary 5.2 and Lemma 5.6, we have the following result.

Proposition 5.7. *Let $\psi = \phi$ be a generic elliptic A-parameter for G. Then there is a decomposition*

$$L^2_{\phi}(G) = \bigoplus_{\pi \in \Pi^{\theta}_{\phi}(G, \epsilon_{\phi})} \pi.$$

Hence, to complete the proof of Theorem 2.6, it remains to describe $\Pi^{\theta}_{\phi_v}(G_v)$ in terms of LLC for G_v. This will be established in the next section.

6. Local comparison

In this section we let F be a local field of characteristic zero. We will compare the packets $\Pi^{\theta}_{\phi}(G)$ and $\Pi^{L}_{\phi}(G)$ for almost all tempered L-parameters ϕ of G.

6.1. Main local theorem

We first recall several notions. Let ϕ be a local L-parameter for G. As explained in [19, Section 8], ϕ can be regarded as a (conjugate) self-dual representation of L_E of a certain parity. If we write ϕ as

$$\phi = \sum_{i} m_i \phi_i$$

for some positive integers m_i and some pairwise distinct irreducible representations ϕ_i of L_E, we say that

- ϕ is *of good parity* if ϕ_i is (conjugate) self-dual of the same parity as ϕ for all i;
- ϕ is *tempered* if ϕ_i is tempered (i.e. has bounded image on the Weil group W_E) for all i;
- ϕ is *almost tempered* if $\phi_i = \phi'_i |\cdot|^{s_i}$ for some tempered representation ϕ'_i of L_E and some $s_i \in \mathbb{R}$ with $|s_i| < 1/2$, for all i.

Note that if ϕ is of good parity, then it is tempered.

Remark 6.1. In fact, the Ramanujan conjecture predicts that all localizations of global generic elliptic A-parameters are tempered. However, we still need to deal with a larger class of L-parameters and not only the tempered ones, since the Ramanujan conjecture has not been proved in general. Thanks to the work of Jacquet–Shalika [36, Corollary 2.5], for our purpose, it is sufficient to do the comparison for almost tempered L-parameters.

Let ϕ be an almost tempered L-parameter for $G = G(V)$. Assume that $\dim V < r$ and let

$$\theta(\phi) = \phi\chi_W^{-1}\chi_V + \chi_V \boxtimes S_{2r-2n+1}. \tag{6.1}$$

Then $\theta(\phi)$ is a local A-parameter for $H = H(W_{(r)})$. Recall that there is an isomorphism between component groups

$$\ell : \mathcal{S}_\phi \to \mathcal{S}_{\theta(\phi)} \to \overline{\mathcal{S}_{\theta(\phi)}}.$$

In the light of Proposition 5.7 and Remark 6.1, to complete the proof of Theorem 2.6, all we need is the following two theorems.

Theorem 6.2. *The local A-packet $\Pi_{\theta(\phi)}(H)$ is multiplicity-free. Hence we can regard $\Pi_{\theta(\phi)}(H)$ as a subset of* $\mathrm{Irr}(H)$.

Theorem 6.3. *There is a commutative diagram*

$$\begin{array}{ccc} \Pi_{\theta(\phi)}(H) & \xrightarrow{\mathcal{J}_{\mathcal{W}'}} & \widehat{\overline{\mathcal{S}_{\theta(\phi)}}} \\ \theta\big\downarrow & & \big\downarrow \ell^*V \\ \bigsqcup \Pi_\phi^L(G') & \xrightarrow{\mathcal{J}_{\mathcal{W}}^L} & \widehat{\mathcal{S}_\phi} \end{array} \tag{6.2}$$

where the disjoint union is taken over all pure inner forms of G, and the arrow θ is a bijection given by the theta lift. More precisely, for $\sigma \in \Pi_{\theta(\phi)}(H)$, we have:

(1) *There exists a unique pure inner form G' of G such that the theta lift $\pi = \theta(\sigma)$ of σ to G' is non-vanishing.*

(2) *π lies in the L-packet $\Pi_\phi^L(G')$, and*

$$\mathcal{J}_{\mathcal{W}}^L(\pi) = \ell^*(\mathcal{J}_{\mathcal{W}'}(\sigma)).$$

Hence $\Pi_\phi^\theta(G) = \Pi_\phi^L(G)$ as sets and $\mathcal{J}_{\mathcal{W}}^L = \mathcal{J}^\theta$.

Remark 6.4. (1) Theorem 6.2 is largely due to Mœglin [55] when F is non-Archimedean, to Mœglin–Renard [56] when $F = \mathbb{C}$ and to Arancibia–Mœglin–Renard [3] and Mœglin–Renard [57–59] when $F = \mathbb{R}$ and ϕ is of good parity. Here we will give a proof of Theorem 6.2 by using theta lifts.

(2) When we are in Case U, we also need to deal with split places where $E \simeq F \times F$. In this case, $G(V)$ and $H(W)$ are isomorphic to general linear groups, and $G(V)$ has no

pure inner form other than itself. We also know that the local L-packet $\Pi_\phi^L(G)$ and the local A-packet $\Pi_{\theta(\phi)}(H)$ are singletons. Therefore, in this case we need to prove that

$$\theta(\pi_\phi) = \sigma_{\theta(\phi)},$$

where π_ϕ and $\sigma_{\theta(\phi)}$ are the irreducible representations of G and H associated to ϕ and $\theta(\phi)$ respectively. This assertion is an analog of Theorem 6.3 for general linear groups. When F is non-Archimedean, it has been proved by Mínguez [52]. When F is Archimedean, it has been proved by Adams–Barbasch [2] for $F = \mathbb{C}$ and by Adams [1] for $F = \mathbb{R}$.

We are going to prove these theorems in the next few subsections.

6.2. *Some local packets*

In this subsection, we review some information about certain local L-packets and A-packets. Let ϕ be an almost tempered L-parameter. We may write it as

$$\phi = \varphi + \phi_0 + (\varphi^c)^\vee, \tag{6.3}$$

where

- ϕ_0 is an L-parameter of good parity for $G_0^* = G(V_0^+)$, where V_0^+ is a c-Hermitian space in the Witt tower containing V^+;
- φ is a sum
$$\varphi = \phi_1|\cdot|^{s_1} + \cdots + \phi_r|\cdot|^{s_r}$$
for some k_i-dimensional irreducible tempered representation ϕ_i of L_E and non-negative real numbers $s_i < 1/2$, such that for each $i \in \{1, \ldots, r\}$, either ϕ_i is not (conjugate) self-dual of the same parity as ϕ, or s_i is positive.

Without loss of generality, we can rearrange the index set $\{1, \ldots, r\}$ so that

$$1/2 > s_1 \geq \cdots \geq s_r \geq 0.$$

We put $k = k_1 + \cdots + k_r$. There is a natural isomorphism $\mathcal{S}_{\phi_0} \simeq \mathcal{S}_\phi$. By the inductive property of LLC for G, we know that $\Pi_\phi^L(G) = \varnothing$ unless the F-rank of G is greater than or equal to k, in which case there exists an F-parabolic subgroup P of G with Levi component $\mathrm{GL}_k(E) \times G_0$, where $G_0 = G(V_0)$ and V_0 is a c-Hermitian space in the Witt tower containing V. Let τ be the irreducible representation of $\mathrm{GL}_k(E)$ associated to φ. Then by [13, Theorem 4.4 (10)], [14, Theorem 2.5.1 (8)] we know that for any $\eta \in \widehat{\mathcal{S}_\phi}$, $\pi = \pi(\phi, \eta)$ is the unique irreducible quotient of

$$\mathrm{Ind}_P^G(\tau \boxtimes \pi_0),$$

where $\pi_0 = \pi(\phi_0, \eta)$, and we regard η as a character of $\mathcal{S}_{\phi_0}$ through the natural isomorphism. In fact, we have more than this:

Lemma 6.5. *The induced representation* $\mathrm{Ind}_P^G(\tau \boxtimes \pi_0)$ *is irreducible for all* $\pi_0 \in \Pi_{\phi_0}(G_0)$. *Hence*

$$\Pi_\phi^L(G) = \{\mathrm{Ind}_P^G(\tau \boxtimes \pi_0) \mid \pi_0 \in \Pi_{\phi_0}(G_0)\}.$$

Proof. When ϕ is tempered, the assertion simply follows from the local intertwining relation for G. So it remains to show the assertion when ϕ is almost tempered but non-tempered. Let $j \in \{1, \dots, r\}$ be the largest integer such that $s_j > 0$. By induction in stages, we may rewrite $\mathrm{Ind}_P^G(\tau \boxtimes \pi_0)$ as a standard module. More precisely, we have

$$\mathrm{Ind}_P^G(\tau \boxtimes \pi_0) = \mathrm{Ind}_{P'}^G(\tau' \boxtimes \pi_0'),$$

where

- P' is a parabolic subgroup of G with Levi component $\mathrm{GL}_{k'}(E) \times G(V_0')$, where V_0' is a c-Hermitian space in the Witt tower containing V, and $k' = k_1 + \cdots + k_j$;
- τ' is the irreducible representation of $\mathrm{GL}_{k'}(E)$ associated to the L-parameter

$$\phi_1|\cdot|^{s_1} + \cdots + \phi_j|\cdot|^{s_j};$$

- π_0' is some irreducible tempered representation of $G(V_0')$ with L-parameter

$$\phi_0' = (\phi_{j+1} + \cdots + \phi_r) + \phi_0 + ((\phi_{j+1} + \cdots + \phi_r)^c)^\vee.$$

Hence it would be sufficient to show the irreducibility of $\mathrm{Ind}_{P'}^G(\tau' \boxtimes \pi_0')$. If F is Archimedean, then this follows from a result of Speh–Vogan [71, Theorem 1.1]; see also [75, Section 8]. If F is non-Archimedean, this has been proved in [8, Proposition 6.7] for Case O and [21, Proposition 9.1, Appendix B] for Case U. ∎

Next we consider certain local A-packets for H. Let

$$W_0 = W_{(r-k)} \quad \text{and} \quad H_0 = H(W_0).$$

Put

$$\theta(\phi_0) = \phi_0\chi_W^{-1}\chi_V + \chi_V \boxtimes S_{2r-2n+1}.$$

Then $\theta(\phi_0)$ is a local A-packet of good parity for H_0. According to the decomposition (6.3) of ϕ, we have a similar decomposition

$$\theta(\phi) = \varphi\chi_W^{-1}\chi_V + \theta(\phi_0) + ((\varphi\chi_W^{-1}\chi_V)^c)^\vee,$$

as well as the natural isomorphism $\mathcal{S}_{\theta(\phi_0)} \simeq \mathcal{S}_{\theta(\phi)}$. Let Q be the standard parabolic subgroup of H with Levi component $\mathrm{GL}_k(E) \times H_0$. Then by the definition and the inductive property of local A-packets (see [4, p. 45, (1.5.1)] and [61, p. 28]), we have

$$\Pi_{\theta(\phi)}(H) = \{\text{Irreducible constituents of } \mathrm{Ind}_Q^H(\tau\chi_W^{-1}\chi_V \boxtimes \sigma_0) \mid \sigma_0 \in \Pi_{\theta(\phi_0)}(H_0)\}.$$

The following irreducibility result for these induced representations will be used in later proofs.

Lemma 6.6. *The induced representation* $\mathrm{Ind}_Q^H(\tau\chi_W^{-1}\chi_V \boxtimes \sigma_0)$ *is irreducible for all irreducible unitary representations* σ_0 *in the* A*-packet* $\Pi_{\theta(\phi_0)}(H_0)$*. Hence as a (multi-) set,*

$$\Pi_{\theta(\phi)}(H) = \{\mathrm{Ind}_Q^H(\tau\chi_W^{-1}\chi_V \boxtimes \sigma_0) \mid \sigma_0 \in \Pi_{\theta(\phi_0)}(H_0)\}.$$

Proof. This assertion has been proved in a more general context by Mœglin in [55, Section 3.2], and [54, Section 5.1] when F is non-Archimedean; and by Mœglin–Renard in [56, Section 6] when $F = \mathbb{C}$. To show the case when $F = \mathbb{R}$, one can appeal to the same argument as in [23, Proposition 3.4]; for a sketch of proof, see Appendix B. ■

6.3. Reduction to the case of good parity

We argue by induction on $\dim V$.

Lemma 6.7. *Theorems 6.2 and 6.3 hold for* $\dim V = 0$*, i.e. in Case* O *or Case* U_0*, and* $n = 0$*.*

Proof. When $\dim V = 0$, we have $\theta(\phi) = \chi_V \boxtimes S_{2r+1}$ (note that if we are in Case O, we have $\chi_V = \mathbb{1}$ by convention), and $\Pi_{\theta(\phi)}(H) = \{\sigma_1\}$, where

$$\sigma_1 = \begin{cases} \text{the trivial representation} & \text{in Case O,} \\ \chi \circ \det & \text{in Case U.} \end{cases}$$

Here in Case U, χ is the character of E^1 corresponding to χ_V by LLC for the torus. It is easy to see that σ_1 is also the theta lift of the trivial representation of G. This completes the proof. ■

From now on, we assume that $\dim V > 0$. We first reduce the proof of Theorems 6.2 and 6.3 to the case of good parity. Given an almost tempered L-parameter ϕ of G, we can write

$$\phi = \varphi + \phi_0 + (\varphi^c)^\vee,$$

where ϕ_0 is the "good parity part" (see (6.3)).

Lemma 6.8. *Assume that Theorems 6.2 and 6.3 hold for* ϕ_0*. Then they also hold for* ϕ*.*

Proof. Let $\sigma \in \Pi_{\theta(\phi)}(H)$. By Lemma 6.6 we know that

$$\sigma = \mathrm{Ind}_Q^H(\tau\chi_W^{-1}\chi_V \boxtimes \sigma_0)$$

for some $\sigma_0 \in \Pi_{\theta(\phi_0)}(H_0)$. Moreover, if we identify $\mathcal{S}_{\theta(\phi)}$ with $\mathcal{S}_{\theta(\phi_0)}$ via the natural isomorphism, then $\mathcal{J}_{\mathscr{W}'}(\sigma) = \mathcal{J}_{\mathscr{W}'}(\sigma_0)$. By our hypothesis, there is exactly one pure inner form G_0' of G_0^* such that

- the theta lift of σ_0 to G_0', denoted by π_0, is non-vanishing;
- $\mathcal{J}_{\mathscr{W}}^L(\pi_0) = \ell^*(\mathcal{J}_{\mathscr{W}'}(\sigma_0))$.

Then it follows from the induction principle for local theta correspondence [41], [2, Corollary 3.21], [63, Theorem 4.5.5], [65, Section 5.2] that there exists a non-zero equivariant map

$$\omega \to \mathrm{Ind}_{P'}^{G'}(\tau \boxtimes \pi_0) \boxtimes \mathrm{Ind}_Q^H(\tau\chi_W^{-1}\chi_V \boxtimes \sigma_0),$$

where G' is a pure inner form of G, and P' is the standard parabolic subgroup of G' with Levi component $\mathrm{GL}_k(E) \times G_0'$. Since $\mathrm{Ind}_{P'}^{G'}(\tau \boxtimes \pi_0)$ is irreducible by Lemma 6.5, this implies that the theta lift of σ to G' is

$$\pi = \mathrm{Ind}_{P'}^{G'}(\tau \boxtimes \pi_0).$$

Moreover, by LLC for G', if we identify $\mathcal{S}_\phi$ with $\mathcal{S}_{\phi_0}$ via the natural isomorphism, then $\mathcal{J}_{\mathscr{W}}^L(\pi) = \mathcal{J}_{\mathscr{W}}^L(\pi_0)$. Hence the theta lift between (G', H) gives us the desired commutative diagram (6.2). Moreover, it follows from the combination of our hypothesis and Lemmas 6.5 and 6.6 that the left vertical arrow θ is a bijection. In particular, $\Pi_{\theta(\phi)}(H)$ is multiplicity-free. This completes the proof. ∎

6.4. *A subdiagram*

By the results in the previous subsection, it remains to prove Theorems 6.2 and 6.3 for L-parameters of good parity. Let ϕ be an L-parameter of good parity for G. Let $\theta(\phi)$ be the A-parameter of H defined in (6.1), and $\phi_{\theta(\phi)}$ the L-parameter associated to $\theta(\phi)$. Since ϕ is of good parity, it is also tempered. Hence the L-packet $\Pi^L_{\phi_{\theta(\phi)}}(H)$ is a subset of the local A-packet $\Pi_{\theta(\phi)}(H)$, and there is a commutative diagram

$$\begin{array}{ccc} \Pi^L_{\phi_{\theta(\phi)}}(H) & \xrightarrow{\mathcal{J}^L_{\mathscr{W}'}} & \widehat{\overline{\mathcal{S}_{\phi_{\theta(\phi)}}}} \\ \big\downarrow & & \big\downarrow \\ \Pi_{\theta(\phi)}(H) & \xrightarrow{\mathcal{J}_{\mathscr{W}'}} & \widehat{\overline{\mathcal{S}_{\theta(\phi)}}} \end{array}$$

By abuse of notation, we shall still denote by ℓ the composition of natural maps

$$\mathcal{S}_\phi \xrightarrow{\ell} \overline{\mathcal{S}_{\theta(\phi)}} \to \overline{\mathcal{S}_{\phi_{\theta(\phi)}}}.$$

Remark 6.9. In our later proofs, we shall also use the fact that the A-packet $\Pi_{\theta(\phi)}(H)$ contains the L-packet $\Pi^L_{\phi_{\theta(\phi)}}(H)$ when ϕ is almost tempered but may not be tempered. We are not very sure whether this has been verified in [4, 61] or not, but this fact can be easily shown based on the tempered case, by using Lemma 6.6.

In this subsection, as the first step toward the good parity case, we prove the following proposition, which will establish a "subdiagram" of Theorem 6.3.

Proposition 6.10. *There is a commutative diagram*

$$\begin{array}{ccc} \Pi^L_{\phi_{\theta(\phi)}}(H) & \xrightarrow{\mathcal{J}^L_{\mathscr{W}'}} & \widehat{\overline{\mathcal{S}_{\phi_{\theta(\phi)}}}} \\ {\scriptstyle\theta}\big\downarrow & & \big\downarrow{\scriptstyle\ell^*} \\ \bigsqcup \Pi^L_\phi(G') & \xrightarrow{\mathcal{J}^L_{\mathscr{W}}} & \widehat{\mathcal{S}_\phi} \end{array}$$

where the disjoint union runs over all pure inner forms of G, and the arrow θ is an injection given by the theta lift. More precisely, for $\sigma \in \Pi^L_{\phi_{\theta(\phi)}}(H)$, we have:

(1) *There exists a unique pure inner form G' of G such that the theta lift $\pi = \theta(\sigma)$ of σ to G' is non-vanishing.*

(2) *π lies in the L-packet $\Pi^L_\phi(G')$, and*

$$\mathcal{J}^L_{\mathscr{W}}(\pi) = \ell^*(\mathcal{J}^L_{\mathscr{W}'}(\sigma)).$$

An ingredient of the proof of this proposition is the so-called Prasad conjecture. It describes the (almost) equal rank theta lift in terms of LLC. We shall briefly recall it.

Let $W_0 = W_{(n)}$ be as defined in (3.1), and $H_0 = H(W_{(n)})$. We shall use G' and H_0' to denote pure inner forms of G and H_0. Consider the theta lift between (G', H_0'), which is in the almost equal rank, with respect to the datum (ψ_F, χ_V, χ_W). To distinguish notations, we use ϑ to denote the theta lift between (G', H_0'). As we will need to use LLC for the group H_0', it is necessary for us to specify which Whittaker datum of H_0 we are using. Following the description in Section 3.5, we can pick up a Whittaker datum of H_0 (depending on the choice of the additive character ψ_F), which we shall also denote by $\mathscr{W}'$ by abuse of notation. Let

$$\vartheta(\phi) = \phi\chi_W^{-1}\chi_V + \chi_V$$

be a tempered L-parameter for H_0. Then *Prasad's conjecture* is the following statement.

Theorem 6.11. (1) *For any pure inner form H_0' of H_0, and any $\sigma_0 \in \Pi^L_{\vartheta(\phi)}(H_0')$, there exists a unique pure inner form G' of G such that the theta lift π of σ_0 to G' is non-zero. Moreover, π is in the L-packet $\Pi^L_\phi(G')$, and*

$$\mathcal{J}^L_{\mathscr{W}}(\pi) = \mathcal{J}^L_{\mathscr{W}'}(\sigma_0)|_{\mathcal{S}_\phi}. \tag{6.4}$$

Here we regard $\mathcal{S}_\phi$ as a subgroup of $\mathcal{S}_{\vartheta(\phi)}$ via the natural embedding $\mathcal{S}_\phi \hookrightarrow \mathcal{S}_{\vartheta(\phi)}$. In particular, if we are in Case U, *then the map*

$$\vartheta : \bigsqcup_{H_0'} \Pi^L_{\vartheta(\phi)}(H_0') \to \bigsqcup_{G'} \Pi^L_\phi(G')$$

given by the theta lift ϑ is surjective.

(2) *Suppose we are in Case* O. *Suppose further that $\chi_W \not\subset \phi$. Then for any pure inner form G' of G, and any $\pi \in \Pi^L_\phi(G')$, both*

$$\begin{cases} \text{the theta lift } \vartheta(\pi) \text{ of } \pi \text{ to } H_0 \text{ and} \\ \text{the theta lift } \vartheta'(\pi\otimes\det) \text{ of } \pi\otimes\det \text{ to } H_0 \end{cases}$$

are non-zero.

(3) *Suppose we are in Case* U, *and F is non-Archimedean. Suppose further that $\chi_W \not\subset \phi$. Let W_0' be the unique skew-Hermitian space over E of the same dimension and the*

opposite sign as W_0, *and* $H_0' = H(W_0')$. *Then for any pure inner form* G' *of* G, *and any* $\pi \in \Pi_\phi^L(G')$, *both*

$$\begin{cases} \text{the theta lift } \vartheta(\pi) \text{ of } \pi \text{ to } H_0 \text{ and} \\ \text{the theta lift } \vartheta'(\pi) \text{ of } \pi \text{ to } H_0' \end{cases}$$

are non-zero.

Proof. This theorem was proved by many people in various cases. When F is non-Archimedean, it is proved by Atobe–Gan [8, Theorem 4.5] in Case O, and by Gan–Ichino [21, Theorem 4.4] in Case U; when F is real, it is essentially proved by Paul [64, Theorem 3.4] [65, Theorems 15, 18]; when F is complex, it is proved by Adams–Barbasch [2, Theorems 2.8, 2.9]. ∎

Remark 6.12. (1) In Case U, the proof of Gan–Ichino [21, Theorem 4.4] uses AMF as input. But one can apply Atobe's method [6, Section 7.1] to this case to avoid using any global method. Indeed, except for (6.4), all other statements in Theorem 6.11 follow from [20, Theorem C.5]. As for (6.4), one can consider the diagram

$$\begin{array}{ccc} \omega \otimes \mathrm{Ind}_Q^{\widetilde{H}_0'}(\tau^c \chi_W^c \boxtimes \sigma_0^\vee) & \xrightarrow{\mathcal{T}} & \mathrm{Ind}_P^{\widetilde{G}'}(\tau \chi_V \boxtimes \pi) \\ \Big\downarrow {\scriptstyle 1 \otimes R(\widetilde{w}', \tau^c \chi_W^c \boxtimes \sigma_0^\vee)} & & \Big\downarrow {\scriptstyle R(\widetilde{w}, \tau\chi_V \boxtimes \pi)} \\ \omega \otimes \mathrm{Ind}_Q^{\widetilde{H}_0'}(\tau^c \chi_W^c \boxtimes \sigma_0^\vee) & \xrightarrow{\mathcal{T}} & \mathrm{Ind}_P^{\widetilde{G}'}(\tau \chi_V \boxtimes \pi) \end{array}$$

constructed in [21, Section 8.2], where τ is an irreducible discrete series representation of some general linear group; $\widetilde{G}'$, $\widetilde{H}_0'$ are some larger unitary groups; P, Q are maximal parabolic subgroups of $\widetilde{G}'$, $\widetilde{H}_0'$; $\mathcal{T}$ is a certain explicit map constructed in [21, Section 8.1]; and finally $R(\widetilde{w}, \tau\chi_V \boxtimes \pi)$, $R(\widetilde{w}', \tau^c \chi_W^c \boxtimes \sigma_0^\vee)$ are normalized intertwining operators. This diagram commutes up to an explicitly computable constant. Let τ run over all discrete series of general linear groups. By the local intertwining relation one can recover $\mathcal{J}_{\mathcal{W}}^L(\pi)$ from the normalized intertwining operators $R(\widetilde{w}, \tau\chi_V \boxtimes \pi)$, whereas the latter can be computed via the diagram above. The readers may also consult our previous paper [14, Corollary 6.3.2] for a similar argument (but in a slightly different setting).

(2) Here is a caveat: Paul's results [64, Theorem 3.4] and [65, Theorems 15, 18] are written in terms of the Harish-Chandra parameters rather than the L-parameters. Therefore we need a "dictionary" between the two languages, so that we can reformulate Paul's results in the form we need. For Case U, such a dictionary has been provided by [7, Appendix A] and [80, Section 3.2]. For Case O, the situation is more complicated. LLC for real even orthogonal groups has not been established in the literature so far, due to their disconnectedness. In Appendix A we will first prove a weaker version of Theorem 6.11 (see Theorem A.1) using Paul's results, and then define LLC for real full even orthogonal groups by using theta lifts. By our construction, Theorem A.1 is upgraded to Theorem 6.11 automatically.

(3) Indeed, this theorem also holds for all generic L-parameters. In particular, it holds for all almost tempered L-parameters.

Now we have three component groups $\mathcal{S}_\phi$, $\mathcal{S}_{\vartheta(\phi)}$, and $\mathcal{S}_{\phi_{\theta(\phi)}}$. The natural embeddings between them give us a commutative diagram

$$\begin{array}{ccc} \mathcal{S}_\phi & \xrightarrow{\quad \ell \quad} & \overline{\mathcal{S}_{\phi_{\theta(\phi)}}} \\ & \searrow \qquad \nearrow & \\ & \overline{\mathcal{S}_{\vartheta(\phi)}} & \end{array}$$

Notice that

$$\phi_{\theta(\phi)} = (|\cdot|^{n-r} + \cdots + |\cdot|^{-1})\chi_V + \vartheta(\phi) + (|\cdot|^1 + \cdots + |\cdot|^{r-n})\chi_V.$$

Hence the natural map $\overline{\mathcal{S}_{\vartheta(\phi)}} \to \overline{\mathcal{S}_{\phi_{\theta(\phi)}}}$ is indeed an isomorphism. The component group $\overline{\mathcal{S}_{\vartheta(\phi)}}$ will serve as a springboard and make our proofs much easier.

Proof of Proposition 6.10. Let σ be an irreducible representation in the L-packet $\Pi^L_{\phi_{\theta(\phi)}}(H)$, corresponding to the character $\eta_\sigma \in \widehat{\overline{\mathcal{S}_{\phi_{\theta(\phi)}}}}$. Let

$$\eta_{\sigma_0} = \eta_\sigma|_{\overline{\mathcal{S}_{\vartheta(\phi)}}},$$

and $\sigma_0 \in \Pi^L_{\vartheta(\phi)}(H_0)$ the irreducible tempered representation corresponding to η_{σ_0}. Then by LLC for H, there is a parabolic subgroup of H, say Q, with Levi component

$$L \simeq \mathrm{GL}_1(E) \times \cdots \times \mathrm{GL}_1(E) \times H_0,$$

so that σ is the unique irreducible quotient of the standard module

$$\mathrm{Ind}^H_Q(\chi_V|\det|^{r-n} \boxtimes \cdots \boxtimes \chi_V|\det|^1 \boxtimes \sigma_0).$$

According to Prasad's conjecture (Theorem 6.11), there exists a unique pure inner form G' of G such that the theta lift π of σ_0 to G' is non-zero. Moreover, π is in the L-packet $\Pi^L_\phi(G')$, corresponding to the character

$$\eta_\pi = \eta_{\sigma_0}|_{\mathcal{S}_\phi}.$$

We claim that the theta lift of π to the group H is just σ. The assertion of Proposition 6.10 follows from this claim.

Now we prove the claim. Indeed, when F is non-Archimedean, the claim follows from [26, Proposition 3.2] directly. When F is Archimedean, by the persistence principle (see [42, Proposition 4.1]) we know that there is a non-zero map

$$\Theta(\pi) \to \mathrm{Ind}^H_{Q'}(\chi_V|\det|^{(n-1-r)/2} \boxtimes \sigma_0),$$

where $\Theta(\pi)$ is the big theta lift of π to H, and Q' is the standard parabolic subgroup of H with Levi component $\mathrm{GL}_{r-n}(E) \times H_0$. Since the dual pair (G', H) is in the stable range, it follows from [51, Theorem A] that $\Theta(\pi) = \theta(\pi)$ is irreducible. Therefore the map above is injective. Applying both the MVW functor and the contragredient functor to the above map, we deduce that $\theta(\pi)$ is a quotient of $\mathrm{Ind}_{Q'}^{H}(\chi_V|\det|^{(r-n+1)/2} \boxtimes \sigma_0)$, which is also the unique quotient of the standard module

$$\mathrm{Ind}_{Q}^{H}(\chi_V|\det|^{r-n} \boxtimes \cdots \boxtimes \chi_V|\det|^{1} \boxtimes \sigma_0).$$

This completes the proof of our claim. ■

6.5. Globalization

To complete the proof of Theorem 6.3, we need to appeal to some global methods. Let ϕ be an L-parameter of good parity for $G = G(V)$. Write

$$\phi = \sum_i \phi_i \tag{6.5}$$

for some (not necessarily distinct) n_i-dimensional irreducible (conjugate) self-dual representations of L_E. We shall globalize ϕ to a generic elliptic A-parameter $\dot{\phi}$, by globalizing each ϕ_i separately.

We first consider the case when ϕ_i is a (conjugate) self-dual character.

Lemma 6.13. *Let $(\dot{F}, \dot{E})$ be a pair of number fields, and u be a place of $\dot{F}$ such that $(\dot{F}_u, \dot{E}_u) \simeq (F, E)$. In particular, $\dot{E} = \dot{F}$ in Case* O *and $\dot{E}/\dot{F}$ is a quadratic extension in Case* U. *Let T be a finite set of places of $\dot{F}$ that contains u. In Case* U, *we also require that $\dot{E}$ is not split at any place $v \in T$. Fix $\kappa = \pm 1$. For each place $v \in T$, let χ_v be a character of $\mathrm{GL}_1(\dot{E}_v)$ which is*

$$\begin{cases} \textit{quadratic} & \textit{in Case } \mathrm{O}, \\ \textit{conjugate self-dual of parity } \kappa & \textit{in Case } \mathrm{U}. \end{cases}$$

Then there exists an automorphic character $\dot{\chi}$ of $\mathrm{GL}_1(\mathbb{A}_{\dot{E}})$, which is also (conjugate) self-dual of the same parity as each χ_v, such that $\dot{\chi}_v = \chi_v$ for all $v \in T$.

Proof. For Case O, this is a special case of the Grunwald–Wang theorem (cf. [5, Chapter X, Theorem 5]). For Case U, it is proved in [32, Lemma 8.8]. ■

Next, we consider the case where $\dim \phi_i = m \geq 2$.

Lemma 6.14. *Let $(\dot{F}, \dot{E})$ be a pair of number fields, and u be a place of $\dot{F}$ such that $(\dot{F}_u, \dot{E}_u) \simeq (F, E)$. In particular, $\dot{E} = \dot{F}$ in Case* O *and $\dot{E}/\dot{F}$ is a quadratic extension in Case* U. *Let T be a finite set of places of $\dot{F}$ containing u. In Case* U, *we also require that $\dot{E}$ is not split at any place $v \in T$. Let $m \geq 2$ be a positive integer and fix $\kappa = \pm 1$. For each place $v \in T$, let ϕ_v be an irreducible m-dimensional*

$$\begin{cases} \textit{othogonal representation of } L_{\dot{E}_v} & \textit{in Case } \mathrm{O}, \\ \textit{conjugate self-dual representation of } L_{\dot{E}_v} \textit{ of parity } \kappa & \textit{in Case } \mathrm{U}. \end{cases}$$

Then there exists an irreducible cuspidal automorphic representation $\dot{\phi}$ of $\mathrm{GL}_m(\mathbb{A}_{\dot{E}})$, which is also (conjugate) self-dual of the same parity as each ϕ_v, such that $\dot{\phi}_v = \phi_v$ for all $v \in T$. Here, we regard ϕ_v as an irreducible representation of $\mathrm{GL}_m(E_v)$ by LLC for GL_m.

Proof. A similar statement is proved in [33, Lemma 6.7] for Case O and [39, Lemma 4.3.1] for Case U. Since our setting is slightly different from theirs, we provide a proof. We first consider the case when m is even in Case O and $\kappa = (-1)^{m-1}$ in Case U. Let $\dot{V}^+$ be the unique m-dimensional c-Hermitian space satisfying the following properties:

- in Case O, for all $v \in S$, the quadratic character associated to $\dot{V}^+_v$ is the same as $\det \phi_v$;
- for all places v of $\dot{F}$, the Hasse–Witt invariant (resp. sign) of $\dot{V}^+_v$ is $+1$, i.e. $\epsilon(\dot{V}^+_v) = 1$.

The existence of such a space is guaranteed by the local-global principle for orthogonal and Hermitian spaces. Let $\dot{G} = G(\dot{V}^+)^0$ be the identity component of the isometry group of $\dot{V}^+$, i.e. $\dot{G} = \mathrm{SO}(\dot{V}^+)$ in Case O, and $\dot{G} = \mathrm{U}(\dot{V}^+)$ in Case U. Since the space $\dot{V}^+$ is maximally split, the group $\dot{G}$ is quasi-split. For each $v \in T$, we pick an irreducible square-integrable representation $\pi_v \in \Pi^L_{\phi_v}(\dot{G}_v)$. We also pick up a finite place w of $\dot{F}$ outside T such that

$$\begin{cases} \dot{V}^+ \text{ is unramified (see [8, Section 2.3])} & \text{in Case O,} \\ \dot{E} \text{ is split at } w & \text{in Case U.} \end{cases}$$

In the setting of [70, Theorem 5.13], we take $S = T \cup \{w\}$ and choose two finite places v_1, v_2 of $\dot{F}$ not contained in S. Then by [70, Theorem 5.13], there exists an irreducible cuspidal automorphic representation $\dot{\pi}$ of $\dot{G}$ such that

- for all $v \in T$, $\dot{\pi}_v = \pi_v$;
- in Case O, $\dot{\pi}_w$ is tempered unramified;
- in Case U, $\dot{\pi}_w$ is a supercuspidal representation of $\dot{G}_w \simeq \mathrm{GL}_m(\dot{F}_w)$.

Based on the results of Arthur [4] and Mok [61], the cuspidal representation $\dot{\pi}$ has an elliptic A-parameter $\dot{\phi}$. The conditions we put on the place w guarantee that $\dot{\phi}$ is generic. Moreover, we know that

$$\dot{\phi}_v = \phi_v$$

for all $v \in S$, since the localization $\dot{\pi}_v = \pi_v$ lies in both $\Pi^L_{\dot{\phi}_v}(\dot{G}_v)$ and $\Pi^L_{\phi_v}(\dot{G}_v)$, and different local L-packets are disjoint from each other. From the irreducibility of ϕ_v, we can further conclude that $\dot{\phi}$ is simple. Therefore, $\dot{\phi}$ is an irreducible cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_{\dot{E}})$ that satisfies all our requirements.

Next we consider the case when m is odd in Case O and $\kappa = (-1)^m$ in Case U. In Case U, we choose a conjugate symplectic character χ_v at each $v \in T$. Then we can globalize ϕ_v by globalizing both $\phi_v \chi_v$ (which is of parity $\kappa = (-1)^{m-1}$ now) and χ_v (using Lemma 6.13). Similarly in Case O, we can twist ϕ_v by a quadratic character and assume $\det \phi_v = 1$ for all $v \in T$. Then a similar argument to the previous case works by replacing $\dot{V}^+$ by $\dot{W}^+$, where $\dot{W}^+$ is the unique $(m-1)$-dimensional symplectic space. This completes the proof of this lemma. ■

As an application of the previous two lemmas, we are now able to globalize an L-parameter ϕ of good parity.

Corollary 6.15. *Let ϕ be an L-parameter of good parity for $G = G(V)$. Then there exists a tuple of data $(\dot{F}, \dot{E}, \dot{V}, \dot{\phi}, u_1, u_2, w)$, where*

- *$\dot{F}$ is a number field, and $\dot{E}$ is either $\dot{F}$ itself or a quadratic extension of $\dot{F}$, according to our cases; if we are in Case* U *and F is non-Archimedean, we may choose $\dot{F}$ to be totally imaginary;*
- *$\dot{V}$ is a vector space over $\dot{E}$, equipped with a non-degenerate Hermitian c-sesquilinear form;*
- *$\dot{\phi}$ is a generic elliptic A-parameter for $\dot{G} = G(\dot{V})$;*
- *u_1, u_2, w are places of $\dot{F}$, and w is finite;*

such that the following conditions hold:

(1) *$(\dot{F}_{u_1}, \dot{E}_{u_1}, \dot{V}_{u_1}, \dot{\phi}_{u_1}) \simeq (\dot{F}_{u_2}, \dot{E}_{u_2}, \dot{V}_{u_2}, \dot{\phi}_{u_2}) \simeq (F, E, V, \phi)$;*

(2) *in Case* U*, $\dot{E}_w / \dot{F}_w$ is a quadratic field extension;*

(3) *$\dot{\phi}_w$ is a discrete L-parameter for $\dot{G}_w = G(\dot{V}_w)$; furthermore, we may choose $\dot{\phi}_w$ so that it does not contain a given character χ_w of $L_{\dot{E}_w}$;*

(4) *the localization maps $\mathcal{S}_{\dot{\phi}} \to \mathcal{S}_{\dot{\phi}_{u_1}}$ and $\mathcal{S}_{\dot{\phi}} \to \mathcal{S}_{\dot{\phi}_{u_2}}$ agree, and they are surjections;*

(5) *at the place w, the localization map $\mathcal{S}_{\dot{\phi}} \to \mathcal{S}_{\dot{\phi}_w}$ is an isomorphism.*

Proof. Firstly we choose a pair $(\dot{E}, \dot{F})$ of number fields, together with three places u_1, u_2, w of $\dot{F}$, satisfying the conditions below:

- $(\dot{E}_{u_1}, \dot{F}_{u_1}) \simeq (\dot{E}_{u_2}, \dot{F}_{u_2}) \simeq (E, F)$;
- in Case O, $\dot{F}_w$ is a finite extension of $\mathbb{Q}_2$ with a sufficiently big residue field;
- in Case U, $\dot{F}_w$ is a finite extension of $\mathbb{Q}_p$ for some $p \neq 2$ with a sufficiently big residue field, and $\dot{E}_w$ is a ramified quadratic field extension of $\dot{F}_w$; if further F is non-Archimedean, then $\dot{F}$ is totally imaginary.

The existence of such a pair of number fields will be proved in Appendix E. We write $\phi = \sum_i \phi_i$ as in (6.5). By the results in Appendices C and D, there are sufficiently many irreducible (conjugate) self-dual representations of $W_{\dot{E}_w}$. Hence we can pick up irreducible (conjugate) self-dual representations $\phi_{w,i}$ of $L_{\dot{E}_w}$ of the same dimension and parity as ϕ_i for each i, such that $\phi_{w,i} \not\simeq \phi_{w,i'}$ whenever $i \neq i'$. Moreover, we can further require that all $\phi_{w,i}$ are different from a given character χ_w of $L_{\dot{E}_w}$.

Let $u = u_1$ and $T = \{u_1, u_2, w\}$. By Lemmas 6.13 and 6.14, we can globalize each ϕ_i to an irreducible cuspidal representation $\dot{\phi}_i$ of $\mathrm{GL}_{n_i}(\mathbb{A}_{\dot{E}})$, which is (conjugate) self-dual of appropriate parity, such that

- $(\dot{\phi}_i)_{u_1} = (\dot{\phi}_i)_{u_2} = \phi_i$;

- $(\dot{\phi}_i)_w = \phi_{w,i}$.

Let

$$\dot{\phi} = \sum_i \dot{\phi}_i.$$

It follows from the local-global principle for orthogonal and Hermitian spaces that there exists a $\dot{V}$ over $\dot{E}$, equipped with a non-degenerate Hermitian c-sesquilinear form, such that

- $\dot{V}_{u_1} \simeq \dot{V}_{u_2} \simeq V$;
- $\dot{\phi}$ is a generic elliptic A-parameter for $\dot{G} = G(\dot{V})$.

Then the tuple of data $(\dot{F}, \dot{E}, \dot{V}, \dot{\phi}, u_1, u_2, w)$ satisfies all our requirements. ∎

6.6. *Multiplicity-freeness*

Let ϕ be an L-parameter of good parity for G and $\theta(\phi)$ be the local A-parameter for H as in (6.1). Recall that Theorem 6.2 asserts that the local A-packet $\Pi_{\theta(\phi)}(H)$ is multiplicity-free, and it has been proved by Mœglin and many others. In this subsection, we shall give an independent and self-contained proof by using global methods and theta lifts.

Let $(\dot{F}, \dot{E}, \dot{V}, \dot{\phi}, u_1, u_2, w)$ be as in Corollary 6.15. In Case U, we fix a trace zero element $\delta \in \dot{E}^\times$. Let $\dot{W}$ be a c-skew-Hermitian space over $\dot{E}$ as in (3.1) which is

$$\begin{cases} 2r\text{-dimensional} & \text{in Case O,} \\ (2r+1)\text{-dimensional} & \text{in Case U}_0, \\ (2r+2)\text{-dimensional} & \text{in Case U}_1, \end{cases}$$

and let $\dot{H} = H(\dot{W})$. We also let $\dot{W}_0$ be a c-skew-Hermitian space over $\dot{E}$ in the Witt tower containing $\dot{W}$, which is

$$\begin{cases} \dim V\text{-dimensional} & \text{in Case O,} \\ (\dim V + 1)\text{-dimensional} & \text{in Case U when } F \text{ is non-Archimedean,} \\ (\dim V - 1)\text{-dimensional} & \text{in Case U when } F \text{ is real.} \end{cases}$$

Let $W_0 = \dot{W}_{0,u_1}$. We put $\dot{H}_0 = H(\dot{W}_0)$ and $H_0 = H(W_0)$. We will use the symbols $\dot{G}'$, $\dot{H}_0'$, and H_0' to denote pure inner forms of $\dot{G}$, $\dot{H}_0$, and H_0 respectively.

Let $\psi_{\dot{F}}$ be a non-trivial additive character of $\mathbb{A}/\dot{F}$, such that $\psi_{\dot{F},u}$ is in the $F^{\times 2}$-orbit of ψ_F for $u \in \{u_1, u_2\}$. We shall use the additive character $\psi_{\dot{F}}$ to fix Whittaker data $\mathscr{W}$ and $\mathscr{W}'$ of $\dot{G}$ and $\dot{H}$ (and also $\dot{H}_0$), as described in Section 3.5. We also globalize characters (χ_V, χ_W) of $E^\times$ to a pair of characters of $\dot{E}^\times \backslash \mathbb{A}_{\dot{E}}^\times$ as follows:

$$\chi_{\dot{V}} = \begin{cases} \text{the quadratic character associated to } \dot{V} & \text{in Case O,} \\ \text{a character such that } \chi_{\dot{V}}|_{\mathbb{A}_{\dot{F}}^\times} = \omega_{\dot{E}/\dot{F}}^{\dim V} & \text{in Case U,} \end{cases}$$

and

$$\chi_{\dot{W}} = \begin{cases} \text{the trivial character of } \dot{F}^\times\backslash\mathbb{A}_{\dot{F}}^\times & \text{in Case O,} \\ \text{a character such that } \chi_{\dot{W}}|_{\mathbb{A}_{\dot{F}}^\times} = \omega_{\dot{E}/\dot{F}}^{\dim W} & \text{in Case U.} \end{cases}$$

We will consider the theta lift between $(\dot{G}', \dot{H})$, which is in the stable range case, with respect to $(\psi_{\dot{F}}, \chi_{\dot{V}}, \chi_{\dot{W}})$. According to Corollary 6.15, we may globalize ϕ suitably so that $\dot{\phi}_w$ does not contain the character $\chi_{\dot{W},w}$, and we will henceforth assume this. Moreover, we will also consider the theta lift between $(\dot{G}', \dot{H}_0')$, which is in the almost equal rank case, with respect to some auxiliary data $(\psi_{\dot{F}}, \chi'_{\dot{V}}, \chi'_{\dot{W}})$. In Case O, there is no flexibility of choosing such data. However in Case U, we shall choose $(\chi'_{\dot{V}}, \chi'_{\dot{W}})$ suitably in later proofs depending on our needs. Indeed, the flexibility of choosing $(\chi'_{\dot{V}}, \chi'_{\dot{W}})$ is a key point in our proof for Case U. To distinguish notations, we use θ to denote the theta lift between $(\dot{G}', \dot{H})$, and use ϑ to denote the theta lift between $(\dot{G}', \dot{H}_0')$. We first show the following:

Lemma 6.16. *Let $\dot{G}'$ be a pure inner form of $\dot{G}$, and $\dot{\pi}$ be an irreducible representation of $\dot{G}'(\mathbb{A})$ such that*

(1) *the L-parameter of $\dot{\pi}_v$ is $\dot{\phi}_v$ for almost all v;*

(2) *$\dot{\pi}_w$ is in the L-packet $\Pi^L_{\dot{\phi}_w}(\dot{G}'_w)$.*

Then $m_{\mathrm{disc}}(\dot{\pi}) \leq 1$. Moreover, if $m_{\mathrm{disc}}(\dot{\pi}) = 1$, then $\dot{\pi}_{u_1} \in \Pi^L_{\phi}(\dot{G}'_{u_1})$.

Proof. If $m_{\mathrm{disc}}(\dot{\pi}) = 0$, then our conclusions hold. So we may assume $m_{\mathrm{disc}}(\dot{\pi}) \geq 1$ in the rest of the proof. Since $\dot{\phi}$ is generic, by Proposition 5.1 we know that

$$m_{\mathrm{cusp}}(\dot{\pi}) = m_{\mathrm{disc}}(\dot{\pi}) = m(\dot{\pi}),$$

and any realization $\mathcal{V}$ of $\dot{\pi}$ in $\mathcal{A}(\dot{G}')$ lies in $\mathcal{A}_{\mathrm{cusp}}(\dot{G}')$. We will prove this lemma by considering (automorphic) theta lifts between $(\dot{G}', \dot{H}_0')$ for pure inner forms $\dot{H}_0'$ of $\dot{H}_0$, with respect to the datum $(\psi_{\dot{F}}, \chi'_{\dot{V}}, \chi'_{\dot{W}})$. We do it case by case, and the choice of $(\chi'_{\dot{V}}, \chi'_{\dot{W}})$ will be specified later in each case.

Case O: In this case, there is no other pure inner form of $\dot{H}_0$, nor the flexibility of choosing the auxiliary datum. By local theta correspondence for unramified representations, we know that $\vartheta(\dot{\pi}_v) \neq 0$ for almost all places v of $\dot{F}$. Hence

$$T' = \left\{v \text{ place of } \dot{F} \mid \vartheta(\dot{\pi}_v) = 0\right\}$$

is a finite set. Let

$$T = \begin{cases} T' & \text{if } |T'| \text{ is even,} \\ T' \cup \{w\} & \text{if } |T'| \text{ is odd} \end{cases}$$

and

$$\dot{\pi} \otimes \det_T = \dot{\pi} \otimes \bigotimes_{v \in T} \det_v.$$

Then

$$m_{\mathrm{cusp}}(\dot{\pi} \otimes \det_T) = m_{\mathrm{cusp}}(\dot{\pi}) \geq 1.$$

Indeed, any realization $\mathcal{V}$ of $\dot{\pi}$ in $\mathcal{A}_{\mathrm{cusp}}(\dot{G}')$ gives a realization

$$\mathcal{V} \otimes \det_T = \left\{ f \otimes \bigotimes_{v \in T} \det_v \;\middle|\; f \in \mathcal{V} \right\}$$

of $\dot{\pi} \otimes \det_T$ in $\mathcal{A}_{\mathrm{cusp}}(\dot{G}')$, and vice versa. It follows from the conservation relation [72] that $\vartheta(\dot{\pi}_v \otimes \det_v) \neq 0$ for all places $v \in T'$. Moreover, since $\mathbb{1} \not\subset \dot{\phi}_w$, it follows from Theorem 6.11 (2) that both $\vartheta(\dot{\pi}_w)$ and $\vartheta(\dot{\pi}_w \otimes \det_w)$ are non-zero. Hence

$$\vartheta((\dot{\pi} \otimes \det_T)_v) \neq 0 \quad \text{for all places } v \text{ of } \dot{F}.$$

For any realization $\mathcal{V}$ of $\dot{\pi} \otimes \det_T$ in $\mathcal{A}_{\mathrm{cusp}}(\dot{G}')$, consider the automorphic theta lift $\vartheta^{\mathrm{aut}}(\mathcal{V})$ of $\mathcal{V}$ to $\dot{H}_0$. Again, since $\mathbb{1} \not\subset \dot{\phi}_w$, we know that $\vartheta(\pi_w)$ and $\vartheta(\pi_w \otimes \det_w)$ are the first occurrences of π_w and $\pi_w \otimes \det_w$. Hence globally, $\vartheta^{\mathrm{aut}}(\mathcal{V})$ is either zero or the first occurrence of $\mathcal{V}$ in the Witt tower containing $\dot{W}_0$. This implies that $\vartheta^{\mathrm{aut}}(\mathcal{V})$ is cuspidal. We would like to show that $\vartheta^{\mathrm{aut}}(\mathcal{V})$ is indeed non-zero. To show this, we investigate the L-function $L(s, \dot{\pi} \otimes \det_T)$.

By a result of Jacquet–Shalika [34], the (full) L-function $L(s, \dot{\phi})$ is holomorphic and non-zero at $s = 1$. Since $\dot{\phi}_v$ is almost tempered for every place v of $\dot{F}$, we know that the local L-factors $L(s, \dot{\phi}_v)$ are holomorphic when $\Re(s) \geq 1/2$. Hence the partial L-functions

$$L^S(s, \dot{\pi} \otimes \det_T) = L^S(s, \dot{\phi})$$

are also holomorphic and non-zero at $s = 1$. Here S is a sufficiently large finite set of places of $\dot{F}$, and $L^S(s, \dot{\pi} \otimes \det_T)$ is the partial L-function of $\dot{\pi} \otimes \det_T$ relative to the standard representation of ${}^L\dot{G}'$. On the other hand, since the local L-factors $L(s, \dot{\pi}_v)$ and $L(s, \dot{\pi}_v \otimes \det_v)$ can never have a zero, the complete L-function $L(s, \dot{\pi} \otimes \det_T)$ is also non-zero at $s = 1$. Finally, we claim that this L-function must be holomorphic at $s = 1$. Suppose on the contrary that it has a pole at $s = 1$. Then [81, Theorem 10.1] asserts that $\mathcal{V}$ has non-zero automorphic theta lift to some symplectic group $H(\dot{W}_-)$, where $\dot{W}_-$ is a symplectic space of dimension strictly less than $\dot{W}_0$. This contradicts the fact that $\vartheta(\pi_w)$ and $\vartheta(\pi_w \otimes \det_w)$ are the first occurrences of π_w and $\pi_w \otimes \det_w$. Hence $L(s, \dot{\pi} \otimes \det_T)$ is holomorphic and non-zero at $s = 1$. It then follows from the Rallis inner product formula [25, Theorem 1.3], [81, Theorem 10.3] that the automorphic theta lift $\vartheta^{\mathrm{aut}}(\mathcal{V})$ is non-vanishing.

Let $\dot{\sigma}_0 = \vartheta^{\mathrm{abs}}(\dot{\pi} \otimes \det_T)$ be the abstract theta lift to $\dot{H}_0'$. Then by multiplicity preservation [18, Proposition 2.6], we have

$$m_{\mathrm{disc}}(\dot{\sigma}_0) \geq m_{\mathrm{cusp}}(\dot{\sigma}_0) \geq m_{\mathrm{cusp}}(\dot{\pi} \otimes \det_T).$$

Also, it follows from local theta correspondence for unramified representations that $\dot{\sigma}_0$ is an irreducible summand of $L^2_{\vartheta(\dot{\phi})}(\dot{H}_0)$, where

$$\vartheta(\dot{\phi}) = \dot{\phi}(\chi'_{\dot{W}})^{-1}\chi'_{\dot{V}} + \chi'_{\dot{V}}.$$

Since $\vartheta(\dot{\phi})$ is generic, AMF for $\dot{H}_0$ (Theorem 3.4) implies that

$$m_{\mathrm{disc}}(\dot{\sigma}_0) = 1.$$

Thus, combining these (in)equalities, we get $m_{\mathrm{disc}}(\dot{\pi}) = 1$. Moreover, by Theorem 3.4, we also have $\dot{\sigma}_{0,u_1} \in \Pi^L_{\vartheta(\dot{\phi})_{u_1}}(\dot{H}'_{0,u_1})$. It then follows from Theorem 6.11 and Remark 2.3 (1) that $\dot{\pi}_{u_1} \in \Pi^L_{\phi}(\dot{G}'_{u_1})$.

Case U_0*, and* F *is non-Archimedean:* Recall that in Corollary 6.15, $\dot{F}$ is chosen to be a totally imaginary number field in this case. We let

$$(\chi'_{\dot{V}}, \chi'_{\dot{W}}) = (\chi_{\dot{V}}, \chi_{\dot{W}}).$$

For each place v of $\dot{F}$, by the conservation relation [72], there is a skew-Hermitian space $W'_{0,v}$ of the same dimension as $\dot{W}_{0,v}$ such that

$$\vartheta(\dot{\pi}_v) \neq 0.$$

Here $\vartheta(\dot{\pi}_v)$ is the theta lift of $\dot{\pi}_v$ to $H'_{0,v}$ with respect to $(\psi_{F,v}, \chi'_{\dot{V},v}, \chi'_{\dot{W},v})$. Since $\chi'_{\dot{W},w} \not\subset \dot{\phi}_w$, it follows from Theorem 6.11 that we will have two choices of the skew-Hermitian space $W'_{0,w}$ at the place w. The flexibility at w allows us to pick these local skew-Hermitian spaces coherently so that they form a global skew-Hermitian space $\dot{W}'_0$ over $\dot{E}$. Let $\dot{H}'_0 = H(\dot{W}'_0)$.

The rest of the proof in this case is similar to Case O. Let $\dot{\sigma}_0 = \vartheta^{\mathrm{abs}}(\dot{\pi})$ be the abstract theta lift to $\dot{H}'_0$. Then one can show that

$$m_{\mathrm{disc}}(\dot{\sigma}_0) \geq m_{\mathrm{cusp}}(\dot{\sigma}_0) \geq m_{\mathrm{cusp}}(\dot{\pi}),$$

and $\dot{\sigma}_0$ is an irreducible summand of $L^2_{\vartheta(\dot{\phi})}(\dot{H}_0)$, where

$$\vartheta(\dot{\phi}) = \dot{\phi}(\chi'_{\dot{W}})^{-1}\chi'_{\dot{V}} + \chi'_{\dot{V}}.$$

Applying Proposition 2.8, we get

$$m_{\mathrm{disc}}(\dot{\sigma}_0) = 1.$$

Combining these (in)equalities, we get $m_{\mathrm{disc}}(\dot{\pi}) = 1$. Moreover, by Proposition 2.8, $\dot{\sigma}_{0,u_1} \in \Pi^L_{\vartheta(\dot{\phi})_{u_1}}(\dot{H}'_{0,u_1})$. It then follows from Theorem 6.11 that $\dot{\pi}_{u_1} \in \Pi^L_{\phi}(\dot{G}'_{u_1})$.

Case U_1*, and* F *is non-Archimedean:* In this case, $\dot{F}$ is a totally imaginary number field. Then the lemma follows from Proposition 2.8 directly.

Case U, and F *is real:* In this case, we argue by induction on $\dim V$. Recall that ϕ is an L-parameter of good parity for G, so it must be of the form

$$\phi = m_1\chi_1 + \cdots + m_r\chi_r,$$

where χ_i is a conjugate self-dual character of $L_{\mathbb{C}} = \mathbb{C}^\times$, and m_i is some positive integer. Recall that in the proof of Corollary 6.15, to globalize the L-parameter ϕ, we globalized each irreducible constituent of ϕ separately, and then added them together. Hence $\dot{\phi}$ is a sum of one-dimensional automorphic characters in this case. We pick up a character $\dot{\chi}$ such that $\dot{\chi} \subset \dot{\phi}$, and set

$$(\chi'_{\dot{V}}, \chi'_{\dot{W}}) = (\chi_{\dot{V}}, \dot{\chi}).$$

Since $\dot{\phi}$ is generic and $\chi'_{\dot{W}} \subset \dot{\phi}$, we know that the function

$$L^S(s, \dot{\pi} \times (\chi'_{\dot{W}})^{-1}) = L^S(s, \dot{\phi}(\chi'_{\dot{W}})^{-1})$$

is holomorphic when $\Re(s) > 1$ and has a pole at $s = 1$, where S is a sufficiently large finite set of places of $\dot{F}$, and $L^S(s, \dot{\pi} \times (\chi'_{\dot{W}})^{-1})$ is the partial L-function associated to $\dot{\pi}$ and $(\chi'_{\dot{W}})^{-1}$. Hence the complete L-function $L(s, \dot{\pi} \times (\chi'_{\dot{W}})^{-1})$ is also holomorphic when $\Re(s) > 1$ and has a pole at $s = 1$, because the local L-factors $L(s, \dot{\pi}_v \times (\chi'_{\dot{W},v})^{-1})$ are holomorphic and non-zero when $\Re(s) > 1/2$. It then follows from the Rallis inner product formula [43, Theorem 7.2.5], [81, Theorem 10.1] that there exists a pure inner form $\dot{H}'_0 = H(\dot{W}'_0)$ of $\dot{H}_0$ such that for any realization $\mathcal{V}$ of $\dot{\pi}$ in $\mathcal{A}_{\mathrm{cusp}}(\dot{G}')$, we have

$$\vartheta^{\mathrm{aut}}(\mathcal{V}) \neq 0,$$

where $\vartheta^{\mathrm{aut}}(\mathcal{V})$ is the automorphic theta lift of $\mathcal{V}$ to $\dot{H}'_0$. Moreover, $\vartheta^{\mathrm{aut}}(\mathcal{V})$ is the first occurrence of $\mathcal{V}$ in the Witt tower containing $\dot{W}'_0$. This implies that $\vartheta^{\mathrm{aut}}(\mathcal{V})$ is cuspidal.

Let $\dot{\sigma}_0 = \vartheta^{\mathrm{abs}}(\dot{\pi})$ be the abstract theta lift to $\dot{H}'_0$. Then by multiplicity preservation [18, Proposition 2.6], we have

$$m_{\mathrm{disc}}(\dot{\sigma}_0) \geq m_{\mathrm{cusp}}(\dot{\sigma}_0) \geq m_{\mathrm{cusp}}(\dot{\pi}).$$

Also, it follows from local theta correspondence for unramified representations that $\dot{\sigma}_0$ is an irreducible summand of $L^2_{\vartheta(\dot{\phi})}(\dot{H}_0)$, where

$$\vartheta(\dot{\phi}) = (\dot{\phi} - \chi'_{\dot{W}})(\chi'_{\dot{W}})^{-1}\chi'_{\dot{V}}.$$

Since $\dot{H}'_0$ is a unitary group of $(\dim V - 1)$-variables, by the induction hypothesis, the lemma holds for $\dot{H}'_0$. Hence

$$m_{\mathrm{disc}}(\dot{\sigma}_0) = 1.$$

Combining these (in)equalities, we get $m_{\mathrm{disc}}(\dot{\pi}) = 1$. Moreover, by the induction hypothesis, $\dot{\sigma}_{0,u_1} \in \Pi^L_{\vartheta(\dot{\phi})_{u_1}}(\dot{H}'_{0,u_1})$. It then follows from Theorem 6.11 that $\dot{\pi}_{u_1} \in \Pi^L_{\phi}(\dot{G}'_{u_1})$. ∎

For any irreducible unitary representation σ of H and any character η of $\overline{S_{\theta(\phi)}}$, we define the multiplicity $m(\sigma, \eta)$ by

$$m(\sigma, \eta) = \dim \mathrm{Hom}_{\overline{S_{\theta(\phi)}} \times H}(\eta \boxtimes \sigma, \Pi_{\theta(\phi)}(H)).$$

Proposition 6.17. (1) *Let σ be an irreducible unitary representation σ of H. Then, for any character η of $\overline{S_{\theta(\phi)}}$, we have*

$$m(\sigma, \eta) \leq 1,$$

with equality for at most one η. Hence $\Pi_{\theta(\phi)}(H)$ is multiplicity-free.

(2) *The theta lift between (G', H) for all pure inner forms G' of G defines an injection*

$$\theta : \Pi_{\theta(\phi)}(H) \to \bigsqcup_{G'} \Pi^L_{\phi}(G'),$$

where the disjoint union is taken over all pure inner forms of G.

Proof. Assume that $m(\sigma, \eta) > 0$ for some η. Let $\dot{\phi}$, $\chi_{\dot{V}}$, $\chi_{\dot{W}}$, $\dot{G}$ and $\dot{H}$ be as given at the beginning of this subsection, and

$$\theta(\dot{\phi}) = \dot{\phi}\chi_{\dot{W}}^{-1}\chi_{\dot{V}} + \chi_{\dot{V}} \boxtimes S_{2r-2n+1}$$

be an elliptic A-parameter for $\dot{H}$. Since $\dot{\phi}$ is generic, it follows from Lemma 5.6 and (5.2) that $\epsilon_{\theta(\dot{\phi})}$ is trivial. We define an abstract irreducible representation $\dot{\sigma} = \bigotimes_v \dot{\sigma}_v$ of $\dot{H}(\mathbb{A})$ as follows:

- $\dot{\sigma}_{u_1} = \dot{\sigma}_{u_2} = \sigma$;
- at a place $v \notin \{u_1, u_2\}$, $\dot{\sigma}_v$ is the irreducible representation in the L-packet $\Pi^L_{\phi_{\theta(\dot{\phi})_v}}(\dot{H}_v)$ associated to the trivial character of $\overline{S_{\phi_{\theta(\dot{\phi})_v}}}$.

By Theorem 3.4, we have an embedding

$$\Big(\bigoplus_{\eta \in \widehat{\overline{S_{\theta(\phi)}}}} (m(\sigma, \eta)\sigma \otimes m(\sigma, \eta)\sigma) \Big) \otimes \Big(\bigotimes_{v \notin \{u_1, u_2\}} \dot{\sigma}_v \Big) \hookrightarrow L^2_{\theta(\dot{\phi})}(\dot{H}).$$

In particular,

$$m_{\text{disc}}(\dot{\sigma}) \geq \sum_{\eta \in \widehat{\overline{S_{\theta(\phi)}}}} m(\sigma, \eta)^2 > 0.$$

Moreover, it follows from Remark 5.3 that there exists a pure inner form $\dot{G}'$ of $\dot{G}$, and an irreducible summand $\dot{\pi}$ of $L^2_{\dot{\phi}}(\dot{G}')$, such that $\dot{\sigma} = \theta^{\text{abs}}(\dot{\pi})$, and

$$m_{\text{disc}}(\dot{\pi}) = m_{\text{disc}}(\dot{\sigma}).$$

It follows from our construction and Proposition 6.10 that $\dot{\pi}_w \in \Pi^L_{\dot{\phi}_w}(\dot{G}'_w)$. Hence

$$m_{\text{disc}}(\dot{\pi}) \leq 1$$

by Lemma 6.16. Combining these (in)equalities, we obtain

$$1 \geq \sum_{\eta \in \widehat{\overline{S_{\theta(\phi)}}}} m(\sigma, \eta)^2.$$

Hence the first statement holds.

For the second statement, notice that if $\sigma \in \Pi_{\theta(\phi)}(H)$, Lemma 6.16 also asserts that

$$\theta(\sigma) = \dot{\pi}_{u_1} \in \Pi_\phi^L(\dot{G}'_{u_1}).$$

Hence it follows from the conservation relation [72] that the theta lift between (G', H) for all pure inner forms G' of G gives a well-defined map

$$\theta : \Pi_{\theta(\phi)}(H) \to \bigsqcup_{G'} \Pi_\phi^L(G'),$$

where the disjoint union is taken over all pure inner forms of G. By Howe duality, this map is an injection. This completes the proof. ■

6.7. The last jigsaw piece

We retain the notations of the last subsection. Having proved Proposition 6.17, we know that $\Pi_\phi^\theta(G) \subset \Pi_\phi^L(G)$ as sets. To finish the proof of Theorem 6.3, we only need to show the following.

Proposition 6.18. *For any pure inner form G' of G, and any irreducible representation π in the L-packet $\Pi_\phi^L(G')$, the theta lift σ of π to H lies in the A-packet $\Pi_{\theta(\phi)}(H)$. Moreover,*

$$\mathcal{J}_{\mathscr{W}}^L(\pi) = \ell^*(\mathcal{J}_{\mathscr{W}'}(\sigma)).$$

With the help of Proposition 6.10, we can first prove Proposition 6.18 for a large class of $\pi \in \Pi_\phi^L(G')$.

Lemma 6.19. *Let G' be a pure inner form of G, and π be an irreducible representation in the L-packet $\Pi_\phi^L(G')$. If the theta lift σ_0 of π to H_0 is non-zero (with respect to the datum (ψ_F, χ_V, χ_W)), then the conclusion of Proposition 6.18 holds for π. In particular, if we are in one of the following cases:*

- *Case* O*;*
- *Case* U*, and F is non-Archimedean,*

and $\chi_W \not\subset \phi$, then the conclusion of Proposition 6.18 holds for any $\pi \in \Pi_\phi^L(G')$.

Proof. The first assertion can be proved exactly as Proposition 6.10. To prove that Proposition 6.18 holds in the cases listed above, one just needs to note that $\vartheta(\pi) \neq 0$ for any $\pi \in \Pi_\phi^L(G')$ in these special cases (see Theorem 6.11). ■

Based on this lemma, we now fill in the last jigsaw piece.

Proof of Proposition 6.18. We will argue case by case. To simplify notations, we let

$$\eta_\pi = \mathcal{J}_{\mathscr{W}}^L(\pi) \quad \text{and} \quad \eta_\sigma = \mathcal{J}_{\mathscr{W}'}(\sigma).$$

Case O: By Lemma 6.19, we only need to consider the case when the theta lift of π to H_0 is zero. It then follows from the conservation relation [72] that the theta lift of $\pi \otimes \det$

to H_0 is non-zero. In particular, Proposition 6.18 holds for $\pi \otimes \det$. Next we appeal to the global method to compare the theta lifts $\theta(\pi)$ and $\theta(\pi \otimes \det)$ of π and $\pi \otimes \det$ to H.

As in the proof of Proposition 6.17, we define an abstract irreducible representation $\dot{\sigma}' = \bigotimes_v \dot{\sigma}'_v$ of $\dot{H}(\mathbb{A})$:

- we set $\dot{\sigma}'_{u_1} = \dot{\sigma}'_{u_2} = \theta(\pi \otimes \det)$;
- at a place $v \notin \{u_1, u_2\}$, $\dot{\sigma}'_v$ is the irreducible representation in the L-packet $\Pi^L_{\phi_{\theta(\dot{\phi})_v}}(\dot{H}_v)$ associated to the trivial character of $\overline{\mathcal{S}_{\phi_{\theta(\dot{\phi})_v}}}$.

By Theorem 3.4, Lemma 5.6 and (5.2), $\dot{\sigma}'$ is a summand of $L^2_{\theta(\dot{\phi})}(\dot{H})$. Remark 5.3 then implies that there exists a pure inner form $\dot{G}'$ of $\dot{G}$, and an irreducible summand $\dot{\pi}'$ of $L^2_{\dot{\phi}}(\dot{G}')$, such that $\dot{\sigma}' = \theta^{\mathrm{abs}}(\dot{\pi}')$. According to the construction, we must have

$$\dot{G}'_{u_1} = \dot{G}'_{u_2} = G' \quad \text{and} \quad \dot{\pi}'_{u_1} = \dot{\pi}'_{u_2} = \pi \otimes \det.$$

Now we define another abstract irreducible representation $\dot{\pi}$ of $\dot{G}'(\mathbb{A})$ by setting $\dot{\pi} = \dot{\pi}' \otimes \det_T$ with $T = \{u_1, w\}$. More precisely,

- at $v \in \{u_1, w\}$, $\dot{\pi}_v = \dot{\pi}'_v \otimes \det_v$;
- at a place $v \notin \{u_1, w\}$, $\dot{\pi}_v = \dot{\pi}'_v$.

Then $\dot{\pi}$ is also an irreducible summand of $L^2_{\dot{\phi}}(\dot{G}')$ such that

$$\dot{\pi}_{u_1} = \pi.$$

Let $\dot{\sigma} = \theta^{\mathrm{abs}}(\dot{\pi})$. We deduce from Corollary 5.2 that $\dot{\sigma}$ is a summand of $L^2_{\theta(\dot{\phi})}(\dot{H})$. Hence by AMF (Theorem 3.3), we have

$$\sigma = \dot{\sigma}_{u_1} \in \Pi_{\theta(\phi)}(H).$$

This proves the first assertion of Proposition 6.18 for Case O.

Next we prove the second assertion. By Theorem 3.4, Lemma 5.6 and (5.2), we have

$$\ell^*(\mathcal{J}_{\mathscr{W}'}(\dot{\sigma})) = 1, \tag{6.6}$$

where

$$\ell^*(\mathcal{J}_{\mathscr{W}'}(\dot{\sigma}))(x) = \prod_v \ell^*_v(\mathcal{J}_{\mathscr{W}'_v}(\dot{\sigma}_v))(x_v)$$

for $x \in \overline{\mathcal{S}_{\dot{\phi}}}$. The character $\ell^*_v(\mathcal{J}_{\mathscr{W}'_v}(\dot{\sigma}_v))$ can be computed explicitly as follows:

- At a place $v \notin \{u_1, u_2, w\}$, $\ell^*_v(\mathcal{J}_{\mathscr{W}'_v}(\dot{\sigma}_v))$ is the trivial character of $\mathcal{S}_{\dot{\phi}_v}$.
- At the place u_2, Proposition 6.18 holds for $\pi \otimes \det$ by hypothesis. Thus

$$\ell^*_{u_2}(\mathcal{J}_{\mathscr{W}'_{u_2}}(\dot{\sigma}_{u_2})) = \mathcal{J}^L_{\mathscr{W}_{u_2}}(\pi \otimes \det) = \eta_\pi \cdot \kappa_\phi.$$

Here we have made use of Remark 2.3 in the last equality.

- At the place w, since $\mathbb{1} \not\subset \dot{\phi}_w$, it follows from Lemma 6.19 that Proposition 6.18, and hence Theorem 6.3, holds for $\dot{\phi}_w$. Thus

$$\ell_w^*(\mathcal{J}_{\mathscr{W}_w'}(\dot{\sigma}_w)) = \mathcal{J}_{\mathscr{W}_w}^L(\dot{\pi}_w' \otimes \det{}_w) = \kappa_{\dot{\phi}_w}.$$

  Here again we have made use of Remark 2.3 in the last equality.

Hence

$$\prod_v \ell_v^*(\mathcal{J}_{\mathscr{W}_v'}(\dot{\sigma}_v))(x) = \eta_\sigma(x_{u_1}) \cdot \eta_\pi(x_{u_2}) \cdot \kappa_\phi(x_{u_2}) \cdot \kappa_{\dot{\phi}_w}(x_w) \tag{6.7}$$

for all $x \in \overline{\mathcal{S}_{\dot{\phi}}}$. On the other hand, it is easy to check that

$$\kappa_\phi(x_{u_2}) \cdot \kappa_{\dot{\phi}_w}(x_w) = 1 \tag{6.8}$$

for all $x \in \overline{\mathcal{S}_{\dot{\phi}}}$. Combining (6.6)–(6.8), we get

$$\eta_\pi(x_{u_2}) = \ell_{u_1}^*(\eta_\sigma)(x_{u_1})$$

for all $x \in \overline{\mathcal{S}_{\dot{\phi}}}$. Finally, since the localization maps $\mathcal{S}_{\dot{\phi}} \to \mathcal{S}_{\dot{\phi}_{u_1}}$ and $\mathcal{S}_{\dot{\phi}} \to \mathcal{S}_{\dot{\phi}_{u_2}}$ agree and are surjective, we deduce that $\eta_\pi = \ell_{u_1}^*(\eta_\sigma)$.

This completes the proof in Case O.

Case U_0*, and* F *is non-Archimedean:* In this case, $\dot{F}$ is a totally imaginary field, W_0 is a $(2n+1)$-dimensional skew-Hermitian space over $\dot{E}$, and $H_0 = \mathrm{U}(W_0)$. For a pure inner form G' of G and an irreducible tempered representation π in the L-packet $\Pi_\phi^L(G')$, Lemma 6.19 asserts that if the theta lift of π to H_0 (with respect to (ψ_F, χ_V, χ_W)) is non-zero, then Proposition 6.18 holds for π. Next we appeal to the global method to reduce the general situation to this known situation.

We pick a pair of characters $(\chi_{\dot{V}}', \chi_{\dot{W}}')$ such that $\chi_{\dot{W},v}' \not\subset \dot{\phi}_v$ for $v \in \{u_1, w\}$. Firstly we use the almost equal rank theta lift to globalize the representation π. As in the proof of Lemma 6.16, let

$$\vartheta(\dot{\phi}) = \dot{\phi}(\chi_{\dot{W}}')^{-1}\chi_{\dot{V}}' + \chi_{\dot{V}}'.$$

We define an irreducible automorphic subrepresentation $\dot{\sigma}_0 = \bigotimes_v \dot{\sigma}_{0,v}$ of $L^2_{\vartheta(\dot{\phi})}(\dot{H}_0)$ as follows:

- at a place $v \notin \{u_1, w\}$, $\dot{\sigma}_{0,v}$ is the irreducible representation in the L-packet $\Pi^L_{\vartheta(\dot{\phi})_v}(\dot{H}_{0,v})$ associated to the trivial character of $\overline{\mathcal{S}_{\vartheta(\dot{\phi})_v}}$;
- at the place u_1, $\dot{\sigma}_{0,u_1} = \vartheta(\pi)$ is the theta lift of π to $\dot{H}_{0,u_1}$, which is non-zero by Theorem 6.11;
- at the place w, $\dot{\sigma}_{0,w}$ is the tempered representation in the L-packet $\Pi^L_{\vartheta(\dot{\phi})_w}(\dot{H}_{0,w})$ corresponding to the character $\eta_{0,w}$, determined by the formula

$$\prod_v \eta_{0,v} = 1,$$

  where $\eta_{0,v} = \mathcal{J}^L_{\mathscr{W}_v'}(\dot{\sigma}_{0,v})$, and we regard $\prod_v \eta_{0,v}$ as a character of the global component group $\mathcal{S}_{\vartheta(\dot{\phi})}$ through the localization maps.

By Theorem 3.4 and (5.2), $\dot{\sigma}_0$ is a summand of $L^2_{\vartheta(\dot{\phi})}(\dot{H}_0)$. Using the same argument as in Lemma 6.16, we can show that there exists a pure inner form $\dot{G}'$ of $\dot{G}$ such that the abstract theta lift $\dot{\pi} = \vartheta^{\mathrm{abs}}(\dot{\sigma}_0)$ of $\dot{\sigma}_0$ to $\dot{G}'$ is non-zero, and

$$m_{\mathrm{disc}}(\dot{\pi}) \geq m_{\mathrm{cusp}}(\dot{\pi}) \geq m_{\mathrm{cusp}}(\dot{\sigma}_0).$$

Hence $\dot{\pi}$ is a summand in $L^2_{\dot{\phi}}(\dot{G}')$, and $\dot{\pi}_v \in \Pi^L_{\dot{\phi}_v}(\dot{G}'_v)$ for all places v of $\dot{F}$. By the conservation relation [72], we know that

$$(\dot{G}'_{u_1}, \dot{\pi}_{u_1}) \simeq (G', \pi).$$

Also, Theorem 6.11 implies that

$$\prod_v \eta_v(x_v) = 1 \tag{6.9}$$

for all $x \in \mathcal{S}_{\dot{\phi}}$, where $\eta_v = \mathcal{J}^L_{\mathscr{W}_v}(\dot{\pi}_v)$. This product is well-defined, since $\eta_v = 1$ for all places $v \notin \{u_1, w\}$.

Next we consider the stable range theta lift of $\dot{\pi}$ to extract some other information. Let $\dot{\sigma} = \theta^{\mathrm{abs}}(\dot{\pi})$. We deduce from J.-S. Li's inequality (Theorem 3.1) that $\dot{\sigma}$ is a summand of $L^2_{\theta(\dot{\phi})}(\dot{H})$. Hence

$$\sigma = \dot{\sigma}_{u_1} \in \Pi_{\theta(\phi)}(H).$$

Also, it follows from Theorem 3.4, Lemma 5.6 and (5.2) that

$$\ell^*(\mathcal{J}_{\mathscr{W}'}(\dot{\sigma})) = 1, \tag{6.10}$$

where

$$\ell^*(\mathcal{J}_{\mathscr{W}'}(\dot{\sigma}))(x) = \prod_v \ell^*_v(\mathcal{J}_{\mathscr{W}'_v}(\dot{\sigma}_v))(x_v)$$

for $x \in \overline{\mathcal{S}_{\dot{\phi}}}$. For all places $v \notin \{u_1, w\}$, we have

$$\ell^*_v(\mathcal{J}_{\mathscr{W}'_v}(\dot{\sigma}_v)) = \eta_v.$$

Indeed, if $\dot{\phi}_v$ is of good parity, then this equality follows from Theorem 6.11 and Lemma 6.19; on the other hand, if $\dot{\phi}_v$ is not of good parity, then the equality follows from our induction hypothesis. At the place w of F, since $\chi_{\dot{W},w} \not\subset \dot{\phi}_w$, it follows from Lemma 6.19 that Theorem 6.3 holds for $\dot{\phi}_w$. Hence

$$\ell^*_w(\mathcal{J}_{\mathscr{W}'_w}(\dot{\sigma}_w)) = \eta_w. \tag{6.11}$$

Combining (6.9)–(6.11), we have

$$\eta_\pi(x_{u_1}) = \ell^*_{u_1}(\eta_\sigma)(x_{u_1}).$$

Since the localization map $\mathcal{S}_{\dot{\phi}} \to \mathcal{S}_{\dot{\phi}_{u_1}}$ is surjective, the desired conclusion for π holds.

Case U_1*, and* F *is non-Archimedean:* The method used in the previous case can also be applied to this case similarly. In fact, this case is even easier, since Theorem 2.8 allows us to globalize $\pi \in \Pi_\phi^L(G')$ directly. We omit the details.

Case U, and F *is real:* The method used in the previous two cases cannot be applied here since the almost equal rank theta lift of $\pi \in \Pi_\phi^L(G')$ to the split Witt tower may vanish. Therefore, we cannot globalize it by using AMF. In this case, we reason by induction on $\dim V$. Then we can use AMF for some smaller unitary groups to help us make the globalization step.

When $\dim V = 0$, this follows from Lemma 6.7. Suppose that for all non-negative integers $m < \dim V$, Proposition 6.18, and hence Theorem 6.3, holds for all real unitary groups of m variables. Now we show that the desired conclusion also holds for all real unitary groups of $\dim V$ variables.

Recall that ϕ is an L-parameter of good parity for G, so it must be of the form

$$\phi = m_1\chi_1 + \cdots + m_r\chi_r,$$

where χ_i is a conjugate self-dual character of $L_{\mathbb{C}} = \mathbb{C}^\times$, and m_i is some positive integer. For the same reason as in the proof of Lemma 6.16, we may assume that $\dot{\phi}$ is a sum of automorphic characters. We pick up a pair of characters $(\chi'_{\dot{V}}, \chi'_{\dot{W}})$ such that $\chi'_{\dot{W}} \subset \dot{\phi}$. Let

$$\vartheta(\dot{\phi}) = (\dot{\phi} - \chi'_{\dot{W}})(\chi'_{\dot{W}})^{-1}\chi'_{\dot{V}}.$$

For any $\pi \in \Pi_\phi^L(G')$, we first use the almost equal rank theta lift to globalize it. By Theorem 6.11, there exists a pure inner form $H'_0 = H(W'_0)$ of H_0 such that the theta lift $\sigma_0 := \vartheta(\pi)$ of π to H'_0 (with respect to $(\psi_{F,v}, \chi'_{\dot{V},v}, \chi'_{\dot{W},v})$) is non-zero. Let $\dot{W}'_0$ be the unique $(\dim V - 1)$-dimensional c-skew-Hermitian space such that it is split at all places of $\dot{F}$ except $\{u_1, w\}$, and the localization of $\dot{W}'_0$ at u_1 is isometric to W'_0. Let $\dot{H}'_0 = H(\dot{W}'_0)$. Now we define an abstract irreducible representation $\dot{\sigma}'_0 = \bigotimes_v \dot{\sigma}'_{0,v}$ of $\dot{H}'_0(\mathbb{A})$:

- at a place $v \notin \{u_1, w\}$, $\dot{\sigma}'_{0,v}$ is the irreducible representation in the L-packet $\Pi^L_{\vartheta(\dot{\phi})_v}(\dot{H}'_{0,v})$ associated to the trivial character of $\mathcal{S}_{\vartheta(\dot{\phi})_v}$;
- at the place u_1, $\dot{\sigma}'_{0,u_1} = \sigma_0$;
- at the place w, $\dot{\sigma}'_{0,w}$ is the irreducible representation in the L-packet $\Pi^L_{\vartheta(\dot{\phi})_w}(\dot{H}'_{0,w})$ corresponding to the character $\eta'_{0,w} \in \widehat{\mathcal{S}_{\vartheta(\dot{\phi})_w}}$ determined by the formula

$$\prod_v \eta'_{0,v} = 1,$$

  where $\eta'_{0,v} = \mathcal{J}^L_{\mathscr{W}'_v}(\dot{\sigma}'_{0,v})$, and we regard $\prod_v \eta'_{0,v}$ as a character of the global component group $\mathcal{S}_{\vartheta(\dot{\phi})}$ through the localization maps. The existence of $\eta'_{0,w}$ is guaranteed by the local-global principle for skew-Hermitian spaces and LLC for unitary groups.

Since $\dot{H}'_0$ is a unitary group of dim $V-1$ variables, by our induction hypothesis, Theorem 6.3 holds for all localizations of $\dot{H}'_0$, hence Theorem 2.6 holds for $\dot{H}'_0$. It then follows that $\dot{\sigma}'_0$ is a summand of $L^2_{\vartheta(\dot{\phi})}(\dot{H}'_0)$. For each place v of $\dot{F}$, by the conservation relation [72], there is a Hermitian space V'_v of the same dimension as V such that

$$\vartheta(\dot{\sigma}'_{0,v}) \neq 0,$$

where $\vartheta(\dot{\sigma}'_{0,v})$ is the theta lift of $\dot{\sigma}'_{0,v}$ to $G'_v = G(V'_v)$ (again with respect to the datum $(\psi_{F,v}, \chi'_{\dot{V},v}, \chi'_{\dot{W},v})$). Since $\dot{\phi}_w$ is discrete, $\vartheta(\dot{\phi})_w$ does not contain the character $\chi'_{\dot{V},w}$. Thus we will have two choices of the skew-Hermitian space V'_w at the place w. The flexibility at w allows us to pick these local Hermitian spaces coherently so that they form a global Hermitian space $\dot{V}'$ over $\dot{E}$. We let $\dot{G}' = G(\dot{V}')$. Let $\dot{\pi} = \vartheta^{\mathrm{abs}}(\dot{\sigma}'_0)$ be the abstract theta lift to $\dot{G}'$. It follows from the same argument as in Lemma 6.16 that

$$m_{\mathrm{disc}}(\dot{\pi}) \geq m_{\mathrm{cusp}}(\dot{\pi}) \geq m_{\mathrm{cusp}}(\dot{\sigma}'_0).$$

Hence $\dot{\pi}$ is a summand in $L^2_{\dot{\phi}}(\dot{G}')$, and $\dot{\pi}_v \in \Pi^L_{\dot{\phi}_v}(\dot{G}'_v)$ for all places v of $\dot{F}$. Then by the conservation relation [72], we know that

$$(\dot{G}'_{u_1}, \dot{\pi}_{u_1}) \simeq (G', \pi).$$

Also, Theorem 6.11 implies that

$$\prod_v \eta_v(x_v) = 1 \tag{6.12}$$

for all $x \in \mathcal{S}_{\vartheta(\dot{\phi})}$. Here $\eta_v = \mathcal{J}^L_{\mathscr{W}_v}(\dot{\pi}_v)$, and we regard $\mathcal{S}_{\vartheta(\dot{\phi})}$ as a subgroup of $\mathcal{S}_{\dot{\phi}}$ via the natural embedding $\mathcal{S}_{\vartheta(\dot{\phi})} \to \mathcal{S}_{\dot{\phi}}$. Since $\eta_v = 1$ for all places $v \notin \{u_1, w\}$, this product is well-defined.

Next we consider the stable range theta lift of $\dot{\pi}$ to extract some other information. Let $\dot{\sigma} = \theta^{\mathrm{abs}}(\dot{\pi})$. We deduce from J.-S. Li's inequality (Theorem 3.1) that $\dot{\sigma}$ is a summand of $L^2_{\theta(\dot{\phi})}(\dot{H})$. Hence

$$\sigma = \dot{\sigma}_{u_1} \in \Pi_{\theta(\phi)}(H).$$

Similar to the previous cases, combining (6.12) and AMF for $\dot{H}$, we get

$$\eta_\pi(x_{u_1}) = \ell^*_{u_1}(\eta_\sigma)(x_{u_1})$$

for all $x \in \mathcal{S}_{\vartheta(\dot{\phi})}$. Since the localization map $\mathcal{S}_{\vartheta(\dot{\phi})} \to \mathcal{S}_{\vartheta(\dot{\phi})_{u_1}}$ is surjective, we have proved that η_π and $\ell^*_{u_1}(\eta_\sigma)$ are equal on the image of the natural embedding $\mathcal{S}_{\vartheta(\dot{\phi})_{u_1}} \hookrightarrow \mathcal{S}_\phi$. Certainly this embedding $\mathcal{S}_{\vartheta(\dot{\phi})_{u_1}} \hookrightarrow \mathcal{S}_\phi$ is not necessarily surjective. But there is nothing to worry about, since we may vary the character $\chi'_{\dot{W}}$. When the character $\chi'_{\dot{W}}$ runs over all irreducible components of $\dot{\phi}$, the image of $\mathcal{S}_{\vartheta(\dot{\phi})_{u_1}}$ will exhaust all elements in $\mathcal{S}_\phi$. Hence the desired conclusion holds for real unitary groups. ∎

Thus, we have finished proving Theorem 2.6.

7. Encore: beyond the generic case

Let F be a number field, and G an even orthogonal or unitary group over F as in the setting of Section 2.1. Let ψ be an elliptic A-parameter for G, and

$$\theta(\psi) = \psi\chi_W^{-1}\chi_V + \chi_V \boxtimes S_{2r-2n+1}$$

an elliptic A-parameter for H. In Section 5.3, we have transferred AMF from $L^2_{\theta(\psi)}(H)$ to $L^2_\psi(G)$ when $\psi = \phi$ is generic. Recall that the key step is to show Proposition 5.1, which implies that J.-S. Li's inequality (Theorem 3.1) is an equality in the generic case. In this section, we want to go one step further beyond the generic case. We would like to propose the following naive conjecture.

Conjecture 7.1. *Let G be an even orthogonal or unitary group, and*

$$\psi = \sum_i \phi_i \boxtimes S_{d_i} \tag{7.1}$$

an elliptic A-parameter for G, where ϕ_i is a cuspidal representation of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$. *Let π be an irreducible representation of $G(\mathbb{A})$ such that the L-parameter of π_v is ϕ_{ψ_v} for almost all v. Then*

$$m_{\mathrm{disc}}(\pi) = m(\pi).$$

We have proved this conjecture in Proposition 5.1 when $\psi = \phi$ is generic. Actually, it is easy to generalize this conjecture to a slightly more general case.

Assumption 7.2. *Let r_V be the Witt index of V. Suppose that in the expression* (7.1), *for any i such that $d_i > 1$, we have $n_i > r_V$.*

Under this weird assumption, we can prove Conjecture 7.1 by using an argument similar to that of Proposition 5.1.

Proposition 7.3. *Suppose that $G = G(V)$ and ψ satisfy Assumption* 7.2. *Let π be an irreducible representation of $G(\mathbb{A})$ such that the L-parameter of π_v is ϕ_{ψ_v} for almost all v. Then*

$$m_{\mathrm{cusp}}(\pi) = m_{\mathrm{disc}}(\pi) = m(\pi).$$

Remark 7.4. A case worth noting is when $r_V = 0$, i.e. $G = G(V)$ is anisotropic. In this case, Assumption 7.2 is automatically satisfied. Indeed, when G is anisotropic, we have $\mathcal{A}_{\mathrm{cusp}}(G) = \mathcal{A}_{\mathrm{disc}}(G) = \mathcal{A}(G)$. Therefore, Proposition 7.3 holds with no extra work needed.

Proof of Proposition 7.3. In the spirit of [22, Proposition 4.1], it suffices to show that for any realization $\mathcal{V} \subset \mathcal{A}(G)$ of π, we have $\mathcal{V} \subset \mathcal{A}_{\mathrm{cusp}}(G)$. Suppose on the contrary that $\mathcal{V} \not\subset \mathcal{A}_{\mathrm{cusp}}(G)$ for some such $\mathcal{V}$. By considering the constant term maps, it follows from [45] that

$$\pi \subset \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \rho$$

for some proper parabolic subgroup P of G with Levi component M, and some irreducible cuspidal automorphic representation ρ of $M(\mathbb{A})$. Suppose that

$$M \simeq \prod_j \mathrm{GL}_{k_j} \times G_0$$

for some $G_0 = G(V_0)$, where V_0 is a space in the Witt tower containing V. Then ρ is of the form

$$\rho \simeq \Big(\boxtimes_j \tau_j\Big) \boxtimes \pi_0$$

for some irreducible cuspidal automorphic representations τ_j and π_0 of GL_{k_j} and G_0, respectively. By Theorem 2.1, π_0 has a weak transfer τ_0 to $\mathrm{GL}_{n_0}(\mathbb{A})$. Then π has a weak transfer to $\mathrm{GL}_n(\mathbb{A})$ of the form

$$\Big(\boxplus_j (\tau_j \boxplus (\tau_j^c)^\vee)\Big) \boxplus \tau_0. \tag{7.2}$$

On the other hand, since the L-parameter of π_v is ϕ_{ψ_v} for almost all v, it follows that π has a weak transfer to $\mathrm{GL}_n(\mathbb{A})$ of the form

$$\boxplus_i (\phi_i|\cdot|^{(d_i-1)/2} \boxplus \cdots \boxplus \phi_i|\cdot|^{-(d_i-1)/2}). \tag{7.3}$$

By the strong multiplicity one theorem [35], the two expressions (7.2) and (7.3) must agree. Hence τ_j in the first expression must have the form $\phi_{i_j}|\cdot|^{s_j}$ for some i_j and $s_j \in \frac{1}{2}\mathbb{Z}$. Note that $k_j \le r_V$. It then follows from Assumption 7.2 that

$$k_j < n_i$$

for any i such that $d_i > 1$. Hence we must have $d_{i_j} = 1, s_j = 0$. This also implies that

$$\phi_{i_j} \boxtimes S_d$$

is not contained in ψ for any $d > 1$. However, $\tau_j = \phi_{i_j}$ occurs with multiplicity at least 2 in (7.2), whereas it occurs with multiplicity 1 in (7.3). This is a contradiction. Hence $\mathcal{V} \subset \mathcal{A}_{\mathrm{cusp}}(G)$ as required. ■

When the pair (G, ψ) does not satisfy Assumption 7.2, the realizations $\mathcal{V} \subset \mathcal{A}(G)$ of π may not lie in $\mathcal{A}_{\mathrm{cusp}}(G)$. To prove Conjecture 7.1, we need some extra inputs. Thanks to the square-integrability criterion [60, I.4.11 Lemma], when $G = G(V)$ is of F-rank 1, we are able to complete the proof.

Proposition 7.5. *Conjecture 7.1 holds if $G = G(V)$ is of F-rank 1, i.e. $r_V = 1$.*

Proof. Here we only handle Case O; the proof in Case U is similar.

Let $\mathcal{V} \subset \mathcal{A}(G)$ be an automorphic realization of π. We need to show that $\mathcal{V}$ is contained in $\mathcal{A}^2(G)$. We may assume that $\mathcal{V}$ is not contained in $\mathcal{A}_{\mathrm{cusp}}(G)$, otherwise it is already contained in $\mathcal{A}^2(G)$. Since G is of F-rank 1, the proper standard parabolic subgroup $P = MN$ of G is unique, with Levi component

$$M \simeq \mathrm{GL}_1 \times G_0,$$

where G_0 is an anisotropic group. Let $\chi|\cdot|^s \boxtimes \pi_0$ be a cuspidal support of π along P, where χ is a unitary automorphic character of GL_1, and π_0 is a cuspidal automorphic representation of G_0. Then it follows from [45] that

$$\pi \hookrightarrow \mathrm{Ind}_P^G(\chi|\cdot|^s \boxtimes \pi_0).$$

Consider the weak transfer of π to $\mathrm{GL}_{2n}(\mathbb{A})$. On the one hand, this weak transfer is represented by the elliptic A-parameter ψ; on the other hand, it also has an expression given by the above embedding. Similar to the proof of Proposition 7.3, by comparing these two expressions, we find that there exists some i such that

$$\chi = \phi_i, \quad d_i \geq 3 \quad \text{and} \quad s = \pm\frac{d_i - 1}{2}.$$

Moreover, the cuspidal automorphic representation π_0 is in the NEC represented by the elliptic A-parameter

$$\psi_0 = \psi - \chi \boxtimes S_{d_i} + \chi \boxtimes S_{d_i - 2}. \tag{7.4}$$

If we can show that $s = -(d_i - 1)/2$, then the square-integrability criterion [60, I.4.11 Lemma] will imply that $\mathcal{V} \subset \mathcal{A}^2(G)$, which will complete the proof. So next we shall prove this by contradiction.

Suppose on the contrary that $s = (d_i - 1)/2$. Then at every unramified place v, we have

$$\pi_v \hookrightarrow \mathrm{Ind}_{P_v}^{G_v}(\chi_v|\cdot|^{(d_i-1)/2} \boxtimes \pi_{0,v}). \tag{7.5}$$

Since $\chi_v|\cdot|^{(d_i-1)/2}$ is not self-dual, it follows from [37, Lemma 3.1.3] that π_v is the unique subrepresentation of $\mathrm{Ind}_{P_v}^{G_v}(\chi_v|\cdot|^{(d_i-1)/2} \boxtimes \pi_{0,v})$. Applying both the MVW functor (see [9, Section 2.7]) and contragredient functors, we know that π_v is also the unique quotient of $\mathrm{Ind}_{P_v}^{G_v}(\chi_v|\cdot|^{-(d_i-1)/2} \boxtimes \pi_{0,v})$. Let K_v be a special maximal compact subgroup of G_v which has good position relative to P_v. Fix a representative $w_v \in K_v$ of the unique non-trivial element in $W_{M_v} = N_{G_v}(M_v)/M_v$. Let

$$\mathcal{M}(s, \chi_v \boxtimes \pi_{0,v}, w_v) : \mathrm{Ind}_{P_v}^{G_v}(\chi_v|\cdot|^s \boxtimes \pi_{0,v}) \to \mathrm{Ind}_{P_v}^{G_v}(\chi_v|\cdot|^{-s} \boxtimes \pi_{0,v}^{w_v})$$

be the unnormalized intertwining operator given by (the meromorphic continuation of) the integral

$$\mathcal{M}(s, \chi_v \boxtimes \pi_{0,v}, w_v) f(g) = \int_{N_v} f(w_v^{-1} n g)\, dn$$

for $f \in \mathrm{Ind}_{P_v}^{G_v}(\chi_v|\cdot|^s \boxtimes \pi_{0,v})$, where $\pi_{0,v}^{w_v}$ is the representation of $G_{0,v}$ on the same space of $\pi_{0,v}$ with the action given by

$$\pi_{0,v}^{w_v}(m) = \pi_{0,v}(w_v^{-1} m w_v).$$

Let $f_{v,s}$ and $f'_{v,-s}$ be the unramified vectors in the spaces $\mathrm{Ind}_{P_v}^{G_v}(\chi_v|\cdot|^s \boxtimes \pi_{0,v})$ and $\mathrm{Ind}_{P_v}^{G_v}(\chi_v|\cdot|^{-s} \boxtimes \pi_{0,v}^{w_v})$ respectively with the normalization

$$f_{v,s}(1_{G_v}) = f'_{v,-s}(1_{G_v}) = 1.$$

Then by the Gindikin–Karpelevich formula [16, p. 141, Theorem 6.7], we have

$$\mathcal{M}(s, \chi_v \boxtimes \pi_0, w_v) f_{v,s} = \frac{L(s, \pi_{0,v} \times \chi_v)}{L(1+s, \pi_{0,v} \times \chi_v)} \cdot f'_{v,-s}.$$

If we write

$$\psi_{0,v} = \sum_j \mu_j \boxtimes S_{d_j}$$

for some unramified characters μ_j (not necessarily unitary), then

$$\frac{L(s, \pi_{0,v} \times \chi_v)}{L(1+s, \pi_{0,v} \times \chi_v)} = \prod_j \frac{L\big(s - \frac{d_j-1}{2}, \mu_j \chi_v\big)}{L\big(s + \frac{d_j+1}{2}, \mu_j \chi_v\big)}. \tag{7.6}$$

Since $\psi_{0,v}$ is the localization of the global elliptic A-parameter ψ_0, each μ_j can be decomposed as

$$\mu_j = \mu'_j |\cdot|^{s}$$

for some unitary character μ'_j and real number $|s_j| < 1/2$. By (7.4), we have $\chi_v \boxtimes S_{d_i-2} \subset \psi_{0,v}$. This implies that the factor (7.6) has a zero at $s = -(d_i - 1)/2$. Hence

$$\mathcal{M}(s, \chi_v \boxtimes \pi_0, w_v) f_{v,s}|_{s=-(d_i-1)/2} = 0. \tag{7.7}$$

Moreover, since π_v is the unique quotient of

$$\mathrm{Ind}_{P_v}^{G_v}(\chi_v|\cdot|^{-(d_i-1)/2} \boxtimes \pi_{0,v}),$$

we know that $\mathrm{Ind}_{P_v}^{G_v}(\chi_v|\cdot|^{-(d_i-1)/2} \boxtimes \pi_{0,v})$ is generated by the unramified vector $f_{v,s}$. By (7.7) and the G_v-equivariance of $\mathcal{M}(s, \chi_v \boxtimes \pi_0, w_v)$, we deduce that $\mathcal{M}(s, \chi_v \boxtimes \pi_0, w_v)$ is holomorphic and zero at $s = -(d_i - 1)/2$, which is impossible (see [77, Theorem VI.1.1 & Remark]). This finishes the proof. ■

Then, by the same proof as for Corollary 5.2, we deduce the following

Corollary 7.6. *Suppose that either*

(1) (G, ψ) *satisfies Assumption 7.2, or*

(2) G *is of F-rank 1, and ψ is any elliptic A-parameter for G.*

Suppose that

$$L^2_\psi(G) = \bigoplus_\pi m_\pi \pi.$$

Then

$$L^2_{\theta(\psi)}(H) = \Big(\bigoplus_\pi m_\pi \theta^{\mathrm{abs}}(\pi)\Big) \oplus \Big({\bigoplus_\sigma}' m_\sigma \sigma\Big),$$

where the second summation on the RHS is over all σ with A-parameter $\theta(\psi)$ and not relevant to G.

Recall that in Section 5.3, we have defined local packets $\Pi^{\theta}_{\psi_v}(G_v)$ for each place v of F, as well as the global packet $\Pi^{\theta}_{\psi}(G,\epsilon_\psi)$. Combining all of these with Lemma 5.6, we deduce the following result.

Theorem 7.7. *Suppose that either*

(1) (G,ψ) *satisfies Assumption* 7.2, *or*

(2) G *is of* F*-rank* 1, *and* ψ *is any elliptic* A*-parameter for* G.

Then there is a decomposition

$$L^2_{\psi}(G) = \bigoplus_{\pi\in\Pi^{\theta}_{\psi}(G,\epsilon_\psi)} \pi.$$

Remark 7.8. In particular, when $G = G(V)$ is of F-rank ≤ 1, we obtain a description of the whole $L^2_{\mathrm{disc}}(G)$. A case worth noting is when G is a unitary group and $G_v \simeq \mathrm{U}_{1,n-1}$ at one real place v. In this case, the description of $L^2_{\mathrm{disc}}(G)$ might have some arithmetic applications to Shimura varieties of type $\mathrm{U}_{1,n-1}$.

These results also motivate us to study these local packets $\Pi^{\theta}_{\psi_v}(G_v)$ at each local place v of F. In particular, we want to show that

- these local packets $\Pi^{\theta}_{\psi_v}(G_v)$ do not depend on the choice of the auxiliary group $H_v = H(W_v)$;
- if G_v is quasi-split, then
$$\Pi^{\theta}_{\psi_v}(G_v) = \Pi^{A}_{\psi_v}(G_v)$$
as representations of $\mathcal{S}_\psi \times G_v$, where $\Pi^{A}_{\psi_v}(G_v)$ is the local A-packet defined by Arthur [4] and Mok [61].

In [12], we prove these when v is a non-Archimedean place of F.

Remark 7.9. (1) In Case O, a large part of these local comparison results have been proved by Mœglin already in a much more general context. In [55], she has constructed a packet $\Pi^{M}_{\psi_v}(G_v)$ explicitly for each ψ_v when v is non-Archimedean. Moreover, she showed that $\Pi^{M}_{\psi_v}(G_v)$ is multiplicity-free, and

$$\Pi^{M}_{\psi_v}(G_v) = \Pi^{A}_{\psi_v}(G_v)$$

as sets if G_v is quasi-split. Using her explicit construction, she studied the Adams conjecture in [53]. It follows from her results that

$$\Pi^{\theta}_{\psi_v}(G_v) = \Pi^{M}_{\psi_v}(G_v)$$

as sets. Hence, compared to her results, the new thing in [12] is that we also compare the "labelings", i.e. the map $\mathcal{J}^{\theta}$.

(2) In Case U, when v is real and the A-parameter ψ_v is Adams–Johnson, these local comparison results have already been proved in [17]. We expect that the method in [17] can also be applied to Case O.

Appendix A. Prasad's conjecture: real even orthogonal-symplectic case

In this appendix, we shall use theta lifts to establish LLC for real full even orthogonal groups, based on Paul's results [65, Theorem 15].

We fix some notation. For $\alpha \in \frac{1}{2}\mathbb{Z}$, we denote by $\chi_{2\alpha}$ the character

$$z \mapsto (z/\bar{z})^{\alpha}$$

of $L_{\mathbb{C}}$, and by $\mathcal{D}_{2\alpha}$ the two-dimensional representation of $L_{\mathbb{R}}$ induced from the character $\chi_{2\alpha}$ of $L_{\mathbb{C}}$. Note that $\mathcal{D}_{2\alpha}$ is irreducible unless $\alpha = 0$. We also retain the notations of Section 6.4. So now V is a $2n$-dimensional orthogonal space over $\mathbb{R}$ with isometry group $G(V)$, and W_0 is a $2n$-dimensional symplectic space over $\mathbb{R}$ with isometry group $H(W_0)$. Pure inner forms of $G(V)$ will be typically denoted by $G(V')$ for some orthogonal space V' (see the beginning of Section 2 for the classification of these V'). We also denote the special even orthogonal group associated to V by $G^0(V)$, which is an index 2 subgroup in $G(V)$. By classical Clifford theory, there is a canonical bijection

$$\iota : \mathrm{Irr}(G(V))/{\sim_{\det}} \to \mathrm{Irr}(G^0(V))/{\sim_{\varepsilon}}$$

given by restriction, where the LHS of the bijection is the set of equivalence classes of irreducible representations of $G(V)$ up to the determinant twist, and the RHS is the set of equivalence classes of irreducible representations of $G^0(V)$ up to the action of the outer automorphism corresponding to an element of $G(V) \setminus G^0(V)$. Given $\pi \in \mathrm{Irr}(G(V))$, we shall use $[\pi]_{\det}$ to denote the equivalence class in $\mathrm{Irr}(G(V))/{\sim_{\det}}$ containing π.

Since $G^0(V)$ and $H(W_0)$ are connected reductive groups over $\mathbb{R}$, by the work of Langlands [46] we have LLC for these two groups, as recalled in Section 2.2. However, instead of original L-parameters for $G^0(V)$, we prefer to use the so-called weak L-parameters which we now describe. Modulo the action of the outer automorphism, we obtain a finite-to-one surjective map

$$\mathcal{L}^0 : \bigsqcup_{V'} \mathrm{Irr}(G^0(V'))/{\sim_{\varepsilon}} \to \Phi^+(G^0(V))/{\sim_{\varepsilon}}, \tag{A.1}$$

from the original LLC for $G^0(V)$, where the disjoint union on the LHS runs over all $2n$-dimensional orthogonal spaces V' of the same discriminant as V, and the RHS is the set of equivalence classes of L-parameters of $G^0(V)$ up to ${}^LG(V) = \mathrm{O}_{2n}(\mathbb{C})$ conjugation. By composing with the standard representation

$$\mathrm{O}_{2n}(\mathbb{C}) \to \mathrm{GL}_{2n}(\mathbb{C}),$$

the set $\Phi^+(G^0(V))/{\sim_{\varepsilon}}$ can be identified with

$$\Phi^+(G(V)) := \{\phi : L_{\mathbb{R}} \to \mathrm{GL}_{2n}(\mathbb{C}) \mid \phi \text{ is semisimple, orthogonal and } \det(\phi) = \chi_V\}.$$

This is the set of weak L-parameters we will make use of.

Consider the theta lift between $(G(V), H(W_0))$, with respect to a non-trivial additive character $\psi_{\mathbb{R}}$ and splitting characters (χ_V, χ_W) as in Section 3.1. Our first goal here is to establish a weaker version of Theorem 6.11, which partially describes this theta lift in terms of (weak) L-parameters for $G^0(V)$ and $H(W_0)$.

Theorem A.1. *Let ϕ be a tempered weak L-parameter for $G^0(V)$, and π an irreducible tempered representation of $G(V)$ such that*

$$\mathcal{L}^0(\iota([\pi]_{\det})) = \phi.$$

Suppose that the theta lift $\vartheta(\pi)$ of π to $H(W_0)$ is non-zero. Then $\vartheta(\pi)$ is tempered and lies in the L-packet associated to

$$\vartheta(\phi) = \phi\chi_W^{-1}\chi_V + \chi_V.$$

Moreover, if $\chi_W \not\subset \phi$, then both

$$\begin{cases} \text{the theta lift } \vartheta(\pi) \text{ of } \pi \text{ to } H(W_0) \text{ and} \\ \text{the theta lift } \vartheta(\pi\otimes\det) \text{ of } \pi\otimes\det \text{ to } H(W_0) \end{cases}$$

are non-zero.

Proof. We first assume that ϕ is a discrete weak L-parameter, i.e. ϕ is multiplicity-free. We can write it as

$$\phi = \mathcal{D}_{2\alpha_1} + \cdots + \mathcal{D}_{2\alpha_n}$$

for some non-negative integers $\alpha_1 > \cdots > \alpha_n \geq 0$. Following Paul [65, Section 3.2], if the Harish-Chandra parameter of π is of the form

$$\lambda = (a_1, a_2, \ldots; b_1, b_2, \ldots),$$

where a_i and b_j are non-negative integers, $a_1 > a_2 > \cdots$ and $b_1 > b_2 > \cdots$, then the infinitesimal character of π is precisely the orbit of λ under the Weyl group action. Since LLC preserves the infinitesimal character, we have

$$\{\alpha_1 \ldots, \alpha_n\} = \{a_1, a_2, \ldots, b_1, b_2, \ldots\}.$$

By [65, Theorem 15], $\vartheta(\pi)$ is the limit of a discrete series, and the Harish-Chandra parameter of $\vartheta(\pi)$ is of the form

$$\lambda' = (a_1, a_2, \ldots; \ldots, -b_2, -b_1).$$

Note that by definition any L-parameter of $H(W_0)$ has trivial determinant. Then again by considering the infinitesimal character, one can see immediately that the L-parameter of π must be $\vartheta(\phi)$ as predicted.

Next we assume that ϕ is tempered but not discrete. It follows from LLC for $G^0(V)$ that any irreducible representation of $G^0(V)$ in $\iota([\pi]_{\det})$ is tempered but not a discrete series, and hence so is π. It is well known that π can be embedded into a parabolic induction of a discrete series representation. Then by the induction principle of local theta correspondence [65, Section 5.2] and the result in the discrete series case, we know that the L-parameter of $\vartheta(\pi)$ is $\vartheta(\phi)$.

Finally, we prove the last statement of the theorem. Let W_{00} be the $(2n-2)$-dimensional symplectic space over $\mathbb{R}$, and $H(W_{00})$ be the corresponding symplectic group. If $\vartheta(\pi)=0$, then by the conservation relation [72] the theta lift of $\pi \otimes \det$ to $H(W_{00})$ is non-zero. By using the same argument as above, we can show that $\chi_W \subset \phi$. Likewise $\vartheta(\pi \otimes \det)=0$ also yields $\chi_W \subset \phi$. This completes the proof. ∎

With this theorem at hand, next we extend LLC for real special even orthogonal groups to full even orthogonal groups using the same idea as in [13, Section 5]. As mentioned before, LLC depends on the choices of a Whittaker datum $\mathscr{W}$ of the quasi-split pure inner form of $G(V)$. Since the construction will also involve LLC for $H(W_0)$, we need to choose a Whittaker datum $\mathscr{W}'$ of $H(W_0)$ as well. We shall make these choices according to the additive character $\psi_{\mathbb{R}}$ (and some other auxiliary data) as explicated in Section 3.5.

Let us deal with tempered representations first. Recall that by weak LLC for $G^0(V)$ we have a finite-to-one surjective map $\mathcal{L}^0$ as in (A.1). We define a map

$$\mathcal{L}: \bigsqcup_{V'} \mathrm{Irr}_{\mathrm{temp}}(G(V')) \to \Phi(G(V))$$

by setting $\mathcal{L}(\pi)=\mathcal{L}^0(\iota([\pi]_{\det}))$ for $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(V'))$. It then follows from the properties of $\mathcal{L}^0$ that $\mathcal{L}$ is a finite-to-one surjective map. For each L-parameter $\phi \in \Phi(G(V))$, to give a parametrization of the fibers

$$\Pi_\phi(G(V')) := \mathcal{L}^{-1}(\phi) \cap \mathrm{Irr}(G(V')),$$

we appeal to the theta lift. Let $\pi \in \Pi_\phi(G(V'))$. Consider the theta lift $\vartheta(\pi)$ of π to $H(W_0)$. There are two possibilities:

- *Case 1:* $\vartheta(\pi)$ is non-zero. Then by Theorem A.1, $\vartheta(\pi) \in \Pi_{\vartheta(\phi)}(H(W_0))$, and $\mathcal{S}_\phi$ can be regarded as a subgroup of $\mathcal{S}_{\vartheta(\phi)}$. In this case we set

$$\eta_\pi := \eta_{\vartheta(\pi)}|_{\mathcal{S}_\phi},$$

  where $\eta_{\vartheta(\pi)} \in \widehat{\mathcal{S}_{\vartheta(\phi)}}$ is the character associated to $\vartheta(\pi)$ by LLC for symplectic groups.
- *Case 2:* $\vartheta(\pi)$ is zero. Then by the conservation relation [72], the theta lift $\vartheta(\pi \otimes \det)$ of $\pi \otimes \det$ to $H(W_0)$ is non-zero. Since $[\pi]_{\det} = [\pi \otimes \det]_{\det}$, it is easy to see from the definition that $\mathcal{L}(\pi)=\mathcal{L}(\pi \otimes \det)$. In the previous case we have already attached a character $\eta_{\pi\otimes\det} \in \widehat{\mathcal{S}_\phi}$ to $\pi \otimes \det$. In this case we set

$$\eta_\pi := \eta_{\pi\otimes\det} \cdot \kappa_\phi,$$

  where $\kappa_\phi \in \widehat{\mathcal{S}_\phi}$ is the character defined in (2.6).

Combining these two cases we obtain a map

$$\mathcal{J}^L_{\mathscr{W}} : \bigsqcup_{V'} \Pi_\phi(G(V')) \to \widehat{\mathcal{S}_\phi}$$

by setting $\mathcal{J}^L_{\mathcal{W}}(\pi) = \eta_\pi$ for $\pi \in \Pi_\phi(G(V'))$. Similar to [13, Proposition 5.10], it follows from Howe duality and the conservation relation [72] that the map $\mathcal{J}^L_{\mathcal{W}}$ is indeed a bijection.

Remark A.2. The following two properties of the map $\mathcal{J}^L_{\mathcal{W}}$ are worth noting.

(1) For any $\pi \in \Pi_\phi(G(V'))$, we have

$$\mathcal{J}^L_{\mathcal{W}}(\pi \otimes \det) = \mathcal{J}^L_{\mathcal{W}}(\pi) \cdot \kappa_\phi.$$

Indeed, if $\vartheta(\pi) = 0$ or $\vartheta(\pi \otimes \det) = 0$, this equality is a direct consequence of the construction. The proof of our main local result, Theorem 6.3 (more precisely, the proof of Proposition 6.18) will only involve this special case. When both $\vartheta(\pi)$ and $\vartheta(\pi \otimes \det)$ are non-zero, we can appeal to the strength of Theorem 2.6 as follows. Similar to Section 6.6, when the L-parameter ϕ is of good parity, one can suitably globalize π to a cuspidal automorphic representation $\dot{\pi}$ with generic A-parameter $\dot{\phi}$ such that

- at a place v, the localizations of $\dot{\pi}$ and $\dot{\phi}$ are π and ϕ;
- at an auxiliary finite place w, the localization map $\mathcal{S}_{\dot{\phi}} \to \mathcal{S}_{\dot{\phi}_w}$ is an isomorphism.

Let $\dot{\pi}'$ be the cuspidal automorphic representation obtained from $\dot{\pi}$ by replacing $\dot{\pi}_v$ and $\dot{\pi}_w$ by $\dot{\pi}_v \otimes \det$ and $\dot{\pi}_w \otimes \det$. Then applying Theorem 2.6 to $\dot{\pi}$ and $\dot{\pi}'$, one gets

$$\eta_{\dot{\pi}_v} \cdot \eta_{\dot{\pi}_w} = \eta_{\dot{\pi}_v \otimes \det} \cdot \eta_{\dot{\pi}_w \otimes \det}.$$

Since the desired equality holds for $\dot{\pi}_w$ (see Remark 2.3 (1)), it also holds for $\pi = \dot{\pi}_v$. For general ϕ the desired conclusion follows from the compatibility of LLC with parabolic inductions.

(2) A priori, the map $\mathcal{J}^L_{\mathcal{W}}$ depends on the choice of the additive character $\psi_{\mathbb{R}}$. However, as suggested by the notation it only depends on the choice of the Whittaker datum $\mathcal{W}$ of the quasi-split pure inner form of $G(V)$ but not on $\psi_{\mathbb{R}}$. This is a consequence of the scaling property of the Weil representation. Similar to [42, II. Corollary 6.2, IV. Proposition 1.9], an easy computation shows that for any $\pi \in \mathrm{Irr}(G(V))$, we have

$$\vartheta_{\psi_{\mathbb{R},a}}(\pi) \simeq \vartheta_{\psi_{\mathbb{R}}}(\pi)^{\delta_a}.$$

Here $a \in \mathbb{R}^\times$, $\psi_{\mathbb{R},a} := \psi_{\mathbb{R}}(a \cdot -)$ and δ_a is an element in $\mathrm{GL}(W_0)$ such that

$$\langle \delta_a(v), \delta_a(v') \rangle_{W_0} = a \cdot \langle v, v' \rangle_{W_0}$$

for any $v, v' \in W_0$. The subscripts "$\psi_{\mathbb{R},a}$" and "$\psi_{\mathbb{R}}$" indicate the additive characters used in the definition of theta lifts. Let $\mathcal{W}'_a$ be the Whittaker datum of $H(W_0)$ determined by $\psi_{\mathbb{R},a}$ as in Section 3.5. Then it follows from [38, Theorem 4.3] that

$$\mathcal{J}^L_{\mathcal{W}'}(\vartheta_{\psi_{\mathbb{R}}}(\pi)) = \mathcal{J}^L_{\mathcal{W}'_a}(\vartheta_{\psi_{\mathbb{R},a}}(\pi)),$$

and in particular these two characters have the same restriction to $\mathcal{S}_\phi$.

After furnishing tempered representations with the maps $\mathcal{L}$ and $\mathcal{J}^L_{\mathscr{W}}$, we can extend these maps to all irreducible representations in a standard manner similar to [11]. Although full even orthogonal groups are disconnected, the Langlands classification is still valid by [65, Section 3.2]. To be more precise, for any irreducible non-tempered representation $\pi \in \mathrm{Irr}(G(V))$, there is a standard module

$$\mathrm{Ind}_P^{G(V)}(\tau_1|\cdot|^{s_1} \boxtimes \cdots \boxtimes \tau_r|\cdot|^{s_r} \boxtimes \pi_0)$$

of $G(V)$, where

- P is a parabolic subgroup of $G(V)$, with Levi component
$$L \simeq \mathrm{GL}_{d_1}(\mathbb{R}) \times \cdots \times \mathrm{GL}_{d_m}(\mathbb{R}) \times G(V_0),$$
where V_0 is some orthogonal space in the Witt tower containing V;
- τ_i is an irreducible (limit of) discrete series of $\mathrm{GL}_{d_i}(\mathbb{R})$, and s_i is a positive real number;
- $\{\tau_i|\cdot|^{s_i}\}_i$ is ordered so that
$$s_1 \geq \cdots \geq s_r > 0;$$
- π_0 is an irreducible tempered representation of $G(V_0)$,

such that π is the unique irreducible quotient of this standard module. Let ϕ_{τ_i} be the L-parameter of τ_i, and $\phi_0 = \mathcal{L}(\pi_0)$. We set

$$\mathcal{L}(\pi) = (\phi_{\tau_1}|\cdot|^{s_1} + \cdots + \phi_{\tau_r}|\cdot|^{s_r}) + \phi_0 + (\phi_{\tau_1}|\cdot|^{s_1} + \cdots + \phi_{\tau_r}|\cdot|^{s_r})^\vee.$$

Then as explicated in [19, Section 8], there is a natural isomorphism $\mathcal{S}_\phi \simeq \mathcal{S}_{\phi_0}$. Let $\eta_0 = \mathcal{J}^L_{\mathscr{W}}(\pi_0)$. Under this identification of component groups, we define

$$\mathcal{J}^L_{\mathscr{W}}(\pi) = \eta_0.$$

Since these P, τ_i, s_i and π_0 are uniquely determined by π, the L-parameter $\mathcal{L}(\pi)$ and the character $\mathcal{J}^L_{\mathscr{W}}(\pi) \in \widehat{\mathcal{S}_\phi}$ are well-defined. We conclude the above discussion as follows.

Theorem A.3. *There is a finite-to-one surjective map*

$$\mathcal{L} : \bigsqcup_{V'} \mathrm{Irr}(G(V')) \to \Phi^+(G(V)),$$

where the disjoint union runs over all $2n$-dimensional orthogonal spaces V' of the same discriminant as V. For each L-parameter $\phi \in \Phi^+(G(V))$, we denote

$$\Pi_\phi(G(V')) := \mathcal{L}^{-1}(\phi) \cap \mathrm{Irr}(G(V')),$$

and we call it the L-packet of G' associated to ϕ. There is a bijection (depending on the choice of the Whittaker datum $\mathscr{W}$)

$$\mathcal{J}^L_{\mathscr{W}} : \bigsqcup_{V'} \Pi_\phi(G(V')) \to \widehat{\mathcal{S}_\phi}.$$

From our construction, one can see immediately that the following holds.

Corollary A.4. *Under LLC provided by Theorem A.3 for real full even orthogonal groups, Theorem 6.11 holds for dual pairs* $(G(V'), H(W_0))$.

Finally, recall that Arthur has already established tempered LLC for quasi-split real full even orthogonal groups, namely a finite-to-one surjective map

$$\mathcal{L}^A : \mathrm{Irr}_{\mathrm{temp}}(G(V^+)) \to \Phi(G(V^+)),$$

together with a bijection (depending on the choice of the Whittaker datum $\mathscr{W}$)

$$\mathcal{J}^A_{\mathscr{W}} : \Pi_\phi(G(V^+)) \to \widehat{\mathcal{S}_\phi}$$

on each fiber $\Pi_\phi(G(V^+))$ of $\phi \in \Phi(G(V^+))$. We can justify our construction of LLC for real full even orthogonal groups by comparing it with Arthur's. Since Arthur's LLC is compatible with LLC for $G^0(V^+)$, for any $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(V^+))$ we have

$$\mathcal{L}^A(\pi) = \mathcal{L}^0(\iota([\pi]_{\det})) = \mathcal{L}(\pi).$$

For any $\phi \in \Phi(G(V^+))$ and $\pi \in \Pi_\phi(G(V^+))$, to compare $\mathcal{J}^A_{\mathscr{W}}(\pi)$ and $\mathcal{J}^L_{\mathscr{W}}(\pi)$, again we appeal to the global method. When the L-parameter ϕ is of good parity, using the same argument as in Section 6.6, we can suitably globalize π to a cuspidal automorphic representation $\dot{\pi}$ of a globally quasi-split even orthogonal group $G(\dot{V}^+)$, with generic A-parameter $\dot{\phi}$, such that at a place v, the localizations of $\dot{\pi}$ and $\dot{\phi}$ are π and ϕ. Then comparing Arthur's original multiplicity formula and our version (Theorem 2.6) for $L^2_{\dot{\phi}}(G(\dot{V}^+))$, we deduce that

$$\mathcal{J}^A_{\mathscr{W}}(\pi) = \mathcal{J}^L_{\mathscr{W}}(\pi).$$

For general ϕ the desired equality follows from the compatibility of LLC with parabolic inductions.

Appendix B. An irreducibility result for some induced representations

In this appendix, we sketch a proof of Lemma 6.6 for $F = \mathbb{R}$. We have $E = \mathbb{R}$ in Case O and $E = \mathbb{C}$ in Case U. We will prove it in a more general context.

Recall that an irreducible representation of L_E is said to be *almost tempered and positive* if it is of the form $\phi|\cdot|^s$, where ϕ is a representation of L_E with bounded image, and $0 < s < 1/2$ is a real number. Let ψ be a local A-parameter for H. We assume

$$\psi = \varphi + \psi_0 + (\varphi^c)^\vee,$$

where

- ψ_0 is a local A-parameter for $H_0 = H(W_0)$, which is of good parity; here W_0 is a c-skew-Hermitian space in the Witt tower containing W;

- φ is a k-dimensional representation of L_E whose irreducible summands are either almost tempered and positive, or tempered but not (conjugate) self-dual with the same parity as ψ.

Let τ be the irreducible representation of $\mathrm{GL}_k(E)$ associated to φ, and Q the standard parabolic subgroup of H with Levi component $L \simeq \mathrm{GL}_k(E) \times H_0$. We shall prove the following theorem.

Theorem B.1. *For any irreducible unitary representation σ_0 in the A-packet $\Pi_{\psi_0}(H_0)$, the induced representation $\mathrm{Ind}_Q^H(\tau \boxtimes \sigma_0)$ is irreducible.*

In [23], Gan–Ichino proved a similar statement for odd orthogonal groups. Mimicking their proof, we briefly describe the strategy to prove the theorem; the readers may consult [23, Section 3I] for full details. An ingredient of the proof is the normalized intertwining operators. Recall that for a real reductive group G, a parabolic subgroup P of G, and an irreducible representation π of the Levi component of P, the induced representation $\mathrm{Ind}_P^G(\pi)$ is called a *standard module* for G if π satisfies certain positivity conditions. For such an induced representation, one can define a (normalized) *intertwining operator*

$$R_{\overline{P}|P}(\pi) : \mathrm{Ind}_P^G(\pi) \to \mathrm{Ind}_{\overline{P}}^G(\pi),$$

where $\overline{P}$ is the parabolic subgroup of G opposite to P, such that the image of $R_{\overline{P}|P}(\pi)$ is the unique irreducible quotient of $\mathrm{Ind}_P^G(\pi)$. In the proof of Theorem B.1, we shall realize the representation $\mathrm{Ind}_Q^H(\tau \boxtimes \sigma_0)$ as the image of a standard module for H.

Firstly, we decompose the representation τ. It follows from our assumptions that we may write φ as a sum of subrepresentations

$$\varphi = \varphi_1 + \cdots + \varphi_r,$$

satisfying the following conditions:

– Suppose we are in Case O.
  - Each φ_i is of the form $\phi_i |\cdot|^{\nu_i}$, where ϕ_i is either sgn^{δ_i} for some $\delta_i \in \mathbb{Z}/2\mathbb{Z}$, or $\mathcal{D}_{2\alpha_i}$ for some $\alpha_i \in \frac{1}{2}\mathbb{Z} \setminus \{0\}$ and ν_i is a complex number.
  - If $\nu_i = 0$ for some i, then $\varphi_i = \mathcal{D}_{2\alpha_i}$ for some $\alpha_i \in \mathbb{Z} + \frac{1}{2}$.

– Suppose we are in Case U.
  - Each φ_i is of the form $\chi_{2\alpha_i} |\cdot|^{\nu_i}$ for some $\alpha_i \in \frac{1}{2}\mathbb{Z}$ and $\nu_i \in \mathbb{C}$.
  - If $\nu_i = 0$ for some i, then $\alpha_i \in \mathbb{Z} + (\dim W)/2$.

In both cases, the summation can be ordered so that

$$1/2 > \Re(\nu_1) \geq \cdots \geq \Re(\nu_r) \geq 0.$$

Let $k_i = \dim \varphi_i$, τ_i be the irreducible representation of $\mathrm{GL}_{k_i}(E)$ corresponding to φ_i by LLC for general linear groups, and $\tau_\varphi = \tau_1 \boxtimes \cdots \boxtimes \tau_r$. It is easy to see that there is a parabolic subgroup Q_φ of $\mathrm{GL}_k(E)$ with Levi component

$$L_\varphi \simeq \mathrm{GL}_{k_1}(E) \times \cdots \times \mathrm{GL}_{k_r}(E)$$

such that

$$\tau \simeq \mathrm{Ind}_{Q_\varphi}^{\mathrm{GL}_k(E)}(\tau_\varphi).$$

Let Q_1 be the parabolic subgroup of H with Levi component $L_1 \simeq L_\varphi \times H_0$ such that $Q_1 \subset Q$, and $Q_1 \cap L = Q_\varphi \times H_0$. Then by induction in stages, we have

$$\mathrm{Ind}_Q^H(\tau \boxtimes \sigma_0) \simeq \mathrm{Ind}_{Q_1}^H(\tau_\varphi \boxtimes \sigma_0).$$

Next we deal with the irreducible representation σ_0. By Langlands' classification, σ_0 is the unique irreducible quotient of a standard module (for H_0)

$$\mathrm{Ind}_{Q_{\psi_0}}^{H_0}(\tau_1' \boxtimes \cdots \boxtimes \tau_m' \boxtimes \sigma_{00}),$$

where

- Q_{ψ_0} is a parabolic subgroup of H_0 with Levi component

$$L_{\psi_0} \simeq \mathrm{GL}_{d_1}(E) \times \cdots \times \mathrm{GL}_{d_m}(E) \times H_{00};$$

  here $H_{00} = H(W_{00})$ for some space W_{00} in the Witt tower containing W_0;
- τ_i' is an irreducible essentially (limit of) discrete series of $\mathrm{GL}_{d_i}(E)$, which is of the form

$$\tau_i' = \tau_i''|\cdot|^{s_i}$$

  for some irreducible (limit of) discrete series τ_i'' of $\mathrm{GL}_{d_i}(E)$ and $s_i > 0$; since σ_0 lies in the local A-packet $\Pi_{\psi_0}(H_0)$, we can further conclude that $s_i \in \frac{1}{2}\mathbb{Z}$;
- $\{\tau_i'\}_i$ is ordered so that

$$s_1 \geq \cdots \geq s_m > 0;$$

- σ_{00} is a tempered representation of H_{00}.

Moreover, if we let $\sigma_{0,\psi_0} = \tau_1' \boxtimes \cdots \boxtimes \tau_m' \boxtimes \sigma_{00}$, and

$$R_{\bar{Q}_{\psi_0}|Q_{\psi_0}}(\sigma_{0,\psi_0}) : \mathrm{Ind}_{Q_{\psi_0}}^{H_0}(\sigma_{0,\psi_0}) \to \mathrm{Ind}_{\bar{Q}_{\psi_0}}^{H_0}(\sigma_{0,\psi_0}) \tag{B.1}$$

is the (normalized) intertwining operator, then σ_0 is just the image of this operator $R_{\bar{Q}_{\psi_0}|Q_{\psi_0}}(\sigma_{0,\psi_0})$.

Now we come to the key step. Applying the functor $\mathrm{Ind}_{Q_1}^H(\tau_\varphi \boxtimes \ \cdot\ )$ to the intertwining map (B.1), we get

$$\mathrm{Ind}_{Q_1}^H(\mathbb{1}_{\tau_\varphi} \boxtimes R_{\bar{Q}_{\psi_0}|Q_{\psi_0}}(\sigma_{0,\psi_0})) : \\ \mathrm{Ind}_{Q_1}^H(\tau_\varphi \boxtimes \mathrm{Ind}_{Q_{\psi_0}}^{H_0}(\sigma_{0,\psi_0})) \to \mathrm{Ind}_{Q_1}^H(\tau_\varphi \boxtimes \mathrm{Ind}_{\bar{Q}_{\psi_0}}^{H_0}(\sigma_{0,\psi_0})).$$

Let Q_2 be the parabolic subgroup of H with Levi component $L_2 \simeq L_\varphi \times L_{\psi_0}$ such that $Q_2 \subset Q_1$ and $Q_2 \cap L_1 = L_\varphi \times Q_{\psi_0}$. Similarly, let Q_2' be the parabolic subgroup of H with the same Levi component as Q_2 such that $Q_2' \subset Q_1$ and $Q_2' \cap L_1 = L_\varphi \times \bar{Q}_{\psi_0}$.

Then, by induction in stages and properties of (normalized) intertwining operators, we can rewrite this intertwining map as

$$R_{Q_2'|Q_2}(\tau_\varphi \boxtimes \sigma_{0,\psi_0}) : \mathrm{Ind}_{Q_2}^H(\tau_\varphi \boxtimes \sigma_{0,\psi_0}) \to \mathrm{Ind}_{Q_2'}^H(\tau_\varphi \boxtimes \sigma_{0,\psi_0}).$$

Since the functor $\mathrm{Ind}_{Q_1}^H(\tau_\varphi \boxtimes \cdot)$ is exact, to show that $\mathrm{Ind}_{Q_1}^H(\tau_\varphi \boxtimes \sigma_0)$ is irreducible, it is sufficient to show that the image of $R_{Q_2'|Q_2}(\tau_\varphi \boxtimes \sigma_{0,\psi_0})$ is irreducible. Let Q_3 be the parabolic subgroup of H with the same Levi component L_3 as Q_2 and Q_2' such that

$$\Re(\omega_{\varphi,\psi_0}) \in \overline{\mathfrak{a}}_{Q_3}^{*,+},$$

where ω_{φ,ψ_0} is the central character of $\tau_\varphi \boxtimes \sigma_{0,\psi_0}$, and $\overline{\mathfrak{a}}_{Q_3}^{*,+}$ is as defined in [23, Section 3H]. Then by the properties of (normalized) intertwining operators, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Ind}_{Q_2}^H(\tau_\varphi \boxtimes \sigma_{0,\psi_0}) & \xrightarrow{R_{Q_2'|Q_2}} & \mathrm{Ind}_{Q_2'}^H(\tau_\varphi \boxtimes \sigma_{0,\psi_0}) \\ {\scriptstyle R_{Q_3|Q_2}}\downarrow & & \downarrow{\scriptstyle R_{\bar{Q}_3|Q_2'}} \\ \mathrm{Ind}_{Q_3}^H(\tau_\varphi \boxtimes \sigma_{0,\psi_0}) & \xrightarrow{R_{\bar{Q}_3|Q_3}} & \mathrm{Ind}_{\bar{Q}_3}^H(\tau_\varphi \boxtimes \sigma_{0,\psi_0}) \end{array}$$

Similar to [23, Lemma 3.10], we have the following lemma.

Lemma B.2. *The (normalized) intertwining operators*

$$R_{Q_3|Q_2}(\tau_\varphi \boxtimes \sigma_{0,\psi_0}) \quad \textit{and} \quad R_{\bar{Q}_3|Q_2'}(\tau_\varphi \boxtimes \sigma_{0,\psi_0})$$

are isomorphisms.

Proof. As in [23, proof of Lemma 3.10], the intertwining operator $R_{Q_3|Q_2}(\tau_\varphi \boxtimes \sigma_{0,\psi_0})$ can be decomposed as the composition of a sequence of intertwining operators

$$R_{Q_3|Q_2} = R_{R_t|R_{t-1}} \circ \cdots \circ R_{R_2|R_1} \circ R_{R_1|R_0},$$

where $R_0 = Q_2, R_1, \ldots, R_t = Q_3$ are parabolic subgroups of H. For each $R_{R_k|R_{k-1}}$ on the RHS, there exist $1 \le i \le r$ and $1 \le j \le m$ such that $R_{R_k|R_{k-1}}$ is essentially (a parabolic induction of) the intertwining operator

$$\mathrm{Ind}_{P_{i,j}}^{\mathrm{GL}_{k_i+d_j}(E)}(\tau_i \boxtimes \tau_j') \to \mathrm{Ind}_{\bar{P}_{i,j}}^{\mathrm{GL}_{k_i+d_j}(E)}(\tau_i \boxtimes \tau_j'),$$

where $P_{i,j}$ is a parabolic subgroup of $\mathrm{GL}_{k_i+d_j}(E)$ with Levi component $M_{i,j} \simeq \mathrm{GL}_{k_i}(E) \times \mathrm{GL}_{d_j}(E)$. It follows from [71, Theorem 6.19] and the conditions on τ_i, τ_j' that $\mathrm{Ind}_{P_{i,j}}^{\mathrm{GL}_{k_i+d_j}(E)}(\tau_i \boxtimes \tau_j')$ is irreducible. Then one can conclude that each $R_{R_k|R_{k-1}}$ is an isomorphism, hence so is their composition $R_{Q_3|Q_2}(\tau_\varphi \boxtimes \sigma_{0,\psi_0})$. Similarly, one can prove that $R_{\bar{Q}_3|Q_2'}(\tau_\varphi \boxtimes \sigma_{0,\psi_0})$ is also an isomorphism. ∎

Therefore, up to isomorphism, the image of $R_{Q_2'|Q_2}(\tau_\varphi \boxtimes \sigma_{0,\psi_0})$ is the same as the image of $R_{\bar{Q}_3|Q_3}(\tau_\varphi \boxtimes \sigma_{0,\psi_0})$. Notice that $\mathrm{Ind}_{Q_3}^H(\tau_\varphi \boxtimes \sigma_{0,\psi_0})$ is already very close to a standard module for H. In general, by "collapsing the same exponents", we can find a parabolic subgroup Q_4 of H with Levi component L_4 such that $Q_3 \subset Q_4$, $L_3 \subset L_4$, and

$$\Re(\omega_{\sigma_\psi}) \in \mathfrak{a}_{Q_4}^{*,+},$$

where ω_{σ_ψ} is the central character of the irreducible representation

$$\sigma_\psi := \mathrm{Ind}_{Q_3 \cap L_4}^{L_4}(\tau_\varphi \boxtimes \sigma_{0,\psi_0}).$$

It follows that $\mathrm{Ind}_{Q_4}^{H}(\sigma_\psi)$ is a standard module for H, and up to isomorphism, the image of $R_{\bar{Q}_3|Q_3}(\tau_\varphi \boxtimes \sigma_{0,\psi_0})$ is the same as the image of

$$R_{\bar{Q}_4|Q_4}(\sigma_\psi) : \mathrm{Ind}_{Q_4}^{H}(\sigma_\psi) \to \mathrm{Ind}_{\bar{Q}_4}^{H}(\sigma_\psi),$$

hence is irreducible. This completes the proof.

Appendix C. On irreducible self-dual Galois representations

In this appendix, we consider irreducible self-dual representations of the Weil group of a local field. The results in this appendix supplement our proof of Corollary 6.15 in Case O.

Let F be a non-Archimedean local field with characteristic zero. Let W_F be the Weil group of F. We use $\mathrm{Irr}^{\star,d}(F)$ (resp. $\mathrm{Irr}^{S,d}(F)$, $\mathrm{Irr}^{O,d}(F)$) to denote the irreducible self-dual (resp. symplectic, orthogonal) representations of W_F of dimension d.

Lemma C.1. *Suppose that F is a finite extension of $\mathbb{Q}_p$. Let π be a uniformizer of F, and k be the residue field of F. Then*

$$F^\times \simeq \pi^{\mathbb{Z}} \times k^\times \times \mu_{p^\infty}(F) \times \mathbb{Z}_p^d,$$

where $\mu_{p^\infty}(F)$ is the group of roots of unity of p-power order in F, and d is the degree of F over $\mathbb{Q}_p$.

Proof. See [62, Chapter 2.5]. ■

Lemma C.2. *Let E be a finite unramified extension of F, and $s = \mathrm{Frob}_F \in W_F$. Then E is a cyclic extension over F, and*

$$W_F / W_E \simeq \langle \bar{s} \rangle,$$

where $\bar{s}$ is the image of s in $\mathrm{Gal}(E/F)$. Moreover, the diagram

$$\begin{array}{ccc} W_E & \xrightarrow{\mathrm{Ad}(s)} & W_E \\ {\scriptstyle r_E}\downarrow & & \downarrow{\scriptstyle r_E} \\ E^\times & \xrightarrow{\bar{s}} & E^\times \end{array}$$

commutes, where r_E is the reciprocity law homomorphism of class field theory.

Proof. See [74, p. 4]. ■

Theorem C.3. *For any positive integer n, there exist infinitely many symplectic (resp. orthogonal) irreducible representations of W_F of dimension $2n$.*

Proof. Let E_0 be the unique unramified extension of F of degree n, and E be the unique unramified quadratic extension of E_0. Then E/F is also unramified. We have

$$E^\times/\mathrm{Nm}_{E/E_0}(E^\times) \simeq (\mathbb{Z}/2\mathbb{Z}) \times (k_E^\times/k_{E_0}^\times) \times (\mu_{p^\infty}(E)/\mu_{p^\infty}(E_0)) \times (\mathbb{Z}_p^{dn} \times T),$$

where T is some finite torsion group. Fix a primitive element x in $k_E^\times$, and a primitive (q^n+1)-th root of unity ζ. Let

$$X_\zeta : k_E^\times/k_{E_0}^\times \to \mathbb{C}^\times, \quad \overline{x} \mapsto \zeta.$$

Then for any character χ of $\mathbb{Z}_p^{dn} \times T$,

$$\mathrm{sgn} \boxtimes X_\zeta \boxtimes \mathbb{1} \boxtimes \chi \quad (\text{resp. } \mathbb{1} \boxtimes X_\zeta \boxtimes \mathbb{1} \boxtimes \chi)$$

gives a character $\widetilde{\chi}$ of $E^\times$, which satisfies

- $\widetilde{\chi}|_{E_0^\times} = \omega_{E/E_0}$ (resp. $\mathbb{1}$);
- $\widetilde{\chi}^{\overline{s}^i} \neq \widetilde{\chi}$ for any $1 \leq i < 2n$.

Therefore $\mathrm{Ind}_{W_E}^{W_F} \widetilde{\chi}$ is symplectic (resp. orthogonal) and irreducible of dimension $2n$. This construction gives an injection

$$\widehat{\mathbb{Z}_p^{dn}}/\sim \; \to \mathrm{Irr}^{S,2n}(F) \text{ (resp. } \mathrm{Irr}^{O,2n}(F)).$$

Here we regard a character of $\mathbb{Z}_p^{dn}$ as a character of $\mathbb{Z}_p^{dn} \times T$ which is trivial on the torsion group T, and define $\chi_1 \sim \chi_2$ if there is some $1 \leq i \leq 2n$ such that $\widetilde{\chi}_1^{\overline{s}^i} = \widetilde{\chi}_2$. Since $\widehat{\mathbb{Z}_p^{dn}}/\sim$ is an infinite set, we are done. ∎

Now we consider irreducible self-dual representations of W_F with arbitrary dimension.

Proposition C.4. *Suppose F is a finite extension of $\mathbb{Q}_p$ and $p \neq 2$. Then there is no irreducible self-dual representation of W_F with odd dimension greater than* 1.

Proof. See [66, Proposition 4]. ∎

We now assume that F is a finite extension of $\mathbb{Q}_2$ with residue field k_F. Let d be the degree of F over $\mathbb{Q}_2$, and d_u be the degree of k_F over $\mathbb{F}_2$.

Theorem C.5. *Let N be an arbitrary positive integer. Suppose*

$$2^d > N.$$

Then

$$|\mathrm{Irr}^{\star,1}(F)| > N.$$

Proof. Just notice that

$$F^\times/(F^\times)^2 \simeq (\mathbb{Z}/2\mathbb{Z}) \times \big(\mu_{2^\infty}(F)/(\mu_{2^\infty}(F))^2\big) \times (\mathbb{Z}/2\mathbb{Z})^d.$$ ∎

Theorem C.6. *Fix a positive integer n. Let N be an arbitrary positive integer. Suppose*

$$\frac{2^{d_u}-1}{n} > N.$$

Then

$$|\mathrm{Irr}^{\star,n}(F)| > N.$$

Proof. Let E be the unique unramified extension of F of degree n. Then

$$E^\times \simeq \pi^{\mathbb{Z}} \times k_E^\times \times U_E^1$$

and

$$U_E^1/U_E^2 \simeq k_E, \quad \text{where} \quad U_E^i = 1 + \pi^i \mathcal{O}_E,$$

and $\mathcal{O}_E$ is the ring of integers of E. Fix $x \in k_E$ such that $\{x, \overline{s}(x), \overline{s}^2(x), \dots\}$ is a basis of k_E over k_F. Then

$$k_E = k \cdot x + k \cdot \overline{s}(x) + \cdots + k \cdot \overline{s}^{n-1}(x),$$

and the Pontryagin dual of k_E can be identified with the set of n-tuples of characters of k_F by

$$(\chi_1, \dots, \chi_n) \mapsto \big(\lambda \cdot \overline{s}^i(x) \mapsto \chi_i(\lambda)\ ,\ \lambda \in k\big).$$

Under this identification, $\overline{s}$ acts on the Pontryagin dual of k_E by

$$\overline{s} : (\chi_1, \dots, \chi_n) \mapsto (\chi_n, \chi_1, \dots, \chi_{n-1}).$$

We have injective maps

$$\widehat{k_F} \setminus \{\mathbb{1}\} \to \widehat{k_E} \to \widehat{E^\times},$$

where the first map is given by $\chi \mapsto (\chi, \mathbb{1}, \dots, \mathbb{1})$, and the second map is induced by the natural projection

$$E^\times \to U_E^1 \to U_E^1/U_E^2 \simeq k_E.$$

We denote the image of χ in $\widehat{E^\times}$ by $\widetilde{\chi}$. By our construction,

- $\widetilde{\chi}$ is quadratic;
- $\widetilde{\chi}^{\overline{s}^i} \neq \widetilde{\chi}$ for any $1 \leq i < n$.

Therefore $\mathrm{Ind}_{W_E}^{W_F} \widetilde{\chi}$ is self-dual and irreducible of dimension n. This construction gives an injection

$$(\widehat{k_F} \setminus \{\mathbb{1}\})/\!\sim\; \to \mathrm{Irr}^{\star,n}(F).$$

Here we define $\chi_1 \sim \chi_2$ if there is some $1 \leq i \leq n$ such that $\widetilde{\chi}_1^{\overline{s}^i} = \widetilde{\chi}_2$. Notice that there are at most n elements in each equivalence class. Hence the LHS of this injection has at least

$$\frac{2^{d_u}-1}{n}$$

elements. By our assumption, we are done. ∎

Appendix D. On irreducible conjugate self-dual Galois representations

In this appendix, we consider irreducible conjugate self-dual representations of the Weil group of a local field. The results in this appendix supplement our proof of Corollary 6.15 in Case U.

Let F be a non-Archimedean local field with characteristic zero, and E be a quadratic field extension of F. Let W_F and W_E be the Weil groups of F and E respectively. We fix an $s \in W_F \backslash W_E$. We use $\mathrm{Irr}_F^{S,d}(E)$ (resp. $\mathrm{Irr}_F^{O,d}(E)$) to denote the irreducible conjugate symplectic (resp. conjugate orthogonal) representation of W_E of dimension d.

Lemma D.1. *Suppose that*

(1) *F is a finite extension of $\mathbb{Q}_p$ and $p \neq 2$;*

(2) *E/F is ramified.*

Then for any positive integer n, $E_n := E \otimes_F F_n$ is a ramified quadratic field extension of F_n, where F_n is the unique degree n unramified extension of F.

Proof. By our assumptions, we can choose a uniformizer π of E such that π^2 is a uniformizer of F. Then

$$E \simeq F[x]/(x^2 - \pi^2).$$

Notice that for each positive integer n, π^2 is also uniformizer of F_n, hence the polynomial $x^2 - \pi^2$ is also irreducible in F_n. It then follows that $E_n := E \otimes_F F_n$ is a ramified quadratic field extension of F_n. ■

In the rest of this appendix, we assume that the local fields F and E satisfy the conditions in this lemma. We also retain the notations in the proof of the lemma. Let Γ_n be the Galois group of F_n/F, and k_n the residue field of both F_n and E_n. We denote by d_u the degree of k_1 over $\mathbb{F}_p$. In the spirit of Theorem C.6, we prove the following.

Theorem D.2. *Fix a positive integer n. Let N be an arbitrary positive integer. Suppose*

$$\frac{p^{d_u} - 1}{n} > N.$$

Then, for $\natural \in \{S, O\}$, we have

$$|\mathrm{Irr}_F^{\natural,n}| > N.$$

Proof. By the structure theorem of local fields, we have

$$E_n^\times \simeq \pi^{\mathbb{Z}} \times k_n^\times \times U_{E_n}^1, \quad \mathrm{Nm}_{E_n/F_n}(E_n^\times) \simeq \pi^{2\mathbb{Z}} \times (k_n^\times)^2 \times U_{F_n}^1,$$

where $U_{E_n}^1$ and $U_{F_n}^1$ are as in the proof of Theorem C.6. These two isomorphisms are indeed Γ_n-equivariant. They induce another Γ_n-equivariant isomorphism

$$E_n^\times/\mathrm{Nm}_{E_n/F_n}(E_n^\times) \simeq (\mathbb{Z}/2\mathbb{Z}) \times (k_n^\times/(k_n^\times)^2) \times (U_{E_n}^1/U_{F_n}^1).$$

Notice that $U^1_{F_n} \subset U^2_{E_n}$ as Γ_n-modules. Hence we obtain a Γ_n-equivariant surjection

$$U^1_{E_n}/U^1_{F_n} \to U^1_{E_n}/U^2_{E_n} \simeq k_{E_n}.$$

By the same trick as in the proof of Theorem C.6, for any non-trivial character χ of k_E we can produce a character $\widetilde{\chi}$ of $E_n^\times$ as follows: firstly we fix some $x \in k_{E_n}$ such that $\{\gamma(x) \mid \gamma \in \Gamma_n\}$ is a basis of k_{E_n} over k_E, which allows us to identify the Pontryagin dual of k_{E_n} with the set of n-tuples of characters of k_E by

$$(\chi_\gamma)_{\gamma\in\Gamma_n} \mapsto \big(\lambda \cdot \gamma(x) \mapsto \chi_\gamma(\lambda),\ \lambda \in k_E\big);$$

then, under this identification, χ can be regarded as a character of k_{E_n} via

$$\chi \mapsto (\chi, \mathbb{1}, \dots, \mathbb{1}),$$

which we shall still denote by χ; finally, we pull back the character

$$\mathbb{1} \boxtimes \mathrm{sgn} \boxtimes \chi \quad (\text{resp. } \mathbb{1} \boxtimes \mathbb{1} \boxtimes \chi)$$

of $(\mathbb{Z}/2\mathbb{Z}) \times (k_n^\times/(k_n^\times)^2) \times k_{E_n}$ along the natural projections

$$E_n^\times \to E_n^\times/\mathrm{Nm}_{E_n/F_n}(E_n^\times) \to (\mathbb{Z}/2\mathbb{Z}) \times (k_n^\times/(k_n^\times)^2) \times k_{E_n}.$$

We denote the image of χ in $\widehat{E_n^\times}$ by $\widetilde{\chi}$. By our construction,

- $\widetilde{\chi}$ is conjugate symplectic (resp. conjugate orthogonal) with respect to F_n;
- $\widetilde{\chi}^\gamma \neq \widetilde{\chi}$ for any $1 \neq \gamma \in \Gamma_n$.

Therefore $\mathrm{Ind}_{W_{E_n}}^{W_E} \widetilde{\chi}$ is conjugate symplectic (resp. conjugate orthogonal) and irreducible of dimension n. This construction gives an injection

$$(\widehat{k_E} \setminus \{\mathbb{1}\})/\sim \to \mathrm{Irr}_F^{S,n}(E) \quad (\text{resp. } (\widehat{k_E} \setminus \{\mathbb{1}\})/\sim \to \mathrm{Irr}_F^{\mathrm{O},n}(E)).$$

Here we define $\chi_1 \sim \chi_2$ if there is some $1 \neq \gamma \in \Gamma_n$ such that $\widetilde{\chi}_1^\gamma = \widetilde{\chi}_2$. Notice that there are at most n elements in each equivalence class. Hence the LHS of this injection has at least

$$\frac{p^{d_u} - 1}{n}$$

elements. By our assumption, we are done. ∎

Appendix E. Existence of certain number fields

In this appendix, we prove the existence of certain number fields. The results in this appendix are used in the proof of Corollary 6.15. We start with a well known general result.

Theorem E.1. *Let $\dot{F}$ be a number field, and $v_1, \dots, v_r$ be inequivalent places of $\dot{F}$. Let $F_i = \dot{F}_{v_i}$, and K_i a finite extension of F_i of degree d_i. Set*

$$d = \max\{d_i \mid 1 \le i \le r\}.$$

Then there exists a degree d extension $\dot{K}$ of $\dot{F}$, and places v_i' of $\dot{K}$ above v_i, such that

$$\dot{K}_{v_i'} \simeq K_i \quad \text{as extensions of } F_i \text{ for all } i = 1, \dots, r.$$

Proof. This is a simple application of Krasner's lemma and the weak approximation theorem. Since we do not know a convenient reference, for completeness we briefly sketch the proof.

Since we are considering characteristic zero fields, any finite extension is simple. For each $i = 1, \dots, r$, let $\alpha_i \in K_i$ be such that $K_i = F_i(\alpha_i)$, and g_i' the minimal polynomial of α_i over F_i. Then

$$F_i[x]/(g_i') \simeq K_i.$$

Let $\beta'_{i,1}, \dots, \beta'_{i,d-d_i} \in F_i$ be distinct and such that $g_i'(\beta'_{i,j}) \neq 0$ for all $1 \le j \le d - d_i$. We put

$$f_i(x) = (x - \beta'_{i,1}) \cdots (x - \beta'_{i,d-d_i}) \cdot g_i'(x).$$

By the weak approximation theorem, we can take a monic polynomial $f \in \dot{F}[x]$ of degree d such that for all $1 \le i \le r$, the coefficients of f are arbitrarily close to the coefficients of f_i (with respect to the valuation v_i). Then by Krasner's lemma and some classical analysis, we can take f so close that for all $1 \le i \le r$,

- f can be decomposed as

$$f(x) = (x - \beta_{i,1}) \cdots (x - \beta_{i,d-d_i}) \cdot g_i(x)$$

  for some $\beta_{i,1}, \dots, \beta_{i,d-d_i} \in F_i$ and $g_i \in F_i[x]$;
- there is an isomorphism

$$F_i[x]/(g_i) \simeq F_i[x]/(g_i') \quad \text{as } F_i\text{-algebras}.$$

Note that there exists $i_0 \in \{1, \dots, r\}$ such that $d_{i_0} = d$. It follows that g'_{i_0} is an irreducible polynomial of degree d in $F_i[x]$. Consequently, f is also irreducible in $\dot{F}[x]$. Therefore

$$\dot{K} := \dot{F}[x]/(f)$$

is a field, and one can easily check that $\dot{K}$ satisfies all our requirements. ∎

Now let F be a local field, and let E be either F itself or a quadratic extension of F. Using the theorem above we can prove the existence of a pair of number fields claimed in the proof of Corollary 6.15.

Corollary E.2. *Given a positive integer d and a prime number p, there exists a pair $(\dot{E}, \dot{F})$ of number fields, together with three places u_1, u_2, w of $\dot{F}$, such that*

(1) $(\dot{E}_{u_1}, \dot{F}_{u_1}) \simeq (\dot{E}_{u_2}, \dot{F}_{u_2}) \simeq (E, F)$;

(2) *$\dot{F}_w$ is a finite extension of $\mathbb{Q}_p$, and the degree of the residue field k_w of $\dot{F}_w$ over $\mathbb{F}_p$ is greater than d;*

(3) *if E is a quadratic extension of F, then $\dot{E}_w$ is a ramified quadratic field extension of $\dot{F}_w$; if further F is non-Archimedean, then $\dot{F}$ is totally imaginary.*

Proof. We shall construct the desired number fields from $\mathbb{Q}$. Let

$$v = \begin{cases} \ell & \text{if } F \text{ is a finite extension of } \mathbb{Q}_\ell, \\ \infty & \text{if } F \text{ is Archimedean.} \end{cases}$$

Let $\dot{F}_0$ be a finite extension of $\mathbb{Q}$, together with three places u_1', u_2', w, such that $\dot{F}_{0,u_1'}$ and $\dot{F}_{0,u_2'}$ are subfields of F, and w' is above p. Such an $\dot{F}_0$ clearly exists: we can take $\dot{F}_0$ to be a quadratic or biquadratic extension of $\mathbb{Q}$, depending on whether v equals p or not, such that v is totally split. Let F_w be a finite extension of $\dot{F}_{0,w'}$ such that

$$[k_w : \mathbb{F}_p] > d,$$

where k_w is the residue field of F_w. By the theorem above, there exists a number field $\dot{F}$, together with three places u_1, u_2, w above u_1', u_2', w', such that

$$\dot{F}_{u_1} \simeq \dot{F}_{u_2} \simeq F \quad \text{and} \quad \dot{F}_w \simeq F_w.$$

If further F is non-Archimedean, we can take $\dot{F}$ to be totally imaginary. Indeed, if $\dot{F}$ is not totally imaginary, let R be the set of all real places of $\dot{F}$. Applying the theorem above, we obtain a quadratic extension $\dot{F}'$ of $\dot{F}$ such that u_1, u_2, w split in $\dot{F}'$, and

$$\dot{F}_u' \simeq \mathbb{C}$$

for all $u \in R$. Then $\dot{F}'$ is a totally imaginary field which satisfies our requirements, and we may replace $\dot{F}$ by $\dot{F}'$. Finally, if E is a quadratic extension of F, once again it follows from the theorem above that there exists a number field $\dot{E}$ such that

$$\dot{E}_{u_1} \simeq \dot{E}_{u_2} \simeq E,$$

and $\dot{E}_w$ is a ramified extension of $\dot{F}_w$. ∎

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