J. Eur. Math. Soc. 27, 5155–5171 (2025) DOI 10.4171/JEMS/1565

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Ricci limit flows and weak solutions

Received March 10, 2022; revised October 29, 2024

Abstract. In this paper we reconcile several different approaches to Ricci flow through singularities that have been proposed over the last few years by Kleiner–Lott, Haslhofer–Naber and Bamler. Specifically, we prove that every noncollapsed limit of Ricci flows, as provided by Bamler's precompactness theorem, as well as every singular Ricci flow of Kleiner–Lott, is a weak solution in the sense of Haslhofer–Naber. We also generalize all path-space estimates of Haslhofer–Naber to the setting of noncollapsed Ricci limit flows. The key step to establish these results is a new hitting estimate for Brownian motion. A fundamental difficulty, in stark contrast to all prior hitting estimates in the literature, is the lack of lower heat kernel bounds under Ricci flow. To overcome this, we introduce a novel approach to hitting estimates that compensates for the lack of lower heat kernel bounds by making use of the heat kernel geometry of space-time.

Keywords: Ricci flow, weak solution, Brownian motion, heat kernel estimates.

1. Introduction

A family $(g_t)_{t\in I}$ of Riemannian metrics, say on a closed n-dimensional manifold M, evolves by Ricci flow if

$$\partial_t g_t = -2\,\mathrm{Ric}(g_t). \tag{1.1}$$

In a recent breakthrough [1–3], Bamler established a precompactness and partial regularity theory. The limits provided by his precompactness theorem are so-called metric flows. A metric flow

$$\mathcal{X} = \big(\mathcal{X}, \mathfrak{t}, (d_t)_{t\in I}, (\nu_{x;s})_{x\in\mathcal{X}, s\in I, s\le \mathfrak{t}(x)}\big), \tag{1.2}$$

is given by a set $\mathcal{X}$, a time-function $\mathfrak{t}:\mathcal{X}\to\mathbb{R}$, complete separable metrics d_t on the time-slices $\mathcal{X}_t = \mathfrak{t}^{-1}(t)$, and probability measures $\nu_{x;s}\in\mathcal{P}(\mathcal{X}_s)$ such that the Kolmogorov consistency condition and a certain sharp gradient estimate for the heat flow hold (see

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Mathematics Subject Classification 2020: 53E20 (primary); 58J65 (secondary).

Section 3.1 for details). In particular, any smooth Ricci flow can of course be viewed as metric flow by choosing $\mathcal{X} = M \times I$, defining $\mathfrak{t}$ as the projection on I, letting d_t be the induced metrics on time slices, and setting $\nu_{x;s}$ to be the conjugate heat kernel measure based at $x = (p, t)$, i.e.

$$d\nu_{(p,t);s}(q) = K(p, t; q, s)\, d\mathrm{Vol}_{g_s}(q), \tag{1.3}$$

where $K(p, t; q, s)$ is the heat kernel of the Ricci flow (specifically, K solves the forwards heat equation as a function of (p, t) and the conjugate heat equation as a function of (q, s)).

Under the noncollapsing assumption that the Nash entropy is bounded below, which is of course perfectly natural in light of Perelman's monotonicity formula [9], Bamler proved that the singular set $\mathcal{S} \subset \mathcal{X}$ of the limit flow has parabolic $*$-Minkowski dimension at most $n - 2$.

In a different direction, a notion of weak solutions for the Ricci flow has been proposed a few years earlier by Naber and the second author. Specifically, it has been shown in [6] that a smooth family $(g_t)_{t\in I}$ of Riemannian metrics evolves by Ricci flow if and only if the sharp infinite-dimensional gradient estimate

$$|\nabla_p \mathbb{E}_{(p,t)}[F]| \le \mathbb{E}_{(p,t)}[|\nabla^{\|} F|] \tag{1.4}$$

holds for all cylinder functions F on the path-space of its space-time. Here, $\mathbb{E}_{(p,t)}$ denotes the expectation with respect to the Wiener measure of Brownian motion starting at (p, t), and $\nabla^{\|}$ denotes the parallel gradient, which is defined via a suitable stochastic parallel transport. Based on this characterization it has been proposed that a possibly singular space equipped with a time-function and a linear heat flow should be called a weak solution of the Ricci flow if and only if the sharp infinite-dimensional gradient estimate on path-space holds for almost every point (p, t).

The goal of the present paper is to reconcile these two approaches. As we will explain in detail in Section 3.2, any noncollapsed Ricci limit flow $\mathcal{X}$ can be canonically equipped with a notion of Brownian motion and stochastic parallel transport. For now, let us just mention that Brownian motion X_τ starting at $x \in \mathcal{X}$ is simply characterized by the formula

$$\mathbb{P}_x[X_{\tau_1} \in B_1, \dots, X_{\tau_k} \in B_k] = \int_{B_1\times\cdots\times B_k} d\nu_{x;\mathfrak{t}(x)-\tau_1}(x_1)\dots d\nu_{x_{k-1};\mathfrak{t}(x)-\tau_k}(x_k). \tag{1.5}$$

Using these notions, we can now state our main theorem:

Theorem 1.1 (Ricci limit flows and weak solutions). *Given any noncollapsed Ricci limit flow $\mathcal{X}$, for any regular point $x = (p, t)$ we have the infinite-dimensional gradient estimate*

$$|\nabla_p \mathbb{E}_{(p,t)}[F]| \le \mathbb{E}_{(p,t)}[|\nabla^{\|} F|] \tag{1.6}$$

for all cylinder functions F. In particular, any noncollapsed limit of Ricci flows, as provided by Bamler's precompactness theorem, is a weak solution of the Ricci flow in the sense of Haslhofer–Naber.

In fact, our argument applies to any noncollapsed metric flow that satisfies the partial regularity properties from [2] and solves the Ricci flow equation on its regular part. In particular, viewing any singular Ricci flow of Kleiner–Lott [8] as a metric flow as in [1, Section 3.7], we can confirm a prediction from [6]:

Corollary 1.2 (Singular Ricci flows and weak solutions). *Every singular Ricci flow in the sense of Kleiner–Lott is a weak solution of the Ricci flow in the sense of Haslhofer–Naber.*

As another important consequence of Theorem 1.1 all other path-space estimates for smooth flows from [6] generalize to the path-space of noncollapsed Ricci limit flows as well:

Corollary 1.3 (Estimates on path-space of Ricci limit flows). *The following estimates hold on the path-space of any noncollapsed Ricci limit flow $\mathcal{X}$:*

- *For every cylinder function F the induced martingale F_τ for almost every $(p,t) \in \mathcal{X}$ satisfies the quadratic variation estimate*

$$\mathbb{E}_{(p,t)}\left[\frac{d[F_\bullet]_\tau}{d\tau}\right] \leq 2\mathbb{E}_{(p,t)}[|\nabla_\tau^\| F|^2]. \tag{1.7}$$

- *For almost every $(p,t) \in \mathcal{X}$ the Ornstein–Uhlenbeck operator on path-space, $\mathcal{L}_{\tau_1,\tau_2} = \int_{\tau_1}^{\tau_2} \nabla_\tau^{\|*}\nabla_\tau^\| \, d\tau$, satisfies the log-Sobolev inequality*

$$\mathbb{E}_{(p,t)}[(F^2)_{\tau_2} \log (F^2)_{\tau_2} - (F^2)_{\tau_1} \log (F^2)_{\tau_1}] \leq 4\mathbb{E}_{(p,t)}[\langle F, \mathcal{L}_{\tau_1,\tau_2} F\rangle]. \tag{1.8}$$

- *For almost every $(p,t) \in \mathcal{X}$ the Ornstein–Uhlenbeck operator on path-space satisfies the spectral gap estimate*

$$\mathbb{E}_{(p,t)}[(F_{\tau_2} - F_{\tau_1})^2] \leq 2\mathbb{E}_{(p,t)}[\langle F, \mathcal{L}_{\tau_1,\tau_2} F\rangle]. \tag{1.9}$$

Indeed, once the gradient estimate (1.6) is established, all other path-space estimates follow by arguing similarly to [6, Section 4].

The key for proving Theorem 1.1 is a new hitting estimate for the Ricci flow. For exposition sake, let us first discuss this estimate in the context of smooth Ricci flows. To this end, let $(g_t)_{t\in(t_0-2r^2,t_0]}$ be a Ricci flow on a closed n-dimensional manifold M, and recall that the Nash entropy based at (p_0,t_0) is defined by

$$\mathcal{N}_{(p_0,t_0)}(\tau) := -\int_M \log K(p_0,t_0;\cdot,t_0-\tau)\, d\nu_{(p_0,t_0);t_0-\tau} - \frac{n}{2}(1+\log(4\pi\tau)). \tag{1.10}$$

Recall also that, given any $\varepsilon > 0$, the quantitative singular set is defined by

$$\mathcal{S}_\varepsilon = \{(p,t) : r_{\mathrm{Rm}}(p,t) \leq \varepsilon\}, \tag{1.11}$$

where $r_{\mathrm{Rm}}(p,t)$ is the largest r such that $|\mathrm{Rm}| \leq r^{-2}$ on the backward parabolic ball $P_-(p,t;r)$.

Theorem 1.4 (Hitting estimate for the Ricci flow). *For all* $Y < \infty$, $\delta > 0$, *and* $r_0 \in (0, r/2)$, *there exists a constant* $C = C(n, Y, \delta, r_0, r) < \infty$ *such that if* $(g_t)_{t\in(t_0-2r^2,t_0]}$ *is a Ricci flow with* $\mathcal{N}_{(p_0,t_0)}(r_0^2) \geq -Y$ *and* $r_{\mathrm{Rm}}(p_0, t_0) \geq r_0$, *then Brownian motion* X_τ *starting at* (p_0, t_0) *satisfies*

$$\mathbb{P}_{(p_0,t_0)}\big[X_\tau \text{ hits } \mathcal{S}_\varepsilon \cap P^*(p_0, t_0; r) \text{ for some } \tau \in [0, r^2]\big] \leq C\varepsilon^{2-\delta} \tag{1.12}$$

for all $\varepsilon > 0$.

Heuristically, one can of course easily guess the (almost) quadratic dependence on ε in light of Bamler's codimension-4 partial regularity result and the intuition that the image of Brownian curves is 2-dimensional. Indeed, hitting estimates in related easier situations go back all the way to the classical work of Kakutani [7]. A sharp hitting estimate for Brownian motion in Euclidean space has been obtained by Benjamini–Pemantle–Peres [4]. Recently, in [5] we generalized the Benjamini–Pemantle–Peres estimate to the setting of spaces with Ricci curvature bounded below.

A fundamental new difficulty in the context of Ricci flow, in stark contrast to all prior hitting estimates in the literature, is that the heat kernel only has upper bounds, but no lower bounds. To overcome this, we introduce a novel approach to hitting estimates. Roughly speaking, we compensate for the lack of lower heat kernel bounds by making use of the heat kernel geometry as introduced in [3], including in particular the properties of H_n-centers and P^* parabolic balls (see Section 2.1 for a discussion of these notions).

Our proof of the hitting estimate also carries through in the more general setting of noncollapsed Ricci limit flows. In particular, we obtain the following result:

Corollary 1.5 (Brownian motion on Ricci limit flows). *If* $\mathcal{X}$ *is a noncollapsed Ricci limit flow, and* $x \in \mathcal{R} \subset \mathcal{X}$ *is a regular point, then the Wiener measure* $\mathbb{P}_x$ *of Brownian motion starting at* x *concentrates on the space of continuous space-time curves that stay entirely in the regular part* $\mathcal{R}$.

Using these results, we can then establish the infinite-dimensional gradient estimate on path-space by adapting the argument from [6] to our setting. Specifically, we first consider the O_n-frame bundle $\pi : \mathcal{F} \to \mathcal{R}$ over the regular part. Recalling that this bundle comes with a distribution of horizontal $(n+1)$-planes induced by Hamilton's space-time connection, we can then construct a process $U_\tau \in \mathcal{F}$ as unique horizontal lift of the Brownian motion $X_\tau \in \mathcal{R}$. Thanks to Corollary 1.5 the process U_τ does not explode. This enables us to define the stochastic parallel transport map

$$P_\tau := U_0 U_\tau^{-1} : T_{X_\tau}\mathcal{R}_{\mathfrak{t}(x)-\tau} \to T_x\mathcal{R}_{\mathfrak{t}(x)}, \tag{1.13}$$

which in turn allows us to define the parallel gradient of any cylinder function $F(X) = f(X_{\tau_1}, \ldots, X_{\tau_k})$ by

$$\nabla^{\|} F(X) = \sum_{i=1}^{k} P_{\tau_i}\, \mathrm{grad}^{(i)}_{g_{\mathfrak{t}(x)-\tau_i}} f(X_{\tau_1}, \ldots, X_{\tau_k}). \tag{1.14}$$

Another key step is to show that if v is a heat flow, then its gradient at any regular point $x \in \mathcal{R}$ is given by the Feynman–Kac type representation formula

$$\mathrm{grad}_{g_t} v(x) = \mathbb{E}_x[P_{t-s}\, \mathrm{grad}_{g_{t-s}} v|_{\mathcal{R}_s}], \tag{1.15}$$

where $t = \mathfrak{t}(x)$. To show this, we localize on $\mathcal{X} \setminus \mathcal{S}_\varepsilon$ via a suitable cutoff function, and then take the limit $\varepsilon \to 0$ using Theorem 1.4. Finally, after this is established, we check that the rest of the argument from [6] goes through with minor adaptions.

This article is organized as follows. In Section 2, we prove Theorem 1.4. In Section 3, we prove Theorem 1.1.

2. Hitting estimate for Ricci flow

2.1. Notation and preliminaries

Let $(g_t)_{t\in I}$ be a Ricci flow on a closed n-dimensional manifold M. The *heat kernel* $K(p,t;q,s)$, where $p,q \in M$ and $s < t$ in I, is defined by

$$(\partial_t - \Delta_{g_t})K(\cdot,\cdot;q,s) = 0, \qquad \lim_{t\searrow s} K(\cdot,t;q,s) = \delta_q. \tag{2.1}$$

By duality, as a function of the last two variables this solves the conjugate problem

$$(-\partial_s - \Delta_{g_s} + R_{g_s})K(p,t;\cdot,\cdot) = 0, \qquad \lim_{s\nearrow t} K(p,t;\cdot,s) = \delta_p. \tag{2.2}$$

The *conjugate heat kernel measure* is defined by

$$d\nu_{(p,t);s}(q) = K(p,t;q,s)d\mathrm{Vol}_{g_s}(q), \qquad d\nu_{(p,t);t} = \delta_p. \tag{2.3}$$

Note that this is a probability measure. We often write

$$d\nu_{(p,t);s}(q) = (4\pi\tau)^{-n/2} e^{-f_{(p,t)}(q,s)} d\mathrm{Vol}_{g_s}(q), \tag{2.4}$$

where $\tau = t - s$. In terms of the potential $f_{(p,t)}$ the *pointed Nash entropy* is given by

$$\mathcal{N}_{(p,t)}(\tau) = \int f_{(p,t)}(\cdot, t-\tau)\, d\nu_{(p,t);t-\tau} - \frac{n}{2}. \tag{2.5}$$

By Perelman's monotonicity formula [9], the function $\tau \mapsto \tau\mathcal{N}_{(p,t)}(\tau)$ is concave. We also recall from [3, Proposition 5.2] that $\tau \mapsto \mathcal{N}_{(p,t)}(\tau)$ is nonincreasing, and hence $\mathcal{N}_{(p,t)} \leq 0$, and

$$\frac{d}{d\tau}\mathcal{N}_{(p,t)}(\tau) \geq \min_{q\in M} R(q, t_0 - \tau) - \frac{n}{2\tau}. \tag{2.6}$$

Next, we recall the well known fact (see e.g. [3, Lemma 2.7]) that under Ricci flow the 1-Wasserstein distance between conjugate heat kernel measures is monotone, namely

$$s \mapsto d_{W_1(g_s)}(\nu_{(p_1,t_1);s}, \nu_{(p_2,t_2);s}) \qquad \text{is nondecreasing.} \tag{2.7}$$

Here, by Kantorovich duality, the 1-Wasserstein distance between probability measures is given by

$$d_{W_1(g)}(\mu_1, \mu_2) = \sup \int_M f \, d\mu_1 - \int_M f \, d\mu_2, \tag{2.8}$$

where the supremum is taken over all bounded 1-Lipschitz functions $f : (M, g) \to \mathbb{R}$. Motivated by this, Bamler pointed out that instead of considering *conventional parabolic balls*

$$P(p_0, t_0; r) := B_{g_{t_0}}(p_0; r) \times [t_0 - r^2, t_0 + r^2], \tag{2.9}$$

it is often more useful to consider so-called P^* *parabolic balls* defined by

$$P^*(p_0, t_0; r) := \big\{(p, t) \in M \times [t_0 - r^2, t_0 + r^2] : \\ d_{W_1(g_{t_0 - r^2})}(\nu_{(p_0,t_0);t_0-r^2}, \nu_{(p,t);t_0-r^2}) < r\big\}. \tag{2.10}$$

By [3, Proposition 9.4], P^* parabolic balls satisfy containment principles similar to those for conventional parabolic balls, in particular:

$$\begin{aligned} (p_1, t_1) \in P^*(p_2, t_2; r) &\implies P^*(p_2, t_2; r) \subseteq P^*(p_1, t_1; 2r), \\ (p_1, t_1) \in P^*(p_2, t_2; r) &\implies P^*(p_1, t_1, r') \subseteq P^*(p_2, t_2; r + r'). \end{aligned} \tag{2.11}$$

Moreover, by [3, Theorem 9.8], there is some universal $C < \infty$ such that if $[t_0 - 2r^2, t_0] \subseteq I$, then for all $t' \in [t_0 - r^2, t_0 + r^2]$ the volume of the time t'-slices is bounded by

$$\mathrm{Vol}_{g_{t'}}(P^*(p_0, t_0; r) \cap \{t = t'\}) \leq C e^{\mathcal{N}_{(p_0,t_0)}(r^2)} r^n. \tag{2.12}$$

We will also need the covering result from [3, Theorem 9.11], which says that there is a universal constant $C < \infty$ with the following significance: If $[t_0 - 2r^2, t_0] \subseteq I$, then for any $X \subseteq P^*(p_0, t_0; r)$ and any $\lambda \in (0, 1)$, we can find points $(q_1, s_1), \ldots, (q_N, s_N)$ in X such that

$$X \subseteq \bigcup_{i=1}^{N} P^*(q_i, s_i; \lambda r) \quad \text{and} \quad N \leq C\lambda^{-(n+2)}. \tag{2.13}$$

Now, assuming $[t_0 - 2r^2, t_0] \subseteq I$ and $\mathcal{N}_{(p_0,t_0)}(r^2) \geq -Y$, if we consider the quantitative singular set

$$\mathcal{S}_\varepsilon = \{(p, t) : r_{\mathrm{Rm}}(p, t) \leq \varepsilon\}, \tag{2.14}$$

where $r_{\mathrm{Rm}}(p, t)$ is the largest r such that $|\mathrm{Rm}| \leq r^{-2}$ on $P_-(p, t; r) = B_{g_t}(p; r) \times [t_0 - r^2, t_0]$, then by Bamler's quantitative parabolic $*$-Minkowski codimension-4 bound [2, Theorem 2.25] we can find points $(q_1, s_1), \ldots, (q_N, s_N) \in \mathcal{S}_\varepsilon \cap P^*_-(p_0, t_0; r)$ such that

$$\mathcal{S}_\varepsilon \cap P^*_-(p_0, t_0; r) \subseteq \bigcup_{i=1}^{N} P^*(q_i, s_i; \varepsilon) \quad \text{and} \quad N \leq C\varepsilon^{-(n-2)-\delta}, \tag{2.15}$$

where $C < \infty$ is a constant that only depends on n, Y, r and δ. Note that for smooth flows we could equally well work with two-sided parabolic balls, but for the generalization to noncollapsed limit flows it is better to use backwards parabolic balls $P^*_-(p_0,t_0;r) = P^*(p_0,t_0;r)\cap\{t\le t_0\}$.

Finally, in general there is no containment between P and P^* parabolic balls. However, if we assume $r_{\mathrm{Rm}}(p,t)\ge r$ then by [3, Corollary 9.6] we have

$$P_-(p_0,t_0;\eta r)\subseteq P^*_-(p_0,t_0;r) \quad \text{and} \quad P^*_-(p_0,t_0;\eta r)\subseteq P_-(p_0,t_0;r), \tag{2.16}$$

where $\eta>0$ is a universal constant.

Next, by an important discovery of Bamler [3, Corollary 3.7], under Ricci flow

$$s\mapsto \mathrm{Var}_{g_s}(\nu_{(p_1,t_1);s},\nu_{(p_2,t_2);s}) + H_n s \quad \text{is nondecreasing.} \tag{2.17}$$

Here, $H_n=\pi^2(n-1)/2+4$, and the variance between two probability measure on (M,g) is defined as

$$\mathrm{Var}_g(\mu_1,\mu_2) = \iint_{M\times M} d_g^2(x_1,x_2)\,d\mu_1(x_1)\,d\mu_2(x_2). \tag{2.18}$$

Motivated by this, as in [3, Definition 3.10] a point (q,s) is called an H_n*-center* of (p,t) if $s\le t$ and

$$\mathrm{Var}_{g_s}(\delta_q,\nu_{(p,t);s})\le H_n(t-s). \tag{2.19}$$

As a direct consequence of (2.17), given any (p,t) and $s\le t$, there always exists at least one H_n-center (q,s) of (p,t) and the distance between any two such H_n-centers is bounded by

$$d_s(q,q')\le 2\sqrt{H_n(t-s)}. \tag{2.20}$$

Here and below, d_s denotes the metric distance induced by the Riemannian metric g_s. Moreover, as a direct consequence of the definitions, for any $A<\infty$ one has

$$\nu_{(p,t);s}(B_{g_s}(q,\sqrt{AH_n(t-s)}))\ge 1-A^{-1}. \tag{2.21}$$

Finally, in general there is no universal bound on the distance from H_n-centers to the base point p. However, if we assume for instance $r_{\mathrm{Rm}}(p,t)\ge r$, then by [3, proof of Proposition 9.5] there is a universal $C<\infty$ such that for all H_n-centers (q,s) with $s\in[t-C^{-1}r^2,t)$,

$$d_s(q,p)\le C\sqrt{t-s}. \tag{2.22}$$

To conclude this subsection, let us discuss heat kernel bounds. By [3, Theorem 7.2], if $R\ge R_{\min}$ and $[t-\tau,t]\subseteq I$, then for some $C=C(\tau\cdot R_{\min})<\infty$ we have the upper bound

$$K(p,t;q,t-\tau)\le \frac{C}{\tau^{n/2}}e^{-\mathcal{N}_{(p,t)}(\tau)}e^{-\frac{d_{t-\tau}(p_{t-\tau},q)^2}{10\tau}}, \tag{2.23}$$

where $(p_{t-\tau},t-\tau)$ is any H_n-center of (p,t). In general, there are no corresponding lower bounds.

2.2. *Proof of the hitting estimate*

In this subsection, we prove Theorem 1.4. By time translation and parabolic rescaling we may assume that $t_0=0$ and $r=1$, i.e. it suffices to prove the following.

Theorem 2.1 (Hitting estimate for the Ricci flow; restated). *For all $Y < \infty$, $\delta > 0$, and $r_0 \in (0, 1/2)$, there exists a constant $C = C(n, Y, \delta, r_0) < \infty$, such that if $(g_t)_{t\in(-2,0]}$ is a Ricci flow with $\mathcal{N}_{(p_0,0)}(r_0^2) \geq -Y$ and $r_{\mathrm{Rm}}(p_0, 0) \geq r_0$, then Brownian motion X_τ starting at $(p_0, 0)$ satisfies*

$$\mathbb{P}_{(p_0,0)}\big[X_\tau \text{ hits } \mathcal{S}_\varepsilon \cap P^*(p_0, 0; 1) \text{ for some } \tau \in [0, 1]\big] \leq C\varepsilon^{2-\delta} \tag{2.24}$$

for all $\varepsilon > 0$.

Proof. To begin with, let us observe that since the flow is defined on the interval $(-2, 0]$, the maximum principle for the evolution of scalar curvature under Ricci flow implies

$$R \geq -n/2 \quad \text{for } t \in [-1, 0]. \tag{2.25}$$

Together with (2.6) and the assumption $\mathcal{N}_{(p_0,0)}(r_0^2) \geq -Y$ this yields

$$\mathcal{N}_{(p_0,0)}(1) \geq -C(r_0, Y). \tag{2.26}$$

Hence, we have all the estimates from the previous subsection, which depend on a lower scalar bound and/or a lower entropy bound, at our disposal. In the following, we will simply write C for constants that only depend on n, Y, δ and r_0, and are allowed to change from line to line. Also, we can assume throughout that $\varepsilon \leq r_0/10$, since otherwise there is nothing to prove.

As above, denote by X_τ Brownian motion on our Ricci flow starting at $(p_0, 0)$. Given any closed subset $\mathcal{A} \subseteq M \times [-1, 0]$, we consider the hitting time

$$\tau_{\mathcal{A}} := \inf\{\tau > 0 : X_\tau \in \mathcal{A}\} \in [0, \infty]. \tag{2.27}$$

Note that $\tau_{\mathcal{A}} \wedge 1 := \min(\tau_{\mathcal{A}}, 1)$ is a stopping time. Let μ be the distribution of $X_{\tau_{\mathcal{A}}\wedge 1}$, i.e. set

$$\mu(\mathcal{A}') := \mathbb{P}_{(p_0,0)}[X_{\tau_{\mathcal{A}}\wedge 1} \in \mathcal{A}'] \tag{2.28}$$

for any Borel set $\mathcal{A}' \subseteq \mathcal{A}$. Observe that

$$\mathbb{P}_{(p_0,0)}[X_\tau \in \mathcal{A} \text{ for some } 0 \leq \tau \leq 1] = \mu(\mathcal{A}). \tag{2.29}$$

In the following, we write $\mathcal{A}'_s := \mathcal{A}' \cap \{t = s\}$ for the time-slices. Our first goal is to show the following:

Claim 2.1 (Hitting distribution). *The hitting distribution measure μ satisfies*

$$\int_{-1}^0 \int_{\mathcal{A}'_s} \int_{\mathcal{A}\cap\{t\geq s\}} K(p, t; q, s)\, d\mu(p, t)\, d\mathrm{Vol}_{g_s}(q)\, ds \leq \int_{-1}^0 \int_{\mathcal{A}'_s} \frac{C}{(-s)^{n/2}} e^{-\frac{d_s(p_s,q)^2}{10(-s)}}\, d\mathrm{Vol}_{g_s}(q)\, ds, \tag{2.30}$$

where (p_s, s) is any H_n-center of $(p_0, 0)$.

Proof of Claim 2.1. Consider the expected occupancy time

$$\mathbb{E}_{(p_0,0)}\Big[\int_0^1 1_{\{X_\tau \in \mathcal{A}'\}}\, d\tau\Big] = \int_{-1}^0 \int_{\mathcal{A}'_s} K(p_0,0;q,s)\, d\mathrm{Vol}_{g_s}(q)\, ds. \tag{2.31}$$

By the upper heat kernel bound (2.23), remembering also (2.25) and (2.26), we can estimate

$$\mathbb{E}_{(p_0,0)}\Big[\int_0^1 1_{\{X_\tau \in \mathcal{A}'\}}\, d\tau\Big] \le \int_{-1}^0 \int_{\mathcal{A}'_s} \frac{C}{(-s)^{n/2}} e^{-\frac{-d_s(p_s,q)^2}{10(-s)}}\, d\mathrm{Vol}_{g_s}(q)\, ds, \tag{2.32}$$

where (p_s, s) is any H_n-center of $(p_0, 0)$. On the other hand, we can also compute the expected occupancy time of $\mathcal{A}'$ by conditioning on $X_{\tau_{\mathcal{A}}\wedge 1}$. Specifically, observing that $X_{(\tau_{\mathcal{A}}\wedge 1)+\tau}$ is a Brownian motion with initial distribution μ, and using the strong Markov property, we infer that

$$\mathbb{E}_{(p_0,0)}\Big[\int_0^1 1_{\{X_\tau \in \mathcal{A}'\}}\, d\tau\Big] \ge \int_{\mathcal{A}} \int_{-1}^t \int_{\mathcal{A}'_s} K(p,t;q,s)\, d\mathrm{Vol}_{g_s}(q)\, ds\, d\mu(p,t). \tag{2.33}$$

Changing the order of integration, and combining the above inequalities, the claim follows. ∎

We now fix

$$\mathcal{A} := \{x \in P_-^*(p_0, 0; 1) : \varepsilon/2 \le r_{\mathrm{Rm}}(x) \le \varepsilon\}. \tag{2.34}$$

Since $r_{\mathrm{Rm}}(p_0, 0) \ge 10\varepsilon$ at the initial point, and $r_{\mathrm{Rm}} = \varepsilon$ on the support of μ, we see that

$$\mu(\mathcal{A}) = \mathbb{P}_{(p_0,0)}\big[X_\tau \text{ hits } \mathcal{S}_\varepsilon \cap P^*(p_0, 0; 1) \text{ for some } \tau \in [0, 1]\big]. \tag{2.35}$$

In the standard proof in the elliptic setting (see e.g. our prior paper [5]), the next step would be to estimate the capacity-type integral $\iint_{A\times A} K\, d\mu\, d\mu$, which however only works if A is a subset of a fixed space. In our current space-time setting, we consider instead the averaged quantity

$$\mathcal{I} := \int_{\mathcal{A}} \int_{P_-^*(q,s;4\eta\varepsilon)} \int_{\mathcal{A}\cap\{t\ge s'\}} K(p,t;q',s')\, d\mu(p,t)\, d\mathrm{Vol}_{g_{s'}}(q') ds'\, d\mu(q,s), \tag{2.36}$$

where $\eta > 0$ is a small constant to be chosen below. Using Claim 2.1 we can estimate

$$\mathcal{I} \le \int_{\mathcal{A}} \int_{P_-^*(q,s;4\eta\varepsilon)} \frac{C}{(-s')^{n/2}} e^{-\frac{d_{s'}(p_{s'},q')^2}{10(-s')}}\, d\mathrm{Vol}_{g_{s'}}(q')\, ds'\, d\mu(q,s), \tag{2.37}$$

where $(p_{s'}, s')$ is any H_n-center of $(p_0, 0)$ as above. To proceed, we observe that if $(q, s) \in \mathrm{spt}(\mu)$ and $(q', s') \in P_-^*(q, s; 4\eta\varepsilon)$, then fixing $\eta = \eta(n)$ small enough we have the bound

$$\frac{1}{(-s')^{n/2}} e^{-\frac{d_{s'}(p_{s'},q')^2}{10(-s')}} \le C. \tag{2.38}$$

Indeed, for sufficiently small η, if $-s' \le \eta r_0^2$ then using in particular (2.16) and (2.22) we see that $d_{s'}(p_{s'}, q') \ge \eta r_0$, and consequently the left hand side of (2.38) is bounded by some $C = C(r_0) < \infty$. On the other hand, if $-s' \geq \eta r_0^2$ then the left hand side is clearly bounded by $(\eta r_0)^{-n/2}$. Together with the bound (2.12) for the volume of P^* parabolic balls, this yields

$$\mathcal{I} \leq C\varepsilon^{n+2}\mu(\mathcal{A}). \tag{2.39}$$

Next, we would like to bound our quantity $\mathcal{I}$ from below, by estimating the contribution close to the diagonal. Specifically, let us consider $P_i^* = P^*(p_i, t_i; \eta\varepsilon)$ for some $(p_i, t_i) \in \mathcal{A}$. Recall that if $(p,t) \in \mathrm{spt}(\mu)$, then $r_{\mathrm{Rm}}(p,t) = \varepsilon$. Together with (2.22), we thus infer that there is some universal $A \in (1, \infty)$ with the following significance: If $(p,t), (q,s) \in P_i^* \cap \mathrm{spt}(\mu)$ satisfy $t \leq s$, then for each $s' \in [t - A^{-1}(\eta\varepsilon)^2, t]$ there is an H_n-center $(p_{s'}, s')$ of (p,t) such that

$$B_{g_{s'}}(p_{s'}, \sqrt{2H_n(t-s')}) \subseteq P_-^*(q, s; 4\eta\varepsilon). \tag{2.40}$$

Combined with (2.21) this implies

$$\int_{P_-^*(q,s;4\eta\varepsilon)} K(p,t;q',s')\, d\mathrm{Vol}_{g_{s'}}(q')\, ds' \geq \frac{1}{2}\int_{t-A^{-1}(\eta\varepsilon)^2}^{t} ds' \geq C^{-1}\varepsilon^2. \tag{2.41}$$

This yields

$$\begin{aligned}\int_{P_i^*}\int_{P_-^*(q,s;4\eta\varepsilon)}\int_{P_i^*\cap\{t\geq s'\}} & K(p,t;q',s')\, d\mu(p,t)\, d\mathrm{Vol}_{g_{s'}}(q')\, ds'\, d\mu(q,s) \\ &\geq C^{-1}\varepsilon^2 \int_{P_i^*}\int_{P_i^*} 1_{\{t\leq s\}}\, d\mu(p,t)\, d\mu(q,s) \geq C^{-1}\frac{\varepsilon^2}{2}\mu(P_i^*)^2. \end{aligned}\tag{2.42}$$

Now, let $P_i^* = P^*(p_i, t_i; \eta\varepsilon)$, where $(p_i, t_i) \in \mathcal{A}$ for $i = 1, \ldots, N$, be a covering of $\mathcal{A}$ with minimal covering number $N = N(\mathcal{A}, \eta\varepsilon)$, i.e.

$$N = \min\Big\{n : \text{there are } (p_1,t_1), \ldots, (p_n,t_n) \in \mathcal{A} \text{ with } \mathcal{A} \subseteq \bigcup_i P^*(p_i, t_i; \eta\varepsilon)\Big\}. \tag{2.43}$$

Observe that, thanks to minimality, the covering multiplicity is uniformly bounded. Indeed, if $P^*(p_{i_1}, t_{i_1}; \eta\varepsilon), \ldots, P^*(p_{i_m}, t_{i_m}; \eta\varepsilon)$ from a minimal covering intersect at some point (p,t), then by the containment relations (2.11), these P^* parabolic balls are contained in $P^*(p,t;2\eta\varepsilon)$, and together with the covering result from (2.13) this implies that m is bounded by some universal constant. Together with (2.42) we thus infer that

$$\mathcal{I} \geq C^{-1}\varepsilon^2 \sum_{i=1}^{N} \mu(P_i^*)^2. \tag{2.44}$$

Combined with the elementary inequality

$$\mu(\mathcal{A})^2 \leq \Big(\sum_{i=1}^{N}\mu(P_i^*)\Big)^2 \leq N\sum_{i=1}^{N}\mu(P_i^*)^2, \tag{2.45}$$

and the upper bound from (2.39), this yields

$$\mu(\mathcal{A}) \le CN\varepsilon^n. \tag{2.46}$$

Finally, by Bamler's quantitative parabolic $*$-Minkowski codimension-4 bound from (2.15) we have

$$N \le C\varepsilon^{-(n-2)-\delta}, \tag{2.47}$$

and remembering (2.35) we thus conclude that

$$\mathbb{P}_{(p_0,0)}\big[X_\tau \text{ hits } \mathcal{S}_\varepsilon \cap P^*(p_0,0;1) \text{ for some } \tau \in [0,1]\big] \le C\varepsilon^{2-\delta}. \tag{2.48}$$

This finishes the proof of the theorem. ∎

Corollary 2.2 (Occupancy time). *Under the same assumption as in Theorem 2.1, we have*

$$\mathbb{E}_{(p_0,0)}\left[\int_0^1 1_{\{X_\tau \in \mathcal{S}_\varepsilon \cap P^*(p_0,0;1)\}} d\tau\right] \le C(n,Y,\delta,r_0)\varepsilon^{4-\delta}. \tag{2.49}$$

Proof. By definition of Brownian motion,

$$\mathbb{E}_{(p_0,0)}\left[\int_0^1 1_{\{X_\tau \in \mathcal{S}_\varepsilon \cap P^*(p_0,0;1)\}}\, d\tau\right] = \int_{\mathcal{S}_\varepsilon \cap P^*_-(p_0,0;1)} K(p_0,0;q,s)\, d\mathrm{Vol}_{g_s}(q)\, ds. \tag{2.50}$$

Similarly to (2.38) we have the estimate

$$\sup_{(q,s)\in \mathcal{S}_\varepsilon \cap P^*_-(p_0,0;1)} K(p_0,0;q,s) \le C. \tag{2.51}$$

Now, by Bamler's quantitative parabolic $*$-Minkowski codimension-4 bound from (2.15) the set $\mathcal{S}_\varepsilon \cap P^*_-(p_0,t_0;1)$ can be covered by $C\varepsilon^{-n+2-\delta}$ P^* parabolic balls of radius ε centered at $(q_i,s_i) \in \mathcal{S}_\varepsilon \cap P^*_-(p_0,0;1)$. Moreover, by (2.12) the space-time volume of each P^* parabolic ball in the covering is bounded by $C\varepsilon^{n+2}$. Combining the above facts yields the assertion. ∎

3. Ricci limit flows and weak solutions

3.1. Preliminaries on Ricci limit flows

As in [1, Definition 3.2] a *metric flow over* $I \subseteq \mathbb{R}$,

$$\mathcal{X} = \big(\mathcal{X}, \mathfrak{t}, (d_t)_{t\in I}, (\nu_{x;s})_{x\in\mathcal{X}, s\in I, s\le \mathfrak{t}(x)}\big), \tag{3.1}$$

consists of a set $\mathcal{X}$, a time-function $\mathfrak{t} : \mathcal{X} \to \mathbb{R}$, complete separable metrics d_t on the time-slices $\mathcal{X}_t = \mathfrak{t}^{-1}(t)$, and probability measures $\nu_{x;s} \in \mathcal{P}(\mathcal{X}_s)$, such that

- $\nu_{x;\mathfrak{t}(x)} = \delta_x$ for all $x \in \mathcal{X}$, and for all $t_1 \le t_2 \le t_3$ in I and all $x \in \mathcal{X}_{t_3}$ we have the Kolmogorov consistency condition

$$\nu_{x;t_1} = \int_{\mathcal{X}_{t_2}} \nu_{\cdot;t_1}\, d\nu_{x;t_2}; \tag{3.2}$$

- for all $s < t$ in I, any $T > 0$, and any $T^{-1/2}$-Lipschitz function $f_s : \mathcal{X}_s \to \mathbb{R}$, setting $v_s = \Phi \circ f_s$, where $\Phi : \mathbb{R} \to (0, 1)$ denotes the antiderivative of $(4\pi)^{-1}e^{-x^2/4}$, the function
$$v_t : \mathcal{X}_t \to \mathbb{R}, \quad x \mapsto \int_{\mathcal{X}_s} v_s \, d\nu_{x;s} \tag{3.3}$$
is of the form $v_t = \Phi \circ f_t$ for some $(t - s + T)^{-1/2}$-Lipschitz function $f_t : \mathcal{X}_t \to \mathbb{R}$.

In particular, on any metric flow we always have a *heat flow* of integrable functions and a *conjugate heat flow* of probability measures, which are defined for $s \leq \mathfrak{t}(x)$ via the formulas
$$v_{\mathfrak{t}(x)}(x) := \int_{\mathcal{X}_s} v_s \, d\nu_{x;s}, \quad \mu_s := \int_{\mathcal{X}_t} \nu_{x;s} \, d\mu_{\mathfrak{t}(x)}(x) \, . \tag{3.4}$$
We recall from [1, Definitions 3.30, 4.25] that a metric flow $\mathcal{X}$ is called *H-concentrated* if for all $s \leq t$ in I and all $x_1, x_2 \in \mathcal{X}_t$,
$$\mathrm{Var}(\nu_{x_1;s}, \nu_{x_2;s}) \leq d_t^2(x_1, x_2) + H(t - s), \tag{3.5}$$
and is called *future continuous at $t_0 \in I$* if for all conjugate heat flows $(\mu_t)_{t \in I'}$ with finite variance and $t_0 \in I'$, the function $t \mapsto \int_{\mathcal{X}_t} \int_{\mathcal{X}_t} d_t \, d\mu_t \, d\mu_t$ is right continuous at t_0.

As in [1, Definition 5.1] a *metric flow pair* over an interval I consists of a metric flow $\mathcal{X}$ over $I' \subseteq I$ with $|I \setminus I'| = 0$, and a conjugate heat flow $(\mu_t)_{t \in I'}$ on $\mathcal{X}$ with $\mathrm{spt}(\mu_t) = \mathcal{X}_t$ for all $t \in I'$.

Now, any sequence $(M^i, (g_t^i)_{t \in I^i}, p^i)$ of pointed Ricci flows on closed n-dimensional manifolds, where $I^i = (-T^i, 0]$ for ease of notation, can be viewed as sequence of metric flow pairs by considering the associated metric flows $\mathcal{X}^i = M^i \times I^i$ and the conjugate heat flows $(\mu_t^i) = (\nu_{(p^i,0);t})_{t \in I^i}$. By Bamler's compactness theory [1] after passing to a subsequence we have $\mathbb{F}$-convergence on compact time intervals to a metric flow pair $(\mathcal{X}, (\nu_{x_\infty;t})_{t \in (-T_\infty, 0]})$, where $\mathcal{X}$ is a future continuous, H_n-concentrated metric flow of full support over $(-T_\infty, 0]$, and $T_\infty = \lim_{i \to \infty} T^i \in (0, \infty]$.

We will assume throughout that the sequence of Ricci flows is *noncollapsed*, that is, there are constants $\tau_0 > 0$ and $Y_0 < \infty$ such that
$$\mathcal{N}_{(p_i,0)}(\tau_0) \geq -Y_0. \tag{3.6}$$
Then, by Bamler's partial regularity theory [2] we have the decomposition
$$\mathcal{X} \setminus \{x_\infty\} = \mathcal{R} \cup \mathcal{S} \tag{3.7}$$
into the regular and singular parts, where the singular part $\mathcal{S}$ has parabolic $*$-Minkowski dimension at most $n-2$. Furthermore, the $\mathbb{F}$-convergence is smooth on the regular part $\mathcal{R}$, and the regular part can be equipped with a unique structure of a Ricci flow space-time,
$$\mathcal{R} = (\mathcal{R}, \mathfrak{t}, \partial_\mathfrak{t}, g), \tag{3.8}$$
as introduced by Kleiner–Lott [8]. Hence, $\mathcal{R}$ is a smooth $(n+1)$-manifold, the time-function $\mathfrak{t} : \mathcal{R} \to (-T_\infty, 0)$ is smooth without critical points, $\partial_\mathfrak{t}$ is a vector field on $\mathcal{R}$ satisfying $\partial_\mathfrak{t} \mathfrak{t} = 1$, and $g = (g_t)_{t \in (-T_\infty, 0)}$ is a smooth inner product on $\ker(d\mathfrak{t}) \subset T\mathcal{R}$

satisfying the Ricci flow equation

$$\mathcal{L}_{\partial_{\mathfrak{t}}} g = -2\,\mathrm{Ric}(g). \tag{3.9}$$

3.2. Brownian motion and stochastic parallel transport

In this subsection, we explain that every noncollapsed Ricci limit flow can be canonically equipped with a notion of Brownian motion and stochastic parallel transport. In the following $\mathcal{X}$ denotes any noncollapsed Ricci limit flow, as in the previous subsection. Recall in particular that its regular part $\mathcal{R} \subset \mathcal{X}$ has the structure of a Ricci flow space-time.

Definition 3.1 (Brownian motion). *Brownian motion* $\{X_\tau\}_{\tau\in[0,T_\infty-|\mathfrak{t}(x)|)}$ starting at $x \in \mathcal{X}$ is defined by

$$\mathbb{P}_x[X_{\tau_1} \in B_1, \ldots, X_{\tau_k} \in B_k] = \int_{B_1\times\cdots\times B_k} d\nu_{x;\mathfrak{t}(x)-\tau_1}(x_1)\ldots d\nu_{x_{k-1};\mathfrak{t}(x)-\tau_k}(x_k) \tag{3.10}$$

for any Borel sets $B_i \subseteq \mathcal{X}_{\mathfrak{t}(x)-\tau_i}$ and any times $0 \le \tau_1 < \cdots < \tau_k < T_\infty - |\mathfrak{t}(x)|$.

Thanks to the Kolmogorov consistency condition (3.2), there indeed exists a unique such probability measure by the Kolmogorov extension theorem. A priori the probability measure is defined on the infinite product space $\prod_{\tau\in[0,T_\infty-|\mathfrak{t}(x)|)} \mathcal{X}_{\mathfrak{t}(x)-\tau}$, but we will see now that for $x \in \mathcal{R}$ it actually concentrates on the space of continuous space-time curves that stay entirely in the regular part:

Lemma 3.2 (Concentration on regular part). *For any $x \in \mathcal{R}$ we have*

$$\mathbb{P}_x\big[X_\tau \textit{ hits } \mathcal{S} \textit{ for some } \tau \in [0, T_\infty - |\mathfrak{t}(x)|)\big] = 0. \tag{3.11}$$

Proof. In the proof of Theorem 1.4 we only used the relation between the Wiener measure and the heat kernel, which now holds true by Definition 3.1, and Bamler's estimates that we recalled in Section 2.1, which as explained in [1, 2] hold for limit flows as well. Thus, the hitting estimate (1.12) holds for Ricci limit flows. In particular, applying (1.12) with $\varepsilon_j = j^{-1}$ yields (3.11).

Let us elaborate on a few technical points: The lower scalar bound (2.25) was only used to derive the Nash entropy bound (2.26) and to get a uniform constant in the heat kernel upper bound (2.23). In the setting of this subsection, one has instead a lower scalar bound along the sequence of smooth flows, and can then pass the Nash entropy bound and the heat kernel upper bound to the limit flow using the definition of $\mathbb{F}$-convergence and [2, Theorem 2.10]. Furthermore, recall that we defined r_{Rm} by taking the supremum over backward parabolic balls $P_-(p,t;r)$, which is slightly more restrictive than the definition of r'_{Rm} used in [2, Theorem 2.30]. Hence, (2.15) indeed holds for noncollapsed limit flows. ∎

By Lemma 3.2 the process X_τ stays entirely in $\mathcal{R}$ and can be described in terms of the smooth geometry of $\mathcal{R}$. In particular, almost surely X_τ is a continuous space-time curve satisfying $\mathfrak{t}(X_\tau) = \mathfrak{t}(x) - \tau$.

Our next goal is to construct stochastic parallel transport, by adapting the construction from [6] to the setting of Ricci flow space-times. Let Y be a spatial vector field over $\mathcal{R}$, and let $x \in \mathcal{R}$. The *covariant spatial derivative* in direction $X \in T_x\mathcal{R}_{\mathfrak{t}(x)}$ is defined as

$$\nabla_X Y = \nabla_X^{g_{\mathfrak{t}(x)}} Y, \tag{3.12}$$

using the Levi–Civita connection of the metric $g_{\mathfrak{t}(x)}$. Define the *covariant time derivative* by

$$\nabla_{\mathfrak{t}} Y = \partial_{\mathfrak{t}} Y + \tfrac{1}{2}\mathcal{L}_{\partial_{\mathfrak{t}}} g(Y,\cdot)^{\sharp_g}, \tag{3.13}$$

and observe that with this definition the connection is metric, namely $\frac{d}{dt}|Y|_g^2 = 2\langle Y, \nabla_{\mathfrak{t}} Y\rangle$. Next, consider the O_n-bundle $\pi : \mathcal{F} \to \mathcal{R}$ whose fibres $\mathcal{F}_x$ are given by the orthogonal maps $u : \mathbb{R}^n \to (T_x\mathcal{R}_{\mathfrak{t}(x)}, g_{\mathfrak{t}(x)})$, and where O_n acts from the right via composition. For any spatial vector $X \in T_x\mathcal{R}_{\mathfrak{t}(x)}$ its horizontal lift X^* is simply given as horizontal lift with respect to Levi–Civita connection of the metric $g_{\mathfrak{t}(x)}$. In particular, we have n canonical horizontal vector fields

$$H_i(u) = (ue_i)^*, \tag{3.14}$$

where $u \in \mathcal{F}$, and $e_1, \ldots, e_n$ denotes the standard basis in $\mathbb{R}^n$. Furthermore, denote by $D_{\mathfrak{t}}$ the horizontal lift of the time vector field $\partial_{\mathfrak{t}}$. Similarly to [6, Lemmas 3.1, 3.3], covariant derivatives of spatial tensor fields on $\mathcal{R}$ can be expressed in terms of horizontal derivatives of the associated equivariant functions on the frame bundle. For example, identifying spatial vector fields Y on $\mathcal{R}$ with equivariant functions $\tilde{Y} : \mathcal{F} \to \mathbb{R}^n$ via $\tilde{Y}(u) = u^{-1}Y(\pi u)$, we have

$$\widetilde{\nabla_{\mathfrak{t}} Y} = D_{\mathfrak{t}}\tilde{Y}. \tag{3.15}$$

Now, given any initial frame $u \in \mathcal{F}_x$, there exists a unique horizontal lift U_τ of X_τ, i.e. a horizontal process U_τ starting at $U_0 = u$ such that $\pi(U_\tau) = X_\tau$. Concretely, using the Eells–Elworthy–Malliavin formalism, similarly to [6, Section 3.2], this process is given as the solution of the stochastic differential equation

$$dU_\tau = -D_{\mathfrak{t}}(U_\tau)d\tau + \sum_{i=1}^n H_i(U_\tau)\circ dW_\tau^i, \quad U_0 = u, \tag{3.16}$$

where $\circ d$ denotes the Stratonovich differential, and we use the normalization

$$dW_\tau^i dW_\tau^j = 2\delta_{ij}d\tau. \tag{3.17}$$

Since we have seen in Lemma 3.2 that X_τ stays entirely in the regular part $\mathcal{R} = \pi(\mathcal{F})$, the solution of (3.16) does not explode, i.e. we have $U_\tau \in \mathcal{F}$ for all $\tau \in [0, T_\infty - |\mathfrak{t}(x)|)$.

Definition 3.3 (Stochastic parallel transport). The family of isometries

$$P_\tau := U_0 U_\tau^{-1} : T_{X_\tau}\mathcal{R}_{\mathfrak{t}(x)-\tau} \to T_x\mathcal{R}_{\mathfrak{t}(x)}, \tag{3.18}$$

where U_τ is the horizontal lift of X_τ, is called *stochastic parallel transport*.

Note that, by equivariance under the O_n-action, P_τ does not depend on the choice of $u \in \mathcal{F}_x$.

3.3. Gradient estimate on path-space

In this final subsection, we prove that every noncollapsed Ricci limit flow $\mathcal{X}$ is a weak solution in the sense of Haslhofer–Naber. Recall that a *cylinder function* is a function of the form

$$F(X) = f(X_{\tau_1}, \dots, X_{\tau_k}), \tag{3.19}$$

where $f : \mathcal{X}_{\mathfrak{t}(x)-\tau_1} \times \cdots \times \mathcal{X}_{\mathfrak{t}(x)-\tau_k}$ is a Lipschitz function with compact support, for some given times $0 \le \tau_1 < \cdots < \tau_k < T_\infty - |\mathfrak{t}(x)|$. The *parallel gradient* $\nabla^{\|} F(X) \in T_x\mathcal{R}_{\mathfrak{t}(x)}$ is defined by

$$\nabla^{\|} F(X) = \sum_{i=1}^{k} P_{\tau_i} \operatorname{grad}^{(i)}_{g_{\mathfrak{t}(x)-\tau_i}} f(X_{\tau_1}, \dots, X_{\tau_k}), \tag{3.20}$$

where $\operatorname{grad}^{(i)}$ denotes the gradient with respect to the i-th entry, and $P_{\tau_i} : T_{X_{\tau_i}}\mathcal{R}_{\mathfrak{t}(x)-\tau_i} \to T_x\mathcal{R}_{\mathfrak{t}(x)}$ denotes stochastic parallel transport (see Definition 3.3). The goal of this subsection is to prove the following.

Theorem 3.4 (Gradient estimate). *For any $x \in \mathcal{R}$ we have the gradient estimate*

$$|\operatorname{grad}_{g_{\mathfrak{t}(x)}} \mathbb{E}_x[F]| \le \mathbb{E}_x[|\nabla^{\|} F|] \tag{3.21}$$

for all cylinder functions F. In particular, $\mathcal{X}$ is a weak solution of the Ricci flow in the sense of Haslhofer–Naber.

Proof. Suppose first $k = 1$. Then, by the definition of Brownian motion from (3.10), the expectation on the left hand side is given by the heat flow, namely

$$\mathbb{E}_x[F] = v(x), \tag{3.22}$$

where v is the heat flow from (3.4) with initial condition f at time $\mathfrak{t}(x) - \tau_1$. Observe that the gradient of v satisfies

$$\nabla_{\mathfrak{t}} \operatorname{grad}_g v = \Delta_g \operatorname{grad}_g v \tag{3.23}$$

on $\mathcal{R} \cap \mathfrak{t}^{-1}((\mathfrak{t}(x) - \tau_1, \mathfrak{t}(x)])$, by virtue of the Ricci flow equation (3.9). The key to proceed is the following claim:

Claim 3.1 (Feynman–Kac type representation formula). *For any $x \in \mathcal{R}$ we have*

$$\operatorname{grad}_{g_{\mathfrak{t}(x)}} v(x) = \mathbb{E}_x[P_{\tau_1} \operatorname{grad}_{g_{\mathfrak{t}(x)-\tau_1}} f]. \tag{3.24}$$

Proof of Claim 3.1. Set $Y = \operatorname{grad}_g v$, and consider the associated equivariant function $\tilde{Y}(u) = u^{-1}Y(\pi u)$. Using (3.15) we see that the lift of the evolution equation (3.23) is given by

$$D_{\mathfrak{t}}\tilde{Y} = \Delta_H \tilde{Y}, \tag{3.25}$$

where $\Delta_H = \sum_{i=1}^{n} H_i H_i$ denotes the horizontal Laplacian.

Now, for any $\varepsilon > 0$, as before denote by $\mathcal{S}_\varepsilon \subseteq \mathcal{X}$ the space-time points with curvature scale less than ε. Let $\eta_\varepsilon : \mathcal{X} \to [0,1]$ be a cutoff function with $\eta_\varepsilon = 1$ on $\mathcal{X} \setminus \mathcal{S}_\varepsilon$ and $\eta_\varepsilon = 0$ on $\mathcal{S}_{\varepsilon/2}$, and such that

$$\varepsilon|\nabla\eta_\varepsilon| + \varepsilon^2|\nabla^2\eta_\varepsilon| + \varepsilon^2|\partial_{\mathfrak{t}}\eta_\varepsilon| \leq C. \tag{3.26}$$

Set $\tilde{\eta}_\varepsilon := \eta_\varepsilon \circ \pi$, and consider the truncated function

$$\tilde{Y}^\varepsilon := \tilde{\eta}_\varepsilon \tilde{Y}. \tag{3.27}$$

Similarly to [6, proof of Proposition 3.7] the Ito formula on the frame bundle takes the form

$$d\varphi(U_\tau) = \sum_{i=1}^n H_i\varphi(U_\tau)dW_\tau^i + (\Delta_H\varphi - D_{\mathfrak{t}}\varphi)(U_\tau)d\tau. \tag{3.28}$$

To proceed, recall that by assumption, f is Lipschitz with compact support, and that thanks to (3.3) this Lipschitz bound is preserved under the flow (see [1, Proposition 3.12]); namely, $|Y| \leq \mathrm{Lip}(f)$. Together with standard interior estimates we thus get

$$\sup_{\mathcal{X}}|Y| + \varepsilon \sup_{\mathcal{X}\setminus\mathcal{S}_{\varepsilon/2}} |\nabla Y| \leq C. \tag{3.29}$$

Hence, using (3.25) and (3.26), we infer that

$$\begin{aligned} d\tilde{Y}^\varepsilon(U_\tau) &= \sum_{i=1}^n H_i\tilde{Y}^\varepsilon(U_\tau)dW_\tau^i + ((\widetilde{\Delta\eta_\varepsilon} - \widetilde{\partial_{\mathfrak{t}}\eta_\varepsilon})\tilde{Y})(U_\tau)d\tau + 2\sum_{i=1}^n(H_i\tilde{\eta}_\varepsilon H_i\tilde{Y})(U_\tau)d\tau \\ &= \text{martingale} + E_\varepsilon\, d\tau, \end{aligned} \tag{3.30}$$

where the error term satisfies

$$|E_\varepsilon| \leq \frac{C}{\varepsilon^2} 1_{\{X_\tau\in\mathcal{S}_\varepsilon\setminus\mathcal{S}_{\varepsilon/2}\}}. \tag{3.31}$$

This implies

$$|\tilde{Y}^\varepsilon(u) - \mathbb{E}_u[\tilde{Y}^\varepsilon(U_{\tau_1})]| \leq \frac{C}{\varepsilon^2}\mathbb{E}_x\left[\int_0^{\tau_1} 1_{\{X_\tau\in\mathcal{S}_\varepsilon\setminus\mathcal{S}_{\varepsilon/2}\}}\, d\tau\right]. \tag{3.32}$$

By Corollary 2.2 we have

$$\mathbb{E}_x\left[\int_0^{\tau_1} 1_{\{X_\tau\in\mathcal{S}_\varepsilon\setminus\mathcal{S}_{\varepsilon/2}\}}\, d\tau\right] \leq C\varepsilon^{4-\delta}. \tag{3.33}$$

Moreover, using again Theorem 1.4, and remembering also the Lipschitz estimate from (3.3), we see that

$$\lim_{\varepsilon\to 0}\mathbb{E}_u[\tilde{Y}^\varepsilon(U_{\tau_1})] = \mathbb{E}_u[\tilde{Y}(U_{\tau_1})]. \tag{3.34}$$

Also, since $u \in \mathcal{F}_x$, where $x \in \mathcal{R}$, we have

$$\lim_{\varepsilon\to 0}\tilde{Y}^\varepsilon(u) = \tilde{Y}(u). \tag{3.35}$$

Combining the above facts, we conclude that

$$\tilde{Y}(u) = \mathbb{E}_u[\tilde{Y}(U_{\tau_1})]. \tag{3.36}$$

Pushing down via π establishes the claim. ∎

Continuing the proof of the theorem, by Claim 3.1 and the definition of the parallel gradient from (3.20) we thus have

$$\operatorname{grad}_{g_{\mathfrak{t}(x)}} \mathbb{E}_x[F] = \mathbb{E}_x[\nabla^{\parallel} F], \tag{3.37}$$

provided F is a 1-point cylinder function. Arguing by induction on k, similarly to [6, proof of Theorem 4.2], where we now use Claim 3.1 instead of [6, Proposition 3.36], we see that the gradient formula (3.37) holds for k-point cylinder functions as well. This implies the assertion of the theorem. ∎

Funding. This first author has been supported by NRF of Korea grant No. 2022R1C1C1013511. The second author has been supported by an NSERC Discovery Grant and a Sloan Research Fellowship.

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