J. Noncommut. Geom. (Online first)
DOI 10.4171/JNCG/629

Quantum Mycielski graphs

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Abstract. The classical Mycielski transformation allows for constructing from a given graph a new one with an arbitrarily large chromatic number but preserving the size of the largest clique contained in it. This particular construction and its specific generalizations were widely discussed in graph theory literature. Here we propose an analog of these transformations for quantum graphs and study how they affect the (quantum) chromatic number as well as the clique numbers associated with them.

1. Introduction

Properties and characteristics of classical graph coloring were widely studied from different perspectives, starting from their fundamental aspects, through applications in several branches of science and engineering, as well as in multiple daily life problems [15]. The typical parameters one uses to characterize a given graph G are related to a number of colors that can be used to label either its vertices or edges according to a certain set of rules. The most common characteristic is called the chromatic number $\chi(G)$ and is defined as the minimal number of colors that could be used to label the vertices of G in such a way that none of the edges has the same colors associated with its two endpoints. The problem of determining this quantity for a generic graph is known to be NP-hard [14, 22]. Yet another NP-hard parameter containing information about the structure of a given graph G is its clique number $\omega(G)$, which determines the largest subgraph K_n of G whose every vertex shares an edge with any other vertex of this subgraph, the complete graph of n vertices. In particular, if $\omega(G) = 2$, then the graph is triangle-free (i.e., there is no closed loop formed out of three vertices). One could ask if starting from a triangle-free connected graph, we can arbitrarily enlarge the chromatic number by adding a certain number of vertices and edges, but at the same time, no triangle is generated as a subgraph, and the resulting graph remains connected. The affirmative answer to this question was given by Mycielski [21]. The resulting transformation that from a given graph G produces a new graph $\mu(G)$ such that $G \subseteq \mu(G)$, $\omega(G) = \omega(\mu(G))$ and $\chi(\mu(G)) = \chi(G) + 1$ is referred to as the Mycielski transformation. This construction was later generalized in [25, 27], and for it, the notion of generalized Mycielski transformation, Stiebitz transformation or (higher) cones over a graph is used.

Mathematics Subject Classification 2020: 46L05 (primary); 81P45 (secondary).
Keywords: quantum graphs, chromatic numbers, Mycielski transformation.

One area of intriguing applications of graph theory is the information theory [10, 24], where, for a given noisy classical channel $\Phi_{cl} : A \to \mathrm{Prob}(B)$ between two persons, Alice and Bob – where $\mathrm{Prob}(B)$ denotes the set of probability distributions over the set B – one can associate the so-called confusability graph. This graph is defined as a complement of the distinguishability graph whose vertices are elements of the set A, and there exists an edge between two vertices $a_1, a_2 \in A$ if and only if for all $b \in B$ we have $\Phi_{cl}(b \mid a_1)\Phi_{cl}(b \mid a_2) = 0$, where $\Phi_{cl}(b \mid a)$ stands for the conditional probability that Bob will receive $b \in B$ provided that Alice had sent $a \in A$. The fact that there is a connection between classical channels and graphs led to an intriguing possibility of associating to a quantum channel Φ_q an object that will mimic the behavior of classical graphs – the quantum graph. This can be done either by directly using the associated Kraus operators defining the completely positive trace-preserving map Φ_q or by using them to construct certain operator systems (spaces) and study their properties. The concept of quantum graphs already appeared in [11]; however, the relation to quantum information, in particular the problem of zero-error correction, significantly accelerated the interest in quantum graphs [6, 9]. Several formulations, which under some conditions turned out to be equivalent, were proposed. For a neat survey, we refer the reader to [7]. The two most widely used are:

(I) approach based on quantum relations and their formulations in terms of operator systems (spaces) [29, 30],

(II) formulation by mimicking in the quantum world the properties of adjacency matrices [3, 13, 20].

Let us again consider the aforementioned noisy classical channel Φ_{cl} and now promote Alice and Bob to be players for the following game [24]. Introducing an external referee, say, Charlie, one can consider the scenario in which Charlie chooses a value c from its own set C and, with a conditional probability $P(x, u \mid c)$, sends a message $x \in X$ to Alice and a value $u \in B$ to Bob. If Alice is equipped with a map $f : X \to A$, called strategy, she can use it to produce $a = f(x)$ and send this value to Bob using the channel Φ_{cl} so that he will receive a value $b \in B$ with probability $\Phi_{cl}(b|a)$. The goal of the players is to decode the initial value $c \in C$. With a set X, one can associate the so-called characteristic graph defined in terms of the conditional probability $P(x, u \mid c)$, and it turns out [24] that the existence of this type of scenario with a winning strategy is equivalent to the existence of a homomorphism between such characteristic graph and the distinguishability one associated with the channel Φ_{cl}. This observation suggests that certain characteristics of graphs could be potentially defined in terms of the existence of a winning strategy for certain types of games. In particular, this is true for the chromatic number. One can consider a two-player game in which Charlie chooses two vertices, say v and w, of a given graph G and sends one of them to Alice and the second one to Bob. They have to respond with numbers α, β from a fixed set $\{1, \ldots, c\}$ according to the following rules:

- $v = w \Rightarrow \alpha = \beta$,

- vw is an edge $\Rightarrow \alpha \neq \beta$.

The lack of communication between Alice and Bob is assumed during the game, but they can agree on a strategy before the game starts. It turns out that such a winning strategy exists if and only if $c \geq \chi(G)$. This motivated the concept of quantum chromatic number introduced in [5], where the players are no longer assumed to be classical but they are allowed to share some entangled quantum state Ψ and, instead of providing answers, they are performing quantum measurements. More precisely, the players' strategy is mathematically modeled by positive operator-valued measures (POVMs) $(E_{v\alpha})_{\alpha=1,\dots,c}$ and $(F_{v\alpha})_{\alpha=1,\dots,c}$, respectively, and it is a winning strategy if

- $\forall v \in V_G \; \forall \alpha \neq \beta \; \langle \Psi | E_{v\alpha} \otimes F_{v\beta} | \Psi \rangle = 0$,
- $\forall vw \in E_G \;\; \forall \alpha \; \langle \Psi | E_{v\alpha} \otimes F_{w\alpha} | \Psi \rangle = 0$.

The quantum chromatic number is defined, by analogy to its classical counterpart, as the minimal number for which there exists a winning strategy for this non-local game; see also [23].

Both classical and quantum chromatic numbers, together with the notion of (quantum) coloring, can also be defined in the framework of quantum graphs [4,12]. It makes sense to ask about analogs of Mycielski's theorem in the quantum setting. In this paper, we propose such a generalization of the Mycielski transformation for quantum graphs and study how it affects parameters like (quantum) chromatic numbers as well as (several versions of) clique numbers. In Section 2, we describe in detail the Mycielski transformation for classical graphs and discuss its generalization into cones over a graph. In Section 3, we briefly recall the notion of a quantum graph and establish the notation widely used in the rest of the paper. We introduce the (generalized) Mycielski transformation for quantum graphs in Section 4 and show that indeed the resulting object is a well-defined quantum graph. Next, we study in Section 5 how this transformation affects the (quantum) chromatic number. In particular, in Proposition 5.3 we demonstrate that, in contrast to the classical case, for quantum graphs the (quantum) chromatic number satisfies only inequalities $\chi(\mathscr{G}) \leq \chi(\mu(\mathscr{G})) \leq \chi(\mathscr{G}) + 1$, instead of the equality $\chi(\mu(\mathscr{G})) = \chi(\mathscr{G}) + 1$. The impact on the (different versions of) clique numbers is discussed in Section 6, where we show that analogous to the classical situation result holds, that is, the clique number is invariant under the Mycielski transformation also for quantum graphs. Finally, in Section 7 we briefly comment on other graph parameters and collect a series of open questions as well as potential future research directions.

2. Mycielski transformation for classical graphs

Let G be an undirected graph with a given set of vertices V_G and the set of edges E_G. The Mycielski transformation [21] of G, denoted by $\mu(G)$, is the graph with the set of vertices $V_{\mu(G)} = \{\bullet\} \sqcup V_G \sqcup V_G$ such that every vertex from the second copy of V_G is connected with the distinguished vertex $\bullet$, and if $\{v_i, v_j\} \in E_G$ and u_i denotes the copy of v_i in the second summand, then $\{v_i, v_j\} \in E_{\mu(G)}$, and both $\{v_i, u_j\}$ and $\{v_j, u_i\}$ are edges in $\mu(G)$. In other words, the adjacency matrix $A_{\mu(G)}$ is of the form

$$A_{\mu(G)} = \begin{pmatrix} 0 & \vec{0}^T & \vec{1}^T \\ \vec{0} & A_G & A_G \\ \vec{1} & A_G & \mathbb{0} \end{pmatrix}, \tag{2.1}$$

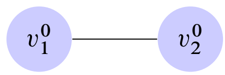
where, for $a \in \{0, 1\}$, $\vec{a}^T$ denotes the row vector $(a, \ldots, a)$ of length $|V_G|$ and $\vec{a}$ stands for its column version. We stress that, in this construction, vertex u_i is not adjacent to u_j. The Mycielski transformation $\mu(G)$ is a special case of so-called generalized Mycielski transformation $\mu_r(G)$ (or r-Mycielskian) for $r = 1$, also known as cones over a graph G [25, 27]. If the vertex set of G is $V^0 = \{v_1^0, v_2^0, \ldots, v_n^0\}$ and the edge set of G is denoted by E_0, then the vertex set of r-Mycielskian is

$$\{\bullet\} \sqcup V^0 \sqcup V^1 \sqcup \cdots \sqcup V^r,$$

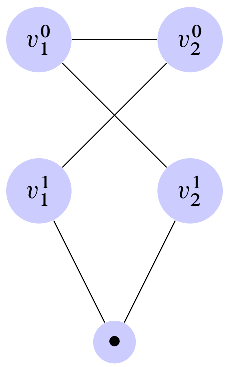
where $V^i = \{v_1^i, v_2^i, \ldots, v_n^i\}$ is the distinct copy of V^0 and the edge set is given by

$$E = E_0 \cup \bigcup_{i=0}^{r-1} \{v_j^i v_{j'}^{i+1} : v_j^0 v_{j'}^0 \in E_0\} \cup \{v_1^r \bullet, v_2^r \bullet, \ldots, v_n^r \bullet\}.$$

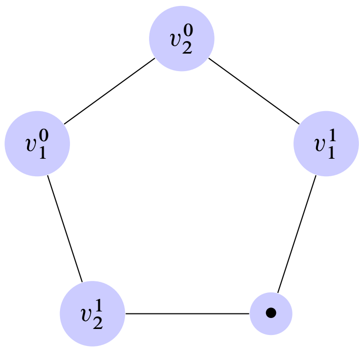
We illustrate this construction in a particular example. Let $G = K_2$, that is,



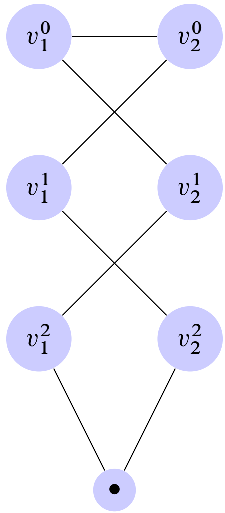
The Mycielski transformation $\mu_1(G)$ of this graph is



Notice that this graph is simply



Next, we observe that $\mu_2(G)$ is of the form



We also notice that $\mu_1(\mu_1(G)) \neq \mu_2(G)$.

3. Quantum graphs

Let us recall here the definition of a quantum graph [7, Definition 2.4].

Definition 3.1. A triple $\mathcal{G} = (\mathbb{V}, \psi, A)$ is a quantum graph, if

- $\mathbb{V}$ is a finite quantum space; the corresponding (finite-dimensional) C*-algebra will be denoted by $\mathrm{C}(\mathcal{G})$;
- $\psi : \mathrm{C}(\mathcal{G}) \to \mathbb{C}$ is a faithful state; the GNS space will be denoted by $L^2(\mathcal{G})$; the multiplication map when viewed as a linear map $L^2(\mathcal{G}) \otimes L^2(\mathcal{G}) \to L^2(\mathcal{G})$ will be denoted by m; since $\mathbb{V}$ is finite and ψ is faithful, $\mathrm{C}(\mathcal{G})$ and $L^2(\mathcal{G})$ can be identified as vector spaces; the map $\mathbb{C} \ni \lambda \mapsto \lambda \mathbb{1}_{\mathrm{C}(\mathcal{G})} \in L^2(\mathcal{G})$ will be denoted by η; note incidentally that $\eta^*(x) = \psi(x)$ for all $x \in L^2(\mathcal{G})$;

- ψ is a δ-form, that is, $mm^* = \delta^2 \,\mathrm{id}_{L^2(\mathcal{G})}$;
- A, called quantum adjacency matrix, is a self-adjoint map $A : L^2(\mathcal{G}) \to L^2(\mathcal{G})$ such that

$$A = \delta^{-2} m(A \otimes A)m^*. \tag{3.1}$$

$$A = (\mathrm{id} \otimes \eta^* m)(\mathbb{1} \otimes A \otimes \mathbb{1})(m^* \eta \otimes \mathrm{id}). \tag{3.2}$$

If, moreover, $m(A \otimes \mathbb{1})m^* = \delta^2 \mathbb{1}$, then the quantum graph $\mathcal{G}$ is called reflexive. On the other hand, if $m(A \otimes \mathbb{1})m^* = 0$, then the quantum graph is called irreflexive.

We will often deal with a collection of quantum graphs and, to avoid confusion, subscripts indicating the corresponding quantum graph will be added to the state, adjacency matrix, multiplication map, etc., and, for example, we shall write $\mathcal{G} = (\mathbb{V}_{\mathcal{G}}, \psi_{\mathcal{G}}, A_{\mathcal{G}})$; $\dim \mathrm{C}(\mathcal{G})$ will be denoted by $|\mathcal{G}|$. From now on, we assume that the state ψ is tracial. The non-tracial situation is more subtle, and we postpone a detailed discussion for the future.

The C*-algebra $\mathrm{C}(\mathcal{G})$ will be viewed as a C*-algebra of operators acting on $L^2(\mathcal{G})$, and $\mathrm{C}(\mathcal{G})'$ will denote its commutant

$$\mathrm{C}(\mathcal{G})' = \{T \in B(L^2(\mathcal{G})) : Ta = aT \text{ for all } a \in \mathrm{C}(\mathcal{G})\}.$$

The map

$$P : B(L^2(\mathcal{G})) \ni X \longmapsto \delta^{-2} m(A \otimes X)m^* \in B(L^2(\mathcal{G})) \tag{3.3}$$

is a projection satisfying

$$P(XT) = P(X)T, \quad P(TX) = TP(X), \quad P(X^*) = P(X)^*$$

for all $X \in B(L^2(\mathcal{G}))$ and $T \in \mathrm{C}(\mathcal{G})'$. In particular, the image S of P

$$S = P(B(L^2(\mathcal{G}))) \tag{3.4}$$

is a self-adjoint operator subspace in $B(L^2(\mathcal{G}))$ which is also a bimodule over $\mathrm{C}(\mathcal{G})'$; see [12]. Since $\mathrm{C}(\mathcal{G})$ is finite-dimensional, it is also a von Neumann algebra, thus the triple $(S, \mathrm{C}(\mathcal{G}), B(L^2(\mathcal{G})))$ is a quantum graph in the sense of Weaver [29, 30]. We shall usually write $S_{\mathcal{G}}$ for the self-adjoint subspace $S \subset B(L^2(\mathcal{G}))$ described above. Note that if $\mathcal{G}$ is reflexive, then $S_{\mathcal{G}}$ is an operator system. The operator space S_G for a classical graph G is spanned by the matrix units $e_{ij} \in \mathsf{Mat}_n$ corresponding to edges $(ij) \in E_G$; see Section 6.

4. Mycielski transformation for quantum graphs

Before defining the (generalized) Mycielski transformation, we need to fix a notation: given linear maps $T_i : V_i \to W$ for $i = 1, 2, \ldots, r$, the unique linear map $T : \bigoplus_{i=1}^r V_i \to W$ such that $T|_{V_i} = T_i$ for $i = 1, 2, \ldots, r$ will be denoted by $\bigoplus_{i=1}^r T_i$.

Definition 4.1. The Mycielski transformation $\mu(\mathscr{G})$ of a quantum graph $\mathscr{G} = (\mathbb{V}_{\mathscr{G}}, \psi_{\mathscr{G}}, A_{\mathscr{G}})$ is a triple $(\mathbb{V}_{\mu(\mathscr{G})}, \psi_{\mu(\mathscr{G})}, A_{\mu(\mathscr{G})})$ where

- $\mathbb{V}_{\mu(\mathscr{G})} = \bullet \sqcup \mathbb{V}_{\mathscr{G}} \sqcup \mathbb{V}_{\mathscr{G}}$ (i.e., the corresponding C*-algebra is $\mathbb{C} \oplus \mathrm{C}(\mathscr{G}) \oplus \mathrm{C}(\mathscr{G})$),
- $\psi_{\mu(\mathscr{G})}\left(\begin{smallmatrix}\lambda\\ x\\ y\end{smallmatrix}\right) = \frac{1}{1+2\delta^2}\big(\lambda + \delta^2\psi_{\mathscr{G}}(x) + \delta^2\psi_{\mathscr{G}}(y)\big)$,
- $A_{\mu(\mathscr{G})} : L^2(\mu(\mathscr{G})) \to L^2(\mu(\mathscr{G}))$ is defined by

$$A_{\mu(\mathscr{G})}\begin{pmatrix}\lambda\\ x\\ y\end{pmatrix} = \begin{pmatrix}\delta^2\psi_{\mathscr{G}}(y)\\ A_{\mathscr{G}}(x+y)\\ \lambda\mathbb{1}_{\mathrm{C}(\mathscr{G})} + A_{\mathscr{G}}(x)\end{pmatrix} \tag{4.1}$$

  for all $\lambda \in \mathbb{C}$ and $x, y \in L^2(\mathscr{G})$. Note that we identify $L^2(\mu(\mathscr{G}))$ with $\mathbb{C} \oplus L^2(\mathscr{G}) \oplus L^2(\mathscr{G})$, with the scalar product on the latter given by

$$\left\langle \begin{pmatrix}\lambda\\ x\\ y\end{pmatrix} \middle| \begin{pmatrix}\mu\\ u\\ v\end{pmatrix} \right\rangle = \frac{1}{1+2\delta^2}\big(\overline{\lambda}\mu + \delta^2\langle x|u\rangle + \delta^2\langle y|v\rangle\big).$$

Instead of showing that this construction indeed defines a quantum graph, we first introduce the quantum analog of the generalized Mycielski graphs. We will show that the generalized Mycielski transformation for quantum graphs produces new quantum graphs. In particular, it will follow that the Mycielski transformation as defined in Definition 4.1 also has this property.

Definition 4.2. Let a triple $\mathscr{G} = (\mathbb{V}_{\mathscr{G}}, \psi_{\mathscr{G}}, A_{\mathscr{G}})$ be a quantum graph, and let $r \geq 1$. The $r-1$-Mycielski transformation $\mu_{r-1}(\mathscr{G})$ of $\mathscr{G}$ is a triple $(\mathbb{V}_{\mu_{r-1}(\mathscr{G})}, \psi_{\mu_{r-1}(\mathscr{G})}, A_{\mu_{r-1}(\mathscr{G})})$ where

- $\mathbb{V}_{\mu_{r-1}(\mathscr{G})} = \bullet \sqcup \mathbb{V}_{\mathscr{G}} \sqcup \cdots \sqcup \mathbb{V}_{\mathscr{G}}$, with r copies of $\mathbb{V}_{\mathscr{G}}$, (i.e., the C*-algebra of $\mu_{r-1}(\mathscr{G})$ is $\mathbb{C} \oplus \mathrm{C}(\mathscr{G})^{\oplus r}$),
- $\psi_{\mu_{r-1}}(\mathscr{G}) : L^2(\mu_{r-1}(\mathscr{G})) \to \mathbb{C}$ is defined by:

$$\psi_{\mu_{r-1}(\mathscr{G})}\begin{pmatrix}\lambda\\ x_1\\ \vdots\\ x_r\end{pmatrix} = \frac{1}{1+r\delta^2}\left(\lambda + \delta^2\sum_{k=1}^{r}\psi_{\mathscr{G}}(x_k)\right),$$

- $A_{\mu_{r-1}(\mathcal{G})} : L^2(\mu_{r-1}(\mathcal{G})) \to L^2(\mu_{r-1}(\mathcal{G}))$ is defined by

$$A_{\mu_{r-1}(\mathcal{G})} \begin{pmatrix} \lambda \\ x_1 \\ \vdots \\ x_r \end{pmatrix} = \begin{pmatrix} \delta^2 \psi_{\mathcal{G}}(x_r) \\ A_{\mathcal{G}}(x_1 + x_2) \\ A_{\mathcal{G}}(x_1 + x_3) \\ \vdots \\ A_{\mathcal{G}}(x_k + x_{k+2}) \\ \vdots \\ A_{\mathcal{G}}(x_{r-2} + x_r) \\ \lambda \mathbb{1}_{C(\mathcal{G})} + A_{\mathcal{G}}(x_{r-1}) \end{pmatrix} \tag{4.2}$$

for all $\lambda \in \mathbb{C}$ and $x_1, \ldots, x_r \in L^2(\mathcal{G})$. Here $L^2(\mu_{r-1}(\mathcal{G}))$ is identified with $\mathbb{C} \oplus \bigoplus_{k=1}^r L^2(\mathcal{G})$. Observe incidentally that

$$\iota_1^* A_{\mu_{r-1}(\mathcal{G})} \iota_1 = A_{\mathcal{G}}. \tag{4.3}$$

Let us right away observe that $\mu_1(\mathcal{G})$ as described in Definition 4.2 coincides with Mycielskian $\mu(\mathcal{G})$ of $\mathcal{G}$ as defined in Definition 4.1, and for that consistency reason we have chosen the above indexing of the generalized Mycielski graphs. We remark that $\mu_{r-1}(\mathcal{G})$ contains $r-1$ duplicate copies of the original graph together with the original one as well as an extra vertex, denoted by $\bullet$. Note also that for every $r \geq 1$, the scalar product on $L^2(\mu_{r-1}(\mathcal{G}))$ is given by

$$\left\langle \begin{pmatrix} \lambda \\ x_1 \\ \vdots \\ x_r \end{pmatrix} \middle| \begin{pmatrix} \mu \\ y_1 \\ \vdots \\ y_r \end{pmatrix} \right\rangle = \frac{1}{1 + r\delta^2} \left(\overline{\lambda} \mu + \delta^2 \sum_{k=1}^r \langle x_k | y_k \rangle \right).$$

For $k = 1, \ldots, r$, let ι_k be the isometric embedding of $L^2(\mathcal{G})$ into $L^2(\mu_{r-1}(\mathcal{G}))$

$$\iota_k : L^2(\mathcal{G}) \ni x \longmapsto \sqrt{\frac{1 + r\delta^2}{\delta^2}} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ x \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in L^2(\mu_{r-1}(\mathcal{G})),$$

where the element x is embedded into the kth summand of $\bigoplus_{k=1}^{r} L^2(\mathscr{G})$. Note that

$$\iota_k^* \begin{pmatrix} x_0 \\ \vdots \\ x_k \\ \vdots \\ x_r \end{pmatrix} = \sqrt{\frac{\delta^2}{1+r\delta^2}}\, x_k.$$

We also define the isometric embedding ι_0 of $\mathbb{C}$ into $L^2(\mu_{r-1}(\mathscr{G}))$

$$\iota_0 : \mathbb{C} \ni \lambda \mapsto \sqrt{1+r\delta^2} \begin{pmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in L^2(\mu_{r-1}(\mathscr{G})),$$

and we have

$$\iota_0^* \begin{pmatrix} \lambda \\ x_1 \\ \vdots \\ x_r \end{pmatrix} = \sqrt{\frac{1}{1+r\delta^2}}\, \lambda.$$

For every $k, l = 1, \dots, r$, we have $\iota_l^* \iota_k = \delta_{kl}\, \mathrm{id}_{L^2(\mathscr{G})}$, and $\iota_0^* \iota_0 = \mathrm{id}_{\mathbb{C}}$. Furthermore,

$$\sum_{j=0}^{r} \iota_j \iota_j^* = \mathrm{id}_{L^2(\mu_{r-1}(\mathscr{G}))}.$$

Denoting $m_0 = m_\bullet$ and $m_j = \delta^{-1} m_{\mathscr{G}}$ for $j = 1, \dots, r$, with $m_\bullet$ being the usual multiplication on $\mathbb{C}$ and $m_{\mathscr{G}}$ the multiplication map from Definition 3.1, we have

$$m_{\mu_{r-1}(\mathscr{G})} = \sqrt{1+r\delta^2} \sum_{k=0}^{r} \iota_k m_k (\iota_k^* \otimes \iota_k^*), \tag{4.4}$$

and

$$m^*_{\mu_{r-1}(\mathscr{G})} = \sqrt{1+r\delta^2} \sum_{k=0}^{r} (\iota_k \otimes \iota_k) m_k^* \iota_k^*.$$

The quantum adjacency matrix can be written as

$$A_{\mu_{r-1}(\mathscr{G})} = \delta \iota_r \eta_{\mathscr{G}} \iota_0^* + \delta \iota_0 \eta_{\mathscr{G}}^* \iota_r^* + \iota_1 A_{\mathscr{G}} \iota_1^* + \sum_{k=1}^{r-1} \big(\iota_k A_{\mathscr{G}} \iota_{k+1}^* + \iota_{k+1} A_{\mathscr{G}} \iota_k^* \big). \tag{4.5}$$

From this formula, we deduce that $A_{\mu_{r-1}(\mathscr{G})}$ is a self-adjoint operator on $L^2(\mu_{r-1}(\mathscr{G}))$. Let us prove that $\psi_{\mu_{r-1}(\mathscr{G})}$ is a $\sqrt{1+r\delta^2}$-form.

Proposition 4.3. *The equation*

$$m_{\mu_{r-1}(\mathcal{G})} m^*_{\mu_{r-1}(\mathcal{G})} = (1 + r\delta^2)\,\mathrm{id}_{L^2(\mu_{r-1}(\mathcal{G}))}$$

holds, and in particular $\psi_{\mu_{r-1}(\mathcal{G})}$ *is a* $\sqrt{1+r\delta^2}$*-form.*

Proof. We compute

$$\begin{aligned} m_{\mu_{r-1}(\mathcal{G})} m^*_{\mu_{r-1}(\mathcal{G})} &= (1+r\delta^2)\iota_0 m_\bullet m_\bullet^* \iota_0^* + \frac{1+r\delta^2}{\delta^2}\sum_{k=1}^r \iota_k m_{\mathcal{G}} m_{\mathcal{G}}^* \iota_k^* \\ &= (1+r\delta^2)\iota_0\iota_0^* + \frac{1+r\delta^2}{\delta^2}\delta^2 \sum_{k=1}^r \iota_k\iota_k^* = (1+r\delta^2)\,\mathrm{id}_{L^2(\mu_{r-1}(\mathcal{G}))}. \end{aligned}$$

It remains to check that $\psi_{\mu_{r-1}(\mathcal{G})}$ is faithful.

Suppose that $\psi_{\mu_{r-1}(\mathcal{G})}$ vanishes on $(|\lambda|^2, x_1^* x_1, \dots, x_r^* x_r)^T$, for some $\lambda \in \mathbb{C}$ and $x_k \in L^2(\mathcal{G})$, $k = 1, \dots, r$. That is, $|\lambda|^2 + \delta^2 \sum_{k=1}^r \psi_{\mathcal{G}}(x_k^* x_k) = 0$. Since $\psi_{\mathcal{G}}$ was faithful, we have $\lambda = 0$ and $x_k = 0$ and thus $\psi_{\mu_{r-1}(\mathcal{G})}$ is faithful. ■

Proposition 4.4. *For any quantum graph* $\mathcal{G}$ *and any* $r \geq 1$*, the generalized Mycielski transformation* $\mu_{r-1}(\mathcal{G})$ *is a quantum graph. Moreover, if* $\mathcal{G}$ *was irreflexive (resp. reflexive), then* $\mu_{r-1}(\mathcal{G})$ *is so.*

Proof. To prove the claim, we have to verify that $\mu_{r-1}(\mathcal{G})$ fulfills all the axioms from Definition 3.1. First, by Proposition 4.3 we know that $\psi_{\mu_{r-1}(\mathcal{G})}$ is a $\sqrt{1+r\delta^2}$-form. Moreover, $A_{\mu_{r-1}(\mathcal{G})} \in B(L^2(\mu_{r-1}(\mathcal{G})))$ is a self-adjoint operator; linearity and self-adjointness follows immediately from equation (4.5).

Next, we compute

$$\begin{aligned} & m_{\mu_{r-1}(\mathcal{G})}(A_{\mu_{r-1}(\mathcal{G})} \otimes A_{\mu_{r-1}(\mathcal{G})}) m^*_{\mu_{r-1}(\mathcal{G})} \\ &\quad = (r\delta^2+1)\left(\iota_0 m_\bullet(\iota_0^*\otimes\iota_0^*) + \frac{1}{\delta}\sum_{k=1}^r \iota_k m_{\mathcal{G}}(\iota_k^*\otimes\iota_k^*)\right) \\ &\qquad \times (A_{\mu_{r-1}(\mathcal{G})} \otimes A_{\mu_{r-1}(\mathcal{G})})\left((\iota_0\otimes\iota_0)m_\bullet^*\iota_0^* + \frac{1}{\delta}\sum_{k=1}^r (\iota_k\otimes\iota_k)m_{\mathcal{G}}^*\iota_k^*\right). \end{aligned}$$

Since $m_{\mu_{r-1}(\mathcal{G})}$ and $m^*_{\mu_{r-1}(\mathcal{G})}$ contain only tensors of the form $\iota_j^* \otimes \iota_j^*$ and $\iota_j \otimes \iota_j$ ($j = 0, \dots, r$), respectively (i.e., diagonal in the subscripts), the only non-zero contribution from $A_{\mu_{r-1}(\mathcal{G})} \otimes A_{\mu_{r-1}(\mathcal{G})}$ is

$$\begin{aligned} & \delta^2(\iota_r\otimes\iota_r)(\eta_{\mathcal{G}}\otimes\eta_{\mathcal{G}})(\iota_0^*\otimes\iota_0^*) + \delta^2(\iota_0\otimes\iota_0)(\eta_{\mathcal{G}}^*\otimes\eta_{\mathcal{G}}^*)(\iota_r^*\otimes\iota_r^*) \\ &\quad + (\iota_1\otimes\iota_1)(A_{\mathcal{G}}\otimes A_{\mathcal{G}})(\iota_1^*\otimes\iota_1^*) \\ &\quad + \sum_{k=1}^{r-1}\big[(\iota_k\otimes\iota_k)(A_{\mathcal{G}}\otimes A_{\mathcal{G}})(\iota_{k+1}^*\otimes\iota_{k+1}^*) \\ &\qquad\qquad + (\iota_{k+1}\otimes\iota_{k+1})(A_{\mathcal{G}}\otimes A_{\mathcal{G}})(\iota_k^*\otimes\iota_k^*)\big]. \end{aligned}$$

Therefore,

$$
\begin{aligned}
& m_{\mu_{r-1}(\mathscr{G})}(A_{\mu_{r-1}(\mathscr{G})} \otimes A_{\mu_{r-1}(\mathscr{G})})m^*_{\mu_{r-1}(\mathscr{G})} \\
& \quad = (1 + r\delta^2)\Big\{\delta\iota_0 m_\bullet(\eta^*_{\mathscr{G}} \otimes \eta^*_{\mathscr{G}})m^*_{\mathscr{G}}\iota^*_r + \delta\iota_r m_{\mathscr{G}}(\eta_{\mathscr{G}} \otimes \eta_{\mathscr{G}})m^*_\bullet\iota^*_0 \\
& \qquad + \delta^{-2}\iota_1 m_{\mathscr{G}}(A_{\mathscr{G}} \otimes A_{\mathscr{G}})m^*_{\mathscr{G}}\iota^*_1 \\
& \qquad + \delta^{-2}\sum_{k,p=1}^{r}\sum_{l=1}^{r-1}\iota_k m_{\mathscr{G}}(\iota^*_k \otimes \iota^*_k) \\
& \qquad \cdot [(\iota_l \otimes \iota_l)(A_{\mathscr{G}} \otimes A_{\mathscr{G}})(\iota^*_{l+1} \otimes \iota^*_{l+1}) \\
& \qquad\quad + (\iota_{l+1} \otimes \iota_{l+1})(A_{\mathscr{G}} \otimes A_{\mathscr{G}})(\iota^*_l \otimes \iota^*_l)](\iota_p \otimes \iota_p)m^*_{\mathscr{G}}\iota^*_p\Big\}.
\end{aligned}
$$

This can be further simplified into

$$
\begin{aligned}
& m_{\mu_{r-1}(\mathscr{G})}(A_{\mu_{r-1}(\mathscr{G})} \otimes A_{\mu_{r-1}(\mathscr{G})})m^*_{\mu_{r-1}(\mathscr{G})} \\
& \quad = (1 + r\delta^2)\Big\{\delta\iota_0 m_\bullet(\eta^*_{\mathscr{G}} \otimes \eta^*_{\mathscr{G}})m^*_{\mathscr{G}}\iota^*_r + \delta\iota_r m_{\mathscr{G}}(\eta_{\mathscr{G}} \otimes \eta_{\mathscr{G}})m^*_\bullet\iota^*_0 \\
& \qquad + \delta^{-2}\iota_1 m_{\mathscr{G}}(A_{\mathscr{G}} \otimes A_{\mathscr{G}})m^*_{\mathscr{G}}\iota^*_1 \\
& \qquad + \delta^{-2}\sum_{k=1}^{r-1}(\iota_k m_{\mathscr{G}}(A_{\mathscr{G}} \otimes A_{\mathscr{G}})m^*_{\mathscr{G}}\iota^*_{k+1} \\
& \qquad\quad + \iota_{k+1} m_{\mathscr{G}}(A_{\mathscr{G}} \otimes A_{\mathscr{G}})m^*_{\mathscr{G}}\iota^*_k)\Big\}.
\end{aligned}
$$

Since $m_{\mathscr{G}}(A_{\mathscr{G}} \otimes A_{\mathscr{G}})m^*_{\mathscr{G}} = \delta^2 A_{\mathscr{G}}$ and $m_{\mathscr{G}}(\eta_{\mathscr{G}} \otimes \eta_{\mathscr{G}})m^*_\bullet = \eta_{\mathscr{G}}$, we get

$$
m_{\mu_{r-1}(\mathscr{G})}(A_{\mu_{r-1}(\mathscr{G})} \otimes A_{\mu_{r-1}(\mathscr{G})})m^*_{\mu_{r-1}(\mathscr{G})} = (r\delta^2 + 1)A_{\mu_{r-1}(\mathscr{G})}.
$$

To show that

$$
\begin{aligned}
& (\mathrm{id} \otimes \eta^*_{\mu_{r-1}(\mathscr{G})} m_{\mu_{r-1}(\mathscr{G})})(\mathrm{id} \otimes A_{\mu_{r-1}(\mathscr{G})} \otimes \mathrm{id}) \\
& \quad \cdot (m^*_{\mu_{r-1}(\mathscr{G})}\eta_{\mu_{r-1}(\mathscr{G})} \otimes \mathrm{id}) = A_{\mu_{r-1}(\mathscr{G})},
\end{aligned} \tag{4.6}
$$

we first notice that all the maps involved in the definition of $A_{\mu_{r-1}(\mathscr{G})}$ (cf. equation (4.5)) are $*$-maps, $A_{\mu_{r-1}}(\mathscr{G})$ is so. Therefore, [18, Lemma 2.13] together with the self-adjointness of $A_{\mu_{r-1}(\mathscr{G})}$ demonstrates that equation (4.6) holds.

In order to check the (ir)reflexivity of $\mu_{r-1}(\mathcal{G})$, we compute

$$\begin{aligned}
& m_{\mu_{r-1}(\mathcal{G})}(A_{\mu_{r-1}(\mathcal{G})} \otimes \mathrm{id}) m^*_{\mu_{r-1}(\mathcal{G})} \\
&\quad = (1+r\delta^2)\Big[\iota_0 m_\bullet(\iota_0^* \otimes \iota_0^*) + \frac{1}{\delta}\sum_{k=1}^{r} \iota_k m_{\mathcal{G}}(\iota_k^* \otimes \iota_k^*)\Big] \\
&\qquad \times \Big[\delta \iota_r \eta_{\mathcal{G}} \iota_0^* \otimes \mathrm{id} + \delta \iota_0 \eta_{\mathcal{G}}^* \iota_r^* \otimes \mathrm{id} + \iota_1 A_{\mathcal{G}} \iota_1^* \otimes \mathrm{id} \\
&\qquad\qquad + \sum_{k=1}^{r} (\iota_k A_{\mathcal{G}} \iota_{k+1}^* \otimes \mathrm{id} + \iota_{k+1} A_{\mathcal{G}} \iota_k^* \otimes \mathrm{id})\Big] \\
&\qquad \times \Big[(\iota_0 \otimes \iota_0) m_\bullet^* \iota_0^* + \frac{1}{\delta}\sum_{k=1}^{r} (\iota_k \otimes \iota_k) m_{\mathcal{G}}^* \iota_k^*\Big] \\
&\quad = \frac{1+r\delta^2}{\delta^2} \iota_1 m_{\mathcal{G}}(A_{\mathcal{G}} \otimes \mathrm{id}) m_{\mathcal{G}}^* \iota_1^* = \begin{cases} 0, & \mathcal{G}\text{-irreflexive}, \\ (1+r\delta^2)\,\mathrm{id}, & \mathcal{G}\text{-reflexive}. \end{cases}
\end{aligned}$$
■

5. Chromatic numbers under Mycielski transformation

Definition 5.1 ([12, Definition 2.16]). Let $\mathcal{G}$ be an irreflexive quantum graph and $S_{\mathcal{G}}$ the associated operator space as described in equation (3.4). We say that $\mathcal{G}$ possesses a quantum c-coloring if there exist a finite von Neumann algebra $\mathcal{N}$ and a partition of unity $\{P_a\}_{a=1}^c \subset \mathrm{C}(\mathcal{G}) \otimes \mathcal{N}$, that is, a set of orthogonal projections satisfying $\sum_{a=1}^c P_a = \mathbb{1}_{\mathrm{C}(\mathcal{G})\otimes\mathcal{N}}$, and such that

$$\forall_{1\le a\le c}\ \forall_{X\in S_{\mathcal{G}}} : P_a(X \otimes \mathbb{1}_{\mathcal{N}})P_a = 0.$$

The quantum chromatic number is defined as (cf. [4])

$$\chi_q(\mathcal{G}) = \min\{c \in \mathbb{N} : \exists\, c\text{-coloring with } \dim(\mathcal{N}) < \infty\}, \tag{5.1}$$

while the classical chromatic number for a quantum graph is

$$\chi_{\mathrm{loc}}(\mathcal{G}) = \min\{c \in \mathbb{N} : \exists\, c\text{-coloring with } \dim(\mathcal{N}) = 1\}. \tag{5.2}$$

We remark that in the case of a classical graph understood as a quantum one, the above definitions coincide with the classical ones; see [12, Example 2.18]. For a classical graph G, it is known that $\chi_{\mathrm{loc}}(\mu(G)) = \chi_{\mathrm{loc}}(G) + 1$. We now study this type of relation in the quantum framework. We begin with the following observation.

Lemma 5.2. *Let $\mathcal{G}$ be an irreflexive quantum graph. Then $\{P_a\}_{a=1}^c \subset \mathrm{C}(\mathcal{G}) \otimes \mathcal{N}$ is a quantum c-coloring for $\mathcal{G}$ iff for all $a \in \{1,\dots,c\}$ we have $P_a(A_{\mathcal{G}} \otimes \mathbb{1}_{\mathcal{N}})P_a = 0$.*

Proof. Suppose first that $\{P_a\}_{a=1}^c$ is a quantum c-coloring for $\mathcal{G}$ and let $X \in S(\mathcal{G})$. Then $X = m(A_{\mathcal{G}} \otimes Y)m^*$ for some $Y \in B(L^2(\mu_{r-1}(\mathcal{G})))$. Taking $Y = \delta^{-2}A_{\mathcal{G}}$, we have $X = A_{\mathcal{G}}$, so that $P_a(A_{\mathcal{G}} \otimes I_{\mathcal{N}})P_a = 0$ for all $a \in \{1, \dots, c\}$.

For the converse implication, let us note that for all $T \in \mathrm{C}(\mathcal{G})$, all $R \in \mathcal{N}$, all $a, b \in L^2(\mathcal{G})$ and all $x \in L^2(\mathcal{N})$, we have $(T \otimes R)m_{12}(a \otimes b \otimes x) = Tab \otimes Rx = m_{12}(T \otimes \mathbb{1}_{\mathrm{C}(\mathcal{G})} \otimes R)(a \otimes b \otimes x)$. In particular,

$$P_a(m_{12}(A \otimes Y \otimes \mathbb{1}_{\mathcal{N}})m_{12}^*)P_a = m_{12}(P_a)_{13}(A \otimes Y \otimes \mathbb{1}_{\mathcal{N}})(P_a)_{13}m_{12}^*,$$

and hence if $P_a(A \otimes \mathbb{1}_{\mathcal{N}})P_a = 0$, then $P_a(X \otimes \mathbb{1}_{\mathcal{N}})P_a = 0$ for all $X \in S(\mathcal{G})$. ∎

Proposition 5.3. *For every irreflexive quantum graph $\mathcal{G}$, any $r \geq 1$ and any $t \in \{\mathrm{loc}, q\}$,*

$$\chi_t(\mathcal{G}) \leq \chi_t(\mu_{r-1}(\mathcal{G})) \leq \chi_t(\mathcal{G}) + 1.$$

Proof. For the first inequality, the proof for $t = \mathrm{loc}$ is based on the observation that if $\{Q_a\}_{a=1}^c \in \mathrm{C}(\mu_{r-1}(\mathcal{G}))$ is a c-coloring for $\mu_{r-1}(\mathcal{G})$, then $\{P_a = \iota_1^* Q_a \iota_1\}_{a=1}^c$ is a c-coloring for $\mathcal{G}$. Obviously, $\{P_a\}_{a=1}^c$ forms a partition of unity. The identity $Q_a A_{\mu_{r-1}(\mathcal{G})} Q_a = 0$ yields that

$$0 = \iota_1\iota_1^* Q_a A_{\mu_{r-1}(\mathcal{G})} Q_a \iota_1\iota_1^* = Q_a \iota_1\iota_1^* A_{\mu_{r-1}(\mathcal{G})} \iota_1\iota_1^* Q_a,$$

where in the second equality we use the fact that $\iota_1\iota_1^*$ is a central projection in $\mathrm{C}(\mu_{r-1}(\mathcal{G}))$, whereas $Q_a \in \mathrm{C}(\mu_{r-1}(\mathcal{G}))$. Thus we conclude that

$$0 = \iota_1^* Q_a \iota_1\iota_1^* A_{\mu_{r-1}(\mathcal{G})} \iota_1\iota_1^* Q_a \iota_1 = P_a A_{\mathcal{G}} P_a,$$

where in the last equality we use equation (4.3). Similar proof works for $t = q$.

For the second part, the proof for $t = q$ is an amplification of the proof for $t = \mathrm{loc}$ which we give below. We show that if $\{P_k\}_{k=1}^c \subset \mathrm{C}(\mathcal{G})$ is a c-coloring of $\mathcal{G}$, then there exists a $c+1$-coloring $\{Q_k\}_{k=0}^c \subset \mathrm{C}(\mu_{r-1}(\mathcal{G}))$ of $\mu_{r-1}(\mathcal{G})$. The main idea behind the proof is to mimic the construction known from classical graph theory in the quantum setting. For a classical graph G, the proper coloring of $\mu(G)$ is produced by coloring the distinguished vertex $\bullet$ with a new color and leaving each copy of G in $\mu(G)$ with its original coloring.

Viewing $\mathrm{C}(\mu_{r-1}(\mathcal{G}))$ as an algebra acting on $L^2(\mu_{r-1}(\mathcal{G}))$, we define self-adjoint projections $Q_0 = \iota_0\iota_0^*$ and $Q_i = \sum_{j=1}^r \iota_j P_i \iota_j^*$ for $i = 1, 2, \dots, c$ which incidentally form a partition of unity of $\mathrm{C}(\mu_{r-1}(\mathcal{G}))$:

$$\sum_{i=0}^r Q_0 = \iota_0\iota_0^* + \sum_{i,j=1}^r \iota_j P_i \iota_j^* = \iota_0\iota_0^* + \sum_{j=1}^r \iota_j\iota_j^* = \mathbb{1}.$$

In order to conclude that $\{Q_k\}_{k=0}^c$ is $c+1$-coloring of $\mu_{r-1}(\mathcal{G})$, we compute

$$Q_0 A_{\mu_{r-1}(\mathcal{G})} Q_0 = \iota_0\iota_0^* A_{\mu_{r-1}(\mathcal{G})} \iota_0\iota_0^* = 0,$$

and, for $i = 1, \ldots, c$,

$$Q_i A_{\mu_{r-1}(\mathcal{G})} Q_i = \iota_1 P_i A_{\mathcal{G}} P_i \iota_1^* + \sum_{j=1}^{r-1} (\iota_j P_i A_{\mathcal{G}} P_i \iota_{j+1}^* + \iota_{j+1} P_i A_{\mathcal{G}} P_i \iota_j^*) = 0,$$

using the fact that $\{P_k\}_{k=1}^c \subset \mathrm{C}(\mathcal{G})$ is a c-coloring of $\mathcal{G}$. Lemma 5.2 concludes the proof. ■

For classical graphs, it is known that in contrast to the case $r = 2$, the generalized Mycielski construction μ_{r-1} for $r > 2$ may leave classical chromatic number unchanged [26]. Let us restrict our attention to the $r = 2$-case for quantum graph $\mathcal{G}$ and suppose that $\{Q_k\}_{k=0}^{c'}$ is a $c' + 1$-coloring of $\mu(\mathcal{G})$. Since $\mathrm{C}(\mu(\mathcal{G})) = \mathbb{C} \oplus \mathrm{C}(\mathcal{G}) \oplus \mathrm{C}(\mathcal{G})$, we can, without loss of generality, assume that $Q_0 = (1, Q_{01}, Q_{02})$ and $Q_k = (0, Q_{k1}, Q_{k2})$ for $k = 1, 2, \ldots, c'$. Our goal is to prove that under a mild commutativity condition $\chi_{\mathrm{loc}}(\mu(\mathcal{G})) = \chi_{\mathrm{loc}}(\mathcal{G}) + 1$. First, we observe the following lemma.

Lemma 5.4. *With the notation above, $Q_{02} = 0$.*

Proof. We compute

$$0 = Q_0 A_{\mu(\mathcal{G})} Q_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = Q_0 A_{\mu(\mathcal{G})} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = Q_0 \begin{pmatrix} 0 \\ 0 \\ \mathbb{1}_{\mathrm{C}(\mathcal{G})} \end{pmatrix} = Q_{02}, \tag{5.3}$$

what completes the proof. ■

Proposition 5.5. *Let $\mathcal{G}$ be a quantum graph with $c' + 1 = \chi_{\mathrm{loc}}(\mu(\mathcal{G}))$ and suppose that there exists $c' + 1$-coloring $\{Q_k\}_{k=0}^{c'}$ of $\mu(\mathcal{G})$ with the property that $Q_{01} Q_{l2} = Q_{l2} Q_{01}$ for all $l \in \{1, \ldots, c'\}$. Then $\chi_{\mathrm{loc}}(\mathcal{G}) = c'$.*

Proof. By Proposition 5.3, it is enough to construct a c'-coloring $\{Q_l\}_{l=1}^{c'}$ of $\mathcal{G}$. By Lemma 5.4, we can assume that $Q_0 = (1, Q_{01}, 0)$ and $Q_k = (0, Q_{k1}, Q_{k2})$ for $k = 1, \ldots, c'$. Note that $Q_{12} + \cdots + Q_{c'2} = \mathbb{1}$. We define $P_l = Q_{l1} + Q_{01} Q_{l2}$ and note that

$$P_1 + \cdots + P_{c'} = \mathbb{1}.$$

Moreover, under the assumed condition, we have

$$P_l^2 = P_l = P_l^*.$$

Next, we compute assuming that $l > 0$,

$$0 = Q_l A_{\mu(\mathcal{G})} Q_l \begin{pmatrix} \lambda \\ x \\ y \end{pmatrix} = Q_l A_{\mu(\mathcal{G})} \begin{pmatrix} 0 \\ Q_{l1} x \\ Q_{l2} y \end{pmatrix} = \begin{pmatrix} 0 \\ Q_{l1} A_{\mathcal{G}} (Q_{l1} x + Q_{l2} y) \\ Q_{l2} A_{\mathcal{G}} Q_{l1} x \end{pmatrix}$$

for all $x, y \in L^2(\mathscr{G})$. Hence we have $Q_{l2} A_{\mathscr{G}} Q_{l1} = 0 = Q_{l1} A_{\mathscr{G}} Q_{l1}$ for all $l > 0$. Next,

$$Q_0 A_{\mu(\mathscr{G})} Q_0 \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = Q_0 A_{\mu(\mathscr{G})} \begin{pmatrix} 1 \\ Q_{01} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ Q_{01} A_{\mathscr{G}} Q_{01} x \\ 0 \end{pmatrix}$$

for all $x \in L^2(\mathscr{G})$. Hence $Q_{l2} A_{\mathscr{G}} Q_{l1} = 0 = Q_{l1} A_{\mathscr{G}} Q_{l1}$ for all $l \geq 0$ which used in the next computation (under the assumption $Q_{01} Q_{l2} = Q_{l2} Q_{01}$) yields

$$\begin{aligned} P_l A_{\mathscr{G}} P_l &= (Q_{l1} + Q_{01} Q_{l2}) A_{\mathscr{G}} (Q_{l1} + Q_{l2} Q_{01}) \\ &= Q_{l1} A_{\mathscr{G}} Q_{l1} + Q_{l1} A_{\mathscr{G}} Q_{l2} Q_{01} + Q_{01} Q_{l2} A_{\mathscr{G}} Q_{l1} \\ &\quad + Q_{l2} Q_{01} A_{\mathscr{G}} Q_{01} Q_{l2} = 0. \end{aligned}$$

Thus if $Q_{01} Q_{l2} = Q_{l2} Q_{01}$ for all $l \in \{0, 1, \ldots, c'\}$, then $\{P_l\}_{l=1}^{c'}$ is a c'-coloring. ■

Question 5.6. Is there a quantum graph $\mathscr{G}$ for which $\chi_{\mathrm{loc}}(\mu(\mathscr{G})) = \chi_{\mathrm{loc}}(\mathscr{G})$?

We thank David E. Roberson for pointing out to us the following fact.

Remark 5.7. There exists a classical graph G such that $\chi_q(\mu(G)) = \chi_q(G)$. Indeed, let G_{13} be the graph introduced in [17] and defined as follows. Consider a three-dimensional cube centered on the origin of $\mathbb{R}^3$ and identify vector v with $-v$. The set of vertices V for G_{13} consists of vectors (after the aforementioned identification) representing midpoints of the faces (three of them), midpoints of the edges (six of them) and vertices of the cube (four of them). The pair $\{u, v\}$ forms an edge in G_{13} if and only if v and u are orthogonal as vectors in $\mathbb{R}^3$. Adding a single vertex and connecting it with all the vertices from G_{13}, one gets a new graph, called G_{14}. By [17, Lemma 6], we have $\chi_q(G_{13}) = \chi_q(G_{14})$. On the other hand, by construction there is a morphism $\mu(G_{13}) \to G_{14}$, so that $\chi_q(\mu(G_{13})) \leq \chi_q(G_{14})$ by [5, Proposition 2]. This shows that $\chi_q(\mu(G_{13})) = \chi_q(G_{13})$.

6. Clique numbers

Let us recall that with an irreflexive graph G with $|V(G)| = n$, we associate an operator space of the form

$$S_G = \mathrm{span}\{|e_i\rangle\langle e_j| : e_i \sim e_j\} \subseteq B(\mathbb{C}^n)$$

with $\{e_i\}$ being the standard basis of $\mathbb{C}^n$, and $|e_i\rangle\langle e_j|$ denoting the rank-1 operator in $B(\mathbb{C}^n)$ given by $|e_i\rangle\langle e_j|(e_k) = \delta_{jk}|e_i\rangle$. The clique number $\omega(G)$ of G, which is the size of the largest complete graph contained in G, can be equivalently defined [10, 24] as the maximal cardinality of a set K for which we can find a collection of non-zero vectors $\{\psi_k \in \ell_n^2 \setminus \{0\} : k \in K\}$ such that $|\psi_i\rangle\langle\psi_j| \in S_G$ for $i, j, k \in K$ and $i \neq j$, that is,

$$\begin{aligned} \omega(G) = \max\{|\{\psi_k \in \ell_n^2 : k \in K\}| : \psi_k \neq 0, |\psi_i\rangle\langle\psi_j| \in S_G \\ \text{for } i, j, k \in K \text{ and } i \neq j\}. \end{aligned} \tag{6.1}$$

This definition can be adapted to the context of a finite quantum graph $\mathcal{G}$: the clique number for a quantum graph $\mathcal{G} = (\mathbb{V}, \psi, A)$ is given by

$$\omega(\mathcal{G}) = \max\{|\{\psi_k \in L^2(\mathcal{G}) : k \in K\}| : \psi_k \neq 0, |\psi_i\rangle\langle\psi_j| \in S_{\mathcal{G}} \\ \text{for } i, j, k \in K \text{ and } i \neq j\}. \tag{6.2}$$

Definition 6.1 ([24, Definition 7]). We say that there exists a homomorphism between quantum graphs $\mathcal{G}$ and $\mathcal{F}$, and write $\mathcal{G} \to \mathcal{F}$, if there exist a Hilbert space H and an isometry $\mathcal{J} : L^2(\mathcal{G}) \to L^2(\mathcal{F}) \otimes \mathsf{H}$ such that $\mathcal{J} S_{\mathcal{G}} \mathcal{J}^* \subseteq S_{\mathcal{F}} \otimes B(\mathsf{H})$.

By [24, Theorem 8], every homomorphism in the above sense between classical graphs corresponds to a homomorphism between these graphs.

Definition 6.2 ([24, Section III, p. 6]). We say that $\mathcal{G}$ is a quantum subgraph of $\mathcal{F}$ if there exists an isometry $\mathcal{J} : L^2(\mathcal{G}) \to L^2(\mathcal{F})$ such that $\mathcal{J} S_{\mathcal{G}} \mathcal{J}^* \subseteq S_{\mathcal{F}}$.

Remark 6.3. Let us note that $\mathcal{G}$ is a quantum subgraph of Mycielskian $\mu_{r-1}(\mathcal{G})$, for any $r \geq 1$, where the isometry $J : L^2(\mathcal{G}) \to L^2(\mu_{r-1}(\mathcal{G}))$ is given by $J(x) = \iota_1(x)$.

Remark 6.4. The clique number can also be defined using the notion of graph homomorphism from a complete graph, that is, by $\omega(\mathcal{G}) = \max\{|K_n| : K_n \to \mathcal{G}\}$ [24]. It is easy to check that the above definition agrees with (6.2) [24, Theorem 13]. Moreover, if there is a morphism $\mathcal{G} \to \mathcal{F}$, then $\omega(\mathcal{G}) \leq \omega(\mathcal{F})$. In particular, $\omega(\mathcal{G}) \leq \omega(\mu_{r-1}(\mathcal{G}))$ for every $r \geq 1$.

Proposition 6.5. *Let $\mathcal{G}$ be a quantum graph. Then $\omega(\mu(\mathcal{G})) = \omega(\mathcal{G})$.*

Proof. By Remark 6.4, we have $\omega(\mathcal{G}) \leq \omega(\mu(\mathcal{G}))$.

In the proof of the converse inequality, let us use the following notation: remembering that as vector spaces we have $L^2(\mu(\mathcal{G})) = \mathbb{C} \oplus L^2(\mathcal{G}) \oplus L^2(\mathcal{G})$, the components of $\varphi \in L^2(\mu(\mathcal{G}))$ will be denoted by $\varphi^0, \varphi^1, \varphi^2$, respectively. Similarly, every $X \in B(L^2(\mu(\mathcal{G})))$ can be written in a matrix form

$$X = \begin{pmatrix} X_{00} & X_{01} & X_{02} \\ X_{10} & X_{11} & X_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix},$$

where $X_{00} \in \mathbb{C}$, $X_{ij} \in B(L^2(\mathcal{G}))$ and $X_{i0}, X_{0i}^* \in L^2(\mathcal{G})$ for $i, j \in \{1, 2\}$.

Suppose that the clique number $\omega(\mu(\mathcal{G})) = |K|$ where K is a finite set such that there is a set of non-zero vectors $\{\psi_k \in L^2(\mu(\mathcal{G})) : k \in K\}$ satisfying $|\psi_i\rangle\langle\psi_j| \in S_{\mu(\mathcal{G})}$ for $i, j \in K$ and $i \neq j$.

Let $i \neq j$. Since $|\psi_i\rangle\langle\psi_j| \in S_{\mu(\mathcal{G})}$, there exists $X^{(ij)} \in B(L^2(\mu(\mathcal{G})))$ such that

$$|\psi_i\rangle\langle\psi_j| = \frac{2\delta^2+1}{\delta^2} \cdot \begin{pmatrix} 0 & 0 & m_\bullet(\eta_{\mathcal{G}}^* \otimes X_{02}^{(ij)})m_{\mathcal{G}}^* \\ 0 & \frac{1}{\delta^2}m_{\mathcal{G}}(A_{\mathcal{G}} \otimes X_{11}^{(ij)})m_{\mathcal{G}}^* & \frac{1}{\delta^2}m_{\mathcal{G}}(A_{\mathcal{G}} \otimes X_{12}^{(ij)})m_{\mathcal{G}}^* \\ m_{\mathcal{G}}(\eta_{\mathcal{G}} \otimes X_{20}^{(ij)})m_\bullet^* & \frac{1}{\delta^2}m_{\mathcal{G}}(A_{\mathcal{G}} \otimes X_{21}^{(ij)})m_{\mathcal{G}}^* & 0 \end{pmatrix}, \tag{6.3}$$

where the matrix structure on the right of equation (6.3) is a direct consequence of Definition 4.1 of $A_{\mu(\mathcal{G})}$. On the other hand, we have

$$\begin{aligned} |\psi_i\rangle\langle\psi_j| &= \begin{pmatrix} |\psi_i^0\rangle\langle\psi_j^0| & |\psi_i^0\rangle\langle\psi_j^1| & |\psi_i^0\rangle\langle\psi_j^2| \\ |\psi_i^1\rangle\langle\psi_j^0| & |\psi_i^1\rangle\langle\psi_j^1| & |\psi_i^1\rangle\langle\psi_j^2| \\ |\psi_i^2\rangle\langle\psi_j^0| & |\psi_i^2\rangle\langle\psi_j^1| & |\psi_i^2\rangle\langle\psi_j^2| \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & |\psi_i^0\rangle\langle\psi_j^2| \\ 0 & |\psi_i^1\rangle\langle\psi_j^1| & |\psi_i^1\rangle\langle\psi_j^2| \\ |\psi_i^2\rangle\langle\psi_j^0| & |\psi_i^2\rangle\langle\psi_j^1| & 0 \end{pmatrix}, \end{aligned} \tag{6.4}$$

and looking at the lower-right corner of the right-hand side of (6.4), we conclude that for every $i \neq j$ either $\psi_i^2 = 0$ or $\psi_j^2 = 0$. Suppose that there is i_0 such that $\psi_{i_0}^2 \neq 0$ (say, without loss of generality, we can take $i_0 = 1$). Then $\psi_j^2 = 0$ for every $j \neq 1$. Consider now $i \neq j$ and $i, j \neq 1$. In this case,

$$|\psi_i\rangle\langle\psi_j| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & |\psi_i^1\rangle\langle\psi_j^1| & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.5}$$

and therefore $|\psi_i^1\rangle\langle\psi_j^1| \neq 0$ for all $i \neq j$ and $i, j \neq 1$ and hence $\{\psi_1^2, \psi_2^1, \psi_3^1, \dots, \psi_{|K|}^1\}$ is a witness of a clique of size $|K|$ in $\mathcal{G}$. If $\psi_i^2 = 0$ for all i, then equation (6.5) holds for all i, j. Hence $\psi_i^1 \neq 0$ for all i, and therefore $\{\psi_1^1, \psi_2^1, \psi_3^1, \dots, \psi_{|K|}^1\}$ is a witness of a clique of size $|K|$ in $\mathcal{G}$ and this ends the proof. ∎

The techniques used in the proof of Proposition 6.5 lead to the following generalization.

Proposition 6.6. *Let $\mathcal{G}$ be a quantum graph and $r \geq 1$. Then $\omega(\mu_{r-1}(\mathcal{G})) = \omega(\mathcal{G})$.*

Proof. The proof is inductive with respect to r entering Proposition 6.5. For the simplicity of the notation, we shall discuss the inductive step from $r = 2$ to $r = 3$. The general case is done similarly. Suppose that the clique number $\omega(\mu_{r-1}(\mathcal{G})) = |K|$ where K is a finite set analogous to the one in the proof of Proposition 6.5.

Let $i \neq j$. In a completely similar manner to before,

$$|\psi_i\rangle\langle\psi_j| = \begin{pmatrix} 0 & 0 & 0 & |\psi_i^0\rangle\langle\psi_j^3| \\ 0 & |\psi_i^1\rangle\langle\psi_j^1| & |\psi_i^1\rangle\langle\psi_j^2| & 0 \\ 0 & |\psi_i^2\rangle\langle\psi_j^1| & 0 & |\psi_i^2\rangle\langle\psi_j^3| \\ |\psi_i^3\rangle\langle\psi_j^0| & 0 & |\psi_i^3\rangle\langle\psi_j^2| & 0 \end{pmatrix}. \tag{6.6}$$

Looking at the bottom-right corner, we see that there are two options:

- either for all $i \in \{1,2,3\}$ we have $\psi_i^3 = 0$, and therefore

$$|\psi_i\rangle\langle\psi_j| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |\psi_i^1\rangle\langle\psi_j^1| & |\psi_i^1\rangle\langle\psi_j^2| & 0 \\ 0 & |\psi_i^2\rangle\langle\psi_j^1| & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.7}$$

  which can be dealt as in the case $r = 2$, thereby leading to the clique witness for $\mathcal{G}$: $\{\psi_{l_1}^{i_1}, \ldots, \psi_{l_K}^{i_K}\}$ where in this case $i_1, \ldots, i_K \in \{1, \ldots, K\}$ or
- there exists i_0 such that $\psi_{i_0}^r \neq 0$, and then $\psi_j^r = 0$ for every $j \neq i_0$. For $i \neq j$ such that $i, j \neq i_0$, we then have

$$|\psi_i\rangle\langle\psi_j| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |\psi_i^1\rangle\langle\psi_j^1| & |\psi_i^1\rangle\langle\psi_j^2| & 0 \\ 0 & |\psi_i^2\rangle\langle\psi_j^1| & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.8}$$

  and again we choose the clique witness for $\mathcal{G}$: $\left\{\psi_3^{i_0}, \ldots, \psi_{l_K}^{i_K}\right\}$ where in this case $i_2, \ldots, i_K \neq i_0$. ■

The concept of a complete graph admits a quantum version with a quantum space playing the role of a space of vertices. Thus the clique number of a given (quantum) graph admits a quantum version that measures the maximal size of a complete quantum subgraph.

Definition 6.7. Given a finite quantum space $\mathbb{V}$ and a δ-form $\psi : \mathrm{C}(\mathbb{V}) \to \mathbb{C}$, we define a quantum graph $(\mathbb{V}, \psi, A)$ where $A : L^2(\mathbb{V}, \psi) \to L^2(\mathbb{V}, \psi)$ is given by

$$Ax = \delta^2 \psi(x) \mathbb{1}_{\mathrm{C}(\mathbb{V})} - x.$$

This quantum graph is denoted $\mathcal{K}_{\mathbb{V},\psi}$ and referred to as a complete quantum $(\mathbb{V}, \psi)$-graph.

Remark 6.8. We shall often skip ψ and instead writing $\mathcal{K}_{\mathbb{V},\psi}$ we use $\mathcal{K}_{\mathbb{V}}$ referring to it as a complete quantum $\mathbb{V}$-graph.

Definition 6.9. The quantum clique number for a quantum graph $\mathcal{G}$ is given by

$$\omega_q(\mathcal{G}) = \max\{|\mathcal{K}_{\mathbb{V}}| : \mathcal{K}_{\mathbb{V}} \to \mathcal{G}\}. \tag{6.9}$$

Remark 6.10. Analogously to Remark 6.4, we see that if there exists a morphism $\mathcal{G} \to \mathcal{F}$, then $\omega_q(\mathcal{G}) \leq \omega_q(\mathcal{F})$. In particular, if $\mathcal{G}$ is a quantum subgraph of $\mathcal{F}$, then $\omega_q(\mathcal{G}) \leq \omega_q(\mathcal{F})$ and hence $\omega_q(\mathcal{G}) \leq \omega_q(\mu_{r-1}(\mathcal{G}))$ for every $r \geq 1$.

Question 6.11. Is the opposite inequality also true? If not, what is the minimal counterexample?

7. Outlook and open questions

We have defined the Mycielski transformation (and its generalized versions) for quantum graphs and demonstrated how it affects their certain parameters, in particular, (quantum) clique numbers as well as (quantum) chromatic numbers. In contrast to the classical counterpart, we were not able to prove that this transformation automatically enlarges the chromatic number by one. Though we were not able to explicitly construct an example that violates the aforementioned equality, we believe that in general, this equality is not true.

To a classical graph G, one can also associate the Lovász number $\overline{\vartheta}(G)$ [16], which satisfies the monotonicity condition, that is, the existence of a graph homomorphism $G \to F$ implies $\overline{\vartheta}(G) \leq \overline{\vartheta}(F)$ [8, Section 4]. Moreover, $\omega(G) \leq \overline{\vartheta}(G) \leq \chi_{\text{loc}}(G)$. The generalization of the Lovász number into the framework of quantum graphs was proposed in [10] (see also [24, Definition 6]). By [24, Theorem 19], if for two irreflexive quantum graphs $\mathcal{G}$ and $\mathcal{F}$ we have $\mathcal{G} \to \mathcal{F}$, then $\overline{\vartheta}(\mathcal{G}) \leq \overline{\vartheta}(\mathcal{F})$, so that $\overline{\vartheta}(\mathcal{G}) \leq \overline{\vartheta}(\mu_{r-1}(\mathcal{G}))$. In the world of quantum graphs, there are plenty of other variants of clique number and chromatic number, for which one could ask similar questions as discussed in this work. In Question 6.11, we formulated one such example for the quantum clique number, and below we will also discuss the Motzkin–Straus version of this graph characteristic. The cases with other variants lead to intriguing open questions; however, they are beyond the scope of the current paper.

Yet another aspect that we aim to investigate in the forthcoming publication is to study how the proposed Mycielski transformation for quantum graphs affects their (quantum) groups of symmetries [1, 2, 28].

In addition to the aforementioned open questions, we formulate below further potentially intriguing problems motivated by the known results in classical graph theory.

7.1. Quantum versions of Motzkin–Straus clique number

The clique number of a classical graph $G = (V, E)$ can also be computed using the Motzkin–Straus characterization [19]:

$$1 - \frac{1}{\omega(G)} = \max\left\{\langle v, A_G v\rangle : v \in \mathbb{R}_+^{|V|}, \quad \sum_{i=1}^{|V|} v_i = 1\right\}. \tag{7.1}$$

We now mimic this characterization in the quantum setting. For a given convex closed cone in $\mathcal{S} \subseteq L^2(\mathcal{G})$, we can define the following clique numbers.

Definition 7.1. The Motzkin–Straus clique number $\omega_{\mathcal{S}}(\mathcal{G})$ for quantum graph $\mathcal{G}$ and convex closed cone $\mathcal{S} \subseteq L^2(\mathcal{G})$ is defined through

$$1 - \frac{1}{\omega_{\mathcal{S}}(\mathcal{G})} = \max_{v \in \mathcal{S}} \langle v, A_{\mathcal{G}} v\rangle. \tag{7.2}$$

Question 7.2. Characterize (if they exist) cones that correspond to the clique numbers for quantum graphs defined in Section 6. Which of the Motzkin–Straus clique numbers are preserved by the Mycielski transformation?

7.2. Quantum version of Stiebitz theorem

For a given classical graph G, $n \geq 1$ and $r_j \geq 1$ for $j = 1, \ldots, n$, we define

$$\mu_{\{r_1,\ldots,r_n\}}(G) = \mu_{r_n-1}(\cdots \mu_{r_2-1}(\mu_{r_1-1}(G))\cdots). \tag{7.3}$$

For $n = 0$, however, we identify $\{r_1, \ldots, r_n\}$ with $\emptyset$ and put $\mu_\emptyset(G) = G$. For $k \geq 2$, we then define

$$\mathcal{M}_k = \{\mu_{\{r_1,\ldots,r_{k-2}\}}(K_2) : r_j \geq 1,\ j = 1, \ldots, k-2\}, \tag{7.4}$$

that is, it is the set of all generalized Mycielski transformations of K_2 obtained from $k-2$ consecutive applications of $\mu_{r-1}(\cdot)$ with possibly different r's in every iteration, and the following holds.

Theorem 7.3 ([25]). *For every $G \in \mathcal{M}_k$, we have $\chi_{\text{loc}}(G) \geq k$.*

Let $\mathcal{K}_n$ be the quantum complete graph on Mat_n equipped with the tracial δ-form ψ_n, and define

$$\mathbb{M}_k = \{\mu_{\{r_1,\ldots,r_{k-2}\}}(\mathcal{K}_2) : r_j \geq 1,\ j = 1, \ldots, k-2\}. \tag{7.5}$$

Question 7.4. For which type of (quantum) chromatic numbers do we have $\chi_\bullet(\mathcal{G}) \geq k$ for all $\mathcal{G} \in \mathbb{M}_k$?

Acknowledgments. We thank David E. Roberson for his helpful comments. We acknowledge an anonymous referee for valuable comments on the content of the manuscript and suggestions for improving it.

Funding. The work of AB was partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – EXC-2111 – 390814868.

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Received 15 March 2024; revised 30 April 2025.

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