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Mathematical Physics. – *Existence of solutions for fractional Klein–Gordon–Maxwell equations involving Hardy–Sobolev critical exponents*, by Zhenyu Guo and Xin Zhang, accepted on 17 June 2025.

Abstract. – This paper is dedicated to investigating the existence of solutions for fractional Klein–Gordon–Maxwell equations involving Hardy–Sobolev critical exponents. By means of the Mountain Pass Theorem, infinitely many critical points and the least energy solution with positive energy for the subcritical system are obtained. Then we prove that there are at least two different solutions for the critical system by the Ekeland variational principle.

Keywords. – existence of solutions, Klein–Gordon–Maxwell equations, Hardy–Sobolev critical exponents.

Mathematics Subject Classification 2020. – 35B33.

1. Introduction and preliminaries

During the past few decades, scientists have been exploring fractional calculus as a tool for developing more sophisticated mathematical models that can accurately describe various systems. In particular, a great attention has been focused on the study of problems involving the fractional Laplacian from a pure mathematical point of view as well as from concrete applications. Moreover, the fractional Laplacian and more general nonlocal operators of elliptic type have been widely studied in many fields such as optimization, finance, phase transitions, stratified materials, anomalous diffusion and so on (see [3, 8]). In particular, from a probabilistic point of view, the fractional Laplace operator is the infinitesimal generator of a Lévy process. For more details and applications, see [2, 3, 5].

The fractional Laplacian can be defined in $\mathbb{R}^N$ in many equivalent ways. However, when these definitions are restricted to bounded domains, the associated boundary conditions lead to distinct operators. Those different, but yet equivalent, definitions for $(-\Delta)^s$ allow us to take very different approaches in solving related problems, creating a fruitful interplay between e.g. variational techniques, the theory of pseudo-differential operators, functional analysis and potential theory.

In [13], authors proved the existence of infinitely many solutions and least energy solutions for the following nonhomogeneous Klein–Gordon equation coupled with

Born–Infeld equations:

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\omega > 0$ is a constant, $V \in C(\mathbb{R}^3, \mathbb{R})$ and $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ is super-linear.

Motivated by the study of solitary waves of the nonlinear Klein–Gordon equation interacting with an electromagnetic field, Benci and Fortunato in [1] derived a model that is described by the following elliptic system:

$$\begin{cases} -\Delta u + \left[m_0^2 - (\omega+\phi)^2\right]u = |u|^{p-2}u, \\ \Delta\phi = (\omega+\phi)u^2, \end{cases} \tag{1.2}$$

where m_0 and ω are real constants. They proved the existence of infinitely many radially symmetric solutions $(u,\phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ for the above system when $|m_0| > |\omega|$ and for sub-critical exponents p satisfying $4 < p < 2^*$. In [4], Cassani investigated the critical case for the problem, i.e.,

$$p = 2^* = \frac{2N}{N-2},$$

the critical Sobolev exponent. Firstly, Cassani used a Pohozaev-type argument, which points out an invariance property for the problem (1.2), to prove the non-existence of solutions with a suitable decay at infinity, and in particular it turns out to be the case of radially symmetric solutions. Then Cassani replaced the first equation of the system (1.2), adding a lower-order perturbation, by the following:

$$-\Delta u + \left[m_0^2 - (\omega+\phi)^2\right]u = \mu|u|^{p-2}u + |u|^{2^*-2}u, \tag{1.3}$$

where $\mu > 0$ and $4 \le p < 6 = 2^*$. In this case, he recovered a Mountain-Pass type solution for the equation (1.3) and the second equation of system (1.2).

In [10], Jin and Fang studied the fractional elliptic equation with critical Hardy–Sobolev nonlinearity

$$(-\Delta)^s u + a(x)u = \frac{|u|^{2_\alpha^*-2}u}{|x|^\alpha} + k(x)|u|^{q-2}u, \tag{1.4}$$

where $u \in H^s(\mathbb{R}^N)$, $2 < q < 2_s^*$, $0 < s < 1$, $N > 4s$, $0 < \alpha < 2s$, $2_\alpha^* = \frac{2(N-\alpha)}{N-2s}$ is the critical Hardy–Sobolev exponents, $2_s^* = \frac{2N}{N-2s}$ is the critical Sobolev exponent, $a(x), k(x) \in C(\mathbb{R}^N)$. $(-\Delta)^s$ with $s \in (\frac{1}{2}, 1)$ stands for the fractional Laplacian. Through a compactness analysis of the associated functional, they obtained the existence of positive solutions under certain assumptions on $a(x), k(x)$.

In this paper, firstly we study existence of solutions for the following fractional Klein–Gordon–Maxwell equation:

$$\begin{cases} (-\Delta)^s u + \left[m^2 - (\omega+\phi)^2\right]u = \mu \dfrac{|u|^{p-2}u}{|x|^\alpha}, & x \in \mathbb{R}^3, \\ (-\Delta)^s \phi + u^2\phi = -\omega u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.5}$$

where $\mu > 0$, m and ω are real constants, $2 < p < 2_\alpha^* = \frac{2(3-\alpha)}{3-2s}$, $0 < \alpha < 2$. $2_s^* = \frac{6}{3-2s}$ is the fractional critical Sobolev exponent and 2_α^* is the Hardy–Sobolev exponent.

Then we investigate the following system involving a Hardy–Sobolev critical exponent:

$$\begin{cases} (-\Delta)^s u + \left[m^2 - (\omega+\phi)^2\right]u = \mu \dfrac{|u|^{q-2}u}{|x|^\alpha} + \dfrac{|u|^{p-2}u}{|x|^\alpha} + \dfrac{|u|^{2_\alpha^*-2}u}{|x|^\alpha}, & x \in \mathbb{R}^3, \\ (-\Delta)^s \phi + u^2\phi = -\omega u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.6}$$

where $1 < q < 2, 2 < p < 2_\alpha^*$.

There are two ways to define fractional Sobolev spaces. One is via the Gagliardo seminorm

$$H^s(\mathbb{R}^3) := \left\{ u \in L^2(\mathbb{R}^3) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{3}{2}+s}} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \right\},$$

the other one is via Fourier transform

$$\hat{H}^s(\mathbb{R}^3) := \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left(1 + |\xi|^{2s}\right) \left|\mathcal{F}u(\xi)\right|^2 \mathrm{d}\xi < +\infty \right\}$$

and $H^s(\mathbb{R}^3) = \hat{H}^s(\mathbb{R}^3)$. In the present paper, as the norm of fractional Sobolev spaces, we define

$$\|u\|_{H^s}^2 := \int_{\mathbb{R}^3} (m^2 - \omega^2) u^2 \mathrm{d}x + \frac{C_{3,s}}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\left|u(x) - u(y)\right|^2}{|x-y|^{3+2s}} \mathrm{d}x\, \mathrm{d}y,$$

where $|m| > |\omega|$. The fractional Laplacian is defined by

$$\begin{aligned} (-\Delta)^s u(x) &= C_{3,s}\, \mathrm{P.V.} \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x-y|^{3+2s}} \mathrm{d}y \\ &= C_{3,s} \lim_{\varepsilon \to 0^+} \int_{B_\varepsilon^c(x)} \frac{u(x) - u(y)}{|x-y|^{3+2s}} \mathrm{d}y \\ &= -\frac{1}{2} C_{3,s} \int_{\mathbb{R}^3} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{3+2s}} \mathrm{d}y \\ &= \mathcal{F}^{-1}\left(|\xi|^{2s} \mathcal{F}u(\xi)\right), \end{aligned}$$

where

$$C_{3,s} = \left(\int_{\mathbb{R}^3} \frac{1-\cos(\zeta_1)}{|\zeta|^{3+2s}} \mathrm{d}\zeta \right)^{-1},$$

and P.V. is the Cauchy principle value.

Consider the Sobolev space

$$D^{s,2}(\mathbb{R}^3) := \left\{ u \in L^{2_s^*}(\mathbb{R}^3) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{3}{2}+s}} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ by means of the norm

$$\|u\|_{D^{s,2}}^2 := \frac{C_{3,s}}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\, \mathrm{d}y.$$

Throughout this paper, we denote the $L^p(\mathbb{R}^3, |x|^{-\alpha}\mathrm{d}x)$ norm by

$$|u|_{p,\alpha} := \left(\int_{\mathbb{R}^3} \frac{|u|^p}{|x|^\alpha} \mathrm{d}x \right)^{\frac{1}{p}}$$

and $L^p(\mathbb{R}^3, \mathrm{d}x)$ norm by $|u|_p := (\int_{\mathbb{R}^3} |u|^p \mathrm{d}x)^{\frac{1}{p}}$. The fractional Hardy–Sobolev best constant is given by

$$S_s := \inf_{u \in D^{s,2}(\mathbb{R}^3), u \neq 0} \frac{\|u\|_{D^{s,2}}^2}{|u|_{2_\alpha^*}^2}.$$

2. Main results

The energy functional related to system (1.5) is the following:

$$F(u,\phi) = \frac{1}{2}\|u\|_{H^s}^2 - \frac{1}{2}\|\phi\|_{D^{s,2}}^2 - \frac{1}{2}\int_{\mathbb{R}^3} (2\omega\phi + \phi^2)u^2 \mathrm{d}x - \frac{\mu}{p}\int_{\mathbb{R}^3} \frac{|u|^p}{|x|^\alpha} \mathrm{d}x. \tag{2.1}$$

However, for (1.6), one has

$$\widetilde{F}(u,\phi) = \frac{1}{2}\|u\|_{H^s}^2 - \frac{1}{2}\|\phi\|_{D^{s,2}}^2 - \frac{1}{2}\int_{\mathbb{R}^3} (2\omega\phi + \phi^2)u^2 \mathrm{d}x - \frac{\mu}{q}\int_{\mathbb{R}^3} \frac{|u|^q}{|x|^\alpha} \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} \frac{|u|^p}{|x|^\alpha} \mathrm{d}x - \frac{1}{2_\alpha^*}\int_{\mathbb{R}^3} \frac{|u|^{2_\alpha^*}}{|x|^\alpha} \mathrm{d}x. \tag{2.2}$$

Critical points of F and $\widetilde{F}$ correspond to the solutions of (1.5) and (1.6), respectively.

REMARK 2.1. The functional F and $\widetilde{F}$ are strongly indefinite, i.e., unbounded from below and from above on infinite-dimensional subspaces. In order to avoid this indefiniteness, which rules out many of the usual tools of critical point theory, a reduction method is performed in [1] which we now recall.

LEMMA 2.1. *For every $u \in H^s(\mathbb{R}^3)$,*

(i) *there exists a unique function $\phi = \Phi(u) \in D^{s,2}(\mathbb{R}^3)$ which solves the second equation of system (1.5);*

(ii) *if u is radially symmetric, then $\Phi(u)$ is radial too;*

(iii) *$\Phi(u)(x) \leq 0$, moreover, $\Phi(u)(x) \geq -\omega$, if $u(x) \neq 0$ and $\omega > 0$.*

PROOF. The proof of the first result is proved in [11, Lemma 2.1]. The second one, though not explicitly stated, is proved in [7, Lemma 5], where these two equations are different, but the method is the same. In the following, we begin to prove the third result.

Multiplying the second equation of problem (1.5) by $\Phi^+(u) = \max\{\Phi(u), 0\}$, we get

$$-\|\Phi^+(u)\|^2_{D^{s,2}} = \omega \int_{\mathbb{R}^3} \Phi^+(u)u^2 \mathrm{d}x + \int_{\mathbb{R}^3} u^2\big(\Phi^+(u)\big)^2 \mathrm{d}x \geq 0,$$

so that $\Phi^+(u) \equiv 0$.

If we multiply the second equation of problem (1.5) by $(\omega + \Phi(u))^-$, one has

$$\int_{\{x:\Phi(u)<-\omega\}} \big|(-\Delta)^{\frac{s}{2}}\Phi(u)\big|^2 \mathrm{d}x = -\int_{\{x:\Phi(u)<-\omega\}} \big(\omega + \Phi(u)\big)^2 u^2 \mathrm{d}x,$$

so that $(\omega + \Phi(u))^- = 0$ where $u(x) \neq 0$. ∎

LEMMA 2.2. *The map Φ is C^1 and $G_\phi = \{(u,\phi) \in H^s(\mathbb{R}^3) \times D^{s,2}(\mathbb{R}^3) \mid F'_\phi(u,\phi) = 0\}$.*

PROOF. From the second equation of system (1.5), one gets that

$$-\|\Phi(u)\|^2_{D^{s,2}} = \int_{\mathbb{R}^3} \big(\omega + \Phi(u)\big)\Phi(u)u^2 \mathrm{d}x = \int_{\mathbb{R}^3} \omega\Phi(u)u^2 \mathrm{d}x + \int_{\mathbb{R}^3} \Phi^2(u)u^2 \mathrm{d}x. \tag{2.3}$$

In addition, according to (2.1), one gets that

$$F'_\phi\big(u, \Phi(u)\big) = -\|\Phi(u)\|^2_{D^{s,2}} - \int_{\mathbb{R}^3} \omega\Phi(u)u^2 \mathrm{d}x - \int_{\mathbb{R}^3} \Phi^2(u)u^2 \mathrm{d}x. \tag{2.4}$$

Obviously, substituting (2.3) into (2.4), one gets

$$F'_\phi\big(u, \Phi(u)\big) = 0 \quad \text{for any } (u,\phi) \in H^s(\mathbb{R}^3) \times D^{s,2}(\mathbb{R}^3).$$

Thus,

$$F'\big(u,\Phi(u)\big) = F'_u\big(u,\Phi(u)\big) + F'_\phi\big(u,\Phi(u)\big)\Phi'(u) = F'_u\big(u,\Phi(u)\big). \qquad \blacksquare$$

Define $I(u) := F(u,\Phi(u))$, and if $u, v \in H^s(\mathbb{R}^3)$, one gets that

$$I'(u)v = \langle u,v\rangle_{H^s} + \int_{\mathbb{R}^3}\left[\big(m^2 - \big(\omega+\Phi(u)\big)^2\big)uv - \mu\frac{|u|^{p-2}}{|x|^\alpha}uv\right]\mathrm{d}x. \tag{2.5}$$

LEMMA 2.3. *The following statements are equivalent:*

(i) $(u,\phi) \in H^s(\mathbb{R}^3)\times D^{s,2}(\mathbb{R}^3)$ *is a solution of system* (1.5)*;*

(ii) u *is a critical point for* I *and* $\phi = \Phi(u)$.

PROOF. (ii)⇒(i) Obviously.

(i)⇒(ii) Suppose $F'_u(u,\phi)$ and $F'_\phi(u,\phi)$ denote the partial derivatives of F at $(u,\phi) \in H^s(\mathbb{R}^3)\times D^{s,2}(\mathbb{R}^3)$. Then for every $v \in H^s(\mathbb{R}^3)$ and $\psi \in D^{s,2}(\mathbb{R}^3)$, one gets that

$$F'_u(u,\phi)[v] = \langle u,v\rangle_{H^s} + \int_{\mathbb{R}^3}\left[\big(m^2 - (\omega+\phi)^2\big)uv - \mu\frac{|u|^{p-2}}{|x|^\alpha}uv\right]\mathrm{d}x, \tag{2.6}$$

$$F'_\phi(u,\phi)[\psi] = -\langle\phi,\psi\rangle_{D^{s,2}} - \int_{\mathbb{R}^3}\omega\psi u^2\mathrm{d}x - \int_{\mathbb{R}^3}\phi\psi u^2\mathrm{d}x. \tag{2.7}$$

By the standard computations, we can prove that $F'_u(u,\phi)$ and $F'_\phi(u,\phi)$ are continuous. From (2.6) and (2.7), it is easy to obtain that its critical points are solutions of system (1.5), and by (i) of Lemma 2.1, one has $\phi = \Phi(u)$. $\blacksquare$

Our main results are the following.

THEOREM 2.1. *Let* $\omega > 0$*, and if one of the following conditions is satisfied:*

(1) $2^*_\alpha \le 4 : 2 < p < 2^*_\alpha$ *and* $(\frac{p}{2}-1)m^2 > \omega^2$*, or*

(2) $2^*_\alpha > 4 : 2 < p < 4$ *and* $(\frac{p}{2}-1)m^2 > \omega^2$*, or* $4 \le p < 2^*_\alpha$ *and* $|m| > |\omega|$*,*

then the functional I *of system* (1.5) *has infinitely many critical points that have radial symmetry.*

THEOREM 2.2. *Under the assumptions of Theorem* 2.1*, if* $(\frac{1}{2}-\frac{1}{p})m^2 > (\frac{1}{2}+\frac{1}{p})\omega^2$*, then system* (1.5) *possesses a least energy solution with positive energy.*

THEOREM 2.3. *Under the assumptions of Theorem* 2.1*, there exists a constant* $m_1 > 0$ *such that system* (1.6) *admits at least two different solutions* (u,ϕ) *satisfying* $\|u\|_{H^s} < +\infty$*,* $\|\phi\|_{D^{s,2}} < +\infty$*, when* $0 < \mu < m_1$.

REMARK 2.2. Denote by C_i the Sobolev embedding constant for the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^i(\mathbb{R}^3, |x|^{-\alpha}\mathrm{d}x)$ for $2 \le i \le 2^*_\alpha$.

3. The proof of Theorem 2.1

Since system (1.5) is set on $\mathbb{R}^3$, it is well known that the Sobolev embedding

$$H^s(\mathbb{R}^3) \hookrightarrow L^p\big(\mathbb{R}^3, |x|^{-\alpha}\mathrm{d}x\big) \quad (2 \le p \le 2^*_\alpha)$$

is not compact and then it is usually difficult to prove that a Palais–Smale sequence is strongly convergent when we seek solutions of problem (1.5) by variational methods. A standard tool to overcome the problem is to restrict ourselves to radial functions; namely, we look at the functional I on the subspace

$$H^s_r(\mathbb{R}^3) = \big\{u \in H^s(\mathbb{R}^3) \mid u(x) = u\big(|x|\big)\big\}$$

and

$$D^{s,2}_r(\mathbb{R}^3) = \big\{u \in D^{s,2}(\mathbb{R}^3) \mid u(x) = u\big(|x|\big)\big\}$$

compactly embedded in $L^p_r(\mathbb{R}^3, |x|^{-\alpha}\mathrm{d}x)$ for $2 < p < 2^*_\alpha$ and $L^p_r(\mathbb{R}^3, \mathrm{d}x)$ for $2 < p < 2^*_s$. By standard arguments, one sees that a critical point $u \in H^s_r(\mathbb{R}^3)$ for the functional $I|_{H^s_r(\mathbb{R}^3)}$ is also a critical point of I.

Lemma 3.1. *If $2 < p < 2^*_\alpha$ and $(\frac{p}{2} - 1)m^2 > \omega^2$, then $I|_{H^s_r(\mathbb{R}^3)}$ satisfies the Palais–Smale condition.*

Proof. Suppose that $\{u_n\}$ in $H^s_r(\mathbb{R}^3)$ is such that $I'(u_n) \to 0$ and $I(u_n) \to c$, for a positive c.

If $2^*_\alpha > 4$, we have the following cases.

Case (i): $2 < p < 4$. According to (2.1), (2.3) and (2.5), there exists a constant $\varepsilon > 0$, and one obtains that

$$\begin{aligned}
&p(c+1) + \varepsilon\|u_n\|_{H^s_r}\\
&\quad \ge pI(u_n) - I'(u_n)(u_n)\\
&\quad = \left(\frac{p}{2} - 1\right)\|u_n\|^2_{H^s_r} - \left(\frac{p}{2} - 2\right)\int_{\mathbb{R}^3} \omega\Phi(u_n)u_n^2\mathrm{d}x + \int_{\mathbb{R}^3} \Phi^2(u_n)u_n^2\mathrm{d}x\\
&\quad \ge \left(\frac{p}{2} - 1\right)\frac{C_{3,s}}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\big(u_n(x) - u_n(y)\big)^2}{|x-y|^{3+2s}}\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^3} \Phi^2(u_n)u_n^2\mathrm{d}x\\
&\qquad + \int_{\mathbb{R}^3}\left[\left(\frac{p}{2} - 1\right)(m^2 - \omega^2) + \left(\frac{p}{2} - 2\right)\omega^2\right]u_n^2\mathrm{d}x\\
&\quad \ge C\|u_n\|^2_{H^s_r}.
\end{aligned}$$

It follows that $\{u_n\}$ is bounded in $H^s_r(\mathbb{R}^3)$ as $2 < p < 4$.

Case (ii): $4 \le p < 2_\alpha^*$. By (2.1), (2.3) and (2.5), one gets that

$$\begin{aligned}(c+1)+\varepsilon\|u_n\|_{H_r^s} &\ge I(u_n)-\frac{1}{p}I'(u_n)(u_n)\\ &=\left(\frac12-\frac1p\right)\|u_n\|_{H_r^s}^2+\left(\frac12+\frac1p\right)\int_{\mathbb{R}^3}\Phi^2(u_n)u_n^2\mathrm{d}x\\ &\quad+\frac12\left\|\Phi(u_n)\right\|_{D_r^{s,2}}^2+\frac2p\int_{\mathbb{R}^3}\omega\Phi(u_n)u_n^2\mathrm{d}x\\ &=\left(\frac12-\frac1p\right)\|u_n\|_{H_r^s}^2+\left(\frac12-\frac1p\right)\int_{\mathbb{R}^3}\Phi^2(u_n)u_n^2\mathrm{d}x\\ &\quad+\left(\frac12-\frac2p\right)\left\|\Phi(u_n)\right\|_{D_r^{s,2}}^2\\ &\ge C\|u_n\|_{H_r^s}^2.\end{aligned}$$

Therefore, $\{u_n\}$ is bounded in $H_r^s(\mathbb{R}^3)$ as $4 \le p < 2_\alpha^*$.

If $2_\alpha^* \le 4$, the proof process is the same as Case (i).

Moreover, according to (2.3), one has

$$\left\|\Phi(u_n)\right\|_{D_r^{s,2}}^2=-\int_{\mathbb{R}^3}\omega\Phi(u_n)u_n^2\mathrm{d}x-\int_{\mathbb{R}^3}\Phi^2(u_n)u_n^2\mathrm{d}x.$$

Then by Hölder inequality and Sobolev inequality, one obtains that

$$\begin{aligned}\left\|\Phi(u_n)\right\|_{D_r^{s,2}}^2 &\le -\int_{\mathbb{R}^3}\omega\Phi(u_n)u_n^2\mathrm{d}x\\ &\le|\omega|\left(\int_{\mathbb{R}^3}\left|\Phi(u_n)\right|^{2_s^*}\mathrm{d}x\right)^{\frac{1}{2_s^*}}\left(\int_{\mathbb{R}^3}|u_n|^{\frac{2\cdot 2_s^*}{2_s^*-1}}\mathrm{d}x\right)^{\frac{2_s^*-1}{2_s^*}}\\ &=|\omega|\left(\int_{\mathbb{R}^3}\left|\Phi(u_n)\right|^{\frac{6}{3-2s}}\mathrm{d}x\right)^{\frac{3-2s}{6}}\left(\int_{\mathbb{R}^3}|u_n|^{\frac{12}{3+2s}}\mathrm{d}x\right)^{\frac{3+2s}{6}}\\ &\le C\left\|\Phi(u_n)\right\|_{D_r^{s,2}}\|u_n\|_{H_r^s}^2.\end{aligned}$$

Thus, $(\Phi(u_n))_n$ is also bounded in $D_r^{s,2}(\mathbb{R}^3)$.

We know a sequence $\{u_n\}$ in $H_r^s(\mathbb{R}^3)$, which satisfies $I(u_n)\to c$, $I'(u_n)\to 0$, and $\sup\|u_n\|_{H_r^s}<+\infty$. Going if necessary to a subsequence, we assume that

$$u_n \rightharpoonup u \quad \text{in } H_r^s(\mathbb{R}^3).$$

Since the embedding $H_r^s(\mathbb{R}^3)\hookrightarrow L_r^p(\mathbb{R}^3,|x|^{-\alpha}\mathrm{d}x)$ is compact for any $p\in(2,2_\alpha^*)$, we have

$$u_n \to u \quad \text{in } L_r^p\left(\mathbb{R}^3,|x|^{-\alpha}\mathrm{d}x\right). \tag{3.1}$$

Moreover, likewise, for $p \in (2, 2_s^*)$, we also get that

$$u_n \to u \quad \text{in } L_r^p(\mathbb{R}^3, \mathrm{d}x). \tag{3.2}$$

According to (2.5), one obtains that

$$\begin{aligned} I'(u_n)(u_n - u) &= \int_{\mathbb{R}^3} \big((-\Delta)^{\frac{s}{2}} u_n \cdot (-\Delta)^{\frac{s}{2}}(u_n - u) + \big[m^2 - \big(\omega + \Phi(u_n)\big)^2\big] u_n(u_n - u)\big)\mathrm{d}x \\ &\quad - \mu \int_{\mathbb{R}^3} \frac{|u_n|^{p-2}}{|x|^\alpha} u_n(u_n - u)\mathrm{d}x. \end{aligned}$$

Similarly, one gets that

$$\begin{aligned} I'(u)(u_n - u) &= \int_{\mathbb{R}^3} \big((-\Delta)^{\frac{s}{2}} u \cdot (-\Delta)^{\frac{s}{2}}(u_n - u) + \big[m^2 - \big(\omega + \Phi(u)\big)^2\big] u(u_n - u)\big)\mathrm{d}x \\ &\quad - \mu \int_{\mathbb{R}^3} \frac{|u|^{p-2}}{|x|^\alpha} u(u_n - u)\mathrm{d}x. \end{aligned}$$

By $I'(u)v = 0$, we easily get that

$$\begin{aligned} &\int_{\mathbb{R}^3} \Big(\big|(-\Delta)^{\frac{s}{2}}(u_n - u)\big|^2 + (m^2 - \omega^2)(u_n - u)^2\Big)\mathrm{d}x \\ &\quad = \big\langle I'(u_n) - I'(u), u_n - u\big\rangle + \mu \int_{\mathbb{R}^3} \bigg(\frac{|u_n|^{p-2}}{|x|^\alpha} u_n - \frac{|u|^{p-2}}{|x|^\alpha} u\bigg)(u_n - u)\mathrm{d}x \\ &\qquad + 2\omega \int_{\mathbb{R}^3} \big(\Phi(u_n)u_n - \Phi(u)u\big)(u_n - u)\mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^3} \big(\Phi^2(u_n)u_n - \Phi^2(u)u\big)(u_n - u)\mathrm{d}x. \end{aligned} \tag{3.3}$$

It is clear that

$$\big\langle I'(u_n) - I'(u), u_n - u\big\rangle \to 0, \quad \text{as } n \to \infty. \tag{3.4}$$

Furthermore, in view of (3.1), we have

$$\begin{aligned} \mu \int_{\mathbb{R}^3} \frac{|u_n|^{p-1}}{|x|^\alpha}(u_n - u)\mathrm{d}x &\le \mu \bigg(\int_{\mathbb{R}^3} \frac{|u_n|^p}{|x|^\alpha}\mathrm{d}x\bigg)^{\frac{p-1}{p}} \bigg(\int_{\mathbb{R}^3} \frac{|u_n - u|^p}{|x|^\alpha}\mathrm{d}x\bigg)^{\frac{1}{p}} \\ &= \mu |u_n|_{p,\alpha}^{p-1} |u_n - u|_{p,\alpha} \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{3.5}$$

Similarly, we also obtain that

$$\mu \int_{\mathbb{R}^3} \frac{|u|^{p-1}}{|x|^\alpha}(u_n - u)\mathrm{d}x \to 0, \quad \text{as } n \to \infty. \tag{3.6}$$

Thus, combining (3.5) and (3.6), one gets that

$$\begin{aligned}(3.7)\quad &\mu\int_{\mathbb{R}^3}\Big(\frac{|u_n|^{p-2}}{|x|^\alpha}u_n-\frac{|u|^{p-2}}{|x|^\alpha}u\Big)(u_n-u)\mathrm{d}x\\ &=\mu\int_{\mathbb{R}^3}\frac{|u_n|^{p-1}}{|x|^\alpha}(u_n-u)\mathrm{d}x-\mu\int_{\mathbb{R}^3}\frac{|u|^{p-1}}{|x|^\alpha}(u_n-u)\mathrm{d}x\to 0,\quad \text{as } n\to\infty.\end{aligned}$$

By Hölder inequality and Sobolev inequality, one has

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}\big(\Phi(u_n)-\Phi(u)\big)u_n(u_n-u)\mathrm{d}x\Big| &\le \big|\big(\Phi(u_n)-\Phi(u)\big)(u_n-u)\big|_2|u_n|_2\\ &\le \big|\Phi(u_n)-\Phi(u)\big|_{\frac{6}{3-2s}}|u_n-u|_{\frac{3}{s}}|u_n|_2\\ &\le C\big\|\Phi(u_n)-\Phi(u)\big\|_{D_r^{s,2}}|u_n-u|_{\frac{3}{s}}\|u_n\|_{H_r^s}.\end{aligned}$$

According to (3.2), one gets $\int_{\mathbb{R}^3}(\Phi(u_n)-\Phi(u))u_n(u_n-u)\mathrm{d}x\to 0$, as $n\to\infty$.

Moreover,

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}\Phi(u)(u_n-u)^2\mathrm{d}x\Big| &\le \big|\Phi(u)\big|_{\frac{6}{3-2s}}|u_n-u|_{\frac{3}{s}}|u_n-u|_2\\ &\le C\big\|\Phi(u)\big\|_{D_r^{s,2}}|u_n-u|_{\frac{3}{s}}\|u_n-u\|_{H_r^s}\to 0,\quad \text{as } n\to\infty.\end{aligned}$$

Thus, we get that

$$\begin{aligned}(3.8)\qquad &\int_{\mathbb{R}^3}\big(\Phi(u_n)u_n-\Phi(u)u\big)(u_n-u)\mathrm{d}x\\ &=\int_{\mathbb{R}^3}\big(\Phi(u_n)-\Phi(u)\big)u_n(u_n-u)\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^3}\Phi(u)(u_n-u)^2\mathrm{d}x\to 0,\quad \text{as } n\to\infty.\end{aligned}$$

Furthermore, by Hölder inequality and Sobolev inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^3}\big(\Phi^2(u_n)u_n-\Phi^2(u)u\big)(u_n-u)\mathrm{d}x\\ &=\int_{\mathbb{R}^3}\big(\Phi^2(u_n)-\Phi^2(u)\big)u_n(u_n-u)\mathrm{d}x+\int_{\mathbb{R}^3}\Phi^2(u)(u_n-u)(u_n-u)\mathrm{d}x\\ &\le \big|\Phi(u_n)+\Phi(u)\big|_{\frac{6}{2-s}}\big|\Phi(u_n)-\Phi(u)\big|_{\frac{6}{2-s}}|u_n|_3|u_n-u|_{\frac{3}{s}}\\ &\quad+\big|\Phi(u)\big|^2_{\frac{12}{3-2s}}|u_n-u|_{\frac{3}{s}}|u_n-u|_2\\ &\le C\Big(\big\|\Phi(u_n)+\Phi(u)\big\|_{D_r^{s,2}}\big\|\Phi(u_n)-\Phi(u)\big\|_{D_r^{s,2}}\|u_n\|_{H_r^s}|u_n-u|_{\frac{3}{s}}\Big)\\ &\quad+C\Big(\big\|\Phi(u)\big\|^2_{D_r^{s,2}}|u_n-u|_{\frac{3}{s}}\|u_n-u\|_{H_r^s}\Big).\end{aligned}$$

By (3.2), one has

$$\int_{\mathbb{R}^3} \big(\Phi^2(u_n)u_n - \Phi^2(u)u\big)(u_n - u)\mathrm{d}x \to 0, \quad \text{as } n \to \infty. \tag{3.9}$$

Therefore, according to (3.4)–(3.9), we obtain that

$$\begin{aligned}\|u_n - u\|^2_{H^s_r} &= \int_{\mathbb{R}^3} \Big(\big|(-\Delta)^{\frac{s}{2}}(u_n - u)\big|^2 + (m^2 - \omega^2)(u_n - u)^2\Big)\mathrm{d}x \\ &\to 0, \quad \text{as } n \to \infty.\end{aligned} \tag{3.10}$$

Thus, $\{u_n\}$ has a strongly convergent subsequence in $H^s_r(\mathbb{R}^3)$. ■

Now we verify that $I\big|_{H^s_r(\mathbb{R}^3)}$ satisfies the geometrical hypothesis of the $\mathbb{Z}_2$ version of the Mountain Pass Theorem (see [12, Theorem 9.12]).

LEMMA 3.2 ($\mathbb{Z}_2$-Mountain Pass Theorem). *Let E be a Banach space with $\dim(E) = \infty$. Assume that $B \in C^1(E, \mathbb{R}^3)$ and B satisfies the Palais–Smale condition and let $B(0) = 0$. Suppose that*

(i) *there exists $\rho > 0$ and $\beta > 0$ such that $B(u) \geq \beta$ for $\forall u$ with $\|u\|_{H^s_r} = \rho$;*

(ii) *for every finite-dimensional subspace X of E, there exists $R(X)$ such that $B(u) \leq 0$ if $\|u\|_{H^s_r} \geq R$.*

Then B has an unbounded sequence of critical values.

PROOF. We observe that $I(0) = 0$. Moreover, from (2.1) and (2.3), one has

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2_{H^s_r} + \frac{1}{2}\|\phi\|^2_{D^{s,2}_r} + \frac{1}{2}\int_{\mathbb{R}^3} \phi^2 u^2 \mathrm{d}x - \frac{\mu}{p}\int_{\mathbb{R}^3} \frac{|u|^p}{|x|^\alpha}\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2_{H^s_r} - \frac{\mu}{p}\int_{\mathbb{R}^3} \frac{|u|^p}{|x|^\alpha}\mathrm{d}x \geq \frac{1}{2}\|u\|^2_{H^s_r} - \frac{C_p\mu}{p}\|u\|^p_{H^s_r}, \quad 2 < p < 2^*_\alpha. \end{aligned} \tag{3.11}$$

Therefore, there exists $\rho > 0$ and small enough such that

$$\inf_{\|u\|_{H^s_r} = \rho} I(u) \geq \beta > 0.$$

Next we prove the second condition of Lemma 3.2. It is obvious that

$$\begin{aligned} \lim_{t\to+\infty} I(tu) &= \frac{t^2}{2}\|u\|^2_{H^s_r} - t^2\int_{\mathbb{R}^3} \omega\Phi(tu)u^2\mathrm{d}x - \frac{t^2}{2}\int_{\mathbb{R}^3} \Phi^2(tu)u^2\mathrm{d}x \\ &\quad - \frac{1}{2}\big\|\Phi(tu)\big\|^2_{D^{s,2}_r} - \frac{\mu t^p}{p}\int_{\mathbb{R}^3} \frac{|u|^p}{|x|^\alpha}\mathrm{d}x \\ &\leq \frac{t^2}{2}\|u\|^2_{H^s_r} - \omega t^2\int_{\mathbb{R}^3} \Phi(tu)u^2\mathrm{d}x - \frac{\mu t^p}{p}\int_{\mathbb{R}^3} \frac{|u|^p}{|x|^\alpha}\mathrm{d}x \\ &\leq \frac{t^2}{2}\left(\|u\|^2_{H^s_r} + \int_{\mathbb{R}^3} 2\omega^2 u^2\mathrm{d}x\right) - \frac{\mu t^p}{p}\int_{\mathbb{R}^3} \frac{|u|^p}{|x|^\alpha}\mathrm{d} \to -\infty, \end{aligned}$$

which implies that $I(u) \to -\infty$, as $\|u\|_{H_r^s} \to \infty$. Therefore, if R is big enough and $\|u\|_{H_r^s} \geq R$, then $I(u) \leq 0$.

We have thus verified all the conditions of Lemma 3.2, proving that I has an unbounded sequence of critical values. ■

4. The proof of Theorem 2.2

Set

$$\mathcal{M} = \{u \in H_r^s(\mathbb{R}^3) \setminus \{0\} : I'(u) = 0\} \quad \text{and} \quad l = \inf_{u \in \mathcal{M}} I(u).$$

We say that $u \in H_r^s(\mathbb{R}^3)$ is a least energy solution for system (1.5) if $u \in \mathcal{M}$ and $I(u) = l$.

Lemma 4.1. *$\varphi = \Phi(u)$ and $\Phi(u_n) \to \Phi(u)$ in $D_r^{s,2}(\mathbb{R}^3)$.*

Proof. First we prove the uniqueness. For every fixed $u \in H_r^s(\mathbb{R}^3)$, we consider the following minimizing problem $\inf_{\phi \in D_r^{s,2}} E_u(\phi)$, where $E_u : D_r^{s,2} \to \mathbb{R}$ defined as the energy functional of the second equation in system (1.5):

$$E_u(\phi) = \frac{1}{2}\|\phi\|^2_{D_r^{s,2}} + \int_{\mathbb{R}^3} \omega \phi u^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \phi^2 u^2 \mathrm{d}x.$$

In fact, by the proof of [15, Lemma 2.1], we know that

$$\Phi(u_n) \to \varphi, \quad \text{locally uniformly in } \mathbb{R}^3,$$

so we obtain that

$$\int_{\mathbb{R}^3} \Phi(u_n) u_n^2 \mathrm{d}x \to \int_{\mathbb{R}^3} \varphi u^2 \mathrm{d}x, \quad \int_{\mathbb{R}^3} \Phi^2(u_n) u_n^2 \mathrm{d}x \to \int_{\mathbb{R}^3} \varphi^2 u^2 \mathrm{d}x.$$

From the weak lower semicontinuity of the norm in $D_r^{s,2}(\mathbb{R}^3)$ and the convergence above, one has

$$E_u(\varphi) \leq \liminf_{n\to\infty} E_{u_n}\big(\Phi(u_n)\big) \leq \liminf_{n\to\infty} E_{u_n}\big(\Phi(u)\big) = E_u\big(\Phi(u)\big),$$

and then by Lemma 2.3, $\varphi = \Phi(u)$.

Next we prove that $\{\Phi(u_n)\}$ converges strongly in $D_r^{s,2}(\mathbb{R}^3)$. Since $\Phi(u_n)$ and $\Phi(u)$ satisfy the second equation in problem (1.5),

$$\begin{cases} \langle \Phi(u_n), \psi \rangle_{D_r^{s,2}} = -\displaystyle\int_{\mathbb{R}^3} \big[\omega u_n^2 \psi + \Phi(u_n) u_n^2 \psi\big] \mathrm{d}x, \\ \langle \Phi(u), \psi \rangle_{D_r^{s,2}} = -\displaystyle\int_{\mathbb{R}^3} \big[\omega u^2 \psi + \Phi(u) u^2 \psi\big] \mathrm{d}x, \end{cases}$$

then we take the difference for Φ, and one obtains that

$$\langle \Phi(u_n)-\Phi(u),\psi\rangle_{D_r^{s,2}}=-\int_{\mathbb{R}^3}\left[\omega(u_n^2-u^2)\psi+\big(\Phi(u_n)u_n^2-\Phi(u)u^2\big)\psi\right]\mathrm{d}x \tag{4.1}$$

for $\psi\in D_r^{s,2}(\mathbb{R}^3)$. Thus,

$$\begin{aligned}\langle \Phi(u_n)-\Phi(u),\psi\rangle_{D_r^{s,2}}+\int_{\mathbb{R}^3}\left[u_n^2\big(\Phi(u_n)-\Phi(u)\big)\psi\right]\mathrm{d}x+\int_{\mathbb{R}^3}(u_n^2-u^2)\Phi(u)\psi\,\mathrm{d}x&\\ =-\omega\int_{\mathbb{R}^3}(u_n^2-u^2)\psi\,\mathrm{d}x,\quad \psi\in D_r^{s,2}(\mathbb{R}^3).&\end{aligned} \tag{4.2}$$

By Hölder inequality and Sobolev inequality, testing with $\psi=(\Phi(u_n)-\Phi(u))$, the following holds:

$$\begin{aligned}\big\|\Phi(u_n)-\Phi(u)\big\|^2_{D_r^{s,2}}&=-\omega\int_{\mathbb{R}^3}(u_n^2-u^2)\big(\Phi(u_n)-\Phi(u)\big)\mathrm{d}x\\&\quad-\int_{\mathbb{R}^3}u_n^2\big(\Phi(u_n)-\Phi(u)\big)^2\mathrm{d}x\\&\quad-\int_{\mathbb{R}^3}(u_n^2-u^2)\Phi(u)\big(\Phi(u_n)-\Phi(u)\big)\mathrm{d}x\\&\le|\omega|\int_{\mathbb{R}^3}|u_n^2-u^2|\big|\Phi(u_n)-\Phi(u)\big|\mathrm{d}x\\&\quad+\int_{\mathbb{R}^3}|u_n^2-u^2|\big|\Phi(u)\big|\big|\Phi(u_n)-\Phi(u)\big|\mathrm{d}x\\&\le|\omega|\big|\Phi(u_n)-\Phi(u)\big|_{\frac{6}{3-2s}}|u_n^2-u^2|_{\frac{6}{3+2s}}\\&\quad+|u_n^2-u^2|_2\big|\Phi(u)\big|_{\frac{6}{3-2s}}\big|\Phi(u_n)-\Phi(u)\big|_{\frac{3}{s}}\\&\le C|u_n-u|_{\frac{12}{3+2s}}+C|u_n-u|_4.\end{aligned}$$

Since $u_n\rightharpoonup u$ in $H_r^s(\mathbb{R}^3)$, $u_n\to u$ in $L_r^p(\mathbb{R}^3)$ $(2<p<2_s^*)$, one has $\Phi(u_n)\to\Phi(u)$ strongly in $D_r^{s,2}(\mathbb{R}^3)$. ∎

Lemma 4.2. *Under the assumptions of Theorem 2.2, $\mathcal{M}\neq\emptyset$, and there exists a constant $\xi>0$ such that $I(u)\ge\xi$, $\forall u\in\mathcal{M}$.*

Proof. By Lemma 3.1, there exists a sequence $\{u_n\}\subset H_r^s(\mathbb{R}^3)$ satisfying $I(u_n)\to c$, $I'(u_n)\to 0$ and $\|u_n\|_{H_r^s}\le M$ for a constant $M>0$. Up to a subsequence, we have

$$\begin{aligned}u_n&\rightharpoonup\tilde{u}\quad\text{in } H_r^s(\mathbb{R}^3),\\u_n&\to\tilde{u}\quad\text{in } L_r^p(\mathbb{R}^3)\text{ for } 2<p<2_s^*.\end{aligned}$$

Suppose by contradiction that $\tilde{u} = 0$. We get that

$$\begin{aligned} c + o(1) &= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle \\ &= -\frac{1}{2}\|\Phi(u_n)\|^2_{D_r^{s,2}} - \mu\left(\frac{1}{p} - \frac{1}{2}\right)\int_{\mathbb{R}^3} \frac{|u_n|^p}{|x|^\alpha}\mathrm{d}x \\ &\le C\|u_n\|^p_{H_r^s} \to 0, \end{aligned}$$

which contradicts with $c > 0$; thus, $\tilde{u} \neq 0$.

Moreover, we have that $\{\Phi(u_n)\}$ is bounded in $D_r^{s,2}(\mathbb{R}^3)$. Up to a subsequence, we suppose that there exists $\varphi \in D_r^{s,2}(\mathbb{R}^3)$ such that $\Phi(u_n) \rightharpoonup \varphi$, $\varphi = \Phi(u)$ and $\Phi(u_n) \to \Phi(u)$ in $D_r^{s,2}(\mathbb{R}^3)$. Then it follows from (3.8) and (3.9), for any $v \in H_r^s(\mathbb{R}^3)$, that

$$\langle I'(\tilde{u}), v\rangle = \lim_{n\to\infty}\langle I'(u_n), v\rangle = 0.$$

Thus, $\tilde{u} \neq 0$ and $I'(\tilde{u}) = 0$, which means $\mathcal{M} \neq \emptyset$. According to (2.3), (2.5) and $\langle I'(u), u\rangle = 0$ for every $u \in \mathcal{M}$, one gets that

$$\begin{aligned} \|u\|^2_{H_r^s} &\le \|u\|^2_{H_r^s} - \int_{\mathbb{R}^3} \omega\phi u^2\mathrm{d}x + \|\phi\|^2_{D_r^{s,2}} \\ &= \mu\int_{\mathbb{R}^3} \frac{|u|^p}{|x|^\alpha}\mathrm{d}x \le C\|u\|^p_{H_r^s}, \quad 2 < p < 2^*_\alpha. \end{aligned}$$

This implies that there exists a constant ξ_0 such that

$$\|u\|_{H_r^s} \ge \xi_0 > 0, \quad \forall u \in \mathcal{M},$$

where $\xi_0 = (\frac{1}{C})^{\frac{1}{p-2}}$. For $u \in \mathcal{M}$, it follows that

$$\begin{aligned} I(u) &= I(u) - \frac{1}{p}\langle I'(u), u\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2_{H_r^s} + \frac{1}{2}\|\phi\|^2_{D_r^{s,2}} + \frac{2}{p}\int_{\mathbb{R}^3} \omega\phi u^2\mathrm{d}x + \left(\frac{1}{2} + \frac{1}{p}\right)\int_{\mathbb{R}^3} \phi^2 u^2\mathrm{d}x \\ &\ge \left(\frac{1}{2} - \frac{1}{p}\right)\frac{C_{3,s}}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}}\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^3}\left[\left(\frac{1}{2} - \frac{1}{p}\right)(m^2 - \omega^2)u^2 - \frac{2}{p}\omega^2u^2\right]\mathrm{d}x \\ &\ge C\|u\|^2_{H_r^s} \ge C\xi_0^2 =: \xi. \end{aligned}$$

This shows $I(u) \ge \xi > 0$, $\forall u \in \mathcal{M}$. ∎

PROOF OF THEOREM 2.2. According to Lemma 4.2, we have $l > 0$. Let $\{u_n\} \subset \mathcal{M}$ and $I(u_n) \to l$. By similar arguments of Lemma 3.1, there exists $\bar{u} \in H_r^s(\mathbb{R}^3) \setminus \{0\}$ such that

$$\begin{aligned} u_n &\to \bar{u} && \text{in } H_r^s(\mathbb{R}^3), \\ \Phi(u_n) &\to \Phi(\bar{u}) && \text{in } D_r^{s,2}(\mathbb{R}^3). \end{aligned}$$

Moreover, $I'(\bar{u}) = 0$ and $I(\bar{u}) \geq l$. It follows from the weak semicontinuity of norm and Fatou's lemma that

$$\begin{aligned} l &= \lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} \Big[I(u_n) - \frac{1}{p}\big\langle I'(u_n), u_n \big\rangle \Big] \\ &= \lim_{n\to\infty} \Big[\Big(\frac{1}{2} - \frac{1}{p}\Big)\|u_n\|_{H_r^s}^2 + \Big(\frac{1}{2} - \frac{1}{p}\Big)\int_{\mathbb{R}^3} \Phi^2(u_n)u_n^2 dx \\ &\qquad\qquad + \Big(\frac{1}{2} - \frac{2}{p}\Big)\|\Phi(u_n)\|_{D_r^{s,2}}^2 \Big] \\ &\geq \Big[\Big(\frac{1}{2} - \frac{1}{p}\Big)\|\bar{u}\|_{H_r^s}^2 + \Big(\frac{1}{2} - \frac{1}{p}\Big)\int_{\mathbb{R}^3} \Phi^2(\bar{u})\bar{u}^2 dx + \Big(\frac{1}{2} - \frac{2}{p}\Big)\|\Phi(\bar{u})\|_{D_r^{s,2}}^2 \Big] \\ &= I(\bar{u}) - \frac{1}{p}\big\langle I'(\bar{u}), \bar{u} \big\rangle \geq l. \end{aligned}$$

This shows that $\bar{u} \in \mathcal{M}$ and $I(\bar{u}) = l$. ■

5. THE PROOF OF THEOREM 2.3

Here we have lemmas similar to Lemmas 2.1, 2.2, 2.3. Define the functional of system (1.6) as follows:

$$\begin{aligned} J(u) := \tilde{F}(u,\phi) &= \frac{1}{2}\|u\|_{H^s}^2 - \frac{1}{2}\|\phi\|_{D^{s,2}}^2 - \frac{1}{2}\int_{\mathbb{R}^3} (2\omega\phi + \phi^2)u^2 dx \\ &\quad - \frac{\mu}{q}\int_{\mathbb{R}^3} \frac{|u|^q}{|x|^\alpha} dx - \frac{1}{p}\int_{\mathbb{R}^3} \frac{|u|^p}{|x|^\alpha} dx - \frac{1}{2_\alpha^*}\int_{\mathbb{R}^3} \frac{|u|^{2_\alpha^*}}{|x|^\alpha} dx. \end{aligned} \tag{5.1}$$

If $u, v \in H^s(\mathbb{R}^3)$, we have

$$\begin{aligned} J'(u)v &= \langle u, v\rangle_{H^s} + \int_{\mathbb{R}^3} \big(m^2 - (\omega + \Phi(u))^2\big) uv dx \\ &\quad - \int_{\mathbb{R}^3} \Big[\mu\frac{|u|^{q-2}}{|x|^\alpha}uv + \frac{|u|^{p-2}}{|x|^\alpha}uv + \frac{|u|^{2_\alpha^*-2}}{|x|^\alpha}uv \Big] dx. \end{aligned} \tag{5.2}$$

LEMMA 5.1. *Under the assumptions of Theorem 2.3, there exist some constants* $\rho_1, \beta_1, m_1 > 0$ *such that* $J(u)|_{\|u\|_{H^s} = \rho_1} \geq \beta_1 > 0$ *for all* μ *satisfying* $0 < \mu < m_1$.

PROOF. From (2.2), one obtains that

$$J(u)=\frac{1}{2}\|u\|_{H^s}^2-\frac{1}{2}\|\phi\|_{D^{s,2}}^2-\int_{\mathbb{R}^3}\omega\phi u^2\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^3}\phi^2u^2\mathrm{d}x \\ -\frac{\mu}{q}\int_{\mathbb{R}^3}\frac{|u|^q}{|x|^\alpha}\mathrm{d}x-\frac{1}{p}\int_{\mathbb{R}^3}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x-\frac{1}{2_\alpha^*}\int_{\mathbb{R}^3}\frac{|u|^{2_\alpha^*}}{|x|^\alpha}\mathrm{d}x. \tag{5.3}$$

Substituting (2.3) into (5.3), we have

$$\begin{aligned}J(u)&=\frac{1}{2}\|u\|_{H^s}^2+\frac{1}{2}\|\phi\|_{D^{s,2}}^2+\frac{1}{2}\int_{\mathbb{R}^3}\phi^2u^2\mathrm{d}x-\frac{\mu}{q}\int_{\mathbb{R}^3}\frac{|u|^q}{|x|^\alpha}\mathrm{d}x\\&\quad-\frac{1}{p}\int_{\mathbb{R}^3}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x-\frac{1}{2_\alpha^*}\int_{\mathbb{R}^3}\frac{|u|^{2_\alpha^*}}{|x|^\alpha}\mathrm{d}x\\&\geq\frac{1}{2}\|u\|_{H^s}^2-\frac{\mu}{q}\int_{\mathbb{R}^3}\frac{|u|^q}{|x|^\alpha}\mathrm{d}x-\frac{1}{p}\int_{\mathbb{R}^3}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x-\frac{1}{2_\alpha^*}\int_{\mathbb{R}^3}\frac{|u|^{2_\alpha^*}}{|x|^\alpha}\mathrm{d}x\\&\geq\frac{1}{2}\|u\|_{H^s}^2-\frac{C_q\mu}{q}\|u\|_{H^s}^q-\frac{C_p}{p}\|u\|_{H^s}^p-\frac{1}{2_\alpha^*}S_s^{-\frac{2_\alpha^*}{2}}\|u\|_{H^s}^{2_\alpha^*}\\&=\|u\|_{H^s}^q\left(\frac{1}{2}\|u\|_{H^s}^{2-q}-\frac{C_p}{p}\|u\|_{H^s}^{p-q}-\frac{1}{2_\alpha^*}S_s^{-\frac{2_\alpha^*}{2}}\|u\|_{H^s}^{2_\alpha^*-q}-\frac{C_q\mu}{q}\right),\quad 1<q<2.\end{aligned}$$

Set $g_1(\varrho)=\frac{1}{2}\varrho^{2-q}-\frac{C_p}{p}\varrho^{p-q}-\frac{1}{2_\alpha^*}S_s^{-\frac{2_\alpha^*}{2}}\varrho^{2_\alpha^*-q}$, $\varrho\geq 0$. Consider

$$\begin{aligned}g_1'(\varrho)&=\frac{2-q}{2}\varrho^{1-q}-\frac{C_p(p-q)}{p}\varrho^{p-q-1}-\frac{2_\alpha^*-q}{2_\alpha^*}S_s^{-\frac{2_\alpha^*}{2}}\varrho^{2_\alpha^*-q-1}\\&=\varrho^{1-q}\left(\frac{2-q}{2}-\frac{C_p(p-q)}{p}\varrho^{p-2}-\frac{2_\alpha^*-q}{2_\alpha^*}S_s^{-\frac{2_\alpha^*}{2}}\varrho^{2_\alpha^*-2}\right).\end{aligned}$$

Evidently, when $\varrho\geq 0$ is small enough, $g_1'(\varrho)$ is greater than 0, and $g_1(\varrho)$ increases monotonically. When $\varrho\geq 0$ is large enough, $g_1'(\varrho)$ is less than 0, and $g_1(\varrho)$ decreases monotonically. Therefore, there exists a maximum point b such that

$$g_1'(b)=b^{1-q}\left(\frac{2-q}{2}-\frac{C_p(p-q)}{p}b^{p-2}-\frac{2_\alpha^*-q}{2_\alpha^*}S_s^{-\frac{2_\alpha^*}{2}}b^{2_\alpha^*-2}\right)=0.$$

Obviously, we observe that $b=0$ or $b>0$, while $b=0$ is impossible. Then $b>0$, and next we obtain $\max_{\varrho\geq 0}g_1(\varrho)=g_1(b)>0$. Choosing that $\|u\|_{H^s}=b:=\rho_1$, we deduce, for all μ satisfying $0<\mu<\frac{qg_1(\rho_1)}{2C_q}:=m_1$, that

$$J(u)\big|_{\|u\|_{H^s}=\rho_1}\geq\rho_1^q\cdot\left(g_1(\rho_1)-\frac{C_q\mu}{q}\right)\geq\frac{\rho_1^q\cdot g_1(\rho_1)}{2}:=\beta_1>0,$$

and the proof is completed. ∎

LEMMA 5.2. *Under the assumptions of Theorem 2.3, there exists a function* $\eta_1 \in H^s(\mathbb{R}^3)$ *with* $\|\eta_1\|_{H^s} > \rho_1$ *such that* $J(\eta_1) < 0$.

PROOF. It is easy to obtain

$$\begin{aligned}\lim_{t\to+\infty} J(tu) &= \frac{t^2}{2}\|u\|^2_{H^s} - t^2\int_{\mathbb{R}^3}\omega\Phi(tu)u^2\mathrm{d}x - \frac{t^2}{2}\int_{\mathbb{R}^3}\Phi^2(tu)u^2\mathrm{d}x \\ &\quad - \frac{1}{2}\|\Phi(tu)\|^2_{D^{s,2}} - \frac{\mu t^q}{q}\int_{\mathbb{R}^3}\frac{|u|^q}{|x|^\alpha}\mathrm{d}x - \frac{t^p}{p}\int_{\mathbb{R}^3}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x \\ &\quad - \frac{t^{2^*_\alpha}}{2^*_\alpha}\int_{\mathbb{R}^3}\frac{|u|^{2^*_\alpha}}{|x|^\alpha}\mathrm{d}x \\ &\le \frac{t^2}{2}\|u\|^2_{H^s} - \omega t^2\int_{\mathbb{R}^3}\Phi(tu)u^2\mathrm{d}x - \frac{\mu t^q}{q}\int_{\mathbb{R}^3}\frac{|u|^q}{|x|^\alpha}\mathrm{d}x \\ &\quad - \frac{t^p}{p}\int_{\mathbb{R}^3}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x - \frac{t^{2^*_\alpha}}{2^*_\alpha}\int_{\mathbb{R}^3}\frac{|u|^{2^*_\alpha}}{|x|^\alpha}\mathrm{d}x \\ &\le \frac{t^2}{2}\left(\|u\|^2_{H^s} + \int_{\mathbb{R}^3}2\omega^2u^2\mathrm{d}x\right) - \frac{t^p}{p}\int_{\mathbb{R}^3}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x - \frac{t^{2^*_\alpha}}{2^*_\alpha}\int_{\mathbb{R}^3}\frac{|u|^{2^*_\alpha}}{|x|^\alpha}\mathrm{d}x \\ &\to -\infty,\end{aligned}$$

which implies that $J(u) \to -\infty$, as $\|u\|_{H^s} \to \infty$.

The lemma is proved by taking $\eta_1 = tu$ with $t > 0$ large enough and $u \neq 0$. Therefore, we know that there exists $\eta_1 \in H^s(\mathbb{R}^3)$, $\|\eta_1\|_{H^s} > \rho_1$ such that $J(\eta_1) < 0$. ■

Similarly, we look at the functional I on the subspace $H^s_r(\mathbb{R}^3) = \{u \in H^s(\mathbb{R}^3) \mid u(x) = u(|x|)\}$ and $D^{s,2}_r(\mathbb{R}^3) = \{u \in D^{s,2}(\mathbb{R}^3) \mid u(x) = u(|x|)\}$ compactly embedded in $L^p_r(\mathbb{R}^3, |x|^{-\alpha}\mathrm{d}x)$ for $2 < p < 2^*_\alpha$ and $L^p_r(\mathbb{R}^3, \mathrm{d}x)$ for $2 < p < 2^*_s$.

LEMMA 5.3. *Under the assumptions of Theorem 2.3, if* $\{u_n\} \in H^s_r(\mathbb{R}^3)$ *is a bounded Palais–Smale sequence of* J*, then* $\{u_n\}$ *has a strongly convergent subsequence in* $H^s_r(\mathbb{R}^3)$.

PROOF. The proof is similar to the proof of Lemma 3.1. ■

PROOF OF THEOREM 2.3. The proof is divided into two steps.

Step 1. There exists $u_1 \in H^s_r(\mathbb{R}^3)$ such that $J'(u_1) = 0$ and $J(u_1) < 0$.

We choose a function $v \in H^s_r(\mathbb{R}^3)$. Since $\Phi(u) \ge -\omega$, one has

$$\begin{aligned}J(tv) &= \frac{t^2}{2}\|v\|^2_{H^s_r} - t^2\int_{\mathbb{R}^3}\omega\Phi(tv)v^2\mathrm{d}x - \frac{t^2}{2}\int_{\mathbb{R}^3}\Phi^2(tv)v^2\mathrm{d}x - \frac{1}{2}\|\Phi(tv)\|^2_{D^{s,2}_r} \\ &\quad - \frac{\mu t^q}{q}\int_{\mathbb{R}^3}\frac{|v|^q}{|x|^\alpha}\mathrm{d}x - \frac{t^p}{p}\int_{\mathbb{R}^3}\frac{|v|^p}{|x|^\alpha}\mathrm{d}x - \frac{t^{2^*_\alpha}}{2^*_\alpha}\int_{\mathbb{R}^3}\frac{|v|^{2^*_\alpha}}{|x|^\alpha}\mathrm{d}x \\ &\le \frac{t^2}{2}\|v\|^2_{H^s_r} + \int_{\mathbb{R}^3}t^2\omega^2v^2\mathrm{d}x - \frac{\mu t^q}{q}\int_{\mathbb{R}^3}\frac{|v|^q}{|x|^\alpha}\mathrm{d}x < 0\end{aligned} \tag{5.4}$$

for $t > 0$ small enough. Thus, we have $c_1 = \inf\{J(u) : u \in \overline{B}_{\rho_1}\} < 0$, where $\rho_1 > 0$ is given by Lemma 5.1, $B_{\rho_1} = \{u \in H^s_r(\mathbb{R}^3) : \|u\|_{H^s_r} < \rho_1\}$ (see [6]). By Ekeland's variational principle [9], let $\varepsilon = \frac{1}{n} > 0$, $\delta = \|\zeta - u_n\|_{H^s_r} > 0$ for all $\zeta \in \overline{B}_{\rho_1}$. Then there exists a sequence $\{u_n\} \subset \overline{B}_{\rho_1}$ such that

$$c_1 \le J(u_n) \le c_1 + \varepsilon$$

and

$$c_1 + \frac{\varepsilon}{\delta}\|\zeta - u_n\|_{H^s_r} \le J(\zeta) + \frac{\varepsilon}{\delta}\|\zeta - u_n\|_{H^s_r}.$$

Then we obtain that

$$J(\zeta) \ge J(u_n) - \frac{\varepsilon}{\delta}\|\zeta - u_n\|_{H^s_r} \quad \text{for all } \zeta \in \overline{B}_{\rho_1}.$$

Obviously, in view of Lemma 5.1, $u_n \in \overline{B}_{\rho_1}$, for n large enough. Thus, for any $\phi \in H^s_r(\mathbb{R}^3)$ with $\|\phi\|_{H^s_r} = 1$, we can take $t > 0$ such that $(u_n + t\phi) \in \overline{B}_{\rho_1}$ for n large enough. Then we have

$$\frac{J(u_n + t\phi) - J(u_n)}{t} \ge -\frac{1}{n}.$$

Letting $t \to 0$, we get that

$$\langle J'(u_n), \phi\rangle \ge -\frac{1}{n}.$$

We replace ϕ by $-\phi$ in the above inequality, and then it follows that

$$\langle J'(u_n), -\phi\rangle \ge -\frac{1}{n},$$

i.e.,

$$\langle J'(u_n), \phi\rangle \le \frac{1}{n}.$$

Thus, one obtains that

$$\left|\langle J'(u_n), \phi\rangle\right| \le \frac{1}{n},$$

which implies $J'(u_n) \to 0$ as $n \to \infty$.

Hence, we conclude that $\{u_n\}$ is a bounded PS sequence of J for c_1. Therefore, by Lemma 5.3, we get that there exists a function $u_1 \in H^s_r(\mathbb{R}^3)$ such that $J'(u_1) = 0$ and $J(u_1) = c_1 < 0$.

Step 2. There exists $u_2 \in H^s_r(\mathbb{R}^3)$ such that $J'(u_2) = 0$ and $J(u_2) > 0$.

From Lemmas 5.1, 5.2 and the Mountain Pass Theorem [14], there is a sequence $\{u_n\} \subset H^s_r(\mathbb{R}^3)$ such that

$$J(u_n) \to c_2 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J\big(\gamma(t)\big) \ge \beta_1 > 0$$

and

$$J'(u_n) \to 0,$$

where $\Gamma = \{\gamma \in C([0,1], H_r^s(\mathbb{R}^3)) : \gamma(0) = 0,\ \gamma(1) = \eta_1\}$. From Lemma 5.3, we only need to prove that $\{u_n\}$ is bounded in $H_r^s(\mathbb{R}^3)$.

If $2_\alpha^* > 4$, we have the following cases.

Case (i): $2 < p < 4$. From (2.3) and (5.3), one has

$$J(u) = \frac{1}{2}\|u\|_{H_r^s}^2 - \frac{1}{2}\int_{\mathbb{R}^3} \omega\Phi(u)u^2 \mathrm{d}x - \frac{\mu}{q}\int_{\mathbb{R}^3} \frac{|u|^q}{|x|^\alpha}\mathrm{d}x \\ - \frac{1}{p}\int_{\mathbb{R}^3} \frac{|u|^p}{|x|^\alpha}\mathrm{d}x - \frac{1}{2_\alpha^*}\int_{\mathbb{R}^3} \frac{|u|^{2_\alpha^*}}{|x|^\alpha}\mathrm{d}x.$$

Then according to $\Phi(u_n) \geq -\omega$, $1 < q < 2$, we get that

$$\begin{aligned} &p(c_2+1) + \|u_n\|_{H_r^s} \geq pJ(u_n) - \langle J'(u_n), u_n\rangle \\ &= \left(\frac{p}{2}-1\right)\|u_n\|_{H_r^s}^2 - \left(\frac{p}{2}-2\right)\int_{\mathbb{R}^3} \omega\Phi(u_n)u_n^2\mathrm{d}x + \int_{\mathbb{R}^3}\Phi^2(u_n)u_n^2\mathrm{d}x \\ &\quad + \mu\left(1-\frac{p}{q}\right)\int_{\mathbb{R}^3}\frac{|u_n|^q}{|x|^\alpha}\mathrm{d}x + \left(1-\frac{p}{2_\alpha^*}\right)\int_{\mathbb{R}^3}\frac{|u_n|^{2_\alpha^*}}{|x|^\alpha}\mathrm{d}x \\ &\geq \left(\frac{p}{2}-1\right)\int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2\mathrm{d}x + \mu\left(1-\frac{p}{q}\right)\int_{\mathbb{R}^3}\frac{|u_n|^q}{|x|^\alpha}\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3}\left[\left(\frac{p}{2}-1\right)(m^2-\omega^2) + \left(\frac{p}{2}-2\right)\omega^2\right]u_n^2\mathrm{d}x \\ &\geq C\|u_n\|_{H_r^s}^2 + \mu\left(1-\frac{p}{q}\right)C_q\|u_n\|_{H_r^s}^q \end{aligned}$$

for n large enough. It follows that $\{u_n\}$ is bounded in $H_r^s(\mathbb{R}^3)$.

Case (ii): $4 \leq p < 2_\alpha^*$. From (2.3) and (5.3), one has

$$J(u) = \frac{1}{2}\|u\|_{H_r^s}^2 + \frac{1}{2}\int_{\mathbb{R}^3}\Phi^2(u)u^2\mathrm{d}x + \frac{1}{2}\|\Phi(u)\|_{D_r^{s,2}}^2 \\ - \frac{\mu}{q}\int_{\mathbb{R}^3}\frac{|u|^q}{|x|^\alpha}\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3}\frac{|u|^p}{|x|^\alpha}\mathrm{d}x - \frac{1}{2_\alpha^*}\int_{\mathbb{R}^3}\frac{|u|^{2_\alpha^*}}{|x|^\alpha}\mathrm{d}x.$$

Then by (5.2), one obtains that

$$\begin{aligned} &(c_2+1) + o(1)\|u_n\|_{H_r^s} \geq J(u_n) - \frac{1}{p}\langle J'(u_n), u_n\rangle \\ &= \left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|_{H_r^s}^2 + \left(\frac{1}{2}+\frac{1}{p}\right)\int_{\mathbb{R}^3}\Phi^2(u_n)u_n^2\mathrm{d}x + \frac{1}{2}\|\Phi(u_n)\|_{D_r^{s,2}}^2 \\ &\quad + \mu\left(\frac{1}{p}-\frac{1}{q}\right)\int_{\mathbb{R}^3}\frac{|u_n|^q}{|x|^\alpha}\mathrm{d}x + \frac{2}{p}\int_{\mathbb{R}^3}\omega\Phi(u_n)u_n^2\mathrm{d}x + \left(\frac{1}{p}-\frac{1}{2_\alpha^*}\right)\int_{\mathbb{R}^3}\frac{|u_n|^{2_\alpha^*}}{|x|^\alpha}\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|^2_{H^s_r} + \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^3}\Phi^2(u_n)u_n^2\mathrm{d}x + \left(\frac{1}{2}-\frac{2}{p}\right)\|\Phi(u_n)\|^2_{D^{s,2}_r} \\
&\quad + \mu\left(\frac{1}{p}-\frac{1}{q}\right)\int_{\mathbb{R}^3}\frac{|u_n|^q}{|x|^\alpha}\mathrm{d}x + \left(\frac{1}{p}-\frac{1}{2^*_\alpha}\right)\int_{\mathbb{R}^3}\frac{|u_n|^{2^*_\alpha}}{|x|^\alpha}\mathrm{d}x \\
&\geq C\|u_n\|^2_{H^s_r} + \mu\left(\frac{1}{p}-\frac{1}{q}\right)C_q\|u_n\|^q_{H^s_r}
\end{aligned}
$$

for n large enough. Therefore, it follows that $\{u_n\}$ is bounded in $H^s_r(\mathbb{R}^3)$.

If $2^*_\alpha \leq 4$, the proof process is the same as Case (i). The proof is completed. ∎

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Zhenyu Guo
School of Mathematics, Liaoning Normal University
116029 Dalian, P. R. China
guozy@163.com

Xin Zhang
Tongliao New City No. 1 Middle School
028000 Tongliao, P. R. China
zhang_xinbb@163.com