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A regularity property of fractional Brownian sheets

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Abstract. A function f defined on $[0,1]^d$ is called strongly chargeable if there is a continuous vector field v such that $f(x_1,\dots,x_d)$ equals the flux of v through the rectangle $[0,x_1]\times\cdots\times[0,x_d]$ for all $(x_1,\dots,x_d)\in[0,1]^d$. In other words, f is the primitive of the divergence of a continuous vector-field. We prove that the sample paths of the Brownian sheet with $d\geq 2$ parameters are almost surely not strongly chargeable. On the other hand, those of the fractional Brownian sheet of Hurst parameter $(H_1,\dots,H_d)$ are shown to be almost surely strongly chargeable whenever $H_1+\cdots+H_d>d-1$.

1. Introduction

In order to motivate our results regarding multidimensional Brownian sheets, we start with a few remarks about the 1-dimensional Brownian motion. We recall that a Gaussian space is an infinite dimensional separable Hilbert space $E\subset L^2(\Omega,\mathcal{B},\mathbb{P})$ containing only centered Gaussian variables where $(\Omega,\mathcal{B},\mathbb{P})$ is a large enough probability space. There then exists a Gaussian noise, i.e., a Hilbert space isomorphism $G\colon L^2([0,1])\to E$ and setting $W_t=G(\mathbb{1}_{[0,t]})$, $0\leq t\leq 1$, is a way of defining the standard Brownian motion $(W_t)_{0\leq t\leq 1}$. Here, $\mathbb{1}_{[0,t]}$ denotes the indicator function of the interval $[0,t]$. It is now trivial to check that the covariance $\mathbb{E}(W_sW_t)=\min(s,t)$. By an application of Kolmogorov's continuity theorem, one may assume that the function $t\mapsto W_t(\omega)$ is continuous for all $\omega\in\Omega$.

We say that $K\subset[0,1]$ is a *figure* if $K=\bigcup_{i=1}^p[s_i,t_i]$ for some finitely many pairwise nonoverlapping intervals $[s_1,t_1],\dots,[s_p,t_p]$ and we let $\mathcal{F}([0,1])$ denote the set of figures. The increment of the Brownian motion W on K is defined as

$$\Delta_W K=\sum_{i=1}^p(W_{t_i}-W_{s_i})=G(\mathbb{1}_K).$$

Even though $G(\mathbb{1}_A)$ is defined for every measurable subset A of $[0,1]$, it is not the case that, almost surely the increment $\mathcal{F}([0,1])\to\mathbb{R}\colon K\mapsto\Delta_W K$ extends to a (signed)

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Borel measure on [0, 1]. Therefore, it makes sense to study the pathwise regularity of $K \mapsto \Delta_W K$ as a function of figures.

The increment is readily finitely additive, i.e.,

$$\Delta_W(K_1 \cup K_2) = \Delta_W(K_1) + \Delta_W(K_2)$$

whenever K_1 and K_2 are nonoverlapping. Furthermore, one easily checks that, owing to the continuity of $t \mapsto W_t$ the increment possesses the following continuity property: If $(F_n)_n$ is a sequence of figures whose (Lebesgue) measure tends to zero and whose number of components is uniformly bounded, then $\Delta_W(F_n) \to 0$. A finitely additive function $\mu: \mathcal{F}([0,1]) \to \mathbb{R}$ satisfying this continuity property is called a *charge* and it is easy to show that the space of charges $CH([0,1])$ is isomorphic to $C_0([0,1])$, the space of continuous functions vanishing at 0, by means of associating with $f \in C_0([0,1])$ the charge $\mu_f: \bigcup_{i=1}^p [s_i, t_i] \mapsto \sum_{i=1}^p (f(t_i) - f(s_i))$. Thus, it appears that the domain of a charge $\mu_f \in CH([0,1])$ can be extended to include all functions of bounded variation, referring to the Lebesgue–Stieltjes integral,

$$\mu_f : BV\big([0,1]\big) \to \mathbb{R} : u \mapsto \int f du.$$

In this functional analytic context, we think of charges as the members of the dual of $BV([0,1])$ with respect to some appropriate topology.

Finally, we recall a second point of view on Brownian motion, namely the Lévy–Ciesielski construction. We denote by $h_{n,k}$ the Haar function supported in the interval $[k2^{-n}, (k+1)2^{-n}]$ so that the sequence $(h_{n,k})_{n,k}$ is a Hilbertian basis of $L^2([0,1])$. The Faber–Schauder basis $(f_{n,k})_{n,k}$ of $C_0([0,1])$ is then obtained as a sequence of indefinite integrals of the former, $f_{n,k}(t) = \int_0^t h_{n,k}$. It was introduced by Faber in [10]. Next, one can define the Brownian motion W by its decomposition in the Faber–Schauder basis with an independent sequence of Gaussian centered coefficients $(A_{n,k})_{n,k}$:

$$W_t = \sum_{n,k} A_{n,k} f_{n,k}(t).$$

The advantage of this point of view is that one can study the regularity of a continuous function $\sum_{n,k} A_{n,k} f_{n,k}$ according to the asymptotic behavior of its sequence of coefficients $(A_{n,k})_{n,k}$, for instance whether it is Hölder continuous [6]. See [7] for applications to probability. We are now ready to make sense of the corresponding observations for multidimensional stochastic processes.

Here, $d \geq 2$ is an integer. Given $H = (H_1, \ldots, H_d) \in (0,1)^d$ we say that a Gaussian centered random process $(W^H_{t_1,\ldots,t_d})_{0 \leq t_i \leq 1}$ is a fractional Brownian sheet of

Hurst multiparameter H if the covariance

$$\mathbb{E}\big(W^H_{s_1,\dots,s_d} W^H_{t_1,\dots,t_d}\big) = \prod_{i=1}^d \frac{s_i^{2H_i} + t_i^{2H_i} - |t_i - s_i|^{2H_i}}{2}.$$

When $H = (1/2, \dots, 1/2)$ we recover the standard Brownian sheet and we sometimes simply write $(W_{t_1,\dots,t_d})_{0\le t_i\le 1}$ with no reference to H. In that case,

$$\mathbb{E}(W_{s_1,\dots,s_d} W_{t_1,\dots,t_d}) = \prod_{i=1}^d \min(s_i, t_i).$$

As in case $d = 1$, $(W_{t_1,\dots,t_d})_{0\le t_i\le 1}$ can be defined by means of a d-dimensional Gaussian noise. According to Kolmogorov's continuity theorem, one may assume that every sample of the function $(t_1, \dots, t_d) \mapsto W^H_{t_1,\dots,t_d}$ is continuous.

We say that $K \subset [0, 1]^d$ is a *rectangle* if $K = \prod_{i=1}^d [s_i, t_i]$ is a Cartesian product of compact intervals and we define *figures* to be the unions of finitely many rectangles (which we can assume, without loss of generality, are pairwise nonoverlapping, i.e., the Lebesgue measure of their intersection vanishes). The set of figures is denoted by $\mathcal{F}([0, 1]^d)$. We also let $C_0([0, 1]^d)$ be the space of continuous functions defined on $[0, 1]^d$ that vanish at $x = (x_1, \dots, x_d)$ if at least one $x_i = 0$. With $f \in C_0([0, 1]^d)$ is associated its increment $\Delta_f K$ on a rectangle K, whose definition we recall now only when $d = 2$:

$$\Delta_f\big([s_1, t_1] \times [s_2, t_2]\big) = f(t_1, t_2) - f(s_1, t_2) - f(t_1, s_2) + f(s_1, s_2).$$

Since Δ_f is finitely additive on the set of rectangles, it extends uniquely to a finitely additive function of figures, still denoted Δ_f. In general, Δ_f does not extend to a (signed) Borel measure on $[0, 1]^d$. An interesting question consists of making sense of extra regularity properties of Δ_X when $X_{t_1,\dots,t_d}$ is a multidimensional stochastic process with almost sure continuous realizations. This is what we do in this paper in case $X = W^H$.

A d-dimensional *charge* is a finitely additive function $\mu \colon \mathcal{F}([0, 1]^d) \to \mathbb{R}$ satisfying the following continuity property: If $(F_n)_n$ is a sequence of figures such that $|F_n| \to 0$ and $\sup_n \|F_n\| \to 0$ then $\mu(F_n) \to 0$. Here, $|F_n|$ is the Lebesgue measure of the figure F_n and $\|F_n\| = \mathcal{H}^{d-1}(\partial F_n)$ is the $(d-1)$-dimensional Hausdorff measure of its boundary. Note that, when $d = 1$ the term $\mathcal{H}^0(\partial F)$ equals twice the number of components of the figure F, making clear the analogy with the higher dimensional case. The space of d-dimensional charges is denoted $CH([0, 1]^d)$. In fact, $\mu \in CH([0, 1]^d)$ extends (additively and continuously) to a larger collection of sets A than figures, called sets of finite perimeter, i.e., whose indicator function is $\mathbb{1}_A \in BV([0, 1]^d)$. Here, $BV([0, 1]^d)$ is the space of functions of bounded variation in the sense of De Giorgi, i.e., those functions $u \in L^1([0,1]^d)$ whose distributional gradient is a vector-valued measure Du of finite variation. See Section 3 for relevant information about functions of bounded variation and charges.

Recall that $d \geq 2$. We prove that:

- The increment Δ_W of the sample paths of the Brownian sheet W are almost surely not a charge;
- The increment Δ_{W^H} of the sample paths of the fractional Brownian sheet are almost surely a charge provided $H_1 + \cdots + H_d > d - 1$.

In fact, we prove more than this. In order to state our results, we need to introduce the notion of a *strong charge*. We consider the continuous linear embedding $T: L^d([0,1]^d) \to BV([0,1]^d)^*$ defined by $T_f(u) = \int fu$. We let $SCH([0,1]^d)$ be the closure of the range of T and we call its members the d-dimensional strong charges. One can show that $SCH([0,1]^d)$ is a predual of $BV([0,1]^d)$, Theorem 4.1. The strong charges are exactly the linear functionals $\alpha: BV([0,1]^d) \to \mathbb{R}$ associated with a continuous vector-field $v: [0,1]^d \to \mathbb{R}^d$ in the following way: $\alpha(u) = -\int \langle v, Du \rangle$ for each $u \in BV([0,1]^d)$, Theorem 4.3. With each strong charge α, one can associate a (unique) charge μ_α by means of the formula $\mu_\alpha(A) = \alpha(\mathbb{1}_A)$ but the converse is not true.

We identify a Schauder basis of $SCH([0,1]^d)$ and we establish useful criteria for corresponding random series to be strong charges. These apply to the Brownian and fractional Brownian sheet. We start by describing multidimensional Haar functions (see Subsection 5.4). These are tensor products of their 1-dimensional analogues. Specifically, we let

$$g_{n,k,r} = 2^{nd/2} \sum_{\ell=0}^{2^d-1} (A_d)_{r,\ell} \mathbb{1}_{K_{n+1,2^d k+\ell}}$$

where the orthogonal 2^d-dimensional square matrix A_d is the Kronecker product of $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, d times with itself, and the dyadic cubes $K_{n,j}$ have been numbered in an appropriate way where n denotes their generation, i.e., $K_{n,j}$ has side length 2^{-n}. One shows that the sequence $(g_{n,k,r})_{n,k,r}$ is a Hilbertian basis of $L^2([0,1]^d)$. Similarly to the 1-dimensional case, one then wants to consider "indefinite integrals" of these Haar functions in some sense, as did Faber. We introduce the strong charges $T_{g_{n,k,r}}$ (recall the embedding T from the previous paragraph) as indefinite integrals of the Haar functions $g_{n,k,r}$ and we prove these constitute a Schauder basis of $SCH([0,1]^d)$, Theorem 5.1. In fact, we show that if $\alpha \in SCH([0,1]^d)$ then

$$\alpha = \sum_{n,k,r} \alpha(g_{n,k,r}) T_{g_{n,k,r}}.$$

The strong charges $T_{g_{n,k,r}}$ behave somewhat like a wavelet basis in the sense that they have nonoverlapping supports, $\operatorname{spt}(T_{g_{n,k,r}}) = K_{n,k}$. Taking advantage of this, we are able to determine whether some series $\sum_{n,k,r} a_{n,k,r} T_{g_{n,k,r}}$ converge or not to a strong charge, according to the asymptotic behavior of their coefficients $(a_{n,k,r})_{n,k,r}$, Corollary 6.3.

Now let $f \in C_0([0,1]^d)$ and Δ_f be its increment. If $\alpha = \Delta_f$ were a strong charge then the coefficients in the above convergent series would be

$$\Delta_f(g_{n,k,r}) = 2^{nd/2} \sum_{\ell=0}^{2^d-1} (A_d)_{r,\ell} \Delta_f\big(\mathbb{1}_{K_{n+1,2^d k+\ell}}\big).$$

In case f is W^H, the increments $\Delta_{W^N}(K_{n,j})$ on dyadic cubes $K_{n,j}$ are random variables whose asymptotics as (n, j) grows can be controlled. Together with the quantitative criteria evoked at the end of last paragraph, we are then able to establish that:

- The increment Δ_{W^H} of the sample paths of the fractional Brownian sheet W^H are almost surely not a charge if $\bar{H} \le \frac{d-1}{d}$, Theorem 10.5;
- The increment Δ_{W^H} of the sample paths of the fractional Brownian sheet W^H are almost surely a strong charge if $\bar{H} > \frac{d-1}{d}$, Theorem 10.2;

where

$$\bar{H} = \frac{H_1 + \cdots + H_d}{d}.$$

Charges are natural integrators in non-absolute integration theories, see for example [16]. We already mentioned that strong charges act on BV functions. Thus, in case $\bar{H} > \frac{d-1}{d}$, it makes sense to integrate BV functions with respect to the charge Δ_{W^H} representing the variations of the fractional Brownian sheet. Such an integral is understood in a pathwise sense. The possibility to extend this integral to a full-fledged Young integral, where the integrand is allowed to be a Hölder continuous function, is investigated in [4].

2. Notations

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers. We work in an ambient space whose dimension is an integer $d \ge 1$, typically $(0,1)^d$, $[0,1]^d$, or $\mathbb{R}^d$. The Euclidian norm of $x \in \mathbb{R}^d$ is denoted by $|x|$.

The closure, the interior, and the topological boundary of a set $E \subset \mathbb{R}^d$ are denoted $\operatorname{cl} E$, $\operatorname{int} E$, and ∂E, respectively. The indicator function of E is $\mathbb{1}_E$. The symmetric difference of two sets $E_1, E_2 \subset \mathbb{R}^d$ is written $E_1 \triangle E_2$.

Unless otherwise specified, the expressions "measurable", "almost all", as well as "almost everywhere" tacitly refer to the Lebesgue measure. The Lebesgue (outer) measure of a set $E \subset \mathbb{R}^d$ is simply written $|E|$. Two subsets $E_1, E_2 \subset \mathbb{R}^d$ are said to be almost disjoint whenever $|E_1 \cap E_2| = 0$. If $U \subset \mathbb{R}^d$ is a measurable set and $1 \leq p \leq \infty$, the corresponding Lebesgue spaces are denoted $L^p(U)$. Here, U is endowed with its Lebesgue σ-algebra and the Lebesgue measure. The L^p norm is written $\|\cdot\|_p$. The notation $\|\cdot\|_\infty$ might also refer to the supremum norm in the space of continuous functions. The integral of a function f with respect to the Lebesgue measure is simply written $\int f$, with no mention of the Lebesgue measure. In case another measure is used, it is clear from the notation. The $(d-1)$-dimensional Hausdorff measure (defined on Borel subsets of $\mathbb{R}^d$) is denoted $\mathcal{H}^{d-1}$ and the corresponding L^p spaces are written $L^p(U;\mathcal{H}^{d-1})$, where U is a Borel subset of $\mathbb{R}^d$.

The topological dual of a Banach space X is X^*. Unless otherwise specified, the operator norm of a continuous linear map T between normed spaces is written $\|T\|$.

3. Preliminaries on BV functions, BV sets, and charges

3.1. BV functions

We start by introducing all the necessary definitions and results concerning functions of bounded variation. For more insight about these, we refer to the book of Evans and Gariepy [9].

Let $U \subset \mathbb{R}^d$ be an open set. The variation of a Lebesgue integrable function $u: U \to \mathbb{R}$ over an open subset $V \subset U$ is the quantity

$$\|Du\|(V) = \sup\left\{\int_V u \operatorname{div} v : v \in C_c^1(V;\mathbb{R}^d) \text{ and } |v(x)| \leq 1 \text{ for all } x \in V\right\} \tag{3.1}$$

where $C_c^1(V;\mathbb{R}^d)$ denotes the space of continuously differentiable compactly supported vector fields on V.

The function u is said to be of bounded variation whenever $\|Du\|(U) < \infty$. In this case, the vector-valued Riesz representation theorem can be used to prove that the distributional gradient of u is an $\mathbb{R}^d$-valued Borel measure denoted Du. Its total variation measure is denoted $\|Du\|$; it is a finite Borel measure whose values on open subsets V of U is given by formula (3.1).

The set of (equivalence classes of) functions of bounded variation on U is denoted $BV(U)$. It is a Banach space under the norm $\|u\|_{BV} = \|u\|_1 + \|Du\|(U)$. In the sequel, the following results are used.

Theorem (Compactness theorem, [9, 5.2.3, Theorem 4] and [9, 5.2.1, Theorem 1]). *Let $U \subset \mathbb{R}^d$ be a bounded Lipschitz open set and (u_n) be a bounded sequence in*

$BV(U)$. There is a subsequence (u_{n_k}) and a function $u \in BV(U)$ such that $u_{n_k} \to u$ in $L^1(U)$. Furthermore, $\|Du\|(U) \leq \liminf \|Du_{n_k}\|(U)$.

One of the consequences of the compactness theorem is that the closed unit ball of $BV(U)$ is compact when given the L^1-topology. This is a strong indication that $BV(U)$ is a dual Banach space. Indeed, this is proven in Theorem 4.1 (for the case $U = (0,1)^d$, but the proof applies to any bounded Lipschitz open set as well), see also [3, Remark 3.12] for another point of view.

Theorem (Sobolev–Poincaré inequality, [19, Theorem 5.11.1]). *Let $U \subset \mathbb{R}^d$ be a connected bounded Lipschitz open set and $\gamma \in BV(U)^*$ be a continuous linear functional such that $\gamma(\mathbb{1}_U) = 1$. There is a constant $C = C(U,\gamma) \geq 0$ such that for all $u \in BV(U)$,*

$$\|u - \gamma(u)\|_{d/(d-1)} \leq C\|Du\|(U).$$

In the above statement as well as in the remaining part of this paper, when $d = 1$ we agree that $d/(d-1) = \infty$. We apply the Sobolev–Poincaré inequality to the case where

$$\gamma(u) = \frac{1}{|U|}\int_U u \text{ for all } u \in BV(U) \tag{3.2}$$

and more specifically to domains that are d-dimensional open cubes, that is, sets of the form $U = \prod_{i=1}^d (a_i, b_i)$, where $|b_1 - a_1| = \cdots = |b_d - a_d| > 0$. A simple scaling argument shows that, in case of the averaging functional γ defined in (3.2), the Poincaré constant is the same for all d-dimensional open cubes. We denote this constant by C_P.

Theorem (Trace theorem, [9, 5.3, Theorem 1]). *If $U \subset \mathbb{R}^d$ is a bounded Lipschitz open set, there is a continuous linear operator* $\operatorname{tr}\colon BV(U) \to L^1(\partial U; \mathcal{H}^{d-1})$ *such that for all $u \in BV(U)$ and $v \in C^1(\mathbb{R}^d;\mathbb{R}^d)$,*

$$\int_U u \operatorname{div} v = \int_{\partial U} \operatorname{tr}(u) v \cdot n_U \, d\mathcal{H}^{d-1} - \int v \cdot dDu$$

where n_U denotes the normal outer unit vector field defined $\mathcal{H}^{d-1}$-almost everywhere on ∂U.

A useful corollary of the trace theorem is the following result.

Theorem (Extension theorem, [9, 5.4, Theorem 1]). *If $U \subset \mathbb{R}^d$ is a bounded Lipschitz open set and $u \in BV(U)$, define the function*

$$Eu\colon x \mapsto \begin{cases} u(x) & \text{if } x \in U \\ 0 & \text{if } x \notin U. \end{cases}$$

Then $Eu \in BV(\mathbb{R}^d)$ and

$$\|D(Eu)\|(\mathbb{R}^d) = \|Du\|(U) + \int_{\partial U} |\operatorname{tr} u| d\mathcal{H}^{d-1}.$$

3.2. BV sets and charges

The *perimeter* of a measurable subset A of $\mathbb{R}^d$ is the extended real number $\|A\| = \|D\mathbb{1}_A\|(\mathbb{R}^d)$. Usually, A is said to be a set of finite perimeter (or a Caccioppoli set) whenever $\|A\| < \infty$. However, in this paper, we rather follow Pfeffer's terminology [17]: we say that A is a BV*-set* whenever A is bounded, measurable and $\|A\| < \infty$.

We let $\mathcal{BV}(A)$ be the set of BV-subsets of A. It is endowed with the following notion of convergence: a sequence (B_n) is said to w^**-converge* to B whenever

$$\sup \|B_n\| < \infty \text{ and } \lim |B_n \triangle B| = 0.$$

There exists a topology on $\mathcal{BV}(A)$ that is compatible with this notion of convergent sequences (see [17, p. 33 and p. 42]), but we do not use it in the present paper.

A *charge on A* is a function $\mu\colon \mathcal{BV}(A) \to \mathbb{R}$ that satisfies the following proporties:

(A) Finite additivity: $\mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2)$ whenever $B_1, B_2 \in \mathcal{BV}(A)$ are almost disjoint;

(B) Continuity with respect to w^*-convergence: if a sequence (B_n) w^*-converges to B, then $\mu(B_n) \to \mu(B)$.

We observe that a charge necessarily vanishes on negligible sets. This can be seen as a consequence of either (A) or (B). The linear space of charges on A is denoted $CH(A)$. It is worth mentioning that our notation μ is not meant to suggest that charges are measures – indeed, some are not. However, absolutely continuous measures are charges.

The structure of 1-dimensional BV-sets is strikingly simple and this allows for an easy description of charges in case $A = [0,1]$. Indeed, the elements of $\mathcal{BV}([0,1])$ are, up to negligible sets, the disjoint unions of compact intervals. From (B), the function $v\colon [0,1] \to \mathbb{R}$ defined by $v(x) = \mu([0,x])$ is continuous and vanishes at 0. Reciprocally, to any function v belonging to the space $C_0([0,1])$ of continuous functions on $[0,1]$ vanishing at 0, we associate the charge Δ_v that maps any disjoint union of compact intervals to

$$\Delta_v\colon \bigcup_{i=1}^{p} [a_i, b_i] \mapsto \sum_{i=1}^{p} v(b_i) - v(a_i)$$

and such that $\Delta_v(B_1) = \Delta_v(B_2)$ whenever $B_1, B_2 \in \mathcal{BV}([0,1])$ are almost disjoint, i.e., $|B_1 \triangle B_2| = 0$. Thus, $v \mapsto \Delta_v$ is a bijection from $C_0([0,1])$ to $CH([0,1])$.

3.3. The Banach space $C_0([0,1]^d)$

The multidimensional generalization of the space $C_0([0,1])$ is the space $C_0([0,1]^d)$ of continuous functions on $[0,1]^d$ that vanish on the coordinate hyperfacets

$$\bigcup_{i=1}^{d}\big\{(x_1,\dots,x_d)\in[0,1]^d : x_i=0\big\}. \tag{3.3}$$

We equip this space with the maximum norm $\|\cdot\|_\infty$.

3.4. Chargeability

Among the subsets of $[0,1]^d$, we consider some that are more regular than BV sets. We describe these here.

A *dyadic cube* is a set of the type $\prod_{i=1}^{d}[2^{-n}k_i, 2^{-n}(k_i+1)]$, where $n\geq 0$ and $0\leq k_1,\dots,k_d\leq 2^n-1$ are integers. Such a dyadic cube has side length 2^{-n} and we say that it is *of generation* n. Thus, in our terminology, dyadic cubes are subsets of $[0,1]^d$.

A *rectangle* is a set with non-empty interior of the form $\prod_{i=1}^{d}[a_i,b_i]$. We are also interested in *(rectangular) figures* in $[0,1]^d$, i.e., subsets of $[0,1]^d$ that can be written as finite unions of rectangles. The collection of such sets is denoted $\mathcal{F}([0,1]^d)$. Note that each $F\in\mathcal{F}([0,1]^d)$ is a BV-set and that $\|F\|=\mathcal{H}^{d-1}(\partial F)$.

We can now define the increments of a function $f\in C_0([0,1]^d)$ on a rectangle $\prod_{i=1}^{d}[a_i,b_i]\subset[0,1]^d$ by means of the formula

$$\Delta_f\Big(\prod_{i=1}^{d}[a_i,b_i]\Big)=\sum_{(c_i)\in\prod_{i=1}^{d}\{a_i,b_i\}}(-1)^{\delta_{a_1,c_1}}\dots(-1)^{\delta_{a_d,c_d}}f(c_1,\dots,c_d) \tag{3.4}$$

where for any reals t,t', the number $\delta_{t,t'}$ is 1 or 0 according to whether $t=t'$ or not. In the 2-dimensional case, we recover the well-known rectangular increment $\Delta_f([a_1,b_1]\times[a_2,b_2])=f(b_1,b_2)-f(a_1,b_2)-f(b_1,a_2)+f(a_1,a_2)$.

One checks that if a rectangle K can be split as the union of two almost disjoint rectangles $K=K_1\cup K_2$, then $\Delta_f(K)=\Delta_f(K_1)+\Delta_f(K_2)$. Based on this observation and the fact that any figure can be written as a finite union of pairwise almost disjoint rectangles, Δ_f has a unique extension to $\mathcal{F}([0,1]^d)$, that satisfies the finite additivity property (A) of the Subsection 3.2, restricted to the subcollection $\mathcal{F}([0,1]^d)$ of $\mathcal{BV}([0,1]^d)$. This extension is still denoted Δ_f.

Now, we say that the function f is *chargeable* whenever $\Delta_f:\mathcal{F}([0,1]^d)\to\mathbb{R}$ has an extension to $\mathcal{BV}([0,1]^d)$ that is a charge. We state below an approximation theorem of De Giorgi (see [17, Proposition 1.10.3] for a proof), that implies that this extension is necessarily unique. In intuitive terms, the chargeability of f allows to make sense of increments of f over arbitrary BV-sets. The discussion in 3.2 shows that all continuous functions on $[0,1]$ vanishing at 0 are chargeable. In fact, chargeability is a regularity property that differs from continuity only in dimension ≥ 2. We prove later that the sample paths of the Brownian sheet are almost surely not chargeable.

Functions that are chargeable can be thought of as being the indefinite integrals of charges, by integration on rectangles. This statement is made clear by the elementary Proposition 3.2. The class of BV-subsets of $[0,1]^d$ has better properties than that of rectangular figures. It is invariant under biLipschitz transformations and its definition does not rely on a specific choice of a basis in $\mathbb{R}^d$. We uphold the thesis that, whenever a function f is chargeable, the charge Δ_f is a more fundamental object than f itself.

Theorem 3.1 (De Giorgi approximation). *There is a constant $C \geq 0$, depending only on d, such that, for any BV-subset $B \subset [0,1]^d$, there exists a sequence (F_n) of figures in $[0,1]^d$ such that*

$$\sup \|F_n\| \leq C\|B\| \text{ and } \lim |F_n \triangle B| = 0.$$

In particular, (F_n) w^*-converges to B.

Proposition 3.2. *A function $f \in C_0([0,1]^d)$ is chargeable if and only if there exists a charge μ on $[0,1]^d$ such that*

$$f(x_1,\dots,x_d) = \mu\left(\prod_{i=1}^{d}[0,x_i]\right), \qquad 0 \leq x_1,\dots,x_d \leq 1. \tag{3.5}$$

In this case, one has $\mu = \Delta_f$.

Proof. If f is chargeable, then Δ_f is a charge and

$$f(x_1,\dots,x_d) = \Delta_f\big([0,x_1]\times\cdots\times[0,x_d]\big),$$

as f vanishes on (3.3). Conversely, suppose the existence of a charge μ such that (3.5) holds and consider a rectangle $K = \prod_{i=1}^{d}[a_i,b_i]$. Then, by the finite additivity of μ, one obtains

$$f(x_1,\dots,x_{d-1},b_d) - f(x_1,\dots,x_{d-1},a_d) = \mu\left(\prod_{i=1}^{d-1}[0,x_i]\times[a_d,b_d]\right)$$

for any $0 \leq x_1,\dots,x_{d-1} \leq 1$. Repeating this process one step further, one gets

$$\sum_{c_{d-1},c_d} (-1)^{\delta_{a_{d-1},c_{d-1}}}(-1)^{\delta_{a_d,c_d}} f(x_1,\dots,x_{d-2},c_{d-1},c_d) = \mu\left(\prod_{i=1}^{d-2}[0,x_i]\times[a_{d-1},b_{d-1}]\times[a_d,b_d]\right)$$

where c_{d-1} ranges over $\{a_{d-1}, b_{d-1}\}$ and c_d over $\{a_d, b_d\}$. Continuing further, we obtain $\Delta_f(K) = \mu(K)$. By finite additivity of both Δ_f and μ, we deduce that $\mu = \Delta_f$ on $\mathcal{F}([0,1]^d)$, which yields the result. ∎

3.5. The space $BV\big([0,1]^d\big)$

We let $BV([0,1]^d)$ be the subspace of $BV(\mathbb{R}^d)$ that consists of those functions u such that $u = 0$ almost everywhere on $\mathbb{R}^d \setminus [0,1]^d$, equipped with the norm inherited from $BV(\mathbb{R}^d)$.

In fact, the extension operator $E\colon BV((0,1)^d) \to BV([0,1]^d)$ provides an isomorphim, whose reciprocal is the restriction operator. A measurable function $u\colon \mathbb{R}^d \to \mathbb{R}$ belongs to $BV([0,1]^d)$ if and only if $u = 0$ almost everywhere outside $[0,1]^d$ and the restriction of u to $(0,1)^d$ belongs to $BV((0,1)^d)$. This means that the norms $\|\cdot\|_{BV(\mathbb{R}^d)}$ and

$$u \mapsto \int_{(0,1)^d} |u| + \|Du\|\big((0,1)^d\big)$$

are equivalent on $BV([0,1]^d)$.

We claim that $u \mapsto \|Du\|(\mathbb{R}^d)$ is yet another norm on $BV([0,1]^d)$ that is equivalent to the two norms above. To prove this claim, let $\gamma\colon BV((0,1)^d) \to \mathbb{R}$ be the map defined by

$$\gamma(\tilde{u}) = \frac{1}{2d} \int_{\partial(0,1)^d} \operatorname{tr} \tilde{u} \, d\mathcal{H}^{d-1}, \qquad \tilde{u} \in BV\big((0,1)^d\big).$$

It is continuous, by the continuity of tr. By the Sobolev–Poincaré inequality, there is a constant C such that $\|\tilde{u} - \gamma(\tilde{u})\|_{d/(d-1)} \le C\|D\tilde{u}\|((0,1)^d)$ for all $\tilde{u}$. Furthermore, one has, by the extension theorem,

$$\|Du\|(\mathbb{R}^d) = \|D\tilde{u}\|\big((0,1)^d\big) + \int_{\partial(0,1)^d} |\operatorname{tr} \tilde{u}| d\mathcal{H}^{d-1} \tag{3.6}$$

where $\tilde{u} \in BV((0,1)^d)$ denotes here the restriction of u. One notices that

$$\begin{aligned}
\|u\|_1 = \|\tilde{u}\|_1 &\le \|\tilde{u} - \gamma(\tilde{u})\|_1 + \frac{1}{2d}\|\operatorname{tr}\tilde{u}\|_1 \\
&\le \|\tilde{u} - \gamma(\tilde{u})\|_{d/(d-1)} + \frac{1}{2d}\|\operatorname{tr}\tilde{u}\|_1 \\
&\le \max\Big(C, \frac{1}{2d}\Big)\big(\|D\tilde{u}\|\big((0,1)^d\big) + \|\operatorname{tr}\tilde{u}\|_1\big)
\end{aligned}$$

Thus by (3.6), one obtains $\|u\|_{BV} = \|u\|_1 + \|Du\|(\mathbb{R}^d) \le C'\|Du\|(\mathbb{R}^d)$ for some constant $C' > 0$. The upper bound $\|Du\|(\mathbb{R}^d) \le \|u\|_{BV}$ is trivial.

For the definition of strong charge functionals in the next section, it is slightly more convenient to work in $BV([0,1]^d)$ rather than the isomorphic space $BV((0,1)^d)$. One could similarly define a Banach space $BV(A)$, where A is a BV-set, and develop a theory of strong charge functionals on A.

4. Strong charge functionals and strong charges

4.1. The space of strong charge functionals

To each function $f \in L^d([0,1]^d)$, we associate the functional

$$T_f : u \mapsto \int_{[0,1]^d} f u$$

defined on $BV([0,1]^d)$. It is continuous. Indeed, denoting $\bar{u}$ the average value of u on $[0,1]^d$, we have, by the Hölder and Sobolev–Poincaré inequalities,

$$\begin{aligned} |T_f(u)| &= \left| \int_{[0,1]^d} f(u - \bar{u}) + \left(\int_{[0,1]^d} f \right) \bar{u} \right| \\ &\leq \|f\|_d \left(\int_{(0,1)^d} |u - \bar{u}|^{d/(d-1)} \right)^{(d-1)/d} + \|f\|_d \|u\|_1 \\ &\leq \|f\|_d \big(C_P \|Du\| \big((0,1)^d\big) + \|u\|_1 \big) \\ &\leq \max(C_P, 1) \|f\|_d \|u\|_{BV}. \end{aligned}$$

Furthermore, this computation shows that the map $T: L^d([0,1]^d) \to BV([0,1]^d)^*$ sending f to T_f is continuous.

We define the Banach space $SCH([0,1]^d)$ as the closure of the range of the operator T in $BV([0,1]^d)^*$. Elements thereof are called *strong charge functionals*. The choice of the terminology and the link with charges defined in the previous section is explained in Subsection 4.4. With this definition, $SCH([0,1]^d)$ turns out to be a predual of $BV([0,1]^d)$, as we prove now.

Theorem 4.1 (Duality theorem). *The canonical map*

$$\Upsilon : BV\big([0,1]^d\big) \to SCH\big([0,1]^d\big)^*$$

(that sends u to the functional $\alpha \mapsto \alpha(u)$) is an isomorphism of Banach spaces.

Proof. The canonical map Υ can be seen as the composition of the evaluation map $BV([0,1]^d) \to BV([0,1]^d)^{**}$ with the adjoint of the injection map $SCH([0,1]^d) \to BV([0,1]^d)^*$, and, therefore, is continuous. Next, we prove separately that Υ is one-to-one and onto.

If u belongs to the kernel of Υ, then

$$T_f(u) = \int_{[0,1]^d} fu = 0$$

for all $f \in L^d([0,1]^d)$. From this it can be deduced that $u = 0$ a.e. on $[0,1]^d$, hence $u = 0$ in $BV([0,1]^d)$. Therefore, Υ is an injection.

Now let us take $\gamma \in SCH([0,1]^d)^*$. As $\gamma \circ T$ belongs to the dual of $L^d([0,1]^d)$, there is a function $u \in L^{d/(d-1)}([0,1]^d)$ such that

$$\gamma(T_f) = \int_{[0,1]^d} fu$$

for all $f \in L^d([0,1]^d)$. We extend u to $\mathbb{R}^d$ by zero, and we wish to prove that u belongs to $BV([0,1]^d)$. To this end, we consider a vector field $v \in C_c^1(\mathbb{R}^d;\mathbb{R}^d)$ such that $|v| \le 1$ on $\mathbb{R}^d$. Call g the restriction of $\operatorname{div} v$ to $[0,1]^d$. First, note that

$$T_g(\varphi) = \int_{[0,1]^d} \varphi \operatorname{div} v = \int_{\mathbb{R}^d} \varphi \operatorname{div} v \le \|D\varphi\|(\mathbb{R}^d) \le \|\varphi\|_{BV}$$

for all $\varphi \in BV([0,1]^d)$. This establishes that $\|T_g\| \le 1$. Then, we observe that

$$\int_{[0,1]^d} u \operatorname{div} v = \gamma(T_g) \le \|\gamma\|.$$

As v is arbitrary, this proves that $\|Du\|(\mathbb{R}^d) \le \|\gamma\| < \infty$ and so $u \in BV([0,1]^d)$.

The continuous maps γ and $\Upsilon(u)$ coincide on $T(L^d([0,1]^d))$, a dense subspace of $SCH([0,1]^d)$. On this account, we infer that $\gamma = \Upsilon(u)$. So, Υ is onto. Finally, we apply the open mapping theorem to conclude that Υ^{-1} is continuous as well. ■

The next proposition characterizes weak* convergence of sequences in the space $BV([0,1]^d)$ (weak* convergence with respect to the duality between $SCH([0,1]^d)$ and $BV([0,1]^d)$). This is the same convergence that appears in the BV compactness theorem.

Proposition 4.2. *A sequence (u_n) in $BV([0,1]^d)$ weak* converges to u if and only if it is bounded and $u_n \to u$ in L^1.*

Proof. Of course, a weak* convergent sequence (u_n) is bounded by the uniform boundedness principle. To prove that $u_n \to u$ in $L^1([0,1]^d)$, it suffices to remark that any subsequence of (u_n) has a subsequence converging to u in $L^1(\mathbb{R}^d)$. Indeed, by the compactness theorem (that we may apply to the functions u_n restricted to a bounded Lipschitz open neighborhood of $[0,1]^d$), it is possible to extract, from any subsequence (u_{n_k}) of (u_n), a subsequence still denoted (u_{n_k}) that L^1-converges to some $v \in BV([0,1]^d)$. For any $f \in L^\infty([0,1]^d)$, we have $T_f(u_{n_k}) \to T_f(v)$, whereas we also have $T_f(u_{n_k}) \to T_f(u)$ by weak* convergence. Hence $T_f(u) = T_f(v)$. As $f \in L^\infty([0,1]^d)$ is arbitrary, we deduce that $u = v$ and this finishes the first part of the proof.

Conversely, we observe that the space $L^\infty([0,1]^d)$ is dense in $L^d([0,1]^d)$, and therefore $T(L^\infty([0,1]^d))$ is dense in $SCH([0,1]^d)$. Owing to the boundedness of (u_n), it is sufficient to remark that $T_f(u_n) \to T_f(u)$ for any $f \in L^\infty([0,1]^d)$. ■

4.2. The operator $\mathfrak{div}$

To each continuous vector field $v \in C([0,1]^d; \mathbb{R}^d)$, we associate the functional $\mathfrak{div}\, v : BV([0,1]^d) \to \mathbb{R}$ defined by

$$(\mathfrak{div}\, v)(u) = -\int_{[0,1]^d} v \cdot dDu.$$

We call $\mathfrak{div}\, v$ the divergence of v. This terminology is justified by the fact that

$$\forall u \in BV\big([0,1]^d\big), \qquad (\mathfrak{div}\, v)(u) = \int_{[0,1]^d} u \operatorname{div} v \tag{4.1}$$

whenever $v \in C^1([0,1]^d; \mathbb{R}^d)$. To prove (4.1), one may extend v to a C^1 vector field on $\mathbb{R}^d$, and then apply the trace theorem for a domain U that is a bounded Lipschitz open neighborhood of $[0,1]^d$.

Let us check that $\mathfrak{div}\, v \in SCH([0,1]^d)$ whenever $v \in C([0,1]^d; \mathbb{R}^d)$. For $\varepsilon > 0$, choose a smooth vector field $w : \mathbb{R}^d \to \mathbb{R}^d$ such that $|v(x) - w(x)| \leq \varepsilon$ for all $x \in [0,1]^d$. Then

$$(\mathfrak{div}\, v)(u) = -\int_{[0,1]^d} u \operatorname{div} w - \int_{[0,1]^d} (v - w) \cdot dDu.$$

Call g the restriction of $\operatorname{div} w$ to $[0,1]^d$. The preceding equality implies that

$$\big|\mathfrak{div}\, v(u) - T_g(u)\big| = \left|\int_{[0,1]^d} (v - w) \cdot dDu\right| \leq \varepsilon \|u\|_{BV}.$$

As u is arbitrary, this ensures that $\mathfrak{div}\, v$ is a continuous linear functional and that $\|\mathfrak{div}\, v - T_g\| \leq C\varepsilon$. This shows that $\mathfrak{div}\, v$ belongs to the closure of $T(L^d([0,1]^d))$, i.e., it is a strong charge functional.

Thus, we have defined a linear map $\mathfrak{div} : C([0,1]^d; \mathbb{R}^d) \to SCH([0,1]^d)$. On top of that, it is continuous, as can be seen from the inequality

$$|(\mathfrak{div}\, v)(u)| \leq \|v\|_\infty \|Du\|\big([0,1]^d\big) \leq \|v\|_\infty \|u\|_{BV}.$$

In fact, each strong charge is the divergence of a continuous vector field, as we show now.

Theorem 4.3 (Representation of strong charge functionals). *The operator* $\mathfrak{div}$ *is onto.*

Proof. The following proof is based on the results of [8] and [5], adapted to the present formalism. First, we prove that the range of $\mathfrak{div}$ is dense in $SCH([0,1]^d)$. Let $g\colon [0,1]^d \to \mathbb{R}$ be a smooth function. We prove that $T_g = \mathfrak{div}\, v$ for some continuous vector field v. As $C^\infty([0,1]^d)$ is dense in $L^d([0,1]^d)$, this establishes our claim. Let w be a solution of the Poisson equation $\Delta w = g$ on $[0,1]^d$ and set $v = \nabla w$. As v is of class C^∞ on $[0,1]^d$, we have

$$T_g(u) = \int_{[0,1]^d} ug = \int_{[0,1]^d} u \operatorname{div} v = (\mathfrak{div}\, v)(u)$$

for all $u \in BV([0,1]^d)$, which shows that $T_g = \mathfrak{div}\, v$.

We conclude the proof by showing that the range of $\mathfrak{div}$ is closed. By [14, Theorem 3.1.21], it suffices to show that the range of the adjoint map $\mathfrak{div}^*$ is closed in $SCH([0,1]^d)^*$.

In the following diagram, we let $M([0,1]^d;\mathbb{R}^d)$ be the Banach space of $\mathbb{R}^d$-valued Borel measures on $[0,1]^d$ normed by total variation. It is isomorphic to the dual of $C([0,1]^d;\mathbb{R}^d)$, by the Riesz representation theorem. The vertical arrows Υ and Φ are the obvious isomorphisms and we define $U = \Phi^{-1} \circ \mathfrak{div}^* \circ \Upsilon$, so that the diagram is commutative.

$$\begin{array}{ccc} SCH([0,1]^d)^* & \xrightarrow{\ \mathfrak{div}^*\ } & C([0,1]^d;\mathbb{R}^d)^* \\ \Upsilon\big\uparrow & & \big\uparrow\Phi \\ BV([0,1]^d) & \xrightarrow{\ \ U\ \ } & M([0,1]^d;\mathbb{R}^d) \end{array}$$

For all $u \in BV([0,1]^d)$, we have $(\mathfrak{div}^* \circ \Upsilon)(u) = \Upsilon(u) \circ \mathfrak{div}$. Evaluating at the function $v \in C([0,1]^d;\mathbb{R}^d)$ yields

$$(\mathfrak{div}^* \circ \Upsilon)(u)(v) = \Upsilon(u)(\mathfrak{div}\, v) = (\mathfrak{div}\, v)(u) = -\int_{[0,1]^d} v \cdot dDu.$$

On the other hand,

$$(\Phi \circ U)(u)(v) = \int v \cdot dU(u).$$

Since the preceding equalities hold for every v, we deduce that the measure $U(u)$ is the restriction of $-Du$ to $[0,1]^d$. Therefore, its total variation is

$$\|U(u)\| = \|Du\|\big([0,1]^d\big) = \|Du\|(\mathbb{R}^d).$$

We recall from Subsection 3.5 that $u \mapsto \|Du\|(\mathbb{R}^d)$ is a norm on $BV([0,1]^d)$ that is equivalent to $\|\cdot\|_{BV}$. This completes the proof. ∎

4.3. Remark in dimension 1

In dimension $d = 1$, the space $SCH([0,1])$ is isometric to $C_0([0,1])$. Indeed, we already know that the continuous map $\mathfrak{div}\colon C_0([0,1]) \to SCH([0,1])$ is onto. We define the map

$$\Pi\colon SCH\big([0,1]\big) \to C_0\big([0,1]\big)$$

that sends α to $x \mapsto \alpha(\mathbb{1}_{[0,x]})$. We claim that $\Pi \circ \mathfrak{div} = \mathrm{id}$ and, thus, that $\mathfrak{div}$ is an isomorphism whose inverse is Π. Indeed, letting $v \in C_0([0,1])$ we have for all $x \in [0,1]$,

$$\Pi(\mathfrak{div}\, v)(x) = (\mathfrak{div}\, v)(\mathbb{1}_{[0,x]}) = -\int_{[0,1]} v\, dD\mathbb{1}_{[0,x]} = v(x) - v(0) = v(x).$$

4.4. Strong charges

To each strong charge functional α, we associate the map $\mathcal{S}(\alpha)\colon \mathcal{BV}([0,1]^d) \to \mathbb{R}$

$$\mathcal{S}(\alpha) : B \mapsto \alpha(\mathbb{1}_B).$$

It is clearly finitely additive, by the linearity of α, and continuous with respect to w^*-convergence, by Proposition 4.2. Thus, we have defined a linear map

$$\mathcal{S}\colon SCH\big([0,1]^d\big) \to CH\big([0,1]^d\big).$$

Charges that belong to the range of $\mathcal{S}$ are called *strong*, for historical reasons. We admit for now that $\mathcal{S}$ is injective – a fact that is proved momentarily in Corollary 4.5. As a consequence, strong charge functionals and strong charges are basically the same objects, under different disguises. However, the formalism of strong charge functionals makes it readily clear that $SCH([0,1]^d)$ is a Banach space and this allows us to introduce a functional analytic approach in the next section.

In dimension $d = 1$, charges on $[0,1]$ are automatically strong. Indeed, let μ be any charge on $[0,1]$ and $v\colon x \mapsto \mu([0,x])$ its associated continuous function, then it is easy to check that $\mu = \mathcal{S}(\mathfrak{div}\, v)$.

The structure of BV-sets in $[0,1]^d$ is fully understood. They have a Borel measurable *reduced* boundary $\partial_* B$, defined in a measure-theoretic way, on which a normal unit outer vector field n_B is defined $\mathcal{H}^{d-1}$-almost everywhere and they satisfy a generalized Gauss–Green theorem [9, 5.9, Theorem 1]

$$\int_B \operatorname{div} \varphi = \int_{\partial_* B} \varphi \cdot n_B \, d\mathcal{H}^{d-1},$$

for all $\varphi \in C^1([0,1]^d;\mathbb{R}^d)$. In particular, if a strong charge functional α is the divergence of $v \in C([0,1]^d;\mathbb{R}^d)$, a density argument shows that

$$\alpha(\mathbb{1}_B) = \int_{\partial_* B} v \cdot n_B \, d\mathcal{H}^{d-1}.$$

In terms of charges, a charge μ is strong if and only if there exists a continuous vector field $v \in C([0,1]^d;\mathbb{R}^d)$ such that

$$\mu(B) = \int_{\partial_* B} v \cdot n_B \, d\mathcal{H}^{d-1}, \qquad B \in \mathcal{BV}\big([0,1]^d\big).$$

A function $f \in C_0([0,1]^d)$ is called *strongly chargeable* whenever the increment map Δ_f extends (uniquely) to a strong charge. For example, any continuous function $f \in C_0([0,1])$ is strongly chargeable. This only happens in dimension $d = 1$.

We close this section with an approximation lemma whose full strength is proven useful in Section 5, in the proof of Theorem 5.1. A dyadic partition $\mathcal{P}$ of $[0,1]^d$ is a finite set of pairwise almost disjoint dyadic cubes in $I = [0,1]^d$ such that

$$\bigcup\{K : K \in \mathcal{P}\} = [0,1]^d.$$

We do not require that the cubes in $\mathcal{P}$ are of the same generation.

Lemma 4.4. *Let* $(\mathcal{P}_n)$ *be a sequence of dyadic partitions of* $[0,1]^d$ *whose meshes tend to* 0*, i.e.,*

$$\lim_{n\to\infty} \max\{\operatorname{diam} K : K \in \mathcal{P}_n\} = 0.$$

Let $u \in BV([0,1]^d)$. *For each* n*, define the function*

$$u_n = \sum_{K \in \mathcal{P}_n} \bar{u}_K \mathbb{1}_K, \qquad \text{where } \bar{u}_K = \frac{1}{|K|}\left(\int_K u\right).$$

Then $u_n \to u$ *weakly* in* $BV([0,1]^d)$.

Proof. In regard with Proposition 4.2, we need to prove that $u_n \to u$ in L^1 and that $\sup_n \|Du_n\|(\mathbb{R}^d) < \infty$. That $u_n \to u$ in L^1 is routinely proven by approximating u by a continuous function v on $[0,1]^d$ and using the uniform continuity of v.

Thus, we concentrate our efforts on the second part. By the continuity of the trace operator, there is a constant $C \geq 0$ such that

$$\int_{\partial((0,1)^d)} |\operatorname{tr}\varphi| d\mathcal{H}^{d-1} \leq C\left(\int_{(0,1)^d} |\varphi| + \|D\varphi\|\big((0,1)^d\big)\right),$$

for all $\varphi \in BV((0,1)^d)$. Then a scaling argument shows that

$$\int_{\partial K} |\operatorname{tr}\varphi| d\mathcal{H}^{d-1} \leq \frac{C}{\operatorname{diam} K}\int_{\operatorname{int} K} |\varphi| + C\|D\varphi\|(\operatorname{int} K)$$

whenever $K \subset [0,1]^d$ is a dyadic cube and $\varphi \in BV(\operatorname{int} K)$.

Now, we fix n. For each cube $K \in \mathcal{P}_n$, we call $u_{|\operatorname{int} K} \in BV(\operatorname{int} K)$ the restriction of u to int K and $v_K = (u - \bar{u}_K)\mathbb{1}_K \in BV([0,1]^d)$. By the extension theorem,

$$\|Dv_K\|(\mathbb{R}^d) = \|Du\|(\operatorname{int} K) + \int_{\partial K} |\operatorname{tr}(u_{|\operatorname{int} K} - \bar{u}_K)| \, d\mathcal{H}^{d-1}.$$

Hence, with the help of the preceding inequality and the Hölder inequality, we deduce that

$$\begin{aligned}\|Dv_K\|(\mathbb{R}^d) &\leq (1+C)\|Du\|(\operatorname{int} K) + \frac{C}{\operatorname{diam} K}\int_K |u - \bar{u}_K| \\ &\leq (1+C)\|Du\|(\operatorname{int} K) + \frac{C}{\sqrt{d}}\left(\int_K |u - \bar{u}_K|^{d/(d-1)}\right)^{1-1/d}.\end{aligned}$$

By the Sobolev–Poincaré inequality on a cube, one has

$$\left(\int_K |u - \bar{u}_K|^{d/(d-1)}\right)^{1-1/d} \leq C_P \|Du\|(\operatorname{int} K).$$

where C_P is the Poincaré constant, hence,

$$\|Dv_K\|(\mathbb{R}^d) \leq C'\|Du\|(\operatorname{int} K) \tag{4.2}$$

for some constant C' (depending solely on d). Finally, one notices that

$$\begin{aligned}\|Du_n\|(\mathbb{R}^d) &\leq \left\| D \sum_{K\in\mathcal{P}_n} v_K \right\|(\mathbb{R}^d) + \|Du\|(\mathbb{R}^d) \\ &\leq \sum_{K\in\mathcal{P}_n} \|Dv_K\|(\mathbb{R}^d) + \|Du\|(\mathbb{R}^d) \\ &\leq (1+C')\|Du\|(\mathbb{R}^d)\end{aligned}$$

In the last inequality, we used (4.2) and the fact that the interiors of the cubes $K \in \mathcal{P}_n$ are pairwise disjoint. ∎

Corollary 4.5. *The map $\mathcal{S}\colon SCH([0,1]^d) \to CH([0,1]^d)$ is injective.*

Proof. Let $\alpha \in SCH([0,1]^d)$ be in the kernel of $\mathcal{S}$ and $u \in BV([0,1]^d)$. Then, for every dyadic cube K, one has $\alpha(\mathbb{1}_K) = \mathcal{S}(\alpha)(K) = 0$. For any integer n, consider the collection $\mathcal{P}_n$ of all dyadic cubes of generation n and define the sequence of approximating functions $u_n = \sum_{K\in\mathcal{P}_n} \bar{u}_K \mathbb{1}_K$ as in the preceding lemma. Then $\alpha(u_n) = \sum_{K\in\mathcal{P}_n} \bar{u}_K \alpha(\mathbb{1}_K) = 0$, for all n. Moreover, $u_n \to u$ in the weak* topology. But α belongs to the predual of $BV([0,1]^d)$, consequently, $\alpha(u) = \lim \alpha(u_n) = 0$. This is true for any u, so, we infer that the kernel of $\mathcal{S}$ is trivial. ∎

5. The Faber–Schauder basis in $SCH\left([0,1]^d\right)$

5.1. Schauder basis

We recall that a sequence (e_n) with terms in an infinite-dimensional Banach space X is called a Schauder basis (or simply basis) of X if for each $x \in X$, there is a unique sequence (a_n) of scalars such that

$$x = \sum_{n=0}^{\infty} a_n e_n \tag{5.1}$$

strongly in X.

If the convergence in (5.1) is required to be weak instead of strong, then we say that (e_n) is a weak basis. The weak basis theorem (see for instance [15, Theorem 5.3] for a proof) asserts that a weak basis is in fact a Schauder basis.

5.2. Haar basis

An example of a Schauder basis is provided by the system of Haar functions described here. It is a basis in every $L^p([0,1])$ space, where $1 \le p < \infty$. Set the functions $h_{-1} = 1$,

$$h_{0,0}\colon x \in [0,1] \mapsto \begin{cases} 1 & \text{if } 0 \le x < 1/2 \\ -1 & \text{if } 1/2 \le x \le 1 \end{cases}$$

(one may extend $h_{0,0}$ to $\mathbb{R}$ by zero if needed in the subsequent formulae) and then, for every integer $n \ge 1$ and $k = 0, \ldots, 2^n - 1$,

$$h_{n,k}\colon x \mapsto 2^{n/2} h_{0,0}(2^n x - k) = \begin{cases} 2^{n/2} & \text{if } 2^{-n}k \le x < 2^{-n}k + 2^{-n-1} \\ -2^{n/2} & \text{if } 2^{-n}k + 2^{-n-1} \le x < 2^{-n}(k+1) \\ 0 & \text{otherwise.} \end{cases}$$

The Haar basis is the sequence $h_{-1}, h_{0,0}, h_{1,0}, h_{1,1}, h_{2,0}, h_{2,1}, h_{2,2}, h_{2,3}, \ldots$ (indices are ordered lexicographically). With our normalization choice this is an orthonormal basis in $L^2([0,1])$.

5.3. One-dimensional Faber–Schauder system

By definition, the Faber–Schauder functions are the indefinite integrals of the Haar functions, that is,

$$f_{-1}\colon x \mapsto \int_0^x h_{-1}, \qquad f_{n,k}\colon x \mapsto \int_0^x h_{n,k}$$

for $n \geq 0$ and $k = 0, \dots, 2^n - 1$. It was first proven in [18] that the f_{-1}, $f_{0,0}$, $f_{1,0}$, $f_{1,1}$, $f_{2,0}, \dots$ constitutes a Schauder basis of $C_0([0,1])$. We recall our claim that the map Π introduced in Subsection 4.3 is an isomorphism between $SCH([0,1])$ and $C_0([0,1])$. As $f_{-1} = \Pi(T_{h_{-1}})$ and $f_{n,k} = \Pi(T_{h_{n,k}})$ for all indices n, k, we can assert that $T_{h_{-1}}, T_{h_{0,0}}, T_{h_{1,0}}, T_{h_{1,1}}, T_{h_{2,0}}, \dots$ is a Schauder basis of $SCH([0,1])$.

5.4. Multidimensional Haar basis

Define the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and let A_d be the matrix of order 2^d that is the Kronecker product of A with itself d times, i.e., $A_1 = A$ and

$$A_{d+1} = \begin{pmatrix} A_d & A_d \\ A_d & -A_d \end{pmatrix}, \qquad \text{for } d \geq 1.$$

By induction, A_d is easily seen to be a symmetric matrix such that $(A_d)^2 = 2^d I_{2^d}$ (where I_{2^d} denotes the identity matrix of order 2^d). In other words, $2^{-d/2} A_d$ is an orthogonal matrix. Subsequently, the entries of the matrix A_d are written $(A_d)_{r,\ell}$, where the row and column numbers r and ℓ range over $\{0, \dots, 2^d - 1\}$.

For all $n \geq 0$, we let $K_{n,k}$, $k = 0, \dots, 2^{nd} - 1$, be the collection of all dyadic cubes in $[0,1]^d$ of side-length 2^{-n}. We further require that, for all indices n and k, the cubes $K_{n+1,2^d k}, K_{n+1,2^d k+1}, \dots, K_{n+1,2^d k+2^d-1}$ are the 2^d subcubes of $K_{n,k}$ of side $2^{-(n+1)}$.

We are now able to build the Haar basis. The first Haar function is the exceptional one

$$g_{-1} \colon (x, y) \in [0,1]^d \mapsto 1.$$

Then, we define, for all $n \geq 0$, $k \in \{0, \dots, 2^{nd} - 1\}$ and $r \in \{1, \dots, 2^d - 1\}$:

$$g_{n,k,r} = 2^{nd/2} \sum_{\ell=0}^{2^d-1} (A_d)_{r,\ell} \mathbb{1}_{K_{n+1,2^d k+\ell}}. \tag{5.2}$$

We refer to n, k and r as the generation number, the cube number and the type number of $g_{n,k,r}$. By construction, $g_{n,k,r} = 0$ almost everywhere outside the cube $K_{n,k}$. In fact, the support of $g_{n,k,r}$ is $K_{n,k}$ and the support of g_{-1} is $K_{0,0}$. We also note that the average value of $g_{n,k,r}$ is 0 (this is because the r-th line of A_d is orthogonal to the zeroth line, which is filled with ones).

The Haar basis is (indices are ordered lexicographically):

$$g_{-1}, g_{0,0,1}, \dots, g_{0,0,2^d-1}, g_{1,0,1}, \dots, g_{1,2^d-1,2^d-1}, g_{2,0,1}, \dots$$

Claim. *The Haar functions are orthonormal in* $L^2([0,1]^d)$.

Those functions are indeed appropriately normalized; and the cases worth considering in proving that these functions are pairwise orthogonal are:

- The case of two functions $g_{n,k,r}$ and $g_{n',k',r'}$ with $n < n'$: if $K_{n',k'} \subset K_{n,k}$, then

$$\int g_{n,k,r} g_{n',k',r'} = \pm 2^{nd/2} \int g_{n',k',r'} = 0.$$

  Otherwise, $|K_{n',k'} \cap K_{n,k}| = 0$, thus, $g_{n,k,r} g_{n',k',r'} = 0$ a.e., which implies that $g_{n,k,r}$ and $g_{n',k',r'}$ are orthogonal.
- The case of two functions $g_{n,k,r}$ and $g_{n,k,r'}$ of the same generation and same cube numbers but different type numbers $r \neq r'$:

$$\int g_{n,k,r} g_{n',k',r'} = \frac{1}{2^d} \sum_{\ell=0}^{2^d-1} (A_d)_{r,\ell} (A_d)_{r',\ell} = 0.$$

  Here, we use that the matrix $2^{-d/2} A_d$ is orthogonal.

Claim. *The indicator function* $\mathbb{1}_{K_{n,k}}$ *is a linear combination of* g_{-1} *and the functions* $g_{n',k',r}$ *corresponding to generation numbers* $n' \in \{0, 1, \dots, n-1\}$ *and cube numbers* $k' = \lfloor 2^{-d(n-n')} k \rfloor$ *(this is equivalent to* $K_{n',k'} \supset K_{n,k}$*), and arbitrary type numbers* $r \in \{1, \dots, 2^d - 1\}$.

This claim is proven by induction on n. The base case is straightforward as $\mathbb{1}_{K_{0,0}} = g_{-1}$. Regarding the induction step, we note that

$$\begin{pmatrix} 2^{nd/2} \mathbb{1}_{K_{n,k}} \\ g_{n,k,1} \\ \vdots \\ g_{n,k,2^d-1} \end{pmatrix} = 2^{nd/2} A_d \begin{pmatrix} \mathbb{1}_{K_{n+1,2^d k}} \\ \mathbb{1}_{K_{n+1,2^d k+1}} \\ \vdots \\ \mathbb{1}_{K_{n+1,2^d k+2^d-1}} \end{pmatrix} \tag{5.3}$$

The conclusion follows from the invertibility of A_d.

The main result of this section is that, by antidifferentiating (i.e., by applying T to) the Haar functions in the space of strong charge functionals, we obtain a Schauder basis of $SCH([0,1]^d)$, see Theorem 5.1. In view of Subsection 5.3, this basis is analogous to the 1-dimensional Faber–Schauder basis. It may be worth noting that, like bases of wavelets, the Faber–Schauder basis we obtain is "localized" in the sense that the supports of its members are controlled; in fact, supp $T_{g_{n,k,r}} = K_{n,k}$.

We warn the reader that a Schauder basis does not need to be unconditional, that is, the order of summation matters. In fact, the 1-dimensional Faber–Schauder basis of $C_0([0,1])$ is not unconditional, see [2].

Theorem 5.1. *The sequence $T_{g_{-1}}, T_{g_{0,0,1}}, T_{g_{0,0,2}}, \dots$ is a Schauder basis of the space $SCH([0,1]^d)$, with respect to which each strong charge functional α is decomposed as follows:*

$$\alpha = \alpha(g_{-1})T_{g_{-1}} + \alpha(g_{0,0,1})T_{g_{0,0,1}} + \alpha(g_{0,0,2})T_{g_{0,0,2}} + \cdots .$$

(The convergence occurs, of course, in $BV([0,1]^d)^*$.)

Proof. First, we prove the uniqueness part in the definition of a Schauder basis. Suppose a strong charge functional α has a decomposition

$$\begin{aligned} \alpha = a_{-1}T_{g_{-1}} + a_{0,0,1}T_{g_{0,0,1}} + a_{0,0,2}T_{g_{0,0,2}} + \cdots + a_{0,0,2^d-1}T_{g_{0,0,2^d-1}} \\ + a_{1,0,1}T_{g_{1,0,1}} + \cdots + a_{1,2^d-1,2^d-1}T_{g_{1,2^d-1,2^d-1}} + \cdots \end{aligned} \tag{5.4}$$

Applying (5.4) at $g_{n,k,r}$, we get $a_{n,k,r} = \alpha(g_{n,k,r})$ by the orthonormality of the Haar functions. Likewise, $a_{-1} = \alpha(g_{-1})$.

We turn to the existence. We fix a strong charge functional $\alpha \in SCH([0,1]^d)$. Our goal is to prove that (5.4) holds weakly, when the coefficients a_{-1} and $a_{n,k,r}$ are taken as in the first part of the proof. The weak basis theorem then implies the desired result. Therefore, we consider a function $u \in BV([0,1]^d)$ and wish to prove that

$$\begin{aligned} &\alpha(u) \\ &= a_{-1}T_{g_{-1}}(u) + a_{0,0,1}T_{g_{0,0,1}}(u) + a_{0,0,2}T_{g_{0,0,2}}(u) + \cdots + a_{0,0,2^d-1}T_{g_{0,0,2^d-1}}(u) \\ &\quad + a_{1,0,1}T_{g_{1,0,1}}(u) + \cdots + a_{1,2^d-1,2^d-1}T_{g_{1,2^d-1,2^d-1}}(u) + \cdots \end{aligned} \tag{5.5}$$

For all n, k, we define the truncated sum

$$\alpha_{n,k} = a_{-1}T_{g_{-1}} + \cdots + a_{n,k,2^d-1}T_{g_{n,k,2^d-1}} .$$

We prove that $\alpha_{n,k}(u) \to \alpha(u)$ (as usual, we equip the set of legal couples of indices (n,k) with the lexicographical order). The sequence $(\alpha_{n,k}(u))$ is merely a subsequence of the sequence of partial sums in (5.5) and we deal with this issue at the end of this proof. Define

$$\mathcal{G}_{n,k} = \left\{g_{-1}, g_{0,0,1}, \dots, g_{n-1,2^{(n-1)d},2^d-1}, g_{n,0,1}, \dots, g_{n,k,2^d-1}\right\}$$

so that

$$\alpha_{n,k} = \sum_{g \in \mathcal{G}_{n,k}} \alpha(g)T_g .$$

Define also the dyadic partition of $[0,1]^d$

$$\mathcal{P}_{n,k} = \left\{K_{n,k'} : k+1 \le k' \le 2^{nd}-1\right\} \cup \left\{K_{n+1,k'} : 0 \le k' \le 2^d k + 2^d - 1\right\}.$$

Reasoning as in the first part of the proof, $\alpha_{n,k}(g) = \alpha(g)$ for all $g \in \mathcal{G}_{n,k}$. Also, with the help of the second claim of Subsection 5.4, we have

$$\operatorname{span} \mathcal{G}_{n,k} \supset \operatorname{span}\{\mathbb{1}_K : K \in \mathcal{P}_{n,k}\}. \tag{5.6}$$

The two points above guarantee that $\alpha(\mathbb{1}_K) = \alpha_{n,k}(\mathbb{1}_K)$ for all $K \in \mathcal{P}_{n,k}$.

Next, we define

$$u_{n,k} = \sum_{K \in \mathcal{P}_{n,k}} \frac{1}{|K|}\left(\int_K u\right)\mathbb{1}_K.$$

As each function $g \in \mathcal{G}_{n,k}$ is constant a.e. on the dyadic cubes in $\mathcal{P}_{n,k}$, we clearly have $T_g(u) = T_g(u_{n,k})$. Therefore,

$$\begin{aligned}\alpha_{n,k}(u) &= \alpha_{n,k}(u_{n,k}) \\ &= \sum_{K \in \mathcal{P}_{n,k}} \frac{1}{|K|}\left(\int_K u\right)\alpha_{n,k}(\mathbb{1}_K) \\ &= \sum_{K \in \mathcal{P}_{n,k}} \frac{1}{|K|}\left(\int_K u\right)\alpha(\mathbb{1}_K) \\ &= \alpha(u_{n,k}).\end{aligned}$$

Lemma 4.4 applies, therefore, $u_{n,k} \to u$ weakly* by Proposition 4.2, from which we deduce that $\alpha(u_{n,k}) \to \alpha(u)$.

To finish the proof, we ought to show that the sequence of partial sums in the right-hand side of (5.5) tends to 0. Any partial sum in this sequence differs from some $\alpha_{n,k}(u)$ considered above by at most $2^d - 1$ terms of the type

$$a_{n,k,r} T_{g_{n,k,r}}(u) = \alpha(g_{n,k,r}) \int g_{n,k,r} u = \alpha\left(\left(\int g_{n,k,r} u\right) g_{n,k,r}\right).$$

In other words, we must show that for all $\alpha \in SCH([0,1]^d)$ and all $u \in BV([0,1]^d)$, $a_{n,k,r} T_{g_{n,k,r}}(u) \to 0$. Fixing $u \in BV([0,1]^d)$, this is equivalent to showing that the sequence $((\int g_{n,k,r} u) g_{n,k,r})$ weakly* converges to 0 (with respect to the duality set forth in Theorem 4.1), according to the equation above. By Proposition 4.2, this is in turn equivalent to showing that

$$\left\|\left(\int g_{n,k,r} u\right) g_{n,k,r}\right\|_1 \to 0 \tag{5.7}$$

and

$$\sup\left\|\left(\int g_{n,k,r} u\right) g_{n,k,r}\right\|_{BV} < \infty. \tag{5.8}$$

It is useful to note that, for all d:

$$\|g_{n,k,r}\|_1 = 2^{nd/2}|K_{n,k}| = 2^{nd/2}2^{-nd} = 2^{-nd/2}$$

and a similar computation yields

$$\|g_{n,k,r}\|_d = 2^{nd/2}|K_{n,k}|^{1/d} = 2^{nd/2}2^{-n} = 2^{n(d/2-1)}.$$

Consequently,

$$\left\| \left(\int g_{n,k,r} u \right) g_{n,k,r} \right\|_1 \leq \|u\|_{d/(d-1)} \|g_{n,k,r}\|_d \|g_{n,k,r}\|_1 = 2^{-n} \|u\|_{d/(d-1)} \to 0.$$

Accordingly, (5.7) holds for all d. Next, one observes after (5.2) that $g_{n,k,r}$ has bounded variation and

$$\|Dg_{n,k,r}\|(\mathbb{R}^d) \leq 2^{nd/2} 2^d 2d \left(\frac{1}{2^{n+1}} \right)^{d-1} \leq C 2^{n(1-d/2)},$$

from which we infer that

$$\left\| \left(\int g_{n,k,r} u \right) g_{n,k,r} \right\|_{BV} \leq C \|g_{n,k,r}\|_d \|u\|_{d/(d-1)} \|Dg_{n,k,r}\|(\mathbb{R}^d) \leq C \|u\|_{d/(d-1)}.$$

This proves (5.8) and concludes the proof. ■

5.5. Remark on charge functionals

We briefly outline how it is possible to define a notion of charge functional similar to that of strong charge functional, thereby endowing $CH(A)$ with a Banach space structure. First, define the space $BV_\infty(A)$ of measurable functions $u: \mathbb{R}^d \to \mathbb{R}$ that are essentially bounded, that vanish almost everywhere outside of A, and that have bounded variation. This space is normed by $\|u\|_{BV_\infty} = \|u\|_\infty + \|Du\|(\mathbb{R}^d)$. We define the linear map $T: L^1(A) \to BV_\infty(A)^* : f \mapsto T_f$ by

$$T_f(u) = \int_A fu, \qquad f \in L^1(A),\ u \in BV_\infty(A).$$

The space of charge functionals is the closure of $T(L^1(A))$ in $BV_\infty(A)^*$. This space is a canonical predual of $BV_\infty(A)$. Charge functionals are in bijection with charges on A: if α is a charge functional, then $B \mapsto \alpha(\mathbb{1}_B)$ is a charge. In case $A = [0,1]^d$, the image of the Haar basis of $L^1([0,1]^d)$ under T is a Schauder basis of the space of charge functionals, similar to the Faber–Schauder basis of $SCH([0,1]^d)$.

6. Criteria for strong chargeability

Let $f \in C_0([0,1]^d)$, we define $\lambda_{-1}(f) = \Delta_f([0,1]^d) = f(1,\ldots,1)$ (the last equality is a consequence of (3.4)) and, for all relevant indices n, k and r,

$$\lambda_{n,k,r}(f) = 2^{nd/2} \sum_{\ell=0}^{2^d-1} (A_d)_{r,\ell} \Delta_f(K_{n+1,2^d k+\ell}). \tag{6.1}$$

The maps λ_{-1} and $\lambda_{n,k,r}$ so defined are continuous linear functionals on $C_0([0,1]^d)$. It is clear that if f is strongly chargeable, then Δ_f is by definition a strong charge and $\lambda_{n,k,r}(f)$ are the coefficients of the strong charge functional $\mathcal{S}^{-1}(\Delta_f)$ in the Faber–Schauder basis (see Paragraph 4.4 for the definition of $\mathcal{S}$). From this observation, we derive Theorem 6.1, that equates strong chargeability with the convergence of a series in $SCH([0,1]^d)$.

Theorem 6.1. *A function $f \in C_0([0,1]^d)$ is strongly chargeable if and only if the Faber–Schauder series*

$$\lambda_{-1}(f) T_{g_{-1}} + \lambda_{0,0,1}(f) T_{g_{0,0,1}} + \cdots \tag{6.2}$$

converges in $SCH([0,1]^d)$.

Proof. The direct implication follows from the arguments above. Conversely, suppose the series (6.2) is convergent and denote its sum by α. Then

$$\alpha(g_{-1}) = \alpha(\mathbb{1}_{K_{0,0}}) = \lambda_{-1}(f) = \Delta_f(K_{0,0}) \text{ and } \alpha(g_{n,k,r}) = \lambda_{n,k,r}(f)$$

by Theorem 5.1.

We now prove by induction on n that $\alpha(\mathbb{1}_{K_{n,k}}) = \Delta_f(K_{n,k})$ for all indices $k = 0,\ldots,2^{nd}-1$. The base case $n=0$ is already treated. Suppose the result is valid for a generation number $n \geq 0$. Fix a cube number k. By applying α to (5.3), we get

$$\begin{pmatrix} 2^{nd/2}\Delta_f(K_{n,k}) \\ \lambda_{n,k,1}(f) \\ \vdots \\ \lambda_{n,k,2^d-1}(f) \end{pmatrix} = 2^{nd/2} A_d \begin{pmatrix} \alpha(\mathbb{1}_{K_{n+1,2^d k}}) \\ \alpha(\mathbb{1}_{K_{n+1,2^d k+1}}) \\ \vdots \\ \alpha(\mathbb{1}_{K_{n+1,2^d k+2^d-1}}) \end{pmatrix}$$

On the other hand, from the definition (6.1) of the $\lambda_{n,k,r}$ functionals, we have

$$\begin{pmatrix} 2^{nd/2}\Delta_f(K_{n,k}) \\ \lambda_{n,k,1}(f) \\ \vdots \\ \lambda_{n,k,2^d-1}(f) \end{pmatrix} = 2^{nd/2} A_d \begin{pmatrix} \Delta_f(K_{n+1,2^d k}) \\ \Delta_f(K_{n+1,2^d k+1}) \\ \vdots \\ \Delta_f(K_{n+1,2^d k+2^d-1}) \end{pmatrix}$$

As A_d is invertible, this ends the proof by induction.

Let $(x_1, \dots, x_d) \in [0,1]^d$ be a point whose coordinates are dyadic numbers. Then, we can write $K = \prod_{i=1}^{d} [0, x_i]$ as a finite union of almost disjoint dyadic cubes. Using the result above and finite additivity, we derive that $\mathcal{S}^{-1}(\alpha)(K) = \alpha(\mathbb{1}_K) = \Delta_f(K)$. Thus, by (3.4), $\mathcal{S}^{-1}(\alpha)(K) = f(x_1, \dots, x_d)$. When $(x_1, \dots, x_d) \in [0,1]^d$ is arbitrary, we use a simple density argument (and the continuity of f and α) to justify that $\mathcal{S}^{-1}(\alpha)(K) = f(x_1, \dots, x_d)$ holds as well. Therefore, f is chargeable, by Proposition 3.2, and $\Delta_f = \mathcal{S}^{-1}(\alpha)$, which ensures that f is strongly chargeable. ■

The 1-dimensional Faber–Schauder functions $f_{n,k}$ (see Subsection 5.3) have the pleasant property of having localized supports. This helps estimate the norm of a linear combination of $f_{n,k}$ functions, for a fixed generation number $n \geq 0$. Indeed, we clearly have

$$\left\| \sum_{k=0}^{2^n-1} a_k f_{n,k} \right\|_\infty = \frac{1}{2^{n/2+1}} \max_{0 \leq k \leq 2^n-1} |a_k|.$$

The following Proposition 6.2 is a subtler multidimensional analogue. Together with Theorem 6.1, it allows to state strong chargeability or non-strong chargeability criteria of practical use in Corollary 6.3.

Proposition 6.2. *There is a positive constant C such that for all $n \geq 0$ and scalars $(a_{k,r})$, we have*

$$\frac{1}{2^{n(d/2+1)}C} \max_{1 \leq r \leq 2^d-1} \sum_{k=0}^{2^{nd}-1} |a_{k,r}| \leq \left\| \sum_{k=0}^{2^{nd}-1} \sum_{r=1}^{2^d-1} a_{k,r} T_{g_{n,k,r}} \right\|$$
$$\leq C 2^{n(d/2-1)} \max_{k,r} |a_{k,r}|.$$

Proof. First, we prove the upper bound. Let $u \in BV([0,1]^d)$. For any cube number k and type number r, we have

$$a_{k,r} T_{g_{n,k,r}}(u) = a_{k,r} \int g_{n,k,r} u = a_{k,r} \int g_{n,k,r}(u - \bar{u}_{K_{n,k}})$$

where $\bar{u}_{K_{n,k}}$ is the average value of u on the $K_{n,k}$. By the Hölder inequality and the Poincaré inequality, we obtain that

$$\left| a_{k,r} T_{g_{n,k,r}}(u) \right| \leq |a_{k,r}| \|g_{n,k,r}\|_d \left(\int_{K_{n,k}} |u - \bar{u}_{K_{n,k}}|^{d/(d-1)} \right)^{1-1/d}$$
$$\leq C_P 2^{n(d/2-1)} |a_{k,r}| \, \|Du\|(\operatorname{int} K_{n,k})$$

where C_P is the Poincaré constant for d-dimensional cubes. It follows that

$$\begin{aligned}\left(\sum_{k=0}^{2^{nd}-1}\sum_{r=1}^{2^d-1} a_{k,r} T_{g_{n,k,r}}\right)(u) \\ &\le (2^d-1)2^{n(d/2-1)} C_P\left(\max_{k,r}|a_{k,r}|\right)\sum_{k=0}^{2^{nd}-1}\|Du\|(\operatorname{int} K_{n,k}) \\ &\le (2^d-1)2^{n(d/2-1)} C_P\left(\max_{k,r}|a_{k,r}|\right)\|u\|_{BV}.\end{aligned}$$

Next, we turn to the lower bound. Let $r \in \{1,\dots,2^d-1\}$. Define

$$u = \sum_{k=0}^{2^{nd}-1} \varepsilon_{k,r} g_{n,k,r}$$

where $\varepsilon_{k,r} \in \{-1,1\}$ are chosen so that $\varepsilon_{k,r}a_{k,r} = |a_{k,r}|$. First, we have $\|u\|_1 = 2^{nd/2}$ and

$$\begin{aligned}\|Du\|(\mathbb{R}^d) &\le \sum_{k=0}^{2^{nd}-1}\|Dg_{n,k,r}\|(\mathbb{R}^d) \\ &\le 2^{nd}2^{nd/2}\frac{\|Dg_{0,0,r}\|(\mathbb{R}^d)}{2^{n(d-1)}} \\ &\le 2^{n(d/2+1)}\max\big(\|Dg_{0,0,1}\|(\mathbb{R}^d),\dots,\|Dg_{0,0,2^d-1}\|(\mathbb{R}^d)\big).\end{aligned}$$

Hence, $\|u\|_{BV} \le 2^{n(d/2+1)}C'$, for some constant C'. As the functions $(g_{n,k,r'})$ are pairwise orthogonal in $L^2([0,1]^d)$, we have

$$\left(\sum_{k=0}^{2^{nd}-1}\sum_{r'=1}^{2^d-1} a_{k,r'} T_{g_{n,k,r'}}\right)(u) = \sum_{k=0}^{2^{nd}-1}|a_{k,r}|.$$

Hence, we infer

$$\sum_{k=0}^{2^{nd}-1}|a_{k,r}| \le 2^{n(d/2+1)}C'\left\|\sum_{k=0}^{2^{nd}-1}\sum_{r'=1}^{2^d-1} a_{k,r'} T_{g_{n,k,r'}}\right\|. \qquad \blacksquare$$

Corollary 6.3. *Let $f \in C_0([0,1]^d)$.*

(A) *The condition*

$$\limsup_{n\to\infty}\frac{1}{2^{n(d/2+1)}}\max_{1\le r\le 2^d-1}\sum_{k=0}^{2^{nd}-1}|\lambda_{n,k,r}(f)| > 0$$

implies that f is not strongly chargeable.

(B) *The condition*

$$\sum_{n=0}^{\infty} 2^{n(d/2-1)} \max_{k,r} |\lambda_{n,k,r}(f)| < \infty$$

implies that f is strongly chargeable.

Proof. Condition (A) implies that the sequence of partial sums in (6.2) is not Cauchy. Condition (B) implies that it is. ■

7. Sample paths of the Brownian sheet

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and assume it is sufficiently large for the processes below to be defined on it. We recall that the Brownian sheet is a Gaussian centered random process $\{W_{t_1,\ldots,t_d} : (t_1, \ldots, t_d) \in [0,1]^d\}$ with covariance function

$$\Gamma\left((t_1, \ldots, t_d), (t'_1, \ldots, t'_d)\right) = \prod_{i=1}^{d} \min(t_i, t'_i).$$

Such a process exists and one may construct it the following way: Start from a Gaussian noise G from $L^2([0,1]^d)$ to a (centered) Gaussian space E, i.e., an isometry from $L^2([0,1]^d)$ to a closed linear subspace $E \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ which contains only centered Gaussian variables. We refer to [13, 1.4] for the existence of Gaussian noises. One then sets

$$W_{t_1,\ldots,t_d} = G(\mathbb{1}_{[0,t_1]\times\cdots\times[0,t_d]}).$$

This definition leads to the correct covariance function, since

$$\mathbb{E}(W_{t_1,\ldots,t_d} W_{t'_1,\ldots,t'_d}) = \int \mathbb{1}_{[0,t_1]\times\cdots\times[0,t_d]} \mathbb{1}_{[0,t'_1]\times\cdots\times[0,t'_d]} = \prod_{i=1}^{d} \min(t_i, t'_i).$$

Next, we let the reader prove that $\Delta_W(K) = G(\mathbb{1}_K)$ almost surely, for a rectangle $K \subset [0,1]^d$, where $\Delta_W(K)$ is defined as in (3.4). The Brownian sheet admits a continuous modification, by a standard application of Kolmogorov's continuity theorem [12, Theorem 3.23], so we may suppose that $(t_1, \ldots, t_d) \mapsto W_{t_1,\ldots,t_d}(\omega)$ is an element of $C_0([0,1]^d)$ for all $\omega \in \Omega$.

As an application of the results from the previous section, we prove the following theorem. A generalized result is obtained in Section 10 using a more probabilistic argument.

Theorem 7.1. *The sample paths of the Brownian sheet are almost surely not strongly chargeable (for $d \geq 2$).*

Proof. First, we note that $\lambda_{-1}(W) = W_{1,\dots,1} = G(\mathbb{1}_{[0,1]^d}) = G(g_{-1})$. By using (6.1), for n, k we have, almost surely,

$$\begin{aligned}\lambda_{n,k,r}(W) &= 2^{nd/2} \sum_{\ell=0}^{2^d-1} (A_d)_{r,\ell} \Delta_W(K_{n+1,2^d k+\ell}) \\ &= 2^{nd/2} \sum_{\ell=0}^{2^d-1} (A_d)_{r,\ell} G(\mathbb{1}_{K_{n+1,2^d k+\ell}}) \\ &= G(g_{n,k,r}),\end{aligned}$$

by (5.2). As the sequence of Haar functions $g_{-1}, g_{0,0,1}, \dots$ is orthonormal and G is an isometry, we deduce that the random variables $\lambda_{-1}(W), \lambda_{0,0,1}(W), \dots$ are pairwise uncorrelated and follow the standard Gaussian distribution. Since they are jointly Gaussian, we infer that they are independent. For each integer $n \geq 0$, define the random variables

$$T_n = \frac{1}{2^{nd}} \sum_{k=0}^{2^{nd}-1} |\lambda_{n,k,1}(W)| \text{ and } S_n = 2^{n(d/2-1)} T_n = \frac{1}{2^{n(d/2+1)}} \sum_{k=0}^{2^{nd}-1} |\lambda_{n,k,1}(W)|.$$

Each random variable $|\lambda_{n,k,1}(W)|$ follows a half-normal distribution of mean $\sqrt{2/\pi}$ and variance $1 - 2/\pi$. By independence,

$$\left\| T_n - \sqrt{\frac{2}{\pi}} \right\|_2^2 = \operatorname{Var} T_n = \frac{1}{2^{nd}} \left(1 - \frac{2}{\pi}\right) \to 0.$$

We infer the existence of a subsequence (T_{n_k}) that converges almost surely to $\sqrt{2/\pi}$. Hence, $\limsup S_n = \infty$ if $d \geq 3$ and $\limsup S_n \geq \sqrt{2/\pi}$ if $d = 2$. In either case, one has $\limsup S_n > 0$ almost surely, and thus, we may conclude with the help of Corollary 6.3(A). ∎

We could enhance the previous result by demonstrating that the sample paths of the Brownian sheet are almost surely non-chargeable (though this result is still less comprehensive than Theorem 10.5). This could be accomplished by utilizing the Faber–Schauder basis of the space of charge functionals (as seen in Subsection 5.5) rather than that of $SCH([0,1]^d)$. The approach mirrors the methods employed to establish Theorem 7.1.

8. Hölder strong charges

8.1. Definition

The 1-dimensional Faber–Schauder basis serves as a sort of wavelet basis for the space $C_0([0,1])$. Consequently, it provides a way to assess the regularity of a function in $f \in C_0([0,1])$ by examining the rate at which the coefficients from the Faber–Schauder decomposition of f approach 0.

The existence of a Faber–Schauder-type basis in the space $SCH([0,1]^d)$ supports the idea that some further notions of regularity can be formulated for strong charges. In this section, we expound upon a theory of Hölder strong charges.

At first, it is worth noting that if μ is a strong charge on $[0,1]^d$, and $\alpha = \mathcal{S}^{-1}(\mu)$ represents its associated strong charge functional (see Paragraph 4.4 for the definition of $\mathcal{S}$), then the coefficients $a_{n,k,r}$ of α in the Faber–Schauder decomposition depend solely on the values on μ over dyadic cubes, since

$$a_{n,k,r} = \alpha(g_{n,k,r}) = 2^{nd/2} \sum_{\ell=0}^{2^d-1} (A_d)_{r,\ell}\, \mu(K_{n+1,2^d k+\ell}).$$

Taking this into consideration, we introduce the following definition. Let $\gamma \in \left(\frac{d-1}{d}, 1\right)$ and μ be a strong charge on $[0,1]^d$. We say that μ is *γ-Hölder* whenever there is a constant $C \geq 0$ such that $|\mu(K)| \leq C|K|^\gamma$ for all dyadic cubes $K \subset [0,1]^d$.

The reader may find it surprising that we have imposed the restriction

$$\gamma > (d-1)/d$$

on the Hölder exponent. The reason is that there seems to be no meaningful theory for exponents less than or equal to $(d-1)/d$. This is foreshadowed in Proposition 8.1 that provides examples of Hölder charges only for exponents greater than $(d-1)/d$.

8.2. Hölder strong charges in dimension $d = 1$

In dimension $d = 1$, Hölder exponents are allowed to range over $\gamma \in (0,1)$. We claim that a charge μ is γ-Hölder if and only if its associated continuous function $v\colon x \mapsto \mu([0,x])$ is γ-Hölder. This can be proved by elementary methods. In fact, it is well known that a continuous function $v\colon [0,1] \to \mathbb{R}$ is γ-Hölder continuous if and only if there is a constant $C \geq 0$ such that

$$\left| v\left(\frac{k+1}{2^n}\right) - v\left(\frac{k}{2^n}\right) \right| \leq \frac{C}{2^{n\gamma}}$$

for all integers $n \geq 0$ and $0 \leq k \leq 2^n - 1$, see for example [13, Lemma 2.10].

In dimension $d \geq 2$, there is no corresponding result. The Hölder character of strong charges is a new regularity notion that has no counterpart for functions. In fact, it is possible for a γ-Hölder continuous function in $C_0([0,1]^d)$ to be $\tilde{\gamma}$-Hölder strongly chargeable, where $\tilde{\gamma} > \gamma$. The sample paths of the fractional Brownian sheet may exhibit this phenomenon, as discussed in Section 10. For such functions, adopting the point of view of charges leads to a gain in regularity.

Proposition 8.1. *Let $f \in C_0([0,1]^d)$ and $\frac{d-1}{d} < \gamma < 1$. Suppose that there is a constant $C \geq 0$ such that $|\Delta_f(K)| \leq C|K|^\gamma$ for all dyadic cubes K. Then f is γ-Hölder strongly chargeable.*

Proof. By (6.1), we estimate

$$|\lambda_{n,k,r}(f)| \leq C2^{nd/2}2^d\left(\frac{1}{2^{(n+1)d}}\right)^\gamma.$$

Thus,

$$\sum_{n=0}^{\infty} 2^{n(d/2-1)} \max_{k,r} |\lambda_{n,k,r}(f)| \leq C2^{d(1-\gamma)} \sum_{n=0}^{\infty} 2^{n(d-1-d\gamma)} < \infty,$$

because $d-1-d\gamma < 0$. By Corollary 6.3(B), we conclude that f is strongly chargeable. It now follows from the hypothesis that the strong charge Δ_f is γ-Hölder. ∎

9. A Kolmogorov-type chargeability theorem for stochastic processes

Theorem 9.1 (Kolmogorov-type chargeability theorem). *Let X be a random process indexed on $[0,1]^d$ with continuous sample paths. Let $q > 0, C \geq 0, \delta > 0$ such that*

$$\frac{d-1}{d} < \frac{\delta}{q} \leq 1 \tag{9.1}$$

and

$$\mathbb{E}\big(|\Delta_X K|^q\big) \leq C|K|^{1+\delta}$$

for all dyadic cubes K, where $\Delta_X K$ is the (random) increment of X over K, defined by (3.4). Then, a.s., X is γ-Hölder strongly chargeable for any $\frac{d-1}{d} < \gamma < \delta/q$.

Proof. It suffices to fix a Hölder exponent $\frac{d-1}{d} < \gamma < \frac{\delta}{q}$ and prove the apparently weaker statement that, a.s., X is γ-Hölder strongly chargeable.

For any integer $p \geq 0$, we let $\mathcal{K}_p$ be the set of dyadic cubes of generation p. For such a dyadic cube K, we have

$$\mathbb{P}\big(|\Delta_X K| \geq |K|^\gamma\big) \leq \frac{1}{|K|^{q\gamma}}\mathbb{E}\big(|\Delta_X K|^q\big) \leq C\left(\frac{1}{2^{pd}}\right)^{1+\delta-q\gamma}.$$

Hence,

$$\mathbb{P}\left(\exists K \in \mathcal{K}_p : |\Delta_X K| \geq |K|^\gamma\right) \leq C\left(\frac{1}{2^{pd}}\right)^{\delta - q\gamma},$$

as $\#\mathcal{K}_p = 2^{pd}$. It follows that

$$\sum_{p=0}^{\infty} \mathbb{P}\left(\exists K \in \mathcal{K}_p : |\Delta_X K| \geq |K|^\gamma\right) < \infty.$$

By the Borel–Cantelli lemma, we have, a.s.,

$$\sup_{p\geq 0} \sup_{K\in\mathcal{K}_p} \frac{|\Delta_X(K)|}{|K|^\gamma} < \infty.$$

We conclude from Proposition 8.1 that the sample paths of X are almost surely γ-Hölder chargeable. ■

10. Sample paths of the fractional Brownian sheet

For any $h \in (0,1)$, $t, t' \geq 0$, we define

$$\phi^h(t,t') = \frac{|t|^{2h} + |t'|^{2h} - |t-t'|^{2h}}{2}.$$

Let $H_1, \ldots, H_d \in (0,1)$. The *fractional Brownian sheet* of Hurst multiparameter $H = (H_1, \ldots, H_d)$ is a Gaussian centered random process $\{W^H_{t_1,\ldots,t_d} : (t_1, \ldots, t_d) \in [0,1]^d\}$ of covariance function

$$\Gamma\left((t_1, \ldots, t_d), (t'_1, \ldots, t'_d)\right) = \prod_{i=1}^{d} \phi^{H_i}(t_i, t'_i).$$

When $H_1 = \cdots = H_d = 1/2$, we recover the Brownian sheet from Section 7. Again, we suppose that the sample paths of the fractional Brownian sheet are continuous. This is possible because Kolmogorov's continuity theorem applies to this more general case as well. The mean of the Hurst coefficients is written

$$\bar{H} = \frac{H_1 + \cdots + H_d}{d}.$$

This parameter is crucial to determine whether the sample paths of the fractional Brownian sheet are chargeable or not. Theorems 10.2 and 10.5 describe in detail the behavior of these sample paths. We need to compute the variance of the increments in Lemma 10.1. Though this is well known, we include this calculation for the reader's convenience.

Lemma 10.1 (Increments of the fractional Brownian sheet). *Let*

$$K = \prod_{i=1}^{d} [a_i, b_i] \subset [0,1]^d.$$

Then $\Delta_{W^H}(K)$ is a centered Gaussian random variable of variance

$$\prod_{i=1}^{d} |b_i - a_i|^{2H_i}.$$

In particular, this variance is $|K|^{2\bar{H}}$ when K is a cube.

Proof. The random variable $\Delta_{W^H}(K)$ is clearly Gaussian of mean zero, so, we need only compute its variance. We proceed by induction on d. If $d = 1$ then, clearly,

$$\mathbb{E}\big((W_{t_1}^{H_1} - W_{t_1'}^{H_1})^2\big) = \phi^{H_1}(t_1, t_1) - 2\phi^{H_1}(t_1, t_1') + \phi^{H_1}(t_1', t_1') = |t - t'|^{2H_1}.$$

Suppose now that $d \geq 2$ and that the result holds for $d - 1$. We define a process

$$\widetilde{W}_{t_1,\dots,t_{d-1}} = W^H_{t_1,\dots,t_{d-1},b_d} - W^H_{t_1,\dots,t_{d-1},a_d}.$$

Its covariance function is

$$\begin{aligned}
&\widetilde{\Gamma}\big((t_1,\dots,t_{d-1}),(t_1',\dots,t_{d-1}')\big)\\
&\quad = \mathbb{E} W^H_{t_1,\dots,t_{d-1},b_d} W^H_{t_1',\dots,t_{d-1}',b_d} - \mathbb{E} W^H_{t_1,\dots,t_{d-1},b_d} W^H_{t_1',\dots,t_{d-1}',a_d}\\
&\qquad - \mathbb{E} W^H_{t_1,\dots,t_{d-1},a_d} W^H_{t_1',\dots,t_{d-1}',b_d} + \mathbb{E} W^H_{t_1,\dots,t_{d-1},a_d} W^H_{t_1',\dots,t_{d-1}',a_d}\\
&\quad = \left(\prod_{i=1}^{d-1} \phi^{H_i}(t_i, t_i')\right)\big(\phi^{H_d}(b_d, b_d) - 2\phi^{H_d}(a_d, b_d) + \phi^{H_d}(a_d, a_d)\big)\\
&\quad = \left(\prod_{i=1}^{d-1} \phi^{H_i}(t_i, t_i')\right)|b_d - a_d|^{2H_d}.
\end{aligned}$$

Thus, $|b_d - a_d|^{-H_d}\widetilde{W}$ is a fractional Brownian sheet of parameter $(H_1,\dots,H_{d-1})$. Observing that $\Delta_{W^H}(K) = \Delta_{\widetilde{W}}([a_1,b_1]\times\dots\times[a_{d-1},b_{d-1}])$, one can now conclude. ■

Theorem 10.2. *If $\bar{H} > \frac{d-1}{d}$, then, a.s., the sample paths of the fractional Brownian sheet are γ-Hölder strongly chargeable for any $0 < \gamma < \bar{H}$.*

Proof. Let $K \subset [0,1]^d$ be a dyadic cube. By the Gaussian character of the increments $\Delta_{W^H} K$, there is, for each $q > 0$, a constant C_q (not depending on K) such that

$$\mathbb{E}\,(|\Delta_{W^H}(K)|^q) = C_q |K|^{q\bar{H}}.$$

In particular, under the condition that $\bar{H} > (d-1)/d$, the Kolmogorov-type chargeability theorem can be applied for q that is sufficiently large for

$$q\bar{H} > 1 \text{ and } \bar{H} - \frac{1}{q} > \frac{d-1}{d},$$

and we conclude from this that the sample paths of W^H are almost surely γ-Hölder chargeable for any $\gamma < \bar{H}$. ∎

Next, we prove that the sample paths of the fractional Brownian sheet are almost surely not chargeable whenever $\bar{H} \leq \frac{d-1}{d}$. This shows that the condition (9.1) in the chargeability theorem is sharp. The proof is based on ideas in [1, Theorem 1.4.5].

First, we need the following two lemmas.

Lemma 10.3. *Let $X_0, X_1, \dots$ be a sequence of standard normal random variables such that for each integer n, one has* $\lim \operatorname{Cov}(X_n, X_k) = 0$ *as $k \to \infty$. Then*

$$\mathbb{P}\left(X_n \geq 1 \text{ infinitely often}\right) = 1.$$

Proof. Let (Y_1, Y_2) be a Gaussian random vector where $\operatorname{Var} Y_1 = \operatorname{Var} Y_2 = 1$ and $\rho = \operatorname{Cov}(Y_1, Y_2)$. By the Portmanteau lemma, we have

$$\mathbb{P}(Y_1 \geq 1 \text{ and } Y_2 \geq 1) \xrightarrow[\rho \to 0]{} \mathbb{P}(Y_1 \geq 1)\mathbb{P}(Y_2 \geq 1).$$

Let $\varepsilon > 0$. By the preceding paragraph, it is possible to extract a subsequence (X_{n_k}) such that

$$\mathbb{P}(X_{n_k} \geq 1 \text{ and } X_{n_\ell} \geq 1) \leq (1+\varepsilon)\mathbb{P}(X_{n_k} \geq 1)\mathbb{P}(X_{n_\ell} \geq 1)$$

for all distinct integers k, ℓ. It follows that

$$\limsup_{n\to\infty} \frac{\left(\sum_{k=1}^{n} \mathbb{P}(X_{n_k} \geq 1)\right)^2}{\sum_{k=1}^{n}\sum_{\ell=1}^{n} \mathbb{P}(X_{n_k} \geq 1 \text{ and } X_{n_\ell} \geq 1)} \geq \frac{1}{1+\varepsilon}.$$

From the Kochen–Stone lemma [11, Chapter 6, Lemma 4], we deduce that

$$\mathbb{P}\left(X_n \geq 1 \text{ i.o.}\right) \geq \mathbb{P}(X_{n_k} \geq 1 \text{ i.o.}) \geq \frac{1}{1+\varepsilon}$$

and we finally conclude from the arbitrariness of ε. ∎

Lemma 10.4. *Let $K = \prod_{i=1}^{d}[a_i, b_i]$ and $K' = \prod_{i=1}^{d}[a_i', b_i']$ be two rectangles in the cube $[0,1]^d$. Then*

$$\operatorname{Cov}\big(\Delta_{W^H} K, \Delta_{W^H} K'\big) = \frac{1}{2^d}\prod_{i=1}^{d}\big(|b_i' - a_i|^{2H_i} + |b_i - a_i'|^{2H_i} - |a_i' - a_i|^{2H_i} - |b_i - b_i'|^{2H_i}\big).$$

If K and K' are cubes, then

$$\mathrm{Cov}\left(\frac{\Delta_{W^H} K}{|K|^{\bar{H}}}, \frac{\Delta_{W^H} K'}{|K'|^{\bar{H}}}\right) = \frac{1}{2^d} \prod_{i=1}^{d} \frac{|b_i' - a_i|^{2H_i} + |b_i - a_i'|^{2H_i} - |a_i' - a_i|^{2H_i} - |b_i - b_i'|^{2H_i}}{|b_i - a_i|^{H_i} |b_i' - a_i'|^{H_i}}.$$

Proof. Using the formula for increments, one computes

$$\Delta_{W^H} K \Delta_{W^H} K' = \sum_{(c_i) \in \prod_{i=1}^d \{a_i, b_i\}} \sum_{(c_i') \in \prod_{i=1}^d \{a_i', b_i'\}} \left(\prod_{i=1}^{d} (-1)^{\delta_{a_i, c_i} + \delta_{a_i', c_i'}}\right) W^H_{c_1, \ldots, c_d} W^H_{c_1', \ldots, c_d'}.$$

Taking expectations on both sides, one finds

$$\begin{aligned} \mathbb{E}(\Delta_{W^H} K \Delta_{W^H} K') &= \sum_{(c_i) \in \prod_{i=1}^d \{a_i, b_i\}} \sum_{(c_i') \in \prod_{i=1}^d \{a_i', b_i'\}} \prod_{i=1}^{d} (-1)^{\delta_{a_i, c_i} + \delta_{a_i', c_i'}} \phi^{H_i}(c_i, c_i') \\ &= \prod_{i=1}^{d} \left(\phi^{H_i}(b_i, b_i') - \phi^{H_i}(a_i, b_i') - \phi^{H_i}(a_i', b_i) + \phi^{H_i}(a_i, a_i')\right) \\ &= \prod_{i=1}^{d} \frac{|b_i' - a_i|^{2H_i} + |b_i - a_i'|^{2H_i} - |a_i' - a_i|^{2H_i} - |b_i' - b_i|^{2H_i}}{2}. \end{aligned} \tag{10.1}$$

We recall that the random variables $\Delta_{W^H} K$ and $\Delta_{W^H} K'$ are centered. Finally, we get the second equality, in case K and K' are cubes, by dividing both sides in the equality (10.1) by

$$|K|^{\bar{H}} |K'|^{\bar{H}} = \prod_{i=1}^{d} |b_i - a_i|^{H_i} |b_i' - a_i'|^{H_i}. \qquad \blacksquare$$

Theorem 10.5. *If $\bar{H} \leq \frac{d-1}{d}$, then the sample paths of the fractional Brownian sheet are almost surely not chargeable.*

Proof. To any point $x \in [0,1)^{d-1}$ and $p \geq 0$, we associate the dyadic cube $K(x, p)$ in $[0,1]^d$ defined by

$$K(x, p) = \prod_{i=1}^{d} \left[\frac{k_i}{2^p}, \frac{k_i + 1}{2^p}\right], \text{ where } k_i = \lfloor 2^p x_i \rfloor \text{ for } 1 \leq i \leq d-1 \text{ and } k_d = 0.$$

We note that the sequence of cubes $(K(x,p))$ is decreasing and that $(x,0)\in K(x,p)$.

We let

$$A=\left\{(x,\omega)\in[0,1)^{d-1}\times\Omega:\frac{\Delta_{W^{H}(\omega)}K(x,p)}{|K(x,p)|^{\bar H}}\geq 1 \text{ for infinitely many } p\right\}$$

where Ω denotes the underlying sample space on which our process is defined. This set is clearly measurable (with respect to the product σ-algebra). For any $x\in[0,1)^{d-1}$, we define the event $A(x)=\{\omega\in\Omega:(x,\omega)\in A)\}$.

Next, we apply Lemma 10.3 to deduce that $A(x)$ is almost certain. To this end, we first notice that the random variables $\Delta_{W^H}K(x,p)/|K(x,p)|^{\bar H}$ are standard normal variables. It then suffices to establish that, for any integer p,

$$\lim_{q\to\infty}\operatorname{Cov}\left(\frac{\Delta_{W^H}K(x,p)}{|K(x,p)|^{\bar H}},\frac{\Delta_{W^H}K(x,q)}{|K(x,q)|^{\bar H}}\right)=0. \tag{10.2}$$

Suppose $q\geq p$ and write $K(x,p)=\prod_{i=1}^d[a_i,b_i]$ and $K(x,q)=\prod_{i=1}^d[a_i',b_i']$. Then the covariance in (10.2) is given by Lemma 10.4. As $K(x,q)\subset K(x,p)$, we can decompose

$$|b_i'-a_i|=|a_i'-a_i|+|b_i'-a_i'| \text{ and } |b_i-a_i'|=|b_i-b_i'|+|b_i'-a_i'|.$$

Since the function $x\mapsto x^{2H_i}$ is $\min(2H_i,1)$-Hölder continuous on $[0,1]$, we infer the existence of a constant C (depending on $d,H_1,\dots,H_d$) such that

$$\begin{aligned}\operatorname{Cov}&\left(\frac{\Delta_{W^H}K(x,p)}{|K(x,p)|^{\bar H}},\frac{\Delta_{W^H}K(x,q)}{|K(x,q)|^{\bar H}}\right)\\ &\leq C\prod_{i=1}^d\frac{|b_i'-a_i'|^{\min(2H_i,1)}}{|b_i-a_i|^{H_i}|b_i'-a_i'|^{H_i}}\\ &=C\prod_{i=1}^d\frac{|b_i'-a_i'|^{\min(H_i,1-H_i)}}{|b_i-a_i|^{H_i}}\\ &=\frac{C}{|K(x,p)|^{\bar H}}\left(\frac{1}{2^q}\right)^{\min(H_1,1-H_1)+\cdots+\min(H_d,1-H_d)}.\end{aligned}$$

This completes the proof of (10.2), from which we can assert that $\mathbb{P}(A(x))=1$.

By the measurability of A and the Fubini theorem, it follows that almost surely, for almost all $x\in[0,1)^{d-1}$, we have $\Delta_{W^H}K(x,p)\geq|K(x,p)|^{\bar H}$ for infinitely many p.

For any integer $n\geq 0$, we are led to consider the (random) collection $\mathcal{C}_n$ of dyadic cubes K of the form $K=K(x,p)$, where $p\geq n$ and $\Delta_{W^H}K\geq|K|^{\bar H}$. For each such cube K, we let $\tilde K$ denote its "bottom face" $\tilde K=\{y\in[0,1]^{d-1}:(y,0)\in K\}$, which is itself a dyadic cube of $[0,1]^{d-1}$. Its volume is related to that of K by $|\tilde K|=$

$|K|^{(d-1)/d}$. By what precedes, the cubes $\tilde{K}$, where K ranges over $\mathcal{C}_n$, cover a subset of $[0,1)^{d-1}$ of full Lebesgue measure. By considering only maximal cubes within $\mathcal{C}_n$, we can extract a finite subset $\mathcal{D}_n \subset \mathcal{C}_n$ of pairwise disjoint cubes such that

$$\sum_{K \in \mathcal{D}_n} |\tilde{K}| = \sum_{K \in \mathcal{D}_n} |K|^{\frac{d-1}{d}} \geq \frac{1}{2}.$$

We define F_n to be the dyadic figure $F_n = \bigcup_{K \in \mathcal{D}_n} K$. Then

$$\Delta_{W^H} F_n = \sum_{K \in \mathcal{D}_n} \Delta_{W^H} K \geq \sum_{K \in \mathcal{D}_n} |K|^{\bar{H}} \geq \sum_{K \in \mathcal{D}_n} |K|^{\frac{d-1}{d}} \geq \frac{1}{2}. \tag{10.3}$$

On top of that, we have $F_n \subset [0,1]^{d-1} \times [0, 2^{-n}]$, which ensures that $|F_n| \to 0$. Regarding the perimeters, we can estimate

$$\|F_n\| \leq \sum_{K \in \mathcal{D}_n} \|K\| = 2d \sum_{K \in \mathcal{D}_n} |\tilde{K}| \leq 2d \left|[0,1]^{d-1}\right| = 2d.$$

Therefore, we have proved that the sequence (F_n) w^*-converges to $\emptyset$. The lower bound (10.3) shows that Δ_{W^H} is not continuous with respect to w^*-convergence and, therefore, cannot be extended to a charge. As this happens almost surely, the proof is complete. ∎

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