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On the cardinality and dimension of the slices of Okamoto's functions

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Abstract. The graphs of Okamoto's functions, denoted by K_q, are self-affine fractal curves contained in $[0,1]^2$, parameterised by $q \in (1,2)$. In this paper we consider the cardinality and dimension of the intersection of these curves with horizontal lines. Our first theorem proves that if q is sufficiently close to 2, then K_q admits a horizontal slice with exactly three elements. Our second theorem proves that if a horizontal slice of K_q contains an uncountable number of elements then it has positive Hausdorff dimension provided q is in a certain subset of $(1,2)$. Finally, we prove that if q is a k-Bonacci number for some $k \in \mathbb{N}_{\geq 3}$, then the set of $y \in [0,1]$ such that the horizontal slice at height y has $(2m+1)$ elements has positive Hausdorff dimension for any $m \in \mathbb{N}$. We also show that, under the same assumption on q, there is some horizontal slice whose cardinality is countably infinite.

1. Introduction

John Martstrand's 1954 paper [22] provided some of the cornerstone results for the field known as *Fractal Geometry* [14]. Marstrand's paper eventually attracted attention primarily for the projection theorem (Theorem A) and slicing theorem (Theorem B) therein. The original statement of Marstrand's projection theorem imposes the restriction that the planar set E with $\dim_H E = s$ is an s-set. That is, E is required to be both measurable and satisfy $0 < \mathcal{H}^s(E) < \infty$. Davies [7] proved that analytic sets E with $\dim_H E = s$ and satisfying $\mathcal{H}^s(E) = \infty$ contain subsets of arbitrarily large finite $\mathcal{H}^s$-measure, in particular they contain s-sets. This leads to the following version of Marstrand's theorem.

Theorem A (Marstrand's projection theorem). *Let $E \subset \mathbb{R}^2$ be an analytic set with* $\dim_H E = s$.

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(a) *If* $s \leq 1$ *then the Hausdorff dimension of almost every orthogonal projection of* E *is equal to* s.

(b) *If* $s > 1$ *then almost every orthogonal projection of* E *has positive length (and hence Hausdorff dimension 1).*

If E is as above, with $s > 1$, then Marstrand's projection theorem indicates that the projection of E in almost all directions has positive length. However, the Hausdorff dimension of the projection to a line cannot exceed 1 so it is natural to seek a finer description via slices. An interpretation of Theorem B is as a description of how the "surplus dimension" of E is stored in the fibres of the projection. Formally, Theorem B provides information on the typical dimension of the intersection of E with lines in the plane which pass through E. The original statement of the slicing theorem in Marstrand [22] imposed conditions on the planar set E being an s-set, as with the projection theorem above. As mentioned, by Davies' result on analytic sets [7], we are able to reformulate Marstrand's slicing theorem as follows.

Theorem B (Marstrand's slicing theorem). *Let* $E \subset \mathbb{R}^2$ *be analytic with* $\dim_{\mathrm{H}} E = s > 1$. *Then almost every line through* $\mathcal{H}^s$*-almost every point of* E *intersects* E *in a set of Hausdorff dimension* $s-1$ *with finite Hausdorff* $(s-1)$*-dimensional measure.*

Davies [8] went on to prove that if the planar set E is not assumed to be analytic, then Theorem A fails completely. Precisely, a planar set E^* is found which projects to a set of Hausdorff dimension zero in all directions and E^* is *essentially* 2*-dimensional*[1]. Although the results we present in this paper are not linked directly to projections, our work is heavily influenced by Marstrand's closely related slicing theorem. The improvements that have been found for the projection theorem often have analogous consequences for the slicing theorem. For example, the extension to higher dimensions that Mattila [23] found for the projection theorem carries immediately over to the slicing theorem, as does Davies' result that the set E in question need only be analytic.

The sets of measure zero where the conclusions of the theorems fail are known as exceptional sets. Given Theorems A and B, natural questions arise regarding the exceptional sets, for instance, we can ask what the Hausdorff and packing dimension of these sets are in general. The research already published on this topic motivates our own research on the horizontal slices of Okamoto's functions. In the context of Marstrand's projection theorem, Kaufman proved in [19] that the Hausdorff dimension of the set of exceptional directions of E is no larger than $\dim_{\mathrm{H}} E$, and Kaufman

[1]Davies [8] defines a set to be *essentially at least* s*-dimensional* if it cannot be expressed as the countable union of sets of dimension strictly less than s.

and Mattila [20] proved that this bound is sharp. Since the set of directions is an interval, this result is only significant when $\dim_H E \leq 1$. However, in the case $\dim_H E > 1$, Falconer [11] proved that the exceptional set, i.e., the set of directions whose projections give a set of zero Lebesgue measure, has Hausdorff dimension at most $2 - s$. This bound was also proved to be sharp in the same paper. In the context of Theorem B, let $\dim_H E = s$. Let $D \subset S^1$ be the set of exceptional directions, where $d \in D$ if the set of points $e \in E$ for which the line through e in direction d intersects E in a set of Hausdorff dimension $s - 1$ has $\mathcal{H}^s$-measure zero. Orponen [27] proved that $\dim_H D \leq 2 - s$. This result is analogous to the results of Kaufman [19] and Falconer [11] mentioned above.

A conjecture of Furstenberg [16] concerns the Hausdorff dimension of the intersection of sets $A, B \subset [0, 1]$ which are invariant under multiplication maps $T_p : x \mapsto px \mod 1$ and $T_q : x \mapsto qx \mod 1$, respectively. The conjecture states that for all $u, v \in \mathbb{R}$, if p and q are multiplicatively independent (i.e., $\frac{\log p}{\log q}$ is irrational) then

$$\dim_H\big((uA + v) \cap B\big) \leq \max\{0, \dim_H A + \dim_H B - 1\}. \tag{1.1}$$

We can see for $u, v \in \mathbb{R}$ that $(uA + v) \cap B$ is the intersection of $A \times B$ and the line $y = ux + v$. This provides a link to our study of slices. More precisely, Furstenberg's conjecture can be viewed as an improvement on the 'almost all' condition of Marstrand's slicing theorem because it gives a bound for the dimension of *all* slices through $A \times B$, under some constraints on A and B. Recently, in 2019, Shmerkin [30] and Wu [33] independently and by different methods proved a stronger version of Furstenberg's conjecture, with upper box dimension replacing Hausdorff dimension on the left-hand side of (1.1).

In this paper, we study the horizontal intersections of the family of fractal curves known as *Okamoto's functions* [26]. Although some special cases of this family of curves have been studied before, namely by Bourbaki [5] and Perkins [29], research on the whole family of functions was first carried out in 2005 by Okamoto [26].

Let $q \in (1, 2)$ and consider the following iterated function system on $[0, 1]^2$, which we refer to as the *Okamoto IFS*:

$$\begin{aligned}
S_0(x, y) &= \left(\frac{x}{3}, \frac{y}{q}\right), \\
S_1(x, y) &= \left(\frac{x+1}{3}, \left(\frac{2}{q} - 1\right)(1 - y) + \left(1 - \frac{1}{q}\right)\right), \\
S_2(x, y) &= \left(\frac{x+2}{3}, \frac{y}{q} + \left(1 - \frac{1}{q}\right)\right).
\end{aligned}$$

Each S_i is a contraction on $[0, 1]^2$ so by a theorem of Hutchinson [18], there is a unique nonempty compact set K_q with the property that

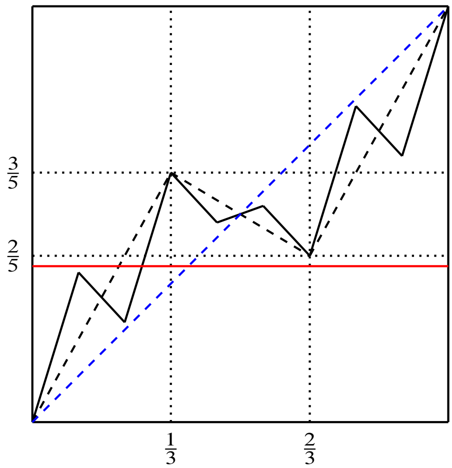
$$K_q = \bigcup_{i\in\{0,1,2\}} S_i(K_q),$$

which is the self-affine set of the Okamoto IFS. For $i \in \{0, 1, 2\}$, define $v_i(x)$ and $u_i(y)$ by the equation $S_i(x, y) = (v_i(x), u_i(y))$, that is, they are the coordinate functions of the Okamoto IFS. For any $y \in [0, 1]$ we define the *horizontal slice at* y of K_q to be the set

$$\mathrm{sl}_q(y) = \{x \in [0, 1] : (x, y) \in K_q\}.$$

For any $q \in (1, 2)$ the set $K_q \subset [0, 1]^2$ can be realised as the graph of the limit of a sequence of piecewise linear continuous functions on [0, 1] (see Figure 1). The family of functions K_q for $q \in (1, 2)$ (known as Okamoto's functions) was first studied by Okamoto [26], where the interest originated in their pathological property of being continuous and nowhere differentiable. In this paper it is shown that if $q \in (1, 3/2)$ then K_q is nowhere differentiable, but if $q \in (3/2, 2)$ then K_q is differentiable at infinitely many points. It is worth emphasising that the family of functions studied by Okamoto in [26] is actually larger than the set we consider. Essentially, they allow the parameter q to range over $(1, \infty)$, however, the limiting functions that appear for $q \in [2, \infty)$ do not have interesting horizontal intersections, so we ignore them here.

Given the x-components of the Okamoto IFS are of the form $\frac{x+i}{3}$ for $i \in \{0, 1, 2\}$, one can expect information on the ternary expansion of x to tell us about the local behaviour of the point $(x, y) \in K_q$. Allaart [1] proved for any $q \in (1, 2)$, that the derivative of K_q at x is infinite if and only if the number of 1s in the ternary expansion of x is finite and some technical limiting condition on q and the ternary expansion of x holds. Moreover, we have the satisfying polarity property that if the number of 1s in the ternary expansion of x is even then the derivative is $+\infty$, while if it is negative, the derivative is $-\infty$. In this paper, Allaart also made use of base q expansion theory to prove results on the Hausdorff dimension of the set of points x for which Okamoto's function has infinite derivative. We remark here that both ternary and base q expansions also play a key role in our work in this paper on horizontal intersections of K_q. More recently, building on this work, the derivatives of Okamoto's functions with respect to its defining parameter q were studied by Dalaklis et al. [6]. Again, the ternary expansion of x and in particular the limiting behaviour of the number of 1s in the ternary expansion of x play a key role in the main theorem of this paper.



(a) The first two iterations of the IFS construction of the graph of Okamoto's function, K_q, where $q = 5/3$.

(b) The red line $y = 3/8$ has unique intersection with K_q, i.e., $\mathrm{sl}_{5/3}(3/8)$ has a unique element.

Figure 1. Constructing K_q for $q = 5/3$.

We prove in Corollary 2.8 that for all $q \in (1, 2)$, almost all $y \in [0, 1]$ are such that $\mathrm{sl}_q(y)$ is uncountable. This result motivates our interest in which parameters $q \in (1, 2)$ admit values $y \in [0, 1]$ such that $\mathrm{sl}_q(y)$ is finite or at most countably infinite. In the context of Theorem B, the set $\mathrm{sl}_q(y)$ is the intersection of a horizontal line with the planar set K_q, so for typical y has Hausdorff dimension $\dim_{\mathrm{H}} K_q - 1$. Bárány (private communication) has found the following results which link in with the work we present in this paper. Let $s = 1 + \log_3(\frac{4}{q} - 1)$ be the affinity dimension [12] of the Okamoto IFS, and let $\dim_{\mathrm{B}}$ and $\dim_{\mathrm{A}}$ denote the box [13] and Assouad [15] dimensions, respectively. Then there is a set E with $\dim_{\mathrm{H}} E = 0$ such that for all $q \in (1, 2) \setminus E$, the following are true:

(a) $\dim_{\mathrm{H}} K_q = \dim_{\mathrm{B}} K_q = \dim_{\mathrm{A}} K_q = s$.

(b) For all $y \in [0, 1]$, $\dim_{\mathrm{H}}(\mathrm{sl}_q(y)) \leq s - 1$.

(c) Lebesgue almost every $y \in [0, 1]$ has $\dim_{\mathrm{H}}(\mathrm{sl}_q(y)) = s - 1$.

The fact that $\mathrm{sl}_q(y)$ is typically uncountable tells us that the set of y for which $\mathrm{sl}_q(y)$ is finite or countable is anomalous. Despite the question of whether K_q is an s-set for any s being open, it follows easily from the definition of K_q that all projections of K_q have Hausdorff dimension 1 and indeed are all intervals.

In Subsection 2.2 we prove the existence of a bijection between $\mathrm{sl}_q(y)$ and the set of allowable sequences of maps in a well-understood system. An important property of this bijection is that if $(f_{i_j})_{j=1}^{\infty}$ is one of the aforementioned allowable sequences of maps, then the sequence $(i_j)_{j=1}^{\infty}$ is the ternary expansion of some $x \in \mathrm{sl}_q(y)$.

Moreover, this bijection allows us to restate our theorems on $\mathrm{sl}_q(y)$ in a form that allows for the introduction of techniques and ideas from base q expansions (also known as β expansions).

Let $k \in \mathbb{N}$. A generalisation of the golden ratio, $G \approx 1.618$, which satisfies $G^2 - G - 1 = 0$ is the real number $q_k \in (1,2)$ we call the *k-Bonacci number* which satisfies

$$q^k - q^{k-1} - \cdots - q - 1 = 0. \tag{1.2}$$

We note that $q_2 = G$ is the golden ratio and $q_1 = 1$ is ignored.

Our main results are stated below.

Theorem 1. *If $q \in (q_9, 2) = (1.99803..., 2)$ then there is some $y \in [0,1]$ such that $|\mathrm{sl}_q(y)| = 3$.*

Theorem 2. *Let $q \in (1,2)$ be such that for all $y \in [0,1]$, $|\mathrm{sl}_q(y)| \in \{1, 2^{\aleph_0}\}$. Then if $\mathrm{sl}_q(y)$ is uncountable, there exists an $s > 0$ depending only on q such that*

$$\dim_{\mathrm{H}}\big(\mathrm{sl}_q(y)\big) \geq s.$$

Let $\mathcal{T}$ be the set of transcendental numbers and let $q_{\aleph_0} \approx 1.64541$ be the root of $x^6 = x^4 + x^3 + 2x^2 + x + 1$ in $(1,2)$. It was shown by the first author in [4] that $q_{\aleph_0}$ is the smallest $q \in (G,2)$ with the property that there exists $x \in I_q$ with countably many base q expansions. Let q_{KL} be the Komornik-Loreti constant [21], defined to be the smallest base q for which 1 has a unique base q expansion. We prove that Corollary 2 follows from Theorem 2 in Section 4, before proving an equivalent version of Theorem 2.

Corollary 2. *Let $q \in (1, q_{\aleph_0}) \setminus \{G\}$ or $q \in \mathcal{T} \cap (q_{\aleph_0}, q_{\mathrm{KL}})$. Then if $\mathrm{sl}_q(y)$ is uncountable there is some $s > 0$ depending only on q such that $\dim_{\mathrm{H}}(\mathrm{sl}_q(y)) \geq s$.*

In the context of base q expansions, Sidorov [32, Theorem 2.1] proved the following dichotomy. If $q \in \mathcal{T} \cap (G, q_{\mathrm{KL}})$ then any $x \in [0, \frac{1}{q-1}]$ has either a unique base q expansion or uncountably many of them. Corollary 2 strengthens this result in the sense that if q is restricted to the same set, then we know that for any $y \in [0,1]$, $\mathrm{sl}_q(y)$ has either a unique element or it has positive Hausdorff dimension.

Theorem 3. *Let $\{q_i\}_{i=3}^{\infty}$ be the set of k-Bonacci numbers excluding G. If $q = q_i$ for some $i \in \{3, 4, \ldots\}$ then the following are true.*

(1) *There exists an $\epsilon > 0$ such that if*

$$Y_m = \big\{y \in [0,1] : |\mathrm{sl}_q(y)| = 2m + 1\big\},$$

then for all $m \in \mathbb{N}$, $\dim_{\mathrm{H}}(Y_m) > \epsilon$.

(2) *There exists* $y_{\aleph_0} \in [0,1]$ *such that the cardinality of* $\mathrm{sl}_q(y_{\aleph_0})$ *is countably infinite.*

In Section 3, we prove Theorem 1. In Section 4, we prove Theorem 2 and its Corollary 2. In Section 5, we prove Theorem 3 along with the result that there is some $y \in [0,1]$ such that $\mathrm{sl}_q(y)$ has two elements if and only if 1 has a unique base q expansion (Theorem 5.5).

2. Background and preliminaries

In this section, we present the necessary background theory required for the proofs of Theorems 1, 2 and 3. In particular, we prove Lemma 2.5 which provides an important correspondence allowing us to effectively study both the cardinality and dimension of $\mathrm{sl}_q(y)$.

2.1. Background theory

Let $q \in (1,2)$ and let K_q be the graph of Okamoto's function with parameter q as defined above. Define $\{0,1,2\}^* = \cup_{k\in\mathbb{N}_{\geq 0}}\{0,1,2\}^k$ and $\{0,1\}^* = \cup_{k\in\mathbb{N}_{\geq 0}}\{0,1\}^k$. The following proposition is a consequence of [13, Theorem 9.1].

Proposition 2.1. *A point* $(x,y) \in K_q$ *if and only if there is a sequence of maps of the Okamoto IFS* $(S_{i_j})_{j=1}^{\infty} \in \{S_0, S_1, S_2\}^{\mathbb{N}}$ *such that for all* $k \in \mathbb{N}_{\geq 0}$,

$$(x,y) \in S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_k}\big([0,1]^2\big), \tag{2.1}$$

which is equivalent to both

$$x \in v_{i_1} \circ v_{i_2} \circ \cdots \circ v_{i_k}\big([0,1]\big), \tag{2.2}$$

and

$$y \in u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_k}\big([0,1]\big), \tag{2.3}$$

holding for all $k \in \mathbb{N}_{\geq 0}$.

Note that for $k = 0$, the sequence $(i_j)_{j=1}^{k} \in \{0,1,2\}^*$ is the empty word and the corresponding sequence of maps $S_{i_1} \circ \cdots \circ S_{i_k}$ is just the identity map. Therefore, in the case $k = 0$, the proposition states that $(x,y) \in [0,1]^2$.

For each $q \in (1,2)$, let $I_q = [0, \frac{1}{q-1}]$ and $J_q = [\frac{1}{q}, \frac{1}{q(q-1)}]$. In the literature on base q expansions (e.g., [32]) J_q is commonly referred to as the *switch region*. The map $\pi_3 : \{0,1,2\}^{\mathbb{N}} \to [0,1]$ is defined by

$$\pi_3\big((i_j)_{j=1}^{\infty}\big) = \sum_{j=1}^{\infty} i_j 3^{-j},$$

and for $q \in (1,2)$, the map $\pi_q : \{0,1\}^{\mathbb{N}} \to I_q$ is defined by

$$\pi_q\big((i_j)_{j=1}^{\infty}\big) = \sum_{j=1}^{\infty} i_j q^{-j}.$$

These are the *projection maps*. The domain of π_q can be extended to $\{-1,0,1\}^{\mathbb{N}}$ in the obvious way and in this case the codomain is $I_q^* = [-\frac{1}{q-1}, \frac{1}{q-1}]$. We need this extension of π_q in Section 3.

Given a finite sequence $(a_j)_{j=1}^k \in \{0,1\}^*$, we define the associated *cylinder* by

$$\big[(a_j)_{j=1}^k\big] = \big\{(i_j)_{j=1}^{\infty} \in \{0,1\}^{\mathbb{N}} : i_j = a_j \text{ for } 1 \le j \le k\big\}.$$

Similarly, if $(a_j)_{j=1}^k \in \{0,1,2\}^*$ then the associated cylinder is defined to be

$$\big[(a_j)_{j=1}^k\big] = \big\{(i_j)_{j=1}^{\infty} \in \{0,1,2\}^{\mathbb{N}} : i_j = a_j \text{ for } 1 \le j \le k\big\}.$$

If $k = 0$ we define the empty cylinder to be the whole sequence space: $[e] = \{0,1\}^{\mathbb{N}}$ or $[e] = \{0,1,2\}^{\mathbb{N}}$ where e represents the empty word. It is clear from context which one of $\{0,1\}^{\mathbb{N}}$ or $\{0,1,2\}^{\mathbb{N}}$ $[e]$ represents. The following lemma is a simple consequence of the definition of the projection map π_q and the existence of at least one base q expansion for any point in I_q, which Parry showed in his seminal paper [28].

Lemma 2.2. *If $(a_j)_{j=1}^k \in \{0,1\}^*$ is arbitrary then $\pi_q[(a_j)_{j=1}^k]$ is the interval*

$$\big[\pi_q\big((a_j)_{j=1}^k 0^{\infty}\big), \pi_q\big((a_j)_{j=1}^k 1^{\infty}\big)\big] = \left[\sum_{j=1}^{k} a_j q^{-j}, \sum_{j=1}^{k} a_j q^{-j} + q^{-k}\left(\frac{1}{q-1}\right)\right].$$

Let $(a_j)_{j=1}^k, (b_j)_{j=1}^l \in \{0,1,2\}^*$ be two finite ternary sequences. We say that $(a_j)_{j=1}^k$ is a *prefix* of $(b_j)_{j=1}^l$ if $k \le l$ and $a_j = b_j$ for all $0 \le j \le k$. The sequence $(a_j)_{j=1}^k$ is a *strict prefix* of $(b_j)_{j=1}^l$ if it is a prefix and they are not equal as sequences. Elements of $\{0,1,2\}^{\mathbb{N}}$ can be prefixed by elements of $\{0,1,2\}^*$ in the expected way and the definitions hold for elements of $\{0,1\}^*$ too.

The proof of the following lemma is straightforward and omitted.

Lemma 2.3. *Let $(a_j)_{j=1}^k, (b_j)_{j=1}^l \in \{0,1\}^*$ with $(a_j)_{j=1}^k$ a prefix of $(b_j)_{j=1}^l$. Then $\pi_q[(b_j)_{j=1}^l] \subset \pi_q[(a_j)_{j=1}^k]$. The same conclusion holds for the elements of $\{0,1,2\}^*$ and the map π_3.*

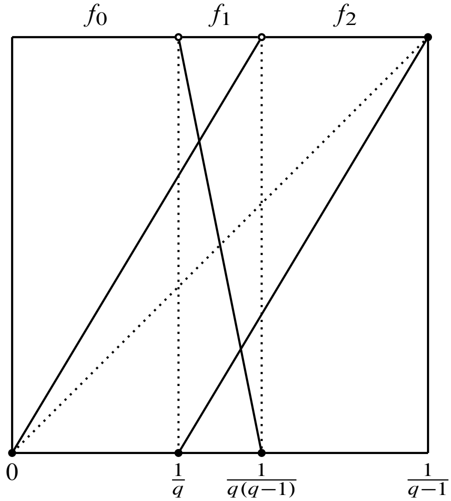


Figure 2. The set of maps $E_q = \{f_0, f_1, f_2\}$ on I_q.

For $q \in (1,2)$, we define the set of maps $E_q = \{f_0, f_1, f_2\}$ (Figure 2) where f_0, f_1 and f_2 are given by

$$f_0 : \left[0, \frac{1}{q(q-1)}\right) \to \left[0, \frac{1}{q-1}\right); \qquad f_0(x) = qx,$$
$$f_1 : \left(\frac{1}{q}, \frac{1}{q(q-1)}\right] \to \left[0, \frac{1}{q-1}\right); \qquad f_1(x) = \frac{1}{q-1} - \frac{1}{2-q}(qx-1),$$
$$f_2 : \left[\frac{1}{q}, \frac{1}{q-1}\right] \to \left[0, \frac{1}{q-1}\right]; \qquad f_2(x) = qx - 1.$$

For any $(i_j)_{j=1}^k \in \{0,1,2\}^*$ let $f_{i_k} \circ \cdots \circ f_{i_1} = f_{i_1 \ldots i_k}$ and write $(f_{i_j})_{j=1}^\infty$ for an arbitrary element of $\{f_0, f_1, f_2\}^{\mathbb{N}}$. In particular if $k = 0$ then $f_{i_1 \ldots i_k}$ is defined to be the identity map. For any $x \in I_q$, we define the *orbit space* of x in E_q by

$$\Omega_{E_q}(x) = \left\{(f_{i_j})_{j=1}^\infty \in \{f_0, f_1, f_2\}^{\mathbb{N}} : f_{i_1 \ldots i_k}(x) \in I_q \ \forall k \in \mathbb{N}_{\geq 0}\right\}.$$

Let us note here that implicit in the definition of the orbit space is the fact that $f_{i_1 \ldots i_k}(x) \in I_q$ only makes sense when $f_{i_1 \ldots i_{l-1}}(x)$ is in the domain of f_{i_l} for all $1 \leq l \leq k$. If this property fails for some $k \in \mathbb{N}$ then $(f_{i_j})_{j=1}^\infty$ is not an element of $\Omega_{E_q}(x)$. The half-open intervals that form the domains of f_0 and f_1 prevent points other than $\frac{1}{q-1}$ from mapping to $\frac{1}{q-1}$, which restricts the possible sequences of maps in the orbit space as we see in Subsection 2.2.

Lemma 2.4. *Let $x \in I_q$, then $(f_{i_j})_{j=1}^\infty \in \Omega_{E_q}(x)$ is unique if and only if $f_{i_1 \ldots i_k}(x) \in I_q \setminus J_q$ for all $k \in \mathbb{N}_{\geq 0}$.*

Proof. Let $x \in I_q$. The sequence $(f_{i_j})_{j=1}^{\infty} \in \Omega_{E_q}(x)$ is unique if and only if the following condition holds.

$$\text{For all } k \in \mathbb{N}_{\geq 0}, \text{ there is a unique choice of } i_{k+1} \in \{0, 1, 2\} \text{ such that } f_{i_1 \dots i_k}(x) \text{ is in the domain of } f_{i_{k+1}}. \tag{A}$$

By inspection of the domains of the maps in E_q, condition (A) is equivalent to $f_{i_1 \dots i_k}(x) \in I_q \setminus J_q$ for all $k \in \mathbb{N}_{\geq 0}$ and hence the lemma. ∎

2.2. Bijection lemma

Let $x \in [0, 1]$. We define the *restricted ternary expansion* (RTE) of x to be the unique sequence $(i_j)_{j=1}^{\infty} \in \{0, 1, 2\}^{\mathbb{N}}$ with $\pi_3((i_j)_{j=1}^{\infty}) = x$ such that $(i_j)_{j=k}^{\infty} \neq 2^{\infty}$ for any $k \in \mathbb{N}$ unless $x = 1$ in which case $(i_j)_{j=1}^{\infty} = 2^{\infty}$. By observing that $v_i(x) = \frac{x+i}{3}$ for $i \in \{0, 1, 2\}$, it is clear that given $x \in [0, 1]$, $(i_j)_{j=1}^{\infty} \in \{0, 1, 2\}^{\mathbb{N}}$ satisfies (2.2) for all $k \in \mathbb{N}_{\geq 0}$ if and only if $x = \pi_3((i_j)_{j=1}^{\infty})$. The following lemma allows us to transpose questions about $\mathrm{sl}_q(y)$ into questions about $\Omega_{E_q}(\frac{y}{q-1})$. In Lemma 2.7, we prove an inequality between the cardinality of $\Omega_{E_q}(x)$ and the number of base q expansions of x for any $x \in I_q$. Hence, this transposition to working with $\Omega_{E_q}(\frac{y}{q-1})$ allows us to use existing work on base q expansion theory. The details of this are contained in Subsection 2.4.

Lemma 2.5. *Let $q \in (1, 2)$ and $y \in [0, 1]$, then there is a bijection between $\mathrm{sl}_q(y)$ and $\Omega_{E_q}(\frac{y}{q-1})$ given by $x \leftrightarrow (f_{i_j})_{j=1}^{\infty}$ where $(i_j)_{j=1}^{\infty}$ is the RTE of x.*

The proof of the lemma concerns the following two closely related sets of maps on $[0, 1]$. Define $F_q = \{u_0^{-1}, u_1^{-1}, u_2^{-1}\}$ by

$$\begin{aligned}
&u_0^{-1} : [0, 1/q] \to [0, 1]; && u_0^{-1}(y) = qy,\\
&u_1^{-1} : [1 - 1/q, 1/q] \to [0, 1]; && u_1^{-1}(y) = \frac{1 - qy}{2 - q},\\
&u_2^{-1} : [1 - 1/q, 1] \to [0, 1]; && u_2^{-1}(y) = qy + (1 - q).
\end{aligned}$$

The corresponding orbit space is

$$\Omega_{F_q}(y) = \left\{(u_{i_j}^{-1})_{j=1}^{\infty} \in \{u_0^{-1}, u_1^{-1}, u_2^{-1}\}^{\mathbb{N}} : u_{i_1 \dots i_k}^{-1}(y) \in [0, 1] \text{ for all } k \in \mathbb{N}_{\geq 0}\right\},$$

and has the property that for a given $y \in [0, 1]$, $(i_j)_{j=1}^{\infty}$ solves (2.3) for all $k \in \mathbb{N}_{\geq 0}$ if and only if $(u_{i_j}^{-1})_{j=1}^{\infty} \in \Omega_{F_q}(y)$.

Let $\tilde{F}_q = \{\tilde{u}_0^{-1}, \tilde{u}_1^{-1}, \tilde{u}_2^{-1}\}$ where we make the requirement that the maps $\tilde{u}_0^{-1}$ and $\tilde{u}_1^{-1}$ are defined on half-open intervals but are otherwise identical to the corresponding maps in F_q:

$$\tilde{u}_0^{-1} : [0, 1/q) \to [0, 1); \qquad \tilde{u}_0^{-1}(y) = qy,$$
$$\tilde{u}_1^{-1} : (1 - 1/q, 1/q] \to [0, 1); \qquad \tilde{u}_1^{-1}(y) = \frac{1 - qy}{2 - q},$$
$$\tilde{u}_2^{-1} : [1 - 1/q, 1] \to [0, 1]; \qquad \tilde{u}_2^{-1}(y) = qy + (1 - q).$$

The orbit space of $\tilde{F}_q$ is

$$\Omega_{\tilde{F}_q}(y) = \left\{(\tilde{u}_{i_j}^{-1})_{j=1}^{\infty} \in \{\tilde{u}_0^{-1}, \tilde{u}_1^{-1}, \tilde{u}_2^{-1}\}^{\mathbb{N}} : \tilde{u}_{i_1 \dots i_k}^{-1}(y) \in [0, 1] \text{ for all } k \in \mathbb{N}_{\geq 0}\right\}.$$

The purpose of making the restriction from F_q to $\tilde{F}_q$ is to guarantee that if $(i_j)_{j=1}^{\infty} \in \{0, 1, 2\}^{\mathbb{N}}$ is not an RTE then $(\tilde{u}_{i_j}^{-1})_{j=1}^{\infty}$ is not an element of $\Omega_{\tilde{F}_q}(y)$ for any $y \in [0, 1]$.

Proof of Lemma 2.5. We first show there is a bijection $\mathrm{sl}_q(y) \leftrightarrow \Omega_{\tilde{F}_q}(y)$ given by $x \leftrightarrow (\tilde{u}_{i_j}^{-1})_{j=1}^{\infty}$ where $(i_j)_{j=1}^{\infty}$ is the RTE of x, before proving that $(\tilde{u}_{i_j}^{-1})_{j=1}^{\infty} \leftrightarrow (f_{i_j})_{j=1}^{\infty}$ is a bijection between $\Omega_{\tilde{F}_q}(y)$ and $\Omega_{E_q}(\frac{y}{q-1})$ for all $y \in [0, 1]$.

Let $q \in (1, 2)$, let $x \in \mathrm{sl}_q(y)$ for some $y \in [0, 1]$ and let $(i_j)_{j=1}^{\infty}$ be the RTE of x. By Proposition 2.1, $x \in \mathrm{sl}_q(y)$ if and only if $u_{i_1 \dots i_k}^{-1}(y) \in [0, 1]$ for all $k \in \mathbb{N}_{\geq 0}$, i.e., $(u_{i_j}^{-1})_{j=1}^{\infty} \in \Omega_{F_q}(y)$.

Suppose $y = 1$, if $(u_{i_j}^{-1})_{j=1}^{\infty} \in \Omega_{F_q}(1)$ then $(i_j)_{j=1}^{\infty} = 2^{\infty}$, so $(\tilde{u}_{i_j}^{-1})_{j=1}^{\infty} \in \Omega_{\tilde{F}_q}(1)$. If $y \in [0, 1)$ then since $(i_j)_{j=1}^{\infty}$ is an RTE, it avoids the tail 2^{∞}. Thus, $u_{i_1 \dots i_k}^{-1}(y) \neq 1$ for any $k \in \mathbb{N}_{\geq 0}$. To see what the consequences of this are, we consider the inverse images of 1. There are three cases corresponding to the three distinct maps of F_q.

(1) $u_0^{-1}(1/q) = 1$.

(2) $u_1^{-1}(1 - 1/q) = 1$.

(3) $u_2^{-1}(1) = 1$.

We can ignore the final case since we know $u_{i_1 \dots i_k}^{-1}(y) \neq 1$ for any $k \in \mathbb{N}_{\geq 0}$. The first case means that if $u_{i_1 \dots i_k}^{-1}(y) = 1/q$ for some $k \in \mathbb{N}_{\geq 0}$ then $i_{k+1} \neq 0$. If this were the case then $(i_j)_{j=1}^{\infty}$ would have tail 02^{∞}, contradicting the fact that it is an RTE. Similarly, in the second case, if $u_{i_1 \dots i_k}^{-1}(y) = 1 - 1/q$ for some $k \in \mathbb{N}_{\geq 0}$ then $i_{k+1} \neq 1$ to avoid $(i_j)_{j=1}^{\infty}$ having the tail 12^{∞}. Therefore, this restriction is equivalent to removing $1/q$ from the domain of u_0^{-1} and removing $1 - 1/q$ from the domain of u_1^{-1}. One can observe that this is exactly the restriction imposed on F_q to transform it into $\tilde{F}_q$. Hence if $y \in [0, 1)$ then $(\tilde{u}_{i_j}^{-1})_{j=1}^{\infty} \in \Omega_{\tilde{F}_q}(y)$. We get that $(\tilde{u}_{i_j}^{-1})_{j=1}^{\infty} \in \Omega_{\tilde{F}_q}(y)$ for all $y \in [0, 1]$ and conclude that we have a map $\mathrm{sl}_q(y) \to \Omega_{\tilde{F}_q}(y)$ given by $x \mapsto (\tilde{u}_{i_j}^{-1})_{j=1}^{\infty}$ where $(i_j)_{j=1}^{\infty}$ is the RTE of x.

Let $(\tilde{u}_{i_j}^{-1})_{j=1}^{\infty} \in \Omega_{\tilde{F}_q}(y)$ for some point $y \in [0, 1]$. Because 1 is not in the image of $\tilde{u}_0^{-1}$ or $\tilde{u}_1^{-1}$, $(\tilde{u}_{i_j}^{-1})_{j=1}^{\infty}$ has the property that $\tilde{u}_{i_1 \dots i_k}^{-1}(y) \neq 1$ for any $k \in \mathbb{N}_{\geq 0}$ unless

$y = 1$. If $y = 1$ then $(i_j)_{j=1}^{\infty} = 2^{\infty}$ and if $y \in [0, 1)$, there is no integer $k \in \mathbb{N}$ such that $(i_j)_{j=k}^{\infty} = 2^{\infty}$. In either case $(i_j)_{j=1}^{\infty}$ is an RTE. If $(\tilde{u}_{i_j}^{-1})_{j=1}^{\infty} \in \Omega_{\tilde{F}_q}(y)$ then $(u_{i_j}^{-1})_{j=1}^{\infty} \in \Omega_{F_q}(y)$ since the domain of $\tilde{u}_i^{-1}$ is contained in the domain of u_i^{-1} for $i \in \{0, 1, 2\}$. Then, by Proposition 2.1, $(i_j)_{j=1}^{\infty}$ is the RTE of some $x \in \mathrm{sl}_q(y)$. This proves that there is a bijection $\mathrm{sl}_q(y) \leftrightarrow \Omega_{\tilde{F}_q}(y)$ given by $x \leftrightarrow (\tilde{u}_{i_j}^{-1})_{j=1}^{\infty}$ where $(i_j)_{j=1}^{\infty}$ is the RTE of x.

It remains to show that the map $(\tilde{u}_{i_j}^{-1})_{j=1}^{\infty} \leftrightarrow (f_{i_j})_{j=1}^{\infty}$ is a bijection between $\Omega_{\tilde{F}_q}(y)$ and $\Omega_{E_q}(\frac{y}{q-1})$ for all $y \in [0, 1]$. For each $i \in \{0, 1, 2\}$, and for any $y \in [0, 1]$, it can be checked that

$$\frac{1}{q-1}\tilde{u}_i^{-1}(y) = f_i\left(\frac{y}{q-1}\right).$$

Moreover, for all $i \in \{0, 1, 2\}$, if the domain of f_i is D_i and the domain of $\tilde{u}_i^{-1}$ is C_i then $D_i = \frac{1}{q-1}C_i$. Hence, for all $k \in \mathbb{N}_{\geq 0}$ and any $y \in [0, 1]$,

$$\tilde{u}_{i_1\ldots i_k}^{-1}(y) \in [0, 1] \iff \frac{1}{q-1}\left[\tilde{u}_{i_1\ldots i_k}^{-1}(y)\right] \in I_q \iff f_{i_1\ldots i_k}\left(\frac{y}{q-1}\right) \in I_q.$$

So $(\tilde{u}_{i_j}^{-1})_{j=1}^{\infty} \in \Omega_{\tilde{F}_q}(y) \iff (f_{i_j})_{j=1}^{\infty} \in \Omega_{E_q}(y)$. Therefore, given $q \in (1, 2)$ and any $y \in [0, 1]$, the map $\mathrm{sl}_q(y) \leftrightarrow \Omega_{E_q}(\frac{y}{q-1})$ given by $x \leftrightarrow (f_{i_j})_{j=1}^{\infty}$ where $(i_j)_{j=1}^{\infty}$ is the RTE of x, is a bijection. ■

2.3. Base q expansions

In this subsection we introduce base q expansions [28] and state Theorem 2.6 which, alongside Lemma 2.5, puts $\mathrm{sl}_q(y)$ in the context of base q expansions.

Let $q \in (1, 2)$ and $x \in I_q$. Then we have that a base q expansion of x is a sequence $(i_j)_{j=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ such that $x = \sum_{j=1}^{\infty} i_j q^{-j}$, that is, such that $x = \pi_q((i_j)_{j=1}^{\infty})$. We define the *base q dynamics* (Figure 3) to be the set of maps $\hat{E}_q = \{\hat{f}_0, \hat{f}_1\}$ where

$$\hat{f}_0 : \left[0, \frac{1}{q(q-1)}\right] \to I_q; \qquad \hat{f}_0(x) = qx,$$
$$\hat{f}_1 : \left[\frac{1}{q}, \frac{1}{q-1}\right] \to I_q; \qquad \hat{f}_1(x) = qx - 1.$$

For any $x \in I_q$, the orbit space of x in $\hat{E}_q$ is given by

$$\Omega_{\hat{E}_q}(x) = \left\{(\hat{f}_{i_j})_{j=1}^{\infty} \in \{\hat{f}_0, \hat{f}_1\}^{\mathbb{N}} : \hat{f}_{i_1\ldots i_k}(x) \in I_q \text{ for all } k \in \mathbb{N}_{\geq 0}\right\}.$$

For any $x \in I_q$, it is a simple exercise to show that $(\hat{f}_{i_j})_{j=1}^{\infty} \in \Omega_{\hat{E}_q}(x)$ if and only if $\pi_q((i_j)_{j=1}^{\infty}) = x$ (c.f. [3]). We say that $x \in I_q$ has a unique base q expansion if $\Omega_{\hat{E}_q}(x)$ has only one element.

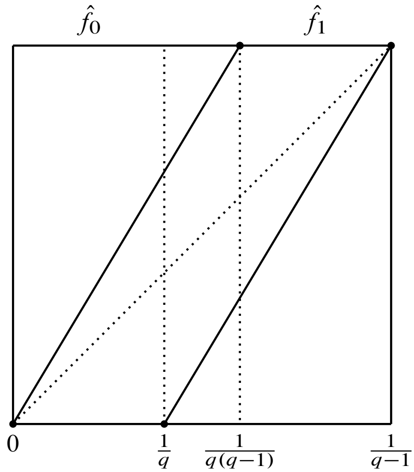


Figure 3. The base q dynamics $\hat{E}_q = \{\hat{f}_0, \hat{f}_1\}$ on I_q.

Table 1 lists some notation for $\hat{E}_q$ in line with Sidorov [32] on the left-hand side. On the right-hand side is the notation for the corresponding sets in E_q, in the context of the self-affine sets K_q. We emphasise here that the notation $\mathcal{V}_q^{(k)}$ used here is *not* in line with [32].

$\hat{E}_q$
$\mathcal{B}_k = \{q \in (1,2) : \exists\, x \in I_q \text{ such that } \lvert\Omega_{\hat{E}_q}(x)\rvert = k\}$
$\mathcal{B}_{\aleph_0} = \{q \in (1,2) : \exists\, x \in I_q \text{ such that } \lvert\Omega_{\hat{E}_q}(x)\rvert = \aleph_0\}$
$\mathcal{U}_q = \{x \in I_q : \lvert\Omega_{\hat{E}_q}(x)\rvert = 1\}$
$\mathcal{U}_q^{(k)} = \{x \in I_q : \lvert\Omega_{\hat{E}_q}(x)\rvert = k\}$
$\mathcal{U}_q^{(\aleph_0)} = \{x \in I_q : \lvert\Omega_{\hat{E}_q}(x)\rvert = \aleph_0\}$
$\mathcal{U} = \{q \in (1,2) : \lvert\Omega_{\hat{E}_q}(1)\rvert = 1\}$

E_q
$\mathcal{C}_k = \{q \in (1,2) : \exists\, x \in I_q \text{ such that } \lvert\Omega_{E_q}(x)\rvert = k\}$
$\mathcal{C}_{\aleph_0} = \{q \in (1,2) : \exists\, x \in I_q \text{ such that } \lvert\Omega_{E_q}(x)\rvert = \aleph_0\}$
$\mathcal{V}_q = \{x \in I_q : \lvert\Omega_{E_q}(x)\rvert = 1\}$
$\mathcal{V}_q^{(k)} = \{x \in I_q : \lvert\Omega_{E_q}(x)\rvert = k\}$
$\mathcal{V}_q^{(\aleph_0)} = \{x \in I_q : \lvert\Omega_{E_q}(x)\rvert = \aleph_0\}$
$\mathcal{V} = \{q \in (1,2) : \lvert\Omega_{E_q}(1)\rvert = 1\}$

Table 1. Notation for $\hat{E}_q$ and E_q.

The following result which combines [10, Theorem 3] and [31, Theorem 1] alongside Lemma 2.5 has immediate implications for the cardinality of $\mathrm{sl}_q(y)$ as shown in Subsection 2.4.

Theorem 2.6. (1) *If $q \in (1, G)$ then $\Omega_{\hat{E}_q}(x)$ is uncountably infinite for all $x \in I_q^\circ$.*

(2) *If $q = G$ then there is a countably infinite set of points $x \in I_q^\circ$ such that $\Omega_{\hat{E}_q}(x)$ is countably infinite and for all other $x \in I_q^\circ$, $\Omega_{\hat{E}_q}(x)$ is uncountably infinite.*

(3) *If $q \in (G, 2)$ then the set of points $x \in I_q^\circ$ such that $\Omega_{\hat{E}_q}(x)$ has a unique element is at least countably infinite. The set of points $x \in I_q^\circ$ such that $\Omega_{\hat{E}_q}(x)$ is uncountably infinite has full Lebesgue measure.*

2.4. Implications of the bijection lemma

In this subsection we use existing results from the theory of base q expansions to prove some elementary results on $\mathrm{sl}_q(y)$.

If $(\hat{f}_{i_j})_{j=1}^\infty \in \Omega_{\hat{E}_q}(x)$ for some $x \in I_q$ then define $D : \{0,1\}^{\mathbb{N}} \to \{0,1,2\}^{\mathbb{N}}$ by $D((i_j)_{j=1}^\infty) = (2i_j)_{j=1}^\infty$ unless $(i_j)_{j=1}^\infty = (01^\infty)$ in which case $D(01^\infty) = (10^\infty)$. Then it can be checked that $(\hat{f}_{i_j})_{j=1}^\infty \in \Omega_{\hat{E}_q}(x)$ implies that $(f_{d(i_j)})_{j=1}^\infty \in \Omega_{E_q}(x)$ where $D((i_j)_{j=1}^\infty) = (d(i_j))_{j=1}^\infty$.

Lemma 2.4 and the existence of the map D with the aforementioned property prove the following lemma.

Lemma 2.7. *Let $x \in I_q$, then $|\Omega_{E_q}(x)| \geq |\Omega_{\hat{E}_q}(x)|$ and $|\Omega_{\hat{E}_q}(x)| = 1$ if and only if $|\Omega_{E_q}(x)| = 1$.*

An immediate consequence of Lemma 2.7 is that $\mathcal{U} = \mathcal{V}$. Although the set $\mathcal{U}$ has generated significant interest within base q expansion theory (see, e.g., [10, 21]), we do not consider it here until the final result of the paper (Theorem 5.5).

In conjunction with Lemmas 2.5 and 2.7, Theorem 2.6 has the following corollary on slices.

Corollary 2.8. (1) *If $q \in (1, G)$ then for any $y \in (0,1)$, $\mathrm{sl}_q(y)$ is uncountably infinite. In fact, a consequence of* [2, Theorem 1.4] *is that $\mathrm{sl}_q(y)$ has positive Hausdorff dimension under these hypotheses.*

(2) *If $q = G$ then there is a countably infinite set of points $y \in (0,1)$ such that $\mathrm{sl}_q(y)$ is at least countably infinite and for all other $y \in (0,1)$ $\mathrm{sl}_q(y)$ is uncountably infinite.*

(3) *Let $q \in (G, 2)$. The set of points $y \in (0,1)$ such that $\mathrm{sl}_q(y)$ has a unique element is at least countably infinite. The set of points $y \in (0,1)$ such that $\mathrm{sl}_q(y)$ is uncountably infinite has full Lebesgue measure.*

3. Proof of Theorem 1

Using Lemma 2.5, Theorem 1 is equivalent to the claim that if $q \in (q_9, 2)$ then there is some $x \in I_q$ such that $|\Omega_{E_q}(x)| = 3$. That is, with the above notation, Theorem 1 is equivalent to Theorem 3.1 below.

Theorem 3.1. $(q_9, 2) \subset \mathcal{C}_3$.

We say that a sequence (finite or infinite) $(i_j)_{j=1}^k \in \{0,1\}^*$ *avoids* $(a_j)_{j=1}^l \in \{0,1\}^*$ if there does not exist $m \in \mathbb{N}$ such that $(i_j)_{j=m}^{m+l-1} = (a_j)_{j=1}^l$.

Define

$$\mathcal{S}^k = \left\{(i_j)_{j=1}^\infty \in \{0,1\}^{\mathbb{N}} : (i_j)_{j=1}^\infty \text{ avoids } (01^k) \text{ and } (10^k)\right\},$$

and notice that if a finite sequence is a prefix of an element of $\mathcal{S}^k$, then it avoids (01^k) and (10^k). The following lemma is a consequence of [17, Lemma 4].

Lemma 3.2. *If* $q \in (q_k, 2)$ *then* $\pi_q(\mathcal{S}^k) \subset \mathcal{U}_q$.

In particular, we use the fact that if $q \in (q_9, 2)$ then $\pi_q((i_j)_{j=1}^\infty) \in \mathcal{U}_q$ for any sequence $(i_j)_{j=1}^\infty \in \mathcal{S}^9$. We sketch the proof of Theorem 3.1.

(1) Suppose that for each $q \in (q_9, 2)$ there is some set $A_q \subset \mathcal{S}^9$ such that

$$\pi_q(A_q), \pi_q(A_q) - 1 \subset \pi_q(\mathcal{S}^9)$$

and A_q also satisfies

$$\big((2-q)\pi_q(\mathcal{S}^9) + 1\big) \cap \pi_q(A_q) \neq \emptyset. \tag{3.1}$$

Taking $qy \in ((2-q)\pi_q(\mathcal{S}^9) + 1) \cap \pi_q(A_q)$, we have

$$\begin{aligned} f_0(y) &= qy \in \pi_q(A_q) \subset \pi_q(\mathcal{S}^9), \\ f_1(y) &= \frac{1}{q-1} - \frac{qy-1}{2-q} \in \pi_q(\mathcal{S}^9), \\ f_2(y) &= qy - 1 \in \pi_q(A_q) - 1 \subset \pi_q(\mathcal{S}^9). \end{aligned}$$

In the second line we used

$$qy \in (2-q)\pi_q(\mathcal{S}^9) + 1 \iff \frac{qy-1}{2-q} \in \pi_q(\mathcal{S}^9),$$

and the fact that $x \in \pi_q(\mathcal{S}^k)$ if and only if $\frac{1}{q-1} - x \in \pi_q(\mathcal{S}^k)$ for all $k \in \mathbb{N}$. So under the assumptions on $\pi_q(A_q)$ we have $f_0(y), f_1(y), f_2(y) \in \pi_q(\mathcal{S}^9)$.

(2) By Lemma 3.2, $q \in (q_9, 2)$ implies that $\pi_q(\mathcal{S}^9) \subset \mathcal{U}_q$. Let $q \in (q_9, 2)$, Item 1 implies that if A_q exists then there exists $y \in \mathcal{V}_q^{(3)}$ and hence $(q_9, 2) \subset \mathcal{C}_3$. Therefore to prove the theorem, for each $q \in (q_9, 2)$ we need to construct A_q such that $\pi_q(A_q), \pi_q(A_q) - 1 \subset \pi_q(\mathcal{S}^9)$ and (3.1) holds.

(3) The claim that each A_q satisfies equation (3.1) is proved using Newhouse's theorem [24].

(4) The property that $\pi_q(A_q), \pi_q(A_q) - 1 \subset \pi_q(\mathcal{S}^9)$ follows from our construction of A_q.

The majority of the work is in proving that the structure of $\pi_q(A_q)$ is amenable to an application of Newhouse's theorem. This involves calculating the thickness of $\pi_q(A_q)$ and showing it is interleaved with $(2-q)\pi_q(\mathcal{S}^9) + 1$.

3.1. Results on projections of sequences

This subsection introduces some useful results on the projections of cylinders and inequalities involving k-Bonacci numbers.

Lemma 3.3. *Let $k \in \mathbb{N}_{\geq 2}$ then*

(1) (a) $0 < (2-q) < q^{-k}$ *if* $q \in (q_k, 2)$,
(b) $q^{-k-1} \leq (2-q) < q^{-k}$ *if* $q \in (q_k, q_{k+1}]$.

(2) (a) $0 < q^k - q^{k-1} - \cdots - q - 1 < 1$ *if* $q \in (q_k, 2)$,
(b) $0 < q^k - q^{k-1} - \cdots - q - 1 \leq \frac{1}{q}$ *if* $q \in (q_k, q_{k+1}]$.

Proof. It can be checked that $q^{k+1} > k$ whenever $k \in \mathbb{N}_{\geq 2}$ and $q \in (G, 2)$, and it follows that $1 > kq^{-k-1}$ under the same conditions. Observe that

$$\left|\frac{\mathrm{d}}{\mathrm{d}q}(2-q)\right| = |-1| > |-kq^{-k-1}| = \left|\frac{\mathrm{d}}{\mathrm{d}q}(q^{-k})\right|,$$

so $(2-q)$ is monotone decreasing faster than q^{-k}. Therefore, if $q = q_k$ solves $2 - q = q^{-k}$ then both 1(a) and 1(b) would follow. Let $q = q_k$, then q satisfies

$$q^k - q^{k-1} - \cdots - q - 1 = 0,$$

so

$$\begin{aligned} 0 &= q(q^k - q^{k-1} - \cdots - q - 1) \\ &= q^{k+1} - 2q^k + (q^k - q^{k-1} - \cdots - q^2 - q) \\ &= q^{k+1} - 2q^k + 1 = q^k(q-2) + 1, \end{aligned}$$

so $(2 - q_k) = q_k^{-k}$.

We prove 2(a) by induction. Since $G^2 - G - 1 = 0$, $2^2 - 2 - 1 = 1$ and $q^2 - q - 1$ is increasing for $q \in (G, 2)$, we know that the claim holds for $q \in (G, 2)$. For the induction, assume that $q \in (q_{k-1}, 2)$ implies that $0 < q^{k-1} - q^{k-2} - \cdots - q - 1 < 1$ for some $k \in \mathbb{N}_{\geq 3}$. Let $q \in (q_k, 2)$, and set

$$f(q) = q^k - q^{k-1} - \cdots - q - 1,$$

then

$$\frac{\mathrm{d}f}{\mathrm{d}q} = kq^{k-1} - (k-1)q^{k-2} - \cdots - 2q - 1 > k(q^{k-1} - q^{k-2} - \cdots - q - 1).$$

Since $q_{k-1} < q_k < 2$, we know by assumption that if $q \in (q_k, 2)$ then

$$0 < q^{k-1} - q^{k-2} - \cdots - q - 1 < 1$$

so $\frac{\mathrm{d}f}{\mathrm{d}q} > 0$. Hence, $f(q)$ is increasing with q on the interval $(q_k, 2)$. By observing that $f(q_k) = 0$ and $f(2) = 1$, we have shown that $q \in (q_k, 2)$ implies that $0 < f(q) < 1$ and by induction we are done.

For 2(b), the above proof of 2(a) shows that $f(q)$ is increasing for all $q \in (q_k, 2)$, and $\frac{1}{q}$ is obviously decreasing with q. It follows easily from the definition that $q = q_{k+1}$ solves $q^k - q^{k-1} - \cdots - q - 1 = \frac{1}{q}$, which completes the proof. ∎

Lemma 3.4. *If $q \in (q_k, 2)$ for some $k \in \mathbb{N}_{\geq 2}$ then*

(1) $\pi_q(01^k0^\infty) < \pi_q(10^\infty) < \pi_q(01^\infty) < \pi_q(10^k1^\infty)$,

(2) $\pi_q((01^{k-1})^\infty) < \pi_q(10^\infty)$ *and symmetrically* $\pi_q(01^\infty) < \pi_q((10^{k-1})^\infty)$.

Proof. For part (1), if $q \in (q_k, 2)$ Lemma 3.3 implies that $q^k - q^{k-1} - \cdots - q - 1 > 0$, so

$$q^{-2} + q^{-3} + \cdots + q^{-k-1} < q^{-1}, \tag{3.2}$$

and hence $\pi_q(01^k0^\infty) < \pi_q(10^\infty)$. Since $a < b \implies \frac{1}{q-1} - a > \frac{1}{q-1} - b$ we have the symmetric result, $\pi_q(01^\infty) < \pi_q(10^k1^\infty)$. $q \in (1, 2)$ implies that

$$\pi_q(10^\infty) < \pi_q(01^\infty)$$

which gives the complete inequality.

For the second part, we observe that a consequence of (3.2) is that for any $m \in \mathbb{N}$,

$$q^{-mk-2} + q^{-mk-3} + \cdots + q^{-(m+1)k-1} < q^{-mk-1}. \tag{3.3}$$

Using (3.2) and (3.3),

$$\begin{aligned}\pi_q(10^\infty) &= q^{-1} > (q^{-2}+q^{-3}+\cdots+q^{-k}) + q^{-k-1}\\ &> (q^{-2}+q^{-3}+\cdots+q^{-k}) + (q^{-k-2}+q^{-k-3}+\cdots+q^{-2k}) + q^{-2k-1}\\ &\ \vdots\\ &> \sum_{m=0}^{\infty}(q^{-mk-2}+q^{-mk-3}+\cdots+q^{-(m+1)k})\\ &= \pi_q((01^{k-1})^\infty),\end{aligned}$$

where at each step we have replaced the final term q^{-mk-1} with the finite sum

$$(q^{-mk-2}+\cdots+q^{-(m+1)k-1}),$$

which is smaller by (3.3). By symmetry we also have $\pi_q(01^\infty) < \pi_q((10^{k-1})^\infty)$. ∎

Let $(i_j)_{j=1}^k, (i_j')_{j=1}^k \in \{0,1\}^k$ be arbitrary distinct finite sequences. We write $(i_j)_{j=1}^k \prec (i_j')_{j=1}^k$ if $i_p = 0$ and $i_p' = 1$ where $p \in \mathbb{N}$ is the smallest number with $i_p \neq i_p'$. This is the natural ordering of the sequence space with the obvious extension to infinite sequences in $\{0,1\}^{\mathbb{N}}$. Let $k \in \mathbb{N}_{\geq 0}$ and let $(i_j)_{j=1}^k, (i_j')_{j=1}^k \in \{0,1\}^k$ be two finite binary strings of length k. We say that $(i_j)_{j=1}^k$ and $(i_j')_{j=1}^k$ are *lexicographically consecutive* if they are consecutive as binary numbers. That is, if

$$\sum_{j=1}^k i_j 2^{k-j} + 1 = \sum_{j=1}^k i_j' 2^{k-j} \quad \text{or} \quad \sum_{j=1}^k i_j' 2^{k-j} + 1 = \sum_{j=1}^k i_j 2^{k-j}.$$

Lemma 3.5. *Let $q \in (q_k, 2)$ and let $(i_j)_{j=1}^l, (i_j')_{j=1}^l$ be prefixes of elements of $\mathcal{S}^k$ such that $(i_j)_{j=1}^l \prec (i_j')_{j=1}^l$. Then $\pi_q((i_j)_{j=1}^l 0^\infty) < \pi_q((i_j')_{j=1}^l 0^\infty)$ and $\pi_q((i_j)_{j=1}^l 1^\infty) < \pi_q((i_j')_{j=1}^l 1^\infty)$.*

Proof. Let $q \in (q_k, 2)$ and let $(i_j)_{j=1}^l \prec (i_j')_{j=1}^l$ where both sequences are prefixes of elements of $\mathcal{S}^k$. Since both $(i_j)_{j=1}^l 0^\infty$ and $(i_j')_{j=1}^l 0^\infty$ are greedy expansions, as defined in Section 2 of [9], by [9, Proposition 2.5] we know that

$$\pi_q\big((i_j)_{j=1}^l 0^\infty\big) < \pi_q\big((i_j')_{j=1}^l 0^\infty\big).$$

This inequality immediately tells us that $\pi_q((i_j)_{j=1}^l 1^\infty) < \pi_q((i_j')_{j=1}^l 1^\infty)$. ∎

Recall by Lemma 2.2 that $\pi_q((i_j)_{j=1}^l 0^\infty)$ and $\pi_q((i_j)_{j=1}^l 1^\infty)$ are the left and right endpoints of the interval $\pi_q[(i_j)_{j=1}^l]$.

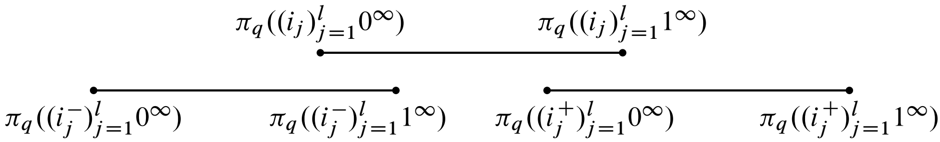


Figure 4. In the context of Lemma 3.6, $\pi_q[(i_j^-)_{j=1}^l]$ and $\pi_q[(i_j^+)_{j=1}^l]$ intersect if and only if $(i_j^-)_{j=1}^l \prec (i_j^+)_{j=1}^l$ are lexicographically consecutive.

Lemma 3.6. *Let $q \in (q_k, 2)$, $l \in \mathbb{N}$ and let $(i_j)_{j=1}^l, (i_j')_{j=1}^l$ be prefixes of elements of $\mathcal{S}^k$. Then $\pi_q[(i_j)_{j=1}^l] \cap \pi_q[(i_j')_{j=1}^l] \neq \emptyset$ if and only if $(i_j)_{j=1}^l$ and $(i_j')_{j=1}^l$ are lexicographically consecutive.*

Proof. Let us first show the reverse direction that lexicographically consecutive sequences subject to the hypotheses of the lemma project to intervals which intersect.

Let $q \in (q_k, 2)$ and let $(i_j)_{j=1}^l, (i_j')_{j=1}^l$ be prefixes of elements of $\mathcal{S}^k$. Suppose $(i_j)_{j=1}^l, (i_j')_{j=1}^l$ are lexicographically consecutive with $(i_j)_{j=1}^l \prec (i_j')_{j=1}^l$, then there is some $p \in \mathbb{N}$ such that $(i_j)_{j=1}^{l-p} = (i_j')_{j=1}^{l-p}$, $i_{l-p+1} = 0, i_{l-p+2} = \cdots = i_l = 1$, and $i_{l-p+1}' = 1, i_{l-p+2}' = \cdots = i_l' = 0$. So $(i_j)_{j=1}^l$ and $(i_j')_{j=1}^l$ are of the form

$$(i_j)_{j=1}^l = (i_j)_{j=1}^{l-p} 0 1^{p-1},$$

and

$$(i_j')_{j=1}^l = (i_j)_{j=1}^{l-p} 1 0^{p-1}.$$

Since $(i_j)_{j=1}^l$ and $(i_j')_{j=1}^l$ avoid (01^k) and (10^k), we know that $1 \le p \le k$. To prove $\pi_q[(i_j)_{j=1}^l] \cap \pi_q[(i_j')_{j=1}^l] \neq \emptyset$ it suffices to show that the left endpoint of $\pi_q[(i_j)_{j=1}^l]$ is less than the left endpoint of $\pi_q[(i_j')_{j=1}^l]$ which in turn is less than the right endpoint of $\pi_q[(i_j)_{j=1}^l]$ (see Figure 4). This is equivalent to the sequence of inequalities

$$\pi_q\big((i_j)_{j=1}^l 0^\infty\big) < \pi_q\big((i_j')_{j=1}^l 0^\infty\big) < \pi_q\big((i_j)_{j=1}^l 1^\infty\big).$$

The first inequality is immediate from Lemma 3.5 and the second is equivalent to $\pi_q(10^\infty) < \pi_q(01^\infty)$ which is true for all $q \in (1,2)$. This proves the reverse direction.

For the forwards direction of the proof, we show that sequences $(i_j^-)_{j=1}^l \prec (i_j^+)_{j=1}^l$ prefixing elements of $\mathcal{S}^k$ which are not lexicographically consecutive project to disjoint cylinders. In this case, by Lemma 3.5, the left and right endpoints are well ordered as follows:

$$\pi_q\big((i_j^-)_{j=1}^l 0^\infty\big) < \pi_q\big((i_j^+)_{j=1}^l 0^\infty\big),$$

and

$$\pi_q\big((i_j^-)_{j=1}^l 1^\infty\big) < \pi_q\big((i_j^+)_{j=1}^l 1^\infty\big).$$

It therefore suffices to show that if $(i_j^-)_{j=1}^l, (i_j^+)_{j=1}^l$ prefix elements of $\mathcal{S}^k$ and are such that $(i_j^-)_{j=1}^l \prec (i_j)_{j=1}^l$ are lexicographically consecutive and $(i_j)_{j=1}^l \prec (i_j^+)_{j=1}^l$ are lexicographically consecutive for some $(i_j)_{j=1}^l \in \{0,1\}^l$, then the right endpoint of $(i_j^-)_{j=1}^l$ is less than the left endpoint of $(i_j^+)_{j=1}^l$ (see Figure 4). That is, we aim to prove the inequality

$$\pi_q\big((i_j^-)_{j=1}^l 1^\infty\big) < \pi_q\big((i_j^+)_{j=1}^l 0^\infty\big). \tag{3.4}$$

It can be checked that $i_l^- = i_l^+$ and that $(i_j^-)_{j=1}^{l-1}$ and $(i_j^+)_{j=1}^{l-1}$ are lexicographically consecutive with $(i_j^-)_{j=1}^{l-1} \prec (i_j^+)_{j=1}^{l-1}$. Hence $(i_j^-)_{j=1}^l$ and $(i_j^+)_{j=1}^l$ are of the form

$$(i_j^-)_{j=1}^l = (i_j^-)_{j=1}^{l-p-1} 01^{p-1} i_l^-,$$

and

$$(i_j^+)_{j=1}^l = (i_j^-)_{j=1}^{l-p-1} 10^{p-1} i_l^-.$$

As in the proof of the reverse direction, we know that $p \leq k-1$ since by assumption $(i_j^-)_{j=1}^l$ and $(i_j^+)_{j=1}^l$ avoid (01^k) and (10^k). Substituting the above expressions for $(i_j^-)_{j=1}^l$ and $(i_j^+)_{j=1}^l$ back into (3.4), the inequality we now aim to prove is

$$\pi_q\big((i_j^-)_{j=1}^{l-p-1} 01^{p-1} i_l^- 1^\infty\big) < \pi_q\big((i_j^-)_{j=1}^{l-p-1} 10^{p-1} i_l^- 0^\infty\big),$$

which we simplify by restricting to the tails to give

$$\pi_q(01^{p-1} i_l^- 1^\infty) < \pi_q(10^{p-1} i_l^- 0^\infty). \tag{3.5}$$

By inspection, the left-hand side of (3.5) is maximal at $p = k-1$ and the right-hand side is minimal at $p = k-1$. It therefore suffices to consider only $p = k-1$. Setting $i_l^- = 0$, (3.5) becomes

$$\pi_q(01^{k-2}01^\infty) < \pi_q(10^\infty).$$

We have that

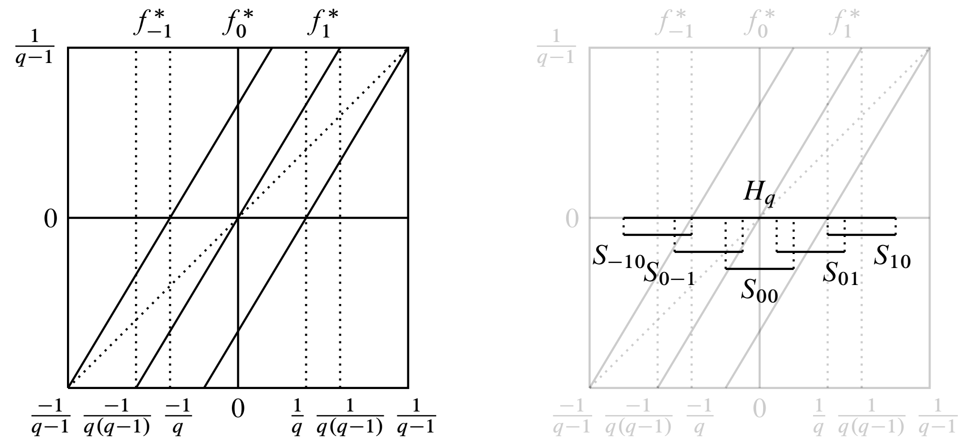
$$\pi_q(01^{k-2}01^\infty) < \pi_q\big(01^{k-2}(10^{k-1})^\infty\big) < \pi_q\big((01^{k-1})^\infty\big) < \pi_q(10^\infty),$$

where in the first and third inequalities we have used Lemma 3.4 and in the second inequality we have used[2] $\pi_q((0^{k-1}1)^\infty) < \pi_q((01^{k-1})^\infty)$. Therefore (3.4) holds when $i_l^- = 0$ and the case for $i_l^- = 1$ is similar. This proves the forwards direction and the lemma holds. ∎

[2]This inequality follows from the fact that

$$\pi_q\big((01^{k-1})^\infty\big) = \pi_q\big((0^{k-1}1)^\infty\big) + \pi_q\big((01^{k-2}0)^\infty\big)$$

and $\pi_q((01^{k-2}0)^\infty)$ is clearly positive.



(a) The extension of the base q expansion given by the maps E_q^*. One can see the top right quadrant resembles the standard base q expansion on I_q if f_{-1}^* is ignored.

(b) Interval H_q with overlapping subintervals $S_w(H_q)$ for $w \in W_2$. Note that the interval H_q and the subintervals are not drawn to scale.

Figure 5. The maps E_q^* and the interval H_q.

3.2. Extension of the base q expansion

In this subsection, we define a set of maps $E_q^* = \{f_{-1}^*, f_0^*, f_1^*\}$ on $I_q^* = [-\frac{1}{q-1}, \frac{1}{q-1}]$ which extends the standard base q expansion on I_q from the alphabet $\{0, 1\}$ to the alphabet $\{-1, 0, 1\}$.

Define the set of maps $E_q^* = \{f_{-1}^*, f_0^*, f_1^*\}$ on the interval I_q^* (see Figure 5a) by

$$f_{-1}^* : \left[-\frac{1}{q-1}, \frac{2-q}{q(q-1)}\right] \to I_q^*; \qquad f_{-1}^*(x) = qx + 1,$$

$$f_0^* : \left[-\frac{1}{q(q-1)}, \frac{1}{q(q-1)}\right] \to I_q^*; \qquad f_0^*(x) = qx,$$

$$f_1^* : \left[\frac{-(2-q)}{q(q-1)}, \frac{1}{q-1}\right] \to I_q^*; \qquad f_1^*(x) = qx - 1.$$

For any $x \in I_q$, the orbit space of x in E_q^* is given by

$$\Omega_{E_q^*}(x) = \left\{(f_{i_j}^*)_{j=1}^{\infty} \in \{f_{-1}^*, f_0^*, f_1^*\}^{\mathbb{N}} : f_{i_1 \ldots i_k}^*(x) \in I_q^* \text{ for all } k \in \mathbb{N}_{\geq 0}\right\}.$$

Since ternary expansions do not index the orbit space of this extension of the base q dynamics, we impose no restrictions on the domains of the maps to be half-open intervals.

As for the standard base q expansion, it is straightforward to show that $(f_{i_j}^*)_{j=1}^{\infty} \in \Omega_{E_q^*}(x)$ if and only if $\pi_q((i_j)_{j=1}^{\infty}) = x$ for any $x \in I_q^*$. We note that in the case of the extended base q dynamics E_q^* the associated sequence space is $\{-1,0,1\}^{\mathbb{N}}$ rather than $\{0,1\}^{\mathbb{N}}$.

Define $H_q = [-\frac{q}{q^2-1}, \frac{q}{q^2-1}] \subset I_q^*$ and $W_2 = \{(-10),(0\,-1),(00),(01),(10)\}$. For $x \in I_q^*$, define $S_i(x) = f_i^{*-1}(x)$ for $i \in \{-1,0,1\}$, and for $w = i_1 \dots i_n \in \{-1,0,1\}^n$, define $S_w(x) = f_{i_1}^{*-1} \circ \cdots \circ f_{i_n}^{*-1}(x)$.

Lemma 3.7. *For any $q \in (1,2)$,*

$$H_q = \bigcup_{w \in W_2} S_w(H_q).$$

Proof. The proof is a straightforward calculation and comparison of endpoints of intervals (see Figure 5b). Notice that

$$S_{00}(H_q) = f_0^{*-1}(f_0^{*-1}(H_q)) = \left[-\frac{1}{q}\left(\frac{1}{q^2-1}\right), \frac{1}{q}\left(\frac{1}{q^2-1}\right)\right],$$
$$S_{01}(H_q) = f_0^{*-1}(f_1^{*-1}(H_q)) = \left[\frac{1}{q^2}\left(\frac{q^2-q-1}{q^2-1}\right), \frac{1}{q^2}\left(\frac{q^2+q-1}{q^2-1}\right)\right],$$
$$S_{10}(H_q) = f_1^{*-1}(f_0^{*-1}(H_q)) = \left[\frac{1}{q}\left(\frac{q^2-2}{q^2-1}\right), \left(\frac{q}{q^2-1}\right)\right],$$
$$S_{0-1}(H_q) = f_0^{*-1}(f_{-1}^{*-1}(H_q)) = \left[\frac{1}{q^2}\left(\frac{-q^2-q+1}{q^2-1}\right), \frac{1}{q^2}\left(\frac{-q^2+q+1}{q^2-1}\right)\right],$$
$$S_{-10}(H_q) = f_{-1}^{*-1}(f_0^{*-1}(H_q)) = \left[\left(-\frac{q}{q^2-1}\right), \frac{1}{q}\left(\frac{2-q^2}{q^2-1}\right)\right].$$

$S_{00}(H_q) \cap S_{01}(H_q) \neq \emptyset$ because

$$\frac{1}{q}\left(\frac{1}{q^2-1}\right) > \frac{1}{q^2}\left(\frac{q^2-q-1}{q^2-1}\right),$$
$$\iff q^2-2q-1<0,$$

which is true for all $q \in (1,2)$, and $S_{01}(H_q) \cap S_{10}(H_q) \neq \emptyset$ because

$$\frac{1}{q^2}\left(\frac{q^2+q-1}{q^2-1}\right) > \frac{1}{q}\left(\frac{q^2-2}{q^2-1}\right),$$
$$\iff q^3-q^2-3q+1<0,$$

which is also true for all $q \in (1,2)$. By symmetry $S_{00}(H_q) \cap S_{0-1}(H_q) \neq \emptyset$ and $S_{0-1}(H_q) \cap S_{-10}(H_q) \neq \emptyset$. The left and right endpoints of H_q are preserved by the maps $S_{-10}(H_q)$ and $S_{10}(H_q)$ respectively, hence the lemma. ∎

The following lemma follows easily from Lemma 3.7.

Lemma 3.8. *If $x \in H_q$ then there is some $(f^*_{i_j})_{j=1}^\infty \in \Omega_{E_q^*}(x)$ such that $(i_j)_{j=1}^\infty \in W_2^{\mathbb{N}}$ and $f^*_{i_1\dots i_k}(x) \in H_q$ for all $k \in 2\mathbb{N}$.*

Proof. If $x \in H_q$ then $x \in S_w(H_q)$ for some $w \in W_2$ by Lemma 3.7. Let $w_1 = i_1 i_2$ then $f^*_{i_2}(f^*_{i_1}(x)) \in H_q$. Repeating the above argument, we generate a sequence of words $(w_j)_{j=1}^\infty = (i_j)_{j=1}^\infty$ where $w_j \in W_2$ for all $j \in \mathbb{N}$ and $f^*_{i_1\dots i_k} \in H_q$ for all $k \in 2\mathbb{N}$. By construction it follows that $(f^*_{i_j})_{j=1}^\infty \in \Omega_{E_q^*}(x)$. ∎

The following lemma guarantees the existence of an orbit of 1 whose associated sequence has a tail in $W_2^{\mathbb{N}}$. This is important for the definition of the *fixed expansion of* 1 in the next subsection to be valid.

Lemma 3.9. *If $q \in (1,2)$ then there is some $M \in \mathbb{N}_{\geq 0}$ depending only on q and some sequence $(w_j)_{j=1}^\infty \in W_2^{\mathbb{N}}$ such that $(i_j)_{j=1}^\infty = 1^M (w_j)_{j=1}^\infty$ satisfies $(f^*_{i_j})_{j=1}^\infty \in \Omega_{E_q^*}(1)$. Moreover, if $q \in (1, G]$ then $M = 0$ and if $q \in (q_k, q_{k+1}]$ for $k \in \mathbb{N}_{\geq 2}$ then $M = k$.*

Proof. Since $q_1 = 1$ and $\lim_{n\to\infty} q_n = 2$, we know that $q \in (1,2)$ implies that $q \in (q_k, q_{k+1}]$ for some $k \in \mathbb{N}$. By Lemma 3.8 it suffices to show, for each $q \in (1,2)$, that there is some $M \in \mathbb{N}_{\geq 0}$ such that $(f_1^*)^M(1) \in H_q$. Recall that $q_2 = G$. If $q \in (1, G]$ then $0 < 1 \leq \frac{q}{q^2-1}$ so $1 \in H_q$ and by setting $M = 0$ we are done. Let $q \in (q_k, q_{k+1}]$ for some $k \geq 2$, and observe $(f_1^*)^k(1) = q^k - q^{k-1} - \dots - q - 1$. By Lemma 3.3, we know that if $q \in (q_k, q_{k+1}]$ then $0 < (f_1^*)^k(1) \leq \frac{1}{q}$. Finally, $\frac{1}{q} < \frac{q}{q^2-1}$ for all $q \in (G, 2)$ so setting $M = k$, $(f_1^*)^M(1) \in H_q$ and we are done. ∎

3.3. Construction of A_q

In this subsection, we construct a family of sequences A_q for each $q \in (q_9, 2)$ with the property that whenever $(a_j)_{j=1}^\infty \in A_q$ we can guarantee that

$$\pi_q\big((a_j)_{j=1}^\infty\big), \pi_q\big((a_j)_{j=1}^\infty\big) - 1 \in \mathcal{U}_q.$$

This is the content of Proposition 3.10 below.

For each $q \in (q_9, 2)$, fix

$$(c_j)_{j=1}^\infty = 1^M (c_j)_{j=M+1}^\infty \in \{-1, 0, 1\}^{\mathbb{N}},$$

where $1 = \pi_q((c_j)_{j=1}^\infty)$ and $(c_j)_{j=M+1}^\infty \in W_2^{\mathbb{N}}$ is as described in Lemma 3.9. The sequence $(c_j)_{j=1}^\infty$ is referred to as the *fixed expansion of* 1. We emphasise here that the sequence $(c_j)_{j=1}^\infty$ with these properties is not necessarily unique, but by fixing one such sequence for each $q \in (q_9, 2)$, the definition of the fixed expansion of 1 is valid.

Let J be the set of zeros of the sequence $(c_j)_{j=1}^{\infty}$:

$$J = \{j \in \mathbb{N} : c_j = 0\}.$$

We can enumerate J in the obvious way: $J = \{j_0, j_1, \ldots\}$ where $j_0 < j_1 < \cdots$. Define

$$J_{\text{fixed},1} = \{j_m \in J : m = 4k+1 \text{ for some } k \in \mathbb{N}\},$$

$$J_{\text{fixed},0} = \{j_m \in J : m = 4k+3 \text{ for some } k \in \mathbb{N}\},$$

and

$$J_{\text{free}} = \{j_m \in J : m \text{ is even}\},$$

and call an index in J_{free} a *free zero* of $(c_j)_{j=1}^{\infty}$. If $j \in J_{\text{fixed},1} \cup J_{\text{fixed},0}$ or if j is such that $c_j \in \{-1, 1\}$ then j is a *fixed index*, otherwise j is a free zero.

Given $q \in (q_9, 2)$ and $1^M (c_j)_{j=M+1}^{\infty}$ the fixed expansion of 1, the set A_q consists of the sequences $(a_j)_{j=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ which satisfy the following properties:

(1) $a_j = 1$ if $c_j = 1$,

(2) $a_j = 0$ if $c_j = -1$,

(3) $a_j = 1$ if $j \in J_{\text{fixed},1}$ and

(4) $a_j = 0$ if $j \in J_{\text{fixed},0}$.

Notice that there are no restrictions on the value of a_j if $j \in J_{\text{free}}$ and later we use this fact along with the bounded distance between zeros of $(c_j)_{j=1}^{\infty}$ to show that the thickness of $\pi_q(A_q)$ can be bounded below. We can now state the key proposition of this subsection.

Proposition 3.10. *If $q \in (q_9, 2)$ then $\pi_q(A_q), \pi_q(A_q) - 1 \subset \mathcal{U}_q$.*

Proposition 3.10 is a consequence of the following two lemmas.

Lemma 3.11. *If $q \in (q_9, 2)$ then $A_q \subset \mathcal{S}^9$ and $\pi_q(A_q) \subset \mathcal{U}_q$.*

Proof. We start with the first containment. Let $q \in (q_9, 2)$ and let $1^M (c_j)_{j=M+1}^{\infty}$ be the fixed expansion of 1. Since each $w \in W_2$ has length two and at least one 0, we know that for any $m \in \mathbb{N}$, $(2m+1)$ consecutive indices of $(c_j)_{j=M+1}^{\infty}$ must contain at least m 0s. In particular, each set of nine consecutive indices must contain at least four 0s. By inspection of the sets $J_{\text{fixed},0}$ and $J_{\text{fixed},1}$, we see that in every set of four consecutive 0s of $(c_j)_{j=M+1}^{\infty}$, $\{j_m, j_{m+1}, j_{m+2}, j_{m+3}\}$, we have at least one element of $J_{\text{fixed},0}$ and at least one element of $J_{\text{fixed},1}$. Hence, for $j > M$, $\{a_j, \ldots, a_{j+8}\}$ contains at least one 0 and at least one 1. Since $(a_j)_{j=1}^{M} = 1^M$ and $(a_j)_{j=M+1}^{\infty}$ avoids the strings (1^9) and (0^9), we know that $(a_j)_{j=1}^{\infty}$ avoids (01^9) and (10^9), i.e., $A_q \subset \mathcal{S}^9$.

Given $q \in (q_9, 2)$, Lemma 3.2 then provides the conclusion that $\pi_q(A_q) \subset \mathcal{U}_q$. ∎

Lemma 3.12. *If $q \in (q_9, 2)$ then $\pi_q(A_q) - 1 \subset \mathcal{U}_q$.*

Proof. Let $q \in (q_9, 2)$ and let $1^M (c_j)_{j=M+1}^{\infty}$ be the fixed expansion of 1. For each $(a_j)_{j=1}^{\infty} \in A_q$ define a sequence $(b_j)_{j=1}^{\infty}$ by declaring that $b_j = a_j - c_j$ for all $j \in \mathbb{N}$. Therefore $(b_j)_{j=1}^{\infty}$ satisfies

(1) $b_j = 0$ if $c_j = 1$,

(2) $b_j = 1$ if $c_j = -1$,

(3) $b_j = a_j$ if $c_j = 0$.

Since $(a_j)_{j=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$, we know that $(b_j)_{j=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$. Moreover, by inspection of the proof of Lemma 3.11, since $b_j = a_j$ whenever $c_j = 0$, we know that $(b_j)_{j=M+1}^{\infty}$ also avoids 0^9 and 1^9. Noting that $(b_j)_{j=1}^{M} = 0^M$, we can say that $(b_j)_{j=1}^{\infty}$ avoids (01^9) and (10^9), so $(b_j)_{j=1}^{\infty} \in \mathcal{S}^9$. Since $q \in (q_9, 2)$, by Lemma 3.2 we know that $\pi_q((b_j)_{j=1}^{\infty}) \in \mathcal{U}_q$.

By definition of $(b_j)_{j=1}^{\infty}$ and by the additive property of π_q, we have that $(a_j)_{j=1}^{\infty}$, $(b_j)_{j=1}^{\infty}$ and $(c_j)_{j=1}^{\infty}$ satisfy

$$1 = \pi_q\big((c_j)_{j=1}^{\infty}\big) = \pi_q\big((a_j)_{j=1}^{\infty}\big) - \pi_q\big((b_j)_{j=1}^{\infty}\big).$$

So for each $(a_j)_{j=1}^{\infty} \in A_q$ there is some uniquely defined $(b_j)_{j=1}^{\infty}$ such that

$$\pi_q\big((b_j)_{j=1}^{\infty}\big) = \pi_q\big((a_j)_{j=1}^{\infty}\big) - 1.$$

Since $q \in (q_9, 2)$, $\pi_q((b_j)_{j=1}^{\infty}) \in \mathcal{U}_q$ and hence $\pi_q(A_q) - 1 \subset \mathcal{U}_q$. ∎

Lemmas 3.11 and 3.12 together prove Proposition 3.10.

3.4. The structure of $\pi_q(A_q)$

In this subsection, we prove Proposition 3.13. This thickness estimate forms a key part of the application of Newhouse's theorem [24] to prove Theorem 3.1. The following definition of *thickness* is due to Newhouse [25]. We note that an interleaving condition is also required for Newhouse's theorem and this is addressed in Subsection 3.5.

Let $C \subset \mathbb{R}$ be a compact set such that the complement of C in $\mathbb{R}$ is the union of two disjoint unbounded open intervals with a countable collection of open intervals we refer to as *gaps*. Denote this countable collection of gaps by $\{G_n : n \in \mathbb{N}\}$ and let $|G_m| \geq |G_n|$ whenever $m < n$. To each gap G_n in the complement of C we associate a left and right *bridge*. Given a gap G_n the left bridge, L_n, is defined to be the interval to the left of G_n whose right endpoint is the left endpoint of G_n and whose left endpoint is the right endpoint of the gap immediately to the left of G_n and at least as big. The right bridge, R_n, is defined similarly. So for a given gap G_n, we know the gaps contained in the bridges L_n and R_n are strictly smaller than G_n. The thickness of the compact set C is given by

$$\tau(C) = \inf\left\{\min\left\{\frac{|L_n|}{|G_n|}, \frac{|R_n|}{|G_n|}\right\} : n \in \mathbb{N}\right\}.$$

Note that the unbounded components of the complement of C in $\mathbb{R}$ are not considered when dealing with thickness, so we ignore these in what follows. We are now able to state the key proposition of this subsection.

Proposition 3.13. *If $q \in (q_9, 2)$ then $\tau(\pi_q(A_q)) > q^{-5}$.*

Let $q \in (q_9, 2)$ and let $1^M(c_j)_{j=M+1}^{\infty} \in \{-1, 0, 1\}^{\mathbb{N}}$ be the fixed expansion of 1, defined in Subsection 3.3. Recall that M and the expansion $1^M(c_j)_{j=M+1}^{\infty}$ are defined such that $(c_j)_{j=M+1}^{\infty} \in W_2^{\mathbb{N}}$ and $\pi_q(1^M(c_j)_{j=M+1}^{\infty}) = 1$. For every $k \in \mathbb{N}$ we define the set of length k prefixes of sequences in A_q by A_q^k, that is

$$A_q^k = \left\{(a_j)_{j=1}^k \in \{0,1\}^k : \exists (a'_j)_{j=1}^{\infty} \in A_q \text{ such that } a'_j = a_j \ \forall 1 \le j \le k\right\},$$

and write

$$\pi_q[A_q^k] = \left\{\pi_q\left[(a_j)_{j=1}^k\right] : (a_j)_{j=1}^k \in A_q^k\right\},$$

so elements of $\pi_q[A_q^k]$ are intervals. By Lemma 3.11, $q \in (q_9, 2)$ implies that $A_q \subset \mathcal{S}^9$, so it is immediate that any $(a_j)_{j=1}^k \in A_q^k$ avoids (10^9) and (01^9).

The lemma below is used frequently throughout the proofs of the remaining lemmas of this section.

Lemma 3.14. *Let $q \in (q_9, 2)$.*

(1) *$(a_j)_{j=1}^k \in A_q^k$ if and only if $\pi_q[(a_j)_{j=1}^k] \cap \pi_q(A_q) \neq \emptyset$.*

(2) *If $(a_j)_{j=1}^k \notin A_q^k$ is such that there are sequences $(a_j^-)_{j=1}^k, (a_j^+)_{j=1}^k \in A_q^k$ with the property that $(a_j^-)_{j=1}^k \prec (a_j)_{j=1}^k$ and $(a_j)_{j=1}^k \prec (a_j^+)_{j=1}^k$, then*

$$\pi_q[(a_j)_{j=1}^k] \subset \operatorname{conv}(\pi_q(A_q)) \setminus \pi_q(A_q).$$

(3) *If $(a_j)_{j=1}^{k-2}0a_k, (a_j)_{j=1}^{k-2}1a_k \in A_q^k$ then $\pi_q[(a_j)_{j=1}^{k-2}a_k\overline{a_k}] \cap \pi_q[(a_j)_{j=1}^{k-2}0a_k] \neq \emptyset$ and $\pi_q[(a_j)_{j=1}^{k-2}a_k\overline{a_k}] \cap \pi_q[(a_j)_{j=1}^{k-2}1a_k] \neq \emptyset$.*

Proof. Let $q \in (q_9, 2)$.

(1) For the forwards implication, if $(a_j)_{j=1}^k \in A_q^k$ then by definition there exists some $(a_j)_{j=1}^{\infty} \in A_q$ which satisfies $\pi_q((a_j)_{j=1}^{\infty}) \in \pi_q[(a_j)_{j=1}^k] \cap \pi_q(A_q)$. For the reverse implication, suppose $\pi_q((a'_j)_{j=1}^{\infty}) \in \pi_q[(a_j)_{j=1}^k] \cap \pi_q(A_q)$. Then $(a'_j)_{j=1}^{\infty} \in A_q$ and by Lemma 3.11, $\pi_q(A_q) \subset \mathcal{U}_q$. Hence $(a'_j)_{j=1}^k = (a_j)_{j=1}^k$ and so $(a_j)_{j=1}^k \in A_q^k$.

(2) The first part of the hypotheses guarantees that $\pi_q[(a_j^-)_{j=1}^k] \cap \pi_q(A_q) \neq \emptyset$ and $\pi_q[(a_j^+)_{j=1}^k] \cap \pi_q(A_q) \neq \emptyset$. Since $(a_j)_{j=1}^k \notin A_q^k$, $(a_j)_{j=1}^k$ is not a prefix of any element of A_q we know that $\pi_q[(a_j)_{j=1}^k] \cap \pi_q(A_q) = \emptyset$. Therefore $(a_j^-)_{j=1}^k$ and $(a_j^+)_{j=1}^k$ are prefixes of sequences $(a_j^-)_{j=1}^\infty, (a_j^+)_{j=1}^\infty \in A_q$ which satisfy

$$\pi_q\big((a_j^-)_{j=1}^\infty\big) < \pi_q\big((a_j)_{j=1}^k 0^\infty\big) < \pi_q\big((a_j)_{j=1}^k 1^\infty\big) < \pi_q\big((a_j^+)_{j=1}^\infty\big).$$

This inequality proves that the interval $\pi_q[(a_j)_{j=1}^k]$ is bounded on both sides by elements of $\pi_q(A_q)$, and hence $\pi_q[(a_j)_{j=1}^k] \subset \operatorname{conv}(\pi_q(A_q)) \setminus \pi_q(A_q)$.

(3) Notice that if a_k is 0 or 1, in either case the inequality

$$\pi_q\big((a_j)_{j=1}^{k-2} 0 a_k 1^\infty\big) > \pi_q\big((a_j)_{j=1}^{k-2} a_k \overline{a_k} 0^\infty\big)$$

reduces to $\pi_q(01^\infty) > \pi_q(10^\infty)$, which we know is true for all $q \in (1,2)$. Therefore $\pi_q[(a_j)_{j=1}^{k-2} a_k \overline{a_k}] \cap \pi_q[(a_j)_{j=1}^{k-2} 0 a_k] \neq \emptyset$. A similar argument shows that $\pi_q[(a_j)_{j=1}^{k-2} a_k \overline{a_k}] \cap \pi_q[(a_j)_{j=1}^{k-2} 1 a_k] \neq \emptyset$ which completes the proof. ∎

A *level k gap* is a gap G which contains the interval (which we show to exist) between $\pi_q[(a_j)_{j=1}^{k-2} 0 a_k]$ and $\pi_q[(a_j)_{j=1}^{k-2} 1 a_k]$ for some $(a_j)_{j=1}^{k-2} 0 a_k, (a_j)_{j=1}^{k-2} 1 a_k \in A_q^k$. Given a gap G, we denote the left and right bridges of G by L_G and R_G, respectively.

We prove the following claims which allow us to bound $\tau(\pi_q(A_q))$ from below.

(1) All gaps of $\pi_q(A_q)$ are open.

(2) Level k gaps exist if and only if $k-1$ is a free zero (which requires $k \geq M+1$).

(3) A level k gap cannot be a level l gap for any $l \neq k$.

(4) All gaps of $\pi_q(A_q)$ are level k gaps for some k.

(5) If G is a level k gap and H is a level l gap for some $l > k$ then $|H| < |G|$. Moreover, $q^{-k} < |G| < q^{-k+1}$.

(6) If G is a level k gap then $|L_G|, |R_G| > q^{-k-4}$.

These steps lead us to conclude that $\tau(\pi_q(A_q)) > q^{-5}$ (Proposition 3.13). This bound is then used in an application of Newhouse's theorem to prove (3.1) from which Theorem 3.1 follows. The items in the list are proved sequentially in the next six lemmas.

Lemma 3.15. *All gaps of $\pi_q(A_q)$ are open.*

Proof. Let $(i_j)_{j=1}^{\infty}, (i'_j)_{j=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ and consider the metric on $\{0,1\}^{\mathbb{N}}$ given by

$$d((i_j)_{j=1}^{\infty}, (i'_j)_{j=1}^{\infty}) = \begin{cases} 2^{-n} & \text{where } n = \min\{j \in \mathbb{N} : i_j \neq i'_j\}, \\ 0 & \text{if } (i_j)_{j=1}^{\infty} = (i'_j)_{j=1}^{\infty}. \end{cases}$$

A_q is compact with respect to the topology generated by this metric, and π_q is continuous. Therefore $\pi_q(A_q)$ is compact with respect to the normal topology on $\mathbb{R}$, generated by the open intervals, and hence its complement is open. The gaps of $\pi_q(A_q)$ form a subset of the complement, so all the gaps are necessarily open. ∎

Let $\overline{x}$ be the reflection of x, i.e., if $x = (i_j)_{j=1}^{k} \in \{0,1\}^*$ then

$$\overline{x} = (\overline{i}_j)_{j=1}^{k} = (1 - i_j)_{j=1}^{k}.$$

In particular if $x \in \{0,1\}$ then $\overline{x} = 1 - x$ and the reflection of the empty word is just itself.

Lemma 3.16. *Let $q \in (q_9, 2)$, then level k gaps exist if and only if $k-1$ is a free zero.*

Proof. Observe that the existence of sequences $(a_j)_{j=1}^{k-2}0a_k$ and $(a_j)_{j=1}^{k-2}1a_k$ as elements of A_q^k is equivalent to $k-1$ being a free zero. Hence it suffices to show that if $(a_j)_{j=1}^{k-2}0a_k$ and $(a_j)_{j=1}^{k-2}1a_k$ are elements of A_q^k then $\pi_q[(a_j)_{j=1}^{k-2}0a_k]$ and $\pi_q[(a_j)_{j=1}^{k-2}1a_k]$ do not intersect and the interval between them is contained in a gap.

Suppose that $(a_j)_{j=1}^{k-2}0a_k, (a_j)_{j=1}^{k-2}1a_k \in A_q^k$ then since $(a_j)_{j=1}^{k-2}0a_k \prec (a_j)_{j=1}^{k-2}1a_k$ are not lexicographically consecutive, Lemma 3.6 implies that $\pi_q[(a_j)_{j=1}^{k-2}0a_k]$ and $\pi_q[(a_j)_{j=1}^{k-2}1a_k]$ do not intersect. Recall from Subsection 3.3 that if $k-1$ is a free zero, then k must be a fixed index, so $(a_j)_{j=1}^{k-2}a_k\overline{a_k} \notin A_q^k$ because the kth entry is not a_k. It can be easily checked that $(a_j)_{j=1}^{k-2}0a_k \prec (a_j)_{j=1}^{k-2}a_k\overline{a_k}$ are lexicographically consecutive and that $(a_j)_{j=1}^{k-2}a_k\overline{a_k} \prec (a_j)_{j=1}^{k-2}1a_k$ are lexicographically consecutive. So, by Lemma 3.14, $\pi_q[(a_j)_{j=1}^{k-2}a_k\overline{a_k}]$ is a gap contained in $\operatorname{conv}(\pi_q(A_q)) \setminus \pi_q(A_q)$.

Again by Lemma 3.14, the set $\pi_q[(a_j)_{j=1}^{k-2}a_k\overline{a_k}]$ intersects both $\pi_q[(a_j)_{j=1}^{k-2}0a_k]$ and $\pi_q[(a_j)_{j=1}^{k-2}1a_k]$ and must therefore contain the interval between them, which completes the proof. ∎

Note that by the definition of the fixed expansion of 1, the first free zero cannot occur before index $M+1$, hence we guarantee there are no level k gaps for $k \leq M+1$. To prove a level k gap cannot be a level l gap for any $l \neq k$, we prove that the gap which contains the interval between $\pi_q[(a_j)_{j=1}^{k-2}0a_k]$ and $\pi_q[(a_j)_{j=1}^{k-2}1a_k]$ is determined uniquely by the sequence $(a_j)_{j=1}^{k-2} \in A_q^{k-2}$. This is the content of the following lemma.

Lemma 3.17. *Let $q \in (q_9, 2)$. Let G be the gap which contains the interval between $\pi_q[(a_j)_{j=1}^{k-2} 0 a_k]$ and $\pi_q[(a_j)_{j=1}^{k-2} 1 a_k]$ for some $k \in \mathbb{N}$ and some*

$$(a_j)_{j=1}^{k-2} 0 a_k, (a_j)_{j=1}^{k-2} 1 a_k \in A_q^k.$$

Let H be the gap which contains the interval between the points $\pi_q[(b_j)_{j=1}^{l-2} 0 b_l]$ and $\pi_q[(b_j)_{j=1}^{l-2} 1 b_l]$ for some $l \in \mathbb{N}$ and $(b_j)_{j=1}^{l-2} 0 b_l$, $(b_j)_{j=1}^{l-2} 1 b_l \in A_q^l$. Then if $(a_j)_{j=1}^{k-2} \neq (b_j)_{j=1}^{l-2}$ then $G \cap H = \emptyset$.

Proof. Assume the hypotheses of the lemma and that $k, l \in \mathbb{N}$ with $k \leq l$. Suppose that $(a_j)_{j=1}^{k-2} \prec (b_j)_{j=1}^{k-2}$. It is clear that $G \subset \pi_q[(a_j)_{j=1}^{k-2}]$ and $H \subset \pi_q[(b_j)_{j=1}^{l-2}] \subset \pi_q[(b_j)_{j=1}^{k-2}]$. Since $k-1$ is a free zero, $k-2$ is a fixed index so $a_{k-2} = b_{k-2}$ and therefore $(a_j)_{j=1}^{k-2}$ and $(b_j)_{j=1}^{k-2}$ are not lexicographically consecutive. By Lemma 3.6 we know that $\pi_q[(a_j)_{j=1}^{k-2}] \cap \pi_q[(b_j)_{j=1}^{k-2}] = \emptyset$ so $G \cap H = \emptyset$.

Suppose instead that $(a_j)_{j=1}^{k-2} = (b_j)_{j=1}^{k-2}$. In order for the premise $(a_j)_{j=1}^{k-2} \neq (b_j)_{j=1}^{l-2}$ to hold, we require that $k < l$. Notice that since $l-1$ is a free zero and k is a fixed index, we require $l \geq k+2$. Without loss of generality[3], let $(b_j)_{j=1}^{k} = (a_j)_{j=1}^{k-2} 1 a_k$. Therefore

$$\pi_q\big[(b_j)_{j=1}^{l-2} 0 b_l\big] \subset \pi_q\big[(b_j)_{j=1}^{l-2}\big] \subset \pi_q\big[(a_j)_{j=1}^{k-2} 1 a_k\big]. \tag{3.6}$$

By Lemma 3.14, $(b_j)_{j=1}^{l-2} 0 b_l \in A_q^l$ so there exists some $x \in \pi_q[(b_j)_{j=1}^{l-2} 0 b_l] \cap \pi_q(A_q)$. Using (3.6), we know that $x \in \pi_q[(a_j)_{j=1}^{k-2} 1 a_k] \cap \pi_q(A_q)$. Since G contains the interval between $\pi_q[(a_j)_{j=1}^{k-2} 0 a_k]$ and $\pi_q[(a_j)_{j=1}^{k-2} 1 a_k]$ and G is a connected component of $\operatorname{conv}(\pi_q(A_q)) \setminus \pi_q(A_q)$, we know that $y < x$ for all $y \in G$.

Similarly, since H contains the interval between the points $\pi_q[(b_j)_{j=1}^{l-2} 0 b_l]$ and $\pi_q[(b_j)_{j=1}^{l-2} 1 b_l]$ we know that $x < z$ for all $z \in H$ (see Figure 6). Since $x \in \pi_q(A_q)$ we conclude that $G \cap H = \emptyset$. ∎

Next, we show that every gap is a level k gap for some $k \in \mathbb{N}$.

Lemma 3.18. *Let $q \in (q_9, 2)$. If G is a gap then G is a level k gap for some $k \in \mathbb{N}$.*

Proof. Let $q \in (q_9, 2)$ and let $1^M (c_j)_{j=M+1}^{\infty}$ be the fixed expansion of 1. Let G be a gap and let $k_G \in \mathbb{N}$ be the largest number such that there exists an element $(a_j)_{j=1}^{k_G-2} \in A_q^{k_G-2}$ with the property that $G \subset \pi_q[(a_j)_{j=1}^{k_G-2}]^\circ$. We show that k_G exists and claim that G is a level k_G gap.

[3] If instead we assume that $(b_j)_{j=1}^{k} = (a_j)_{j=1}^{k-2} 0 a_k$ then (3.6) becomes $\pi_q[(b_j)_{j=1}^{l-2} 1 b_l] \subset \pi_q[(b_j)_{j=1}^{l-2}] \subset \pi_q[(a_j)_{j=1}^{k-2} 0 a_k]$ and the rest of the argument is similar with H disjoint and to the left of G instead of disjoint and to the right of G.

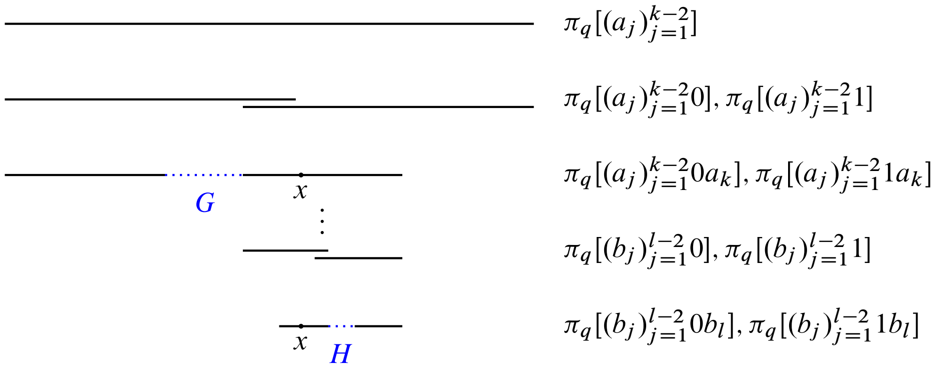


Figure 6. The projections of cylinders used to show $G \cap H = \emptyset$ in the proof of Lemma 3.17. The sets G and H contain the intervals represented by the dotted lines above them. The point $x \in \pi_q(A_q)$ is an element of neither G nor H but necessarily lies between them.

Recall that j_0 is the smallest free zero of $1^M(c_j)_{j=M+1}^{\infty}$, so let $(a_j)_{j=1}^{j_0-1}$ be the unique element of $A_q^{j_0-1}$, then $(a_j)_{j=1}^{j_0-1}$ is a prefix of every element of A_q. So $\pi_q(A_q) \subset \pi_q[(a_j)_{j=1}^{j_0-1}]$. The endpoints of $\pi_q[(a_j)_{j=1}^{j_0-1}]$ are not in $\pi_q(A_q)$ so

$$\operatorname{conv}\big(\pi_q(A_q)\big) \subset \pi_q\big[(a_j)_{j=1}^{j_0-1}\big]^{\circ}.$$

Hence all gaps are contained in $\pi_q[(a_j)_{j=1}^{j_0-1}]^{\circ}$ and so for any gap G, k_G is at least $j_0 + 1$. A calculation shows that $|\pi_q[(a_j)_{j=1}^{k_G-2}]| = q^{-k_G+2}(\frac{1}{q-1})$ which approaches 0 as $k_G \to \infty$. Since every gap G has positive length, k_G is bounded above for every gap G. This shows that for any gap G, k_G exists, is finite and is unique since it is maximal within a finite set.

Let G be a gap and $k_G \in \mathbb{N}$ as above. We show that $k_G - 1$ is a free zero by contradiction. Suppose $k_G - 1$ is a fixed index and let $a_{k_G-1} \in \{0, 1\}$ be such that $(a_j)_{j=1}^{k_G-2} a_{k_G-1} \in A_q^{k_G-1}$ so $(a_j)_{j=1}^{k_G-2}\overline{a_{k_G-1}} \notin A_q^{k_G-1}$. We know that

$$G \subset \pi_q\big[(a_j)_{j=1}^{k_G-2}\big]^{\circ}$$

and

$$\pi_q\big[(a_j)_{j=1}^{k_G-2}\big]^{\circ} = \pi_q\big[(a_j)_{j=1}^{k_G-2} a_{k_G-1}\big]^{\circ} \cup \pi_q\big[(a_j)_{j=1}^{k_G-2}\overline{a_{k_G-1}}\big]^{\circ}$$

Therefore, since $G \not\subset \pi_q[(a_j)_{j=1}^{k_G-2} a_{k_G-1}]^{\circ}$ we know that

$$G \cap \pi_q\big[(a_j)_{j=1}^{k_G-2}\overline{a_{k_G-1}}\big] \neq \emptyset.$$

Because $(a_j)_{j=1}^{k_G-2}\overline{a_{k_G-1}} \notin A_q^{k_G-1}$, Lemma 3.14 tells us that

$$\pi_q\big[(a_j)_{j=1}^{k_G-2}\overline{a_{k_G-1}}\big] \cap \pi_q(A_q) = \emptyset$$

so we know that $\pi_q[(a_j)_{j=1}^{k_G-2}\overline{a_{k_G-1}}] \subset G$. Observe that

$$\pi_q\big((a_j)_{j=1}^{k_G-2}(\overline{a_{k_G-1}})^\infty\big) \in \pi_q\big[(a_j)_{j=1}^{k_G-2}\overline{a_{k_G-1}}\big]$$

is an extremal point of $\pi_q[(a_j)_{j=1}^{k_G-2}]$ so

$$\pi_q\big((a_j)_{j=1}^{k_G-2}(\overline{a_{k_G-1}})^\infty\big) \in \pi_q\big[(a_j)_{j=1}^{k_G-2}\big] \setminus \pi_q\big[(a_j)_{j=1}^{k_G-2}\big]^\circ$$

which contradicts $G \subset \pi_q[(a_j)_{j=1}^{k_G-2}]^\circ$. Hence $k_G - 1$ is a free zero.

We next show that $\pi_q[(a_j)_{j=1}^{k_G-2}\overline{a_{k_G}a_{k_G}}] \cap G = \emptyset$. We show that this implies that G contains the interval between $\pi_q[(a_j)_{j=1}^{k_G-2}0a_{k_G}]$ and $\pi_q[(a_j)_{j=1}^{k_G-2}1a_{k_G}]$, proving G is a level k_G gap. Since $k_G - 1$ is a free zero $k_G - 2$ and k_G are fixed indices so let $a_{k_G} \in \{0, 1\}$ be such that $(a_j)_{j=1}^{k_G-2}0a_{k_G}, (a_j)_{j=1}^{k_G-2}1a_{k_G} \in A_q^{k_G}$. Notice that by Lemma 3.14, $\pi_q[(a_j)_{j=1}^{k_G-2}\overline{a_{k_G}a_{k_G}}] \cap \pi_q(A_q) = \emptyset$ because $(a_j)_{j=1}^{k_G-2}\overline{a_{k_G}a_{k_G}} \notin A_q^{k_G}$. Suppose $\pi_q[(a_j)_{j=1}^{k_G-2}\overline{a_{k_G}a_{k_G}}] \cap G \neq \emptyset$, then since G is a gap and

$$\pi_q\big[(a_j)_{j=1}^{k_G-2}\overline{a_{k_G}a_{k_G}}\big] \cap \pi_q(A_q) = \emptyset,$$

this is equivalent to $\pi_q[(a_j)_{j=1}^{k_G-2}\overline{a_{k_G}a_{k_G}}] \subset G$. In this case,

$$\pi_q\big((a_j)_{j=1}^{k_G-2}(\overline{a_{k_G}})^\infty\big) \in G,$$

but

$$\pi_q\big((a_j)_{j=1}^{k_G-2}(\overline{a_{k_G}})^\infty\big) \in \pi_q\big[(a_j)_{j=1}^{k_G-2}\big] \setminus \pi_q\big[(a_j)_{j=1}^{k_G-2}\big]^\circ,$$

which contradicts $G \subset \pi_q[(a_j)_{j=1}^{k_G-2}]^\circ$. Hence, $\pi_q[(a_j)_{j=1}^{k_G-2}\overline{a_{k_G}a_{k_G}}] \cap G = \emptyset$.

Consider the union

$$\pi_q\big[(a_j)_{j=1}^{k_G-2}\big] = \bigcup_{\delta \in \{a_{k_G}a_{k_G}, a_{k_G}\overline{a_{k_G}}, \overline{a_{k_G}}a_{k_G}, \overline{a_{k_G}a_{k_G}}\}} \pi_q\big[(a_j)_{j=1}^{k_G-2}\delta\big].$$

Since $\pi_q[(a_j)_{j=1}^{k_G-2}\overline{a_{k_G}a_{k_G}}] \cap G = \emptyset$, we know that

$$G \subset \bigcup_{\delta \in \{a_{k_G}a_{k_G}, a_{k_G}\overline{a_{k_G}}, \overline{a_{k_G}}a_{k_G}\}} \pi_q\big[(a_j)_{j=1}^{k_G-2}\delta\big].$$

We know that $\{a_{k_G}a_{k_G}, \overline{a_{k_G}}a_{k_G}\} = \{0a_{k_G}, 1a_{k_G}\}$ and $(a_j)_{j=1}^{k_G-2}a_{k_G}\overline{a_{k_G}} \notin A_q^{k_G}$. Since G is an interval which is not contained in either

$$\pi_q\big[(a_j)_{j=1}^{k_G-2}0a_{k_G}\big]^\circ \text{ or } \pi_q\big[(a_j)_{j=1}^{k_G-2}1a_{k_G}\big]^\circ,$$

we know $\pi_q[(a_j)_{j=1}^{k_G-2} a_{k_G}\overline{a_{k_G}}] \cap G \neq \emptyset$. Again, by Lemma 3.14 we know that

$$\pi_q\big[(a_j)_{j=1}^{k_G-2} a_{k_G}\overline{a_{k_G}}\big] \cap \pi_q(A_q) = \emptyset.$$

Since G is a gap, the fact that $\pi_q[(a_j)_{j=1}^{k_G-2} a_{k_G}\overline{a_{k_G}}] \cap G \neq \emptyset$ tells us that

$$\pi_q\big[(a_j)_{j=1}^{k_G-2} a_{k_G}\overline{a_{k_G}}\big] \subset G.$$

By inspection, either $a_{k_G}a_{k_G} \prec a_{k_G}\overline{a_{k_G}} \prec \overline{a_{k_G}}a_{k_G}$ or $\overline{a_{k_G}}a_{k_G} \prec a_{k_G}\overline{a_{k_G}} \prec a_{k_G}a_{k_G}$ and in each case both pairs $\{a_{k_G}a_{k_G}, a_{k_G}\overline{a_{k_G}}\}$ and $\{a_{k_G}\overline{a_{k_G}}, \overline{a_{k_G}}a_{k_G}\}$ are lexicographically consecutive. Therefore by Lemma 3.14,

$$\pi_q\big[(a_j)_{j=1}^{k_G-2} a_{k_G}\overline{a_{k_G}}\big] \cap \pi_q\big[(a_j)_{j=1}^{k_G-2} a_{k_G}a_{k_G}\big] \neq \emptyset,$$

and

$$\pi_q\big[(a_j)_{j=1}^{k_G-2} a_{k_G}\overline{a_{k_G}}\big] \cap \pi_q\big[(a_j)_{j=1}^{k_G-2} \overline{a_{k_G}}a_{k_G}\big] \neq \emptyset.$$

Since $\pi_q[(a_j)_{j=1}^{k_G-2} a_{k_G}\overline{a_{k_G}}] \subset G$ and G is an interval, we know G contains the interval between $\pi_q[(a_j)_{j=1}^{k_G-2} 0 a_{k_G}]$ and $\pi_q[(a_j)_{j=1}^{k_G-2} 1 a_{k_G}]$. Hence G is a level k_G gap. ∎

Lemma 3.19. *Let $q \in (q_9, 2)$. If G is a level k gap then $q^{-k} < |G| < q^{-k+1}$. Moreover, if H is a level l gap then $|G| > |H|$ whenever $k < l$.*

Proof. Let $q \in (q_9, 2)$ and let G be a level k gap. Suppose G is the level k gap which contains the interval between $\pi_q[(a_j)_{j=1}^{k-2} 0 a_k]$ and $\pi_q[(a_j)_{j=1}^{k-2} 1 a_k]$ for some $(a_j)_{j=1}^{k-2} 0 a_k, (a_j)_{j=1}^{k-2} 1 a_k \in A_q^k$. We find lower and upper bounds on $|G|$ by inspecting further restrictions on the endpoints of G.

Let J_1, J_2, J_3 be successive free zeros such that $J_1 = k-1$. Let α_1 and α_2 be the binary strings between the pairs of free zeros J_1, J_2 and J_2, J_3 respectively. That is $\alpha_1 = (a_j)_{j=J_1+1}^{J_2-1}$ and $\alpha_2 = (a_j)_{j=J_2+1}^{J_3-1}$. Since J_1, J_2, J_3 are successive free zeros, there are no free zeros in the ranges $J_1+1, \ldots, J_2-1$ and $J_2+1, \ldots, J_3-1$, so α_1 and α_2 are fixed since they are indexed by fixed indices. There is at least one and at most four fixed indices between free zeros of any sequence in A_q, so $1 \le |\alpha_1|, |\alpha_2| \le 4$. Therefore, for any $a_{J_1}, a_{J_2} \in \{0,1\}$ we know that

$$(a_j)_{j=1}^{k-2} a_{J_1}\alpha_1 a_{J_2}\alpha_2 \in A_q^{k+|\alpha_1|+|\alpha_2|}.$$

By the above construction, we have the equality:

$$\begin{aligned}\big\{(a_j)_{j=1}^{k-2} a_{J_1}\alpha_1 a_{J_2}\alpha_2 \in A_q^{k+|\alpha_1|+|\alpha_2|} : a_{J_1}, a_{J_2} \in \{0,1\}\big\} \\ = \big\{(a'_j)_{j=1}^{k+|\alpha_1|+|\alpha_2|} \in A_q^{k+|\alpha_1|+|\alpha_2|} : (a'_j)_{j=1}^{k-2} = (a_j)_{j=1}^{k-2}\big\}. \quad (3.7)\end{aligned}$$

In words, (3.7) is saying that the expression $(a_j)_{j=1}^{k-2} a_{J_1} \alpha_1 a_{J_2} \alpha_2$ describes all sequences in $A_q^{k+|\alpha_1|+|\alpha_2|}$ prefixed by $(a_j)_{j=1}^{k-2}$.

We know G contains the interval between $\pi_q[(a_j)_{j=1}^{k-2} 0 a_k]$ and $\pi_q[(a_j)_{j=1}^{k-2} 1 a_k]$, therefore, G contains the interval between the rightmost element of $\pi_q[A_q^{k+|\alpha_1|+|\alpha_2|}]$ contained in $\pi_q[(a_j)_{j=1}^{k-2} 0 a_k]$ and the leftmost element of $\pi_q[A_q^{k+|\alpha_1|+|\alpha_2|}]$ contained in $\pi_q[(a_j)_{j=1}^{k-2} 1 a_k]$. By Equation (3.7), the rightmost element of $\pi_q[A_q^{k+|\alpha_1|+|\alpha_2|}]$ contained in $\pi_q[(a_j)_{j=1}^{k-2} 0 a_k]$ is given by $\pi_q[(a_j)_{j=1}^{k-2} 0 \alpha_1 1 \alpha_2]$ and the leftmost element of $\pi_q[A_q^{k+|\alpha_1|+|\alpha_2|}]$ contained in $\pi_q[(a_j)_{j=1}^{k-2} 1 a_k]$ is given by $\pi_q[(a_j)_{j=1}^{k-2} 1 \alpha_1 0 \alpha_2]$. Moreover, we know that G is contained in the convex hull of the union of these intervals, that is,

$$G \subset \operatorname{conv}\left(\pi_q\left[(a_j)_{j=1}^{k-2} 0 \alpha_1 1 \alpha_2\right] \cup \pi_q\left[(a_j)_{j=1}^{k-2} 1 \alpha_1 0 \alpha_2\right]\right),$$

because both $\pi_q[(a_j)_{j=1}^{k-2} 0 \alpha_1 1 \alpha_2]$ and $\pi_q[(a_j)_{j=1}^{k-2} 1 \alpha_1 0 \alpha_2]$ are elements of the set $\pi_q[A_q^{k+|\alpha_1|+|\alpha_2|}]$.

The above argument finds an interval which is contained in G and an interval which contains G. This provides both upper and lower bounds on $|G|$ (see Figure 7). The containments described above give the following inequality,

$$\begin{aligned} \pi_q\left((a_j)_{j=1}^{k-2} 1 \alpha_1 0 \alpha_2 0^\infty\right) - \pi_q\left((a_j)_{j=1}^{k-2} 0 \alpha_1 1 \alpha_2 1^\infty\right) &\le |G| \\ &\le \pi_q\left((a_j)_{j=1}^{k-2} 1 \alpha_1 0 \alpha_2 1^\infty\right) - \pi_q\left((a_j)_{j=1}^{k-2} 0 \alpha_1 1 \alpha_2 0^\infty\right), \end{aligned}$$

which algebraically gives,

$$\begin{aligned} q^{-(k-2)}\left[q^{-1} - q^{-2-|\alpha_1|} - q^{-2-|\alpha_1|-|\alpha_2|}\left(\frac{1}{q-1}\right)\right] &\le |G| \\ &\le q^{-(k-2)}\left[q^{-1} - q^{-2-|\alpha_1|} + q^{-2-|\alpha_1|-|\alpha_2|}\left(\frac{1}{q-1}\right)\right]. \end{aligned} \tag{3.8}$$

For the lower bound, we observe that the left-hand side of (3.8) is smallest when the negative terms are largest, i.e., when $|\alpha_1| = |\alpha_2| = 1$. Then

$$\begin{aligned} |G| &\ge q^{-k+2}\left[q^{-1} - q^{-3} - q^{-4}\left(\frac{1}{q-1}\right)\right] \\ &> q^{-k+1}\left[1 - \frac{25}{8} q^{-3}\right] \end{aligned}$$

where the last inequality comes from the two implications $q < 2 \implies q^{-3} < 2q^{-4}$ and $q > q_9 \implies \frac{1}{q-1} < \frac{9}{8}$. Now using $q > q_9 \implies q^{-3} < \frac{1}{7}$ and $q^{-1} < \frac{31}{56}$ we have

$$|G| > q^{-k}.$$

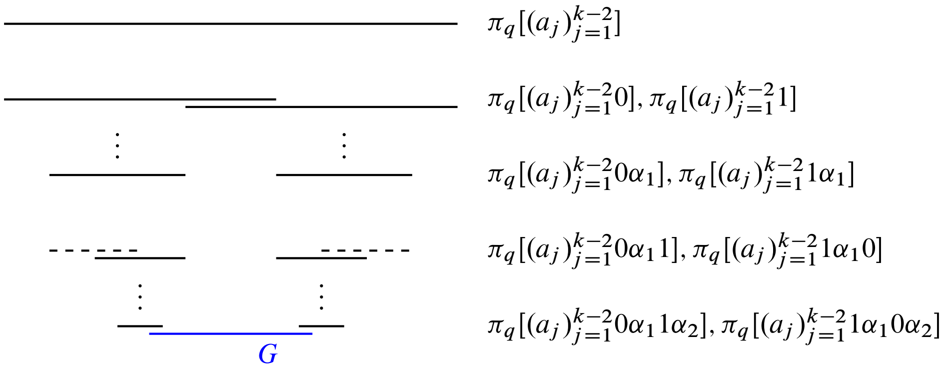


Figure 7. Upper and lower bounds for $|G|$ in the context of Lemma 3.19.

For the upper bound, the right-hand inequality of (3.8) is equivalent to

$$|G| \leq q^{-k+2}\left[q^{-1} + q^{-2-|\alpha_1|}\left(\frac{q^{-|\alpha_2|}}{q-1} - 1\right)\right].$$

The product $q^{-2-|\alpha_1|}(\frac{q^{-|\alpha_2|}}{q-1} - 1)$ is negative so $|G| < q^{-k+1}$. Therefore, $q^{-k} < |G| < q^{-k+1}$. If H is a level l gap for some $l > k$ then since $l-1$ is a free zero, $l \geq k+2$ so $|H| < q^{-l+1} \leq q^{-k-1} < q^{-k} < |G|$ and the lemma is proved. ■

Lemma 3.19 allows us to estimate the length of the bridge either side of some gap G. If G is a level k gap of $\pi_q(A_q)$, we know by Lemma 3.19 that if $l > k$ then all level l gaps are smaller than G. Let H be the gap at least as large and immediately to the left of G. Then H is a level m gap for some $m \leq k$. Therefore to bound L_G below it suffices to find a lower bound on the distance between G and H.

Lemma 3.20. *Let $q \in (q_9, 2)$. If G is a level k gap of $\pi_q(A_q)$ then the bridges associated with G on the left and the right, L_G and R_G, have diameter $|L_G|, |R_G| > q^{-k-4}$.*

Proof. Let $k-1$ be a free zero and let $(a_j)_{j=1}^{k-2}0a_k, (a_j)_{j=1}^{k-2}1a_k \in A_q^k$. We know by Lemma 3.17, there is a unique level k gap G which contains the interval between $\pi_q[(a_j)_{j=1}^{k-2}0a_k]$ and $\pi_q[(a_j)_{j=1}^{k-2}1a_k]$. Given G, it is sufficient to bound the diameter of the left bridge $|L_G|$ from below because the same bound on the diameter of the right bridge $|R_G|$ follows by symmetry. To bound $|L_G|$ from below we seek an interval immediately to the left of G which does not intersect any gap larger than G, whose length we bound below.

We start by proving that $\pi_q[(a_j)_{j=1}^{k-2}0a_k]$ does not contain a level l gap for any $l \leq k$. Let H be a level l gap for some $l \leq k$. Then there are elements $(b_j)_{j=1}^{l-2}0b_l$, $(b_j)_{j=1}^{l-2}1b_l \in A_q^l$ such that the gap H contains the interval between $\pi_q[(b_j)_{j=1}^{l-2}0b_l]$

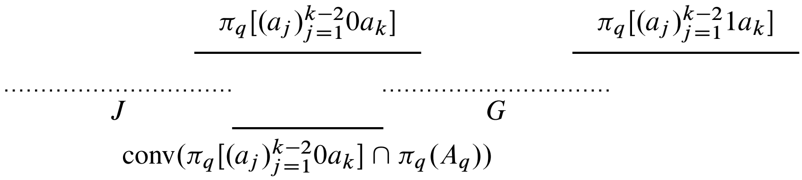


Figure 8. Constructing the subset $\mathrm{conv}(\pi_q[(a_j)_{j=1}^{k-2}0a_k] \cap \pi_q(A_q)) \subset \pi_q[(a_j)_{j=1}^{k-2}0a_k]$ which does not intersect any gap with diameter greater than or equal to $|G|$. J is defined to be the gap which contains the point $\pi_q((a_j)_{j=1}^{k-2}0a_k0^\infty)$ which may satisfy $|J| \geq |G|$.

and $\pi_q[(b_j)_{j=1}^{l-2}1b_l]$, and this uniquely determines H. By the proof of Lemma 3.18, H contains the interval $\pi_q[(b_j)_{j=1}^{l-2}b_l\overline{b_l}]$ and since $l \leq k$, we have the inequality

$$|H| \geq \left|\pi_q\left[(b_j)_{j=1}^{l-2}b_l\overline{b_l}\right]\right| \geq \left|\pi_q\left[(a_j)_{j=1}^{k-2}0a_k\right]\right|.$$

Hence H cannot be contained in $\pi_q[(a_j)_{j=1}^{k-2}0a_k]$.

By Lemmas 3.18 and 3.19, this implies that any gap contained in $\pi_q[(a_j)_{j=1}^{k-2}0a_k]$ is smaller than G. We emphasise that it is possible for a gap with diameter greater than or equal to $|G|$ to *intersect* $\pi_q[(a_j)_{j=1}^{k-2}0a_k]$ but no such gap is *contained* in $\pi_q[(a_j)_{j=1}^{k-2}0a_k]$. Removing the elements of $\pi_q[(a_j)_{j=1}^{k-2}0a_k]$ which are not elements of $\pi_q(A_q)$ and taking the convex hull removes the possibility of intersecting with a gap of diameter greater than or equal to $|G|$ (see Figure 8). That is, the set

$$\mathrm{conv}\left(\pi_q\left[(a_j)_{j=1}^{k-2}0a_k\right] \cap \pi_q(A_q)\right)$$

does not intersect any gap with diameter greater than or equal to $|G|$. Therefore a lower bound on $|\pi_q[(a_j)_{j=1}^{k-2}0a_k] \cap \pi_q(A_q)|$ is a lower bound on $|L_G|$. It now suffices to show that $|\pi_q[(a_j)_{j=1}^{k-2}0a_k] \cap \pi_q(A_q)| > q^{-k-4}$.

As in the proof of Lemma 3.19, let J_1, J_2, J_3, J_4 be successive free zeros such that $J_1 = k-1$ and let $\alpha_1, \alpha_2, \alpha_3$ be the fixed binary strings between the pairs of free zeros J_1, J_2 and J_2, J_3 and J_3, J_4 respectively. That is $\alpha_1 = (a_j)_{j=J_1+1}^{J_2-1}$, $\alpha_2 = (a_j)_{j=J_2+1}^{J_3-1}$ and $\alpha_3 = (a_j)_{j=J_3+1}^{J_4-1}$ are fixed because there are no free zeros in the ranges $J_1+1, \ldots, J_2-1$ and $J_2+1, \ldots, J_3-1$ and $J_3+1, \ldots, J_4-1$. There is at least one and at most four fixed indices between free zeros of any sequence in A_q, so $1 \leq |\alpha_1|, |\alpha_2|, |\alpha_3| \leq 4$. As before, the implication of this is that $\alpha_1, \alpha_2, \alpha_3$ are such that for any $a_{J_1}, a_{J_2}, a_{J_3} \in \{0, 1\}$, $(a_j)_{j=1}^{k-2}a_{J_1}\alpha_1 a_{J_2}\alpha_2 a_{J_3}\alpha_3 \in A_q^{k+1+|\alpha_1|+|\alpha_2|+|\alpha_3|}$. Using this, we know that $\pi_q[(a_j)_{j=1}^{k-2}0\alpha_1 0\alpha_2 0\alpha_3]$ and $\pi_q[(a_j)_{j=1}^{k-2}0\alpha_1 1\alpha_2 1\alpha_3]$ are subintervals of $\pi_q[(a_j)_{j=1}^{k-2}0a_k]$ and intersect $\pi_q(A_q)$, so the distance between them is a lower bound for $|L_G|$. This distance is given by

$$\begin{aligned}|L_G| &\geq \pi_q\big((a_j)_{j=1}^{k-2}0\alpha_1 1\alpha_2 1\alpha_3 0^\infty\big) - \pi_q\big((a_j)_{j=1}^{k-2}0\alpha_1 0\alpha_2 0\alpha_3 1^\infty\big)\\ &= q^{-(k-1+|\alpha_1|)}\left[q^{-1} + q^{-2-|\alpha_2|} - q^{-2-|\alpha_2|-|\alpha_3|}\left(\frac{1}{q-1}\right)\right]\\ &= q^{-k-|\alpha_1|}\left[1 + q^{-1-|\alpha_2|}\left(1 - \frac{q^{-|\alpha_3|}}{q-1}\right)\right].\end{aligned}$$

We aim to minimize this expression under the constraints $1 \leq |\alpha_1|, |\alpha_2|, |\alpha_3| \leq 4$. Since $|\alpha_3| \geq 1$, the term in round brackets is positive which implies the term in square brackets is positive. This gives $|\alpha_1| = 4$ and $|\alpha_2| = 4$. $|\alpha_3| = 1$ minimizes the term in round brackets and hence the whole expression. This gives,

$$|L_G| \geq q^{-k-4}\left[1 + q^{-5}\left(1 - \frac{q^{-1}}{q-1}\right)\right] > q^{-k-4},$$

which completes the proof. ∎

Recall from Lemma 3.19 that if G is a level k gap then $|G| < q^{-k+1}$. We combine this with Lemma 3.20 to find a lower bound on the thickness of $\pi_q(A_q)$.

Proof of Proposition 3.13. By Lemma 3.18, we can rewrite the definition of the thickness of $\pi_q(A_q)$ in terms of level k gaps. That is, we seek a lower bound on

$$\tau\big(\pi_q(A_q)\big) = \inf\left\{\min\left\{\frac{|L_G|}{|G|}, \frac{|R_G|}{|G|}\right\} : k \in \mathbb{N}, G \text{ a level } k \text{ gap}\right\}.$$

If G is a level k gap then $|L_G|, |R_G| > q^{-k-4}$ by Lemma 3.20 and $|G| < q^{-k+1}$ by Lemma 3.19. Therefore

$$\min\left\{\frac{|L_G|}{|G|}, \frac{|R_G|}{|G|}\right\} > \frac{q^{-k-4}}{q^{-k+1}} = q^{-5}.$$

Since this is independent of $k \in \mathbb{N}$, we conclude that $\tau(\pi_q(A_q)) > q^{-5}$. ∎

3.5. Interleaving

Let C_1 and C_2 be two compact subsets of $\mathbb{R}$ then C_1 and C_2 are *interleaved* if $\mathrm{conv}(C_1) \not\subset \mathbb{R} \setminus C_2$ and $\mathrm{conv}(C_2) \not\subset \mathbb{R} \setminus C_1$. This is often written more directly as neither set is contained in a gap of the other. For C_1 and C_2 to be interleaved, it is sufficient to show that $C_2 \subset \mathrm{conv}(C_1)$ and $|C_2| \geq |G|$ where G is the largest interval in $\mathrm{conv}(C_1) \setminus C_1$.

In this subsection, we prove the following proposition, which is the consequence of the two lemmas which follow it.

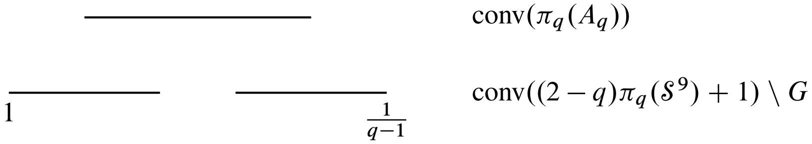


Figure 9. $\pi_q(A_q)$ and $(2-q)\pi_q(\mathcal{S}^9)+1$ are interleaved. Here, G denotes the largest gap of $((2-q)\pi_q(\mathcal{S}^9)+1)$.

Proposition 3.21. *The sets* $(2-q)\pi_q(\mathcal{S}^9)+1$ *and* $\pi_q(A_q)$ *are interleaved.*

As shown in [32] and Subsection 3.4 respectively, the sets $\pi_q(\mathcal{S}^k)$ and $\pi_q(A_q)$ are each the complement of an interval with a collection of infinitely many gaps of decreasing length. We show that $\operatorname{conv}((2-q)\pi_q(\mathcal{S}^9)+1)$ contains $\pi_q(A_q)$ and that the largest gap of $((2-q)\pi_q(\mathcal{S}^9)+1)$ is smaller than $|\pi_q(A_q)|$, which is sufficient for the sets to be interleaved. Figure 9 shows $\operatorname{conv}(\pi_q(A_q))$ and $\operatorname{conv}((2-q)\pi_q(\mathcal{S}^9)+1)$ with the largest gap removed. Notice that $\pi_q(\mathcal{S}^9)$ contains 0 and $\frac{1}{q-1}$ because $0=\pi_q(0^\infty)\in\pi_q(\mathcal{S}^9)$ and $\frac{1}{q-1}=\pi_q(1^\infty)\in\pi_q(\mathcal{S}^9)$. Since 0 and $\frac{1}{q-1}$ are extremal points of $\pi_q(\mathcal{S}^9)$, it is easy to check that $\operatorname{conv}((2-q)\pi_q(\mathcal{S}^9)+1)=[1,\frac{1}{q-1}]$.

Lemma 3.22. *If* $q\in(q_9,2)$ *then* $\pi_q(A_q)\subset[1,\frac{1}{q-1}]$.

Proof. Let $q\in(q_9,2)$ and $(c_j)_{j=1}^\infty\in\{-1,0,1\}^{\mathbb{N}}$ be the fixed expansion of 1 (see Subsection 3.3). Let $(a_j)_{j=1}^\infty\in A_q$ and $(b_j)_{j=1}^\infty$ be the sequence with $c_j=a_j-b_j$ for all $j\in\mathbb{N}$. Using that $\pi_q((c_j)_{j=1}^\infty)=1$, $\pi_q((b_j)_{j=1}^\infty)>0$ and $\pi_q((a_j)_{j=1}^\infty),\pi_q((b_j)_{j=1}^\infty)\in\pi_q(\mathcal{S}^9)$, we know that $\pi_q((a_j)_{j=1}^\infty)>1$. Lexicographically, $(a_j)_{j=1}^\infty\in A$ satisfies $(a_j)_{j=1}^\infty\prec 1^\infty$, which implies $\pi_q((a_j)_{j=1}^\infty)<\frac{1}{q-1}$. Therefore $\pi_q(A_q)\subset[1,\frac{1}{q-1}]$. ∎

Lemma 3.23. *If* $q\in(q_9,2)$ *the diameter of the largest gap of* $((2-q)\pi_q(\mathcal{S}^9)+1)$ *is smaller than* $|\pi_q(A_q)|$.

Proof. The largest gap of $((2-q)\pi_q(\mathcal{S}^9)+1)$ has the same diameter as the largest gap of $(2-q)\pi_q(\mathcal{S}^9)$. We prove the lemma by bounding this gap above and showing that this upper bound is less than a lower bound for $|\pi_q(A_q)|$. We start by finding a lower bound for $|\pi_q(A_q)|$.

Let $q\in(q_9,2)$. Recall from Subsection 3.2 that

$$H_q=\left[-\frac{q}{q^2-1},\frac{q}{q^2-1}\right].$$

If $q_k<q\le q_{k+1}$ then by Lemma 3.3,

$$0<f_1^k(1)=q^k-q^{k-1}-\cdots-q-1\le\frac{1}{q}<\frac{q}{q^2-1},$$

so $f_1^k(1) \in H_q$. Let s be the smallest index such that $f_1^s(1) \in H_q$ (so $s \le k$). Let $(c_j)_{j=1}^{\infty}$ be the fixed expansion of 1. By inspection of the words $w \in W_2$, if $c_{s+1} \neq 0$ then $c_{s+2} = 0$, so if j_0 is the first zero (which is a free zero) of $(c_j)_{j=1}^{\infty}$ then $j_0 \le s+2$. Since $s \le k$, then $j_0 \le k+2$.

Let $(j_{2k})_{k=0}^{\infty} = J_{\text{free}}$ be the sequence of free zeros of $(c_j)_{j=1}^{\infty}$ and for $l \in \mathbb{N}$ let

$$\alpha_l = (a_j)_{j=j_{2(l-1)}+1}^{j_{2l}-1}$$

be the values of every sequence in A_q at the fixed indices between $j_{2(l-1)}$ and j_{2l}. Since there is at least one free zero in every set of five consecutive indices after j_0, and free zeros never occupy consecutive indices, we know that $1 \le |\alpha_l| \le 4$ for every $l \in \mathbb{N}$. By definition,

$$A_q = \left\{(a_j)_{j=1}^{j_0-1} a_{j_0}\alpha_1 a_{j_2}\alpha_2 a_{j_4}\ldots : a_{j_{2k}} \in \{0,1\}\ \forall k \in \mathbb{N}_{\ge 0}\right\},$$

so $\pi_q(A_q)$ is bounded below and above by

$$L = \pi_q\big((a_j)_{j=1}^{j_0-1} 0\alpha_1 0\alpha_2 0\ldots\big) \quad\text{and}\quad U = \pi_q\big((a_j)_{j=1}^{j_0-1} 1\alpha_1 1\alpha_2 1\ldots\big)$$

respectively, and $L, U \in \pi_q(A_q)$. Hence

$$\begin{aligned}|\pi_q(A_q)| = U - L &= q^{-j_0+1}\big[q^{-1} + q^{-2-|\alpha_1|} + q^{-3-|\alpha_1|-|\alpha_2|} + \cdots\big]\\ &\ge q^{-k-1}\big[q^{-1} + q^{-6} + q^{-11} + \cdots\big] = q^{-k-2}\left[\frac{q^5}{q^5-1}\right]\\ &> q^{-k-2}.\end{aligned}$$

We compare this lower bound for $|\pi_q(A_q)|$ with an upper bound for the largest gap of $(2-q)\pi_q(\mathcal{S}^9)$. Let $q \in (q_9, 2)$ and let G be the largest gap of $\pi_q(\mathcal{S}^9)$. A consequence of the arguments in [32, Proof of Lemma 4.1] is that $|G| < q^{-8}$. If $q_k < q \le q_{k+1}$ then $q^{-k-1} \le (2-q) < q^{-k}$ by Lemma 3.3. Therefore, the diameter of the largest gap of $(2-q)\pi_q(\mathcal{S}^9) + 1$ is bounded above by $(2-q)|G| < q^{-k-8}$.

Hence if $q \in (q_9, 2)$ and $q \in (q_k, q_{k+1}]$ then

$$|\pi_q(A_q)| > q^{-k-2} > q^{-k-8} > (2-q)|G|. \qquad \blacksquare$$

We now observe that Proposition 3.21 follows from Lemmas 3.22 and 3.23 combined with the fact that $\mathrm{conv}(\pi_q(\mathcal{S}^9)) = [1, \frac{1}{q-1}]$.

3.6. Proof of Theorem 3.1

Proof of Theorem 3.1. Let $q \in (q_9, 2)$. By Newhouse's theorem ([25, Lemma 4]), in order for two compact subsets of $\mathbb{R}$ to have nonempty intersection, it is sufficient to show that the product of the thicknesses of the sets is greater than 1 and that the sets are interleaved, that is, neither set lies entirely within a gap of the other. Proposition 3.21 shows that $(2-q)\pi_q(\mathcal{S}^9)+1$ and $\pi_q(A_q)$ are interleaved and Proposition 3.13 shows that $\tau(\pi_q(A_q)) > q^{-5}$. In [32] it is shown that $\tau(\pi_q(\mathcal{S}^k)) > q^{k-3}$ whenever $q_k < q < 2$ for all $k \geq 3$. Hence, for $k = 9$ we can say that $\tau(\pi_q(\mathcal{S}^9)) > q^6$ whenever $q \in (q_9, 2)$. Because thickness is preserved under affine transformations, we also know that $\tau((2-q)\pi_q(\mathcal{S}^9)+1) > q^6$. Hence, we can conclude that

$$\tau\big((2-q)\pi_q(\mathcal{S}^9)+1\big) \times \tau\big(\pi_q(A_q)\big) > q^6 \times q^{-5} > 1$$

and by Newhouse's theorem, $((2-q)\pi_q(\mathcal{S}^9)+1) \cap \pi_q(A_q) \neq \emptyset$.

By Lemma 3.2, since $q \in (q_9, 2)$, we know

$$\big((2-q)\pi_q(\mathcal{S}^9)+1\big) \cap \pi_q(A_q) \neq \emptyset,$$

implies that

$$\big((2-q)\mathcal{U}_q+1\big) \cap \pi_q(A_q) \neq \emptyset.$$

Hence it is sufficient to show that $((2-q)\mathcal{U}_q+1) \cap \pi_q(A_q) \neq \emptyset$ implies that $q \in \mathcal{C}_3$.

If $((2-q)\mathcal{U}_q+1) \cap \pi_q(A_q) \neq \emptyset$ then let $qy \in ((2-q)\mathcal{U}_q+1) \cap \pi_q(A_q)$ for some $y \in [\frac{1}{q}, \frac{1}{q(q-1)}]$. Then[4]

$$\frac{1}{q-1} - \frac{qy-1}{2-q} \in \mathcal{U}_q,\ qy \in \pi_q(A_q) \subset \mathcal{U}_q \ \text{ and } \ qy-1 \in \pi_q(A_q)-1 \subset \mathcal{U}_q$$

by construction. Hence $f_0(y), f_1(y), f_2(y) \in \mathcal{U}_q$, so y has three orbits, i.e., $y \in \mathcal{V}_q^{(3)}$ and so $q \in \mathcal{C}_3$. ∎

4. Proof of Theorem 2

In this section, we prove Theorem 2 and Corollary 2. We start by proving that Theorem 2 is equivalent to Theorem 4.1 and that Corollary 2 follows from Theorem 4.1. We then show that Theorem 4.1 is equivalent to Theorem 4.2 before proving Theorem 4.2.

Let $a = (a_j)_{j=1}^k \in \{0,1,2\}^*$ be a finite ternary sequence, then $(f_{a_j})_{j=1}^k$ is the associated finite sequence of maps. We write f_a to denote the finite composition of maps $f_{a_1 \ldots a_k}$. When we do not state it explicitly, $|a|$ denotes the length of the sequence a, so in this case $|a| = k$. Given $q \in (1,2)$, $x \in I_q$, we define

$$\Omega_{E_q}^k(x) = \big\{(f_{i_j})_{j=1}^k \in \{f_0, f_1, f_2\}^k : f_{i_1 \ldots i_l}(x) \in I_q \text{ for all } 1 \leq l \leq k\big\},$$

[4]Recall that $x \in \mathcal{U}_q$ if and only if $\frac{1}{q-1} - x \in \mathcal{U}_q$ and we know that $\frac{qy-1}{2-q} \in \mathcal{U}_q$.

$$\Omega^*_{E_q}(x) = \bigcup_{k\in\mathbb{N}_{\geq 0}} \Omega^k_{E_q}(x),$$

and the set

$$\mathcal{D} = \{q \in (1,2) : |\Omega_{E_q}(x)| \in \{1, 2^{\aleph_0}\}\ \forall x \in I_q\}.$$

If $q \in \mathcal{D}$ then by Lemma 2.5, $|\mathrm{sl}_q(y)| \in \{1, 2^{\aleph_0}\}$ for all $y \in [0,1]$. Therefore, Theorem 2 is equivalent to Theorem 4.1.

Theorem 4.1. *Let $q \in \mathcal{D}$ and let $y \in [0,1]$. If $\mathrm{sl}_q(y)$ is uncountable then there is some $s > 0$ depending only on q such that $\dim_{\mathrm{H}}(\mathrm{sl}_q(y)) \geq s$.*

The implications of Corollary 2.8, [32, Theorem 2.1] and [4, Corollary 1.3] are that $(1, q_{\aleph_0}) \setminus \{G\} \subset \mathcal{D}$ and $\mathcal{T} \cap (q_{\aleph_0}, q_{\mathrm{KL}}) \subset \mathcal{D}$. Hence, Corollary 2 follows from Lemma 2.5 and Theorem 4.1. Some work is required to prove that Theorem 4.1 is equivalent to Theorem 4.2.

Theorem 4.2. *Let $q \in \mathcal{D}$. If $x \in J_q$ then there is some $s > 0$ depending only on q such that $\dim_{\mathrm{H}}(\mathrm{sl}_q(x(q-1))) \geq s$.*

Proof of Theorem 4.1 $\iff$ Theorem 4.2. The forwards implication is more straightforward so we prove this first. Observe that $x \in I_q \iff x(q-1) \in [0,1]$. Let $x \in J_q$ and let $q \in \mathcal{D}$, then we have the following sequence of implications.

$$x \in J_q \overset{\text{Lemma 2.4}}{\Longrightarrow} |\Omega_{E_q}(x)| \geq 2 \overset{q\in\mathcal{D}}{\Longrightarrow} \Omega_{E_q}(x) \text{ uncountable} \overset{\text{Lemma 2.5}}{\iff} \mathrm{sl}_q(x(q-1)) \text{ uncountable.} \tag{4.1}$$

The above chain of implications shows that the hypotheses of Theorem 4.1 hold if the hypotheses of Theorem 4.2 hold, and since the conclusions of the theorems are the same, we have proved the forwards implication.

For the reverse implication, let $q \in \mathcal{D}$ and $x(q-1) \in [0,1]$ satisfy $\mathrm{sl}_q(x(q-1))$ is uncountable. By Lemma 2.5, this is equivalent to $\Omega_{E_q}(x)$ is uncountable. By Lemma 2.4, this implies that there is a finite sequence of maps $(f_{a_j})_{j=1}^k \in \Omega^*_{E_q}(x)$ which satisfies $f_{a_1\ldots a_k}(x) \in J_q$. Applying Theorem 4.2, we conclude that there is some $s > 0$ depending only on q such that $\dim_{\mathrm{H}}(\mathrm{sl}_q(f_{a_1\ldots a_k}(x)(q-1))) \geq s$. Hence it suffices to show, for any finite sequence of maps $(f_{a_j})_{j=1}^k \in \Omega^*_{E_q}(x)$, that

$$\dim_{\mathrm{H}}\big(\mathrm{sl}_q(x(q-1))\big) \geq \dim_{\mathrm{H}}\big(\mathrm{sl}_q(f_{a_1\ldots a_k}(x)(q-1))\big).$$

Let $k \in \mathbb{N}_{\geq 0}$, $x \in I_q$ and let $(f_{a_j})_{j=1}^k \in \Omega^*_{E_q}(x)$ be a finite sequence of maps. By the definition of $\Omega_{E_q}(x)$,

$$(f_{a_j})_{j=1}^k (f_{i_j})_{j=1}^\infty \in \Omega_{E_q}(x) \iff (f_{i_j})_{j=1}^\infty \in \Omega_{E_q}(f_{a_1\ldots a_k}(x)).$$

Hence, by Lemma 2.5,

$$\pi_3\big((i_j)_{j=1}^{\infty}\big) \in \mathrm{sl}_q(f_{a_1\ldots a_k}(x)(q-1)) \iff \pi_3((a_j)_{j=1}^{k}(i_j)_{j=1}^{\infty}) \in \mathrm{sl}_q(x(q-1)).$$

That is, for some affine map $F : [0,1] \to [0,1]$ which maps the point $\pi_3((i_j)_{j=1}^{\infty})$ to $\pi_3((a_j)_{j=1}^{k}(i_j)_{j=1}^{\infty})$,

$$F(\mathrm{sl}_q(f_{a_1\ldots a_k}(x)(q-1))) \subset \mathrm{sl}_q(x(q-1)),$$

so $\dim_{\mathrm{H}}(\mathrm{sl}_q(x(q-1))) \geq \dim_{\mathrm{H}}(\mathrm{sl}_q(f_{a_1\ldots a_k}(x)(q-1)))$. ∎

We outline the logic of the proof of Theorem 4.2.

(1) Let $q \in \mathcal{D}$ and let $x \in J_q$. If there is some set $\mathcal{R} \subset \{0,1,2\}^{\mathbb{N}}$ such that $\pi_3(\mathcal{R}) \subset \mathrm{sl}_q(x(q-1))$ then it suffices to prove that $\dim_{\mathrm{H}}(\pi_3(\mathcal{R})) \geq s$ for some $s > 0$. The set $\mathcal{R}$ depends on both q and x but s depends only on q.

(2) For all $k \in \mathbb{N}_{\geq 0}$ we construct the sets $\mathcal{R}^k \subset \{0,1,2\}^{\mathbb{N}}$ in terms of a function $A : \{0,1\}^* \to \{0,1,2\}^*$ written $A(\epsilon) = b^{\epsilon} = (b_j^{\epsilon})_{j=1}^{N_\epsilon}$ which depends on x and q. Precisely, for all $k \in \mathbb{N}_{\geq 0}$, $\mathcal{R}^k$ is the union of cylinders $\cup_{\epsilon \in \{0,1\}^k}[b^{\epsilon}]$. So elements of $\mathcal{R}^k$ are sequences in $\{0,1,2\}^{\mathbb{N}}$ which are prefixed by b^{ϵ} for some $\epsilon \in \{0,1\}^k$.

(3) Define $\mathcal{R} = \cap_{k\in\mathbb{N}_{\geq 0}} \mathcal{R}^k$. Therefore, the set $\pi_3(\mathcal{R})$ is the countable intersection over $k \in \mathbb{N}_{\geq 0}$ of sets $\pi_3(\mathcal{R}^k)$. That is,

$$\pi_3(\mathcal{R}) = \bigcap_{k\in\mathbb{N}_{\geq 0}} \bigcup_{\epsilon\in\{0,1\}^k} \pi_3[b^{\epsilon}].$$

It follows from the properties of the function A that the $\mathcal{R}^k$ sets are nested and nonempty so the above intersection is nonempty.

(4) It is a consequence of the construction of the sets $\mathcal{R}^k$ that for each $k \in \mathbb{N}_{\geq 0}$, if $[b^{\epsilon}] \subset \mathcal{R}^k$ then $(f_{b_j^{\epsilon}})_{j=1}^{N_\epsilon} \in \Omega_{E_q}^*(x)$. So if $(i_j)_{j=1}^{\infty} \in \mathcal{R}$, we know that $(f_{i_j})_{j=1}^{\infty} \in \Omega_{E_q}(x)$ which provides the property that $\pi_3(\mathcal{R}) \subset \mathrm{sl}_q(x(q-1))$ by Lemma 2.5.

(5) A measure μ is defined on $\pi_3(\mathcal{R})$ by declaring the value of $\mu(\pi_3[b^{\epsilon}])$ for all $\epsilon \in \{0,1\}^*$. We show that this measure μ satisfies the following property:

$$\text{For every set } U \subset \mathbb{R},\ \mu(U) \leq 4|U|^s \text{ where } s = \frac{\log 2}{M \log 3}, \tag{P}$$

where $M \in \mathbb{N}$ is a constant depending only on q which arises from the construction of the $\mathcal{R}^k$ sets. (We emphasise that the constant M in this section is different from the M which appears in the definition of the fixed expansion of 1 in the previous section.) Via the mass distribution principle [13], the existence of such a measure proves that $\mathcal{H}^s(\pi_3(\mathcal{R})) \geq \frac{1}{4}$ and $\dim_{\mathrm{H}}(\pi_3(\mathcal{R})) \geq s$.

(6) The existence of a measure μ with property (P) relies on the fact that for any $l \in \mathbb{N}_{\geq 0}$, $\pi_3(\mathcal{R}^l)$ is the union of 2^l intervals with disjoint interiors and of diameter at least 3^{-lM}. Using this fact, it is possible to prove that there exists a measure μ such that whenever $lM \leq k < (l+1)M$ and $\epsilon \in \{0,1\}^k$, $\mu(\pi_3[b^\epsilon]) \leq 2(2^{-\frac{k}{M}})$.

(7) The existence of the $\mathcal{R}^k$ sets with the required property described in Item 6 makes use of branching tree constructions and the compactness of the interval J_q.

The majority of the work of the proof is in the construction of the $\mathcal{R}^k$ sets.

4.1. Branching trees

In this subsection, we prove the following proposition. To do this, we follow [4] by constructing the *branching tree* and the *infinite branching tree* for any $x \in I_q$.

Proposition 4.3. *Let $q \in \mathcal{D}$. For every $x \in J_q$ there are two distinct finite sequences of maps in $\Omega^*_{E_q}(x)$ which map x into J_q°.*

The branching tree constructions below provide a useful visual aid when thinking about the orbit spaces $\Omega_{E_q}(x)$ for $x \in I_q$. This is because for any $x \in I_q$, the set of infinite paths in the branching tree of x is in bijection with $\Omega_{E_q}(x)$. We show how the structure of the infinite branching tree for $x \in I_q$ allows us to distinguish between the cases when $\Omega_{E_q}(x)$ is at most countably infinite and when it is uncountable. Recall from (4.1) that the hypotheses of Theorem 4.2 were shown to imply that $\Omega_{E_q}(x)$ is uncountable. We use the uncountability of $\Omega_{E_q}(x)$ to conclude that the infinite branching tree of x has a certain structure. This forms the first step towards constructing the set $\mathcal{R} \subset \{0,1,2\}^{\mathbb{N}}$ alluded to in the outline above. Before defining the branching trees, we need the following definitions.

Let $x \in I_q$ and let $b = (b_j)_{j=1}^N \in \{0,1,2\}^*$ be a finite ternary sequence. If b is such that $(f_{b_j})_{j=1}^N \in \Omega^*_{E_q}(x)$ and $f_b(x) \in J_q$ then $(f_{b_j})_{j=1}^N$ is a *branching sequence* for x, $f_b(x)$ is a *branching point* of x, and N is the *branching length* of $(f_{b_j})_{j=1}^N$. In particular, if $x \in J_q$ then if $e \in \{0,1,2\}^*$ is the empty word, f_e is the identity map so $f_e(x) = x \in J_q$ is a branching point and the empty word is a branching sequence for x. A *minimal branching sequence* for x is a branching sequence for x such that there does not exist a branching sequence for x with smaller branching length. If $|\Omega_{E_q}(x)| > 1$ then it is obvious that there is a branching sequence and hence also a minimal branching sequence for x.

If $x \in I_q$ and $b = (b_j)_{j=1}^N \in \{0,1,2\}^*$ is such that $(f_{b_j})_{j=1}^N$ is a branching sequence for x such that there are at least two indices $i \in \{0,1,2\}$ for which the orbit space $\Omega_{E_q}(f_i(f_b(x)))$ is infinite, then $(f_{b_j})_{j=1}^N$ is said to be a *doubly infin-*

ite branching sequence for x, $f_b(x)$ is a *doubly infinite branching point* of x and as before, N is the branching length of $(f_{b_j})_{j=1}^N$. A *minimal doubly infinite branching sequence* for x is a doubly infinite branching sequence for x such that there does not exist a doubly infinite branching sequence for x with smaller branching length.

4.1.1. Branching tree $\mathcal{T}(x)$. We define the *branching tree*, $\mathcal{T}(x)$, for any $x \in I_q$.

Let $x \in I_q$. Suppose x satisfies $|\Omega_{E_q}(x)| = 1$ then the branching tree for x is defined to be an infinite horizontal line, representing the unique orbit of x in E_q. If $|\Omega_{E_q}(x)| > 1$ then there is a minimal branching sequence $(f_{b_j})_{j=1}^N$ for x, which gives the branching point $f_b(x)$. To construct the branching tree $\mathcal{T}(x)$, the transformation f_b is represented by a finite horizontal line which then *bifurcates* or *trifurcates* into two or three branches respectively, as follows. If $f_b(x) \in J_q^\circ$ then we have a trifurcation and if $f_b(x) \in \partial J_q$ then we have a bifurcation. In the first case, the three branches of the trifurcation correspond to the images $\{f_0(f_b(x)), f_1(f_b(x)), f_2(f_b(x))\}$ which we define to be the *roots* of the branches. In the second case, if $f_b(x) = \frac{1}{q}$ then the two branches of the bifurcation correspond to the images $\{f_0(f_b(x)), f_2(f_b(x))\}$ and if $f_b(x) = \frac{1}{q(q-1)}$ then they correspond to $\{f_1(f_b(x)), f_2(f_b(x))\}$. This is a formal way of saying that the branching points have two or three branches which correspond to those maps in $\{f_0, f_1, f_2\}$ which satisfy the property that $f_b(x)$ is in the domain of f_i, and these points $f_i(f_b(x))$ are the roots of the branches. Recall that f_1 is not defined at $\frac{1}{q}$ and f_0 is not defined at $\frac{1}{q(q-1)}$, hence there are only two branches of $\mathcal{T}(x)$ at these points.

If for some $i \in \{0, 1, 2\}$, $f_i(f_b(x))$ satisfies $|\Omega_{E_q}(f_i(f_b(x)))| = 1$ then the branch with root $f_i(f_b(x))$ is extended by an infinite horizontal line. If $|\Omega_{E_q}(f_i(f_b(x)))| > 1$ then $f_i(f_b(x))$ has a unique minimal branching sequence $(f_{c_j})_{j=1}^L$ so we extend the branch with root $f_i(f_b(x))$ by a finite horizontal line segment which then bifurcates or trifurcates according to the value of $f_c(f_i(f_b(x)))$ as before. These rules are repeatedly applied to successive branches of the construction which results in a tree we call the *branching tree* of x and denote by $\mathcal{T}(x)$. See Figure 10 for a diagram of a branching tree.

For any branching point $f_b(x)$ of $\mathcal{T}(x)$, the branch with root $f_i(f_b(x))$ is the branching tree $\mathcal{T}(f_i(f_b(x)))$ and is a subtree (in the graph theoretical sense) of $\mathcal{T}(x)$. The branches of $\mathcal{T}(x)$ are in one-to-one correspondence with points of the form $f_i(f_b(x))$ where $f_b(x)$ is a branching point of x, $i \in \{0, 1, 2\}$ and $f_i(f_b(x)) \in I_q$. We remark that, from the definition of the branching tree, there is a bijection between the space of infinite paths in $\mathcal{T}(x)$ and the orbit space $\Omega_{E_q}(x)$. We say that a tree or a branch is finite if it contains finitely many branching points and infinite otherwise.

Observe that $|\Omega_{E_q}(f_2(\frac{1}{q}))| = |\Omega_{E_q}(f_1(\frac{1}{q(q-1)}))| = 1$ and $\frac{1}{q}, \frac{1}{q(q-1)}$ are not in the domains of f_1 and f_0, respectively. This means that if $f_b(x) \in \partial J_q$, there is at most one index $i \in \{0, 1, 2\}$ such that $\Omega_{E_q}(f_i(f_b(x)))$ is infinite. With Lemma 2.4, this

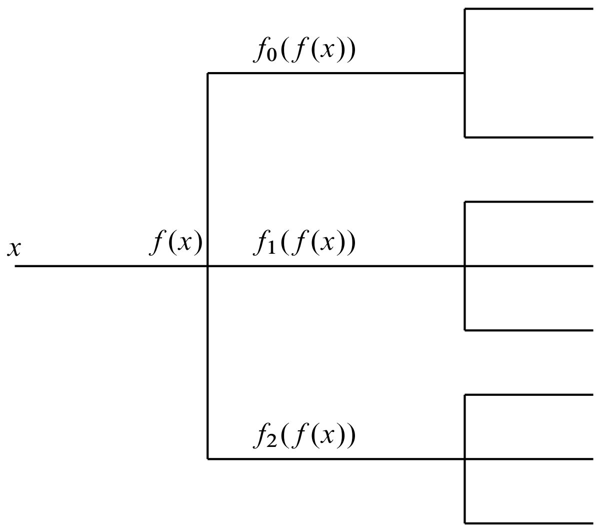


Figure 10. The start of the branching tree $\mathcal{T}(x)$ of some $x \in I_q$.

implies that if $(f_{b_j})_{j=1}^N$ is a doubly infinite branching sequence for x then $f_b(x) \in J_q^\circ$. Heuristically, we are aiming to show that the number of branches grows sufficiently quickly, so this kind of branching can be ignored. In fact, any finite branches can be ignored. This motivates the definition of the *infinite branching tree* below.

4.1.2. Infinite branching tree $\mathcal{T}_\infty(x)$. The infinite branching tree for $x \in I_q$ ignores all finite branching behaviour in the sense that only doubly infinite branching points generate branching points in this tree. We now define the *infinite branching tree*, $\mathcal{T}_\infty(x)$, for any $x \in I_q$.

Suppose $x \in I_q$ has the property that either $\Omega_{E_q}(x)$ is finite or, for each branching point $f_b(x)$ of x, we have at least two indices $i \in \{0, 1, 2\}$ such that $\Omega_{E_q}(f_i(f_b(x)))$ is finite, then the *infinite branching tree for* x, $\mathcal{T}_\infty(x)$, is defined to be an infinite horizontal line. This is equivalent to x admitting no doubly infinite branching sequence and so there does not exist a doubly infinite branching point of x.

Suppose this is not the case, then there is a minimal doubly infinite branching sequence $(f_{b_j})_{j=1}^N$ for x and $f_b(x)$ is a doubly infinite branching point. Let $(f_{b_j})_{j=1}^N$ be a minimal doubly infinite branching sequence, then the infinite branching tree for x consists of a finite horizontal line segment corresponding to the transformation f_b which then bifurcates or trifurcates depending on whether there are two or three indices $i \in \{0, 1, 2\}$ such that $\Omega_{E_q}(f_i(f_b(x)))$ is infinite. In each case, the branches of the bifurcation or trifurcation correspond to the images $f_i(f_b(x))$ that satisfy $\Omega_{E_q}(f_i(f_b(x)))$ is infinite, and as before, the points $f_i(f_b(x))$ are defined to be the roots of the branches. Each branch is then extended by the same rules ad infinitum.

We have the analogous observation that for any branching point $f_b(x)$ of $\mathcal{T}_\infty(x)$, the branch with root $f_i(f_b(x))$ is the infinite branching tree $\mathcal{T}_\infty(f_i(f_b(x)))$ and is a subtree of $\mathcal{T}_\infty(x)$. Similarly, the branches of $\mathcal{T}_\infty(x)$ are in one-to-one correspondence with points of the form $f_i(f_b(x))$ where $f_b(x)$ is a doubly infinite branching point of x and $\Omega_{E_q}(f_i(f_b(x)))$ is infinite.

4.1.3. Null infinite points. It is possible that $x \in I_q$ satisfies $\Omega_{E_q}(x)$ is infinite but that for each branching point $f_b(x)$ of x, there is exactly one index $i \in \{0, 1, 2\}$ such that $\Omega_{E_q}(f_i(f_b(x)))$ is infinite. In this case, $\mathcal{T}(x)$ is infinite, $\mathcal{T}_\infty(x)$ is an infinite horizontal line and x is said to be a *null infinite point*. If x is a null infinite point then $|\Omega_{E_q}(x)| = \aleph_0$. See Figure 11 for a depiction of the branching tree of a null infinite point.

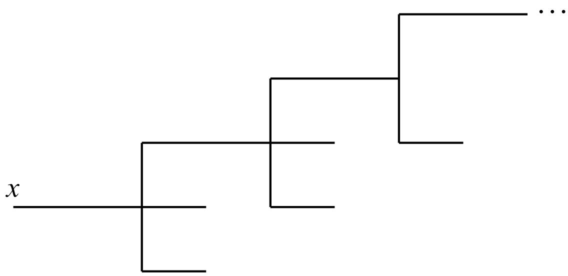


Figure 11. The branching tree $\mathcal{T}(x)$ of a null infinite point x. All branching points generate exactly one infinite branch.

4.1.4. Proof of Proposition 4.3. For the following proof, we require the above definitions and remarks to deal with the subtle difference between the cases when $\Omega_{E_q}(x)$ is countably infinite and when it is uncountably infinite. Precisely, we require the conclusion that x admits a doubly infinite branching sequence from the hypothesis that $\Omega_{E_q}(x)$ is uncountable.

Proof of Proposition 4.3. Let $q \in \mathcal{D}$ and let $x \in J_q$, so $|\Omega_{E_q}(x)| > 1$. Since $q \in \mathcal{D}$, $|\Omega_{E_q}(x)| = 2^{\aleph_0}$ so x is not a null infinite point. Therefore, x admits a minimal doubly infinite branching sequence, $(f_{b_j})_{j=1}^N \in \Omega^*_{E_q}(x)$ such that $f_b(x) \in J_q^\circ$ and there are at least two indices $i_0, i_1 \in \{0, 1, 2\}$ such that $\Omega_{E_q}(f_{i_0}(f_b(x)))$ and $\Omega_{E_q}(f_{i_1}(f_b(x)))$ are infinite. Again using $q \in \mathcal{D}$, this implies that

$$\left|\Omega_{E_q}(f_{i_0}(f_b(x)))\right| = \left|\Omega_{E_q}(f_{i_1}(f_b(x)))\right| = 2^{\aleph_0}$$

so neither $f_{i_0}(f_b(x))$ nor $f_{i_1}(f_b(x))$ are null infinite points. Therefore, both points $f_{i_0}(f_b(x))$ and $f_{i_1}(f_b(x))$ admit minimal doubly infinite branching sequences.

Let $h^0 = (h^0_j)_{j=1}^{K_0}$ and $h^1 = (h^1_j)_{j=1}^{K_1}$ be elements of $\{0,1,2\}^*$ such that the associated finite sequences of maps

$$(f_{h^0_j})_{j=1}^{K_0} \in \Omega^*_{E_q}(f_{i_0}(f_b(x))) \text{ and } (f_{h^1_j})_{j=1}^{K_1} \in \Omega^*_{E_q}(f_{i_1}(f_b(x)))$$

are minimal doubly infinite branching sequences for both $f_{i_0}(f_b(x))$ and $f_{i_1}(f_b(x))$. Therefore $f_{h^0} \circ f_{i_0} \circ f_b(x) \in J_q^\circ$ and $f_{h^1} \circ f_{i_1} \circ f_b(x) \in J_q^\circ$. Let $b^0 = (b^0_j)_{j=1}^{N_0}$ and $b^1 = (b^1_j)_{j=1}^{N_1}$ be elements of $\{0,1,2\}^*$ with

$$b^0 = (b_j)_{j=1}^{N} i_0 (h^0_j)_{j=1}^{K_0} \text{ and } b^1 = (b_j)_{j=1}^{N} i_1 (h^1_j)_{j=1}^{K_1}.$$

Therefore $f_{b^0} = f_{h^0} \circ f_{i_0} \circ f_b$ and $f_{b^1} = f_{h^1} \circ f_{i_1} \circ f_b$. We know that $f_{b^0}(x)$, $f_{b^1}(x) \in J_q^\circ$ and $(f_{b^0_j})_{j=1}^{N_0}, (f_{b^1_j})_{j=1}^{N_1} \in \Omega^*_{E_q}(x)$ by construction and the proposition holds. ∎

4.2. Positive Hausdorff dimension

In this subsection, we construct $\mathcal{R} \subset \{0,1,2\}^{\mathbb{N}}$ and a measure μ on $\pi_3(\mathcal{R})$ which satisfies the hypotheses of the mass distribution principle and we use this to prove Theorem 4.2. We emphasise that $\mathcal{R}$ depends upon the fixed $q \in \mathcal{D}$ and $x \in J_q$ which are chosen arbitrarily. We construct the constant $M \in \mathbb{N}$ and the map $A : \{0,1\}^* \to \{0,1,2\}^*$ before proving their required properties in Proposition 4.4. Recall the definitions of *prefix* and *strict prefix* from Subsection 2.1

Let $q \in \mathcal{D}$ and let $x \in J_q$. By Proposition 4.3, there are two doubly infinite branching sequences $(f_{b^0_j})_{j=1}^{N_0}, (f_{b^1_j})_{j=1}^{N_1} \in \Omega^*_{E_q}(x)$ such that $f_{b^0}(x), f_{b^1}(x) \in J_q^\circ$. The maps f_0, f_1, f_2 are continuous, so any finite composition of these maps is also continuous. This allows us to fix an open interval U_x containing x such that $f_{b^0}(U_x)$, $f_{b^1}(U_x) \subset J_q^\circ$. Since $x \in J_q$ was arbitrary, this generates an open cover $\{U_x\}_{x \in J_q}$ of J_q. By compactness of the closed interval J_q, this open cover must admit a finite subcover. That is, there is some finite subset $F \subset J_q$ such that $\{U_{x'}\}_{x' \in F}$ is a finite open cover of J_q. For each $x' \in F$, we can choose a pair of doubly infinite branching sequences $(f_{b^0_j})_{j=1}^{N_0}, (f_{b^1_j})_{j=1}^{N_1}$ and define $M_{x'} = \max\{N_0, N_1\}$. We subsequently define $M = \max_{x' \in F}\{M_{x'}\}$.

Let $x \in J_q$ be arbitrary. Then $x \in U_{x'}$ for some $x' \in F$. and there are doubly infinite branching sequences $(f_{b^0_j})_{j=1}^{N_0}, (f_{b^1_j})_{j=1}^{N_1} \in \Omega^*_{E_q}(x)$ for x as to $\max\{N_0, N_1\} \le M$. We call such a pair of sequences of maps the *branching pair* for x. Although the branching pair of some $x \in J_q$ may not be unique, by fixing a branching pair for each $x \in J_q$, the definition is valid. We emphasise that since $x \in J_q$ was arbitrary,

M is independent of x and depends only on q. Since $f_{b^0}(x), f_{b^1}(x) \in J_q^\circ$, by Proposition 4.3 there are branching pairs for $f_{b^0}(x)$ and $f_{b^1}(x)$. That is, there are finite sequences of maps $(f_{c_j^0})_{j=1}^{K_0}, (f_{c_j^1})_{j=1}^{K_1} \in \Omega_{E_q}^*(f_{b_0}(x))$ and $(f_{d_j^0})_{j=1}^{L_0}, (f_{d_j^1})_{j=1}^{L_1} \in \Omega_{E_q}^*(f_{b_1}(x))$ with $K_0, K_1, L_0, L_1 \leq M$ such that $f_{c^0} \circ f_{b^0}(x)$, $f_{c^1} \circ f_{b^0}(x)$, $f_{d^0} \circ f_{b^1}(x)$, $f_{d^1} \circ f_{b^1}(x) \in J_q^\circ$. Let $(b_j^{00})_{j=1}^{N_{00}}, (b_j^{01})_{j=1}^{N_{01}}, (b_j^{10})_{j=1}^{N_{10}}, (b_j^{11})_{j=1}^{N_{11}} \in \{0,1,2\}^*$ be the finite sequences such that

$$
\begin{aligned}
f_{b^{00}} &= f_{c^0} \circ f_{b^0}, \\
f_{b^{01}} &= f_{c^1} \circ f_{b^0}, \\
f_{b^{10}} &= f_{d^0} \circ f_{b^1}, \\
f_{b^{11}} &= f_{d^1} \circ f_{b^1}.
\end{aligned}
$$

Then, by construction, $(f_{b_j^{00}})_{j=1}^{N_{00}}, (f_{b_j^{01}})_{j=1}^{N_{01}}, (f_{b_j^{10}})_{j=1}^{N_{10}}, (f_{b_j^{11}})_{j=1}^{N_{11}} \in \Omega_{E_q}^*(x)$. Again using Proposition 4.3, we can argue that since, $f_{b^\epsilon}(x) \in J_q^\circ$ for all $\epsilon \in \{0,1\}^2$, each of these points has a branching pair. In general, for $\epsilon \in \{0,1\}^k$ and $B \in \{0,1\}$ we inductively define $(f_{b_j^{\epsilon B}})_{j=1}^{N_{\epsilon B}} \in \{f_0, f_1, f_2\}^*$ to be a finite sequence of maps which satisfies $f_{b^{\epsilon B}} = f_{g^B} \circ f_{b^\epsilon}$ where $f_{g^0}, f_{g^1} \in \Omega_{E_q}^*(f_{b^\epsilon}(x))$ is the branching pair for $f_{b^\epsilon}(x)$. This defines $(f_{b_j^\epsilon})_{j=1}^{N_\epsilon}$, and by construction $(f_{b_j^\epsilon})_{j=1}^{N_\epsilon} \in \Omega_{E_q}^*(x)$ for all $\epsilon \in \{0,1\}^*$.

In summary for every $\epsilon \in \{0,1\}^*$, we have an associated finite ternary sequence $b^\epsilon = (b_j^\epsilon)_{j=1}^{N_\epsilon} \in \{0,1,2\}^*$ with the property that $(f_{b_j^\epsilon})_{j=1}^{N_\epsilon} \in \Omega_{E_q}^*(x)$. We write this mapping as $A : \{0,1\}^* \to \{0,1,2\}^*$ where $A(\epsilon) = b^\epsilon = (b_j^\epsilon)_{j=1}^{N_\epsilon}$. Note that if ϵ is the empty word then b^ϵ is the empty word. Moreover, the map A depends on $x \in J_q$ and $q \in \mathcal{D}$ because the branching pairs in the construction of b^ϵ depend on x and q.

Proposition 4.4. *For any $q \in \mathcal{D}$, the constant $M \in \mathbb{N}$ and the map $A : \{0,1\}^* \to \{0,1,2\}^*$ satisfy the following properties for any $x \in J_q$.*

(1) *If $\epsilon \in \{0,1\}^*$ then $(f_{b_j^\epsilon})_{j=1}^{N_\epsilon} \in \Omega_{E_q}^*(x)$.*

(2) *If $\epsilon, \epsilon' \in \{0,1\}^*$, then b^ϵ is a prefix of $b^{\epsilon'}$ if and only if ϵ is a prefix of ϵ'.*

(3) *If ϵ and ϵ' are distinct elements of $\{0,1\}^k$ for some $k \in \mathbb{N}_{\geq 0}$ then $\pi_3[b^\epsilon]$ and $\pi_3[b^{\epsilon'}]$ intersect in at most one point.*

(4) *For all $k \in \mathbb{N}_{\geq 0}$, if $\epsilon \in \{0,1\}^k$ then $N_\epsilon \leq kM$.*

Proof. Let $q \in \mathcal{D}$, $x \in J_q$ for every part of this proof.

(1) This follows easily from the construction of $(f_{b_j^\epsilon})_{j=1}^{N_\epsilon}$ for each $\epsilon \in \{0,1\}^*$.

(2) Let $\epsilon, \epsilon' \in \{0,1\}^*$. If $\epsilon = \epsilon'$ the result is obvious. Assume that ϵ is a *strict* prefix of ϵ'. Suppose that $B \in \{0,1\}$ is such that ϵB is a prefix of ϵ' and recall

by the inductive definition of b^ϵ that $f_{b^{\epsilon B}} = f_{g^B} \circ f_{b^\epsilon}$ where f_{g^0}, f_{g^1} is the branching pair of $f_{b^\epsilon}(x)$. This equation can be rewritten as

$$f_{b_1^{\epsilon B}\dots b_{N_{\epsilon B}}^{\epsilon B}} = f_{g^B} \circ f_{b_1^\epsilon \dots b_{N_\epsilon}^\epsilon}.$$

Hence $(b_j^{\epsilon B})_{j=1}^{N_\epsilon} = (b_j^\epsilon)_{j=1}^{N_\epsilon}$ so b^ϵ is a prefix of $b^{\epsilon B}$. If $\epsilon' = \epsilon B$ then we are done. If not then ϵB is a strict prefix of ϵ' and we repeat the above argument. Performing this argument finitely many times proves the result in the general case that ϵ is a strict prefix of ϵ'.

Suppose ϵ is not a prefix of ϵ' and vice versa, then there is some $l \in \mathbb{N}$ such that the length l prefixes of ϵ and ϵ' are distinct but the length $l-1$ prefixes coincide. Without loss of generality, let $\delta 0$ and $\delta 1$ be the length l prefixes of ϵ and ϵ', respectively. Then $b^{\delta 0}$ and $b^{\delta 1}$ satisfy $f_{b^{\delta 0}} = f_{g^0} \circ f_{b^\delta}$ and $f_{b^{\delta 1}} = f_{g^1} \circ f_{b^\delta}$ where f_{g^0}, f_{g^1} is the branching pair for $f_{b^\delta}(x)$. Since f_{g^0}, f_{g^1} is a branching pair, we know that g^0 is not a prefix of g^1 and vice versa. Therefore $b^{\delta 0}$ is not a prefix of $b^{\delta 1}$ and vice versa. Since $b^{\delta 0}$ and $b^{\delta 1}$ are prefixes of b^ϵ and $b^{\epsilon'}$ respectively, we can conclude that b^ϵ is not a prefix of $b^{\epsilon'}$ and vice versa.

(3) It can be easily checked that intervals of the form $\pi_3[b]$ and $\pi_3[b']$ for $b, b' \in \{0,1,2\}^*$ are either nested or are disjoint apart from possibly at a single point. They can only be nested if b is a prefix of b' or vice versa. If $\epsilon, \epsilon' \in \{0,1\}^k$ are distinct then ϵ is not a prefix of ϵ' and vice versa so by Item 2, b^ϵ is not a prefix of $b^{\epsilon'}$ and vice versa. Therefore $\pi_3[b^\epsilon]$ and $\pi_3[b^{\epsilon'}]$ are not nested and intersect in at most a single point.

(4) We prove the claim by induction. Let $(f_{b_j^0})_{j=1}^{N_0}, (f_{b_j^1})_{j=1}^{N_1} \in \Omega_{E_q}^*(x)$ be the branching pair for x, so $N_0, N_1 \leq M$. Therefore, the claim holds for $\epsilon \in \{0,1\}$. Given $\epsilon \in \{0,1\}^k$ and $b^\epsilon = (b_j^\epsilon)_{j=1}^{N_\epsilon}$, assume that $N_\epsilon \leq kM$. By definition, for any $B \in \{0,1\}$, $f_{b^{\epsilon B}} = f_{g^B} \circ f_{b^\epsilon}$ where f_{g^0}, f_{g^1} is the branching pair for $f_{b^\epsilon}(x)$. We know $|g^0|, |g^1| \leq M$ and $N_{\epsilon B} = |g^B| + N_\epsilon$ so $N_{\epsilon B} \leq (k+1)M$. Since every $\delta \in \{0,1\}^{k+1}$ is of the form ϵB for some $\epsilon \in \{0,1\}^k$ and some $B \in \{0,1\}$, the proof is complete. ■

Fix some $q \in \mathcal{D}$ and $x \in J_q$, on which the construction of the set $\mathcal{R}$ implicitly depends due to the dependence of the map A on q and x. For each $k \in \mathbb{N}_{\geq 0}$, define the set $\mathcal{R}^k$ by

$$\mathcal{R}^k = \bigcup_{\epsilon \in \{0,1\}^k} [b^\epsilon].$$

Note that the sequences b^ϵ which define the cylinders of $\mathcal{R}^k$ need not have length k but rather are the images under A of the binary strings of length k. Recall that

$A(\epsilon) = b^\epsilon = (b_j^\epsilon)_{j=1}^{N_\epsilon}$ so the cylinder $[b^\epsilon] \subset \mathcal{R}^k$ has length N_ϵ. Define also

$$\mathcal{R} = \bigcap_{k \in \mathbb{N}_{\geq 0}} \mathcal{R}^k.$$

That is, $\mathcal{R}$ is the set of infinite sequences $(i_j)_{j=1}^\infty \in \{0, 1, 2\}^\mathbb{N}$ with the property that for infinitely many $l \in \mathbb{N}$, $(i_j)_{j=1}^l \in \mathcal{R}^k$ for some $k \in \mathbb{N}$. Equivalently, $\mathcal{R}$ is the set of infinite sequences $(i_j)_{j=1}^\infty$ such that for all $k \in \mathbb{N}_{\geq 0}$ there is some $\epsilon \in \{0, 1\}^*$ with the property that b^ϵ is a prefix of $(i_j)_{j=1}^k$ and for some $B \in \{0, 1\}$, $(i_j)_{j=1}^k$ is a strict prefix of $b^{\epsilon B}$. We emphasise that $\mathcal{R}$ is implicitly dependent upon the fixed $q \in \mathcal{D}$ and $x \in J_q$ due to the dependence of the map A on these values. By the second part of Proposition 4.4, the sets $\mathcal{R}^k$ are nested, and since $\mathcal{R}^k$ is nonempty for all $k \in \mathbb{N}_{\geq 0}$, we know that $\mathcal{R}$ is nonempty. Using the fact that $(f_{b_j^\epsilon})_{j=1}^{N_\epsilon} \in \Omega_{E_q}^*(x)$, a simple consequence of the definition of the orbit space is that if $(i_j)_{j=1}^k$ is a prefix of b^ϵ then $(f_{i_j})_{j=1}^k \in \Omega_{E_q}^*(x)$.

By the above remarks, it follows that $(f_{i_j})_{j=1}^\infty \in \Omega_{E_q}(x)$ for all $(i_j)_{j=1}^\infty \in \mathcal{R}$. Hence, by Lemma 2.5, $\pi_3(\mathcal{R}) \subset \mathrm{sl}_q(x(q-1))$. Moreover, we know that

$$\pi_3(\mathcal{R}) = \bigcap_{k \in \mathbb{N}_{\geq 0}} \bigcup_{\epsilon \in \{0,1\}^k} \pi_3[b^\epsilon],$$

so we define a measure μ on $\pi_3(\mathcal{R})$ by defining the values of $\mu(\pi_3[b^\epsilon])$ for all $\epsilon \in \{0, 1\}^*$. Given $\epsilon \in \{0, 1\}^k$, define

$$\mu\big(\pi_3[b^\epsilon]\big) = 2^{-k}.$$

By Proposition 4.4, we know that if $\epsilon, \epsilon' \in \{0, 1, 2\}^k$ are distinct then $\pi_3[b^\epsilon]$ and $\pi_3[b^{\epsilon'}]$ intersect in at most one point, and therefore

$$\mu\big(\pi_3[b^\epsilon] \cup \pi_3[b^{\epsilon'}]\big) = \mu\big(\pi_3[b^\epsilon]\big) + \mu\big(\pi_3[b^{\epsilon'}]\big).$$

Then

$$\mu\big(\pi_3(\mathcal{R}^k)\big) = \sum_{\epsilon \in \{0,1\}^k} \mu\big(\pi_3[b^\epsilon]\big) = 1,$$

since $|\{0, 1\}^k| = 2^k$. It follows that $\mu(\pi_3(\mathcal{R})) = 1$. It is a consequence of Lemma 2.3 that if b^ϵ is a prefix of $(i_j)_{j=1}^l \in \{0, 1, 2\}^*$ then $\pi_3[(i_j)_{j=1}^l] \subset \pi_3[b^\epsilon]$ so

$$\mu\big(\pi_3[(i_j)_{j=1}^l]\big) \leq \mu\big(\pi_3[b^\epsilon]\big).$$

Proposition 4.5. *The measure μ has the property that for all $U \subset \mathbb{R}$, $\mu(U) \leq 4|U|^s$ where $s = \frac{\log 2}{M \log 3}$.*

Proof. We start by bounding the measure of intervals of the form $\pi_3[(i_j)_{j=1}^l]$ where $(i_j)_{j=1}^l \in \{0,1,2\}^*$ before generalising to arbitrary intervals in $\mathbb{R}$.

Let $(i_j)_{j=1}^l \in \{0,1,2\}^*$ be such that $(i_j)_{j=1}^l$ is not a prefix of b^ϵ for any $\epsilon \in \{0,1\}^*$. Then $(i_j)_{j=1}^l$ is not a prefix of any sequence in $\mathcal{R}$ and hence $\pi_3[(i_j)_{j=1}^l] \cap \pi_3(\mathcal{R})$ is either empty or consists of a single point, so $\mu(\pi_3[(i_j)_{j=1}^l]) = 0$. Suppose instead that there is some $k \in \mathbb{N}_{\geq 0}$, $\epsilon \in \{0,1\}^k$ and some $B \in \{0,1\}$ with the property that $(i_j)_{j=1}^l$ is a strict prefix of $b^{\epsilon B}$. We can choose k to be minimal so that b^ϵ is a prefix of $(i_j)_{j=1}^l$. Therefore $\mu(\pi_3[(i_j)_{j=1}^l]) \leq \mu(\pi_3[b^\epsilon]) \leq 2^{-k}$. Since $(i_j)_{j=1}^l$ is a strict prefix of $b^{\epsilon B}$ we know, using the third part of Proposition 4.4, that $l < N_{\epsilon B} \leq (k+1)M$, so $2^{-k} = 2(2^{-k-1}) \leq 2(2^{-l/M})$. Therefore, for any $(i_j)_{j=1}^l \in \{0,1,2\}^*$, $\mu(\pi_3[(i_j)_{j=1}^l]) \leq 2(2^{-l/M})$.

Let $U \subset \mathbb{R}$ satisfy $3^{-l-1} \leq |U| < 3^{-l}$ for some $l \in \mathbb{N}$. Then since $|\pi_3[(i_j)_{j=1}^l]| = 3^{-l}$ for any $(i_j)_{j=1}^l \in \{0,1,2\}^l$, the set U intersects at most two intervals of the form $\pi_3[(i_j)_{j=1}^l]$, so $\mu(U) \leq 4(2^{-l/M})$. Solving $2^{-l/M} = (3^{-l-1})^{s_l}$ gives $s_l = \frac{l}{l+1} \cdot \frac{\log 2}{M \log 3} < s = \frac{\log 2}{M \log 3}$. Therefore $\mu(U) \leq 4(3^{-l-1})^s \leq 4|U|^s$. ∎

Note that since s is a constant depending only on M, and M is dependent only on q, we know that s is independent of $x \in J_q$ and depends only on $q \in \mathcal{D}$.

Proof of Theorem 4.2. By the mass distribution principle [13], Proposition 4.5, yields that $\mathcal{H}^s(\pi_3(\mathcal{R})) \geq \frac{\mu(\pi_3(\mathcal{R}))}{4} = \frac{1}{4}$ and $\dim_H(\pi_3(\mathcal{R})) \geq s$. Given $q \in \mathcal{D}$ and $x \in J_q$, we know that $\pi_3(\mathcal{R}) \subset \mathrm{sl}_q(x(q-1))$ so we conclude that $\dim_H(\mathrm{sl}_q(x(q-1)) \geq s$ which proves Theorem 4.2. ∎

5. The special case of the k-Bonacci numbers

In this section, we prove Theorem 3 as well as an additional result (Theorem 5.5) in the special case where q is a k-Bonacci number for some $k \in \mathbb{N}_{\geq 3}$. We recall the notation from Subsection 2.3. By Lemma 2.5, Theorem 3 is equivalent to the following theorem.

Theorem 5.1. *Let $\{q_i\}_{i=3}^\infty$ be the set of k-Bonacci numbers excluding G.*

(1) *If $q \in \{q_i\}_{i=3}^\infty$ then for all $m \in \mathbb{N}$, $\dim_H(\mathcal{V}_q^{(2m+1)}) > \epsilon > 0$ where ϵ depends only on q.*

(2) $\{q_i\}_{i=3}^\infty \subset \mathcal{C}_{\aleph_0}$.

Note that $\{q_i\}_{i=3}^\infty \subset \mathcal{C}_{2m+1}$ is an immediate implication of part (1) of the theorem. In order to prove the theorem, we need the following lemma on the cardinalities of orbit spaces. We refer the reader to Figure 12 for an illustration of what happens at q_3.

Lemma 5.2. *Let $q \in (G, 2)$ and let $\alpha \in \{0,1\}^{\mathbb{N}}$ be arbitrary. If $x_n = \pi_q(1^n 0\alpha)$ for some $n \in \mathbb{N}$ then $|\Omega_{E_q}(x_n)| = |\Omega_{E_q}(x_{n-1})| = \cdots = |\Omega_{E_q}(x_1)|$.*

Proof. Let $\alpha \in \{0,1\}^{\mathbb{N}}$ be fixed and $x_n = \pi_q(1^n 0\alpha)$ for all $n \in \mathbb{N}$. It is obvious that $q^{-1} + q^{-2} \leq x_2 < x_3 < \ldots$ and since $q \in (G, 2)$ implies that $q^{-1} + q^{-2} > \frac{1}{q(q-1)}$, we know that

$$x_n \in \left(\frac{1}{q(q-1)}, \frac{1}{q-1}\right], \tag{5.1}$$

for all $n \geq 2$. Equation (5.1) implies that f_2 is the only map in $\{f_0, f_1, f_2\}$ that satisfies $f_i(x_n) \in I_q$. By observing that $f_2(x_n) = x_{n-1}$, it is clear that since (5.1) holds for all $n \geq 2$, $|\Omega_{E_q}(x_n)| = |\Omega_{E_q}(x_{n-1})| = \cdots = |\Omega_{E_q}(x_1)|$. ∎

Recall the reflection notation $\overline{x}$ from Subsection 3.4. We observe that since $\frac{1}{q-1} = \pi_q(1^\infty)$,

$$\pi_q(\overline{x}) = \frac{1}{q-1} - \pi_q(x). \tag{5.2}$$

Lemma 5.3. *Let $k \geq 3$, $q = q_k$, and $x_m = \pi_q(1(0^k)^m\alpha)$ where $m \in \mathbb{N}$ and $\alpha \in \{0,1\}^{\mathbb{N}}$ is arbitrary. Then $f_1(x_m) = \pi_q((1^k)^{m-1}\overline{\alpha})$.*

Proof. First, observe that for any x in the domain of f_1, that $f_1(x) = \frac{1}{q-1} - \frac{1}{2-q} f_2(x)$. If $q = q_k$ for some $k \geq 3$ then by the proof of Lemma 3.3, $\frac{1}{2-q} = q^k$. From the relation $f_2(x_m) = \pi_q((0^k)^m\alpha)$ we know that $\frac{1}{2-q} f_2(x_m) = \pi_q((0^k)^{m-1}\alpha)$. Using (5.2), $f_1(x_m) = \frac{1}{q-1} - \pi_q((0^k)^{m-1}\alpha) = \pi_q((1^k)^{m-1}\overline{\alpha})$. ∎

Before proving Theorem 5.1, we require the following definition and lemma. For all $k \in \mathbb{N}$, define

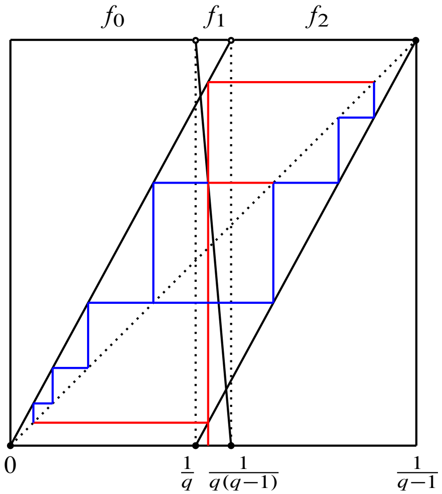
$$\hat{\mathcal{S}}^k = \{(i_j)_{j=1}^\infty \in \{0,1\}^{\mathbb{N}} : (i_j)_{j=1}^\infty \text{ avoids } (01^k) \text{ and } (10^k) \text{ and does not end with } (01^{k-1})^\infty \text{ or } (10^{k-1})^\infty\},$$

and

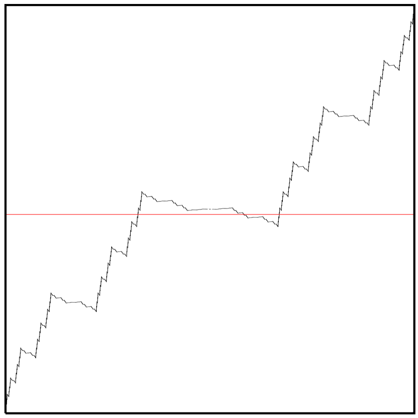
$$\tilde{\mathcal{S}}^k = \{(i_j)_{j=1}^\infty \in \{0,1\}^{\mathbb{N}} : (i_j)_{j=1}^\infty \text{ avoids } (0^k) \text{ and } (1^k) \text{ and does not end with } (01^{k-1})^\infty \text{ or } (10^{k-1})^\infty\},$$

and note that $\tilde{\mathcal{S}}^k \subset \hat{\mathcal{S}}^k \subset \mathcal{S}^k$ so $\pi_q(\tilde{\mathcal{S}}^k) \subset \pi_q(\hat{\mathcal{S}}^k) \subset \pi_q(\mathcal{S}^k)$ for each $k \in \mathbb{N}$. Note also that $\delta \in \mathcal{S}^k$ if and only if $\overline{\delta} \in \mathcal{S}^k$ and the same holds for $\hat{\mathcal{S}}^k$ and $\tilde{\mathcal{S}}^k$.

By [17, Lemma 4], if $q = q_k$, then $\pi_q(\hat{\mathcal{S}}^k) = \mathcal{U}_q$ and so if $q \in (q_k, 2)$ then $\pi_q(\hat{\mathcal{S}}^k) \subset \mathcal{U}_q$. This follows from the fact that $\mathcal{U}_q$ is increasing with q in the sense



(a) The three orbits of the point x_1.



(b) A horizontal slice with three elements.

Figure 12. E_q and the corresponding fractal K_q for $q = q_3$.

that if $\pi_q((i_j)_{j=1}^{\infty}) \in \mathcal{U}_q$ for some $q \in (1,2)$ then $\pi_{q'}((i_j)_{j=1}^{\infty}) \in \mathcal{U}_{q'}$ for all $q' \in (q,2)$ [17, Lemma 4]. Observe that if $\delta \in \tilde{\mathcal{S}}^k$ then $0^n\delta \in \hat{\mathcal{S}}^k$ for all $n \in \mathbb{N}$. This proves the following lemma.

Lemma 5.4. *Let $q \in (q_k, 2)$ and $\delta \in \tilde{\mathcal{S}}^k$. Then $|\Omega_{E_q}(\pi_q(0^n\delta))| = |\Omega_{E_q}(\pi_q(\delta))| = 1$ for all $n \in \mathbb{N}$.*

Since $\pi_q(\mathcal{S}^k) \setminus \pi_q(\hat{\mathcal{S}}^k)$ and $\pi_q(\hat{\mathcal{S}}^k) \setminus \pi_q(\tilde{\mathcal{S}}^k)$ have countably many points, we know that for any $q \in (1,2)$,

$$\dim_{\mathrm{H}}\big(\pi_q(\mathcal{S}^k)\big) = \dim_{\mathrm{H}}\big(\pi_q(\hat{\mathcal{S}}^k)\big) = \dim_{\mathrm{H}}\big(\pi_q(\tilde{\mathcal{S}}^k)\big).$$

It was shown in [17, Theorem 2] that $\dim_{\mathrm{H}}(\pi_q(\mathcal{S}^k)) > 0$ for all $q \in (q_{\mathrm{KL}}, 2)$, $k \in \mathbb{N}$ and hence $\dim_{\mathrm{H}}(\pi_q(\tilde{\mathcal{S}}^k)) > 0$. We note that $q_3 \approx 1.83929 > q_{\mathrm{KL}} \approx 1.78723$, so the range in which this result holds contains $\{q_i\}_{i=3}^{\infty}$.

Proof of Theorem 5.1. Proof of part (1):

Let $k \geq 3$ and let $q = q_k$. Define

$$X_m = \big\{(1(0^k)^m\delta) \in \{0,1\}^{\mathbb{N}} : \delta \in \tilde{\mathcal{S}}^k\big\}.$$

Since an affine image of $\pi_q(\tilde{\mathcal{S}}^k)$ is contained in $\pi_q(X_m)$ for any $m \in \mathbb{N}$, we know that $\dim_{\mathrm{H}}(\pi_q(X_m)) \geq \dim_{\mathrm{H}}(\pi_q(\tilde{\mathcal{S}}^k)) > \epsilon > 0$ for some ϵ which depends only on q. Hence, it suffices to show that if $x \in \pi_q(X_m)$ then $|\Omega_{E_q}(x)| = 2m + 1$.

Since $q = q_k$ for some $k \geq 3$,

$$q^k = q^{k-1} + \cdots + q + 1,$$

and equivalently

$$q^{-1} = q^{-2} + \cdots + q^{-k-1},$$

which implies that

$$\pi_q(10^k\alpha) = \pi_q(01^k\alpha), \tag{5.3}$$

where $\alpha \in \{0,1\}^{\mathbb{N}}$ is an arbitrary infinite binary sequence. Let $\delta \in \tilde{\mathcal{S}}^k$, and let $x_m = \pi_q(1(0^k)^m\delta)$ for all $m \in \mathbb{N}$. (5.3) implies that $\pi_q(1(0^k)^m\delta) = \pi_q(0(1^k)(0^k)^{m-1}\delta)$ and with Lemma 5.3, we evaluate $f_0(x_m)$, $f_1(x_m)$ and $f_2(x_m)$,

$$\begin{aligned} f_0(x_m) &= \pi_q\big(1^k(0^k)^{m-1}\delta\big),\\ f_1(x_m) &= \pi_q\big((1^k)^{m-1}\overline{\delta}\big),\\ f_2(x_m) &= \pi_q\big((0^k)^m\delta\big), \end{aligned}$$

and proceed by induction. By Lemma 5.4, we can see that

$$|\Omega_{E_q}(f_2(x_m))| = |\Omega_{E_q}(f_1(x_m))| = 1$$

and by Lemma 5.2, $|\Omega_{E_q}(f_0(x_m))| = |\Omega_{E_q}(\pi_q(1(0^k)^{m-1}\delta))| = |\Omega_{E_q}(x_{m-1})|$. Since $|\Omega_{E_q}(x_m)| = |\Omega_{E_q}(f_0(x_m))| + |\Omega_{E_q}(f_1(x_m))| + |\Omega_{E_q}(f_2(x_m))|$, this means that $|\Omega_{E_q}(x_m)| = 2 + |\Omega_{E_q}(x_{m-1})|$. For x_1, Lemma 5.2 shows that

$$\big|\Omega_{E_q}(f_0(x_1))| = |\Omega_{E_q}(f_1(x_1))| = |\Omega_{E_q}(f_2(x_1))\big| = 1.$$

Hence, $|\Omega_{E_q}(x_m)| = 2m + 1$ for all $m \in \mathbb{N}$, or equivalently, $\pi_q(X_m) \subset \mathcal{V}_q^{(2m+1)}$. Therefore, if $q = q_k$ for some $k \geq 3$ and $m \in \mathbb{N}$, since $\pi_q(X_m) \subset \mathcal{V}_q^{(2m+1)}$ and $\dim_{\mathrm{H}}(\pi_q(X_m)) > \epsilon > 0$, we have proved that $\dim_{\mathrm{H}}(\mathcal{V}_q^{(2m+1)}) > \epsilon > 0$.

Proof of part (2):

We show that $x_{\aleph_0} = \pi_q(10^\infty) \in \mathcal{V}_q^{(\aleph_0)}$ for all $q \in \{q_i\}_{i=3}^\infty$. Let $q = q_k$ for some $k \geq 3$. Using (5.3), we know that

$$\pi_q(10^\infty) = \pi_q(01^k0^\infty). \tag{5.4}$$

Since $x_{\aleph_0} = 1/q$, we know that $f_0(x_{\aleph_0}), f_2(x_{\aleph_0}) \in I_q$ but $f_1(x_{\aleph_0})$ is not defined. We count the number of orbits of $f_0(x_{\aleph_0})$ and $f_2(x_{\aleph_0})$. We have that $f_2(x_{\aleph_0}) = \pi_q(0^\infty)$ so $|\Omega_{E_q}(f_2(x_{\aleph_0}))| = 1$. Equation (5.4) implies that $f_0(x_{\aleph_0}) = \pi_q(1^k0^\infty)$. Hence, by Lemma 5.2, $|\Omega_{E_q}(\pi_q(1^k0^\infty))| = |\Omega_{E_q}(\pi_q(10^\infty))| = |\Omega_{E_q}(x_{\aleph_0})|$. Therefore $|\Omega_{E_q}(x_{\aleph_0})| = |\Omega_{E_q}(x_{\aleph_0}) \cup \Omega_{E_q}(f_2(x_{\aleph_0}))|$ so $\Omega_{E_q}(x_{\aleph_0})$ is infinite. At each branching point of $x_{\aleph_0}$ the maps $f_i \in \{f_0, f_1, f_2\}$ which satisfy $f_i(x_{\aleph_0}) \in I_q$ are f_0 and f_2. These maps have the property that $|\Omega_{E_q}(f_0(x_{\aleph_0}))|$ is infinite and

$$\left|\Omega_{E_q}(f_2(x_{\aleph_0})\right| = 1.$$

Recalling the definition from Subsection 4.1.3, we have that $x_{\aleph_0}$ is a null infinite point so $|\Omega_{E_q}(x_{\aleph_0})|$ is countably infinite, that is $x_{\aleph_0} \in \mathcal{V}_q^{(\aleph_0)}$. ■

We note that in general, the sets $\mathcal{C}_{2m}$ for $m \in \mathbb{N}$ are harder to understand because the only points $x \in I_q$ for which it is possible that $|\Omega_{E_q}(x)| = 2$ are $x \in \{\frac{1}{q}, \frac{1}{q(q-1)}\}$. Recalling that $\mathcal{U}$ is the set of $q \in (1,2)$ for which the base q expansion of 1 is unique, we have the following theorem.

Theorem 5.5. $\mathcal{C}_2 = \mathcal{U}$.

Proof. Let $q \in \mathcal{U}$. Then $f_0(\frac{1}{q}) = 1$, $f_1(\frac{1}{q})$ is not defined and $f_2(\frac{1}{q}) = 0$, therefore $|\Omega_{E_q}(\frac{1}{q})| = 2$ and $q \in \mathcal{C}_2$.

Let $q \in \mathcal{C}_2$. If $x \in \mathcal{V}_q^{(2)}$ then by Lemma 2.4 there is some unique finite sequence $(i_j)_{j=1}^k \in \{0,1,2\}^*$ such that $(f_{i_j})_{j=1}^k(x) \in J_q$. Moreover, we can observe that if $(f_{i_j})_{j=1}^k(x) \in J_q^\circ$ then $|\Omega_{E_q}(x)| \geq 3$. Hence

$$(f_{i_j})_{j=1}^k(x) \in \partial J_q = \left\{\frac{1}{q}, \frac{1}{q(q-1)}\right\}.$$

Without loss of generality, if we let $(f_{i_j})_{j=1}^k(x) = \frac{1}{q}$, then $\hat{f}_0(\frac{1}{q}) = f_0(\frac{1}{q}) = 1$ and $f_2(\frac{1}{q}) = 0$. Since $|\Omega_{E_q}(\frac{1}{q})| = 2$, we know that $|\Omega_{E_q}(1)| = 1$ so $|\Omega_{\hat{E}_q}(1)| = 1$ and $q \in \mathcal{U}$. ■

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