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Critical values for intermediate and box dimensions of projections and other images of compact sets

Nicolas Angelini and Ursula Maria Molter

Abstract. Given a compact set $E \subset \mathbb{R}^d$, we investigate for which values of m the equality $\dim_\theta P_V(E) = m$ or $\dim_\theta P_V(E) = \dim_\theta E$ holds for $\gamma_{d,m}$-almost all $V \in G(d,m)$. Our result extends to more general functions, including orthogonal projections and fractional Brownian motion. As a particular case, when $\theta = 1$, the results apply to the box dimension.

1. Introduction

The variation of the dimension of a set with respect to its orthogonal projection onto m-dimensional linear subspaces of $\mathbb{R}^d$ is a classical problem in geometric measure theory, and there is a substantial body of results on this topic. Let $G(d,m)$ denote the set of all m-dimensional subspaces of $\mathbb{R}^d$. The question is: How is $\dim P_V(E)$ related to $\dim E$, where $E \subset \mathbb{R}^d$, $V \in G(d,m)$, and P_V denotes the orthogonal projection onto V?

When dim represents the Hausdorff dimension, the problem has already been studied, yielding the result that if $E \subset \mathbb{R}^d$ is a Borel set and P_V denotes the orthogonal projection onto an m-dimensional linear subspace V with $m \leq d$, then

$$\dim_H P_V E = \min\{\dim_H E, m\} \tag{1.1}$$

for $\gamma_{d,m}$-almost all $V \in G(d,m)$. This result was first obtained by Marstrand [16] for the case $d = 2$, $m = 1$ and later extended to Borel sets $E \subset \mathbb{R}^d$, $d \geq 2$ by Mattila [17].

The lower and upper θ-intermediate dimensions, $\underline{\dim}_\theta$ and $\overline{\dim}_\theta$ respectively, form a continuous family of dimensions that interpolate between the Hausdorff and the box dimensions of bounded sets in $\mathbb{R}^d$. They satisfy, for all $\theta \in (0,1]$, the inequalities $\dim_{\mathrm{H}} E \leq \underline{\dim}_\theta E \leq \overline{\dim}_\theta E \leq \overline{\dim}_{\mathrm{B}} E$ and $\underline{\dim}_\theta E \leq \underline{\dim}_{\mathrm{B}} E$, with $\overline{\dim}_1 E = \overline{\dim}_{\mathrm{B}} E$. For $\theta = 1$, we have $\underline{\dim}_1 E = \underline{\dim}_{\mathrm{B}} E$. These dimensions are continuous

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functions of θ for all $\theta \in (0, 1]$; however, in general, the θ-intermediate dimension does not converge to the Hausdorff dimension as $\theta \to 0$.

For box and θ-intermediate dimensions, general results such as (1.1) do not hold. Nevertheless, as shown in [6] for the box dimension and in [4] for θ-intermediate dimensions, for any fixed $\theta \in (0, 1]$, the value of $\dim_\theta P_V(E)$ remains invariant for $\gamma_{d,m}$-almost every $V \in G(d, m)$. Recall that $\dim_1 P_V(E) = \dim_B P_V(E)$.

These values of the intermediate dimension are referred to by the authors as the *(Upper or Lower) θ-intermediate dimension profiles*, and are denoted by $\overline{\dim}_\theta^m$ and $\underline{\dim}_\theta^m$, respectively. In general, these profiles are difficult to work with.

When dealing with projections, the superscript m in the dimension profiles typically denotes an integer. However, in other contexts – as we will see – the superscript may represent a real number. To distinguish between these cases, from now on we use t for real numbers and m for integers when referring to dimension profiles.

The precise result obtained in [6] and [4] is:

Theorem 1.1. *Let $E \subset \mathbb{R}^d$ be bounded. Then for all $V \in G(d, m)$,*

$$\underline{\dim}_\theta P_V(E) \le \underline{\dim}_\theta^m E \text{ and } \overline{\dim}_\theta P_V(E) \le \overline{\dim}_\theta^m E$$

for all $\theta \in (0, 1]$, and for $\gamma_{d,m}$-almost all $V \in G(d, m)$,

$$\underline{\dim}_\theta P_V(E) = \underline{\dim}_\theta^m E \text{ and } \overline{\dim}_\theta P_V(E) = \overline{\dim}_\theta^m E.$$

The reader can also refer to [13] for additional results on θ-intermediate dimensions of projections.

In the present paper, we study for which values of m, or under which conditions on the set E, results analogous to Marstrand's theorem can be obtained for intermediate and box dimensions.

We introduce the notions of the *upper* and *lower quasi-Hausdorff dimensions* of a set E, denoted by $\overline{\dim}_{\mathrm{qH}} E$ and $\underline{\dim}_{\mathrm{qH}} E$, respectively, as

$$\overline{\dim}_{\mathrm{qH}} E := \lim_{\theta \to 0} \overline{\dim}_\theta E \quad \text{and} \quad \underline{\dim}_{\mathrm{qH}} E := \lim_{\theta \to 0} \underline{\dim}_\theta E.$$

This helps us make the statements and proofs of our results clearer and more straightforward.

We show that, given a compact set $E \subset \mathbb{R}^d$ and a positive integer $m \le d$, the almost sure value of the upper and lower intermediate dimensions of the orthogonal projection onto an m-dimensional subspace satisfies

$$\overline{\dim}_{\mathrm{qH}} P_V E = \min\{m, \overline{\dim}_{\mathrm{qH}} E\} \quad \text{for } \gamma_{d,m}\text{-almost every } V \in G(d, m),$$

and similarly,

$$\underline{\dim}_{\mathrm{qH}} P_V E = \min\{m, \underline{\dim}_{\mathrm{qH}} E\} \quad \text{for } \gamma_{d,m}\text{-almost every } V \in G(d, m),$$

see Theorem 3.1. From this, we deduce:

$$\overline{\dim}_{\mathrm{B}} P_V(E) = m \text{ for } \gamma_{d,m}\text{-almost all } V \in G(d,m) \iff m \le \overline{\dim}_{\mathrm{qH}} E,$$

and

$$\underline{\dim}_{\mathrm{B}} P_V(E) = m \text{ for } \gamma_{d,m}\text{-almost all } V \in G(d,m) \iff m \le \underline{\dim}_{\mathrm{qH}} E.$$

A direct consequence of Theorem 3.1 is that if E satisfies $\underline{\dim}_{\mathrm{B}} E = \underline{\dim}_{\mathrm{qH}} E$, then

$$\underline{\dim}_{\mathrm{B}}(P_V E) = \min\{m, \underline{\dim}_{\mathrm{B}} E\}$$

for almost every $V \in G(d,m)$.

Note that in [12, Corollary 1.3] the authors show that if $\underline{\dim}_{\mathrm{B}} E = \dim_{\mathrm{qA}} E$, then

$$\underline{\dim}_{\mathrm{B}}(P_V E) = \min\{\underline{\dim}_{\mathrm{B}} E, m\}$$

for almost every $V \in G(d,m)$.

Since by Theorem 3.9 we have that, $\underline{\dim}_{\mathrm{B}} E = \dim_{\mathrm{qA}} E$ implies $\underline{\dim}_{\mathrm{B}} E = \underline{\dim}_{\mathrm{qH}} E$, and the converse is not necessarily true, the result proven in this paper is strictly more general than the one in [12]. All the above statements remain valid if one replaces the lower box dimension and the lower quasi-Hausdorff dimension with their upper counterparts.

We also investigate for which values of m the intermediate dimension and the intermediate dimension profiles coincide. We obtain a general lower bound for the profiles in terms of the Assouad and Assouad dimension spectrum $\dim_A^\alpha$:

$$\overline{\dim}_\theta^t E \ge \overline{\dim}_\theta E - \max\bigl\{0, \dim_{\mathrm{A}}^\alpha E - t, (\dim_{\mathrm{A}} E - t)(1-\alpha)\bigr\},$$

for all $\alpha \in (0,1)$, which yields that if $m \ge \dim_{\mathrm{qA}} E$, then for all $\theta \in (0,1]$ we have $\dim_\theta E = \dim_\theta P_V E$. This result was previously known only in the case of the box dimension.

Another known bound that we adapt to intermediate dimensions is the inequality:

$$\overline{\dim}_\theta^s E \ge \frac{\overline{\dim}_\theta^t E}{1 + \bigl(\frac{1}{s} - \frac{1}{t}\bigr)\overline{\dim}_\theta^t E},$$

and we demonstrate with a simple example that this inequality is sharp, with equality holding for certain sets.

Since these profiles are also related to the dimensions of more general families of functions – such as fractional Brownian motion, as shown in [3] – we can extend all our results to such functions.

Theorem 3.1 and its corollaries provide genuinely new insights only in cases where the intermediate dimensions of the set are not continuous at $\theta = 0$. To illustrate this point, we now present an example of a set for which our projection theorems yield information that is not captured by existing results.

Example 1.2. Let $n \geq 2$. For each $1 \leq j < n$, it is not difficult to construct a set $E \subset \mathbb{R}^n$ such that

$$\dim_{\mathrm{H}} E < j \leq \underline{\dim}_{\mathrm{qH}} E.$$

Indeed, let $F_1 \subset \mathbb{R}^j$ be a set with $j - 1 < \dim_{\mathrm{H}} F_1 = \dim_B F_1 \leq j$; for instance, F_1 could be a self-similar set. Define

$$F_2 = \left(\{0\} \cup \left\{\frac{1}{\log(n)}\right\}_{n\in\mathbb{N}}\right)^{n-j} \subset \mathbb{R}^{n-j}.$$

Then F_2 is a countable set such that $\underline{\dim}_\theta F_2 = \overline{\dim}_\theta F_2 = n - j$ for all $\theta \in (0, 1]$. Now define $E = F_1 \times F_2$. Then

$$\dim_{\mathrm{H}} E = \dim_{\mathrm{H}} F_1 \leq j < n - j + \dim_{\mathrm{H}} F_1 = \underline{\dim}_{\mathrm{qH}} E,$$

where the last equality follows from the product rule for intermediate dimensions (see [1, Theorem 5.4]).

Therefore, by Theorem 3.1, we conclude that

$$\underline{\dim}_{\mathrm{qH}} P_V E = \overline{\dim}_{\mathrm{qH}} P_V E = j \quad \text{for } \gamma_{n,j}\text{-almost every } V \in G(n, j).$$

In contrast, Marstrand's projection theorem gives

$$\dim_{\mathrm{H}} P_V E = \dim_{\mathrm{H}} E < j \quad \text{for } \gamma_{n,j}\text{-almost every } V \in G(n, j).$$

2. Preliminaries

Throughout the whole document $B(x, r)$, $r > 0$ denotes the open ball in $\mathbb{R}^d$ with center x and radius r. Given a non-empty set $E \subset \mathbb{R}^d$, $\mathbb{I}_E(x)$ represents the indicator function of E, i.e., $\mathbb{I}_E(x) = 1$ if $x \in E$ and $\mathbb{I}_E(x) = 0$ if $x \notin E$.

We write $|E|$ for the diameter of the set, $\dim_H E$ always represents the Hausdorff dimension while $\overline{\dim}_{\mathrm{B}} E$ and $\underline{\dim}_{\mathrm{B}} E$ represents the upper and lower box dimension respectively, see [5] for more information on these dimensions. $G(d, m)$ denotes the manifold of all m dimensional linear sub-spaces of $\mathbb{R}^d$ and $\gamma_{d,m}$ the Haar measure on $G(d, m)$, see Section 3 on [18] for more information. Given a compact set $E \subset \mathbb{R}^d$ and $V \in G(d, m)$, $P_V(E)$ is the orthogonal projection of E onto V.

The θ intermediate dimensions were introduced in [11] and are defined as follows.

Definition 2.1. Let $F \subseteq \mathbb{R}^d$ be bounded, and let $0 \le \theta \le 1$. We define the *lower θ-intermediate dimension* of F by

$$\underline{\dim}_\theta F = \inf\Big\{s \ge 0 : \forall \varepsilon > 0, \text{ and all } \delta_0 > 0, \text{ there } \exists 0 < \delta \le \delta_0, \text{ and } \{U_i\}_{i\in I} : F \subseteq \cup_{i\in I} U_i : \delta^{1/\theta} \le |U_i| \le \delta \text{ and } \sum_{i\in I} |U_i|^s \le \varepsilon\Big\}.$$

Similarly, we define the *upper θ-intermediate dimension* of F by

$$\overline{\dim}_\theta F = \inf\Big\{s \ge 0 : \forall \varepsilon > 0, \text{ there } \exists \delta_0 > 0, : \forall 0 < \delta \le \delta_0, \text{ there } \exists \{U_i\}_{i\in I} : F \subseteq \cup_{i\in I} U_i : \delta^{1/\theta} \le |U_i| \le \delta \text{ and } \sum_{i\in I} |U_i|^s \le \varepsilon\Big\}.$$

For all $\theta \in (0,1]$ and a compact set $A \subset \mathbb{R}^d$, we have $\dim_H A \le \overline{\dim}_\theta A \le \overline{\dim}_{\mathrm{B}} A$ and similarly with the Lower case replacing $\overline{\dim}_{\mathrm{B}}$ with $\underline{\dim}_{\mathrm{B}}$. This spectrum of dimensions has the property of being continuous for $\theta \in (0,1]$ with $\underline{\dim}_1 E = \underline{\dim}_{\mathrm{B}} E$ and $\overline{\dim}_1 E = \overline{\dim}_{\mathrm{B}} E$, leaving the natural question of the continuity in $\theta = 0$, i.e., $\lim_{\theta\to 0} \underline{\dim}_\theta E = \dim_H E$ or $\lim_{\theta\to 0} \overline{\dim}_\theta E = \dim_H E$, as a problem to study. In general, these equalities are not true. The reader can find more information in [11].

As mentioned in the introduction, we adopt a specific notation for the limiting case of the intermediate dimension as the parameter θ tends to 0. We define the following quantities, which we refer to as the *quasi-Hausdorff dimensions*.

Definition 2.2. Let $E \subset \mathbb{R}^d$ be a non-empty bounded set. We define the *lower quasi-Hausdorff dimension* and the *upper quasi-Hausdorff dimension* of E as

$$\underline{\dim}_{\mathrm{qH}} E := \lim_{\theta\to 0} \underline{\dim}_\theta E \quad \text{and} \quad \overline{\dim}_{\mathrm{qH}} E := \lim_{\theta\to 0} \overline{\dim}_\theta E.$$

If both limits coincide, we say that the *quasi-Hausdorff dimension* of E exists, and we denote it by

$$\dim_{qH} E := \underline{\dim}_{\mathrm{qH}} E = \overline{\dim}_{\mathrm{qH}} E.$$

There is also an equivalent definition of intermediate dimension. Working with it is more beneficial to our objectives. For $E \subset \mathbb{R}^d$ bounded and non-empty, $\theta \in (0,1]$, $r > 0$ and $s \in [0,d]$, define

$$S^s_{r,\theta}(E) = \inf\Big\{\sum_i |U_i|^s : \{U_i\}_i \text{ is a cover of } E \text{ such that } r \le |U_i| \le r^\theta \text{ for all } i\Big\}.$$

Then, we have

$$\underline{\dim}_\theta E = \left(\text{the unique } s \in [0,d] \text{ such that } \liminf_{r\to 0} \frac{\log S^s_{r,\theta}(E)}{-\log(r)} = 0\right)$$

and

$$\overline{\dim}_\theta E = \left(\text{the unique } s \in [0,d] \text{ such that } \limsup_{r\to 0} \frac{\log S^s_{r,\theta}(E)}{-\log(r)} = 0\right).$$

Furthermore, the next result holds.

Lemma 2.3 ([4, Lemma 2.1]). *Let* $\theta \in (0,1]$ *and* $E \subset \mathbb{R}^d$. *For each* $0 < r < 1$,

$$-(s-t) \le \frac{\log S^s_{r,\theta}(E)}{-\log(r)} - \frac{\log S^t_{r,\theta}(E)}{-\log(r)} \le -\theta(s-t) \quad (0 \le t \le s \le d).$$

Remark 2.4. Note that this implies that if $\liminf_{r\to 0} \frac{\log S^s_{r,\theta}(E)}{-\log(r)} \ge 0$ then $s \le \underline{\dim}_\theta E$, and analogously, if $\limsup_{r\to 0} \frac{\log S^s_{r,\theta}(E)}{-\log(r)} \ge 0$ then $s \le \overline{\dim}_\theta E$.

Let $E \subset \mathbb{R}^d$ be a non-empty set and let $N_r(E)$ be the minimum number of sets of diameter r that can cover E. The *Assouad dimension* of E is defined by

$$\dim_{\mathrm{A}} E = \inf\{s : \exists C > 0 \text{ such that } \forall 0 < r < R \text{ and } x \in E, N_r\big(B(x,R) \cap E\big) \le C(R/r)^s\}$$

and the *upper Assouad spectrum* of E, for $\alpha \in (0,1)$ is defined by

$$\overline{\dim}^\alpha_{\mathrm{A}} E = \inf\{t : \exists C > 0 \text{ such that } \forall 0 < r \le R^{1/\alpha} < R < 1 \text{ and } x \in E, N_r\big(B(x,R) \cap E\big) \le C(R/r)^t\}.$$

The upper Assouad spectrum is non-decreasing with respect to α. Finally, the *quasi-Assouad dimension*, introduced in [15], is defined by

$$\dim_{\mathrm{qA}} E = \lim_{\alpha\to 1} \overline{\dim}^\alpha_{\mathrm{A}} E$$

and we have for $\alpha \in (0,1)$

$$\overline{\dim}_{\mathrm{B}} E \le \overline{\dim}^\alpha_{\mathrm{A}} E \le \dim_{\mathrm{qA}} E \le \dim_{\mathrm{A}} E.$$

For background on Assouad-type dimensions, the reader can refer to [14].

Given a compact set $E \subset \mathbb{R}^d$ and $1 \le m \le d$, in [4] the lower and upper θ- intermediate dimension profile, $\underline{\dim}^m_\theta E$ and $\overline{\dim}^m_\theta E$ respectively, were introduced in order to denote the value of the θ intermediate dimension (lower or upper, respectively) of $P_V(E)$, for $\gamma_{d,m}$ a.e. $V \in G(d,m)$.

In fact, the next results holds.

Theorem 2.5 ([4, Theorem 5.1]). *Let $E \subset \mathbb{R}^d$ be bounded. Then, for all $V \in G(d,m)$*

$$\underline{\dim}_\theta P_V E \le \underline{\dim}_\theta^m E \quad \text{and} \quad \overline{\dim}_\theta P_V E \le \overline{\dim}_\theta^m E,$$

for all $\theta \in (0,1]$. Moreover, for $\gamma_{d,m}$-almost all $V \in G(d,m)$,

$$\underline{\dim}_\theta P_V E = \underline{\dim}_\theta^m E \quad \text{and} \quad \overline{\dim}_\theta P_V E = \overline{\dim}_\theta^m E,$$

for all $\theta \in (0,1]$.

Theorem 2.6 ([3, Theorem 3.9]). *Let $E \subset \mathbb{R}^d$ be compact, $m \in \{1,\dots,d\}$ and $0 \le \lambda \le m$, then*

$$\dim_H \left\{ V \in G(d,m) : \overline{\dim}_\theta P_V E < \overline{\dim}_\theta^\lambda E \right\} \le m(d-m) - (m-\lambda)$$

and

$$\dim_H \left\{ V \in G(d,m) : \underline{\dim}_\theta P_V E < \underline{\dim}_\theta^\lambda E \right\} \le m(d-m) - (m-\lambda).$$

Results on the exceptional directions for the box dimension of projections of sets were first obtained in [8]. Later, in [3], the θ intermediate profile was generalized from considering integers m to consider positive real numbers $t > 0$. These new possible values turn out to appear for the case of the almost sure θ intermediate dimension of the image of the set E under the fractional Brownian motion.

We recall the definition of index$-\alpha$ fractional Brownian motion, which, following the notation of [3], we denote by $B_\alpha : \mathbb{R}^d \to \mathbb{R}^m$ for $d,m \in \mathbb{N}$; $B_\alpha = (B_{\alpha,1},\dots,B_{\alpha,m})$, where $B_{\alpha,i} : \mathbb{R}^d \to \mathbb{R}$ for each i. They satisfy:

- $B_{\alpha,i}(0) = 0$;
- $B_{\alpha,i}$ is continuous with probability 1;
- the increments $B_{\alpha,i}(x) - B_{\alpha,i}(y)$ are normally distributed with mean 0 and variance $|x-y|^{2\alpha}$ for all $x,y \in \mathbb{R}^d$.

Moreover, $B_{\alpha,i}$ and $B_{\alpha,j}$ are independent for all $i,j \in \{1,\dots,m\}$.

The next result holds.

Theorem 2.7 ([3, Theorem 3.4]). *Let $\theta \in (0,1]$, $m,d \in \mathbb{N}$, $B_\alpha : \mathbb{R}^d \to \mathbb{R}^m$ be index-α fractional Brownian motion $(0 < \alpha < 1)$ and $E \subset \mathbb{R}^d$ be compact. Then almost surely*

$$\underline{\dim}_\theta B_\alpha(E) = \frac{1}{\alpha}\underline{\dim}_\theta^{m\alpha} E \quad \text{and} \quad \overline{\dim}_\theta B_\alpha(E) = \frac{1}{\alpha}\overline{\dim}_\theta^{m\alpha} E.$$

To define the intermediate dimension profiles, we first need to define the necessary kernel functions.

Definition 2.8. Let $\theta \in (0,1]$, $t > 0$, $0 \le s \le t$ and $0 < r < 1$, define $\phi_{r,\theta}^{s,t}(x)$ by

$$\phi_{r,\theta}^{s,t}(x) = \begin{cases} 1 & \text{if } 0 \le |x| < r \\ \left(\frac{r}{|x|}\right)^s & \text{if } r \le |x| \le r^\theta \\ \frac{r^{\theta(t-s)+s}}{|x|^t} & \text{if } r^\theta \le |x| \end{cases}$$

and

$$C_{r,\theta}^{s,t}(E) = \left(\inf_{\mu \in \mathcal{M}(E)} \iint \phi_{r,\theta}^{s,t}(x-y)\,d\mu(x)\,d\mu(y)\right)^{-1}$$

where $\mathcal{M}(E)$ denotes the set of probability measures supported on E.

The lower and upper θ intermediate dimension profiles of a bounded set $E \subset \mathbb{R}^d$ are defined as follows.

Definition 2.9. Let $t > 0$ and $E \subset \mathbb{R}^d$. The *lower intermediate dimension profile* of E is defined as

$$\underline{\dim}_\theta^t E = \left(\text{The unique } s \in [0,t] \text{ such that } \liminf_{r \to 0} \frac{\log C_{r,\theta}^{s,t}(E)}{-\log r} = s\right)$$

and the *upper intermediate dimension profile* of E is

$$\overline{\dim}_\theta^t E = \left(\text{The unique } s \in [0,t] \text{ such that } \limsup_{r \to 0} \frac{\log C_{r,\theta}^{s,t}(E)}{-\log r} = s\right).$$

Remark 2.10. By [4, Lemma 3.2] and [3, Lemma 2.2], we have that

$$\text{if } 0 \le \liminf_{r \to 0} \frac{\log C_{r,\theta}^{s,t}(E)}{-\log r} - s, \quad \text{then} \quad s \le \underline{\dim}_\theta^t E,$$

and

$$\text{if } 0 \le \limsup_{r \to 0} \frac{\log C_{r,\theta}^{s,t}(E)}{-\log r} - s, \quad \text{then} \quad s \le \overline{\dim}_\theta^t E.$$

Finally, we give a result that we need later.

Corollary 2.11 ([1, Corollary 3.14]). *Let $E \subset \mathbb{R}^d$ bounded. Then*

$$\overline{\dim}_\theta E \ge \frac{\theta d \,\overline{\dim}_{\mathrm{B}} E}{d - (1-\theta)\overline{\dim}_{\mathrm{B}} E}.$$

The same holds replacing $\overline{\dim}_\theta$ with $\underline{\dim}_\theta$ and $\overline{\dim}_{\mathrm{B}}$ with $\underline{\dim}_{\mathrm{B}}$, respectively.

Lemma 2.12 ([11, Proposition 2.3], [1, Lemma 5.2]). *Let E be a compact subset of $\mathbb{R}^d$, let $0 < \theta \le 1$, and suppose $0 < s < \underline{\dim}_\theta E$. There exists a constant $c > 0$ such that for all $r \in (0,1)$ we can find a Borel probability measure μ_r supported on E such that for all $x \in \mathbb{R}^d$ and $r \le \delta \le r^\theta$,*

$$\mu_r\big(B(x,\delta)\big) \le c\delta^s. \tag{2.1}$$

Analogously, if $0 < s < \overline{\dim}_\theta E$, we have that there exists a constant $c > 0$ such that for all $r_0 > 0$ there exists $r \in (0, r_0)$ and a Borel probability measure μ_r supported on E satisfying (2.1) for $r \le \delta \le r^\theta$ and all $x \in \mathbb{R}^d$.

We finish the preliminaries by recalling another family of kernels $\tilde{\phi}^s_{r,\theta}$ on $\mathbb{R}^d$, defined for $0 < r < 1$, $\theta \in [0,1]$ and $0 < s \le m$ in [4] and [10]

$$\tilde{\phi}^s_{r,\theta} = \begin{cases} 1 & \text{if } |x| < r \\ \left(\frac{r}{|x|}\right)^s & \text{if } r \le |x| < r^\theta \\ 0 & \text{if } r^\theta \le |x|. \end{cases}$$

These kernels are very important because they are useful in order to bound the sums of the diameters of coverings. The next result holds.

Lemma 2.13 ([10, Lemma 2.1]). *Let $E \subset \mathbb{R}^d$ be compact, $\theta \in (0,1]$, $0 < r < 1$ and $0 \le s \le d$ and let $\mu \in \mathcal{M}(E)$. Then*

$$S^s_{r,\theta}(E) \ge r^s\Big[\iint \tilde{\phi}^s_{r,\theta}(x-y)\,d\mu(x)\,d\mu(y)\Big]^{-1}.$$

3. Results

Our first result is a version of *Marstrand's theorem* for quasi-Hausdorff dimension. If $\dim_\theta E$ is continuous at $\theta = 0$, then our result recovers the classical Marstrand–Mattila theorem. In general, however, we have

$$\dim_{\mathrm{H}} E \le \underline{\dim}_{\mathrm{qH}} E \le \overline{\dim}_{\mathrm{qH}} E,$$

and these inequalities can be strict.

Theorem 3.1. *Let $E \subset \mathbb{R}^d$ be a compact set and $0 \le m \le d$. Then*

$$\min\{m, \underline{\dim}_{\mathrm{qH}} E\} = \underline{\dim}_{\mathrm{qH}} P_V E \quad \textit{for } \gamma_{d,m}\textit{-almost all } V \in G(d,m),$$

and

$$\min\{m, \overline{\dim}_{\mathrm{qH}} E\} = \overline{\dim}_{\mathrm{qH}} P_V E \quad \textit{for } \gamma_{d,m}\textit{-almost all } V \in G(d,m).$$

Proof. We prove the case of $\underline{\dim}_\theta E$; the argument for $\overline{\dim}_\theta E$ is similar. We formulate the proof in terms of intermediate dimension profiles. That is, we show that

$$\min\{m, \lim_{\theta\to 0} \underline{\dim}_\theta E\} = \lim_{\theta\to 0} \underline{\dim}^m_\theta E.$$

Recall that the intermediate dimension profile gives the almost sure value of the intermediate dimension of orthogonal projections.

Since for all $0 \le m \le d$, we have $\underline{\dim}^m_\theta E \le m$ and $\underline{\dim}^m_\theta E \le \underline{\dim}_\theta E$, it follows that $\lim_{\theta\to 0} \underline{\dim}^m_\theta E$ is always less than or equal to the left-hand side.

We now prove the reverse inequality. Assume, without loss of generality, that $|E| < 1$.

Let $d(E) := \lim_{\theta\to 0} \underline{\dim}_\theta E$, and choose $t < \min\{m, d(E)\}$ and $\varepsilon > 0$ small enough so that $t + \varepsilon < \min\{m, d(E)\}$.

Then there exists $\theta_0 > 0$ such that $t + \varepsilon < \underline{\dim}_\theta E$ for all $\theta \in (0, \theta_0)$.

Let $r > 0$, $\theta \in (0, 1)$, and choose $\theta_1 \in (0, \min\{\theta, \theta_0\})$ such that $|E| < r^{\theta_1}$.

Then, by Lemma 2.12, there exists a constant $c > 0$ and a probability measure μ supported on E such that

$$\mu\big(B(x, u)\big) \le c u^{t+\varepsilon} \tag{3.1}$$

for all $x \in \mathbb{R}^d$ and $u \in [r, r^{\theta_1}]$.

Now, integrating the kernel $\phi^{t,m}_{r,\theta}$ with respect to μ, and using $\mathbb{I}_{\mathrm{A}}$ for the characteristic function of the set A, we have

$$\begin{aligned}
\big(C^{t,m}_{r,\theta}(E)\big)^{-1} &\le \iint \phi^{t,m}_{r,\theta}(x-y)d\mu(y)d\mu(x)\\
&= \int \mu(B(x,r))d\mu(x) + r^t \iint \frac{\mathbb{I}_{(B(x,r^\theta)\setminus B(x,r))}(y)}{|x-y|^t} d\mu(y)d\mu(x)\\
&\quad + r^{\theta(m-t)+t} \iint \frac{\mathbb{I}_{(\mathbb{R}^d\setminus B(x,r^\theta))}(y)}{|x-y|^m} d\mu(y)d\mu(x)\\
&= S_1 + S_2 + S_3.
\end{aligned}$$

By (3.1), we have $S_1 \le c r^{t+\varepsilon} < c r^t$.

For S_2, using a change of variables and again (3.1), we obtain

$$\begin{aligned}
&\int \frac{\mathbb{I}_{(B(x,r^\theta)\setminus B(x,r))}(y)}{|x-y|^t} d\mu(y) = \int \mu\Big\{y : \frac{\mathbb{I}_{(B(x,r^\theta)\setminus B(x,r))}(y)}{|x-y|^t} \ge u\Big\} du\\
&\quad = \int_0^{1/r^{\theta t}} \mu\big(B(x,r^\theta)\setminus B(x,r)\big)du + \int_{1/r^{\theta t}}^{1/r^t} \mu\Big\{y : \frac{1}{|x-y|^t} \ge u\Big\} du\\
&\quad \le c r^{\theta\varepsilon} + t \int_r^{r^\theta} s^{-t-1}\mu\big(B(x,s)\big)ds \le c r^{\theta\varepsilon} + \frac{ct}{\varepsilon}(r^{\theta\varepsilon} - r^\varepsilon)\\
&\quad < c + \frac{ct}{\varepsilon},
\end{aligned}$$

and hence

$$S_2 < \Big(c + \frac{ct}{\varepsilon}\Big) r^t.$$

For S_3, since $|E| < r^{\theta_1}$ and μ is supported on E, we have

$$\begin{aligned}
&\int \frac{\mathbb{I}_{(\mathbb{R}^d \setminus B(x,r^\theta))}(y)}{|x-y|^m} d\mu(y) \\
&\quad = \int_0^{1/r^{\theta m}} \mu\big(B(x,u^{-1/m}) \setminus B(x,r^\theta)\big) du \\
&\quad \le \int_0^{1/|E|^m} 1 du + \int_{1/|E|^m}^{1/r^{\theta m}} \mu\big(B(x,u^{-1/m})\big) du \\
&\quad = |E|^{-m} + m \int_{r^\theta}^{|E|} s^{-m-1} \mu\big(B(x,s)\big) ds \\
&\quad \le |E|^{-m} + cm \int_{r^\theta}^{|E|} s^{t+\varepsilon-m-1} ds \\
&\quad = |E|^{-m} + \frac{cm}{m-(t+\varepsilon)} (r^{\theta(t+\varepsilon-m)} - |E|^{\theta(t+\varepsilon-m)}) \\
&\quad \le |E|^{-m} + \frac{cm}{m-(t+\varepsilon)} r^{\theta(t+\varepsilon-m)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
S_3 &\le |E|^{-m} r^{\theta(m-t)+t} + \frac{cm}{m-(t+\varepsilon)} r^{\theta\varepsilon} r^t \\
&\le \Big(|E|^{-m} + \frac{cm}{m-(t+\varepsilon)}\Big) r^t.
\end{aligned}$$

Letting $C = 3\max\{c + \frac{ct}{\varepsilon}, |E|^{-m} + \frac{cm}{m-(t+\varepsilon)}\}$, we obtain

$$\big(C_{r,\theta}^{t,m}(E)\big)^{-1} \le C r^t \quad \text{which implies} \quad \frac{\log C_{r,\theta}^{t,m}(E)}{-\log r} - t \ge \frac{\log C}{\log r} \text{ for all } r.$$

Taking the lim inf of both sides yields

$$t \le \underline{\dim}_\theta^m E.$$

Letting $t \to \min\{m, d(E)\}$ completes the proof. ∎

Theorem 3.2. *Let $E \subset \mathbb{R}^d$ be compact, $0 < t < d$, and $\theta \in (0,1]$. Then:*

$$\textit{If } \underline{\dim}_\theta^t E = t, \textit{ then } t \le \underline{\dim}_{\mathrm{qH}} E,$$

and

$$\textit{If } \overline{\dim}_\theta^t E = t, \textit{ then } t \le \overline{\dim}_{\mathrm{qH}} E.$$

Proof. Let $0 < \alpha < 1$ and $0 < m \leq d$ such that $t = \alpha m$.

Let $B_\alpha : \mathbb{R}^d \to \mathbb{R}^m$ be the index-α fractional Brownian motion. Then, using Theorem 2.7, we have

$$m = \underline{\dim}_\theta B_\alpha(E) \leq \underline{\dim}_{\mathrm{B}} B_\alpha(E) \leq m,$$

almost surely, and hence $\underline{\dim}_{\mathrm{B}} B_\alpha(E) = m$. Now, using Corollary 2.11, we obtain that $\underline{\dim}_\theta B_\alpha(E) = \underline{\dim}_{\mathrm{B}} B_\alpha(E) = m$ for all $\theta \in (0, 1)$. Using Theorem 2.7 again and the fact that $\underline{\dim}^s_\theta E \leq \underline{\dim}_\theta E$ for all $0 \leq s \leq d$, we have for all $\theta \in (0, 1)$,

$$m = \underline{\dim}_\theta B_\alpha(E) \leq \frac{\underline{\dim}_\theta E}{\alpha}.$$

Letting $\theta \to 0$, we have

$$t = \alpha m \leq \lim_{\theta \to 0} \underline{\dim}_\theta E,$$

which completes the proof for $\underline{\dim}_\theta$. The proof for the $\overline{\dim}_\theta E$ part is analogous. ∎

We have the following corollary.

Corollary 3.3. *Let $E \subset \mathbb{R}^d$ be compact, $\theta \in (0, 1]$, and $0 < t < d$. Then:*

$$\underline{\dim}^t_\theta E = t \iff t \leq \underline{\dim}_{\mathrm{qH}} E$$

and

$$\overline{\dim}^t_\theta E = t \iff t \leq \overline{\dim}_{\mathrm{qH}} E.$$

Proof. This corollary follows directly from the proof of Theorem 3.1, Theorem 3.2, and the fact that $\lim_{\theta \to 0} \overline{\dim}^t_\theta E \leq \overline{\dim}^t_\theta E \leq t$ for all θ, with the same holding when replacing $\overline{\dim}^t_\theta$ with $\underline{\dim}^t_\theta$. ∎

Using Theorem 2.5 and Theorem 2.7, we obtain the following results about orthogonal projections and fractional Brownian motion.

The following corollary was already proved in the case of compact sets whose θ-intermediate dimension is continuous at $\theta = 0$; see [4, Corollary 6.4]. Our result improves upon this by extending it to general compact sets.

Corollary 3.4. *Let $E \subset \mathbb{R}^d$ be compact and let $0 < m < d$. Then:*

$$\underline{\dim}_{\mathrm{B}} P_V(E) = m \text{ for } \gamma_{d,m} - \text{almost all } V \in G(d, m) \iff m \leq \underline{\dim}_{\mathrm{qH}} E$$

and

$$\overline{\dim}_{\mathrm{B}} P_V(E) = m \text{ for } \gamma_{d,m} - \text{almost all } V \in G(d, m) \iff m \leq \overline{\dim}_{\mathrm{qH}} E.$$

Proof. Combine Corollary 3.3 with Theorem 2.5. ∎

Corollary 3.5. *Let $E \subset \mathbb{R}^d$ be compact, $\theta \in (0,1]$, $0 < m \le d$, and $B_\alpha : \mathbb{R}^d \to \mathbb{R}^m$ be an index-α fractional Brownian motion* $(0 < \alpha < 1)$. *Then:*

$$\underline{\dim}_\theta B_\alpha(E) = m \text{ almost surely} \iff \alpha m \le \underline{\dim}_{\mathrm{qH}} E$$

and

$$\overline{\dim}_\theta B_\alpha(E) = m \text{ almost surely} \iff \alpha m \le \overline{\dim}_{\mathrm{qH}} E.$$

Proof. Combine Corollary 3.3 with Theorem 2.7. ∎

Our next theorem is a lower bound for the θ-intermediate dimension profile in terms of the quasi-Assouad spectrum and the Assouad dimension of the set. This result generalizes the result obtained in [12] for the upper and lower box dimensions.

Theorem 3.6. *Let $\theta \in (0,1]$, $\alpha \in (0,1)$, and $E \subset \mathbb{R}^n$ be bounded. Then, if $\underline{\dim}^t_\theta E < t$, we have*

$$\underline{\dim}^t_\theta E \ge \underline{\dim}_\theta E - \max\left\{0, \dim^\alpha_{\mathrm{A}} E - t, (\dim_{\mathrm{A}} E - t)(1-\alpha)\right\}. \tag{3.2}$$

And if $\overline{\dim}^t_\theta E < t$, we have

$$\overline{\dim}^t_\theta E \ge \overline{\dim}_\theta E - \max\left\{0, \dim^\alpha_{\mathrm{A}} E - t, (\dim_{\mathrm{A}} E - t)(1-\alpha)\right\}.$$

Proof. If $\underline{\dim}_\theta E = 0$, there is nothing to prove. If $\underline{\dim}_\theta E = \underline{\dim}_{\mathrm{B}} E \le t$ for all θ, then

$$\underline{\dim}^t_\theta E \ge \lim_{\theta\to 0} \underline{\dim}^t_\theta E = \min\{t, \lim_{\theta\to 0} \underline{\dim}_\theta E\} = \underline{\dim}_{\mathrm{B}} E = \underline{\dim}_\theta E,$$

and the result follows. If $\underline{\dim}_\theta E = \underline{\dim}_{\mathrm{B}} E > t$ for all θ, then

$$\underline{\dim}^t_\theta E \ge \lim_{\theta\to 0} \underline{\dim}^t_\theta E = \min\{t, \lim_{\theta\to 0} \underline{\dim}_\theta E\} = t,$$

which contradicts the hypothesis.

So, suppose that $0 < \underline{\dim}_\theta E < \underline{\dim}_{\mathrm{B}} E$.

Let $A > \dim_{\mathrm{A}} E$ and for $\alpha \in (0,1)$, let $t \ne \dim^\alpha_{\mathrm{A}} E$ and let $A_\alpha > \dim^\alpha_{\mathrm{A}} E$ such that $A_\alpha \ne t$.

By definition, there exists a constant $C > 0$ such that for all $0 < r < R$ and $x \in E$,

$$N_r\big(B(x,R) \cap E\big) \le C\left(\frac{R}{r}\right)^A,$$

and for all $0 < r \le R^{1/\alpha} < R$ and $x \in E$,

$$N_r\big(B(x,R) \cap E\big) \le C\left(\frac{R}{r}\right)^{A_\alpha}.$$

Let $\theta \in (0,1)$, $s < \underline{\dim}_\theta E$ and $s' < s - \max\{0, A_\alpha - t, (A-t)(1-\alpha)\}$. Note that $s' < \underline{\dim}_\theta E$. For $r \in (0,1)$, let μ_r^s be the Frostman measure of the set E for s of Lemma 2.12.

Since μ_r^s are probability measures supported on E, we have

$$\begin{aligned}
\big(C_{r,\theta}^{s',t}(E)\big)^{-1} &\leq \iint \phi_{r,\theta}^{s',t}(x-y)\,d\mu_r^s(y)\,d\mu_r^s(x)\\
&= \int \mu_r^s\big(B(x,r)\big)d\mu_r^s(x) + r^{s'}\iint \frac{\mathbb{I}_{(B(x,r^\theta)\setminus B(x,r))}(y)}{|x-y|^{s'}}d\mu_r^s(y)\,d\mu_r^s(x)\\
&\quad + r^{\theta(t-s')+s'}\iint \frac{\mathbb{I}_{(\mathbb{R}^n\setminus B(x,r^\theta))}(y)}{|x-y|^t}d\mu_r^s(y)\,d\mu_r^s(x)\\
&= S_1 + S_2 + S_3.
\end{aligned}$$

By the Frostman condition of μ_r^s, we have $S_1 \leq c r^s < c r^{s'}$.

For S_2, by a change of variable $u = t^{-1/s'}$, we have

$$\begin{aligned}
&\int \frac{\mathbb{I}_{(B(x,r^\theta)\setminus B(x,r))}(y)}{|x-y|^{s'}}d\mu_r^s(y)\\
&= \int \mu_r^s\left\{y : \frac{\mathbb{I}_{(B(x,r^\theta)\setminus B(x,r))}(y)}{|x-y|^{s'}} \geq t\right\}dt\\
&= \int_0^{1/r^{\theta s'}} \mu_r^s\big(B(x,r^\theta)\setminus B(x,r)\big)dt + \int_{1/r^{\theta s'}}^{1/r^{s'}} \mu_r^s\left\{y : \frac{1}{|x-y|^{s'}} \geq t\right\}dt\\
&\leq c r^{\theta(s-s')} + \int_{1/r^{\theta s'}}^{1/r^{s'}} \mu_r^s\big(B(x,(1/t)^{1/s'})\big)dt\\
&= c r^{\theta(s-s')} + s'\int_r^{r^\theta} u^{-s'-1}\mu_r^s\big(B(x,u)\big)du\\
&\leq c r^{\theta(s-s')} + \frac{cs'}{s-s'}\big(r^{\theta(s-s')} - r^{s-s'}\big) < c + \frac{cs'}{s-s'},
\end{aligned}$$

and then

$$S_2 < \Big(c + \frac{cs'}{s-s'}\Big)r^{s'}.$$

For S_3, we have

$$\begin{aligned}
\int \frac{\mathbb{I}_{(\mathbb{R}^n\setminus B(x,r^\theta))}(y)}{|x-y|^t}d\mu_r^s(y) &= \int_0^\infty \mu_r^s\left\{y : \frac{\mathbb{I}_{(\mathbb{R}^n\setminus B(x,r^\theta))}(y)}{|x-y|^t} \geq w\right\}dw\\
&= \int_0^{r^{-\theta t}} \mu_r^s\big(B(x,w^{-1/t})\big)dw = t\int_{r^\theta}^\infty u^{-t-1}\mu_r^s\big(B(x,u)\big)du\\
&= t\int_{r^\theta}^{r^{\alpha\theta}} u^{-t-1}\mu_r^s\big(B(x,u)\big)du + t\int_{r^{\alpha\theta}}^\infty u^{-t-1}\mu_r^s\big(B(x,u)\big)du\\
&= S_{3,1} + S_{3,2}.
\end{aligned}$$

For $S_{3,2}$, we have

$$\begin{aligned}
t\int_{r^{\alpha\theta}}^{\infty} & u^{-t-1}\mu_r^s\big(B(x,u)\big)du \\
&= t\int_{r^{\alpha\theta}}^{|E|} u^{-t-1}\mu_r^s\big(B(x,u)\big)du + t\int_{|E|}^{\infty} u^{-t-1}\mu_r^s\big(B(x,u)\big)du \\
&\le t\int_{r^{\alpha\theta}}^{\infty} u^{-t-1}\Big(\frac{u}{r^\theta}\Big)^{A_\alpha}\mu_r^s\big(B(x,r^\theta)\big)du + t\int_{|E|}^{\infty} u^{-t-1}du \\
&\le t\int_{r^{\alpha\theta}}^{|E|} u^{-t-1}\Big(\frac{u}{r^\theta}\Big)^{A_\alpha}\mu_r^s\big(B(x,r^\theta)\big)du + |E|^{-t} \\
&\le t\int_{r^{\theta}}^{|E|} u^{-t-1}\Big(\frac{u}{r^\theta}\Big)^{A_\alpha}\mu_r^s\big(B(x,r^\theta)\big)du + |E|^{-t} \\
&\le \frac{ct}{A_\alpha - t} r^{\theta(s-A_\alpha)}\big(|E|^{A_\alpha - t} - r^{\theta(A_\alpha - t)}\big) + |E|^{-t} \\
&= \frac{ct|E|^{A_\alpha - t}}{A_\alpha - t} r^{\theta(s-A_\alpha)} - \frac{ct}{A_\alpha - t} r^{(s-t)} + |E|^{-t}.
\end{aligned}$$

For $S_{3,1}$, first suppose that $t \neq A$. Then,

$$\begin{aligned}
t\int_{r^{\theta}}^{r^{\alpha\theta}} & u^{-t-1}\mu_r^s\big(B(x,u)\big)du \\
&\le t\int_{r^{\theta}}^{r^{\alpha\theta}} u^{-t-1}\Big(\frac{u}{r^\theta}\Big)^{A}\mu_r^s\big(B(x,r^\theta)\big)du \\
&\le \frac{ct}{A-t} r^{\theta(s-A)}\big(r^{\alpha\theta(A-t)} - r^{\theta(A-t)}\big) \\
&= \frac{ct}{A-t} r^{\theta(s-A)+\alpha\theta(A-t)} - \frac{ct}{A-t} r^{\theta(s-t)},
\end{aligned}$$

and then

$$\begin{aligned}
S_3 &\le r^{\theta(t-s')+s'}\Big(\tfrac{ct|E|^{A_\alpha - t}}{A_\alpha - t} r^{\theta(s-A_\alpha)} + \tfrac{ct}{A-t} r^{\theta(s-A)+\alpha\theta(A-t)} \\
&\qquad\qquad - \big(\tfrac{ct}{A-t} + \tfrac{ct}{A_\alpha - t}\big) r^{\theta(s-t)} + |E|^{-t}\Big) \\
&= \big(\tfrac{ct|E|^{A_\alpha - t}}{A_\alpha - t}\big) r^{\theta(s-(A_\alpha - t)-s')} r^{s'} + \big(\tfrac{ct}{A-t}\big) r^{\theta(s-(1-\alpha)(A-t)-s')} r^{s'} \\
&\qquad\qquad - \big(\tfrac{ct}{A-t} + \tfrac{ct}{A_\alpha - t}\big) r^{\theta(s-s')} r^{s'} + |E|^{-t} r^{\theta(t-s')+s'}.
\end{aligned}$$

So, by our choice of s', we have

$$s - (A_\alpha - t) - s' \ge 0 \quad \text{and} \quad s - (1-\alpha)(A-t) - s' \ge 0,$$

and there exists a constant $C > 0$ such that

$$S_3 \le C r^{s'},$$

and then

$$\left(C_{r,\theta}^{s',t}(E)\right)^{-1} \leq \iint \phi_{r,\theta}^{s',t}(x-y)d\mu_r^s(x)d\mu_r^s(y) \leq \left(1+c+\frac{cs'}{s-s'}+C\right)r^{s'}.$$

This implies

$$\liminf_{r\to 0} \frac{\log C_{r,\theta}^{s',t}(E)}{-\log r} - s' \geq 0,$$

and finally, using Remark 2.10,

$$\underline{\dim}_\theta^t E \geq s'.$$

Now, suppose that $A = t$. Then,

$$\begin{aligned} S_{3,1} &= t\int_{r^\theta}^{r^{\alpha\theta}} u^{-t-1}\mu_r^s\big(B(x,u)\big)du \leq ct\int_{r^\theta}^{r^{\alpha\theta}} u^{-t-1}\Big(\frac{u}{r^\theta}\Big)^t r^{\theta s}du \\ &< -ctr^{\theta(s-t)}\theta\log(r), \end{aligned}$$

and since $1 \leq -\log(r)$ and proceeding as before,

$$\left(C_{r,\theta}^{s',t}(E)\right)^{-1} \leq \iint \phi_{r,\theta}^{s',t}(x-y)d\mu_r^s(x)d\mu_r^s(y) \leq -C'\log(r)r^{s'}$$

for some constant $C' > 0$ independent of r, which implies

$$\liminf_{r\to 0} \frac{\log C_{r,\theta}^{s',t}(E)}{-\log r} - s' \geq \liminf_{r\to 0}\frac{\log(\log(r))}{\log(r)} = 0.$$

As before, we conclude that

$$\underline{\dim}_\theta^t E \geq s'.$$

Finally, letting $A_\alpha \to \dim_{\mathrm{A}}^\alpha E$, $A \to \dim_{\mathrm{A}} E$, and $s \to \underline{\dim}_\theta E$, we obtain

$$\underline{\dim}_\theta^t E > s'$$

for all $s' < \underline{\dim}_\theta E - \max\{0, \dim_{\mathrm{A}}^\alpha E - t, (\dim_{\mathrm{A}} E - t)(1-\alpha)$, which implies that

$$\underline{\dim}_\theta^t E \geq \underline{\dim}_\theta E - \max\left\{0, \dim_{\mathrm{A}}^\alpha E - t, (\dim_{\mathrm{A}} E - t)(1-\alpha)\right\}. \qquad (3.3)$$

The case $t = \dim_{\mathrm{A}}^\alpha E$ follows from (3.3) and by the continuity in t of $\underline{\dim}_\theta^t E$; see [3, Corollary 3.8]. The second part of the theorem, concerning the upper θ-intermediate dimension, is similar. ∎

The next corollaries are immediate consequences of the above theorem. The case for $\theta = 1$ was already proved in [12, Corollary 1.3].

Corollary 3.7. *Let* $E \subset \mathbb{R}^d$ *be bounded and suppose that* $\dim_{\mathrm{qA}} E \leq \max\{m, \overline{\dim}_\theta E\}$. *Then,*

$$\overline{\dim}_\theta P_V E = \min\{m, \overline{\dim}_\theta E\},$$

for all $\theta \in (0,1]$ *and* $\gamma_{n,m}$*-almost all* $V \in G(n,m)$*. More generally,*

$$\overline{\dim}_\theta P_V E \geq \overline{\dim}_\theta E - \max\{0, \dim_{\mathrm{qA}} E - m\}, \tag{3.4}$$

for all $\theta \in (0,1]$ *and* $\gamma_{n,m}$*-almost all* $V \in G(n,m)$*. The same conclusion holds with* $\overline{\dim}_\theta$ *replaced with* $\underline{\dim}_\theta$.

Proof. Let $\alpha \to 1$ in Theorem 3.6. ∎

From Theorem 3.1, the following corollary is immediate.

Corollary 3.8. *Let* $E \subset \mathbb{R}^n$ *be a bounded set such that* $\overline{\dim}_{\mathrm{B}} E = \overline{\dim}_{\mathrm{qH}} E$*. Then, for* $\gamma_{n,m}$*-almost every* $V \in G(n,m)$*, we have*

$$\overline{\dim}_{\mathrm{B}}(P_V E) = \min\{m, \overline{\dim}_{\mathrm{B}} E\},$$

and similarly,

$$\underline{\dim}_{\mathrm{B}}(P_V E) = \min\{m, \underline{\dim}_{\mathrm{B}} E\}$$

whenever $\underline{\dim}_{\mathrm{B}} E = \underline{\dim}_{\mathrm{qH}} E$.

By [12, Corollary 1.3], we have that if $\overline{\dim}_{\mathrm{B}} E = \dim_{\mathrm{qA}} E$, then $\overline{\dim}_{\mathrm{B}} P_V E = \min\{\overline{\dim}_{\mathrm{B}} E, m\}$ for a.e. $V \in G(d,m)$, and the same holds for $\underline{\dim}_{\mathrm{B}}$. A natural question is which hypothesis is more general: $\dim_B E = \dim_{\mathrm{qA}} E$ or $\dim_B E = \dim_{qH} E$.

In the following theorem, we prove that if $\overline{\dim}_{\mathrm{B}} E = \dim_{\mathrm{qA}} E$, then $\overline{\dim}_\theta E = \overline{\dim}_{\mathrm{B}} E$, and if $\underline{\dim}_{\mathrm{B}} E = \dim_{\mathrm{qA}} E$, then $\underline{\dim}_\theta E = \underline{\dim}_{\mathrm{B}} E$ for all $\theta \in (0,1]$. The converse is not generally true; in fact, it is not hard to construct a Cantor set $C \subset \mathbb{R}$ such that $\dim_{\mathrm{H}} C = \underline{\dim}_{\mathrm{B}} C < \dim_{qA} C$. Therefore, Corollary 3.8 is more general than the result obtained in [12].

Theorem 3.9. *Let* $E \subset \mathbb{R}^d$ *be bounded. Then, if* $\underline{\dim}_{\mathrm{B}} E = \dim_{\mathrm{qA}} E$*, we have*

$$\underline{\dim}_\theta E = \underline{\dim}_{\mathrm{B}} E \quad \forall \theta \in (0,1],$$

and if $\overline{\dim}_{\mathrm{B}} E = \dim_{\mathrm{qA}} E$*, then*

$$\overline{\dim}_\theta E = \overline{\dim}_{\mathrm{B}} E \quad \forall \theta \in (0,1].$$

Proof. We prove the first case. The case of $\overline{\dim}_\theta$ is analogous.

If $\underline{\dim}_\theta E = 0$ for some θ, then using Lemma 2.11, we have that the intermediate dimension is constantly 0. So suppose that $0 < \underline{\dim}_\theta E$. By [2, Corollary 2.8], we have that if $\underline{\dim}_\theta E = \dim_{\mathrm{A}} E$ for some θ, then the intermediate dimensions are constant and equal to the Assouad dimension. So we suppose that $\underline{\dim}_\theta E < \dim_A E$ for all $\theta \in (0, 1]$.

The idea of the proof is to integrate appropriate kernels $\tilde{\phi}^t_{r,\theta}$ with respect to a Frostman measure and show that this integral is bounded above.

Let $0 < \theta < 1$ and $r \in (0, 1)$. Suppose that $\underline{\dim}_B E = \dim_{qA} E$. Then we have $\underline{\dim}_B E = \dim_A^\alpha E$ for all $\alpha \in [0, 1]$. Using the continuity of the intermediate dimensions, choose ρ and α sufficiently close to 1 such that $\underline{\dim}_\rho E - \dim_A E(1-\alpha) > 0$ and $\rho\underline{\dim}_\rho E - \dim_A^\alpha E(\rho - \theta) > 0$.

Now let $d_\rho, d_\theta, A_\alpha, A_\rho, A$ be such that $0 < d_\rho < \underline{\dim}_\rho E < A < \dim_A E$, $0 < d_\theta < \underline{\dim}_\theta E$, $0 < A_\alpha < \dim_A^\alpha E$, and $0 < A_\rho < \dim_A^\rho E$, and satisfies

$$0 < t := \min\left\{\frac{d_\rho - A(1-\alpha)}{\alpha}, \frac{\rho d_\rho - A_\alpha(\rho - \theta)}{\theta}\right\}.$$

Note that $t < d_\rho$. By the definition of the Assouad dimension and the Upper Assouad spectrum, there exists $C > 0$ such that for all $0 < r < R < 1$ and $x \in E$,

$$N_r\big(B(x, R) \cap E\big) \le C\left(\frac{R}{r}\right)^A, \tag{3.5}$$

and for all $0 < r \le R^{1/\alpha} < R < 1$ and $x \in E$,

$$N_r\big(B(x, R) \cap E\big) \le C\left(\frac{R}{r}\right)^{A_\alpha}, \tag{3.6}$$

and finally, for all $0 < r \le r^{1/\rho} < R < 1$ and $x \in E$,

$$N_r\big(B(x, R) \cap E\big) \le C\left(\frac{R}{r}\right)^{A_\rho}. \tag{3.7}$$

Let μ_r be a Borel probability measure supported on E that satisfies the condition that there exists $C' > 0$ such that

$$\mu_r\big(B(x, \delta)\big) \le C\delta^{d_\rho}$$

for all $\delta \in [r, r^\rho]$ and $x \in \mathbb{R}^d$. We now integrate the kernels $\widetilde{\phi}^t_{r,\theta}$ with respect to μ_r and bound this integral essentially by $r^{t-(\rho-\theta)(A_\rho - t)}$.

Then

$$\begin{aligned}
&\iint \tilde{\phi}^t_{r,\theta}(x-y)d\mu_r(x)d\mu_r(y) \\
&\quad = \iint \mu_r\big(B(x,r)\big)d\mu_r(x) + r^t \iint \frac{\mathbb{I}_{(B(x,r^\theta)\setminus B(x,r))}}{|x-y|^t} d\mu_r(x)d\mu_r(y) \\
&\quad = S_1 + S_2.
\end{aligned}$$

By the Frostman condition of μ_r and since $t < d_\rho$, we have

$$S_1 \le C' r^{d_\rho} < C' r^t \le C' r^{t-(\rho-\theta)(A_\rho - t)}.$$

For S_2, we use inequality (3.7) and the Frostman condition on μ_r since $t < d_\rho$. We have

$$\begin{aligned}
&\int \frac{\mathbb{I}_{(B(x,r^\theta)\setminus B(x,r))}}{|x-y|^t} d\mu_r(y) \\
&\quad = \int_0^\infty \mu_r\Big(\Big\{y : \frac{\mathbb{I}_{(B(x,r^\theta)\setminus B(x,r))}}{|x-y|^t} \ge u\Big\}\Big) du \\
&\quad = \int_0^{1/r^{\theta t}} \mu_r\big(B(x,r^\theta)\setminus B(x,r)\big) du + \int_{1/r^{\theta t}}^{1/r^t} \mu_r\big(B(x,u^{-1/t})\big) du \\
&\quad \le \int_0^{1/r^{\theta t}} \mu_r\big(B(x,r^\theta)\big) du + t\int_r^{r^\theta} u^{-1-t}\mu_r\big(B(x,u)\big) du \\
&\quad \le C\Big(\frac{r^\theta}{r^\rho}\Big)^{A_\rho} \int_0^{1/r^{\theta t}} \mu_r\big(B(x,r^\rho)\big) du + t\int_r^{r^\theta} u^{-1-t}\mu_r\big(B(x,u)\big) du \\
&\quad \le C'C\Big(\frac{r^\theta}{r^\rho}\Big)^{A_\rho} \int_0^{1/r^{\theta t}} r^{\rho d_\rho} du + t\int_r^{r^\theta} u^{-1-t}\mu_r\big(B(x,u)\big) du \\
&\quad \le C'C r^{(\rho-\theta)(t-A_\rho)} + t\int_r^{r^\theta} u^{-1-t}\mu_r\big(B(x,u)\big) du.
\end{aligned}$$

To bound the last term of the sum, we partition the interval of integration $I = (r, r^\theta)$ into three parts. One interval in which we can apply the Frostman condition of μ_r, and for the remaining two intervals, we utilize the properties of the quasi-Assouad spectrum and Assouad dimension. Namely,

$$I = (r, r^\theta) = (r, r^\rho) \cup [r^\rho, r^{\alpha\rho}) \cup [r^{\alpha\rho}, r^\theta).$$

Hence

$$t\int_r^{r^\theta} u^{-1-t}\mu_r\big(B(x,u)\big) du = S_{2,1} + S_{2,2} + S_{2,3}.$$

Using the Frostman condition, it is immediate that

$$S_{2,1} \le \frac{t}{d_\rho - t} r^{\rho(d_\rho - t)} \le \frac{t}{d_\rho - t}.$$

For $S_{2,2}$, we use inequality (3.5), the Frostman condition for μ_r, and the fact that $t \le \frac{d_\rho - A(1-\alpha)}{\alpha}$. We have

$$\begin{aligned}
t\int_{r^\rho}^{r^{\alpha\rho}} u^{-1-t}\mu_r\big(B(x,u)\big) du &\le Ct\int_{r^\rho}^{r^{\alpha\rho}} u^{-1-t}\Big(\frac{u}{r^\rho}\Big)^A \mu_r\big(B(x,r^\rho)\big) du \\
&\le \frac{CC't}{A-t} r^{-\rho(A-d_\rho)} r^{(\alpha\rho)(A-t)} \le \frac{CC't}{A-t}.
\end{aligned}$$

Finally, for $S_{2,3}$, we need to use inequality (3.6) and the fact that $t \leq \frac{\rho d_\rho - A_\alpha(\rho-\theta)}{\theta}$ to obtain

$$t \int_{r^{\alpha\rho}}^{r^\theta} u^{-1-t} \mu_r\big(B(x,u)\big) du \leq \frac{CC't}{A-t}.$$

Choosing $M = 3\max\{\frac{CC't}{A-t}, \frac{t}{d_\rho - t}, CC'\}$, we have

$$S_2 \leq r^t M(r^{(\rho-\theta)(t-A_\rho)} + 1) \leq 2Mr^{t-(\rho-\theta)(A_\rho-t)}.$$

Therefore,

$$\iint \tilde{\phi}^t_{r,\theta}(x-y) d\mu_r(x) d\mu_r(y) \leq 2Mr^{t-(\rho-\theta)(A_\rho-t)}.$$

By Lemma 2.13, this implies $r^{(\rho-\theta)(A_\rho-t)} \leq S^t_{r,\theta}(E)$. Taking logarithms and letting $r \to 0$, we have

$$(\rho-\theta)(t-A_\rho) \leq \liminf_{r\to 0} \frac{\log S^t_{r,\theta}(E)}{-\log r}.$$

Now, since by [4, Lemma 2.1] the right-hand side of the inequality is continuous in t, we can let $\alpha \to 1$, $d_\rho \to \underline{\dim}_\rho E$, and $A_\rho \to \dim^\rho_{\mathrm{A}} E$, obtaining

$$t = \dim_{\mathrm{qA}} E - \frac{\dim_{\mathrm{qA}} E - \underline{\dim}_\rho E}{\theta}.$$

Finally, letting $\rho \to 1$ and using the assumption that $\underline{\dim}_{\mathrm{B}} E = \dim_{\mathrm{qA}} E$, we have

$$0 \leq \liminf_{r\to 0} \frac{\log S^{\underline{\dim}_{\mathrm{B}} E}_{r,\theta}(E)}{-\log r},$$

which implies $\underline{\dim}_{\mathrm{B}} E \leq \underline{\dim}_\theta E$, and the result follows. ∎

Our last theorem is another lower bound for intermediate dimension profiles. This lower bound generalizes the one obtained in [7, Proposition 2.12] for upper and box dimension profiles, although the result in that reference is stated in a different form. After rearranging the inequality presented there, one can recover a similar bound to the one we obtain here for intermediate θ-dimensions. The approach to the proof is also closely related.

Theorem 3.10. *Let $E \subset \mathbb{R}^d$ be bounded, $\theta \in (0,1)$, and $t \leq s$. Then,*

$$\underline{\dim}^t_\theta E \geq \frac{\underline{\dim}^s_\theta E}{1 + (\frac{1}{t} - \frac{1}{s})\underline{\dim}^s_\theta E}$$

and

$$\overline{\dim}^t_\theta E \geq \frac{\overline{\dim}^s_\theta E}{1 + (\frac{1}{t} - \frac{1}{s})\overline{\dim}^s_\theta E}.$$

Proof. This proof is inspired by the argument used in [7, Proposition 2.12], although the details differ due to the generality of the intermediate dimension setting.

Let $d_t < \underline{\dim}^t_\theta E$. Then, we have

$$r^{-d_t} \leq C^{d_t,t}_{r,\theta} \tag{3.8}$$

for all sufficiently small r.

Let $d_s = \dfrac{d_t}{1 + (\frac{1}{s} - \frac{1}{t})d_t}$, $r > 0$, and let $R \geq r$ be such that $R^{d_t/d_s} = r$, i.e.,

$$R = r^{\frac{\theta s + d_s(1-\theta)}{\theta s + d_t(1-\theta\frac{s}{t})}}.$$

Let μ be a Borel probability measure supported on E such that

$$\int \phi^{d_t,t}_{r,\theta}(x-y)d\mu(y) = \big(C^{d_t,t}_{r,\theta}(E)\big)^{-1} \tag{3.9}$$

for μ-almost all x. Now, by integrating, we get:

$$\begin{aligned}\int \phi^{d_s,s}_{r,\theta}(x-y)d\mu(y) &\leq \mu\big(B(x,R)\big) + \int_{R\leq|x-y|\leq R^\theta} \frac{r^{d_s}}{|x-y|^{d_s}}d\mu(y) \\ &\qquad + \int_{|x-y|>R^\theta} \frac{r^{\theta(s-d_s)+d_s}}{|x-y|^s}d\mu(y) \\ &= S_1 + S_2 + S_3.\end{aligned}$$

For S_1, using the definition of R, we obtain

$$\begin{aligned}\mu(B(x,R)) &\leq R^{d_t}\Big(R^{-d_t}\int \phi^{d_t,t}_{R,\theta}(x-y)d\mu(y)\Big) \\ &= r^{d_s}\Big(R^{-d_t}\int \phi^{d_t,t}_{R,\theta}(x-y)d\mu(y)\Big).\end{aligned} \tag{3.10}$$

For S_2, by Hölder's inequality:

$$\begin{aligned}\int_{R\leq|x-y|\leq R^\theta} \frac{r^{d_s}}{|x-y|^{d_s}}d\mu(y) &\leq r^{d_s}\Big(\int \frac{1}{|x-y|^{d_t}}d\mu(y)\Big)^{d_s/d_t} \\ &\leq r^{d_s}\Big(R^{-d_t}\int \phi^{d_t,t}_{R,\theta}(x-y)d\mu(y)\Big)^{d_s/d_t}.\end{aligned} \tag{3.11}$$

Finally, for S_3, again using Hölder:

$$\begin{aligned}\int_{|x-y|>R^\theta} \frac{r^{\theta(s-d_s)+d_s}}{|x-y|^s}d\mu(y) &\leq r^{\theta(s-d_s)+d_s}R^{-\theta s}\Big(\int \frac{R^{\theta t}}{|x-y|^t}d\mu(y)\Big)^{s/t} \\ &= r^{d_s}\Big(R^{-d_t}\int \phi^{d_t,t}_{R,\theta}(x-y)d\mu(y)\Big)^{s/t},\end{aligned} \tag{3.12}$$

by our choice of d_s.

Now, using (3.8), (3.9), and combining (3.10), (3.11), and (3.12), we get:

$$\left(C_{r,\theta}^{d_s,s}(E)\right)^{-1} \leq \int \phi_{r,\theta}^{d_s,s}(x-y)d\mu(y) \leq 3r^{d_s}$$

for all sufficiently small r. Hence,

$$\liminf_{r\to 0} \frac{\log C_{r,\theta}^{d_s,s}(E)}{-\log r} - d_s \geq 0,$$

which implies

$$\underline{\dim}_\theta^s E \geq d_s = \frac{d_t}{1+(\frac{1}{s}-\frac{1}{t})d_t}.$$

The result follows by letting $d_t \to \underline{\dim}_\theta^t E$. The case of the upper intermediate dimension profile is similar. ∎

Remark 3.11. The lower bounds in the previous theorem are attained for some sets E, as shown in the next example.

Let $0 < s \leq t \leq 1$, $F_p = \{\frac{1}{n^p} : n \in \mathbb{N}\}$, and $\theta \in (0,1)$. By Theorem 1.1 in [9] and Theorem 3.4 in [4], we have $\underline{\dim}_\theta^s F_p = \overline{\dim}_\theta^s F_p = \frac{s\theta}{\theta+sp}$ for any $s \in (0,1]$. Therefore, for $s \geq t$, we have

$$\begin{aligned} \frac{\underline{\dim}_\theta^s F_p}{1+(\frac{1}{t}-\frac{1}{s})\underline{\dim}_\theta^s F_p} &= \frac{s\theta}{\theta+sp}\left(1+\left(\frac{1}{t}-\frac{1}{s}\right)\frac{s\theta}{\theta+sp}\right)^{-1} \\ &= \left(\frac{ts\theta}{\theta+sp}\right)\left(\frac{t(\theta+sp)+s\theta-t\theta}{\theta+sp}\right)^{-1} \\ &= \frac{t\theta}{\theta+tp} = \underline{\dim}_\theta^t F_P. \end{aligned}$$

Thus, the lower bound given in Theorem 3.10 is attained for this set.

Moreover, combining Theorem 3.10 and Corollary 3.7, we have for $t \leq \dim_{\mathrm{qA}} E$,

$$\underline{\dim}_\theta^t E \geq \frac{\underline{\dim}_\theta E}{1+(\frac{1}{t}-\frac{1}{\dim_{\mathrm{qA}} E})\underline{\dim}_\theta E}$$

and

$$\overline{\dim}_\theta^t E \geq \frac{\overline{\dim}_\theta E}{1+(\frac{1}{t}-\frac{1}{\dim_{\mathrm{qA}} E})\overline{\dim}_\theta E}. \tag{3.13}$$

One may wonder when does inequality (3.13) improve inequality (3.4) for sets E such that $0 < \overline{\dim}_\theta E < \dim_{\mathrm{qA}} E$ and $0 < t < \dim_{\mathrm{qA}} E$, i.e., when is it true that

$$\overline{\dim}_\theta E - (\dim_{\mathrm{qA}} E - t) < \frac{\overline{\dim}_\theta E}{1+(\frac{1}{t}-\frac{1}{\dim_{\mathrm{qA}} E})\overline{\dim}_\theta E} \leq \overline{\dim}_\theta^t E\ ? \tag{3.14}$$

A direct algebraic manipulation of the last inequality implies that inequality (3.14) holds if and only if

$$t^2(\dim_{qA} E - \overline{\dim}_\theta E) + t\big(2\dim_{qA} E\overline{\dim}_\theta E - (\overline{\dim}_\theta E)^2 - (\dim_{qA} E)^2\big) + \big(\dim_{qA} E(\overline{\dim}_\theta E)^2 - (\dim_{qA} E)^2\overline{\dim}_\theta E\big) < 0.$$

But this is true if and only if

$$t^2 - t(\dim_{qA} E - \overline{\dim}_\theta E) - \dim_{qA} E\overline{\dim}_\theta E < 0.$$

This is a second-degree polynomial in t with zeros at $t = -\overline{\dim}_\theta E$ and $t = \dim_{qA} E$. Therefore, for all $t \in (0, \dim_{qA} E)$, the inequality (3.14) holds for all sets E with $0 < \overline{\dim}_\theta E < \dim_{qA} E$.

We finish this work with a simple corollary that improves Corollary 3.8 in [3].

Corollary 3.12. *Let $E \subset \mathbb{R}^d$ be a bounded set and $\theta \in (0, 1]$ a constant. The functions $f, g : (0, d) \to [0, d]$ defined by*

$$f(t) = \underline{\dim}_\theta^t E$$

and

$$g(t) = \overline{\dim}_\theta^t E$$

are Lipschitz functions.

Proof. For all $0 < t \le d$, we have

$$\underline{\dim}_\theta^t E \le t. \tag{3.15}$$

Now, using Theorem 3.10 and (3.15), we have for $0 < s \le t$,

$$\underline{\dim}_\theta^t E - \underline{\dim}_\theta^s E \le \Big(\frac{\underline{\dim}_\theta^s E\underline{\dim}_\theta^t E}{st}\Big)(t - s) \le t - s. \qquad \blacksquare$$

Final remarks and broader applicability

Broader applicability

We chose to present our results for the case of orthogonal projections and fractional Brownian motion. However, in [3], it is shown that the intermediate dimension profiles also provide meaningful bounds for intermediate dimensions for a more general set of functions.

Definition 3.13. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $(\omega, x) \mapsto f_\omega(x)$ be a $\sigma(\mathcal{F} \times \mathcal{B})$-measurable function, where $\mathcal{B}$ denotes the Borel σ-algebra on $\mathbb{R}^d$. Let $\mathcal{F}_1 = \{f_\omega : E \to \mathbb{R}^m, \omega \in \Omega\}$ be a family of continuous functions, measurable with respect to $\sigma(\{F \times B : F \in \mathcal{F}, B \in \mathcal{B}\})$, that satisfies one of the following conditions:

(1) There exists a constant $c > 0$ such that for all $x, y \in E$ and $r > 0$,

$$P\big(\{\omega : |f_\omega(x) - f_\omega(y)| \le r\}\big) \le c\phi^{m/\gamma, m/\gamma}_{r^\gamma, \theta}(x - y),$$

(2) For all $\omega \in \Omega$, there exists a constant $c_\omega > 0$ such that for all $x, y \in E$,

$$|f_\omega(x) - f_\omega(y)| \le c_\omega |x - y|^{1/\gamma}.$$

Therefore, all our results that involve intermediate dimension profiles could be directly applied to this more general setting, since in [3, Theorems 3.1 and 3.3] it is shown exactly how the intermediate dimension profiles control the dimension of images under functions in the family $\mathcal{F}_1$.

To be more precise: Let $E \subset \mathbb{R}^d$ be compact, $\theta \in (0, 1]$, $\gamma \ge 1$, and $m \in \mathbb{N}$. If $\mathcal{F}_1$ satisfies condition (1), then for P-almost all $\omega \in \Omega$, $f_\omega \in \mathcal{F}_1$ we have

$$\underline{\dim}_\theta f_\omega(E) \ge \gamma \underline{\dim}^{m/\gamma}_\theta E \quad \text{and} \quad \overline{\dim}_\theta f_\omega(E) \ge \gamma \overline{\dim}^{m/\gamma}_\theta E.$$

If $\mathcal{F}_1$ satisfies condition (2), then for all $\omega \in \Omega$, $f_\omega \in \mathcal{F}_1$ we have

$$\underline{\dim}_\theta f_\omega(E) \le \gamma \underline{\dim}^{m/\gamma}_\theta E \quad \text{and} \quad \overline{\dim}_\theta f_\omega(E) \le \gamma \overline{\dim}^{m/\gamma}_\theta E.$$

Size of exceptional directions

We now include two corollaries that quantify the *size* of the set of exceptional directions where the projected dimension drops. These are direct consequences of our main theorem combined with Theorem 2.6, and although the proofs are straightforward, we believe they are of independent interest and help to clarify the scope of the results.

Corollary 3.14. *Let $E \subset \mathbb{R}^d$ be compact and $m \ge \dim_{\mathrm{qA}} E$. Then,*

$$\dim_H \big\{V \in G(d, m) : \overline{\dim}_\theta P_V E < \overline{\dim}_\theta E\big\} \le m(d - m) - (m - \dim_{\mathrm{qA}} E)$$

and

$$\dim_H \big\{V \in G(d, m) : \underline{\dim}_\theta P_V E < \underline{\dim}_\theta E\big\} \le m(d - m) - (m - \dim_{\mathrm{qA}} E).$$

Proof. Apply Corollary 3.7 in Theorem 2.6. ∎

Corollary 3.15. *Let $E \subset \mathbb{R}^d$ be bounded,* $1 \leq m \leq d$, *and* $\lambda \in (0, \overline{\dim}_{\mathrm{qH}} E)$. *Then,*

$$\dim_H \{V \in G(d,m) : \overline{\dim}_\theta P_V E < \lambda\} \leq m(d-m) - (m - \lambda),$$

and if $\lambda \in (0, \underline{\dim}_{\mathrm{qH}} E)$, *then*

$$\dim_H \{V \in G(d,m) : \underline{\dim}_\theta P_V E < \lambda\} \leq m(d-m) - (m - \lambda).$$

Proof. Apply Corollary 3.3 in Theorem 2.6. ∎

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References

[1] A. Banaji, Generalised intermediate dimensions. *Monatsh. Math.* **202** (2023), no. 3, 465–506 Zbl 07750829 MR 4651960

[2] A. Banaji and A. Rutar, Attainable forms of intermediate dimensions. *Ann. Fenn. Math.* **47** (2022), no. 2, 939–960 Zbl 1509.28005 MR 4448745

[3] S. A. Burrell, Dimensions of fractional brownian images. *J. Theoret. Probab.* **35** (2022), no. 4, 2217–2238 Zbl 1526.28003 MR 4509067

[4] S. A. Burrell, K. J. Falconer, and J. M. Fraser, Projection theorems for intermediate dimensions. *J. Fractal Geom.* **8** (2021), no. 2, 95–116 Zbl 1470.28007 MR 4261662

[5] K. Falconer, *Fractal geometry: mathematical foundations and applications*. John Wiley & Sons, Inc., Hoboken, NJ, 2003 MR 2118797

[6] K. Falconer and J. Howroyd, Projection theorems for box and packing dimensions. *Math. Proc. Cambridge Philos. Soc.* **119** (1996), no. 2, 287–295 Zbl 0846.28004 MR 1357045

[7] K. J. Falconer, A capacity approach to box and packing dimensions of projections and other images. In: *Analysis, probability and mathematical physics on fractals*. Fractals Dyn. Math. Sci. Arts Theory Appl. 5, pp. 1–19. World Sci. Publ., Hackensack, NJ, 2020 Zbl 1454.28008 MR 4472245

[8] K. J. Falconer, A capacity approach to box and packing dimensions of projections of sets and exceptional directions. *J. Fractal Geom.* **8** (2021), no. 1, 1–26 Zbl 1473.28007 MR 4226183

[9] K. J. Falconer, Intermediate dimensions: a survey. In: *Thermodynamic formalism*. Lecture Notes in Math. 2290, pp. 469–493. Springer, Cham, 2021 Zbl 1484.28007 MR 4436834

[10] K. J. Falconer, Intermediate dimension of images of sequences under fractional brownian motion. *Statist. Probab. Lett.* **182** (2022), article no. 109300 Zbl 1478.60124 MR 4340821

[11] K. J. Falconer, J. M. Fraser, and T. Kempton, Intermediate dimensions. *Math. Z.* **296** (2020), no. 1-2, 813–830 Zbl 1448.28009 MR 4140764

[12] K. J. Falconer, J. M. Fraser, and P. Shmerkin, Assouad dimension influences the box and packing dimensions of orthogonal projections. *J. Fractal Geom.* **8** (2021), no. 3, 247–259 Zbl 1493.28012 MR 4321217

[13] Z. Feng, Intermediate dimensions under self-affine codings. *Math. Z.* **307** (2024), no. 1, article no. 21 Zbl 07841460 MR 4739520

[14] J. M. Fraser, *Assouad dimension and fractal geometry*. Cambridge Tracts in Math 222, Cambridge University Press, Cambridge, 2021 Zbl 1467.28001 MR 4411274

[15] F. Lü and L.-F. Xi, Quasi-Assouad dimension of fractals. *J. Fractal Geom.* **3** (2016), no. 2, 187–215 Zbl 1345.28019 MR 3501346

[16] J. M. Marstrand, Some fundamental geometrical properties of plane sets of fractional dimensions. *Proc. London Math. Soc. (3)* **4** (1954), no. 1, 257–302 Zbl 0056.05504 MR 0063439

[17] P. Mattila, Hausdorff dimension, orthogonal projections and intersections with planes. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **1** (1975), no. 2, 227–244 Zbl 0348.28019 MR 0409774

[18] P. Mattila, *Geometry of sets and measures in euclidean spaces: fractals and rectifiability*. Cambridge Stud. Adv. Math 44, Cambridge University Press, Cambridge, 1995 Zbl 0911.28005 MR 1333890

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Nicolas Angelini

Departmento de Matemática, Universidad Nacional de San Luis, Avenida Ejército de los Andes 950, 5700 San Luis; Instituto de Matemática Aplicada San Luis, Avenida Italia 1556, 5700 San Luis, Argentina; nicolas.angelini.2015@gmail.com

Ursula Maria Molter

Departamento de Matemática, Universidad de Buenos Aires, Intendente Güiraldes 2160, Ciudad Universitaria, Pab. I, C1428 Ciudad Autónoma de Buenos Aires; Instituto de Investigaciones Matemáticas, Luis A. Santaló UBA-CONICET, Intendente Güiraldes 2160, C1428 Ciudad Autónoma de Buenos Aires, Argentina; umolter@gmail.com, umolter@dm.uba.ar, umolter@conicet.gov.ar