Groups Geom. Dyn. 19 (2025), 1165–1261
DOI 10.4171/GGD/913

Quasi-isometric classification of right-angled Artin groups II: Several infinite out cases

Jingyin Huang

Abstract. We are motivated by the question that for which class of right-angled Artin groups (RAAGs), the quasi-isometric classification coincides with commensurability classification. This is previously known to hold for RAAGs with finite outer automorphism groups. In this paper, we identify two classes of RAAGs, where their outer automorphism groups are allowed to contain adjacent transvections and partial conjugations, hence are infinite. If G belongs to one of these classes, then any other RAAG G' is quasi-isometric to G if and only if G' is commensurable with G. We also show that in such cases, there exists an algorithm to determine whether two RAAGs are quasi-isometric by looking at their defining graphs. Compared to the finite out case, the main issue we need to deal with here is that one may not be able to straighten the quasi-isometries in a canonical way. We introduce a deformation argument, as well as techniques from cubulation to deal with this issue.

Contents

1. Introduction

1.1. Motivation and background

Recall that a *quasi-isometry* $q : X \to Y$ between two metric spaces X and Y is a map such that there exist constants $L, A > 0$ with the following properties:

(1) $L^{-1}d(x,y) - A \leq d(f(x), f(y)) \leq Ld(x,y) + A$ for any $x, y \in X$.

Mathematics Subject Classification 2020: 20F65 (primary); 20F67, 20F69, 05C25 (secondary).
Keywords: quasi-isometric classification, commensurability classification, right-angled Artin group.

(2) Each point in Y is at most distance A from a point in $q(X)$.

Given a quasi-isometry $q : X \to Y$ between two metric spaces, one common scheme of understanding q is the following. In step (1), we specify a collection of subspaces of X and Y such that they are stable under q and encode the coarse intersection pattern of these subspaces of X (or Y) in a combinatorial object C_X (or C_Y). Then q induces an "isomorphism" $q_* : C_X \to C_Y$. In step (2), we understand whether an isomorphism between C_X and C_Y implies if X and Y are isometric or at least share some interesting geometric feature. Here are two examples of this scheme:

- When $X = Y = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ for $n \geq 3$, in step (1), one shows that q preserves the intersection pattern of maximal flats in X, hence induces an automorphism of the spherical building at infinity, which is a simplicial complex encoding the intersecting pattern of these flats. Moreover, this automorphism is continuous with respect to the cone topology. In step (2), we use the fundamental theorem of projective geometry to deduce that such automorphism of the building actually comes from a homothety of X. Then one deduces that every such q is of bounded distance from a homothety. This is a special case of the results in [13, 25].
- When X and Y are the mapping class groups of oriented closed surfaces of genus ≥ 2, q preserves the intersection pattern of Dehn twist flats [1, 19], hence induces an automorphism of the curve complex, which is a simplicial complex encoding the intersecting pattern of the Dehn twist flats. However, Ivanov's theorem tells us that any automorphism of the curve complex is induced by a mapping class, hence q is of bounded distance from a left multiplication.

While for several other classes of groups and spaces, it is tempting to follow such a scheme to study quasi-isometric classification and rigidity, one cannot expect we are always as lucky in step (2) where some analogue of fundamental theorem of projective geometry or Ivanov's theorem would hold.

Now we look at the class of right-angled Artin groups (RAAGs). Given a finite simplicial graph Γ with vertex set $\{v_i\}_{i \in I}$, the RAAG with defining graph Γ, denoted by $G(\Gamma)$, is given by the following presentation:

$$\{v_i, \text{ for } i \in I \mid [v_i, v_j] = 1 \text{ if } v_i \text{ and } v_j \text{ are joined by an edge}\}.$$

For RAAGs, there are no combinatorial objects as in the above two cases such that on one hand they are quasi-isometry invariants, and on the other hand they satisfy a strong analogue of Ivanov's theorem; as if such objects exist, it would imply the quasi-isometry groups of RAAGs are rather small. However, even for the most rigid class of RAAGs, their quasi-isometry groups are quite large [21].

Under certain conditions, quasi-isometries between RAAGs preserve a collection of subspaces, called standard flats. Kim and Koberda [23] introduced the notion of extension complexes, which is a simplicial complex encoding the coarse intersecting pattern of standard flats in RAAGs. They are quasi-isometry invariants for large class of RAAGs.

For RAAGs with finite outer automorphism groups considered in [4, 21], it is proved that any automorphism α of the extension complex induces a canonically defined bijection α' of the underlying RAAG which preserves enough structure for application to quasi-isometric classification. We will refer the map α' as the "reconstruction map", as the map is closely related to reconstructing the RAAG from intrinsic combinatorial structure of its extension complex (the precise definition of α' is in Section 2.6). This is also a natural extension of classical reconstruction problems (e.g., a theorem of Darboux says that any straight line-preserving bijection of Euclidean spaces must be affine) to the context of RAAGs. Note however the map α' is typically very far away from being a left multiplication or an isometry, so this can be viewed as a weak analogue of what happens in step (2) of the two cases discussed above.

For RAAGs with infinite outer automorphism, the situation could be much worse:

(1) It is possible that there are no well-defined reconstruction maps (in the sense of Section 2.6).

(2) Even if the reconstruction map exists, it may not preserve as much structure as before. This is due to the fact that RAAGs may not "branch" as much as symmetric spaces, thick Euclidean buildings or mapping class groups. Extra conditions are needed to make the reconstruction map "nice", and such cases are studied in [4, 21].

In this paper, we show that quasi-isometric classification results could still be obtained despite these two challenges. We study two classes of RAAGs in this paper. The first class is the largest class of RAAGs such that the reconstruction map exists with respect to automorphisms of extension complexes. Then we will introduce another class of RAAGs, where the reconstruction map fails to exist, and indicate how to get around this issue. It turns out that ideas from cubulation theory are relevant.

The previous quasi-isometric classification results of RAAGs fall into two classes with strong contrast in their conclusions. The study [3] identifies a class of RAAGs whose quasi-isometry types do not depend on the defining graphs, while [4] identifies another class of RAAGs such that two RAAGs in this class are quasi-isometric if and only if they are isomorphic. Higher-dimensional generalizations of these two cases are in [2] and [21], respectively. We intend to understand this strong contrast by "interpolating" between these two cases. The classes of RAAGs discussed in this paper serve as an initial step toward this goal.

1.2. Main results and open questions

We denote the RAAG with defining graph Γ by $G(\Gamma)$. Our search for appropriate classes of RAAGs is roughly guided by the outer automorphism group $\mathrm{Out}(G(\Gamma))$. Namely, if a property is true for all elements in $\mathrm{Out}(G(\Gamma))$, then we ask whether it is also true for all quasi-isometries of $G(\Gamma)$. See Section 2.3 for a review of $\mathrm{Out}(G(\Gamma))$. Since we are mainly interested in the case where $\mathrm{Out}(G(\Gamma))$ is infinite, we need to focus on the three types of generators of $\mathrm{Out}(G(\Gamma))$ which are of infinite order, namely the adjacent transvections, non-adjacent transvections and partial conjugations. Adjacent transvections happen inside free abelian subgroups, so they have relatively nice behavior compared other types. We deal with it first.

Definition 1.1. The group $G(\Gamma)$f is of *weak type I* if

(1) Γ is connected and does not contain any separating closed star.

(2) There do not exist vertices $v, w \in \Gamma$ such that $d(v, w) = 2$ and $\Gamma = \mathrm{St}(v) \cup \mathrm{St}(w)$.

We caution the reader that in this paper, the closed star of a vertex v, which we denote by $\mathrm{St}(v)$, is defined to be the full subgraph spanned by v and vertices adjacent to v. This definition is slightly different from the usual one. Similarly, $\mathrm{lk}(v)$ is defined to be the full subgraph spanned by vertices adjacent to v.

It turns out that $G(\Gamma)$ is of weak type I if and only if one can always reconstruct a map from $G(\Gamma)$ to itself from a given isomorphism of its extension complex in the sense of Definition 2.27 (see Theorem 3.23 for a precise statement). In particular, all RAAGs with finite outer automorphism group are of weak type I.

If $G(\Gamma)$ is of weak type I, then $\mathrm{Out}(G(\Gamma))$ does not contain non-adjacent transvections and partial conjugations; however, $\mathrm{Out}(G(\Gamma))$ may contain adjacent transvections. For example, we can take Γ to be the graph which is made of a 5-cycle and a 3-cycle glued along an edge.

Theorem 1.2 (=Theorem 3.22). *If $G(\Gamma)$ and $G(\Gamma')$ are of weak type I, then they are quasi-isometric if and only if they are isomorphic.*

Having weak type I is not a quasi-isometry invariant (cf. [4, Example 1.4]). However, the following weaker version of Theorem 1.2 is true when only $G(\Gamma_1)$ is of weak type I.

Theorem 1.3 (=Theorem 5.10). *Suppose $G(\Gamma_1)$ is of weak type I. Then the following are equivalent:*

(1) *$G(\Gamma_2)$ is quasi-isometric to $G(\Gamma_1)$.*

(2) *$G(\Gamma_2)$ is isomorphic to a subgroup of finite index in $G(\Gamma_1)$.*

(3) *$G(\Gamma_2)$ is isomorphic to a special subgroup of $G(\Gamma_1)$.*

We refer to Section 2.4 for the definition of special subgroups.

Remark 1.4. We now make some comparison with the finite out case.

(1) As we will see later, in general the collection of special subgroups of $G(\Gamma)$ depends on the choice of standard generators of $G(\Gamma)$. However, the isomorphism types of the special subgroups do not depend on the choice of standard generators (by [21, Section 6], all special subgroups are RAAGs and the isomorphism types of their defining graphs only depend on Γ). For example, let $G(\Gamma) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Given a base $e_1, e_2 \in \mathbb{Z} \oplus \mathbb{Z}$, then special subgroups with such choice of standard generators are of form $\{ne_1 + me_2\}_{n,m \in \mathbb{Z}}$. However, no matter what base we choose, all special subgroups are isomorphic to $n\mathbb{Z} \oplus m\mathbb{Z}$. However, we have more rigidity when $\mathrm{Out}(G(\Gamma))$ is finite. In this case, the definition of special subgroup does not depend on the choice of standard generators, and all finite index RAAG subgroups of $G(\Gamma)$ are special subgroups [21, Theorem 1.4].

(2) By [21, Theorem 1.3], $G(\Gamma_2)$ is quasi-isometric to $G(\Gamma_1)$ if and only if their extension complexes (Section 2.3) are isomorphic, given $\mathrm{Out}(G(\Gamma_1))$ is finite. The if only direction is still true in the case of weak type I group, but the other direction is not clear.

Next we deal with partial conjugations. Since if $\mathrm{Out}(G(\Gamma))$ contains partial conjugations, then Γ contains separating closed stars, one may want to cut Γ into good pieces along separating closed stars. However, this is not well defined in general. Then one may try the opposite way and look at graphs obtained by gluing good pieces along vertex stars in a nice way. By studying such examples, we identify the following class of RAAGs.

Definition 1.5. The group $G(\Gamma)$ is of *type II* if Γ is connected, and for every pair of distinct vertices $v, w \in \Gamma$, $\mathrm{lk}(v) \cap \mathrm{lk}(w)$ does not separate Γ.

This condition has a geometric interpretation. Note that $\mathrm{lk}(v)$ corresponds to hyperplanes in the universal covering of the Salvetti complex, as these hyperplanes are invariant under a conjugate of subgroups of $G(\Gamma)$ generated by vertices in $\mathrm{lk}(v)$. So $\mathrm{lk}(v) \cap \mathrm{lk}(w)$ corresponds to the intersection of hyperplanes. Definition 1.5 can be roughly interpreted as "hyperplanes of codimension 2 do not coarsely separate".

A model example is taking Γ to be the union of a 5-cycle and a 6-cycle identified along a closed vertex star. If $G(\Gamma)$ is of type II, then $\mathrm{Out}(G(\Gamma))$ may contain partial conjugations and adjacent transvections, but not non-adjacent transvections.

A similar but different condition, called SIL, has been studied in [8]. The SIL condition also plays a role in the study of right-angled Coxeter groups [10, 11].

Theorem 1.6 (=Theorem 6.36). *If $G(\Gamma_1)$ is a RAAG of type II, then $G(\Gamma_2)$ is quasi-isometric to $G(\Gamma_1)$ if and only if $G(\Gamma_2)$ is commensurable with $G(\Gamma_1)$. Moreover, there exists a RAAG $G(\Gamma)$ such that $G(\Gamma_1)$ and $G(\Gamma_2)$ are isomorphic to special subgroups in $G(\Gamma)$.*

The following is a consequence of Theorems 1.3 and 1.6 and [21, Section 6.3].

Corollary 1.7. *Let $G(\Gamma)$ be a RAAG of type II or weak type I. Then there is an algorithm to determine whether a given RAAG $G(\Gamma')$ is quasi-isometric to $G(\Gamma)$ or not.*

We close this section with several comments and open questions. A RAAG of weak type I is not necessarily of type II. The following class contains RAAGs of both weak type I and type II (see Lemma 3.20).

Definition 1.8. The group $G(\Gamma)$ is said to have *weak type II* if Γ is connected and for vertices $v, w \in \Gamma$ such that $d(v, w) = 2$, $\Gamma \setminus (\mathrm{lk}(v) \cap \mathrm{lk}(w))$ is connected.

It turns out that weak type II is a quasi-isometry invariant for RAAGs (see Corollary 3.18). Though a large portion of our discussion also generalizes to RAAGs of weak type II, the following question remains open.

Question 1.9. Suppose $G(\Gamma)$ is of weak type II and $G(\Gamma')$ is quasi-isometric to $G(\Gamma)$. Is $G(\Gamma')$ commensurable with $G(\Gamma)$?

The techniques in this paper do not seem to apply effectively to the case when there are non-adjacent transvections in the outer automorphism group. Indeed, in this case, there is serious breakdown of the above form of rigidity. For example, there exist two tree RAAGs which are quasi-isometric but not commensurable [3]. This leads to the following question.

Question 1.10. Suppose Γ is connected and does not admit a non-trivial join decomposition. Suppose $\mathrm{Out}(G(\Gamma))$ contains non-trivial non-adjacent transvection. Does there exist Γ' such that $G(\Gamma)$ and $G(\Gamma')$ are quasi-isometric, but not commensurable?

1.3. Comments on the proof

We refer to Section 2.3 for definitions of relevant terms. The Salvetti complex of $G(\Gamma)$ is denoted by $S(\Gamma)$, the universal covering of $S(\Gamma)$ is denoted by $X(\Gamma)$ and flats in $X(\Gamma)$ that cover standard tori in $S(\Gamma)$ are called standard flats. Two standard flats are *coarsely equivalent* if they have finite Hausdorff distance. Let $\mathcal{P}(\Gamma)$ be the extension complex of $X(\Gamma)$. The k-dimensional simplices in $\mathcal{P}(\Gamma)$ are in 1-1 correspondence with coarse equivalent classes of $(k+1)$-dimensional standard flats in $X(\Gamma)$. Thus $\mathcal{P}(\Gamma)$ captures the coarse intersection pattern of standard flats in $X(\Gamma)$. It turns out to be a quasi-isometry invariant for a large class of RAAGs.

Theorem 1.11. *Let $q : G(\Gamma_1) \to G(\Gamma_2)$ be a quasi-isometry. Suppose* $\mathrm{Out}(G(\Gamma_i))$ *does not contain any non-adjacent transvection for $i = 1, 2$. Then q preserves maximal standard flats up to finite Hausdorff distance. Moreover, it induces a simplicial isomorphism $q_* : \mathcal{P}(\Gamma_1) \to \mathcal{P}(\Gamma_2)$.*

The assumption of Theorem 1.11 is motivated by the observation that any automorphism of $G(\Gamma)$ preserves maximal standard flats up to finite Hausdorff distance if and only if there is no non-adjacent transvection in $\mathrm{Out}(G(\Gamma))$. This observation can be easily checked by going through the list at the end of Section 2.3 (note that maximal standard flats are in 1-1 correspondence to maximal standard abelian groups in $G(\Gamma)$).

It is natural to ask whether $\mathcal{P}(\Gamma_1)$ and $\mathcal{P}(\Gamma_2)$ are isomorphic implies that $G(\Gamma_1)$ and $G(\Gamma_2)$ are isomorphic or commensurable. In Example 6.38, we give a pair $G(\Gamma_1)$

and $G(\Gamma_2)$ such that their outer automorphism groups do not contain non-adjacent transvections such that $\mathcal{P}(\Gamma_1)$ and $\mathcal{P}(\Gamma_2)$ are isomorphic, but $G(\Gamma_1)$ and $G(\Gamma_2)$ are not quasi-isometric, hence are not commensurable. In particular, the converse to Theorem 1.11 is not true.

One can try to reconstruct a "straightening" of q from q_* as follows. Pick vertex $x \in X(\Gamma_1)$, and let $\{F_i\}_{i\in I}$ be the collection of maximal standard flats containing x. Under a mild condition on Γ, we have $x = \bigcap_{i\in I} F_i$. Each F_i is associated with a maximal standard flat $F'_i \subset X(\Gamma_2)$ by Theorem 1.11. It is natural to define $\bar{q} : G(\Gamma_1) \to G(\Gamma_2)$ such that $\bar{q}(x) = \bigcap_{i\in I} F'_i$. However, it is possible that $\bigcap_{i\in I} F'_i = \emptyset$.

1.3.1. The weak type I case. It turns out that this is exactly the case that we always have $\bigcap_{i\in I} F'_i \neq \emptyset$. Under an extra mild condition, we can deduce $\bigcap_{i\in I} F'_i$ is actually a point. Then the map $\bar{q}$ is well defined, and it preserves all the maximal standard flats. A priori, $\bar{q}$ may not preserve standard flats which are not maximal, and the key to prove Theorem 1.3 is to deform $\bar{q}$ such that it preserves all standard flats.

A standard flat is *rigid* if $\bar{q}$ sends its vertex set to the vertex set of another standard flat, otherwise it is *non-rigid*. For example, all intersections of maximal standard flats are rigid, but the converse may not be true.

We will deform $\bar{q}$ in an inductive way. The first step is to show one can deform with respect to minimal rigid flats such that any standard flat contained in a minimal rigid flat is preserved by $\bar{q}$. To continue the induction argument, note that inside a (not necessarily minimal) rigid flat, there are directions which are rigid and directions which are not rigid. So we need to perform the deformation such that each move does not undo the previous moves and does not place obstructions to the moves after. The second point is non-trivial, since rigid flats may intersect each other in a complicated pattern. To describe the deformation, we introduce an atlas for $G(\Gamma)$, where the vertex sets of standard flats are consistently labeled by free abelian groups. The detail is discussed in Section 5.

1.3.2. The type II case. In this case, the map $\bar{q}$ may fail to exist. For example, one can take q to be a partial conjugation.

Instead of reconstructing maps, we ask whether one can reconstruct the space $X(\Gamma)$ from $\mathcal{P}(\Gamma)$. Note that $X(\Gamma)$ is a CAT(0) cube complex. In general, the collection of halfspaces in a CAT(0) cube complex, and their intersection pattern contains the complete information needed to reconstruct the complex itself. This can be formalized in the language of pocset (see Definition 2.6 and Theorem 2.8).

Then we ask whether we can put a pocset structure on $\mathcal{P}(\Gamma)$ such that it is the right pocset structure to recover $X(\Gamma)$. This can always be done. Roughly speaking, one can embed $\mathcal{P}(\Gamma)$ into the Tits boundary of $X(\Gamma)$. Moreover, the collection of subsets of $\mathcal{P}(\Gamma)$ which are the intersections of $\mathcal{P}(\Gamma)$ and the Tits boundary of halfspaces of $X(\Gamma)$ has a natural pocset structure.

Briefly speaking, $X(\Gamma)$ is equivalent to $\mathcal{P}(\Gamma)$ with some decorations on $\mathcal{P}(\Gamma)$. In general, these decorations depend on how one embeds $\mathcal{P}(\Gamma)$ into the Tits boundary, so they do not come from intrinsic properties of $\mathcal{P}(\Gamma)$. Thus the rigidity of $X(\Gamma)$ depends on the amount of non-intrinsic decorations we need to put on $\mathcal{P}(\Gamma)$. For example, in the most rigid case when $G(\Gamma)$ has finite outer automorphism group, the amount of decorations needed is minimal. The worst case is when $X(\Gamma)$ is tree, then $\mathcal{P}(\Gamma)$ is just a discrete set.

If $G(\Gamma)$ is of type II, then the amount of extra decorations is reasonably small (see Corollary 3.13 (1) for a precise statement). We prove Theorem 1.6 in two steps. The pocset structure on $\mathcal{P}(\Gamma)$ is defined in terms of a certain partition of $\mathcal{P}(\Gamma)$. First we show it is possible to refine this partition to obtain a new pocset which does not admit any reasonable further refinement. It turns out the new pocset gives rise to another RAAG which is commensurable with the original one. Such RAAGs are called *prime* RAAGs (Definition 6.7). Then we show two prime RAAGs are quasi-isometric if and only if they are isomorphic, which finishes the proof of Theorem 1.6. We caution the reader that in order to avoid some technicality, we work with pocset on Γ rather than $\mathcal{P}(\Gamma)$ in Section 6. However, the idea is similar.

1.4. Organization of the paper

In Section 2, we summarize and generalize several results from [21] about CAT(0) cube complexes, RAAGs and extension complexes. In particular, Theorem 1.11 will be proved in Section 2.2.

In Section 3, we study the structure of the extension complex $\mathcal{P}(\Gamma)$ and prove Theorem 1.2 at the end of Section 3. In Section 4, we prove some extra properties for extension complex for later use in Section 6.

The goal of Section 5 is to prove Theorem 1.3. We will introduce a notion of atlas for RAAG in Section 5.1 and use this in Section 5.2 as an effective language to describe the deformation argument mentioned above.

We prove Theorem 1.6 in Section 6. Section 6 does not depend on Section 5.

1.5. Index of notation

- $\mathrm{St}(v, K)$: the closed star of v in K, or $\mathrm{St}(v)$ if K is clear (Section 2.1).
- $\mathrm{Lk}(x, X)$ or $\mathrm{Lk}(c, X)$: the link of a vertex x or a cell c in a polyhedral complex X (Section 2.1).
- $\Gamma_1 \circ \Gamma_2$: the join of two graphs (Section 2.1).
- $K_1 * K_2$: the join of two simplicial complexes (Section 2.1).
- $V^{\perp}$: collection of vertices which are adjacent to each vertex in V (Section 2.1).
- $G(\Gamma)$ the RAAG with defining graph Γ (Section 2.3).
- $F(\Gamma)$: the flag complex of a simplicial graph Γ.
- $X(\Gamma) \to S(\Gamma)$ the universal covering of the Salvetti complex (Section 2.3).
- Γ_K: the support of a subcomplex K in $X(\Gamma)$ (Definition 2.9).
- V_K: vertex set of Γ_K (Definition 2.9).

- $\mathcal{I}(C_1, C_2) = (Y_1, Y_2)$: nearest point sets between Y_1 and Y_2 defined in Lemma 2.4 and the paragraph after.
- $\mathcal{P}(\Gamma)$: the extension complex (Section 2.3).
- $\Delta(K)$: for each subcomplex $K \subset X(\Gamma)$, $\Delta(K)$ is a subcomplex of $\mathcal{P}(\Gamma)$ defined in the paragraph before Lemma 2.13.
- For a vertex p of $X(\Gamma)$, the map $i_p : F(\Gamma) \to \mathcal{P}(\Gamma)$ is defined right before Lemma 2.14.
- $(F(\Gamma))_p$: the image of i_p (see the paragraph before Lemma 2.14).
- $\pi : \mathcal{P}(\Gamma) \to F(\Gamma)$: label-preserving canonical projection defined in the third paragraph after Lemma 2.12.
- $v(M)$: the set of vertices in a subset M of a complex.
- π_v or $\pi_{\Delta(\ell)}$, where ℓ is a standard line with $\Delta(\ell) = v$: a projection map defined in Definition 3.2.
- $\pi_\ell : X(\Gamma) \to \ell$: CAT(0) projection from $X(\Gamma)$ to a standard geodesic ℓ (Section 3.1).
- P_v: the parallel set of a standard geodesic ℓ with $\Delta(\ell) = v$ (see the paragraph after Example 3.5).
- ∂B: boundary of a subcomplex B, defined in the paragraph after Lemma 3.9.
- $\bar{B}$: the full subcomplex spanned by B and ∂B (Section 4.2).
- Π: this is a map which associated a component of $\mathcal{P}(\Gamma) \setminus \mathrm{St}(v)$ with a component of $F(\Gamma) \setminus \mathrm{St}(\bar{v})$, explained in the paragraph before Definition 6.4.

2. Preliminaries

2.1. Notations and conventions

Notations here are consistent with [21, Section 2.1]. All graphs in this paper are simplicial. The flag complex of a graph Γ is denoted by $F(\Gamma)$, that is, $F(\Gamma)$ is a flag complex such that its 1-skeleton is Γ.

Let K be a polyhedral complex.

(1) By viewing the 1-skeleton of K as a metric graph with edge lengths 1, we obtain a metric defined on the 0-skeleton of K, which we denote by d.

(2) A subcomplex $K' \subset K$ is *full* if K' contains all the subcomplexes of K which have the same vertex set as K'. If K is 1-dimensional, then we also call K' a *full subgraph*.

(3) We use $\circ$ to denote the join of two graphs, namely $\Gamma_1 \circ \Gamma_2$ is a graph obtained by adding to $\Gamma_1 \sqcup \Gamma_2$ an edge between each vertex of Γ and each vertex of Γ_2 and $*$ to denote the join of two polyhedral complexes.

(4) For a set of vertices $V \subset K$, $V^{\perp}$ is defined to be collection of vertices which are adjacent to each vertex in V.

(5) Let $v \in K$ be a vertex. The *link* of v in K, denoted by $\mathrm{lk}(v, K)$ or $\mathrm{lk}(v)$ when K is clear, is defined to be the full subcomplex spanned by $v^{\perp}$. The *closed star* of v in K, denoted by $\mathrm{St}(v, K)$ or $\mathrm{St}(v)$ when K is clear, is defined to be the full subcomplex spanned by $\{v, v^{\perp}\}$.

(6) Let $M \subset K$ be an arbitrary subset. We denote the collection of vertices inside M by $v(M)$.

We will be using the following simple observation repeatedly.

Lemma 2.1. *Let K be a simplicial complex, and let $K^{(1)}$ be the 1-skeleton of K. Suppose $L \subset K$ is a full subcomplex. Then there is a 1-1 correspondence between connected components of $K \setminus L$ and of $K^{(1)} \setminus L^{(1)}$. Moreover, the intersection of each component of $K \setminus L$ with $K^{(1)}$ is a component of $K^{(1)} \setminus L^{(1)}$.*

Let X be a metric space. We use d_H to denote the Hausdorff distance and use $N_R(Y)$ to denote the open R-neighborhood of a subspace $Y \subseteq X$. Two subspaces A and B are *coarsely equivalent* if they have finite Hausdorff distance. The subpace A is *coarsely contained* in B if A is contained in the R-neighborhood of B for some $R > 0$. A subspace $V \subseteq X$ is the *coarse intersection* of subspaces Y_1 and Y_2 if V is at finite Hausdorff distance from $N_R(Y_1) \cap N_R(Y_2)$ for all sufficiently large R. In general, the coarse intersection of two subspaces might not exist.

2.2. CAT(0) spaces and CAT(0) cube complexes

We mention several relevant facts here and refer to [5] and [30] for more background on CAT(0) spaces and CAT(0) cube complexes. The reader can also check [21, Section 2.2].

Let (X, d) be a CAT(0) space, and let $C \subset X$ be a convex subset. We denote the nearest point projection from X to C by $\pi_C : X \to C$. Denote the Tits boundary of X by $\partial_T X$. If $C' \subset X$ be another convex set, then C' is *parallel* to C if $d(\cdot, C)|_{C'}$ and $d(\cdot, C')|_C$ are constant functions. We define the *parallel set* of C, denoted by P_C, to be the union of all convex subsets of X parallel to C.

Now we turn to CAT(0) cube complexes. All cube complexes in this paper are assumed to be finite-dimensional. There are two common metrics on a CAT(0) cube complex X, namely the CAT(0) metric and the ℓ^1-metric. In this paper, we will mainly use the CAT(0) metric unless otherwise specified.

A *geodesic segment, geodesic ray* or *geodesic* in a CAT(0) cube complex X is an isometric embedding of $[a, b]$, $[0, \infty)$ or $\mathbb{R}$ into X with respect to the CAT(0) metric. A *combinatorial geodesic segment, combinatorial geodesic ray* or *combinatorial geodesic* is an ℓ^1-isometric embedding of $[a, b]$, $[0, \infty)$ or $\mathbb{R}$ into $X^{(1)}$ such that its image is a subcomplex.

The collection of convex subcomplexes in a CAT(0) cube complex enjoys the following version of Helly's property [14].

Lemma 2.2. *Let X be a finite-dimensional* CAT(0) *cube complex, and let $\{C_i\}_{i=1}^k$ be a collection of convex subcomplexes. If $C_i \cap C_j \neq \emptyset$ for any $1 \leq i \neq j \leq k$, then $\bigcap_{i=1}^k C_i \neq \emptyset$.*

Lemma 2.3 ([16]). *Let X be a* CAT(0) *cube complex, and let $Y \subset X$ be a convex subcomplex. Then Y is also combinatorially convex in the sense that any combinatorial geodesic segment joining two vertices in Y is contained in Y.*

Lemma 2.4 ([22, Lemma 2.10]). *Let X be a* CAT(0) *cube complex of dimension n, and let C_1, C_2 be convex subcomplexes. Denote $\triangle = d(C_1, C_2)$. Put $Y_1 = \{y \in C_1 \mid d(y, C_2) = \triangle\}$ and $Y_2 = \{y \in C_2 \mid d(y, C_1) = \triangle\}$. Then*

(1) *Y_1 and Y_2 are not empty.*

(2) *Y_1 and Y_2 are convex; π_{C_1} maps Y_2 isometrically onto Y_1, and π_{C_2} maps Y_1 isometrically onto Y_2; the convex hull of $Y_1 \cup Y_2$ is isometric to $Y_1 \times [0, \triangle]$.*

(3) *Y_1 and Y_2 are subcomplexes, and $\pi_{C_2}|_{Y_1}$ is a cubical isomorphism with its inverse given by $\pi_{C_1}|_{Y_2}$.*

(4) *There exists $A = A(\Delta, n, \varepsilon)$ such that if $p_1 \in C_1$, $p_2 \in C_2$ and $d(p_1, Y_1) \geq \varepsilon > 0$, $d(p_2, Y_2) \geq \varepsilon > 0$, then*

$$d(p_1, C_2) \geq \triangle + A \cdot d(p_1, Y_1); \quad d(p_2, C_1) \geq \triangle + A \cdot d(p_2, Y_2). \tag{2.1}$$

The above lemma implies Y_1 and Y_2 are coarsely equivalent, and Y_1 (or Y_2) is the coarse intersection of C_1 and C_2. We use $\mathcal{I}(C_1, C_2) = (Y_1, Y_2)$ to describe this situation, where $\mathcal{I}$ stands for "intersect".

Pick edge $e \subset X$. It turns out the parallel set P_e is a convex subcomplex (actually P_e is made of cubes which contain an edge parallel to e). There is a natural splitting $P_e = e \times h_e$. The *hyperplane dual to e* is defined to be the subset of P_e of form $\{m\} \times h_e$, where m is the middle point of e. Each hyperplane separates X into exactly two connected components. The closure of these components is called *halfspaces*. The sets Y_1 and Y_2 in Lemma 2.4 can be characterized in terms of hyperplanes.

Lemma 2.5 ([21, Lemma 2.6]). *Let X, C_1, C_2, Y_1 and Y_2 be as in Lemma 2.4. Pick an edge e in Y_1 (or Y_2), and let h be the hyperplane dual to e. Then $h \cap C_i \neq \emptyset$ for $i = 1, 2$. Conversely, if a hyperplane h' satisfies $h' \cap C_i \neq \emptyset$ for $i = 1, 2$, then*

$$\mathcal{I}(h' \cap C_1, h' \cap C_2) = (h' \cap Y_1, h' \cap Y_2)$$

and h' comes from the dual hyperplane of some edge e' in Y_1 (or Y_2).

The collection of halfspaces in X contains enough information to recover X. More generally, we can view X as a space with walls, and [17, 29] introduce a way to construct a CAT(0) cube complex from a given space with walls. There are several variations and developments of this construction [9, 20, 27, 28]. Here we follow the construction in [28] (see Sageev's notes [30]).

Definition 2.6 ([30, Definition 1.5]). A *pocset* is a partially ordered set with an involution $A \to A^c$ such that

(1) $A \neq A^c$ and A and A^c are incomparable.

(2) $A \leq B$ implies $B^c \leq A^c$.

Note that the collection of all closed halfspaces in a CAT(0) cube complex forms a pocset. The partial order comes from inclusion of sets, and the involution is defined by mapping a halfspace h to the unique halfspace which intersects h along a hyperplane.

Definition 2.7 ([30, Definition 2.1]). Let P be a pocset. An *ultrafilter* U is a subset of P such that

(1) For all pairs $\{A, A^c\}$ in P, precisely one of them is in U.

(2) If $A \in U$ and $A \leq B$, then $B \in U$.

For example, pick a vertex p in a CAT(0) cube complex X, then the collection of closed halfspaces in X that contains p forms an ultrafilter. Note that if U is an ultrafilter and A is minimal in U with respect to the partial order on P, then $(U \setminus \{A\}) \cup \{A^c\}$ is also an ultrafilter.

Theorem 2.8 ([28]). *If P is a finite pocset, then there is a* CAT(0) *cube complex X such that its vertices are in* 1-1 *correspondence with ultrafilters of P and two vertices U and U' are connected by an edge if and only if*

$$U' = (U \setminus \{A\}) \cup \{A^c\}$$

for some A minimal in U. Moreover, there is a natural pocset isomorphism from P to the pocset of halfspaces in X.

If P is infinite, then similar conclusions hold under the additional assumptions that P is discrete and of finite width (see [28, 30]). However, in this paper we only need the case when P is finite.

2.3. Basics about RAAGs

Pick a finite simplicial graph Γ, and let $G(\Gamma)$ be the RAAG with defining graph Γ. Let $S(\Gamma)$ be the Salvetti complex [7, Section 2.6] of $G(\Gamma)$, which is a non-positively curved cube complex whose 2-skeleton is the presentation complex of $G(\Gamma)$. Denote the universal cover of $S(\Gamma)$ by $X(\Gamma)$. Pick a standard generating set S for $G(\Gamma)$, label the 1-cells in $S(\Gamma)$ by elements in S and choose an orientation for each 1-cell in $S(\Gamma)$. This lifts to orientation and labeling of edges in $X(\Gamma)$ which are invariant under the action $G(\Gamma) \curvearrowright X(\Gamma)$. As $S(\Gamma)$ only has one vertex, we can identify $G(\Gamma)$ as the 0-skeleton of $X(\Gamma)$.

Let $\Gamma' \subset \Gamma$ be a full subgraph. Then the images of the embeddings $G(\Gamma') \to G(\Gamma)$ and $S(\Gamma') \to S(\Gamma)$ are called *standard subgroup* (of type Γ') and *standard subcomplex* (of type Γ'), respectively. *Standard subcomplexes* of $X(\Gamma)$ are lifts of standard subcomplexes of $S(\Gamma)$. When Γ' is a complete subgraph, $G(\Gamma')$ is called a *standard abelian subgroup*, $S(\Gamma')$ is called a *standard torus* and lifts of $S(\Gamma')$ are called *standard flats*. One-dimensional standard flats are also called *standard geodesics*. As we are identifying $G(\Gamma)$ as the 0-skeleton of $X(\Gamma)$, sometimes we will slightly abuse the notation by referring to a left coset of a standard abelian subgroup (resp. standard $\mathbb{Z}$ subgroup) of $G(\Gamma)$ as a standard flat (resp. standard geodesic).

Definition 2.9. For every edge $e \in X(\Gamma)$, there is a vertex in Γ which shares the same label as e. We denote this vertex by V_e. If $K \subset X(\Gamma)$ is a subcomplex (K does not need to be a standard subcomplex), we define V_K to be $\{V_e \mid e \text{ is an edge in } K\}$ and Γ_K to be the full subgraph spanned by V_K. The subgraph Γ_K is called the *support* of K. Pick a vertex $v \in X(\Gamma)$ and a full subgraph $\Gamma' \subset \Gamma$ and denote the unique standard subcomplex with support Γ' that contains v by $K(v, \Gamma')$.

The following two results are strengthened versions of Lemmas 2.4 and 2.5 in the case of coarse intersection of two standard subcomplexes.

Lemma 2.10 ([21, Lemma 3.1]). *Let Γ be a finite simplicial graph, and let K_1, K_2 be two standard subcomplexes of $X(\Gamma)$. If $(Y_1, Y_2) = \mathcal{I}(K_1, K_2)$, then Y_1 and Y_2 are also standard subcomplexes.*

We can compute the supports of Y_1 and Y_2 as follows.

Lemma 2.11 ([21, Corollary 3.2]). *Let K_1, K_2, Y_1 and Y_2 be as above.*

(1) *Let h be a hyperplane separating K_1 and K_2, and let e be an edge dual to h. Then $V_e \in V_{Y_1}^{\perp} = V_{Y_2}^{\perp}$ (see Definition 2.9 for relevant notations).*

(2) *A vertex $v \in \Gamma$ satisfies $v \in V_{Y_1}$ if and only if*

  (a) $v \in V_{K_1} \cap V_{K_2}$.

  (b) *For any hyperplane h' separating K_1 from K_2 and any edge e' dual to h', we have $d(v, V_{e'}) = 1$.*

The proof roughly goes as follows. Pick vertex $x \in K_1$, and let y be the vertex in K_2 closest to x. Let ℓ be a combinatorial geodesic joining x and y. Then there is a combinatorial embedding $Y_1 \times \ell \hookrightarrow X$. Note that parallel edges have the same label. Two edges span a square if and only if their labels are adjacent in Γ. Thus the label of each edge in ℓ is adjacent to the label of every edge in Y_1. If h separates K_1 and K_2, then h must intersect ℓ. Then (1) follows. Suppose $v \in V_{Y_1}$. Since Y_1 and Y_2 are parallel, then $V_{Y_1} = V_{Y_2}$, thus (2a) follows. Lemma 2.11 (2b) is a consequence of (1).

Lemma 2.12. *Let K_1 and K_2 be two standard subcomplexes of $X(\Gamma)$. Then*

(1) *K_1 is coarsely contained in K_2 (cf. Section 2.1) if and only if there is a standard subcomplex $K_2' \subset K_2$ such that K_1 and K_2' are parallel.*

(2) *K_1 is coarsely equivalent to K_2 if and only if K_1 and K_2 are parallel.*

Proof. We only prove (1). (2) is similar. It suffices to prove the only if direction of (1). Let $\mathcal{I}(K_1, K_2) = (J_1, J_2)$. It suffices to prove $J_1 = K_1$. By Lemma 2.4 (4), J_1 and K_1 have finite Hausdorff distance; moreover, Lemma 2.10 implies J_1 is a standard subcomplex of K_1. If $J_1 \subsetneq K_1$, then their supports satisfy $\Gamma_{J_1} \subsetneq \Gamma_{K_1}$. Pick vertex $\bar{v} \in \Gamma_{K_1} \setminus \Gamma_{J_1}$, and let $\ell \subset K_1$ be a standard geodesic line with its support $=\{\bar{v}\}$. Then Lemma 2.11 (2a) implies $\mathcal{I}(J_1, \ell)$ is a pair of points, which implies ℓ is not contained in a bounded neighborhood of K_1 by Lemma 2.4 (4). This contradicts that J_1 and K_1 have finite Hausdorff distance. So we must have $J_1 = K_1$. ■

Let us now recall extension graph and extension complex defined by Kim and Koberda [23], which will be the key quasi-isometry invariants used in this paper. The combinatorial structure of extension complexes is instrumental for studying quasi-isometries between RAAGs. It is worth mentioning that it was known before that the extension graph is a commensurability invariant in certain classes of RAAGs [24]. There is also a related graph called contact graph [15] which is quasi-isometric to the extension graph as proved in [24], but with a quite different combinatorial structure.

Let $\mathcal{P}(\Gamma)$ be the *extension complex* of Γ, which is the flag complex of the extension graph introduced in [23]. We give an alternative but equivalent definition here. The vertices of $\mathcal{P}(\Gamma)$ are in 1-1 correspondence with the parallel classes of standard geodesics in $X(\Gamma)$ (two standard flats are in the same parallel class if they are parallel). Two distinct vertices $v_1, v_2 \in \mathcal{P}(\Gamma)$ are connected by an edge if for $i = 1, 2$, there is a standard geodesic ℓ_i in the parallel class associated with v_i such that ℓ_1 and ℓ_2 span a standard 2-flat. This definition is equivalent to the definition in [23] as explained in [21, Lemma 4.2].

Note that edges in the same standard geodesics of $X(\Gamma)$ have the same label, and edges in parallel standard geodesics also have the same label. This induces a well-defined labeling of vertices in $\mathcal{P}(\Gamma)$ by vertices of Γ. There is a label-preserving simplicial map $\pi : \mathcal{P}(\Gamma) \to F(\Gamma)$, where $F(\Gamma)$ is the flag complex of Γ. Moreover, since $G(\Gamma) \curvearrowright X(\Gamma)$ by label-preserving cubical isomorphisms, we obtain an induced action $G(\Gamma) \curvearrowright \mathcal{P}(\Gamma)$ by label-preserving simplicial isomorphisms.

Note that each complete subgraph in the 1-skeleton of $\mathcal{P}(\Gamma)$ gives rise to a collection of mutually orthogonal standard geodesic lines. Thus there is a 1-1 correspondence between the $(k-1)$-simplexes in $\mathcal{P}(\Gamma)$ and parallel classes of standard k-flats in $X(\Gamma)$

[21, Section 4.1]. For standard flat $F \subset X(\Gamma)$, we denote the simplex in $\mathcal{P}(\Gamma)$ associated with the parallel class containing F by $\Delta(F)$. For a standard subcomplex $K \subset X(\Gamma)$, define $\Delta(K) := \bigcup_{\lambda \in \Lambda} \Delta(F_\lambda)$, where $\{F_\lambda\}_{\lambda \in \Lambda}$ is the collection of standard flats in K.

Lemma 2.13. *Let K_1 and K_2 be two standard subcomplexes. Then*

(1) $\Delta(K_1) = \Delta(K_2)$ *if and only if K_1 and K_2 are parallel.*

(2) $\Delta(K_1) \subset \Delta(K_2)$ *if and only if K_1 is coarsely contained in K_2.*

Proof. It suffices to prove (1). As (2) follows from (1) and Lemma 2.12 (1). Suppose K_1 and K_2 are parallel. Let $F \subset K_1$ be a standard flat. Then F_1 is coarsely contained in K_2; hence by Lemma 2.12, there exists a standard flat $F_2 \subset K_2$ which is parallel to F_1. Thus

$$\Delta(F_1) = \Delta(F_2) \subset \Delta(K_2),$$

and we deduce that $\Delta(K_1) \subset \Delta(K_2)$. Similarly, we know the inclusion on the other direction, hence $\Delta(K_1) = \Delta(K_2)$.

Now we assume $\Delta(K_1) = \Delta(K_2)$. Then each standard flat of K_1 is coarsely contained in K_2. Let $\mathcal{I}(K_1, K_2) = (J_1, J_2)$. Then each standard flat of K_1 is coarsely contained in J_1 by Lemma 2.4 (4). By the same proof as Lemma 2.12, we know that if $J_1 \subsetneq K_1$, then there is a standard geodesic line in K_1 which is not coarsely contained in J_1. Thus $K_1 = J_1$. Similarly, we can prove $K_2 = J_2$. Hence K_1 and K_2 are parallel. ∎

Given arbitrary vertex $p \in X(\Gamma)$, one can obtain a simplicial embedding

$$i_p : F(\Gamma) \to \mathcal{P}(\Gamma)$$

by considering the collection of standard flats passing through p (where $F(\Gamma)$ denotes the flag complex of Γ). We will denote the image of i_p by $(F(\Gamma))_p$. Note that $\pi \circ i_p$ is the identity map, which implies the following lemma.

Lemma 2.14. *The map i_p is an isometric embedding with respect to the combinatorial distance between vertices of $F(\Gamma)$.*

Now we look at the outer automorphism group $\mathrm{Out}(G(\Gamma))$ of $G(\Gamma)$. By [26, 31], $\mathrm{Out}(G(\Gamma))$ is generated by the following four types of elements (we identify the vertex set of Γ with a standard generating set of $G(\Gamma)$):

(1) Given vertex $v \in \Gamma$, sending $v \to v^{-1}$ and fixing all other vertices.

(2) Graph automorphisms of Γ.

(3) If $\mathrm{lk}(w) \subset \mathrm{St}(v)$ for vertices $w, v \in \Gamma$, sending $w \to wv$ and fixing all other vertices induce a group automorphism. It is called a *transvection*. When $d(v, w) = 1$, it is an *adjacent transvection*, otherwise it is a *non-adjacent transvection*.

(4) Suppose $\Gamma \setminus \mathrm{St}(v)$ is disconnected. Then one obtains a group automorphism by picking a connected component C and sending $w \to vwv^{-1}$ for each vertex $w \in C$. It is called a *partial conjugation*.

2.4. Special subgroups of RAAGs

We first consider the special case $G(\Gamma) \cong \mathbb{Z}^n$. Pick a finite rectangle $K \subset \mathbb{Z}^n$. Then the finite index subgroup $H = \bigoplus_{i=1}^n n_i\mathbb{Z}$, where n_i's are the number of vertices along the edges of the rectangle, has K as its fundamental domain. This can be generalized to all RAAGs in the following way. Let $K \subset X(\Gamma)$ be a compact convex subcomplex. Let $\{\ell_i\}_{i=1}^s$ be a maximal collection of standard geodesics such that $\ell_i \cap K \neq \emptyset$ for all i and $\Delta(\ell_i) \neq \Delta(\ell_j)$ for any $i \neq j$. We consider the left action $G(\Gamma) \curvearrowright X(\Gamma)$. For each i, let $g_i \in G(\Gamma)$ be the unique element that translates ℓ_i toward the positive direction with translation length $=1$ (recall that we have oriented edges of $X(\Gamma)$ in a $G(\Gamma)$-invariant way). Let $n_i = |v(K \cap \ell_i)|$.

Theorem 2.15 ([21, Section 6.1]). *Let $G \leq G(\Gamma)$ be the subgroup generated by $\{g_i^{n_i}\}_{i=1}^s$. Then the following hold:*

(1) *The subcomplex K is a "fundamental domain" for G in the sense that for $g_1, g_2 \in G$, $g_1K \cap g_2K \neq \emptyset$ if and only if $g_1 = g_2$. Moreover, the G-orbit of K covers the 0-skeleton of $X(\Gamma)$. Thus $|G(\Gamma) : G| =$ the number of vertices in K.*

(2) *Let Γ' be the 1-skeleton of the full subcomplex of $\mathcal{P}(\Gamma)$ spanned by $\{\Delta(\ell_i)\}_{i=1}^s$. Then G is isomorphic to the RAAG $G(\Gamma')$.*

Such a group G is called a *special subgroup of* $G(\Gamma)$ *(associated with K)*. Note that the definition of special subgroups implicitly depends on the choice of standard generators of $G(\Gamma)$ (we can think $G(\Gamma)$ as a fixed set, and different choices of standard generators give different ways of building $X(\Gamma)$ from $G(\Gamma)$). A subgroup is S-*special* if it is special with respect to the standard generating set S. In most parts of the paper, we fix a standard generating set, so there will be no confusion.

Alternatively, G can be characterized as the fundamental group of the canonical completion [18] of the local isometry $K \hookrightarrow X(\Gamma) \to S(\Gamma)$. However, we will not need this fact.

Let $G(\Gamma) \cong F_2$, the free group with two generators. We take K to be an edge in $X(\Gamma)$. Then the associated special subgroup G is isomorphic to F_3. In this case, if we collapse all the G-translations of K in $X(\Gamma)$, then the resulting space is isomorphic to the Cayley graph for F_3. Note that F_3 is a special subgroup of F_2 in the sense defined above. This specific example can be generalized to all RAAGs in the following way.

Recall that a cellular map between cube complexes is *cubical* (see [6]) if its restriction $\sigma \to \tau$ between cubes factors as $\sigma \to \eta \to \tau$, where the first map $\sigma \to \eta$ is a natural projection onto a face of σ and the second map $\eta \to \tau$ is an isometry.

Theorem 2.16 ([21, Lemma 6.18]). *Let G, Γ and Γ' be as in Theorem 2.15. There is a surjective cubical map $q : X(\Gamma) \to X(\Gamma')$ such that*

(1) *The map q sends standard flats onto standard flats. Moreover, each standard flat in $X(\Gamma')$ is the q-image of some standard flat in $X(\Gamma)$.*

(2) *Pick a vertex $x' \in X(\Gamma')$, then $q^{-1}(x') = g \cdot K$ for some element $g \in G$ (where K is the compact convex subcomplex of $X(\Gamma)$ defined above).*

(3) *The map q is G-equivariant. In particular, q is a quasi-isometry.*

If we identify each left coset of standard $\mathbb{Z}$ subgroup of $G(\Gamma)$ and $G(\Gamma')$ with a copy of $\mathbb{Z}$ (the identification is well defined up to a translation of $\mathbb{Z}$), then Theorem 2.16 implies that the restriction of q to a left coset of standard $\mathbb{Z}$ subgroup takes form

$$q(x) = \lfloor x/d \rfloor + r$$

for some integers d and r which depends on the $\mathbb{Z}$-coset. To see, denote the $\mathbb{Z}$-coset by ℓ, and let $\ell' = q(\ell)$. Theorem 2.16 (3) implies $q|_\ell$ is H-equivariant, where H is the G-stabilizer of ℓ'. As H acts transitively on ℓ' and $(q|_\ell)^{-1}(y)$ is an interval for any $y \in \ell'$, the formula

$$q|_\ell(x) = \lfloor x/d \rfloor + r$$

follows.

We claim the map q in Theorem 2.16 maps parallel standard geodesics to parallel standard geodesics. Indeed, this follows from that q is cubical if two parallel geodesics are contained in the same standard flat. In general, we can interpolate any two parallel standard geodesics in $X(\Gamma)$ with a chain of parallel standard geodesics in $X(\Gamma)$ such that adjacent members in the chain are contained in the same standard flats. Then the claim follows from Theorem 2.16 (1).

Thus q induces a map

$$q_* : (\mathcal{P}(\Gamma))^{(0)} \to (\mathcal{P}(\Gamma'))^{(0)}.$$

Moreover, as q is quasi-isometry, it cannot map a pair of standard geodesics without finite Hausdorff distance to a pair of standard geodesics with finite Hausdorff distance. Thus q_* is injective. From Theorem 2.16 (1) and the definition of $\mathcal{P}(\Gamma)$, we also know that q_* sends adjacent vertices to adjacent vertices. Now the moreover part of Theorem 2.16 (1) implies that q_* is actually a bijection and extends to a simplicial isomorphism $q_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$.

The following lemma can be viewed as a form of converse to Theorem 2.16.

Lemma 2.17. *Suppose $G(\Gamma)$ and $G(\Gamma')$ are two RAAGs with a homomorphism $i : G(\Gamma') \to G(\Gamma)$. Suppose there exists a surjective cubical map $q : X(\Gamma) \to X(\Gamma')$ such that*

(1) *q is a $G(\Gamma')$-equivariant, where the action $G(\Gamma') \curvearrowright X(\Gamma)$ is induced by i and the left action $G(\Gamma) \curvearrowright X(\Gamma)$.*

(2) *The q-image of a standard flat in $X(\Gamma)$ is a standard flat in $X(\Gamma')$ and the restriction of q to a left coset of standard $\mathbb{Z}$ subgroup takes form*

$$q(x) = \lfloor x/d \rfloor + r$$

for some integers $d \geq 1$ and r which depends on the $\mathbb{Z}$-coset.

(3) *q induces a simplicial isomorphism $q_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$.*

(4) *$q^{-1}(x')$ is bounded for some vertex $x' \in X(\Gamma')$.*

Then i is injective, and $i(G(\Gamma'))$ is a special subgroup of $G(\Gamma)$.

Proof. Take a vertex $x' \in X(\Gamma')$ representing the identity element of $G(\Gamma')$. It follows from the proof of Theorem 5.12 (2) of [21] that $K = q^{-1}(x')$ is a convex subcomplex (assumptions (2) and (3) as above are used here). As $G(\Gamma')$ acts transitively on the vertex set of $X(\Gamma')$, by (1) and (4), $q^{-1}(y)$ is bounded for any vertex $y \in X(\Gamma')$. Thus K is bounded, hence compact. Let $\{\ell_i\}_{i=1}^{s}$, $g_i \in G(\Gamma)$ and $n_i = |v(K \cap \ell_i)|$ be as in the beginning of Section 2.4. Let $\ell'_i = q(\ell_i)$, and let v_i be the generator of $G(\Gamma')$ which acts on ℓ'_i by translation. Note that $v_i \neq v_j$ if $i \neq j$ (as $\Delta(\ell_i) \neq \Delta(\ell_j)$ for any $i \neq j$ and q_* is an isomorphism). By assumption (1), $i(v_i) = g_i^{n_i}$. Let H be the standard subgroup of $G(\Gamma')$ generated by $\{v_i\}_{i=1}^{s}$. By [21, Lemmas 6.3 and 6.4], $i|_H$ is injective with finite index image. The lemma would follow if we know $H = G(\Gamma')$. If this is not true, then H is of infinite index in $G(\Gamma')$. Take point $x' \in X(\Gamma')$ and $x \in X(\Gamma)$ with $q(x) = x'$. On one hand, as $i(H)$ is finite index in $G(\Gamma)$, the orbit $i(H) \cdot x$ is cobounded in $X(\Gamma)$ (i.e., $X(\Gamma)$ is contained in a finite neighborhood of $i(H) \cdot x$), then we deduce from that q is equivariant and Lipschitz that $H \cdot x'$ is cobounded in $X(\Gamma')$. On the other hand, this is impossible as H is of infinite index in $G(\Gamma)$. Thus we must have $H = G(\Gamma')$ and the lemma is proved. ■

2.5. Coarse invariants for RAAGs

Here we summarize and generalize some results from [21].

Note that every join decomposition $\Gamma = \Gamma_1 \circ \Gamma_2$ induces a direct sum decomposition $G(\Gamma) = G(\Gamma_1) \oplus G(\Gamma_2)$. The group $G(\Gamma)$ or the graph Γ is called *irreducible* if Γ does not allow a non-trivial join decomposition. There is a well-defined de Rham decomposition of $X(\Gamma)$ induced by the join decomposition of Γ, which is stable under quasi-isometries.

Theorem 2.18 ([21, Theorem 2.9]). *Given $X = X(\Gamma)$ and $X' = X(\Gamma')$, let*

$$X = \mathbb{R}^n \times \prod_{i=1}^{k} X(\Gamma_i)$$

and

$$X' = \mathbb{R}^{n'} \times \prod_{j=1}^{k'} X(\Gamma'_j)$$

be the corresponding de Rham decomposition. If $\phi : X \to X'$ is an (L, A) quasi-isometry, then $n = n'$, $k = k'$, and there exist constants

$$L' = L'(L, A), \quad A' = A'(L, A), \quad D = D(L, A)$$

such that after re-indexing the factors in X', we have (L', A') quasi-isometry

$$\phi_i : X(\Gamma_i) \to X(\Gamma'_i)$$

so that

$$d\left(p' \circ \phi, \prod_{i=1}^{k} \phi_i \circ p\right) < D,$$

where

$$p : X \to \prod_{i=1}^{k} X(\Gamma_i)$$

and

$$p' : X' \to \prod_{i=1}^{k} X(\Gamma'_i)$$

are the projections.

Note that when there is no Euclidean de Rham factor, the above theorem basically says that any quasi-isometry between X and X' is a product of quasi-isometries between their factors.

We are particularly interested in those standard subcomplexes that are stable under quasi-isometries.

Definition 2.19. A subgraph $\Gamma_1 \subset \Gamma$ is *stable in* Γ if Γ_1 is a full subgraph and for any standard subcomplex $K \subset X(\Gamma)$ with $\Gamma_K = \Gamma_1$ and (L, A)-quasi-isometry $q : X(\Gamma) \to X(\Gamma')$, there exists $D = D(L, A, \Gamma_1, \Gamma) > 0$ and standard subcomplex $K' \subset X(\Gamma')$ such that the Hausdorff distance

$$d_H(q(K), K') < D.$$

A standard subcomplex $K \subset X(\Gamma)$ is *stable* if its support is a stable subgraph of Γ.

Remark 2.20. We caution the reader that the above definition of stable standard subcomplex is different from "stable subgroups" defined in [12]. In particular, a stable subcomplex in the sense of Definition 2.19 does not have to be Gromov hyperbolic, and quasi-geodesics connecting points inside a stable standard subcomplex Y with uniform quasi-isometric constants do not have to stay in a uniform neighborhood of Y.

It is clear that the intersection of two stable subgraphs is still a stable subgraph. See [21, Section 3.2] for more properties about stable subgraphs. In this paper, we will use the following two properties repeatedly.

Lemma 2.21 ([21, Lemma 3.24]). *Let Γ be a finite simplicial graph and pick stable subgraphs Γ_1, Γ_2 of Γ. Let $\overline{\Gamma}$ be the full subgraph spanned by V and $V^{\perp}$, where V is the vertex set of Γ_1. If $\Gamma_2 \subset \overline{\Gamma}$, then the full subgraph spanned by the vertices in $\Gamma_1 \cup \Gamma_2$ is stable in Γ.*

Lemma 2.22. *Suppose there is no non-adjacent transvection in* $\mathrm{Out}(G(\Gamma))$. *Then every maximal clique subgraph of* Γ *is stable.*

A slightly weaker reformulation of the above lemma is the following. If each automorphism of $G(\Gamma)$ preserves maximal standard flats up to finite Hausdorff distance, then so does each quasi-isometry of $G(\Gamma)$.

Proof. Let $\Gamma_1 \subset \Gamma$ be a maximal clique. By [21, Theorem 3.35], it suffices to prove for any vertices $v \in \Gamma_1$ and $w \in \Gamma$, $v^{\perp} \in \mathrm{St}(w)$ implies $w \in \Gamma_1$. Note that $v^{\perp} \in \mathrm{St}(w)$ implies $w \in v^{\perp}$ since there is no non-adjacent transvection. Then w and vertices of Γ_1 span a clique in Γ, thus $w \in \Gamma_1$ by the maximality of Γ_1. ∎

Let Δ be the map sending standard subcomplexes of $X(\Gamma_i)$ to subcomplexes of $\mathcal{P}(\Gamma_i)$ defined in Section 2.3. Let $\mathcal{S}(\mathcal{P}(\Gamma))$ be the collection of subcomplexes of $\mathcal{P}(\Gamma)$ which are Δ-images of stable standard subcomplexes in $X(\Gamma)$ (we assume the empty set is also in $\mathcal{S}(\mathcal{P}(\Gamma))$).

Lemma 2.23. *Any quasi-isometry* $q : X(\Gamma_1) \to X(\Gamma_2)$ *induces a well-defined bijection* $\widetilde{q}_* : \mathcal{S}(\mathcal{P}(\Gamma_1)) \to \mathcal{S}(\mathcal{P}(\Gamma_2))$.

Proof. Pick element $M \in \mathcal{S}(\mathcal{P}(\Gamma_1))$, and let $K_1 \subset X(\Gamma_1)$ be a standard subcomplex such that $\Delta(K_1) = M$. Then we define $\widetilde{q}_*$ to be $\Delta(K_2)$, where $K_2 \subset X(\Gamma_2)$ is a standard subcomplex which is at finite Hausdorff distance from $q(K_1)$. Note that K_2 is also stable, so

$$\Delta(K_2) \in \mathcal{S}(P(\Gamma_2)).$$

If we choose another standard subcomplex $K_1' \subset X(\Gamma_1)$ such that $\Delta(K_1') = M$ and choose another standard subcomplex $K_2' \subset X(\Gamma)$ which is at finite Hausdorff distance from $q(K_1')$, then Lemma 2.13 implies K_1 and K_1' are parallel. Hence K_2 and K_2' are coarsely equivalent. Then Lemma 2.12 implies K_2 and K_2' are parallel, and Lemma 2.13 implies $\Delta(K_2) = \Delta(K_2')$. Thus $\widetilde{q}_*$ is well defined. By considering the quasi-isometry inverse of q, we know $\widetilde{q}_*$ is a bijection. ∎

Lemma 2.24. *The set* $\mathcal{S}(\mathcal{P}(\Gamma_1))$ *is closed under intersection. Moreover, for* $M, M' \in \mathcal{S}(\mathcal{P}(\Gamma_1))$, *we have*

(1) $\widetilde{q}_*(M) \cap \widetilde{q}_*(M') = \widetilde{q}_*(M \cap M')$.

(2) $M \subset M'$ *if and only if* $\widetilde{q}_*(M) \subset \widetilde{q}_*(M')$.

Proof. First we show $M \cap M' \in \mathcal{S}(P(\Gamma_1))$. For $i = 1, 2$, let K_1 and K_1' be stable standard subcomplexes of $X(\Gamma_1)$ such that $\Delta(K_1) = M$ and $\Delta(K_1') = M'$. Let $\mathcal{I}(K_1, K_1') = (J_1, J_1')$. Since we already know J_1 is a standard subcomplex by Lemma 2.10, it remains to show $\Delta(J_1) = M \cap M'$ and J_1 is stable.

Since J_1 is coarsely contained in K_1' and K_1, it follows from Lemma 2.13 (2) that $\Delta(J_1) \subset M \cap M'$. Now pick simplex $s \subset M \cap M'$, and let F_s be a standard flat in $X(\Gamma_1)$

with $\Delta(F_s) = s$. Then F_s is coarsely contained in K_1 and K'_1. By Lemma 2.4 (4), for R large enough, we have

$$d_H(J_1, N_R(K_1) \cap N_R(K'_1)) < \infty, \tag{2.2}$$

where d_H is the Hausdorff distance. By taking R large enough, we conclude that F_s is coarsely contained in J_1, thus $s \subset \Delta(J_1)$ by Lemma 2.13 (2) and

$$M \cap M' \subset \Delta(J_1).$$

Now we show J_1 is stable. Let K'_2 and K_2 be standard subcomplexes in $X(\Gamma_2)$ which are Hausdorff close to $q(K'_1)$ and $q(K_1)$, respectively. Let

$$\mathcal{I}(K_2, K'_2) = (J_2, J'_2).$$

Then

$$d_H(J_2, N_R(K_2) \cap N_R(K'_2)) < \infty$$

by Lemma 2.4 (4). This, together with (2.2), implies

$$d_H(q(J_1), J_2) < \infty.$$

Since J_2 is a standard subcomplex by Lemma 2.10, we know J_1 is a stable standard subcomplex. Moreover,

$$\tilde{q}_*(M) \cap \tilde{q}_*(M') = \Delta(K_2) \cap \Delta(K'_2) = \Delta(J_2) = \tilde{q}_*(\Delta(J_1)) = \tilde{q}_*(M \cap M').$$

It remains to prove (2). The only if direction follows from (1) and the if direction follows by considering the quasi-isometry inverse of q. ■

A subcomplex of $\mathcal{P}(\Gamma_1)$ (or $\mathcal{P}(\Gamma_2)$) is *stable* if it is a member of $\mathcal{S}(\mathcal{P}(\Gamma_1))$ (or $\mathcal{S}(\mathcal{P}(\Gamma_2))$). Then the map $\tilde{q}_*$ defined in Lemma 2.23 induces a 1-1 correspondence between stable k-simplexes in $\mathcal{P}(\Gamma_1)$ and stable k-simplexes in $\mathcal{P}(\Gamma_2)$.

The following result is the starting point of this paper.

Theorem 2.25. *Let $q : X(\Gamma_1) \to X(\Gamma_2)$ be a quasi-isometry. Suppose* $\mathrm{Out}(G(\Gamma_1))$ *does not contain any non-adjacent transvection. Then there exists a simplicial embedding $q_* : \mathcal{P}(\Gamma_1) \to \mathcal{P}(\Gamma_2)$ such that for any stable simplex $s \subset \mathcal{P}(\Gamma)$, we have*

$$q_*(s) = \tilde{q}_*(s). \tag{2.3}$$

If we also assume $\mathrm{Out}(G(\Gamma_2))$ *does not contain any non-adjacent transvection, then q_* is a simplicial isomorphism.*

Proof. For $i = 1, 2$, let $\mathcal{V}_i^k$ be the collection of vertices of $\mathcal{P}(\Gamma_i)$ which are inside some stable m-simplex of $\mathcal{P}(\Gamma)$ for $0 \le m \le k$. By Lemma 2.22, $\mathcal{V}_1^{n-1}$ is exactly the 0-skeleton of $\mathcal{P}(\Gamma_1)$, where $n = \dim(X(\Gamma_1)) = \dim(X(\Gamma_2))$.

We first construct q_* from the 0-skeleton of $\mathcal{P}(\Gamma_1)$ to the 0-skeleton of $\mathcal{P}(\Gamma_2)$ inductively as follows: define $q_*(v) = \tilde{q}_*(v)$ for $v \in \mathcal{V}_1^0$ and suppose we have already defined q_* on $\mathcal{V}_1^k$ such that

(∗) For any stable simplex $s \subset \mathcal{P}(\Gamma_1)$, q_* is a bijection from $\mathcal{V}_1^k \cap s$ to $\mathcal{V}_2^k \cap \widetilde{q}_*(s)$.

By Lemmas 2.23 and 2.24, q_* restricted on $\mathcal{V}_1^0$ satisfies (∗) for $k = 0$.

Now we define q_* on $\mathcal{V}_1^{k+1}$. Pick a stable $(k+1)$-simplex $s^{k+1} \subset \mathcal{P}(\Gamma_1)$. If all vertices of s^{k+1} belong to $\mathcal{V}_1^k$, then we do not need to do anything. Otherwise, we pick vertex $v \in s^{k+1} \setminus \mathcal{V}_1^k$. Note that s^{k+1} is the only stable $(k+1)$-simplex of $\mathcal{P}(\Gamma_1)$ that contains v (if there is a distinct stable $(k+1)$-simplex $s_1^{k+1} \subset \mathcal{P}(\Gamma_1)$ with $v \in s_1^{k+1}$, then $v \in s_1^{k+1} \cap s^{k+1}$, which is a stable simplex of dimension $\leq k$ by Lemma 2.24. This implies $v \in \mathcal{V}_1^k$, which is a contradiction). By inductive assumption, vertices in $s^{k+1} \setminus \mathcal{V}_1^k$ and vertices in $\widetilde{q}_*(s^{k+1}) \setminus \mathcal{V}_2^k$ have the same cardinality, so we can choose an arbitrary bijection between them.

Now we have q_* defined on $\mathcal{V}_1^{k+1}$, and it remains to verify (∗). Given a stable simplex $s \subset \mathcal{P}(\Gamma)$, let $\{s_i\}_{i=1}^d$ be the collection of stable $(k+1)$-simplexes of $\mathcal{P}(\Gamma_1)$ such that $s_i \subset s$. By Lemma 2.24 (2), $\{\widetilde{q}_*(s_i)\}_{i=1}^d$ is exactly the collection of stable $(k+1)$-simplexes of $\mathcal{P}(\Gamma_2)$ contained in $\widetilde{q}_*(s)$. Then $\mathcal{V}_1^{k+1} \cap s$ is the vertex set of

$$(\mathcal{V}_1^k \cap s) \cup \left(\bigcup_{i=1}^d s_i\right),$$

and

$$\mathcal{V}_2^{k+1} \cap \widetilde{q}_*(s)$$

is the vertex set of

$$(\mathcal{V}_1^k \cap \widetilde{q}_*(s)) \cup \left(\bigcup_{i=1}^d \widetilde{q}_*(s_i)\right).$$

By our construction of q_*, it maps vertices in s_i bijectively to vertices in $\widetilde{q}_*(s_i)$. Thus q_* maps $\mathcal{V}_1^{k+1} \cap s$ surjectively to $\mathcal{V}_2^{k+1} \cap \widetilde{q}_*(s)$. It remains to check injectivity. Pick two points $v, v' \in \mathcal{V}_1^{k+1} \cap s$. The case $v, v' \in \mathcal{V}_1^k$ follows from induction. Now we consider the case $v, v' \notin \mathcal{V}_1^k$, and they are contained in different s_i's (for simplicity we assume $v \in s_1$ and $v' \in s_2$). By construction of q_*, we have

$$q_*(v) \in \widetilde{q}_*(s_1) \setminus \mathcal{V}_2^k$$

and

$$q_*(v') \in \widetilde{q}_*(s_2) \setminus \mathcal{V}_2^k.$$

If $q_*(v) = q_*(v')$, then $\widetilde{q}_*(s_1) \cap \widetilde{q}_*(s_2)$ contains a point outside $\mathcal{V}_2^k$. On the other hand, by Lemma 2.24 (1),

$$\widetilde{q}_*(s_1) \cap \widetilde{q}_*(s_2) = \widetilde{q}_*(s_1 \cap s_2),$$

which is a stable simplex of dimension $\leq k$. Thus every vertex of

$$\widetilde{q}_*(s_1) \cap \widetilde{q}_*(s_2)$$

is in $\mathcal{V}_2^k$, which is a contradiction. Thus we must have $q_*(v) \neq q_*(v')$ in this case. The other cases are actually simpler and can be handled similarly.

Up to now, we have defined q_* on the 0-skeleton such that for each stable simplex $s \subset \mathcal{P}(\Gamma)$, q_* maps vertices in s bijectively to vertices in $\tilde{q}_*(s)$. Next we show such q_* is injective. Pick distinct vertices v_1, v_2 in $\mathcal{P}(\Gamma_1)$; if $d(v_1, v_2) = 1$, then by applying $(*)$ to the maximal simplex containing v_1 and v_2 (recall that each maximal simplex is stable by our assumption), we have $d(q_*(v_1), q_*(v_2)) = 1$. If $d(v_1, v_2) \geq 2$, let s_i be a maximal simplex containing v_i for $i = 1, 2$. Then

$$\tilde{q}_*(s_1) \cap \tilde{q}_*(s_2) = \tilde{q}_*(s_1 \cap s_2)$$

by Lemma 2.24 (1). By $(*)$, q_* maps vertices in $s_1 \cap s_2$ bijectively to vertices of $\tilde{q}_*(s_1 \cap s_2)$. Since $v_1 \notin s_1 \cap s_2$, we apply $(*)$ to s_1 to deduce that

$$q_*(v_1) \in \tilde{q}_*(s_1) \setminus \tilde{q}_*(s_1 \cap s_2) = \tilde{q}_*(s_1) \setminus (\tilde{q}_*(s_1) \cap \tilde{q}_*(s_2)) = \tilde{q}_*(s_1) \setminus \tilde{q}_*(s_2).$$

On the other hand, applying $(*)$ to s_2 implies $q_*(v_2) \in \tilde{q}_*(s_2)$. Thus

$$q_*(v_1) \neq q_*(v_2).$$

We have already seen if $d(v_1, v_2) = 1$, then $d(q_*(v_1), q_*(v_2)) = 1$. Thus q_* naturally extends to an injective map on the 1-skeleton. Since $\mathcal{P}(\Gamma_1)$ and $\mathcal{P}(\Gamma_2)$ are flag complexes, we can further extend q_* to obtain the required simplicial embedding.

Now we assume $\mathrm{Out}(G(\Gamma_2))$ does not contain non-adjacent transvection. Then $\mathcal{V}_2^{n-1}$ is the 0-skeleton of $\mathcal{P}(\Gamma_2)$. Thus q_* is surjective on 0-skeleton. We claim q_* is an isomorphism between the 1-skeleton of $\mathcal{P}(\Gamma_1)$ and the 1-skeleton of $\mathcal{P}(\Gamma_2)$. It suffices to show if $d(q_*(v_1), q_*(v_2)) = 1$, then $d(v_1, v_2) = 1$. However, this follows by considering a maximal simplex in $\mathcal{P}(\Gamma_2)$ (which is also stable) containing $q_*(v_1)$ and $q_*(v_2)$ and applying $(*)$. Now we know q_* is a simplicial isomorphism on the whole complex since $\mathcal{P}(\Gamma_1)$ and $\mathcal{P}(\Gamma_2)$ are flag complexes. ∎

Corollary 2.26. *Let $q : X(\Gamma_1) \to X(\Gamma_2)$ be a quasi-isometry. Suppose* $\mathrm{Out}(G(\Gamma_1))$ *does not contain any non-adjacent transvection, and let q_* be the map defined in Theorem 2.25. Then for any subcomplex $M \in \mathcal{S}(\mathcal{P}(\Gamma_1))$,*

$$q_*(M) \subset \tilde{q}_*(M) \quad \text{and} \quad M = q_*^{-1}(\tilde{q}_*(M)). \tag{2.4}$$

If we also assume $\mathrm{Out}(G(\Gamma_2))$ *does not contain any non-adjacent transvection, then*

$$q_*(M) = \tilde{q}_*(M) \tag{2.5}$$

for any subcomplex $M \in \mathcal{S}(\mathcal{P}(\Gamma_1))$.

Proof. Since each maximal simplex in $\mathcal{P}(\Gamma_1)$ is stable, the intersection of such simplex with M is also stable by Lemma 2.24. Since $\mathcal{P}(\Gamma_1)$ is a union of its maximal simplexes, M is a union of stable simplexes. Now the first inclusion of (2.4) follows from Lemma 2.24 (2) and (2.3).

Now we prove the second equality of (2.4). Suppose there exists vertex $v \notin M$ such that $q_*(v) \in \widetilde{q}_*(M)$. Let s_v be the minimal stable simplex of $\mathcal{P}(\Gamma_1)$ such that $v \in s_v$ (recall that the collection of stable simplexes is closed under intersection by Lemma 2.24, and any maximal simplex which contains v is stable, so s_v is well defined). By (2.3), $q_*(v) \in \widetilde{q}_*(s_v)$. By Lemma 2.24 (2) and (2.3), $\widetilde{q}_*(s_v)$ is the minimal stable simplex in $\mathcal{P}(\Gamma_2)$ that contains $q_*(v)$. Since $\widetilde{q}_*(s_v) \cap \widetilde{q}_*(M)$ is also a stable simplex containing $q_*(v)$, we have

$$\widetilde{q}_*(s_v) \subset \widetilde{q}_*(M).$$

Thus $s_v \subset M$ by Lemma 2.24 (2), which is contradictory to $v \in M$.

Now we assume $\mathrm{Out}(G(\Gamma_2))$ does not contain non-adjacent transvection. Then $\widetilde{q}_*(M)$ is a union of stable simplexes by the same argument as before. By Lemma 2.24 (2), there is a 1-1 correspondence between stable simplexes in M and stable simplexes in $\widetilde{q}_*(M)$. Thus (2.5) follows from (2.3). ∎

2.6. Visible isomorphisms between extension complexes

It is natural to ask to what extent the converse of Theorem 2.25 is true, namely, suppose $\alpha : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$ is a simplicial isomorphism, does α induce a map from $G(\Gamma) \to G(\Gamma')$? Here is a natural construction. We identify $G(\Gamma)$ and $G(\Gamma')$ with the 0-skeleton of $X(\Gamma)$ and $X(\Gamma')$, respectively. Pick vertex $p \in G(\Gamma)$, let $\{F_i\}_{i=1}^n$ be the collection of maximal standard flats containing p. For each i, let $F'_i \subset X(\Gamma')$ be the unique maximal standard flat such that $\Delta(F'_i) = \alpha(\Delta(F_i))$. One may wish to map p to $\bigcap_{i=1}^n F'_i$, which motivates the following definition.

Definition 2.27. The simplicial isomorphism $\alpha : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$ is defined to be *visible* if $\bigcap_{i=1}^n F'_i \neq \emptyset$ for any $p \in G(\Gamma)$.

If $G(\Gamma)$ has trivial center and α is visible, then it is easy to see α induces a unique map $\alpha_* : G(\Gamma) \to G(\Gamma')$. To see this, recall that by [31], $G(\Gamma)$ has trivial center if and only if Γ is not contained in the closed star of one of its vertices. Let $\{F_i\}_{i=1}^n$ and $\{F'_i\}_{i=1}^n$ be as in the above discussion. Since the intersection of all maximal cliques in Γ is empty, we have $\bigcap_{i=1}^n \Delta(F_i) = \emptyset$, hence $\bigcap_{i=1}^n \Delta(F'_i) = \emptyset$. Thus p is the only point in $\bigcap_{i=1}^n F_i$ and $\bigcap_{i=1}^n F'_i$ contains at most one point. However, the visibility implies $\bigcap_{i=1}^n F'_i$ is exactly one point. We define this point to be $\alpha_*(p)$.

If $G(\Gamma)$ has non-trivial center, then $\bigcap_{i=1}^n F_i$ and $\bigcap_{i=1}^n F'_i$ correspond to cosets of centralizers of $G(\Gamma)$ and $G(\Gamma')$, respectively. The map α only tells us which coset go to which coset. In order to define $\alpha_* : G(\Gamma) \to G(\Gamma')$, we need to choose a map for each coset. Thus α_* is not uniquely defined.

A sufficient condition for α to be visible has been provided previously in [21, Lemma 4.10]. Here we will find a necessary and sufficient condition for the visibility of α.

3. The structure of extension complex

In this section, we introduce the classes of RAAGs we want to investigate, namely weak type II and type II (Definition 3.4), and weak type I (Definition 3.19). The bulk of this section is on the structure of extension complexes for RAAGs of (weak) type II and quasi-isometries invariance of (weak) type II. The application of this to quasi-isometric classification of RAAGs of weak type I will be discussed at the end of the section. In particular, Theorem 1.2 is proved in Section 3.3.

Throughout this section, we identify Γ with the 1-skeleton of $F(\Gamma)$, and we will implicitly use Lemma 2.1 in various places.

3.1. Tiers and branches of the extension complex

Let $\pi : \mathcal{P}(\Gamma) \to F(\Gamma)$ be the label-preserving simplicial map in Section 2.3.

Pick a standard geodesic $\ell \subset X(\Gamma)$, and let $\pi_\ell : X(\Gamma) \to \ell$ be the CAT(0) projection onto ℓ. Suppose $\ell_1 \subset X(\Gamma)$ is a standard geodesic such that $d(\Delta(\ell_1), \Delta(\ell)) \geq 2$. Then $\pi_\ell(\ell_1)$ is a vertex in ℓ by Lemmas 2.10 and 2.11. Moreover, if ℓ_2 is a standard geodesic parallel to ℓ_1, then $\pi_\ell(\ell_1) = \pi_\ell(\ell_2)$ (see [21, Lemma 6.2]). Thus π_ℓ induces a well-defined map $\pi_{\Delta(\ell)}$ from the $v(\mathcal{P}(\Gamma) \setminus \mathrm{St}(\Delta(\ell)))$, the set of vertices in $\mathcal{P}(\Gamma) \setminus \mathrm{St}(\Delta(\ell))$, to $v(\ell)$.

Lemma 3.1 ([21, Lemma 6.2]). *If v_1 and v_2 are in the same connected component of $\mathcal{P}(\Gamma) \setminus \mathrm{St}(\Delta(\ell))$, then $\pi_{\Delta(\ell)}(v_1) = \pi_{\Delta(\ell)}(v_2)$.*

The following definition plays a central role in our understanding of the extension complex.

Definition 3.2. Pick $v \in \mathcal{P}(\Gamma)$, and let $\ell \subset X(\Gamma)$ be a standard geodesic such that $\Delta(\ell) = v$. Let

$$\pi_{\Delta(\ell)} : v(\mathcal{P}(\Gamma) \setminus \mathrm{St}(v)) \to v(\ell)$$

be the map in Lemma 3.1. A *v-tier* is the full subcomplex spanned by $\pi^{-1}_{\Delta(\ell)}(x)$, where x is a vertex in ℓ and x is called the *height* of the v-tier. A *v-branch* is the full subcomplex spanned by vertices in one connected component of $\mathcal{P}(\Gamma) \setminus \mathrm{St}(v)$.

By Lemma 3.1, a v-branch has non-empty intersection with a v-tier if and only if it belongs to the v-tier, thus a v-tier consists of disjoint union of v-branches. Note that a simplicial isomorphism $\alpha : \mathcal{P}(\Gamma_1) \to \mathcal{P}(\Gamma_2)$ will map branches to branches. However, as we will see later (in Section 6.1), α may not map tiers to tiers.

Lemma 3.3. *Suppose $\alpha : \mathcal{P}(\Gamma_1) \to \mathcal{P}(\Gamma_2)$ is a simplicial isomorphism. If the α-image of any v-tier of $\mathcal{P}(\Gamma_1)$ is inside a single $\alpha(v)$-tier of $\mathcal{P}(\Gamma_2)$, then α is visible.*

Proof. Let p, $\{F_i\}_{i=1}^n$ and $\{F'_i\}_{i=1}^n$ be as in Definition 2.27. By Lemma 2.2, it suffices to show $F'_i \cap F'_j \neq \emptyset$ for any $i \neq j$. Suppose α is not visible. Then there exists a hyperplane h separating F'_i and F'_j. Let ℓ' be a standard geodesic dual to h, and let $v' = \Delta(\ell')$.

Note that F_i' does not contain any line which is parallel to ℓ', otherwise h would have non-trivial intersection with F_i'. Let $\pi_{\ell'}(F_i')$ be the CAT(0) projection of F_i' to ℓ'. Then $\pi_{\ell'}(F_i')$ is a point. Similarly, $\pi_{\ell'}(F_j')$ is a point. Since h separates F_i' and F_j', we have

$$\pi_{\ell'}(F_i') \neq \pi_{\ell'}(F_j').$$

The maximality of F_i' and F_j' implies there exist vertices $v_1' \in \Delta(F_i')$ and $v_2' \in \Delta(F_j')$ such that $v_i' \notin \mathrm{St}(v')$ for $i = 1, 2$. Since $\pi_{\ell'}(F_i') \neq \pi_{\ell'}(F_j')$, v_1' and v_2' are in different v'-tiers. On the other hand, we claim $\alpha^{-1}(v_1')$ and $\alpha^{-1}(v_2')$ are in the same $\alpha^{-1}(v')$-tier, which would give a contraction. To see the claim, note that $\alpha^{-1}(v_i')$ can be represented by a standard geodesic $\ell_i \subset F_i$ for $i = 1, 2$. Since both F_i and F_j contain p, we can assume $p \in \ell_1 \cap \ell_2$. Then the CAT(0)-projection of ℓ_1 and ℓ_2 to any standard geodesic representing $\alpha^{-1}(v')$ is the same. Thus $\alpha^{-1}(v_1')$ and $\alpha^{-1}(v_2')$ are in the same $\alpha^{-1}(v')$-tier. ■

The main goal of this subsection is Corollary 3.13, where we characterize v-branches in a v-tier for a certain class of Γ defined as follows.

Definition 3.4. A graph Γ is of *type II* if Γ is connected and for every pair of distinct vertices $v, w \in \Gamma$, $\Gamma \setminus (\mathrm{lk}(v) \cap \mathrm{lk}(w))$ is connected. The graph Γ is said to have *weak type II* if Γ is connected and for vertices $v, w \in \Gamma$ such that $d(v, w) = 2$, $\Gamma \setminus (\mathrm{lk}(v) \cap \mathrm{lk}(w))$ is connected.

We say $\mathcal{P}(\Gamma)$ is of *type II* if $\mathcal{P}(\Gamma)$ is connected and for every pair of distinct vertices $v, w \in \mathcal{P}(\Gamma)$, we know $(\mathcal{P}(\Gamma))^{(1)} \setminus (\mathrm{lk}(v) \cap \mathrm{lk}(w))$ is connected. We define $\mathcal{P}(\Gamma)$ as being *weak type II* in a similar way.

We say $G(\Gamma)$ or $F(\Gamma)$ is of (weak) type II if Γ is of (weak) type II.

Note that Γ is connected if it is of (weak) type II.

Example 3.5. A pentagon is a graph of type II. A slightly more complicated example of graph of type II is a 5-cycle and a 6-cycle identified along a closed star. However, a path of length 3 (or more generally a tree of diameter ≥ 3) is not of weak type II. In the following discussions and proofs, it would be helpful to have these basic examples in mind and compare them.

Let $v \in \mathcal{P}(\Gamma)$ be a vertex, and let $\ell \subset X(\Gamma)$ be a standard geodesic such that $\Delta(\ell) = v$. Define P_v to be the parallel set P_ℓ of ℓ. Note that P_v does not depend on the choice of the standard geodesic ℓ with $\Delta(\ell) = v$. A subset $K \subset P_v$ is *horizontal* if $\pi_\ell(K)$ is a point (where $\pi_\ell : X(\Gamma) \to \ell$ is the CAT(0) projection) and $\pi_\ell(K)$ is called the *height* of K.

Let $\overline{v} \in \Gamma$ be the label of $v \in \mathcal{P}(\Gamma)$. Then P_v is a standard subcomplex whose support (Definition 2.9) is $\mathrm{St}(\overline{v})$.

Lemma 3.6. *Suppose that Γ is of weak type II. Pick vertices $v, w \in \mathcal{P}(\Gamma)$ such that $d(v, w) = 2$. Let $u \in v^\perp \cap w^\perp$. Then there exists a vertex w' such that*

(1) *$d(v, w') = 2$ and $d(u, w') = 1$.*

(2) *w' and w are in the same v-branch.*

(3) *$P_v \cap P_{w'} \neq \emptyset$.*

In particular, every v-branch contains a vertex w' such that $P_{w'} \cap P_v \neq \emptyset$.

Proof. Let B be a v-branch containing w. Pick vertex $x \in P_w \cap P_u$, and let $y \in P_v$ be the nearest vertex to x with respect to the ℓ^1-metric. The existence and uniqueness of such vertex follow from [18, Lemma 13.8]. Let us assume $x \neq y$, otherwise we are done by putting $w' = w$. Let ω be a combinatorial geodesic connecting x and y. We claim $\omega \subset P_u$. By Lemma 2.3, it suffices to show $y \in P_u$. However, this follows by applying [18, Lemma 13.8] with $C' = P_v$ and $C = P_u$.

Let $\{x_i\}_{i=0}^n$ be vertices in ω such that for $0 \leq i \leq n-1$, $[x_i, x_{i+1}]$ is a maximal sub-segment of ω that is contained in a standard geodesic ($x_0 = x$ and $x_n = y$). Denote the standard geodesic containing $[x_i, x_{i+1}]$ by ℓ_i, and let $v_i = \Delta(\ell_i)$ for $0 \leq i \leq n-1$. Since ω is the shortest combinatorial geodesic connecting x and some vertex in P_v, every dual hyperplane of some edge in ω must separate x and P_v. Thus for each i, there exists a hyperplane dual to ℓ_i which does not intersect P_v. It follows that

$$d(v_i, v) \geq 2 \tag{3.1}$$

for all i. Since $\ell_i \subset P_u$, we also have

$$d(v_i, u) = 1. \tag{3.2}$$

Let $\pi : \mathcal{P}(\Gamma) \to F(\Gamma)$ be the projection mentioned at the beginning of this section. Since $\ell_{n-1} \cap P_v \neq \emptyset$, it follows from (3.1) and (3.2) that

$$d(\pi(v_{n-1}), \pi(v)) = 2. \tag{3.3}$$

We claim $v_0 \in B$. Let $K_{x_0} = (F(\Gamma))_{x_0}$ (i.e., K_{x_0} is the subcomplex of $\mathcal{P}(\Gamma)$ made of simplexes which come from standard flats passing x_0, see the paragraph before Lemma 2.14). First we show $K_{x_0} \cap \mathrm{St}(v)$ is contained in the intersection of the links of two vertices. Pick vertex $s \in K_{x_0} \cap \mathrm{St}(v)$, and let ℓ_s be the standard geodesic such that $x_0 \in \ell_s$ and $\Delta(\ell_s) = s$. Let h be a hyperplane dual to ℓ_{n-1} such that it separates x_0 from P_v. Then $h \cap \ell_s = \emptyset$ by (3.1) (note that if $h \cap \ell_s \neq \emptyset$, then $s = v_{n-1}$), hence h separates ℓ_s from P_v. It follows from Lemmas 2.10 and 2.11 that

$$\pi(s) \in (\pi(v_{n-1}))^{\perp} \cap (\pi(v))^{\perp}.$$

Let K be the full subgraph of Γ spanned by $(\pi(v_{n-1}))^{\perp} \cap (\pi(v))^{\perp}$. Then $t \notin \mathrm{St}(v)$ for any vertex $t \in K_{x_0}$ such that $\pi(t) \notin K$.

By (3.3), K does not separate Γ, so if $\pi(w) \notin K$ and $\pi(v_0) \notin K$, then they can be connected by an edge path outside K, which lifts to a path in K_{x_0} connecting w and v_0 outside

St(v), thus $v_0 \in B$. If $\pi(w) \notin K$ and $\pi(v_0) \in K$, then we connect $\pi(w)$ and $\pi(v_{n-1})$ by an edge path outside K, and then connect $\pi(v_{n-1})$ and $\pi(v_0)$ by an edge; this path also lifts to a path in K_{x_0} connecting w and v_0 outside St(v). The other cases can be dealt with in a similar way. We can repeat this process and argue inductively that actually $v_i \in B$ for $0 \leq i \leq n-1$, then the lemma follows by taking $w' = v_{n-1}$. ■

Lemma 3.7. *Suppose* Γ *is of weak type II. Take vertices* $v \in \mathcal{P}(\Gamma)$ *and* $x \in X(\Gamma)$. *If there exist two vertices* $v_1, v_2 \in (F(\Gamma))_x$ *such that they are in different* v*-branches, then* $v \in (F(\Gamma))_x$.

Proof. If this is not true, then $x \notin P_v$. We take a combinatorial geodesic of the shortest length from x to a vertex in P_v and repeat the above argument to see that $(F(\Gamma))_x \setminus \mathrm{St}(v)$ is connected, which contradicts that v_1 and v_2 are in different v-branches. ■

Lemma 3.8. *Suppose* Γ *is an arbitrary finite simplicial graph. Let* $v \in \mathcal{P}(\Gamma)$ *be a vertex, and let* $v_1, v_2 \in \mathcal{P}(\Gamma) \setminus \mathrm{St}(v)$ *be two vertices such that* $P_{v_i} \cap P_v \neq \emptyset$ *for* $i = 1, 2$. *Suppose* $\overline{v} = \pi(v)$. *If* $\pi(v_1)$ *and* $\pi(v_2)$ *are in different connected components of* $F(\Gamma) \setminus \mathrm{St}(\overline{v})$, *then* v_1 *and* v_2 *are in different* v*-branches.*

Proof. For $i = 1, 2$, let ℓ_i be a standard geodesic such that $\Delta(\ell_i) = v_i$ and $\ell_i \cap P_v \neq \emptyset$. We prove the lemma in two steps. In the first step, we will show that if we assume v_1 and v_2 are in the same v-branch, then there exist vertices $x \in \ell_1 \setminus P_v$ and $y \in \ell_2 \setminus P_v$ such that they can be connected by an edge path outside P_v. In the second step, we will show x and y cannot be connected by an edge path outside P_v. Such contradiction will finish the proof.

Step 1. Note that $\ell_1 \cap P_v$ and $\ell_2 \cap P_v$ are of the same height, otherwise v_1 and v_2 are in different v-tiers, hence are in different v-branches. Let $\{w_i\}_{i=1}^n \subset \mathcal{P}(\Gamma) \setminus \mathrm{St}(v)$ be vertices such that $d(w_i, w_{i+1}) = 1$, $w_1 = v_1$ and $w_n = v_2$. Pick $x = x_1$ to be any vertex in $\ell_1 \setminus P_v$. For $1 \leq i \leq n-1$, let r'_i be a standard geodesic with $\Delta(r'_i) = w_i$ and $r'_i \cap P_{w_{i+1}} \neq \emptyset$ (set $r'_n = \ell_2$). Let ω_1 be horizontal edge path in P_{w_1} connecting x_1 and a vertex $x_2 \in r'_1$. Note that $\omega_1 \cap P_v = \emptyset$ since $P_v \cap P_{w_1}$ is either empty or horizontal in P_{w_1}. Let r_2 be the standard geodesic such that $x_2 \in r_2$ and $\Delta(r_2) = w_2$. If $P_{w_2} \cap P_v = \emptyset$ or $P_{w_2} \cap P_v$ and x_2 have different heights in P_{w_2}, then let ω_2 be a horizontal edge path joining x_2 and a vertex $x_3 \in r'_2$. If $P_{w_2} \cap P_v$ and x_2 have the same height in P_{w_2}, then let ω'_2 be an edge in r_2 joining x_2 and another vertex x'_2, and let ω''_2 be a horizontal edge path joining x'_2 and a vertex $x_3 \in r'_2$. Set $\omega_2 = \omega'_2 \cup \omega''_2$; it is clear that $\omega_2 \cap P_v = \emptyset$ in both cases. We can define ω_i and x_{i+1} for $3 \leq i \leq n$ in the same way. Let $y = x_{n+1}$ and the first step follows.

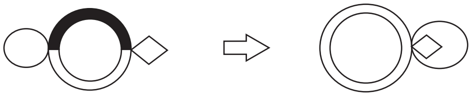
Step 2. Let C_1 be the component of $F(\Gamma) \setminus \mathrm{St}(\overline{v})$ that contains $\pi(v_1)$ and C_2 be the union of all other components. For $i = 1, 2$, let Γ_i be the full subgraph spanned by vertices in

$C_i \cup \mathrm{St}(\bar{v})$. Then $\mathrm{St}(\bar{v}) = \Gamma_1 \cap \Gamma_2$. Let $S(\mathrm{St}(\bar{v}))$ and $S(\mathrm{lk}(\bar{v}))$ be the Salvetti complexes with defining graphs $\mathrm{St}(\bar{v})$ and $\mathrm{lk}(\bar{v})$, respectively. Note that

$$S(\mathrm{St}(\bar{v})) \cong S(\mathrm{lk}(\bar{v})) \times \mathbb{S}^1.$$

Note that $S(\mathrm{St}(\bar{v}))$ sits naturally in $S(\Gamma_1)$ and $S(\Gamma_2)$, we can obtain $S(\Gamma)$ by gluing $S(\Gamma_1)$ and $S(\Gamma_2)$ along $S(\mathrm{St}(\bar{v}))$.

Now we glue $S(\Gamma_1)$ and $S(\Gamma_2)$ in a different way to obtain a new space $\bar{S}(\Gamma)$ as follows. For a reason which will be clear shortly, we assume the $\mathcal{S}^1$ factor in $S(\mathrm{St}(\bar{v}))$ has length $= 4\pi$. Let h be an isometry of $S(\mathrm{St}(\bar{v}))$ which is identity on the $S(\mathrm{lk}(\bar{v}))$ factor and is a rotation of degree $= 2\pi$ on the $\mathbb{S}^1$ factor. Now we glue $S(\Gamma_1)$ and $S(\Gamma_2)$ using the isometry h to obtain $\bar{S}(\Gamma)$. Note that there is a homotopy equivalence $g : \bar{S}(\Gamma) \to S(\Gamma)$ induced by collapsing the interval $[e^{i0}, e^{i2\pi}]$ in the $\mathbb{S}^1$ factor of $S(\mathrm{St}(\bar{v}))$ to one point (see the following picture, where the black part is collapsed). It lifts to a cubical map $\tilde{g} : \bar{X}(\Gamma) \to X(\Gamma)$.



Let $M \subset P_v$ be the standard subcomplex such that $\ell_1 \cap P_v \subset M$ and the support of M satisfies $\Gamma_M = \mathrm{lk}(\bar{v})$. Then there exists a unique hyperplane $\bar{h} \subset \bar{X}(\Gamma)$ such that $\tilde{g}(\bar{h}) = M$. For $i = 1, 2$, let $\bar{\ell}_i \subset \bar{X}(\Gamma)$ be the unique geodesic such that $\tilde{g}(\bar{\ell}_i) = \ell_i$. Then $\bar{\ell}_1$ and $\bar{\ell}_2$ have non-empty intersection with $\tilde{g}^{-1}(P_v)$. Since $\tilde{g}^{-1}(P_v)$ is a lift of $S(\mathrm{St}(\bar{v}))$ in $\bar{X}(\Gamma)$, and $\pi(\Delta(\ell_i)) \in C_i$ for $i = 1, 2$, we know that $\bar{\ell}_1$ and $\bar{\ell}_2$ are separated by $\bar{h}$. Let $\omega = \bigcup_{i=1}^n \omega_i$ be the edge path connecting x and y in the previous step. Note that the inverse image of each edge in $X(\Gamma)$ under $\tilde{g}$ is either an edge or a square; the inverse image of each vertex is either a vertex or an edge. Then $\tilde{g}^{-1}(\omega)$ is a compact connected subcomplex of $X(\bar{\Gamma})$. Since $\omega \cap P_v = \emptyset$, $\tilde{g}^{-1}(\omega) \cap \tilde{g}^{-1}(P_v) = \emptyset$. Hence $\tilde{g}^{-1}(\omega) \cap \bar{h} = \emptyset$. Moreover, $\tilde{g}^{-1}(\omega) \cap \bar{\ell}_i \neq \emptyset$ for $i = 1, 2$, which contradicts the separation property of $\bar{h}$. ■

The following observation follows from Step 2 of the proof of Lemma 3.8.

Lemma 3.9. *Let Γ be arbitrary. Let $\omega \subset X(\Gamma)$ be an edge path joining vertices $x_1, x_2 \in P_v$, and suppose $\omega \setminus \{x_1, x_2\}$ stays inside one component of $X(\Gamma) \setminus P_v$. Then*

(1) *x_1 and x_2 are of the same height in P_v.*

(2) *For $i = 1, 2$, let $e_i \subset \omega$ be the edge containing x_i, and let $\bar{v}_i \in \Gamma$ be the label of e_i. Then $\bar{v}_1$ and $\bar{v}_2$ are in the same component of $\Gamma \setminus \mathrm{St}(\bar{v})$.*

Let $v \in \mathcal{P}(\Gamma)$ be a vertex, and let $\bar{v} = \pi(v) \in \Gamma$. Let C be a component of $\Gamma \setminus \mathrm{St}(\bar{v})$. We define ∂C to be the full subgraph spanned by vertices in $\bar{C} \setminus C$, where $\bar{C}$ is the closure of C. Equivalently, ∂C is the full subgraph spanned by vertices in $\{u \in \Gamma \setminus C \mid \text{there exists vertex } w \in C \text{ such that } d(w, u) = 1\}$. Similarly, for every v-branch

$B \subset \mathcal{P}(\Gamma)$, we define the *boundary* of B, denoted by ∂B, to be the full subcomplex spanned by vertices in

$$\{u \in \mathcal{P}(\Gamma) \setminus B \mid \text{ there exists vertex } w \in B \text{ such that } d(w,u) = 1\}.$$

Equivalently, ∂B is the full subcomplex spanned by

$$\{u \in \mathrm{St}(v) \mid \text{ there exists vertex } w \in B \text{ such that } d(w,u) = 1\}.$$

Such ∂B is called a *v-peripheral subcomplex* of $\mathcal{P}(\Gamma)$. We caution the reader that $B \cup \partial B$ may not be equal to the closure of B.

A subcomplex $K \subset P_v$ is called a *v-peripheral subcomplex (of type ∂C)* if K is a standard subcomplex and $\Gamma_K = \partial C$ for some component C of $\Gamma \setminus \mathrm{St}(\overline{v})$. If the vertex set of ∂C is properly contained in $\overline{v}^{\perp}$, then there are infinitely many v-peripheral subcomplexes of type ∂C which are of the same height.

Example 3.10. We give an example of v-peripheral subcomplex. Let Γ be a pentagon and $\overline{v} \in \Gamma$ be a vertex. Pick a lift $v \in \mathcal{P}(\Gamma)$ of $\overline{v}$. Then P_v is isometric to $\mathbb{R} \times T_4$, where T_4 is the 4-valence tree. Note that $\Gamma \setminus \mathrm{St}(\overline{v})$ only have one component C and $\partial C = \mathrm{lk}(\overline{v})$. So any standard subcomplex of P_v whose support is $\mathrm{lk}(\overline{v})$ is a v-peripheral subcomplex of type ∂C. In our case, v-peripheral subcomplexes are those T_4-slices in $\mathbb{R} \times T_4$. For a given height, there is only one v-peripheral subcomplex of type ∂C.

Lemma 3.11. *Let Γ be arbitrary. Let $x_1, x_2, \overline{v}_1, \overline{v}_2, \omega$ and P_v be as in Lemma 3.9, and let C be the component of $\Gamma \setminus \mathrm{St}(\overline{v})$ containing $\overline{v}_1$ and $\overline{v}_2$. Then x_1 and x_2 are in the same v-peripheral subcomplex of type ∂C.*

Proof. For $i = 1, 2$, let K_i be the v-peripheral subcomplex of type ∂C such that $x_i \in K_i$. Note that K_1 and K_2 are horizontal subcomplexes of $\mathcal{P}(\Gamma)$ of the same height. We argue by contradiction and suppose $K_1 \neq K_2$. Then $K_1 \cap K_2 = \emptyset$. We claim there exists an edge $e \in P_v$ such that its label v_e does not belong to ∂C and the hyperplane dual to e separates K_1 from K_2. To see this, pick vertices $x \in K_1$ and $y \in K_2$ such that

$$d(x, y) = d(K_1, K_2).$$

Let ω_1 be a combinatorial geodesic joining x and y. Then $\omega_1 \subset P_v$ by Lemma 2.3. Moreover, every hyperplane dual to some edge in ω_1 separates K_1 and K_2. Thus there exists an edge $e \in \omega_1$ such that $v_e \notin \partial C$, otherwise we would have $\omega_1 \subset K_1$.

Let h_e be the hyperplane dual to e, and let N_{h_e} be the carrier of h_e. Then h_e separates x_1, x_2, and there exists an edge $e' \subset \omega$ parallel to e (ω is the path in Lemma 3.9). Pick endpoint $y \in e'$, and let $\omega_2 \subset N_{h_e}$ be an edge path of the shortest combinatorial length connecting y and $P_v \cap N_{h_e}$. Let x_3 be the other endpoint of ω_2, and let $e'' \subset \omega_2$ be the edge containing x_3. Then $d(v_{e''}, v_e) = 1$ ($v_{e''}$ is the label of e''), and it follows from $v_e \notin \partial C$ that

$$v_{e''} \notin C. \tag{3.4}$$

Let ω_3 be an edge path connecting x_1 and x_3 obtained by first following ω from x_1 to y, then following ω_2 until x_3. Then applying Lemma 3.9 to ω_3 yields a contradiction to (3.4). ∎

Corollary 3.12. *Suppose Γ is connected. Let v and $\bar{v}$ be as in Lemma 3.8, and pick a component C of $\Gamma \setminus \mathrm{St}(\bar{v})$. Suppose K_1 and K_2 are two distinct v-peripheral subcomplexes of type ∂C and they have the same height. Let $w_1, w_2 \in \mathcal{P}(\Gamma) \setminus \mathrm{St}(v)$ be vertices such that $\pi(w_i) \in C$ for $i = 1, 2$. Suppose $P_{w_i} \cap K_i \neq \emptyset$ for $i = 1, 2$. Then w_1 and w_2 are in different v-branches.*

Proof. For $i = 1, 2$, let ℓ_i be a standard geodesic such that $\ell_i \cap K_i \neq \emptyset$ and $\Delta(\ell_i) = w_i$. If w_1 and w_2 are in the same v-branch, then the argument in the second paragraph of the proof of Lemma 3.8 implies there exists an edge path $\omega \subset X(\Gamma) \setminus P_v$ connecting a vertex in $\ell_1 \setminus P_v$ to a vertex in $\ell_2 \setminus P_v$. Then it follows from Lemma 3.11 that $K_1 = K_2$, which is a contradiction. ∎

Corollary 3.13. *Suppose Γ is of weak type II. Let v and $\bar{v}$ be as before, and let $\ell \subset X(\Gamma)$ be a standard geodesic such that $\Delta(\ell) = v$. Pick vertex $x \in \ell$, then*

(1) *There is a 1-1 correspondence between v-branches in the v-tier of height x and pairs (C, K), where C is a component in $\Gamma \setminus \mathrm{St}(\bar{v})$ and K is a v-peripheral subcomplexes in $X(\Gamma)$ of height x such that $\Gamma_K = \partial C$. Moreover, let B be the v-branch corresponding to (C, K). Then $\partial B = \Delta(K)$.*

Now we assume Γ is of type II, then the following hold:

(2) *For every v-peripheral subcomplex $A \subset \mathcal{P}(\Gamma)$, there exists a unique v-peripheral subcomplex $K \subset X(\Gamma)$ of height x such that $\Delta(K) = A$.*

(3) *Let A be as in (2). Then there are only finitely many v-branches with boundary equal to A in a v-tier.*

(4) *Let $v_1, v_2 \in \mathcal{P}(\Gamma)$ be two different vertices and $B_i \subset \mathcal{P}(\Gamma)$ be a v_i-branch for $i = 1, 2$. Then $B_1 \neq B_2$.*

(5) *Let v_1, v_2 be as above. Then $\mathcal{P}(\Gamma) \setminus (\mathrm{lk}(v_1) \cap \mathrm{lk}(v_2))$ is connected.*

Proof. For $i = 1, 2$, pick pairs (C_i, K_i) as above, let $w_i \in \mathcal{P}(\Gamma)$ be a vertex such that $\pi(w_i) \in C_i$ and $P_{w_i} \cap K_i \neq \emptyset$; we claim w_1 and w_2 are in the same v-branch if and only if $C_1 = C_2$ and $K_1 = K_2$. Assuming the claim, then the first part of (1) follows from Lemma 3.6. The only if direction follows from Lemma 3.8 and Corollary 3.12. For the other direction, pick vertex $x_i \in P_{w_i} \cap K_i$; it suffices to consider the case when x_1 and x_2 are joined by an edge $e \subset K_1$. Let ℓ_e be the standard geodesic containing e, and let $v_e = \Delta(\ell_e)$. Then $\pi(v_e) \in \partial C_1$, and there exists $\bar{u} \in C_1$ such that

$$d(\bar{u}, \pi(v_e)) = 1. \tag{3.5}$$

For $i = 1, 2$, let $\overline{\omega}_i \subset C_1$ be the edge path connecting $\overline{u}$ and $\pi(w_i)$. Then we lift $\overline{\omega}_i$ to an edge path $\omega_i \subset (F(\Gamma))_{x_i}$. Equation (3.5) implies we can concatenate ω_1 and ω_2 to obtain a path connecting w_1 and w_2 outside $\mathrm{St}(v)$.

Now we prove the second statement of (1). Pick pair (C, K) as above and B be the associated v-branch. Since $\Gamma_K = \partial C$, for each standard geodesic $\ell \subset K$, there exists a standard geodesic ℓ' such that

(1) $\pi(\Delta(\ell')) \in C$.

(2) ℓ' and ℓ span a 2-flat.

Thus $\Delta(\ell') \in B$ and $\Delta(\ell) \in \partial B$. Hence $\Delta(K) \subset \partial B$. Now we prove the other direction. Pick $u \in \partial B$. By Lemma 3.6, we can assume there exists $w' \in B$ such that $d(w', u) = 1$ and $P_{w'} \cap P_v \neq \emptyset$. Then $\pi(w') \in C$ by Lemma 3.8, hence $\pi(u) \in \partial C$. Note that $P_{w'} \cap P_u \neq \emptyset$ and $P_v \cap P_u \neq \emptyset$. Thus $P_v \cap P_u \cap P_{w'} \neq \emptyset$ by Lemma 2.2. Pick vertex z in this triple intersection. Then $z \in K$ by Corollary 3.12. Let ℓ_z be a standard geodesic such that $z \in \ell_z$ and $\Delta(\ell_z) = u$. Since $\pi(u) \in \partial C = \Gamma_K$, we have $\ell_z \subset K$. Thus $u \in \Delta(K)$.

The existence in (2) follows from Lemma 3.6 and the above discussion. Let K_1 and K_2 be two v-peripheral subcomplexes of the same height such that $\Delta(K_1) = \Delta(K_2) = A$. Then the Hausdorff distance $d_H(K_1, K_2) < \infty$ by Lemma 2.10. If $K_1 \cap K_2 \neq \emptyset$, then $K_1 = K_2$ since $\Gamma_{K_1} = \Gamma_{K_2}$. Otherwise, there exists a horizontal edge $e \subset P_v$ such that the hyperplane dual to e separates K_1 from K_2 (note that K_1 and K_2 are horizontal). Suppose $\overline{w} \in \Gamma$ is the label of e. Then $d(\overline{w}, \overline{v}) = 1$ and $\Gamma_{K_1} \subset \mathrm{St}(\overline{w}) \setminus \{\overline{w}\}$ by Lemma 2.10. It follows that $\mathrm{lk}(\overline{w}) \cap \mathrm{lk}(\overline{v})$ contains Γ_{K_1}, thus $\mathrm{lk}(\overline{w}) \cap \mathrm{lk}(\overline{v})$ separates $\overline{v}$ from a vertex in a component of $\Gamma \setminus \mathrm{St}(\overline{v})$. This contradicts that Γ has type II. Corollary 3.13 (3) follows from (1) and (2).

To see (4), suppose $B_1 = B_2$. By (1), there exist standard subcomplexes $K_i \subset P_{v_i}$ for $i = 1, 2$ such that $\Delta(K_i) = \partial B_i$. Then K_1 and K_2 are parallel, hence $\Gamma_{K_1} = \Gamma_{K_2}$. Let $\overline{v}_i = \pi(v_i)$ for $i = 1, 2$. Then

$$\Gamma_{K_1} \subset \mathrm{lk}(\overline{v}_1) \cap \mathrm{lk}(\overline{v}_2).$$

By Lemma 3.6, there exists vertex $w \in B_1$ such that $\overline{w} = \pi(w) \in C$, where C is a component of $\Gamma \setminus \mathrm{St}(v_1)$ with $\partial C = \Gamma_{K_1}$. Therefore, Γ_{K_1} separates $\overline{v}_1$ from $\overline{w}$, so does $\mathrm{lk}(\overline{v}_1) \cap \mathrm{lk}(\overline{v}_2)$. This leads to a contradiction in the case $\overline{v}_1 \neq \overline{v}_2$. Suppose $\overline{v}_1 = \overline{v}_2$. Then P_{v_1} and P_{v_2} are standard complexes with the same support. Thus $P_{v_1} \cap P_{v_2} = \emptyset$, otherwise we would have $P_{v_1} = P_{v_2}$ and $v_1 = v_2$. Let h be a hyperplane separating P_{v_1} and P_{v_2} such that the carrier of h intersects P_{v_1}. Then the label of edges dual to h, denoted by $\overline{v}_h$, satisfies $d(\overline{v}_h, \overline{v}_1) \geq 2$. It follows from Lemma 2.11 that $\Gamma_{K_1} \subset \mathrm{lk}(\overline{v}_h)$. Thus $\mathrm{lk}(\overline{v}_1) \cap \mathrm{lk}(\overline{v}_h)$ separates Γ, which is a contradiction.

To see (5), first we assume $d(\overline{v}_1, \overline{v}_2) \neq 0$. Since Γ is of type II, for any component C of $\Gamma \setminus \mathrm{St}(\overline{v}_1)$,

$$\partial C \setminus (\mathrm{lk}(\overline{v}_1) \cap \mathrm{lk}(\overline{v}_2)) \neq \emptyset. \tag{3.6}$$

Let B be a v_1-branch. Then (1) and (2) imply there exist a standard subcomplex $K \subset P_{v_1}$ and a component C' of $\Gamma \setminus \mathrm{St}(\overline{v}_1)$ such that $\partial B = \Delta(K)$ and $\Gamma_K = \partial C'$. Hence $\partial C' =$

$\pi(\partial B) \cap \Gamma$ (recall that we have identified Γ with the 1-skeleton of $F(\Gamma)$). It follows from (3.6) that

$$\partial B \setminus (\mathrm{lk}(v_1) \cap \mathrm{lk}(v_2)) \neq \emptyset,$$

otherwise we would have

$$\partial C' \subset \pi(\partial B) \subset \pi(\mathrm{lk}(v_1) \cap \mathrm{lk}(v_2)) \subset \mathrm{lk}(\bar{v}_1) \cap \mathrm{lk}(\bar{v}_2).$$

Then every vertex in B can be connected to v_1 outside $\mathrm{lk}(v_1) \cap \mathrm{lk}(v_2)$ and (5) follows.

If $\bar{v}_1 = \bar{v}_2$, then $P_{v_1} \cap P_{v_2} = \emptyset$. Hence $d(v_1, v_2) \geq 2$ and

$$\mathrm{lk}(v_1) \cap \mathrm{lk}(v_2) = \mathrm{St}(v_1) \cap \mathrm{St}(v_2).$$

Let h and $\bar{h}_v$ be as in the proof of (4). Let ℓ_h be a standard geodesic dual to ℓ, and let $v_h = \Delta(\ell_h)$. Note that $d(v_h, v_1) \geq 2$ and $\pi(v_h) = \bar{v}_h \neq \bar{v}_1$. It suffices to prove

$$\mathrm{St}(v_1) \cap \mathrm{St}(v_2) \subset \mathrm{St}(v_1) \cap \mathrm{St}(v_h),$$

since this reduces the current case to the previous case. Let k be a vertex in $\mathrm{St}(v_1) \cap \mathrm{St}(v_2)$. Then P_k has non-trivial intersection with both P_{v_1} and P_{v_2}. Since h separates P_{v_1} and P_{v_2}, we have $P_k \cap h \neq \emptyset$. Hence $\ell_h \subset P_k$ and $d(k, v_h) \leq 1$. ∎

3.2. Quasi-isometry invariance of type II and weak type II

The main goal of this subsection is Corollary 3.18, where it is shown that weak type II and type II are quasi-isometry invariants for RAAGs.

Lemma 3.14. *If Γ is of weak type II, then*

(1) *There is no non-adjacent transvection in* $\mathrm{Out}(G(\Gamma))$.

(2) *$\mathcal{P}(\Gamma)$ is of weak type II.*

Proof. (1) follows directly from the definition. To see (2), pick distinct vertices $v_1, v_2 \in \mathcal{P}(\Gamma)$ such that $d(v_1, v_2) = 2$. For $i = 1, 2$, let $\bar{v}_i = \pi(v_i)$. The case $d(\bar{v}_1, \bar{v}_2) = 2$ and $\bar{v}_1 = \bar{v}_2$ has been dealt with in Corollary 3.13 (5). Now we assume $d(\bar{v}_1, \bar{v}_2) = 1$. If $P_{v_1} \cap P_{v_2} \neq \emptyset$, then it is a standard complex whose support is the intersection of the supports of P_{v_1} and P_{v_2}, which is $\mathrm{St}(\bar{v}_1) \cap \mathrm{St}(\bar{v}_2)$. Thus $d(v_1, v_2) = 1$, which yields a contradiction. So $P_{v_1} \cap P_{v_2} = \emptyset$, and we have reduced to second case of Corollary 3.13 (5). ∎

Lemma 3.15 ([21, Lemma 5.1]). *Let Γ be a finite simplicial graph. Pick a vertex $\bar{w} \in \Gamma$, and let $\Gamma_{\bar{w}}$ be the minimal stable subgraph containing $\bar{w}$. Denote $\Gamma_1 = \mathrm{lk}(\bar{w})$ and $\Gamma_2 = \mathrm{lk}(\Gamma_1)$ (see Section 2.1 for definition of links), then either of the following is true:*

(1) *$\Gamma_{\bar{w}}$ is a clique. In this case, $\mathrm{St}(\bar{w})$ is a stable subgraph.*

(2) *Both Γ_1 and the join $\Gamma_1 \circ \Gamma_2$ of Γ_1 and Γ_2 are stable subgraphs of Γ. Moreover, Γ_2 is disconnected.*

Lemma 3.16. *Suppose $\mathcal{P}(\Gamma)$ is of weak type II. Let $q : G(\Gamma) \to G(\Gamma')$ be a quasi-isometry. Then q induces a simplicial isomorphism $q_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$.*

The proof is a variation of [21, Theorem 5.3].

Proof. Let $\Gamma = \Gamma_1 \circ \Gamma_2$ be any join decomposition. Then Γ is of weak type II if and only if each Γ_i is of weak type II. So in light of Theorem 2.18, we only need to focus on the case when Γ is irreducible and is not a clique. In this case, Γ' is also irreducible and is not a clique, hence $\mathrm{diam}(\mathcal{P}(\Gamma)) = \infty$ and $\mathrm{diam}(\mathcal{P}(\Gamma')) = \infty$ [23, Lemma 26 (5)].

Let $q_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$ be the simplicial embedding in Theorem 2.25. Suppose q_* is not surjective. Then there exists a vertex $w' \in \mathcal{P}(\Gamma')$ which is not in the image of q_*. Let $\bar{w}' = \pi(w')$, and let ℓ' be a standard geodesic $\Delta(\ell') = w'$. We apply Lemma 3.15 to $\bar{w}' \in \Gamma'$. If case (1) is true, let F' be the standard flat in $X(\Gamma')$ such that $\ell' \subset F'$ and $\Gamma_{F'} = \Gamma_{w'}$. Since $\Gamma_{w'}$ is stable,

$$w' \in \Delta(F') \subset q_*(\mathcal{P}(\Gamma')),$$

which is a contradiction.

If case (2) is true, let $\Gamma'_1 = \mathrm{lk}(\bar{w}')$ and $\Gamma'_2 = \mathrm{lk}(\Gamma'_1)$. Take K'_1 and K' to be the standard subcomplexes in $X(\Gamma')$ such that (1) $\Gamma_{K'_1} = \Gamma'_1$ and $\Gamma_{K'} = \Gamma'_1 \circ \Gamma'_2$; (2) $\ell' \subset K'$ and $K'_1 \subset K'$. Set $M'_1 = \Delta(K'_1)$ and $M' = \Delta(K')$. Let K'_2 be an orthogonal complement of K'_1 in K', that is, K'_2 is a standard subcomplex such that $\Gamma_{K'_2} = \Gamma'_2$ and $K' = K'_1 \times K'_2$. It follows that M' has a join decomposition $M' = M'_1 * M'_2$ for $M'_2 = \Delta(K'_2)$. By construction, $w' \in M'$ and $\mathrm{lk}(w') = M'_1$.

Since K' and K'_1 are stable, then there exist stable standard subcomplexes K and K_1 in $X(\Gamma)$ such that the Hausdorff distances satisfy $d_H(q(K), K') < \infty$ and $d_H(q(K_1), K'_1) < \infty$. Moreover, by applying Theorem 2.18 to the quasi-isometry between K and K', there exists standard subcomplex $K_2 \in K$ such that $K = K_1 \times K_2$ and K_2 is quasi-isometric to K'_2, thus Γ_{K_2} is also disconnected. Let $M_i = \Delta(K_i) \subset \mathcal{P}(\Gamma)$ for $i = 1, 2$ and $M = M_1 * M_2 = \Delta(K)$. Then $q_*^{-1}(M'_1) = M_1$ by (2.5) and (2.4). Since $\mathrm{lk}(w') = M'_1$ and $w' \notin q_*(\mathcal{P}(\Gamma))$,

$$q_*^{-1}(\mathrm{St}(w')) = M_1. \tag{3.7}$$

Let $I = q_*(\mathcal{P}(\Gamma))$. Then I is D-dense in $\mathcal{P}(\Gamma')$ for some constant $D > 0$, that is, each vertex of $\mathcal{P}(\Gamma)$ is at combinatorial distance $\leq D$ from a vertex in I. To see this, it suffices to show Γ' contains a stable clique, but this follows from the existence of stable clique in Γ.

We claim every w'-tier contains vertices arbitrarily far from w'. To see this, let ℓ' be a standard geodesic such that $\Delta(\ell') = w'$. We consider the action $G(\Gamma) \curvearrowright X(\Gamma)$ by deck transformations and the induced action $G(\Gamma) \curvearrowright \mathcal{P}(\Gamma)$. Then the stabilizer of ℓ' is isomorphic to $\mathbb{Z}$. Moreover, this copy of $\mathbb{Z}$ acts transitively on the collection of w'-tiers. Now the claim follows from $\mathrm{diam}(\mathcal{P}(\Gamma')) = \infty$.

We pick vertices $w'_1, w'_2 \in \mathcal{P}(\Gamma')$ such that they are not in the same w'-tier and $d(w'_i, w') > D + 5$ for $i = 1, 2$. Let $u'_i \in I$ be a vertex such that $d(u'_i, w'_i) \leq D$. Then u'_1 and u'_2 are separated by $\mathrm{St}(w')$ and

$$d(u'_i, I \cap \mathrm{St}(w')) > 4.$$

Define $u_i = q_*^{-1}(u'_i)$, then u_1 and u_2 are in different components of

$$\mathcal{P}(\Gamma) \setminus q_*^{-1}(\mathrm{St}(w')) = \mathcal{P}(\Gamma) \setminus M_1,$$

and

$$d(u_i, M_1) > 4. \tag{3.8}$$

Since Γ_{K_2} is disconnected, there exist vertices $v_1, v_2 \in M_2$ such that $d(v_1, v_2) = 2$. Recall that $M = M_1 * M_2 \subset \mathcal{P}(\Gamma)$, so

$$M_1 \subset \mathrm{lk}(v_1) \cap \mathrm{lk}(v_2).$$

Moreover,

$$d(u_i, \mathrm{lk}(v_1) \cap \mathrm{lk}(v_2)) > 0$$

by (3.8), so u_1 and u_2 are separated by $\mathrm{lk}(v_1) \cap \mathrm{lk}(v_2)$, which is contradictory to our assumption on $\mathcal{P}(\Gamma)$. So q_* must be surjective. ■

Lemma 3.17. *If $\mathcal{P}(\Gamma)$ is of weak type II, then Γ is of weak type II.*

Proof. Suppose Γ is not of weak type II. Then there exist vertices $\bar{v}_1, \bar{v}_2 \in \Gamma$ such that $d(\bar{v}_1, \bar{v}_2) = 2$ and

$$\Gamma \setminus \mathrm{lk}(\bar{v}_1) \cap \mathrm{lk}(\bar{v}_2)$$

is disconnected. Then we can find component C of $\Gamma \setminus \mathrm{St}(\bar{v}_1)$ such that

$$\partial C \subset \mathrm{lk}(\bar{v}_1) \cap \mathrm{lk}(\bar{v}_2).$$

Pick vertex $x_0 \in P_{v_1}$, and let $K \subset P_{v_1}$ be the standard subcomplex with support $= \partial C$ that contains x_0. Pick vertex $\bar{v}_3 \in C$ (it is possible that $\bar{v}_3 = \bar{v}_2$). For $i = 1, 2, 3$, let $v_i \in (F(\Gamma))_{x_0}$ be the lift of $\bar{v}_i$. Then

$$\Delta(K) \subset \mathrm{St}(v_1) \cap \mathrm{St}(v_2) = \mathrm{lk}(v_1) \cap \mathrm{lk}(v_2).$$

Let B be the v_1-branch that contains v_3. We claim $\partial B = \Delta(K)$, which then implies v_1 and B are in different components of $\mathcal{P}(\Gamma) \setminus (\mathrm{lk}(v_1) \cap \mathrm{lk}(v_2))$.

Note that $\Delta(K) \subset \partial B$ follows from the argument in (1) of Corollary 3.13. To see the other direction, pick $w_1 \in \partial B$ and $w_2 \in B$ such that $d(w_1, w_2) = 1$. Let ℓ_3 be the geodesic such that $x_0 \in \ell_3$ and $\Delta(\ell_3) = v_3$. If $P_{w_2} \cap P_{v_1} = \emptyset$, then by the argument in Lemma 3.8, we can find an edge path $\omega \in X(\Gamma) \setminus P_{v_1}$ connecting $x \in \ell_3 \setminus \{x_0\}$ and $y \in P_{w_2} \cap P_{w_1}$. Let $\omega_1 \subset P_{w_1}$ be a horizontal edge path connecting y and some vertex $z \in P_{w_1} \cap P_{v_1}$.

Such path exists since $P_{w_1} \cap P_{v_1}$ contains a standard geodesic ℓ with $\Delta(\ell) = w_1$. We can also assume $\omega_1 \cap P_{v_1} = \{z\}$. Let $e \subset \omega_1$ be the edge containing z and $\overline{v}_e$ be the label of e. Since ω_1 is horizontal in P_{w_1},

$$d(\overline{v}_e, \pi(w_1)) = 1. \tag{3.9}$$

Let ω' be the edge path obtained by (1) going from x_0 to x along ℓ_3; (2) going from x to y along ω; (3) going from y to z along ω_1. By applying Lemmas 3.9 and 3.11 to ω', we have $\overline{v}_e \in C$ and $z \in K$. Hence $\pi(w_1) \in \partial C$ by (3.9). This, together with $z \in P_{w_1}$ and $z \in K$, implies $w_1 \in \Delta(K)$. If $P_{w_2} \cap P_{v_1} \neq \emptyset$, we still have $w_1 \in \Delta(K)$ by the proof of the second statement of Corollary 3.13 (1). Thus $\partial B \subset \Delta(K)$. ■

Actually, the above argument also shows that if $\mathcal{P}(\Gamma)$ is of type II, then Γ is of type II. The following corollary follows from (5) of Corollary 3.13 and Lemmas 3.14, 3.16 and 3.17.

Corollary 3.18. *The graph Γ is of (weak) type II if and only if $\mathcal{P}(\Gamma)$ is of (weak) type II. If $G(\Gamma')$ is quasi-isometric to $G(\Gamma)$, then Γ' is also of (weak) type II.*

3.3. RAAGs of weak type I

In this subsection, we introduce the notion of RAAGs of weak type I and prove quasi-isometric classification results for them.

Definition 3.19. A finite simplicial graph Γ is of *weak type I* if

(1) Γ is of weak type II.

(2) Γ does not contain any separating closed star.

The group $G(\Gamma)$ is of *weak type I* if Γ is of weak type I.

It is immediate from the definition that if $\Gamma = \Gamma_1 \circ \Gamma_2$, then Γ is of weak type I if and only if Γ_1 and Γ_2 are of weak type I.

Lemma 3.20. *The group $G(\Gamma)$ is of weak type I if and only if*

(1) *Γ does not contain any separating closed star.*

(2) *There do not exist vertices $\overline{v}, \overline{w} \in \Gamma$ such that $d(\overline{v}, \overline{w}) = 2$ and $\Gamma = \mathrm{St}(\overline{v}) \cup \mathrm{St}(\overline{w})$.*

Thus Definitions 1.1 and 3.19 are consistent.

Proof. For the only if direction, note that if $\Gamma = \mathrm{St}(\overline{v}) \cup \mathrm{St}(\overline{w})$ with $d(\overline{v}, \overline{w}) = 2$, then $\mathrm{lk}(\overline{v}) \cap \mathrm{lk}(\overline{w})$ separates Γ. For the if direction, we follow the argument in [21, Theorem 5.3]. Suppose there exist vertices $\overline{v}_1$ and $\overline{v}_2$ such that $\mathrm{lk}(\overline{v}_1) \cap \mathrm{lk}(\overline{v}_2)$ separates Γ. Let $\{C_j\}_{j=1}^{d}$ be the connected components of $F(\Gamma) \setminus \mathrm{lk}(\overline{v}_1) \cap \mathrm{lk}(\overline{v}_2)$. Then at most one of C_j is contained in $\mathrm{St}(\overline{v}_1)$. If $d \geq 3$, $\mathrm{St}(\overline{v}_1)$ would separate $F(\Gamma)$, contradiction. Suppose $d = 2$. At least one of C_1 and C_2 is inside $\mathrm{St}(\overline{v}_1)$, otherwise $\mathrm{St}(\overline{v}_1)$ will separate $F(\Gamma)$. We assume $C_1 \subset \mathrm{St}(\overline{v}_1)$. Thus $\overline{v}_2 \in C_2$. Similarly, at least one of C_1 and C_2 is inside $\mathrm{St}(\overline{v}_2)$. So we must have $C_2 \subset \mathrm{St}(\overline{v}_2)$. Hence $F(\Gamma) = \mathrm{St}(\overline{v}_1) \cup \mathrm{St}(\overline{v}_2)$, contradiction again. ■

Theorem 3.21. *Let Γ_1 be of weak type I. Then any simplicial isomorphism $s : \mathcal{P}(\Gamma_1) \to \mathcal{P}(\Gamma_2)$ is visible. In particular, if $q : G(\Gamma_1) \to G(\Gamma_2)$ is a quasi-isometry, then q will induce a visible map $q_* : \mathcal{P}(\Gamma_1) \to \mathcal{P}(\Gamma_2)$. In this case, Γ_2 is of weak type II, hence* $\mathrm{Out}(\Gamma_2)$ *does not contain non-adjacent transvections.*

Proof. Let $p, \{F_i\}_{i=1}^n$ and $\{F'_i\}_{i=1}^n$ be as in Definition 2.27. Suppose s is not visible. Then there exist $i \neq j$ and a hyperplane h separating F'_i and F'_j. Let ℓ' be a standard geodesic dual to h, and let $v' = \Delta(\ell')$. Then there exist $v'_1 \in \Delta(F'_i)$ and $v'_2 \in \Delta(F'_j)$ such that they are in different v'-tiers. Thus $\mathrm{St}(v')$ separates v'_1 from v'_2 by Lemma 3.1. Let $v = s^{-1}(v')$. Then

$$(F(\Gamma))_p \setminus \mathrm{St}(v)$$

is disconnected, hence $v \in (F(\Gamma))_p$ by Lemma 3.7. This would imply $F(\Gamma)$ has a separating closed star, which is a contradiction. The second statement follows from Lemma 3.16. ■

Theorem 3.22. *Suppose $G(\Gamma_1)$ and $G(\Gamma_2)$ are groups of weak type I. Then they are quasi-isometric if and only if they are isomorphic.*

Proof. Let $q : G(\Gamma_1) \to G(\Gamma_2)$ be a quasi-isometry. By Theorem 3.21, q induces a visible simplicial isomorphism $q_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$. Pick vertex $x_1 \in X(\Gamma_1)$. Then the visibility implies

$$q_*((F(\Gamma_1))_{x_1}) \subset (F(\Gamma_2))_{x_2}$$

for some vertex $x_2 \in X(\Gamma_2)$. This induces a graph embedding $\Gamma_1 \to \Gamma_2$. By considering the quasi-isometry inverse of q, we obtain another graph embedding $\Gamma_2 \to \Gamma_1$. Hence $\Gamma_1 \cong \Gamma_2$ and $G(\Gamma_1) \cong G(\Gamma_2)$. ■

Though the definition of weak type I looks technical, it is actually a natural condition to consider for the following reason.

Theorem 3.23. *The following are equivalent:*

(1) *$G(\Gamma)$ is of weak type I.*

(2) *There do not exist vertex $x \in X(\Gamma)$ and vertex $v \in \mathcal{P}(\Gamma)$ such that* $\mathrm{St}(v)$ *separates* $(F(\Gamma))_x$.

(3) *Every element in* $\mathrm{Aut}(\mathcal{P}(\Gamma))$ *is visible.*

We will not need (3) ⇒ (2) in the rest of the paper.

Proof. (1) ⇔ (2) follows from Lemma 3.24 and (2) ⇒ (3) follows from the proof of Theorem 3.21. It suffices to prove (3) ⇒ (2). We argue by contradiction and suppose v_1, v_2 are vertices in different components of $(F(\Gamma))_x \setminus \mathrm{St}(v)$. Then Lemma 3.8 implies v_1 and v_2 are in different v-branches. For $i = 1, 2$, let B_i be the v-branch that contains v_i. Let ℓ be a standard geodesic such that $\Delta(\ell) = v$ and $x \in \ell$ and pick $\alpha \in G(\Gamma)$ to be a non-trivial element such that $\alpha(\ell) = \alpha$. Let $\alpha_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ be the induced map. Then α_* fixes every point in $\mathrm{St}(v)$. Thus there exists $f \in \mathrm{Aut}(\mathcal{P}(\Gamma))$ such that

(1) f fixes every vertex in $\mathcal{P}(\Gamma) \setminus (B_1 \cup \alpha_*(B_1))$.

(2) $f|_{B_1} = \alpha_*|_{B_1}$ and $f|_{\alpha_*(B_1)} = \alpha_*^{-1}|_{\alpha_*(B_1)}$.

We claim f is not visible. To see this, for $i = 1, 2$, pick maximal standard flat F_i such that $x \in F_i$ and $v_i \in \Delta(F_i)$. Then

$$f(\Delta(F_1)) = \alpha_*(\Delta(F_1))$$

and $f(\Delta(F_2)) = \Delta(F_2)$, thus the maximal standard flats corresponding to $f(\Delta(F_1))$ and $f(\Delta(F_2))$ are separated by a hyperplane dual to ℓ, hence have empty intersection. ∎

Lemma 3.24. *The graph Γ is of weak type II if and only if there do not exist vertex $x \in X(\Gamma)$ and vertex $v \in \mathcal{P}(\Gamma) \setminus (F(\Gamma))_x$ such that $(F(\Gamma))_x \setminus \mathrm{St}(v)$ is disconnected.*

Proof. By Lemma 3.7, it suffices to prove the if direction. Suppose Γ is not of weak type II. Let $\{\bar{v}_i\}_{i=1}^3$, $\{v_i\}_{i=1}^3$ and $x_0 \in X(\Gamma)$ be as in Lemma 3.17. For $i = 1, 2$, let ℓ_i be the standard geodesic such that $x_0 \in \ell$ and $\Delta(\ell_i) = v_i$. Pick vertex $x_0' \neq x_0$ in ℓ_1, and let ℓ_2' be the standard geodesic such that $x_0' \in \ell_2'$ and $\pi(\Delta(\ell_2')) = \bar{v}_2$. Then $d(v_2', v_1) = 2$, where $v_2' = \Delta(\ell_2')$, in particular $x_0 \notin P_{v_2'}$, hence $v_2' \notin (F(\Gamma))_{x_0}$.

Since $P_{v_2'}$ and x_0 are separated by some hyperplane dual to ℓ_1, thus by Lemma 2.11,

$$\mathrm{St}(v_2') \cap (F(\Gamma))_{x_0} \subset \mathrm{St}(v_1).$$

Recall that $d(\bar{v}_3, \bar{v}_1) \geq 2$, then $v_3 \in (F(\Gamma))_{x_0} \setminus \mathrm{St}(v_1)$. It follows that $v_3 \notin \mathrm{St}(v_2')$.

We claim that v_3 and v_1 are in different components of $\mathcal{P}(\Gamma) \setminus \mathrm{St}(v_2')$, which then implies $(F(\Gamma))_{x_0} \setminus \mathrm{St}(v_2')$ is disconnected. Lemma 3.17 already implies v_1 and v_3 are separated by $\mathrm{lk}(v_1) \cap \mathrm{lk}(v_2)$. Let $\alpha \in G(\Gamma)$ be the left translation such that $\alpha(x_0) = x_0'$. Then $\alpha(\ell_2) = \ell_2'$. Now we pass to the induced action $G(\Gamma) \curvearrowright \mathcal{P}(\Gamma)$, then $\alpha(v_2) = v_2'$. Since α fixes $\mathrm{St}(v_1)$, we have

$$\mathrm{lk}(v_1) \cap \mathrm{lk}(v_2) = \alpha(\mathrm{lk}(v_1) \cap \mathrm{lk}(v_2)) = \mathrm{lk}(v_1) \cap \mathrm{lk}(v_2').$$

So $\mathrm{lk}(v_1) \cap \mathrm{lk}(v_2')$ separates v_1 from v_3 and the claim follows. ∎

4. Quasi-isometric invariance of v-branches and peripheral subcomplexes

In this section, we collect several observations on quasi-isometric invariance of v-branches and v-peripheral subcomplexes. While the content of this section is closely related to Section 3, it will not be used until Section 6.

4.1. Quasi-isometric invariance of v-branches

Let $G(\Gamma)$ be a RAAG of weak type II, and let $q : G(\Gamma) \to G(\Gamma')$ be a quasi-isometry. Then $G(\Gamma')$ is of weak type II by Corollary 3.18. Then both $\mathrm{Out}(G(\Gamma))$ and $\mathrm{Out}(G(\Gamma'))$ do not admit non-adjacent transvections (Lemma 3.14). Thus Theorem 2.25 implies that q induces a simplicial isomorphism $q_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$. Take a vertex v of $\mathcal{P}(\Gamma)$. Then q_* induces a bijection between v-branches and $q_*(v)$-branches. Later in Section 6, we want to use this bijection as a quasi-isometry invariant of q. However, a priori this is problematic as the simplicial isomorphism q_* is not canonically determined by q (as in the construction of the proof of Theorem 2.25), so strictly speaking, $q_*(v)$ is not even well defined. The goal of this subsection is to clarify this point.

Lemma 4.1. *Suppose Γ is of weak type II. Let $q : G(\Gamma) \to G(\Gamma')$ be a quasi-isometry, and let $\tilde{q}_*$ be as in Lemma 2.23. For $i = 1, 2$, let $(q_i)_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$ be a simplicial isomorphism such that $(q_i)_*(s) = \tilde{q}_*(s)$ for any stable simplex $s \subset \mathcal{P}(\Gamma)$.*

(1) *For any vertex $v \in \mathcal{P}(\Gamma)$, we have $(q_1)_*(\mathrm{St}(v)) = (q_2)_*(\mathrm{St}(v))$.*

(2) *For $i = 1, 2$, we know $d_H(q(P_v), P_{(q_i)_*(v)}) < \infty$.*

(3) *If Γ is of type II and $G(\Gamma)$ is centerless, then $(q_1)_* = (q_2)_*$.*

Proof. Let $\bar{v} \in \Gamma$ be the label of $v \in \mathcal{P}(\Gamma)$. Then it follows from Lemmas 2.22, 3.14 and 3.15 that $\mathrm{St}(\bar{v})$ is a stable subgraph. Thus $\mathrm{St}(v)$ is a stable subcomplex of $\mathcal{P}(\Gamma)$. By Corollary 3.18 and Lemma 3.14, both $\mathrm{Out}(G(\Gamma))$ and $\mathrm{Out}(G(\Gamma'))$ do not admit non-adjacent transvection. Thus Corollary 2.26 implies

$$(q_1)_*(\mathrm{St}(v)) = (q_2)_*(\mathrm{St}(v)) = \tilde{q}_*(\mathrm{St}(v)).$$

Then assertion (1) follows. This also implies assertion (2) as $\Delta(P_v) = \mathrm{St}(v)$ and $\Delta(P_{(q_i)_*(v)}) = \mathrm{St}((q_i)_*(v))$.

Now we prove assertion (3). By Corollary 3.18, Γ' is of type II. As

$$(q_i)_*(\mathrm{St}(v)) = \mathrm{St}((q_i)_*(v)),$$

we know

$$\mathrm{St}(w_1) = \mathrm{St}(w_2),$$

where w_i is defined as $(q_i)_*(v)$. Then

$$\mathrm{St}(\bar{w}_1) = \mathrm{St}(\bar{w}_2).$$

We must have $\bar{w}_1 = \bar{w}_2$, otherwise $\mathrm{lk}(\bar{w}_1) \cap \mathrm{lk}(\bar{w}_2)$ separates $\bar{w}_1$ from any vertex outside $\mathrm{St}(\bar{w}_1)$, which contradicts that Γ' is of type II (note that $G(\Gamma)$ is centerless implies that $G(\Gamma')$ is centerless, so $G(\Gamma')$ contains at least one vertex outside $\mathrm{St}(\bar{w}_1)$). Thus $w_1 = w_2$. ∎

Example 4.2. We give an example of Lemma 4.1 (1). Let Γ be the graph obtained by gluing a pentagon and the 1-skeleton of a 3-simplex along an edge. Let a and b be the two vertices of Γ which is outside the pentagon. Take a quasi-isometry $q : G(\Gamma) \to G(\Gamma)$. Let $v \in \mathcal{P}(\Gamma)$ be a vertex labeled by a, and let s be the maximal simplex of $\mathcal{P}(\Gamma)$ containing v. Then s is a stable subcomplex. We know $q_*(a) \in \widetilde{q}_*(s)$. However, $q_*(a)$ could be either the vertex in $\widetilde{q}_*(s)$ labeled by a or the vertex in $\widetilde{q}_*(s)$ labeled by b. Note that the star of these two possible values of $q_*(a)$ is equal.

We now give an example of Lemma 4.1 (3). Let Γ be the graph obtained by gluing a pentagon and a triangle along an edge. Let a be the vertex of Γ outside the pentagon. Take a quasi-isometry $q : G(\Gamma) \to G(\Gamma)$. Let $v \in \mathcal{P}(\Gamma)$ and $s \subset \mathcal{P}(\Gamma)$ be as before. Then $q_*(a) \in \widetilde{q}_*(s)$. Then $q_*(a)$ can only be the vertex in $\widetilde{q}_*(s)$ labeled by a. Note that if v is not labeled by a, then $\{v\}$ is a stable subcomplex of $\mathcal{P}(\Gamma)$. Then for any standard line ℓ with $\Delta(\ell) = v$, $q(\ell)$ is at finite Hausdorff distance to a standard line ℓ' with $\Delta(\ell') = q_*(v)$. However, if v is labeled by a, in general $q(\ell)$ might not be at finite Hausdorff distance from any standard line, due to the transvection at a, yet $q_*(v)$ is canonically defined.

Returning to the discussion before Lemma 4.1, even though $q_*(v)$ depends on the choices in the proof of Theorem 2.25, Lemma 4.1 implies that the set $\mathrm{St}(q_*(v))$ does not depend on these choices, so is the set of $q_*(v)$-branches.

Corollary 4.3. *Suppose Γ is of type II and $G(\Gamma)$ is centerless. Let $q : G(\Gamma) \to G(\Gamma')$ be a quasi-isometry, and let $\widetilde{q}_*$ be as in Lemma 2.23. Then there is a unique simplicial isomorphism $q_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$ such that $(q_i)_*(s) = \widetilde{q}_*(s)$ for any stable simplex $s \subset \mathcal{P}(\Gamma)$. In particular, for any quasi-isometric inverse $q^{-1} : G(\Gamma') \to G(\Gamma)$, we have $(q^{-1})_* = (q_*)^{-1}$.*

Proof. The first statement of the corollary is an immediate consequence of Lemma 4.1 (3). For the in particular part, note that $\widetilde{q^{-1}}_* \circ \widetilde{q}_*(s) = s$. Then $q_*^{-1} \circ q_*$ must be the identity map by Lemma 4.1 (3) again. Similarly, $q_* \circ q_*^{-1}$ is identity. ∎

4.2. Correspondence between v-branches and subcomplexes of $X(\Gamma)$

The main goal of this subsection is Proposition 4.6, where we establish 1-1 correspondence between v-branches and components of $X(\Gamma) \setminus P_v$ and prove a quasi-isometric invariance result.

Let Γ be an arbitrary simplicial graph (not necessarily of weak type II). Take vertex $v \in \mathcal{P}(\Gamma)$. For any v-branch B, we denote the full subcomplex of $\mathcal{P}(\Gamma)$ spanned by vertices in B and ∂B by $\overline{B}$. For any component L of $X(\Gamma) \setminus P_v$, we use ∂L to denote the full subcomplex of $X(\Gamma)$ spanned by vertices outside L which are adjacent to some vertex in L and use $\overline{L}$ to denote the full subcomplex spanned by vertices in L and ∂L. Note that $\overline{B}$ may not be the closure of B and $\overline{L}$ may not be the closure of L.

For any subcomplex $K \subset X(\Gamma)$, let $\{F_\lambda\}_{\lambda\in\Lambda}$ be the collection of standard flats in K and define $\Delta(K) = \bigcup_{\lambda\in\Lambda} \Delta(F_\lambda)$.

Lemma 4.4. *If K is a convex subcomplex, then $\Delta(K)$ is a full subcomplex of $\mathcal{P}(\Gamma)$.*

Proof. Let $F \subset X(\Gamma)$ be a standard flat such that vertices of $\Delta(F)$ are in $\Delta(K)$. Suppose $(F', K') = \mathcal{I}(F, K)$. Lemma 2.4 (4) implies that every standard geodesic of F is contained in an R-neighborhood of F' for some $R > 0$. However, F' is a convex subcomplex of F, so actually $F = F'$. Hence $\Delta(F) \subset \Delta(K)$. ■

Lemma 4.5. *Let Γ be arbitrary. Pick vertex $v \in \mathcal{P}(\Gamma)$, and let $\bar{v} = \pi(v)$. Let L be a component of $X(\Gamma) \setminus P_v$. Then*

1. *∂L is a v-peripheral subcomplex of $X(\Gamma)$. Moreover, the topological boundary $\partial^{\mathrm{Top}} L$ of L (i.e., $\partial^{\mathrm{Top}} L$ is the closure of L in $X(\Gamma)$ minus L) is contained in ∂L, and $\partial^{\mathrm{Top}} L$ contains the 1-skeleton of ∂L.*
2. *$\bar{L} = L \cup \partial L$, and $\bar{L}$ is a convex subcomplex of $X(\Gamma)$.*

Proof. To see (1), note that Lemma 3.9 implies there exists a component C of $\Gamma \setminus \mathrm{St}(\bar{v})$ such that the label of any edge which connects a vertex in L and a vertex outside L is inside C. Pick a vertex in $X(\Gamma) \setminus L$ which is adjacent to some vertex in L, and let K be the v-peripheral subcomplex of type ∂C that contains this vertex. Then Lemma 3.11 implies vertex set of ∂L is contained in K, hence $\partial L \subset K$. Note that $\partial^{\mathrm{Top}} L$ is a subcomplex whose vertex set is the same as ∂L, hence $\partial^{\mathrm{Top}} L \subset \partial L$. On the other hand, the proof of the first statement of Corollary 3.13 (1) implies every edge of K is contained in $\partial^{\mathrm{Top}} L$. Thus (1) follows.

To see (2), note that $L \cup \partial L$ is a subcomplex. We claim for a vertex $x \in L \cup \partial L$, if a collection of mutual orthogonal edges emanating from x are contained in $L \cup \partial L$, then the cube spanned by these edges is contained in $L \cup \partial L$. This is clear when $x \in L$. The case when $x \in \partial L$ follows from the fact that the labels of these edges are either vertices in C or vertices in ∂C. From the claim we know $L \cup \partial L$ is a locally convex subcomplex of $X(\Gamma)$, in particular it is a full subcomplex, hence (2) follows as locally convex subcomplexes of CAT(0) cube complexes are convex. ■

Proposition 4.6. *Suppose Γ is of weak type II. Pick vertex $v \in \mathcal{P}(\Gamma)$, and let $\bar{v} = \pi(v)$. Let L be a component of $X(\Gamma) \setminus P_v$. Let $q : X(\Gamma) \to X(\Gamma')$ be a quasi-isometry. Then the following hold true:*

1. *There is a 1-1 correspondence between v-branches and components of $X(\Gamma) \setminus P_v$. In particular, there is a unique v-branch B with $\Delta(\bar{L}) = \bar{B}$ and $\Delta(\partial L) = \partial B$.*
2. *There is a component L' of $X(\Gamma') \setminus P_{q_*(v)}$ such that*
$$d_H(q(L), L') < \infty.$$
3. *For any component C of $\Gamma \setminus \mathrm{St}(\bar{v})$, ∂C is a stable subgraph of Γ.*

Proof. We first prove assertion (1). For $i = 1, 2$, let $e_i \subset X(\Gamma)$ be an edge such that one of its endpoints $x_{i1} \in X(\Gamma) \setminus P_v$ and another endpoint $x_{i2} \in P_v$. Let $\bar{v}_i \in \Gamma$ be the label of e_i, and let C_i be the component of $\Gamma \setminus \mathrm{St}(\overline{v})$ that contains $\overline{v}_i$. We claim x_{11} and x_{21} are in the same component of $X(\Gamma) \setminus P_v$ if and only if $C_1 = C_2$ and x_{21} and x_{22} belong to the same v-peripheral subcomplex of ∂C_1. Then we have a 1-1 correspondence between components of $X(\Gamma) \setminus P_v$ and the pair (C, K) as in Corollary 3.13 (1) and the first part of assertion (1) follows. The only if part of the claim follows from Lemmas 3.9 and 3.11. Note that C_1 contains more than one point (otherwise, Γ will not be of weak type II), so the if direction holds in the special case when $\overline{v}_1 = \overline{v}_2$, $x_{21} = x_{22}$ and $x_{11} \neq x_{12}$. The general case follows from the argument in the proof of Corollary 3.13 (1).

Suppose (C, K) is the pair as above corresponding to L. Then the above claim implies $\partial L = K$. Let B be the v-branch corresponding to (C, K). Then $\partial B = \Delta(K) = \Delta(\partial L)$. Now we prove $\Delta(\overline{L}) \subset \overline{B}$. Let $\ell \subset \overline{L}$ be a standard geodesic. If $d(\Delta(\ell), v) \geq 2$, by Lemma 3.6, there exists standard geodesic ℓ_1 such that $\Delta(\ell_1)$ and $\Delta(\ell)$ are in the same v-branch and $\ell_1 \cap P_v \neq \emptyset$. The argument in Lemma 3.8 implies that there exists an edge path $\omega \subset X(\Gamma) \setminus P_v$ connecting a vertex in ℓ and a vertex in ℓ_1, thus

$$\ell_1 \subset \overline{L}.$$

It follows that $\ell_1 \cap K \neq \emptyset$ and $\pi(\Delta(\ell_1)) \subset C$, so

$$\Delta(\ell_1) \in B$$

by Corollary 3.13 (1). Hence

$$\Delta(\ell) \in B.$$

If $d(\Delta(\ell), v) = 1$, since $\overline{L} \cap P_v = K$, we apply Lemma 2.4 (4) with $C_1 = \overline{L}$ and $C_2 = P_v$ to deduce that ℓ stays in the R-neighborhood of K for some $R > 0$, thus

$$\Delta(\ell) \in \Delta(K) = \partial B.$$

Note that $\Delta(\ell) \in \overline{B}$ in both cases, so $\Delta(\overline{L}) \subset \overline{B}$. Now we prove $\overline{B} \subset \Delta(\overline{L})$. Pick vertex $w \in \overline{B}$. If $w \in \partial B$, then we are done by $\partial B = \Delta(\partial L) \subset \Delta(\overline{L})$. Suppose $w \in B$. Pick an edge $e \subset X(\Gamma)$ which connects a point in L and a point outside L, and let ℓ_e be the standard geodesic containing e. Then $\ell_e \subset \overline{L}$ by the discussion in the previous paragraph. Then

$$\Delta(\ell_e) \in \overline{L} \subset \overline{B}.$$

However, $\Delta(\ell_e) \notin \partial B$, hence

$$\Delta(\ell_e) \in B.$$

The argument in Lemma 3.8 implies that there exists an edge path outside P_v connecting a vertex in ℓ_e and a vertex in P_w. Thus $w \in \Delta(\overline{L})$. In summary, each vertex of $\overline{B}$ is in $\Delta(\overline{L})$. Since $\overline{L}$ is convex, $\Delta(\overline{L})$ is a full subcomplex by Lemma 4.4, then $\overline{B} \subset \Delta(\overline{L})$.

To see (2), let $\{\Delta_\lambda\}_{\lambda \in \Lambda}$ be the collection of maximal simplexes in $\mathcal{P}(\Gamma)$ such that $\Delta_\lambda \cap B \neq \emptyset$, and let $\{F_\lambda\}_{\lambda \in \Lambda}$ be the collection of maximal standard flats such that $\Delta(F_\lambda) = \Delta_\lambda$. We claim

$$d_H\left(L, \bigcup_{\lambda \in \Lambda} F_\lambda\right) < \infty,$$

where d_H denotes the Hausdorff distance. Note that $\Delta_\lambda \subset \bar{B}$, hence $F_\lambda \subset \bar{L}$ by assertion (1) and the maximality of F_λ. Pick an arbitrary vertex $x \in L$, and let ℓ_x be a standard geodesic such that $d(\pi(\Delta(\ell_x)), \bar{v}) \geq 2$ and $x \in \ell_x$. Then

$$d(\Delta(\ell_x), v) \geq 2.$$

Hence $\ell_x \cap P_v$ is at most one point. It follows from the proof of assertion (1) that $\ell_x \subset \bar{L}$ and $\Delta(\ell_x) \subset B$. Thus there exists $\lambda_0 \in \Lambda$ such that

$$x \in \ell_x \subset F_{\lambda_0}.$$

Thus L is contained in some neighborhood of $\bigcup_{\lambda \in \Lambda} F_\lambda$. However,

$$d_H(L, \bar{L}) < \infty$$

by Lemma 4.5, hence the claim follows. Let $B' = q_*(B)$ and L' be the component of $X(\Gamma') \setminus P_{q_*(v)}$ corresponding to B' (note that Γ' is also of weak type II by Corollary 3.18). By Lemma 2.22, for each $\lambda \in \Lambda$, there exists a unique maximal standard flat $F'_\lambda \subset X(\Gamma')$ such that

$$d_H(q(F_\lambda), F'_\lambda) < C$$

(C is independent of λ). Note that $\{\Delta(F'_\lambda)\}_{\lambda \in \Lambda}$ is the collection of maximal simplexes of $\mathcal{P}(\Gamma)$ which have non-empty intersection with B'. We argue as before to deduce

$$d_H\Big(L', \bigcup_{\lambda \in \Lambda} F'_\lambda\Big) < \infty.$$

Then

$$d_H(q(L), L') < \infty.$$

Now we prove (3). By Lemma 4.1, $d_H(P_v, P_{q_*(v)}) < \infty$. Let

$$K' = P_{q_*(v)} \cap \bar{L}'.$$

Then K' is a $q_*(v)$-peripheral subcomplex by Lemma 4.5, hence is a standard subcomplex. Recall that $K = P_v \cap L$, so $d_H(q(K), K') < \infty$ by Lemma 2.4 (4). ∎

5. Rigidity and flexibility of RAAG of weak type I

5.1. Motivating discussion and overview

The goal of this section is to understand RAAGs that are quasi-isometric to a given RAAG of weak type I. We will start with a discussion of motivating examples.

We start with the case when Γ is a pentagon. Let $G(\Gamma')$ be a RAAG quasi-isometric to $G(\Gamma)$. This gives a simplicial isomorphism $\mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$ (cf. Theorem 2.25).

We can use this to conjugate the natural action of $G(\Gamma')$ on $\mathcal{P}(\Gamma')$ to another action $G(\Gamma') \curvearrowright \mathcal{P}(\Gamma)$. However, each automorphism of $\mathcal{P}(\Gamma)$ is visible, as for each vertex $v \in \mathcal{P}(\Gamma)$, each v-tier only contains one v-branches. This follows from Corollary 3.13 and Lemma 3.3. Thus each automorphism of $\mathcal{P}(\Gamma)$ gives a bijection of $G(\Gamma)$ that preserves maximal standard flats (recall that we identify $G(\Gamma)$ as the vertex set of $X(\Gamma)$, and we will refer maximal standard flats in $G(\Gamma)$ as the intersections of maximal standard flats in $X(\Gamma)$ with $G(\Gamma)$). As a special feature of the pentagon graph, we know each standard flat in $G(\Gamma)$ is an intersection of maximal standard flats. Thus each automorphism of $\mathcal{P}(\Gamma)$ gives a bijection of $G(\Gamma)$ that sends standard flats to standard flats. We refer to this kind of bijection as *flat-preserving bijections*. The action $G(\Gamma') \curvearrowright \mathcal{P}(\Gamma)$ gives an action $G(\Gamma') \curvearrowright G(\Gamma)$ by flat-preserving bijections. If each flat-preserving bijection of $G(\Gamma)$ were left translations of $G(\Gamma)$, then we can conclude immediately that $G(\Gamma')$ is isomorphic to a finite index subgroup of $G(\Gamma)$. However, this is not the case in general.

Recall that each standard line in $G(\Gamma)$ is a left coset of a standard $\mathbb{Z}$ subgroup. Thus each standard line is labeled by a generator of $G(\Gamma)$, and inherits an order from $\mathbb{Z}$. A flat-preserving bijection of $G(\Gamma)$ is a left translation if and only if it respects the order on each standard line, and respects the labels of standard lines. Thus it is too much to hope that the action $G(\Gamma') \curvearrowright G(\Gamma)$ is by left translations of $G(\Gamma)$. However, if we are able to find a different labeling and ordering of the standard lines of $G(\Gamma)$ such that both of them are invariant $G(\Gamma') \curvearrowright G(\Gamma)$, then the action of $G(\Gamma') \curvearrowright G(\Gamma)$ is conjugate to an action by left translations, which will imply $G(\Gamma')$ is a finite index subgroup of $G(\Gamma)$. The new labeling and ordering need to satisfy some natural consistency conditions for this to work, and the conjugation is via a flat-preserving bijection which connects the new labeling and ordering of standard lines to the old ones.

The way to produce $G(\Gamma')$-invariant labeling and ordering of standard lines of $G(\Gamma)$ is as follows. As the map $\mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$ coming from the quasi-isometry is visible, it gives a map $f : G(\Gamma) \to G(\Gamma')$ which is $G(\Gamma')$-invariant, where the action $G(\Gamma') \curvearrowright G(\Gamma')$ is by left translation. Very roughly speaking, we want to pull back the usual labeling and ordering of standard lines in $G(\Gamma')$ to obtain a $G(\Gamma')$-invariant labeling and ordering of standard lines of $G(\Gamma)$. However, this needs some work as f is not an injective map, so pulling back does not make sense immediately. This gives a rough summary of the strategy in [21] for handling the pentagon case.

Now we consider the general case of RAAGs of weak type I. The example to have in mind is Γ being a pentagon and a triangle glued along an edge. Any quasi-isometry $q : G(\Gamma) \to G(\Gamma')$ still induces a visible simplicial isomorphism $q_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$. Thus as before we have an action $\rho : G(\Gamma') \curvearrowright G(\Gamma)$. The key difference from the previous case is that the action is no longer by flat-preserving bijections. For an element $g' \in G(\Gamma')$, we still know $\rho(g')$ sends maximal standard flats to maximal standard flats. But standard flats which are not maximal might not be preserved, as they are not necessarily intersections of maximal standard flats. For example, let $\overline{v}$ be the vertex in Γ that is not inside the pentagon. Then $\rho(g')$ could send a standard line labeled by $\overline{v}$ to something which is not a standard line. More precisely, suppose the vertices of the triangle Γ_1 in Γ are $\{\overline{v}, \overline{v}_1, \overline{v}_2\}$.

Then $\rho(g')$ sends a standard flat F of type Γ_1 to another 3-dimensional standard flat (as this standard flat is maximal). Moreover, as standard lines of type $\bar{v}_1$ and $\bar{v}_2$ are intersections of maximal standard flats, they are sent to standard lines by $\rho(g')$. As a consequence, those 2-dimensional standard flats of type $\{\bar{v}_1, \bar{v}_2\}$ are also sent to standard flats by $\rho(g')$. We want to think F as being foliated by these 2-dimensional standard flats. The map $\rho(g')$ sends the leaves to parallel standard flats in another 3-dimensional standard flats; however, the $\rho(g')$-images of standard lines of type $\bar{v}$ in F (which are transverse to all the leaves) could be rather arbitrary. So we can think $\rho(g')|_F$ as a leave-preserving shearing map.

For more general Γ of weak type I, we will typically see that $\rho(g')$ preserves some standard flats, but there could also be different types of "shearing" of a family of lower-dimensional standard flats inside a bigger standard flat. Such shearing could happen whenever an adjacent transvection is possible (e.g., in the above example, there is an adjacent transvection sending $\bar{v}$ to $\bar{v}_1$ or $\bar{v}_2$), though the behavior of $\rho(g')$ is generally more complicated than adjacent transvections. Moreover, in general RAAGs of weak type I could allow many adjacent transvections.

The strategy in the case of pentagon no longer works, due to the failure of preservation of standard lines. In the case of weak type I, we will use an atlas system which encodes how various shearing is happening. More precisely, an atlas on $G(\Gamma)$ is a collection of bijections between stable (cf. Definition 2.19) standard flats in $G(\Gamma)$ and $\mathbb{Z}^n$ with suitable n such that these bijections satisfy some natural consistency condition (see Definition 5.2). This generalizes the pentagon case, as each standard line is stable in this case, and we will have bijections between standard lines and $\mathbb{Z}$s, which correspond to the order on standard lines as discussed before.

Now the main point is to construct an atlas on $G(\Gamma)$ which is invariant under the action of $G(\Gamma')$ (cf. Proposition 5.13). Once this is established, we can conclude that the action of $G(\Gamma')$ on $G(\Gamma)$ is conjugated to an action by left translation, which implies that $G(\Gamma')$ is a finite index subgroup of $G(\Gamma)$.

5.2. An atlas for RAAG

Let $G(\Gamma)$ be a RAAG of weak type I with trivial center. We identify $G(\Gamma)$ with the 1-skeleton of $X(\Gamma)$ and define a *standard flat* in $G(\Gamma)$ to be the vertex set of some standard flat in $X(\Gamma)$.

Theorem 3.21 implies there is a homomorphism $s : \mathrm{Aut}(P(\Gamma)) \to \mathrm{Perm}(G(\Gamma))$, where $\mathrm{Perm}(G(\Gamma))$ is the permutation group of elements in $G(\Gamma)$. Note that images of s preserve maximal standard flats. However, this may not be true for all standard flats, since adjacent transvections are allowed in $\mathrm{Out}(G(\Gamma))$.

Let $\mathcal{P}(\Gamma)$ be the extension complex, and let $\pi : \mathcal{P}(\Gamma) \to F(\Gamma)$ be label-preserving simplicial map defined in Section 2.3. Note that for any vertex $x \in X(\Gamma)$, π induces an isomorphism $(F(\Gamma))_x \to F(\Gamma)$. This motivates the following definition.

Definition 5.1 (Coherent labeling). A *coherent labeling* of $G(\Gamma)$ is a simplicial map $L : \mathcal{P}(\Gamma) \to F(\Gamma)$ such that L induces an isomorphism $(F(\Gamma))_x \to F(\Gamma)$ for every vertex $x \in X(\Gamma)$.

Assume $n = \dim(X(\Gamma))$. Let $\mathcal{F}(\Gamma)$ be the collection of stable standard flats in $X(\Gamma)$, and let $\mathcal{F}_k(\Gamma)$ be the collection of k-flats in $\mathcal{F}(\Gamma)$. Define $\mathcal{F}_{<k}(\Gamma) := \bigcup_{i=1}^{k-1} \mathcal{F}_i(\Gamma)$. Here we are considering the set itself, not the coarse equivalent classes of the sets (compared to Theorem 2.25). Recall that we use $v(K)$ to denote the set of vertices in a subset K of some polyhedral complex.

Definition 5.2 (L-atlas). An *L-atlas* is a coherent labeling $L : \mathcal{P}(\Gamma) \to F(\Gamma)$ together with a collection of bijections

$$\{v(F) \to \mathbb{Z}^{v(L(\Delta(F)))}\}_{F \in \mathcal{F}_k(\Gamma), 1 \le k \le n}$$

with the following compatibility condition: pick $F_1 \in \mathcal{F}_m(\Gamma)$ and $F_2 \in \mathcal{F}_\ell(\Gamma)$ with $F_1 \subset F_2$, let $f : v(F_2) \to \mathbb{Z}^{v(L(\Delta(F_2)))}$ and $g : v(F_1) \to \mathbb{Z}^{v(L(\Delta(F_1)))}$ be the associated bijections. Suppose $p : \mathbb{Z}^{v(L(\Delta(F_2)))} \to \mathbb{Z}^{v(L(\Delta(F_1)))}$ is the natural projection. Then

(1) $f(v(F_1))$ is a coset of $\mathbb{Z}^{v(L(\Delta(F_1)))}$ in $\mathbb{Z}^{v(L(\Delta(F_2)))}$.

(2) The following diagram commutes up to translation:

$$\begin{array}{ccc} v(F_1) & \xrightarrow{g} & \mathbb{Z}^{v(L(\Delta(F_1)))} \\ \downarrow i & & \uparrow p \\ v(F_2) & \xrightarrow{f} & \mathbb{Z}^{v(L(\Delta(F_2)))} \end{array}$$

Here i is the inclusion map.

Definition 5.3 (Equivalence and pullback). We say L-atlas $\mathcal{A}_L$ and L'-atlas $\mathcal{A}_{L'}$ are *equal up to translations* if $L = L'$ and the bijections in $\mathcal{A}_L$ and $\mathcal{A}_{L'}$ agree up to translation. We will write $\mathcal{A}_L \overset{e}{=} \mathcal{A}_{L'}$ in this case. Pick $\alpha \in \mathrm{Aut}(\mathcal{P}(\Gamma))$, and let $\alpha_* : G(\Gamma) \to G(\Gamma)$ be the bijection induced by α (cf. Section 2.6 and Theorem 3.23). Recall that α_* preserves stable standard flats. The *pullback* of an L-atlas $\mathcal{A}_L$ under α, denoted by $\alpha^*(\mathcal{A}_L)$, is defined to be the $(L \circ \alpha)$-atlas with its charts being the pullbacks of charts of $\mathcal{A}_L$ under α_*. More precisely, charts of $\alpha^*(\mathcal{A}_L)$ are compositions:

$$\{v(F) \xrightarrow{\alpha_*} \alpha_*(v(F)) \to \mathbb{Z}^{v(L(\Delta(\alpha_*(F))))} = \mathbb{Z}^{v(L \circ \alpha(\Delta(F)))}\}_{F \in \mathcal{F}_k(\Gamma), 1 \le k \le n}.$$

Note that $L(\Delta(\alpha_*(F)))$ and $L \circ \alpha(\Delta(F))$ are the same subset of $F(\Gamma)$.

Remark 5.4. Note that the construction of α_* from α in Section 2.6 only uses the information of what are α-images of maximal simplexes in $\mathcal{P}(\Gamma)$. Thus a priori it could happen that two different elements α_1 and α_2 in $\mathrm{Aut}(\mathcal{P}(\Gamma))$ give the same $\alpha_* : G(\Gamma) \to G(\Gamma)$. It is natural to ask how different is $\alpha_1^*(\mathcal{A}_L)$ from $\alpha_2^*(\mathcal{A}_L)$.

We now clarify this point via the following example. Suppose Γ is obtained by gluing a pentagon and the 1-skeleton of a 3-simplex along an edge. Let a and b be the two vertices of Γ which are outside the pentagon. Let e be an edge in $\mathcal{P}(\Gamma)$ which maps to the edge $\overline{ab} \subset \Gamma$ under the map $\mathcal{P}(\Gamma) \to F(\Gamma)$. We take $\alpha_1 \in \mathrm{Aut}(\mathcal{P}(\Gamma))$ to be the identity map. Take $\alpha_2 \in \mathrm{Aut}(\mathcal{P}(\Gamma))$ to be the automorphism which exchanges the two endpoints of e and fixes all other vertices of $\mathcal{P}(\Gamma)$ pointwise. Then $(\alpha_1)_* = (\alpha_2)_*$ is the identity map and $\alpha_1(\Delta(F)) = \alpha_2(\Delta(F)) = \Delta(F)$ for any $F \in \mathcal{F}(\Gamma)$. Thus all charts in $\alpha_1^*(\mathcal{A}_L)$ and $\alpha_2^*(\mathcal{A}_L)$ are exactly the same. However, $L \circ \alpha_1$ and $L \circ \alpha_2$ are different maps. So

$$\alpha_1^*(\mathcal{A}_L) \overset{e}{\neq} \alpha_2^*(\mathcal{A}_L).$$

Recall that we label each circle in $S(\Gamma)$ by a generator of $G(\Gamma)$ and fix an orientation for each circle. This lifts to $G(\Gamma)$-invariant labeling and orientation of edges in $X(\Gamma)$. Moreover, we have induced action $G(\Gamma) \curvearrowright \mathcal{P}(\Gamma)$ and induced $G(\Gamma)$-invariant labeling of vertices in $\mathcal{P}(\Gamma)$. This leads to a naturally defined L-atlas as follows.

Definition 5.5 (Reference atlas). Let L be the label-preserving map $\pi : \mathcal{P}(\Gamma) \to F(\Gamma)$. For each vertex $u \in \mathcal{P}(\Gamma)$, we pick a standard geodesic $\ell \subset X(\Gamma)$ such that $\Delta(\ell) = u$ and identify vertices of ℓ with $\mathbb{Z}^u$ in an orientation-preserving way. Let $p_u : G(\Gamma) \to \mathbb{Z}^u$ be the map induced by the CAT(0) projection from $G(\Gamma)$ to ℓ (recall that we have identified $G(\Gamma)$ with vertices of $X(\Gamma)$, and Lemma 2.4 implies that the image of each vertex of $X(\Gamma)$ under the CAT(0) projection is a vertex in ℓ). For each standard flat $F \subset X(\Gamma)$, $p_u(v(F))$ is surjective if $u \in \Delta(F)$, otherwise $p_u(v(F))$ is a point. This induces a bijection $\prod_{u\in\Delta(F)} p_u : v(F) \to \mathbb{Z}^{v(\Delta(F))}$, and we define the chart for F to be $\prod_{u\in\Delta(F)} p_u$ post-composed with $\mathbb{Z}^{v(\Delta(F))} \to \mathbb{Z}^{v(L(\Delta(F)))}$. One readily verifies that this atlas $\mathcal{A}_L$ satisfies the above definition of L-atlas; moreover, the diagram in (2) commutes exactly, not up to translations. The following properties are immediate:

(1) $\mathcal{A}_L$ is $G(\Gamma)$-invariant up to translations in the sense that $g^*(\mathcal{A}_L) \overset{e}{=} \mathcal{A}_L$ for all $g \in G(\Gamma)$. Conversely, if $\alpha \in \mathrm{Aut}(\mathcal{P}(\Gamma))$ satisfies $\alpha^*(\mathcal{A}_L) \overset{e}{=} \mathcal{A}_L$, then the induced map $\alpha_* : G(\Gamma) \to G(\Gamma)$ is a left translation.

(2) $\mathcal{A}_L$ is unique up to translations. Since the only ambiguity in the definition of $\mathcal{A}_L$ is the orientation-preserving identification of $v(\ell)$ with $\mathbb{Z}^u$, which is unique up to translations.

The atlas $\mathcal{A}_L$ is called the *reference* atlas.

Lemma 5.6. *Let $G(\Gamma)$ be of weak type I, and pick $F \in \mathcal{F}(\Gamma)$. Then there exist standard flats $\{F_i\}_{i=1}^k$ in F such that F is the convex hull of these flats and each F_i is the intersection of maximal standard flats.*

Proof. Pick vertex $w \in \Gamma$. Let Γ_w be the minimal stable subgraph containing w, and let Γ'_w be the intersection of maximal cliques that contains w. It suffices to show $\Gamma_w = \Gamma'_w$.

Since each maximal clique is stable (Lemmas 2.22 and 3.14), $\Gamma_w \subset \Gamma'_w$. Pick vertex $v \in \Gamma'_w$, then $w^\perp \subset \mathrm{St}(v)$, thus $v \in \Gamma_w$ by [21, Lemma 3.32]. It follows that $\Gamma'_w \subset \Gamma_w$. ■

Suppose $G(\Gamma)$ has weak type I, and it has trivial center. Let $q : G(\Gamma) \to G(\Gamma')$ be a quasi-isometry, and let $s : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$ be an induced simplicial isomorphism (cf. Definition 3.19 and Lemma 3.16). By Theorem 3.21 and Section 2.6, s induces a map $\phi : G(\Gamma) \to G(\Gamma')$.

Lemma 5.7. *There exists $D_0 > 0$ such that $d(q(x), \phi(x)) < D_0$ for any $x \in G(\Gamma)$. Thus ϕ is a quasi-isometry.*

Proof. By Lemma 2.22, Definition 3.19 and Lemma 3.14, each maximal clique subgraph of Γ is stable. Thus there exists $D_1 > 0$ such that for any maximal standard flat $F \subset X(\Gamma)$, there exists a maximal standard flat $F' \subset X(\Gamma')$ such that $d_H(q(F), F') < D_1$. Let $\{F_i\}_{i=1}^n$ and $\{F'_i\}_{i=1}^n$ be as in Section 2.6. Then $d\left(q\left(\bigcap_{i=1}^n F_i\right), \bigcap_{i=1}^n F'_i\right)$ is uniformly upper bounded, which implies the lemma. ■

For every $g' \in G(\Gamma')$, there is left translation $\bar{\phi}_{g'} : G(\Gamma') \to G(\Gamma')$, which gives rise to a simplicial isomorphism $\bar{s}_{g'} : \mathcal{P}(\Gamma') \to \mathcal{P}(\Gamma')$. Let

$$s_{g'} = s^{-1} \circ \bar{s}_{g'} \circ s.$$

Then $s_{g'}$ induces a unique bijection $\phi_{g'} : G(\Gamma) \to G(\Gamma)$ by Theorem 3.21; moreover,

$$\bar{\phi}_{g'} \circ \phi = \phi \circ \phi_{g'}. \tag{5.1}$$

In summary, we have $G(\Gamma')$ that acts on $G(\Gamma')$, $\mathcal{P}(\Gamma')$, $G(\Gamma)$ and $\mathcal{P}(\Gamma)$.

Lemma 5.8. *The map ϕ satisfies the following properties:*

(1) *The map ϕ is surjective. For any $y, y' \in G(\Gamma')$, $|\phi^{-1}(y)| = |\phi^{-1}(y')| < \infty$.*

(2) *For any k and $F \in \mathcal{F}_k(\Gamma)$, there is unique $F' \in \mathcal{F}_k(\Gamma')$ with $\phi(v(F)) = v(F')$. Moreover, let $\mathrm{Stab}(v(F'))$ and $\mathrm{Stab}(v(F))$ be the stabilizer of $v(F')$ and $v(F)$ with respect to the action $G(\Gamma') \curvearrowright G(\Gamma')$ and $G(\Gamma') \curvearrowright G(\Gamma)$, respectively. Then $\mathrm{Stab}(v(F')) = \mathrm{Stab}(v(F))$. In this case, we will write $F' = \phi(F)$ for simplicity.*

(3) *Let $F_1, F_2 \in \mathcal{F}_k(\Gamma)$ be parallel standard flats. Then for vertices $x_1 \in F_1$ and $x_2 \in F_2$,*
$$|\phi^{-1}(\phi(x_1)) \cap F_1| = |\phi^{-1}(\phi(x_2)) \cap F_2|.$$

Proof. Pick a reference point $q \in \mathrm{Im}\,\phi$, and let $K_q = (F(\Gamma'))_q$. Denote the points in $\phi^{-1}(q)$ by $\{p_\lambda\}_{\lambda\in\Lambda}$, and let $K_{p_\lambda} = (F(\Gamma))_{p_\lambda}$. Since $\{\phi(K_{p_\lambda})\}_{\lambda\in\Lambda}$ are distinct subcomplexes of K_q, Λ is a finite set. The other parts of (1) follow from (5.1).

Now we prove (2). It is clear if F is a maximal standard flat. Next we look at the case when $F = \bigcap_{i=1}^h F_i$, where each F_i is a maximal standard flat. Let F'_i be maximal standard flat in $X(\Gamma')$ such that $\Delta(F'_i) = s(\Delta(F_i))$ for $1 \le i \le h$, and let $F' = \bigcap_{i=1}^{h} F'_i$. Then

$$\phi(v(F)) \subset v(F').$$

Note that $\mathrm{Stab}(v(F'))$ (resp. $\mathrm{Stab}(v(F'_i))$) is a conjugate of a standard subgroup of type $\Gamma'_{F'}$ (resp. $\Gamma'_{F'_i}$). Thus

$$\mathrm{Stab}(v(F')) \subset \mathrm{Stab}(v(F_i)).$$

Then the stabilizer $\mathrm{Stab}(v(F'))$ fixes $\Delta(F'_i)$ for all i, hence it fixes Δ_i for all i and $\mathrm{Stab}(v(F')) \subset \mathrm{Stab}(v(F))$. Since $\mathrm{Stab}(v(F'))$ acts on $v(F')$ transitively, (5.1) implies

$$\phi(v(F)) = v(F')$$

and

$$\mathrm{Stab}(v(F)) \subset \mathrm{Stab}(v(F')).$$

Thus $\mathrm{Stab}(v(F')) = \mathrm{Stab}(v(F))$.

In general, by Lemma 5.6, we can assume F is the convex hull of $F_1, F_2 \in \mathcal{F}(\Gamma)$ such that (2) is true for flats in F which are parallel to F_1 or F_2. Let $F'_i = \phi(F_i)$ for $i = 1, 2$. Then $F'_1 \cap F'_2 \neq \emptyset$ and the convex hull of F'_1 and F'_2 is a flat F' (since F is contained in a maximal standard flat, whose image under ϕ is a maximal standard flat containing F'_1 and F'_2). It follows from Lemma 2.21 that $F' \in \mathcal{F}(\Gamma')$. Note that any standard flat that is parallel to F_1 and intersects F_2 is mapped by ϕ to a standard flat that is parallel to F'_1 and intersects F'_2, thus $\phi(v(F)) \subset v(F')$. Let $F_3 \subset F$ be a standard flat parallel to F_1, and let $F'_3 = \phi(F_3)$. Since parallel standard flats in $X(\Gamma')$ have the same stabilizer, we have

$$\mathrm{Stab}(v(F_1)) = \mathrm{Stab}(v(F'_1)) = \mathrm{Stab}(v(F'_3)) = \mathrm{Stab}(v(F_3)).$$

By considering all such F_3's in F, we have

$$\mathrm{Stab}(v(F_1)) \subset \mathrm{Stab}(v(F)).$$

Similarly, $\mathrm{Stab}(v(F_2)) \subset \mathrm{Stab}(v(F))$, thus

$$\begin{aligned}\mathrm{Stab}(v(F')) &= \langle \mathrm{Stab}(v(F'_1)), \mathrm{Stab}(v(F'_2)) \rangle \\ &= \langle \mathrm{Stab}(v(F_1)), \mathrm{Stab}(v(F_2)) \rangle \subset \mathrm{Stab}(v(F)).\end{aligned}$$

Now we can conclude $\phi(v(F)) = v(F')$ as before. It also follows that $\mathrm{Stab}(v(F)) \subset \mathrm{Stab}(v(F'))$, thus

$$\mathrm{Stab}(v(F')) = \mathrm{Stab}(v(F)). \tag{5.2}$$

Now we prove (3). Note that for a pair of parallel standard flats F'_1 and F'_2 in $X(\Gamma')$, there exists $g' \in G(\Gamma')$ such that $g'(v(F'_1)) = v(F'_2)$, so by (5.1), it suffices to prove (3) in the case where

$$\phi(v(F_1)) = \phi(v(F_2)) = v(F').$$

Note that F_1 and F_2 are parallel, and there is an isometry $p : F_1 \to F_2$ induced by the nearest point projection in the ambient CAT(0) cube complex. The map p sends vertices to vertices, hence restricts to a bijection $p : v(F_1) \to v(F_2)$, which we call the *parallelism map* between $v(F_1)$ and $v(F_2)$. Denote

$$p_1 = \phi|_{v(F_1)} : v(F_1) \to v(F')$$

and

$$p_2 = \phi|_{v(F_2)} \circ p : v(F_1) \to v(F').$$

Then there exist L and A such that p_1 and p_2 are (L, A)-quasi-isometries and

$$d(p_1(x), p_2(x)) < A \tag{5.3}$$

for any $x \in v(F_1)$. Pick $y \in v(F')$, and let r_i be the number of points $|p_i^{-1}(y)|$ in $p_i^{-1}(y)$ for $i = 1, 2$ (r_i does not depend on y by previous discussion). We argue by contradiction and assume $r_1 < r_2$. Pick base point $x_0 \in v(F_1)$, let $m = \dim(F_1)$,

$$B_R = B(x_0, R)$$

and

$$K_{i,R} = p_i(B_R)$$

for $i = 1, 2$. Then it follows from (5.3) that

$$\begin{aligned}|K_{1,R}| &\le |N_A(K_{2,R})| = |K_{2,R}| + |N_A(K_{2,R}) \setminus K_{2,R}| \\ &\le |K_{2,R}| + |p_2^{-1}(N_A(K_{2,R}) \setminus K_{2,R})| \le |K_{2,R}| + |B_{LA+A+R} \setminus B_R| \\ &\le |K_{2,R}| + CR^{m-1}(LA + A),\end{aligned}$$

where C is some constant independent of R. Thus by symmetry, we have

$$||K_{1,R}| - |K_{2,R}|| \le CR^{m-1}(LA + A). \tag{5.4}$$

On the other hand, $B_R \subset p_i^{-1}(K_{i,R}) \subset B_{R+A}$ for $i = 1, 2$, thus

$$CR^m \le |p_i^{-1}(K_{i,R})| = r_i|K_{i,R}| \le C(R + A)^m \tag{5.5}$$

for $i = 1, 2$. Now (5.4) and (5.5) imply

$$\begin{aligned}CR^m/r_1 - C(R + A)^m/r_2 &\le |K_{1,R}| - |K_{2,R}| \\ &\le ||K_{1,R}| - |K_{2,R}|| \le CR^{m-1}(LA + A).\end{aligned}$$

Since $r_1 < r_2$, we will get a contradiction when $R \to \infty$. ∎

Lemma 5.9. *Suppose $G(\Gamma)$ has weak type I and trivial center. Given L-atlas $\mathcal{A}_L$ and L'-atlas $\mathcal{A}_{L'}$, there exists $\alpha \in \mathrm{Aut}(\mathcal{P}(\Gamma))$ such that $\alpha^*(\mathcal{A}_{L'}) \overset{e}{=} \mathcal{A}_L$.*

The proof is a variation of [21, Lemma 5.7].

Proof. We prove the first part of the lemma. Pick $v \in G(\Gamma)$, set $\alpha'(e) = v$. For $u \in G(\Gamma)$, pick a word $w_u = a_1 a_2 \cdots a_n$ representing u, let $u_i = a_1 a_2 \cdots a_i$ for $1 \leq i \leq n$ and $u_0 = e$. We define $q_i = \alpha'(a_1 a_2 \cdots a_i) \in G(\Gamma)$ inductively as follows: set $q_0 = v$, and suppose $q_{i-1} = \alpha'(a_1 a_2 \cdots a_{i-1})$ is already defined. Let $F_i \in \mathcal{F}(\Gamma)$ be a standard flat containing u_{i-1} and u_i, and let F'_i be the unique standard flat such that $q_{i-1} \in F'_i$ and

$$L'(\Delta(F'_i)) = L(\Delta(F_i)).$$

There is a natural identification of $g_i : F_i \to F'_i$ via the charts

$$f : F_i \to \mathbb{Z}^{v(L(\Delta(F_i)))}$$

and

$$f' : F'_i \to \mathbb{Z}^{v(L'(\Delta(F'_i)))} = \mathbb{Z}^{v(L(\Delta(F_i)))}$$

such that $g_i = (f')^{-1} \circ f$. Up to post-composing f and f' by translations, we can assume $f_i(u_{i-1}) = q_{i-1}$. Then we define $q_i = f_i(u_i)$. Note that the definition of $q_i = \alpha'(a_1 a_2 \cdots a_i)$ does not depend on the choice of F_i by the compatibility condition (2).

We now claim for any other word w'_u representing u, $\alpha'(w_u) = \alpha'(w'_u)$, hence we have a well-defined map $\alpha' : G(\Gamma) \to G(\Gamma)$. To see this, recall that one can obtain w_u from w'_u by performing the following two basic moves:

(1) $w_1 a a^{-1} w_2 \to w_1 w_2$.

(2) $w_1 a b w_2 \to w_1 b a w_2$ when a and b commute.

It is clear that $\alpha'(w_1 a a^{-1} w_2) = \alpha'(w_1 w_2)$, and it suffices to show $\alpha'(ab) = \alpha'(ba)$, where a and b are mutually commuting generators. Let F be a maximal standard flat that contains e, a and b; we could always choose F in the definition of $\alpha'(ab)$ or $\alpha'(ba)$, thus they are equal.

By switching the role of $\mathcal{A}_L$ and $\mathcal{A}_{L'}$, we can define $\alpha'' : G(\Gamma) \to G(\Gamma)$ which maps v to e in a similar way. It is not hard to check if α' and α'' are inverses of each other. Thus α' is bijective; moreover, α' preserves $\mathcal{F}(\Gamma)$. To define α, pick vertex $w \in \mathcal{P}(\Gamma)$, let Δ be a maximal simplex containing w. Take $F \subset X(\Gamma)$ to be the flat such that $\Delta(F) = \Delta$, and take F' to be the maximal standard flat such that $\alpha(v(F)) = v(F')$; we set $\alpha(w)$ to be the unique point such that $\alpha(w) \in \Delta(F')$ and $L(w) = L'(\alpha(w))$. One readily verifies that $L = L' \circ \alpha$, α' is induced by α, and α' pulls back the charts up to translations, so $\alpha^*(\mathcal{A}_{L'}) \overset{e}{=} \mathcal{A}_L$. ■

5.3. Shearing standard flats

In this subsection, we prove the following theorem.

Theorem 5.10. *Let $G(\Gamma)$ be a group of weak type I. Then the following are equivalent:*

(1) *$G(\Gamma')$ is quasi-isometric to $G(\Gamma)$.*

(2) *$G(\Gamma')$ is isomorphic to a finite index subgroup of $G(\Gamma)$.*

(3) *$G(\Gamma')$ is isomorphic to a special subgroup (Section 2.4) of $G(\Gamma)$.*

Remark 5.11. From Theorem 5.10, we know in particular that a finite index RAAG subgroup H of $G(\Gamma)$ is isomorphic to a special subgroup. However, H might not be a special subgroup of $G(\Gamma)$. Interestingly, under the strong condition that $\mathrm{Out}(G(\Gamma))$ is finite, any finite index RAAG subgroup will automatically be a special subgroup [21, Section 6].

The following is a consequence of Theorem 5.10 and [21, Section 6.3].

Corollary 5.12. *Let $G(\Gamma)$ be a group of weak type I. Then there is an algorithm to determine whether $G(\Gamma')$ and $G(\Gamma)$ are quasi-isometric.*

In the rest of this subsection, we prove Theorem 5.10. Note that it suffices to prove the case when $G(\Gamma)$ has trivial center. Thus from now on, $G(\Gamma)$ is a RAAG of weak type I with trivial center. Let $\mathcal{A}_{L'}$ be the reference atlas for $G(\Gamma')$. Let $q, s, s_{g'}, \overline{s}_{g'}, \phi, \phi_{g'}$ and $\overline{\phi}_{g'}$ be as in the discussion before Lemma 5.8. We will also be using actions of $G(\Gamma')$ on $\mathcal{P}(\Gamma')$, $G(\Gamma')$, $\mathcal{P}(\Gamma)$ and $G(\Gamma)$ discussed over there. A main ingredient of the proof is the following.

Proposition 5.13. *Under the aforementioned setting, there exists a coherent labeling $L : \mathcal{P}(\Gamma) \to F(\Gamma)$ for $G(\Gamma)$ which is invariant under the action $G(\Gamma') \curvearrowright \mathcal{P}(\Gamma)$ and an L-atlas $\overline{\mathcal{A}}_L$ for $G(\Gamma)$ such that*

(1) *$\overline{\mathcal{A}}_L \overset{e}{=} (\phi_{g'})^*(\overline{\mathcal{A}}_L)$ for any $g' \in G(\Gamma')$.*

(2) *(Inverse images are boxes) Given $F \in \mathcal{F}(\Gamma)$, let $F' = \phi(F)$, and let*

$$s_0 : v(L(\Delta(F))) \to v(L'(\Delta(F')))$$

be the bijection induced by s. Suppose $\overline{h}$ and h' are charts for F and F' with respect to $\overline{\mathcal{A}}_L$ and $\mathcal{A}_{L'}$, respectively. Then $\varphi = h' \circ \phi \circ \overline{h}$ admits splitting

$$\varphi = \prod_{w \in v(L(\Delta(F)))} \varphi_w,$$

where

$$\varphi_w : \mathbb{Z}^w \to \mathbb{Z}^{s_0(w)}$$

(where $\mathbb{Z}^w$ denotes the copy of $\mathbb{Z}$ associated with vertex w) is of form

$$\varphi_w(a) = \lfloor a/d_w \rfloor + r_w$$

for some integers r_w and d_w ($d_w > 0$).

Now we prove Theorem 5.10, assuming Proposition 5.13.

Proof of Theorem 5.10. (3) $\Rightarrow$ (1) and (2) $\Rightarrow$ (1) are trivial. Now we look at (1) $\Rightarrow$ (2). By Theorem 2.18, we can assume $G(\Gamma)$ has trivial center. Let $\mathcal{A}_{\text{ref}}$ be the reference atlas for $G(\Gamma)$, and let $\bar{\mathcal{A}}_L$ be as in Proposition 5.13. By Lemma 5.9, there exists simplicial isomorphism $r : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ such that $r^*(\bar{\mathcal{A}}_L) \overset{e}{=} \mathcal{A}_{\text{ref}}$. The $G(\Gamma')$-invariance of $\bar{\mathcal{A}}_L$ implies

$$(r^{-1} \circ s_{g'} \circ r)^*(\mathcal{A}_{\text{ref}}) \overset{e}{=} \mathcal{A}_{\text{ref}}.$$

Let $\phi_r : G(\Gamma) \to G(\Gamma)$ be the map induced by r. Then by Definition 5.5 (1), $\phi_r^{-1} \circ \phi_{g'} \circ \phi_r$ is a left translation of $G(\Gamma)$. Hence we have obtained a faithful action of $G(\Gamma')$ on $G(\Gamma)$ via left translations with finitely many orbits. Thus (2) follows.

(1) $\Rightarrow$ (3). Since the atlas $\bar{A}_L$ satisfies Proposition 5.13 (2), we deduce that for $F \in \mathcal{F}(\Gamma)$, there is $F' \in \mathcal{F}(\Gamma')$ such that the map $\phi \circ \phi_r|_{v(F)} : v(F) \to v(F')$ is surjective, and it maps a collection of boxes in $v(F)$ to single points in $v(F')$. As $\text{Stab}(F')$ acts transitively on $v(F')$, it follows from Proposition 5.13 (1) that all these boxes have the same dimension. As each standard flat is contained in a stable standard flat, we know from Proposition 5.13 (2) that $\phi \circ \phi_r$ sends standard flats to standard flats. Moreover,

- if two elements of $G(\Gamma)$ are adjacent in $X(\Gamma)$, then their $\phi \circ \phi_r$-images are either the same or adjacent;
- given a pair of parallel edges $e_1, e_2 \subset X(\Gamma)$ is parallel, if the $\phi \circ \phi_r(\partial e_1)$ is a single point (∂e_1 denotes the collection of two endpoints of e_1), then the same holds for $\phi \circ \phi_r(\partial e_2)$; if the $\phi \circ \phi_r(\partial e_1) = \partial e_1'$ for an edge $e_1' \subset X(\Gamma')$, then $\phi \circ \phi_r(\partial e_2) = \partial e_2'$ for an edge $e_2' \subset X(\Gamma')$ with e_2' parallel to e_1'.

Now it is clear that $\phi \circ \phi_r$ extends to a cubical map $X(\Gamma) \to X(\Gamma')$. The inverse image of a vertex under this cubical map is a compact subcomplex by Lemma 5.8 (1). Note that $\phi \circ \phi_r$ is $G(\Gamma')$-equivariant, where the $G(\Gamma')$ action on $X(\Gamma)$ is given by $g' \to \phi_r^{-1} \circ \phi_{g'} \circ \phi_r$. As ϕ gives a 1-1 correspondence between maximal standard flats in $X(\Gamma)$ and maximal standard flats in $X(\Gamma')$, and ϕ_r gives a bijection on the set of maximal standard flats in $X(\Gamma)$, we know $\phi \circ \phi_r$ induces an isomorphism of the associated extension complexes in the sense explained after Theorem 2.16. Then Lemma 2.17 implies $G(\Gamma')$ is isomorphic to a special subgroup (note that Lemma 2.17 (2) follows from Proposition 5.13 (2)). ■

In the rest of this subsection, we prove Proposition 5.13. We first arrange the coherent labeling L.

Lemma 5.14. *There exists a coherent labeling $L : \mathcal{P}(\Gamma) \to F(\Gamma)$ for $G(\Gamma)$ which is invariant under the action $G(\Gamma') \curvearrowright \mathcal{P}(\Gamma)$.*

This proof of this lemma is part of the proof of Lemma 5.9 of [21]. We extract here for the convenience of the reader.

Proof. Recall that each vertex of $\mathcal{P}(\Gamma)$ is labeled by a vertex in Γ (cf. Section 2.3), which induces a simplicial map $L_0 : \mathcal{P}(\Gamma) \to F(\Gamma)$. Similarly, we define $L_0' : \mathcal{P}(\Gamma') \to F(\Gamma')$.

Take $g' \in G(\Gamma)$, and let $i_{g'} : F(\Gamma') \to \mathcal{P}(\Gamma')$ be the embedding as in Lemma 2.14. Define

$$L = L_0 \circ s^{-1} \circ i_{g'} \circ L_0' \circ s,$$

which is a simplicial map from $\mathcal{P}(\Gamma)$ to $F(\Gamma)$. Pick arbitrary $g \in G(\Gamma)$. We need to show $L \circ i_g$ is a simplicial isomorphism. Let $K_g = i_g(F(\Gamma))$, and let $g_1' \in G(\Gamma')$ such that $g_1' \cdot \phi(g) = g'$. Then $i_{g'} \circ L_0'|_{s(K_g)} = \bar{s}_{g_1'}|_{s(K_g)}$. Thus

$$L \circ i_g = L_0 \circ s^{-1} \circ i_{g'} \circ L_0' \circ s \circ i_g = L_0 \circ s^{-1} \circ \bar{s}_{g_1'} \circ s \circ i_g = L_0 \circ s_{g_1'} \circ i_g,$$

which is a simplicial isomorphism by Lemma 2.5. It follows that L is a coherent labeling; moreover,

$$\begin{aligned}(s_{g'})^* L &= (L_0 \circ s^{-1} \circ i_q \circ L_0' \circ s) \circ (s^{-1} \circ \bar{s}_{g'} \circ s) = L_0 \circ s^{-1} \circ i_q \circ L_0' \circ \bar{s}_{g'} \circ s \\ &= L_0 \circ s^{-1} \circ i_q \circ L_0' \circ s = L\end{aligned}$$

for any $g' \in G(\Gamma')$, where the third equality follows from $G(\Gamma')$-invariance of L_0'. So L is the required coherent labeling. ∎

It remains to construct an L-atlas $\bar{\mathcal{A}}_L$ for $G(\Gamma)$ satisfying all the requirements. This is the main part of the proof of Proposition 5.13. Note that in light of Lemma 5.14, for verifying Proposition 5.13 (1), it suffices to show charts of $\bar{\mathcal{A}}_L$ are $G(\Gamma')$-invariant up to translations. We will construct $\bar{\mathcal{A}}_L$ by induction on the dimension of charts.

By induction, we assume the charts are already defined for standard flats in $\mathcal{F}(\Gamma)$ of dimension $\leq k-1$ such that the following inductive assumptions hold true:

(1) The charts are compatible and $G(\Gamma')$-invariant up to translations.

(2) (Inverse images are boxes) Given $F \in \mathcal{F}_{<k}(\Gamma)$, let $F' = \phi(F)$, and let

$$s_0 : v(L(\Delta(F))) \to v(L'(\Delta(F')))$$

be the bijection induced by s. Suppose $\bar{h}$ and h' are charts for F and F', respectively. Then $\varphi = h' \circ \phi \circ \bar{h}$ admits splitting

$$\varphi = \prod_{w \in v(L(\Delta(F)))} \varphi_w,$$

where

$$\varphi_w : \mathbb{Z}^w \to \mathbb{Z}^{s_0(w)}$$

is of form

$$\varphi_w(a) = \lfloor a/d_w \rfloor + r_w \quad (a \in \mathbb{Z}^w)$$

for some integers r_w and d_w ($d_w > 0$).

(3) (Extension condition) For $F_1, F_2 \in \mathcal{F}_{<k}(\Gamma)$ such that

$$\phi(v(F_1)) = \phi(v(F_2)) = v(F'),$$

there is a bijection $f : v(F_1) \to v(F_2)$ such that

(a) $\phi(x) = \phi \circ f(x)$ for any $x \in v(F_1)$.

(b) $\bar{h}_2 \circ f \circ \bar{h}_1^{-1}$ is a translation (where $\bar{h}_i : v(F_i) \to \mathbb{Z}^{v(L(\Delta(F_i)))}$ are charts).

(c) Let $F \in \mathcal{F}(\Gamma)$ such that $F_1 \cap F \neq \emptyset$, $F_2 \cap F \neq \emptyset$ and the convex hull of F_1 and F is a flat. Then $f(v(F_1 \cap F)) = v(F_2 \cap F)$.

Remark 5.15. Note that only requiring condition (1) is not enough since the compatibility of existing charts does not imply that we can add more charts in a compatible way to obtain an atlas, thus we need condition (3).

For $i = 1, 2$, let $\varphi_i = h' \circ \phi \circ \bar{h}_i^{-1}$. Then (2) and (3) imply that $\varphi_1^{-1}(y)$ and $\varphi_2^{-1}(y)$ are boxes of the same size for any $y \in \mathbb{Z}^{v(L(\Delta(F')))}$. Thus (3a) and (3b) uniquely determine the map f, and we call f a *chart-induced identification* (CII) between $v(F_1)$ and $v(F_2)$.

In order to define charts for standard flats in $\mathcal{F}_{\leq k}(\Gamma)$, we need a way to "connect" parallel copies of lower-dimensional standard flats in a k-dimensional stable standard flat. For this purpose, we will define CII between parallel standard flats in $\mathcal{F}_{\leq k-1}(\Gamma)$ and prove some basic properties via a sequence of lemmas (Lemmas 5.16–5.20). We will define charts after these preparatory lemmas.

Lemma 5.16. *The map f is* Stab$(v(F'))$*-equivariant.*

Proof. Recall that $\mathrm{Stab}(v(F')) = \mathrm{Stab}(v(F_1)) = \mathrm{Stab}(v(F_2))$ by Lemma 5.8. By (1), the induced action of $\mathrm{Stab}(v(F'))$ on the range of $\bar{h}_1$ (or $\bar{h}_2$) is an action by translations; moreover, this action is completely determined by the size of the box $\varphi_1^{-1}(y)$ (or $\varphi_2^{-1}(y)$). It follows from (3a) and (3b) that $\bar{h}_2 \circ f \circ \bar{h}_1^{-1}$ is $\mathrm{Stab}(v(F'))$-equivariant. Then the lemma follows. ■

Let $F_1, F_2 \in \mathcal{F}_{<k}(\Gamma)$ be two parallel elements. If $F_1' = \phi(F_1)$ and $F_2' = \phi(F_1)$, then F_1' and F_2' are parallel. Let $p : v(F_1') \to v(F_2')$ be the map induced by parallelism, that is, p sends a vertex in F_1' to the nearest point in F_2' (which is a vertex) with respect to the metric on the ambient CAT(0) cube complex. Let g' be the unique element in $G(\Gamma')$ such that $\bar{\phi}_{g'}|_{v(F_1')} = p$. Suppose $F_{21} = \phi_{g'}(F_1)$. Then

$$\phi(F_{21}) = \phi(F_2)$$

by (5.1). We define the *chart-induced identification* (CII)

$$f : v(F_1) \to v(F_2)$$

between $v(F_1)$ and $v(F_2)$ by

$$f = f_1 \circ \phi_{g'},$$

where f_1 is the CII between $v(F_{21})$ and $v(F_2)$.

Lemma 5.17. *The CII map f is* Stab$(v(F_1))$*-equivariant.*

Proof. Note that $\mathrm{Stab}(v(F_1)) = \mathrm{Stab}(v(F_2)) = \mathrm{Stab}(v(F_1')) = \mathrm{Stab}(v(F_2'))$. Since g' commutes with any element in $\mathrm{Stab}(v(F_1))$, $\phi_{g'}$ is $\mathrm{Stab}(v(F_1))$-equivariant. According to Lemma 5.16, f_1 is $\mathrm{Stab}(v(F_1))$-equivariant. Thus the lemma follows. ∎

Lemma 5.18. *The following properties of f are true:*

(1) *The map f satisfies all the properties in inductive assumption (3) with (3a) replaced by $\bar{\phi}_{g'} \circ \phi = \phi \circ f$.*

(2) *The map f is uniquely characterized by the following two properties:*
- *$\bar{h}_2 \circ f \circ \bar{h}_1^{-1}$ is a translation.*
- *$\phi \circ f(x) = p \circ \phi(x)$ for any $x \in v(F_1)$, where $p : v(F_1') \to v(F_2')$ is the parallelism map.*

Proof. We prove the first assertion of the lemma. Conditions (3a) and (3b) follow from inductive assumption (1). It suffices to check (3c). Let $F' = \phi(F)$. Then the convex hull of F' and F_1' (or F_2') is also a flat, thus $\bar{\phi}_{g'}(v(F')) = v(F')$, and (5.2) implies $\phi_{g'}(v(F)) = v(F)$. It follows that

$$\phi_{g'}(v(F \cap F_1)) = v(F \cap F_{21}).$$

Note that F_{21} and F_2 are in the convex hull of F_1 and F, thus

$$f_1(v(F \cap F_{21})) = v(F \cap F_2)$$

by (3c), which implies

$$f(v(F \cap F_1)) = v(F \cap F_2).$$

The second assertion of the lemma follows from the first assertion and inductive assumption (2). ∎

Lemma 5.19. *Let $\{F_i\}_{i=1}^4 \subset \mathcal{F}_{<k}(\Gamma)$ such that F_1, F_2 and F_3 are parallel. Suppose f_{ij} is the CII between $v(F_i)$ and $v(F_j)$ and $\bar{h}_i$ is the chart for F_i. Then*

(1) *$f_{13} = f_{23} \circ f_{12}$.*

(2) *If $F_4 \subset F_1$, then $f_{12}(v(F_4))$ is the vertex set of some standard flat and $f_{12}|_{F_4}$ is the CII between $v(F_4)$ and $f_{12}(v(F_4))$.*

(3) *If $F_i \subset F_4$ for $i = 1, 2$, then f_{12} coincides with the map induced by parallelism between $\bar{h}_4(v(F_1))$ and $\bar{h}_4(v(F_2))$ in $\mathbb{Z}^{v(L(\Delta(F_4)))}$ (note: for the definition of parallelism map, we treat $\mathbb{Z}^{v(L(\Delta(F_4)))}$ as an integer lattice in a Euclidean space and send a point in $\bar{h}_4(v(F_1))$ to the nearest point in $\bar{h}_4(v(F_2))$ with respect to the Euclidean metric).*

(4) *CIIs are $G(\Gamma')$-invariant. Namely, for any $g' \in G(\Gamma')$, the CII between $\phi_{g'}(F_1)$ and $\phi_{g'}(F_2)$ is given by $\phi_{g'} \circ f_{12} \circ \phi_{g'}^{-1}$.*

Proof. The first assertion is a consequence of Lemma 5.18 (2). To see (2), by the compatibility of charts, there is a 1-1 correspondence between standard flats in F_2 that are parallel to F_4 and cosets of $\mathbb{Z}^{v(L(\Delta(F_4)))}$ in $\mathbb{Z}^{v(L(\Delta(F_2)))}$, but $\bar{h}_2(f_{12}(v(F_4)))$ is such a coset by the first item of Lemma 5.18 (2), thus $f_{12}(v(F_4))$ is the vertex set of some standard flat. Lemma 5.18 (2) also implies that $f_{12}|_{F_4}$ is a CII. Assertion (3) follows from inductive assumption (2) and Lemma 5.18 (2). Assertion (4) follows from inductive assumption (1) and Lemma 5.18 (2). ∎

Take $F_1, F \in \mathcal{F}(\Gamma)$ with $\dim(F_1) < k$ and $F_1 \subset F$, and define a map $\pi_1 : v(F) \to v(F_1)$ as follows. For standard flat $K \subset F$ with K parallel to F_1, we set $\pi|_{v(K)} = f_K$, where $f_K : v(K) \to v(F_1)$ is the CII between $v(K)$ and $v(F_1)$. We call π_1 a *chart-induced projection* (CIP).

Lemma 5.20. *Let $F_1' = \phi(F_1)$ and $F' = \phi(F)$. Suppose F_2' is an orthogonal complement of F_1' in F' and $\bar{h}_1$ is the chart for F_1. Then*

(1) *$\pi_1 \circ \bar{h}_1$ is* Stab$(v(F_2'))$*-invariant and* Stab$(v(F_1'))$*-invariant up to translation, hence is* Stab$(v(F'))$*-invariant up to translation.*

(2) *Pick $F_3 \in \mathcal{F}(\Gamma)$ such that $F_3 \subset F_1$. Let $\pi_3 : v(F) \to v(F_3)$ and $\pi_{13} : v(F_1) \to v(F_3)$ be CIPs. Then $\pi_3 = \pi_{13} \circ \pi_1$.*

(3) *Assume* $\dim(F) < k$*, and let $\bar{h}$ be the chart for F. Then π_1 coincides with the map induced by the natural projection from $\bar{h}(F)$ to $\bar{h}(F_1)$ in $\mathbb{Z}^{v(L(\Delta(F)))}$.*

(4) *Suppose π_1' is the orthogonal projection $v(F') \to v(F_1')$. Then*

$$\phi \circ \pi_1(x) = \pi_1' \circ \phi(x)$$

for any $x \in v(F)$.

(5) *Let $F_3 \in \mathcal{F}(\Gamma)$ be a standard flat in F. Then there exists stable standard flat $F_4 \in F_1$ such that $\pi_1(v(F_3)) = v(F_4)$ (F_4 could be a point). Moreover, let $\pi_4 : F \to F_4$ be the CIP. Then*

$$\pi_1|_{v(F_3)} = \pi_4|_{v(F_3)}.$$

Proof. To see (1), note that any element in Stab$(v(F_2'))$ maps F_1' to a flat parallel to F_1', and this map is exactly the parallelism map. It follows from Lemma 5.18 (2) that $\pi_1 \circ \bar{h}_1$ is Stab$(v(F_2'))$-invariant. Lemma 5.17 and inductive assumption (1) imply $\pi_1 \circ \bar{h}_1$ is Stab$(v(F_1'))$-invariant up to translation. Assertion (2) follows from assertion (1) and Lemma 5.19 (2). Assertion (3) follows from Lemma 5.19 (3). Assertion (4) follows from Lemma 5.18 (2). To see (5), we first assume $F_3 \cap F_1 \neq \emptyset$ and take $F_4 = F_1 \cap F_3$, then

$$\pi_1(v(F_3)) = v(F_4)$$

by Lemma 5.18 (1). In general, we pick a standard flat $\widetilde{F}_1$ parallel to F_1 such that $\widetilde{F}_1 \cap F_3 \neq \emptyset$. Let $f_1 : \widetilde{F}_1 \to F_1$ be the CII, and let $\tilde{\pi}_1 : F \to \widetilde{F}_1$ be the CIP. Then

$f_1 \circ \widetilde{\pi}_1 = \pi_1$, which reduces the problem to the previous case. The second assertion in (5) follows from (2). ■

We will construct charts for elements in $\mathcal{F}_k(\Gamma)$ in three steps.

Step 1: We construct charts for a single element in $\boldsymbol{\mathcal{F}_k(\Gamma)}$. Pick a standard k-flat $F \in \mathcal{F}(\Gamma)$ and vertex $p \in F$. Let F_m be the convex hull of all standard flats that are properly contained in F, pass through p and belong to $\mathcal{F}(\Gamma)$. Then $F_m \in \mathcal{F}(\Gamma)$ by Lemma 2.21. We divide the construction into three cases depending on the size of F_m. In each case, we will construct a chart $\overline{h} : v(F) \to \mathbb{Z}^{v(L(\Delta(F)))}$ for F and shall verify

(1) $\overline{h}$ is compatible with charts of elements in $\mathcal{F}_{<k}(\Gamma)$.

(2) $\overline{h}$ is Stab$(v(F'))$-invariant up to translation.

(3) Inductive assumption (2) holds for $\overline{h}$.

Case 1. The flat F_m is a point. Let $F' = \phi(F)$, and let

$$h' : v(F') \to \mathbb{Z}^{v(L'(\Delta(F')))}$$

be the chart for F'. Define

$$h = h' \circ \phi : v(F) \to \mathbb{Z}^{v(L'(\Delta(F')))}.$$

We assign an arbitrary bijection between $v(L'(\Delta(F')))$ and $\{1, 2, \ldots, k\}$ with

$$k = |v(L'(\Delta(F')))|,$$

which leads to an identification of $\mathbb{Z}^{v(L'(\Delta(F')))}$ with $\mathbb{Z}^k$. As h is a map from $v(F)$ to $\mathbb{Z}^k$, we can write

$$h = (h_1, h_2, \ldots, h_k),$$

where each h_i is a coordinate component of h. Denote the identity element in $\mathbb{Z}^{v(L'(\Delta(F')))}$ by $\mathbf{0}$, and let $r = |h^{-1}(\mathbf{0})|$. Since elements in $h^{-1}(\mathbf{0})$ are representatives of the orbits of the action Stab$(v(F')) \curvearrowright v(F)$, there is a natural map $v(F) \to h^{-1}(\mathbf{0})$. By post-composing this map with a bijection

$$h^{-1}(\mathbf{0}) \to \{0, 1, \ldots, r-1\},$$

we obtain a Stab$(v(F'))$-invariant map

$$\chi : v(F) \to \{0, 1, \ldots, r-1\}.$$

Now define

$$\widetilde{h} : v(F) \to \mathbb{Z}^{v(L'(\Delta(F')))}$$

by sending $x \in v(F)$ to

$$(rh_1(x) + \chi(x), h_2(x), \ldots, h_k(x)),$$

then $\widetilde{h}$ is a bijection, and we have the following commutative diagram:

$$\begin{array}{ccccc}
 & & v(F) & \xrightarrow{\quad\phi\quad} & v(F') \\
 & \overline{h}\swarrow & \downarrow \widetilde{h} & \searrow h & \downarrow h' \\
\mathbb{Z}^{v(L(\Delta(F)))} & \xrightarrow{\quad s'\quad} & \mathbb{Z}^{v(L'(\Delta(F')))} & \xrightarrow{\quad\phi'\quad} & \mathbb{Z}^{v(L'(\Delta(F')))}
\end{array}$$

Here ϕ' is the map induced by ϕ, s' is the bijection induced by $s : \Delta(F) \to \Delta(F')$ and $\overline{h} = s'^{-1} \circ \widetilde{h}$. By construction, $\overline{h}$ is $\mathrm{Stab}(v(F'))$-invariant up to translation and satisfies inductive assumption (2). We choose $\overline{h}$ to be the chart for F, which is trivially compatible with the charts already defined.

Case 2. $p \subsetneq F_m \subsetneq F$. Let F' and h be as before, and let $F'_m = \phi(F_m)$. Suppose F_c (or F'_c) is a standard flat which is the orthogonal complement of F_m (or F'_m) in F (or F'). Then we have the following commuting diagram:

$$\begin{array}{ccc}
v(F) & \xrightarrow{\quad h\quad} & \mathbb{Z}^{v(L'(\Delta(F')))} \\
\downarrow \pi & & \downarrow \pi'_c \\
v(F_c) & \xrightarrow{\quad h_c\quad} & \mathbb{Z}^{v(L'(\Delta(F'_c)))}
\end{array}$$

Here π and π'_c are the natural projections. Note that h maps fibers of π to fibers of π'_c, which induces h_c. The action of $\mathrm{Stab}(v(F'_c))$ permutes the fibers of π, which induces an action

$$\mathrm{Stab}(v(F'_c)) \curvearrowright v(F_c).$$

As in Case 1, we can obtain from h_c a bijection

$$\overline{h}_c : v(F_c) \to \mathbb{Z}^{v(L(\Delta(F_c)))}$$

which is $\mathrm{Stab}(v(F'_c))$-invariant up to translation. Then $\overline{h}_c \circ \pi$ is $\mathrm{Stab}(v(F'_m))$-invariant (since $\mathrm{Stab}(v(F'_m))$ stabilizes each fiber of π by (5.2)) and $\mathrm{Stab}(v(F'_c))$-invariant up to translation.

Let $\overline{h}_m$ be the composition

$$v(F) \to v(F_m) \to \mathbb{Z}^{v(L(\Delta(F_m)))}$$

of a CIP with a chart map. Then the map $\overline{h}_m$ is $\mathrm{Stab}(v(F'_m))$-invariant up to translation and $\mathrm{Stab}(v(F'_c))$-invariant by (1) of Lemma 5.20. Now we identify $\mathbb{Z}^{v(L(\Delta(F_m)))}$ and $\mathbb{Z}^{v(L(\Delta(F_c)))}$ as subgroups of $\mathbb{Z}^{v(L(\Delta(F)))}$ and define

$$\overline{h} : v(F) \to \mathbb{Z}^{v(L(\Delta(F)))}$$

by

$$\overline{h} = \overline{h}_c \circ \pi + \overline{h}_m.$$

It is clear that the bijection $\overline{h}$ is $\mathrm{Stab}(v(F'))$-invariant up to translation. We choose $\overline{h}$ to be the chart for F, and the compatibility follows from our construction.

Let s_0 and φ be as in inductive assumption (2). We claim cosets of $\mathbb{Z}^w$ are mapped to cosets of $\mathbb{Z}^{s_0(w)}$ under φ for any $w \in v(L(\Delta(F)))$. If $w \in v(L(\Delta(F_m)))$, then the claim follows from Lemma 5.18 (2) and inductive assumption (2) for F_m. If $w \in v(L(\Delta(F_c)))$, then by Lemma 5.20 (4), $\overline{h}_m$ maps $\mathbb{Z}^w$-cosets to points. The claim follows from the construction of $\overline{h}_c$. Thus φ splits into products and $\overline{h}$ satisfies inductive assumption (2).

Case 3. $F_m = F$. Then there exist standard flats $F_1, F_2 \in \mathcal{F}_{<k}(\Gamma)$ such that F is the convex hull of F_1 and F_2 (a basic case to bear in mind is that $F_1 \cap F_2$ is a single point). Let $F_3 = F_1 \cap F_2$. Suppose $F' = \phi(F)$ and $F'_i = \phi(F_i)$. Take $\overline{h}_i$ to be the charts for F_i for $1 \leq i \leq 3$, and take $\overline{\pi}_i : F \to F_i$ to be the CIP for $1 \leq i \leq 3$. Define

$$\overline{h} : F \to \mathbb{Z}^{v(L(\Delta(F)))}$$

by

$$\overline{h} = \overline{h}_1 \circ \overline{\pi}_1 + \overline{h}_2 \circ \overline{\pi}_2 - \overline{h}_3 \circ \overline{\pi}_3.$$

Then by Lemma 5.20 (1), $\overline{h}$ is Stab($v(F')$)-invariant up to translation.

Lemma 5.21. *The following hold true:*

(1) *The map $\overline{h}$ is a bijection.*

(2) *The map $\overline{h}$ is compatible with charts of elements in $\mathcal{F}_{<k}(\Gamma)$.*

(3) *The map $\overline{h}$ satisfies inductive assumption (2).*

Proof. For assertion (1), note that $\overline{h}$ induces a bijection between standard flats in F which are parallel to F_3 and cosets of $\mathbb{Z}^{v(L(\Delta(F_3)))}$ in $\mathbb{Z}^{v(L(\Delta(F)))}$. It suffices to show for any standard flat $\widetilde{F}_3 \subset F$ parallel to F_3, $\overline{h}$ maps $\widetilde{F}_3$ bijectively to a coset of $\mathbb{Z}^{v(L(\Delta(F_3)))}$. Note that by Lemma 5.18 (2), if we change the standard flats F_1 and F_2 in the definition of $\overline{h}$ to some other flats parallel to them, then $\overline{h}$ would differ by translation, thus we can assume $\widetilde{F}_3 = F_3$. But $\overline{h}$ restricted to F_3 is of form $\overline{h}_1 + \overline{h}_2 - \overline{h}_3$, so what we need to prove is implied by the compatibility condition.

Now we prove assertion (2). Let $F_4 \subset F$ be an element in $\mathcal{F}_{<k}(\Gamma)$, and let $\overline{h}_4$ be its chart. We can assume $F_i \cap F_4 \neq \emptyset$ by moving F_1 and F_2 appropriately as before. For $1 \leq i \leq 3$, let $F_{4i} = F_4 \cap F_i$, let $\overline{\pi}_{4i} : F \to F_{4i}$ be the CIP and let $\overline{h}_{4i}$ be the chart for F_{4i}. By (5) of Lemma 5.20, $\pi_i(F_4) = F_{4i}$ for $1 \leq i \leq 3$ and

$$\overline{h} = \overline{h}_1 \circ \overline{\pi}_{41} + \overline{h}_2 \circ \overline{\pi}_{42} - \overline{h}_3 \circ \overline{\pi}_{43}$$

when restricted on F_4. On the other hand, (3) of Lemma 5.20 and the compatibility condition imply

$$\overline{h}_4 = \overline{h}_{41} \circ \overline{\pi}_{41} + \overline{h}_{42} \circ \overline{\pi}_{42} - \overline{h}_{43} \circ \overline{\pi}_{43}$$

up to translation. Now the compatibility of $\overline{h}_4$ and $\overline{h}$ follows from the compatibility of $\overline{h}_i$ and $\overline{h}_{4i}$ $(1 \leq i \leq 3)$.

For assertion (3), it suffices to show $\overline{h}$ restricted on each $\mathbb{Z}$-coset has the desired property. Let $w \in v(L(\Delta(F)))$, and let K be a $\mathbb{Z}^w$ coset. Then K is contained in either a

$\mathbb{Z}^{v(L(\Delta(F_1)))}$ coset or a $\mathbb{Z}^{v(L(\Delta(F_2)))}$ coset. Since $\bar{h}$ is compatible with other charts, there exists a standard flat $F_5 \subset F$ which is parallel to either F_1 or F_2 such that $K \subset \bar{h}(v(F_5))$. Moreover, if $\bar{h}_5$ is the chart for F_5, then $\bar{h}_5 \circ \bar{h}^{-1}(K)$ is again a $\mathbb{Z}^w$ coset. By applying the inductive assumption to $\bar{h}_5$, we know $\bar{h}|_K$ has the desired behavior. ■

Lemma 5.22. *If we choose different F_1 and F_2 in the definition of $\bar{h}$, then the resulting chart remains the same up to translation.*

Proof. Let $\bar{h}$ be the chart defined above using F_1 and F_2. The lemma follows if we know that for any $F_3 \in \mathcal{F}_{<k}(\Gamma)$, the CIP from $v(F)$ to $v(F_3)$ coincides with the map induced by the natural projection from $\bar{h}(F)$ to $\bar{h}(F_3)$ in $\mathbb{Z}^{v(L(\Delta(F)))}$. To show this property of CIP holds, it suffices to show that for any pair of parallel elements $F_4, F_5 \subset F$ in $\mathcal{F}_{<k}(\Gamma)$, the CII between $v(F_4)$ and $v(F_5)$ coincides with the map induced by parallelism between $\bar{h}(v(F_4))$ and $\bar{h}(v(F_5))$ in $\mathbb{Z}^{v(L(\Delta(F)))}$. This can be proved in the same way as Lemma 5.19 (3), using Lemma 5.21 (2). ■

Step 2: We construct charts for flats with the same ϕ-image as F. We define a graph $\Lambda(F)$. Its vertices are in 1-1 correspondence to standard flats that have the same ϕ-image as F and two vertices are joined by an edge if and only if the corresponding flats are *bolted*, defined as follows.

Definition 5.23. Two parallel elements $H_1, H_2 \in \mathcal{F}(\Gamma)$ are *bolted* if there is $H \in \mathcal{F}(\Gamma)$ such that for $i = 1, 2$, $H \cap H_i \neq \emptyset$, $H \cap H_i \subsetneq H_i$ and the convex hull of H and H_1 is a flat. The standard flat H is called an (H_1, H_2)-*bolt*; we will omit (H_1, H_2) when they are clear.

The main goal of Step 2 is the following.

Lemma 5.24. *Let $F' = \phi(F)$. There exists a collection of charts, one for each vertex in $\Lambda(F)$ such that*

(1) *Each chart is compatible with charts of elements in $\mathcal{F}_{<k}(\Gamma)$.*

(2) *Each chart is* $\mathrm{Stab}(v(F'))$*-invariant up to translation.*

(3) *Inductive assumption (2) holds for each chart.*

(4) *Inductive assumption (3) is satisfied for each pair of charts.*

We choose a representative in each connected component of $\Lambda(F)$ (the representative in the component containing F is chosen to be F), which gives a collection $\{F_\lambda\}_{\lambda\in\Lambda}$. We build a chart $\bar{h}_\lambda$ for each F_λ as in Step 1. By Step 1, Lemma 5.24 (1)–(3) hold for each $\bar{h}_\lambda$. Now we arrange Lemma 5.24 (4) for each pair of charts in $\{\bar{h}_\lambda\}_{\lambda\in\Lambda}$.

Lemma 5.25. *Take $\{H_1, H_2\} \subset \{F_\lambda\}_{\lambda\in\Lambda}$ with their charts $\bar{h}_1$ and $\bar{h}_2$. Then for any $y \in v(F')$, $(\phi \circ \bar{h}_1^{-1})^{-1}(y)$ and $(\phi \circ \bar{h}_2^{-1})^{-1}(y)$ are boxes of the same dimension.*

Proof. Note that H_1 and H_2 must be in the same case of Step 1, so the lemma follows from Lemma 5.8 (3) in Case 1 of Step 1. In Case 2 of Step 1, the lemma is a consequence of Lemma 5.8 (3), and the following observation follows from Lemma 5.18 (2): for any two parallel standard flats $F_3, F_4 \subset \mathcal{F}_{k-1}(\Gamma)$ with charts $\bar{h}_3$ and $\bar{h}_4$, $(\phi \circ \bar{h}_3^{-1})^{-1}(y_3)$ and $(\phi \circ \bar{h}_4^{-1})^{-1}(y_4)$ are boxes of the same dimension where y_i is a point in the image of $\phi \circ \bar{h}_i^{-1}$ for $i = 3, 4$. In Case 3 of Step 1, by Lemma 5.22 (up to a translation) the definition of $\bar{h}_i$ does not depend on the choice of the pair of stable flats in H_i. So we can choose them such that they are parallel to $F_1, F_2 \subset F$ as in Case 3 of Step 1. This, together with the previous observation, implies the lemma. ■

By Lemma 5.25, there exists a unique identification $f : v(H_1) \to v(H_2)$ characterized by (3a) and (3b) of the inductive assumption. Assumption (3c) is trivially true for f, and f is $\mathrm{Stab}(v(F'))$-equivariant since both $\bar{h}_1$ and $\bar{h}_2$ are $\mathrm{Stab}(v(F'))$-invariant up to translation. Thus Lemma 5.24 (4) holds for each pair of charts in $\{\bar{h}_\lambda\}_{\lambda\in\Lambda}$.

It remains to define charts for flats inside one connected component of $\Lambda(F)$, so we assume now $\Lambda(F)$ is connected.

Lemma 5.26. *There is a collection of bijections between each pair of flats in $\Lambda(F)$, which is also called CIIs such that*

(1) *These CIIs are compatible under compositions.*

(2) *Each CII is $\mathrm{Stab}(v(F'))$-equivariant and satisfies inductive assumptions (3a) and (3c).*

(3) *Let $f : H_1 \to H_2$ be a CII between flats H_1 and H_2 in $\Lambda(F)$. Suppose $S_1 \in \mathcal{F}_{<k}(\Gamma)$ be a standard flat in H_1. Then there exists $S_2 \in \mathcal{F}_{<k}(\Gamma)$ parallel to S_1 such that $f(v(S_1)) = v(S_2)$ and $f|_{v(S_1)}$ is the CII between $v(S_1)$ and $v(S_2)$.*

Assuming Lemma 5.26, we can finish the proof of Lemma 5.24 as follows. For any flat H in $\Lambda(F)$, we define the chart of H to be the composition of the CII between H and F, and the chart map of F. This chart satisfies inductive assumption (2) since F also satisfies this condition and the CII satisfies (3a). Recall that the chart of F is compatible with the charts for flats in $\mathcal{F}_{<k}(\Gamma)$, so is the chart of H by Lemma 5.26 (3). Moreover, this chart is $\mathrm{Stab}(v(F'))$-invariant up to translation by Lemma 5.26 (2). Under such definition of charts, the CII between F and another flat in $\Lambda(F)$ trivially satisfies inductive assumption (3b), hence the CII between any two flats in $\Lambda(F)$ satisfies inductive assumption (3b) by Lemma 5.26 (1).

Proof of Lemma 5.26. We will again follow the three cases of how large is F_m in F as in Step 1. In Case 1, we define the CII between any two flats in $\Lambda(F)$ to be the map induced by parallelism, then (1) of Lemma 5.26 is true. Let F_1 and F_2 be a pair of bolted flats, and let H be a bolt. Suppose $f_{12} : v(F_1) \to v(F_2)$ is the CII. Then for $i = 1, 2$, $H \cap F_i$

must be one point, and we denote it by p_i. It is clear that $f_{12}(p_1) = p_2$, thus inductive assumption (3c) follows. Moreover,

$$\begin{aligned}\phi(p_1) &= \phi(v(H) \cap v(F_1)) = \phi(v(H)) \cap \phi(v(F_1)) = \phi(v(H)) \cap \phi(v(F_2)) \\ &= \phi(v(H) \cap v(F_2)) = \phi(p_2) = \phi \circ f_{12}(p_1).\end{aligned}$$

The second and fourth equalities follow from Lemma 5.8 (2). Thus (3a) is true for bolted pair of flats. By moving the bolt H around using the action of $\mathrm{Stab}(F')$, we know f_{12} is $\mathrm{Stab}(F')$-equivariant. The connectivity of $\Lambda(F)$ implies that (3a) and the equivariance are true for all pairs of flats in $\Lambda(F)$. This finishes Case 1.

We need the following terminology before Case 2. Let H be a standard flat. An *H-fiber* is a standard flat parallel to H. Let $H_1, H_2 \in \mathcal{F}(\Gamma)$ be parallel elements that contain H-fibers, and let $p : v(H_1) \to v(H_2)$ be the map induced by parallelism. We say a bijection $f : v(H_1) \to v(H_2)$ is *parallel mod H-fibers* if $f(v(H')) = p(v(H'))$ for any H-fiber H'. For standard flat $S_i \in H_i$, we will write $f(S_1) = S_2$ if $f(v(S_1)) = v(S_2)$.

In Case 2, for flats H_1 and H_2 in $\Lambda(F)$, we define the CII $f : v(H_1) \to v(H_2)$ such that f is parallel mod F_m-fibers, and for each F_m-fiber $T \subset H_1$, $f|_{v(T)}$ is the CII between $v(T)$ and $f(v(T))$. Lemma 5.26 (1) follows from parallelism and Lemma 5.19 (1) for CIIs between F_m-fibers. Let F_1, F_2, f_{12} and H be as in Case 1. Then for $i = 1, 2$, there exist F_m-fibers $F_{im} \subset F_i$ such that $F_i \cap H \subset F_{im}$. Note that

$$f_{12}(v(F_{1m})) = v(F_{2m}),$$

and H is also a bolt for F_{1m} and F_{2m} when $F_1 \cap H \subsetneq F_{1m}$, thus inductive assumption (3c) follows. By Lemma 2.21, we can assume $H \cap F_i$ is actually an F_m-fiber for $i = 1, 2$, then the argument in the previous case implies that the image of any F_m-fiber in F_1 under ϕ and $\phi \circ f_{12}$ is the same. Then (3a) follows since we already know it is true for CIIs between F_m-fibers. The $\mathrm{Stab}(F')$-equivariance follows by applying Lemma 5.19 (4) to CIIs between F_m-fibers. Note that any element of $\mathcal{F}_{<k}(\Gamma)$ that lies in F_1 must stay inside an F_m-fiber, then Lemma 5.26 (3) follows from Lemma 5.19 (2).

In Case 3, let H_1 and H_2 be a bolted pair in $\Lambda(F)$. Pick a vertex $p_0 \in H_1$, and let H be the intersection of all (H_1, H_2)-bolts that contains p_0. Then H is also a bolt. We define the CII

$$f : v(H_1) \to v(H_2)$$

as in Case 2 with F_m-fibers replaced by $H \cap H_1$-fibers. The inductive assumption (3c) for f follows from the minimality of H, and we can prove (3a) and the $\mathrm{Stab}(v(F'))$-equivariance as before.

Now we prove Lemma 5.26 (3) for f. It is clear that S_1 stays inside an $H \cap H_1$-fiber. In general, pick an $H \cap H_1$-fiber T_1 such that

$$T_1 \cap S_1 = S_{11} \neq \emptyset$$

and a standard flat S_{12} which is an orthogonal complement of S_{11} in S_1. Since f is parallel mod T_1-fibers, $f(v(S_1))$ belongs to a $(T_1 \times S_{12})$-fiber R. Suppose $S_{21} = f(S_{11})$ and $T_2 = f(T_1)$. Let $\pi_i : H_i \to T_i$ be the CIP for $i = 1, 2$. Then

$$\pi_1(v(S_1)) = S_{11}$$

by Lemma 5.20 (5), hence

$$\pi_2(f(v(S_1))) = S_{21}$$

by Lemma 5.19 (1). But every two T_1-fibers in R are bolted by S_1-fibers, then the CII between these two T_1-fibers is parallel mod S_{11}-fibers, which implies $f(v(S_1))$ actually stays inside an $(S_{11} \times S_{12})$-fiber. To see the second part of Lemma 5.26 (3), note that S_1 and S_2 are bolted by $H \cap H_1$-fibers, then the CII between them is parallel mod S_{11}-fibers by inductive assumption (3c). Thus the CII coincides with f by Lemma 5.19 (2).

For arbitrary pair H_1 and H_2 in $\Lambda(F)$, we choose an edge path in $\Lambda(F)$ connecting H_1 and H_2, which would induce a CII from H_1 to H_2. This CII will automatically satisfy $\mathrm{Stab}(v(F'))$-equivariance, inductive assumption (3a) and Lemma 5.26 (3), since these properties are true under compositions. For this CII to be well defined, we need to show every edge loop in $\Lambda(F)$ induces the identity map. Let F be a base point in the edge loop, and let

$$f : v(F) \to v(F)$$

be the bijection induced by the edge loop. Pick $F_1, F_2 \in \mathcal{F}_{<k}$ inside F such that their convex hull is F, then it follows from (3c) that for $i = 1, 2$, every CII between two F_i-fibers in F is parallel mod $F_1 \cap F_2$-fibers. We first assume $F_1 \cap F_2$ is a point. By previous discussion, f maps F_i-fiber to F_i-fiber, thus f splits into product $f = f_1 \times f_2$, where $f_i : F_i \to F_i$ are bijections. Moreover, if $g : f(v(F_1)) \to v(F_1)$ is the CII, then

$$g \circ f|_{v(F_1)} = \mathrm{Id}$$

by (1) of Lemma 5.19, thus $f|_{v(F_1)}$ is induced by parallelism and $f_2 = \mathrm{Id}$. Similarly, we can prove $f_1 = \mathrm{Id}$, thus $f = \mathrm{Id}$. In general, we can run the same argument mod $F_1 \cap F_2$-fibers to show that f sends every $F_1 \cap F_2$-fiber to itself, then $f = \mathrm{Id}$ follows by applying Lemma 5.19 (1) to $F_1 \cap F_2$-fibers. ∎

Step 3: We define charts for any element in $\mathcal{F}_k(\Gamma)$. Let F and $F' = \phi(F)$ be as in the previous steps. Let H be an element in $\mathcal{F}_k(\Gamma)$ such that $\phi(H)$ is in the $G(\Gamma')$-orbit of F'. Note that this is equivalent to $L'(\Delta(\phi(H))) = L'(\Delta(F'))$. Pick $g' \in G(\Gamma')$ with $\bar{\phi}_{g'}(\phi(H)) = F'$, then $\phi_{g'}(H)$ is an element in $\Lambda(F)$. We define the chart of H to be the composition of the chart map of $\phi_{g'}(H)$ and $\underline{\phi}_{g'}$. If we choose a different g', the resulting chart would differ by a translation, since $\bar{h}$ is $\mathrm{Stab}(v(F'))$-invariant up to translation. By (5.1), this chart satisfies inductive assumption (2). Moreover, it is compatible with charts of elements in $\mathcal{F}_{<k}(\Gamma)$, since these charts are $G(\Gamma')$-invariant up to translations, and they are compatible with charts of flats in $\Lambda(F)$ by the previous step.

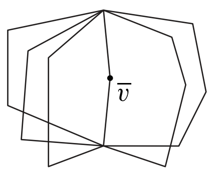
By now, we have defined a $G(\Gamma')$-invariant (up to translations) collection of charts for flats that are $G(\Gamma')$ orbits of flats in $\Lambda(F)$. This collection corresponds to a stable clique of k vertices in Γ', namely the 1-skeleton of $L'(\Delta(F'))$. For each stable k-clique in Γ', we run the same argument to define charts for the corresponding collection of k-flats in $G(\Gamma)$. This gives rise to charts defined for all elements in $\mathcal{F}_k(\Gamma)$ that satisfy all the requirements, hence finishes the induction step. In summary, we have constructed a $G(\Gamma')$-invariant (up to translations) L-atlas $\bar{\mathcal{A}}_L$ such that inductive assumption (2) is true for all charts in this atlas. This finishes the proof of Proposition 5.13.

6. Shuffling tiers and branches

6.1. Motivating discussion and overview

Let $G(\Gamma_1)$ be a RAAG of type II. The goal of this section is to understand the class of RAAGs that are quasi-isometric to $G(\Gamma_1)$. The first motivating example to consider is Γ_1 being a pentagon. In this case, it is known before that any RAAG quasi-isometric to $G(\Gamma_1)$ is isomorphic to a special subgroup of $G(\Gamma_1)$ in the sense of Section 2.4 [21]. A key point in the proof is that any quasi-isometry $q : G(\Gamma_1) \to G(\Gamma_2)$ induces a simplicial isomorphism $\alpha : \mathcal{P}(\Gamma_1) \to \mathcal{P}(\Gamma_2)$ that is visible, as for each vertex $v \in \mathcal{P}(\Gamma_1)$, each v-tier only contains one v-branches. This follows from Corollary 3.13 and Lemma 3.3.

Here is a more interesting example. Suppose $\Lambda_{m,n}$ is obtained by gluing m copies of pentagon and n copies of hexagon along a common closed vertex star. Let $\bar{v}$ be the central vertex of the common vertex star (see the figure below for $\Lambda_{3,2}$). Let $v \in \mathcal{P}(\Gamma)$ be a lift of $\bar{v}$. In this case, each v-tier contains $(m+n)$ v-branches, corresponding to the $(m+n)$ connected components of $\Gamma_1 \setminus \mathrm{St}(\bar{v})$ (cf. Corollary 3.13). Thus a simplicial isomorphism $\alpha : \mathcal{P}(\Gamma_1) \to \mathcal{P}(\Gamma_2)$ does not necessarily send a v-tier to an $\alpha(v)$-tier. However, the hope is that if we post-compose α by a suitable permutation of the $\alpha(v)$-tiers, then it might be possible for modified α to send v-tiers to $\alpha(v)$-tiers, and we could still obtain a visible map.



However, there is an obstruction for this. For example, $G(\Lambda_{2,4})$ is an index 2 special subgroup of $G(\Lambda_{1,2})$ (Section 2.4). If α goes from $\mathcal{P}(\Lambda_{1,2})$ to $\mathcal{P}(\Lambda_{2,4})$, then it is possible to arrange as above such that it sends a v-tier to an $\alpha(v)$-tier. But this is impossible if the domain and range of α are exchanged, simply because in the domain, a v-tier has 6 v-branches, but in range an $\alpha(v)$-tier has 3 v-branches.

This is not the only obstruction. It turns out that as long as in the domain of α, m and n are co-prime, then we can always modify α such that it preserves tiers. We generalize this to define a subclass of type II RAAGs, called prime RAAGs. At this point, we would avoid giving the full details of the definition of prime and simply say that it involves symmetries of v-branches in the sense of Definition 6.1. Some of these symmetries of $\mathcal{P}(\Gamma)$ are normal and expected, and they can be piecewisely defined using the action of $G(\Gamma)$ on $\mathcal{P}(\Gamma)$ (as the map q and q_* in the proof of Lemma 6.3). However, there are other unexpected permutation of branches, which is the base for the definition of prime RAAGs.

The first step of the proof is to show that if two prime RAAGs are quasi-isometric, then they are isomorphic. This is done in Section 6.2.

The second step is to show that any RAAG of type II can be realized as a special subgroup of a prime RAAG. Putting these two steps together, we obtain the main theorem that any RAAG which is quasi-isometric to a RAAG of type II is commensurable with this RAAG. The second step is more involved and takes Sections 6.3–6.5.

Recall from Section 2.4 that if $G(\Gamma)$ were a special subgroup of $G(\Gamma')$, then Γ' is a subgraph of Γ and Γ is obtained by gluing multiple copies of Γ' in a very specific way, and the gluing pattern is encoded in a compact CAT(0) cube complex. More precisely, there is a compact convex subcomplex $K \subset X(\Gamma')$ such that $F(\Gamma)$ is isomorphic to the full subcomplex of $\mathcal{P}(\Gamma)$ spanned by vertices that correspond to standard geodesic lines with non-trivial intersection with K. In particular, $F(\Gamma)$ is a union of subcomplexes of form $(F(\Gamma'))_x$ (the notation $(F(\Gamma'))_x$ is defined before Lemma 2.14), where x ranges over all vertices of K. Each of these subcomplexes is isomorphic to $F(\Gamma')$.

Suppose $G(\Gamma)$ is a RAAG of type II such that it is not prime. This ensures that the branches of $\mathcal{P}(\Gamma)$ have certain kind of unexpected symmetry (as remarked before). The end game is how these symmetries of $\mathcal{P}(\Gamma)$ actually imply that Γ has a very specific structure explained in the previous paragraph. This is done in three sub-steps. First we use the extra symmetries on $\mathcal{P}(\Gamma)$ to build a wall space structure on $F(\Gamma)$. The dual cube complex to this wall space will be the candidate compact cube complex as in the previous paragraph. Moreover, each vertex of this cube complex gives a subcomplex of $F(\Gamma)$, and $F(\Gamma)$ is a union of these subcomplexes (cf. Lemma 6.29). This is done in Section 6.3. However, at this point, it is not clear what the relationship is between these subcomplexes. This is analyzed in Section 6.4, where we also study in detail the pattern in which these subcomplexes are assembled to give $F(\Gamma)$, and how this is related to the dual cube complex. In the last step, namely Section 6.5, we conclude that we indeed construct a prime RAAG such that $G(\Gamma)$ sits inside as a special subgroup.

6.2. Prime RAAG

From now on, we assume $G(\Gamma)$ is a centerless RAAG of type II. We also label and orient edges of $X(\Gamma)$ in a $G(\Gamma)$-invariant way as before (see Section 2.1). The goal of this subsection is to introduce the notion of prime RAAGs and prove Theorem 6.10.

Let $q : G(\Gamma) \to G(\Gamma')$ be a quasi-isometry, and let $q_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$ be the canonical simplicial isomorphism induced by q (cf. Corollary 4.3). Pick vertex $v \in \mathcal{P}(\Gamma)$, then q_*

induces a 1-1 correspondence between v-branches in $\mathcal{P}(\Gamma)$ and $q_*(v)$-branches in $\mathcal{P}(\Gamma')$. This correspondence is the starting point to understand the quasi-isometry q.

Definition 6.1. Let $v \in \mathcal{P}(\Gamma)$ be a vertex. Two v-branches B_1 and B_2 are *quasi-isometrically indistinguishable* (QII) if there exist a quasi-isometry $f : X(\Gamma) \to X(\Gamma)$ and an induced simplicial isomorphism $f_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ such that

(1) f_* fixes every vertex in $\mathcal{P}(\Gamma) \setminus (B_1 \cup B_2)$.

(2) $f_*(B_1) = B_2$ and $f_*(B_2) = B_1$.

Such f or f_* will be called an *elementary permutation*.

Lemma 6.2. *Suppose Γ is of type II. Let f, B_1, B_2 be as in Definition 6.1. Let $q : G(\Gamma) \to G(\Gamma')$ be a quasi-isometry.*

(1) *The map q_* sends QII v-branches to QII $q_*(v)$-branches.*

(2) *Let B_1, B_2, B_3 be mutually different v-branches. If B_1 and B_2 are QII, B_2 and B_3 are QII, then B_1 and B_3 are QII.*

Proof. For assertion (1), let $g = q \circ f \circ q^{-1}$, where q^{-1} is a quasi-isometry inverse of q. As q_* and $(q^{-1})_*$ are inverses of each other by Corollary 4.3, we know that g_* exchanges $q_*(B_1)$ and $q_*(B_2)$ and fixes all other vertices.

For assertion (2), let $h : G(\Gamma) \to G(\Gamma)$ be the quasi-isometry witnessing the QII of B_2 and B_3 with $h_*(B_2) = B_3$. Then h_* sends $\{B_1, B_2\}$ to $\{B_1, B_3\}$. Thus we are done by assertion (1). ∎

Lemma 6.3. *Pick a v-tier T, then for any v-branch B such that $B \not\subseteq T$, there exists a v-branch $B' \subset T$ such that B' and B are QII.*

Proof. Pick standard geodesic $\ell \subset X(\Gamma)$ such that $\Delta(\ell) = v$, and suppose $\pi_{\Delta(\ell)}(B) = x$ and $\pi_{\Delta(\ell)}(T) = x'$ ($\pi_{\Delta(\ell)}$ is the map in Lemma 3.1). Recall that we have an action $G(\Gamma) \curvearrowright X(\Gamma)$, let $\alpha \in G(\Gamma)$ be the element such that α acts by translation along ℓ and $\alpha(x) = x'$. As $B \not\subseteq T$, we know $x \neq x'$, hence α is not the identity element. Note that α induces a simplicial isomorphism $\alpha_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$; moreover, α_* fixes every point in $\mathrm{St}(v)$. Define $B' = \alpha_*(B)$. Let L and L' be the components of $X(\Gamma) \setminus P_v$ corresponding to B and B', respectively (Proposition 4.6). Then $\alpha(L) = L'$. Now we consider the following map $q : X(\Gamma) \to X(\Gamma)$ defined by

$$q(z) = \begin{cases} z & \text{if } z \in X(\Gamma) \setminus (L \cup L'), \\ \alpha(z) & \text{if } z \in L, \\ \alpha^{-1}(z) & \text{if } z \in L'. \end{cases}$$

One readily verifies that q is a quasi-isometry, and Proposition 4.6 implies that q_* satisfies the conditions in Definition 6.1, so B and B' are QII. ∎

Let $\overline{v} = \pi(v)$. It follows from Corollary 3.13 (1) that each v-branch corresponds to a pair (C, K), where C is a component of $F(\Gamma) \setminus \mathrm{St}(\overline{v})$, and K is a v-peripheral subcomplex with support ∂C such that $\partial B = \Delta(K)$. We will denote $C = \Pi(B)$ in such case. As Γ has type II, Corollary 3.13 implies that for each given height, there exists a unique v-branch B' at this height such that $\partial B = \partial B'$ and $\Pi(B') = C$.

Definition 6.4. We define two components C_1 and C_2 of $F(\Gamma) \setminus \mathrm{St}(\overline{v})$ are *quasi-isometrically indistinguishable* (QII) if there exist v-branches B_1 and B_2 which are QII such that $\Pi(B_i) = C_i$ for $i = 1, 2$.

This definition does not depend on the choice of B_1 and B_2 in the sense of the following lemma.

Lemma 6.5. *Suppose C_1 and C_2 are QII components of $F(\Gamma) \setminus \mathrm{St}(\overline{v})$. Then for any pair B'_1 and B'_2 such that $\partial B'_1 = \partial B'_2$ and $\Pi(B'_i) = C_i$, we know B'_1 and B'_2 are QII.*

Proof. We first look at the case $\partial B'_1 = \partial B_1$. It follows from Corollary 3.13 that there is $g \in G(\Gamma)$ with its axis $\ell_g \subset X(\Gamma)$ satisfying $\Delta(\ell_g) = v$ such that $g_*(B_1) = B'_1$. Thus B_1 and B'_1 are QII by Lemma 6.3. Similarly, B_2 and B'_2 are QII. Thus B'_1 and B'_2 are QII by Lemma 6.2 (2). It remains to treat the case $\partial B'_1 \neq \partial B_1$. Let K and K' be the standard subcomplexes in P_v such that $\Delta(K') = \partial B'_1$ and $\Delta(K) = \partial B_1$. Then their supports satisfy $\Gamma_K = \Gamma_{K'}$ by Corollary 3.13. Thus there exists $\alpha \in G(\Gamma)$ such that its action on $G(\Gamma)$ satisfying

$$\alpha(K) = K'$$

and

$$\alpha(P_v) = P_v.$$

Since α is label preserving, we know

$$\Pi(\alpha_*(B_1)) = \Pi(B_1).$$

Also, $\alpha_*(v) = v$, so $\alpha_*(B_1)$ and $\alpha_*(B_2)$ is a pair of QII v-branches with $\partial(\alpha_*(B_1)) = \partial B'_1$. Thus we can conclude the proof by using Lemma 6.2 (2) and the previous case. ∎

Lemma 6.6. *Let C_1, C_2, C_3 be three mutually distinct components of $F(\Gamma) \setminus \mathrm{St}(\overline{v})$. Suppose C_1 and C_2 are QII and C_2 and C_3 are QII. Then C_1 and C_3 are QII.*

Proof. Let B_1 and B_2 be v-branches that are QII with $\Pi(B_i) = C_i$ for $i = 1, 2$. Let B'_2 and B'_3 be v-branches that are QII with $\Pi(B'_i) = C_i$ for $i = 2, 3$. As in the proof of Lemma 6.5, we can find $\alpha \in G(\Gamma)$ such that $\alpha(P_v) = P_v$, $\alpha_*(v) = v$ and $\alpha_*(B'_2) = B_2$. Thus by Lemma 6.2 (1), we can assume without loss of generality that $B_2 = B'_2$. Now the lemma follows from Lemma 6.2 (2). ∎

Lemma 6.6 implies that being QII among components of $F(\Gamma) \setminus \mathrm{St}(\overline{v})$ is an equivalence relation, hence we can divide these components into QII equivalent classes $\{\mathcal{C}_i\}_{i=1}^k$. We associate $\overline{v}$ with a k-tuple of positive integers

$$(n_1, n_2, \ldots, n_k),$$

where each n_i is the number of components of $F(\Gamma) \setminus \mathrm{St}(\overline{v})$ in $\mathcal{C}_i$. The vertex $\overline{v}$ is *prime* if

$$\gcd(n_1, n_2, \ldots, n_k) = 1.$$

It follows from (1) of Corollary 3.13 that if C_1 and C_2 are QII, then $\partial C_1 = \partial C_2$, so every QII class $\mathcal{C}_i$ has a well-defined boundary, which will be denoted by $\partial \mathcal{C}_i$.

Definition 6.7. A RAAG $G(\Gamma)$ is *prime* if and only if $F(\Gamma)$ is of type II and all vertices of $F(\Gamma)$ are prime.

Definition 6.8. For vertex $v \in \mathcal{P}(\Gamma)$, we define the stretch factor of q_* at v as follows. Take a v-branch B, and let $\{B_j\}_{j\in J}$ be the collection of v-branches that are QII to B. Let n be the number of different elements in $\{\Pi(B_j)\}_{j\in J}$. Let n' be the number of different elements in $\{\Pi(q_*(B_j))\}_{j\in J}$. Then the stretch factor of q_* at v is defined to be n'/n.

Lemma 6.9. *The following are true:*

1. *The definition of stretch factor does not depend on the choice of B.*
2. *If $G(\Gamma)$ is prime, then the stretch factor is an integer.*

Proof. Let $\ell \subset X(\Gamma)$ and $\ell' \subset X(\Gamma')$ be standard geodesics such that $\Delta(\ell) = v$ and $\Delta(\ell') = q_*(v)$. Recall that the vertex set $v(\ell)$ has a natural ordering induced from the orientation of edges in $X(\Gamma)$. We identify $v(\ell)$ with $\mathbb{Z}$ in an order-preserving way. Let B and $\{B_j\}_{j\in J}$ be as in Definition 6.8. Suppose

$$\Pi(B) \in \mathcal{C}_i.$$

Then by (1) and (2) of Corollary 3.13, there are exactly n_i elements of $\{B_j\}_{j\in J}$ in a given v-tier, where n_i is the cardinality of $\mathcal{C}_i$. Pick a total order on elements in $\mathcal{C}_i$, and define a total order on J by $j_1 < j_2$ if and only if

$$\pi_v(B_{j_1}) < \pi_v(B_{j_2}) \quad (\pi_v \text{ is the map in Lemma 3.1})$$

or

$$\pi_v(B_{j_1}) = \pi_v(B_{j_2}) \quad \text{and} \quad \Pi(B_{j_1}) < \Pi(B_{j_2}).$$

We identify J with $\mathbb{Z}$ in an order-preserving way, then there is a natural map $g_i : J \to v(\ell)$ induced by π_v. Note that

$$g_i(a) = \lfloor a/n_i \rfloor$$

up to translation.

Let $\{B'_k\}_{k\in K}$ be the collection of $q_*(v)$-branches such that B'_k and $q_*(B)$ are QII. Then

$$\Pi(\{B'_k\}_{k\in K}) = \mathcal{C}'_i$$

for some QII class $\mathcal{C}'_i$ of

$$F(\Gamma') \setminus \mathrm{St}(\overline{v}'),$$

where $\overline{v}' = \pi(q_*(v))$. Note that q_* induces a bijection $f_i : J \to K$. We identify $v(\ell')$ with $\mathbb{Z}$ and K with $\mathbb{Z}$ in the same way as before, and let

$$g'_i : K \to v(\ell')$$

be the natural map given by

$$g'_i(a) = \lfloor a/n'_i \rfloor,$$

where n'_i is the cardinality of $\mathcal{C}'_i$.

We define another map $h_i : v(\ell) \to v(\ell')$ as follows. For $x \in v(\ell)$, pick a B_j such that $\pi_v(B_j) = x$ and define

$$h_i(x) = \pi_{\Delta(\ell')}(q_*(B_j)).$$

Up to bounded distance, the definition of h_i is independent of the choice of B_j. We claim h_i is a quasi-isometry. Pick B_{j_1}, B_{j_2} in $\{B_j\}_{j\in J}$. For $m = 1, 2$, let L_{j_m} be the subset of $X(\Gamma)$ as in Proposition 4.6 such that $\Delta(\overline{L}_{j_m}) = B_{j_m}$. Then

$$d(L_{j_1}, L_{j_2}) = d(\pi_v(B_{j_1}), \pi_v(B_{j_2})).$$

Now it follows from assertions (1) and (3) of Proposition 4.6 that h_i is a quasi-isometry. Note that the following diagram commutes up to bounded distance:

$$\begin{array}{ccc} J & \xrightarrow{f_i} & K \\ \downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle g'_i} \\ v(\ell) & \xrightarrow{h_i} & v(\ell') \end{array}$$

thus f_i is a bijective quasi-isometry from $\mathbb{Z}$ to $\mathbb{Z}$, hence f_i is bounded distance from an isometry and

$$h_i(x) = (n_i/n'_i)x + b$$

up to bounded distance (b is some constant). Now we pick a different QII class of v-branches which gives the QII class $\mathcal{C}_{i'}$ of $F(\Gamma) \setminus \mathrm{St}(\overline{v})$ and define $h_{i'}$ in similar way, then

$$h_i = h_{i'}$$

up to bounded distance, but we also have

$$h_{i'}(x) = (n_{i'}/n'_{i'})x + b'$$

up to bounded distance, so $n_i/n'_i = n_{i'}/n'_{i'}$.

To see (2), note that the previous discussion implies that the multiplication of the stretch factor with each n_i (for $1 \leq i \leq k$) is an integer. Thus the stretch factor must be an integer when $\gcd(n_1, \ldots, n_k) = 1$. ■

In the rest of this subsection, we prove the following.

Theorem 6.10. *If $G(\Gamma)$ and $G(\Gamma')$ are prime RAAGs, then they are quasi-isometric if and only if they are isomorphic.*

Suppose $G(\Gamma)$ and $G(\Gamma')$ are prime RAAGs. Let $q : G(\Gamma) \to G(\Gamma')$ be a quasi-isometry and $q_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$ be the induced simplicial isomorphism. Pick vertex $x \in X(\Gamma)$. Let $K = (F(\Gamma))_x$ and $K' = q_*(K)$. Suppose $\{v_i\}_{i=1}^n$ is the collection of vertices in $\mathcal{P}(\Gamma)$ such that $K \setminus \mathrm{St}(v_i)$ is disconnected, then $v_i \in K$ for all i by Lemma 3.7. Let $v_i' = q_*(v_i)$. Then $\{v_i'\}_{i=1}^n$ are exactly the vertices in $\mathcal{P}(\Gamma')$ such that $K' \setminus \mathrm{St}(v_i')$ is disconnected.

Recall that $\pi : \mathcal{P}(\Gamma') \to F(\Gamma')$ is the canonical projection.

Lemma 6.11. *If for any vertex $v' \in \mathcal{P}(\Gamma)$, all vertices in $K' \setminus \mathrm{St}(v')$ are in the same v'-tier, then $\pi|_{K'}$ is injective. Moreover, $\bigcap_{v \in K'} P_v \neq \emptyset$.*

Proof. Since π maps simplexes to simplexes of the same dimension, it suffices to show there do not exist vertices $w_1, w_2 \in K'$ such that $\pi(w_1) = \pi(w_2)$. Suppose the contrary is true. Then

$$P_{w_1} \cap P_{w_2} = \emptyset.$$

Let h be a hyperplane separating P_{w_1} and P_{w_2}, and let ℓ be a standard geodesic dual to h. Then

$$\pi_{\Delta(\ell)}(w_1) \neq \pi_{\Delta(\ell)}(w_2)$$

($\pi_{\Delta(\ell)}$ is the map in Lemma 3.1), hence w_1 and w_2 are in different $\Delta(\ell)$-tier, which yields a contradiction. The second statement follows from Lemma 2.2 and the previous argument. ■

The following is the main ingredient for Theorem 6.10.

Lemma 6.12. *Under the assumption of Theorem 6.10, it is possible to modify q_* by post-composing q_* with finitely many elementary permutations such that the assumption of Lemma 6.11 is true.*

Assuming Lemma 6.12, we can finish the proof of Theorem 6.10 as follows. Lemmas 6.11 and 6.12 give a simplicial embedding $F(\Gamma) \to F(\Gamma')$. By considering the quasi-isometry inverse, we have a simplicial embedding

$$s' : F(\Gamma') \to F(\Gamma),$$

thus $F(\Gamma)$ and $F(\Gamma')$ have the same number of vertices. Note that $s(F(\Gamma))$ is a full subcomplex of $F(\Gamma')$, so s is actually a simplicial isomorphism and Theorem 6.10 follows. In the rest of this subsection, we prove Lemma 6.12.

Remark 6.13 (Informal discussion of the proof of Lemma 6.12.). We look at some simple special cases of Lemma 6.12 before we discuss the proof in full detail.

The simplest case to consider is that $\{v'_i\}_{i=1}^n$ has only one element. Let $\{B_j\}_{j=1}^k$ be the collection of v_1-branches that have non-trivial intersection with K. Then $\{B_j\}_{j=1}^k$ are in the same v_1-tier (as they have the same height as x with respect to v_1). By Corollary 3.12, $\{B_j\}_{j=1}^k$ are in 1-1 correspondence with connected components of

$$K \setminus \mathrm{St}(v_1) \cong F(\Gamma) \setminus \mathrm{St}(\bar{v}_1),$$

and the correspondence is given by sending B_j to $\Pi(B_j)$. Let $B'_j = q_*(B_j)$. As $G(\Gamma)$ is prime, the stretch factor of q_* at v_1 is an integer ≥ 1. Thus there exists a quasi-isometry g such that $g_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ is a composition of finitely many elementary permutations of v'_1-branches with the property that $\{g_*(B'_j)\}_{j=1}^k$ are in the same v'_1-tier. Thus Lemma 6.12 follows. We refer the procedure in this case as the basic move at v'_1.

The next case to consider is that $\{v'_i\}_{i=1}^n$ has two elements, and their distance is ≥ 2. One naturally wants to first preform the basic move at v'_1 and then perform the basic move at v'_2. The second step will maintain the outcome of the first step by Lemma 6.14, as long as in the second step we send the v'_2-branch that contains v'_1 to itself, but this is always possible to arrange.

A slightly more complicated case to consider is that $\{v'_i\}_{i=1}^n$ has mutual distance ≥ 2. We need to pick the correct order to perform basic moves, namely when we perform the basic move at v'_i, we want to send the v'_i-branch that contains $\{v'_1, \ldots, v'_{i-1}\}$ to itself to maintain the outcome of previous moves. This is possible only if we can order $\{v'_i\}_{i=1}^n$ such that $\{v'_1, \ldots, v'_{i-1}\}$ are in the same v'_i-branch, which motivates the notion of *tight subset* of $\{v'_i\}_{i=1}^n$ as in the proof of Lemma 6.16.

Another case to consider is that $\{v'_i\}_{i=1}^n$ has two elements, and their distance $=1$. Again the key point is to make sure that doing the basic move at v'_2 maintains the outcome of the basic move at v'_1. Take a v'_1-branch B. If $v'_2 \in \partial B$, then doing basic move at v'_2 will send B to itself, as B contains a vertex in $\mathrm{St}(v'_2)$, which is fixed pointwise under the basic move at v'_2. Thus we need to handle the case when $v'_2 \notin \partial B$, which leads to the notion of v'_2-non-crossing as in the proof of Lemma 6.17. This lemma gives a way to correct the situation if doing basic move at v'_2 destroys the outcome of basic move at v'_1.

We start the proof of Lemma 6.12 with two simple observations.

Lemma 6.14. *Assume $d(v_1, v_2) \geq 2$, and let B be any v_1-branch such that $v_2 \notin B$. Then B and v_1 are in the same v_2-branch; in particular, all such B are in the same v_2-branch.*

Proof. Note that $B \cap \mathrm{St}(v_2) = \emptyset$ (otherwise $v_2 \in B$). We also have

$$\partial B \not\subseteq \mathrm{lk}(v_1) \cap \mathrm{lk}(v_2) = \mathrm{St}(v_1) \cap \mathrm{St}(v_2),$$

otherwise B and v_1 are in different connected components of $\mathcal{P}(\Gamma) \setminus (\mathrm{lk}(v_1) \cap \mathrm{lk}(v_2))$, which contradicts Corollary 3.13 (5). As

$$\partial B \subset \mathrm{St}(v_1),$$

we know there is a vertex $w' \in \partial B$ with $w' \notin \mathrm{St}(v_2)$, hence B can be connected to v_1 via w' outside $\mathrm{St}(v_2)$. ∎

From now on, we will write $v_j|_{v_i} v_k$ if v_j and v_k are in different v_i-branches and write $v_j v_k|_{v_i}$ if v_j and v_k are in the same v_i-branch.

Lemma 6.15. *Suppose $F(\Gamma)$ is of type II. Let v_1, v_2, v_3 be vertices in $\mathcal{P}(\Gamma)$. If $v_1|_{v_2} v_3$, then $v_1 v_2|_{v_3}$ and $v_3 v_2|_{v_1}$.*

Proof. Let B be the v_2-branch that contains v_3. Since $\mathcal{P}(\Gamma) \setminus (\mathrm{lk}(v_1) \cap \mathrm{lk}(v_2))$ is connected, we have

$$\partial B \not\subseteq \mathrm{lk}(v_1) \cap \mathrm{lk}(v_2) = \mathrm{St}(v_1) \cap \mathrm{St}(v_2).$$

Then there exists vertex $w \in \partial B$ with $w \notin \mathrm{St}(v_1)$, which implies that v_2 and v_3 can be connected via w outside $\mathrm{St}(v_1)$. ∎

Let $E_n = \{v'_j\}_{j=1}^n$. For $1 \leq i \leq n$, define $E_i = \{v'_j\}_{j=1}^i$ and define $E_0 = \emptyset$.

Lemma 6.16. *It is possible to order the elements $\{v'_i\}_{i=1}^n$ of E_n such that for each $1 \leq i \leq n$ and any $v' \in E_n \setminus E_i$, we have all elements of $E_i \setminus \mathrm{St}(v')$ contained in the same v'-branch.*

Proof. Pick $E \subset \{v'_i\}_{i=1}^n$, and denote $\{v'_i\}_{i=1}^n \setminus E$ by E^c. We say E is *tight* if for any $v'_i \in E^c$, $E \setminus \mathrm{St}(v'_i)$ is inside a v'_i-branch. Pick $v'_i, v'_j \in E^c$, and define $v'_i <_E v'_j$ if and only if there exists $v'_k \in E$ such that v'_j and v'_k are in different v'_i-branches.

We claim that if E is tight, then $\leq_E$ is a partial order on E^c. Now we prove the claim. If $v'_i <_E v'_j$ and $v'_j <_E v'_i$, then there exist v'_{k_1} and v'_{k_2} in E such that

$$v'_j|_{v'_i} v'_{k_1} \quad \text{and} \quad v'_i|_{v'_j} v'_{k_2}.$$

By Lemma 6.15, we have

$$v'_{k_2} v'_j|_{v'_i},$$

so

$$v'_{k_2}|_{v'_i} v'_{k_1},$$

which contradicts the tightness of E. Thus the relation $\leq_E$ is antisymmetric. It remains to check the transitivity. Suppose $v_i' <_E v_j'$ and $v_j' <_E v_k'$ for $v_i', v_j', v_k' \in E^c$. Then there exist v_ℓ' and v_m' in E such that

$$v_\ell'|_{v_i'} v_j' \quad \text{and} \quad v_m'|_{v_j'} v_k'.$$

Since

$$v_\ell' \notin \mathrm{St}(v_j') \quad \text{and} \quad v_m' \notin \mathrm{St}(v_j'),$$

then $v_m' v_\ell'|_{v_j'}$. We also have $v_i' v_\ell'|_{v_j'}$ by Lemma 6.15, so $v_m' v_i'|_{v_j'}$. This, together with $v_m'|_{v_j'} v_k'$, implies $v_i'|_{v_j'} v_k'$, hence we have

$$v_k' v_j'|_{v_i'}.$$

However, we also know $v_\ell'|_{v_i'} v_j'$, so $v_\ell'|_{v_i'} v_k'$ and $v_i' <_E v_k'$.

If E is tight, let $v_i' \in E^c$ be a minimal element in E^c with respect to $\leq_E$. Then $E \cup \{v_i'\}$ is also tight. Let $E_1 = \{v_1'\}$. E_1 is clearly tight, so it is possible to add a vertex in E_1^c to E_1 to obtain a tight set E_2. By repeating this process for $n-1$ times, we obtain a filtration $E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{n-1} \subsetneq E_n = \{v_i'\}_{i=1}^n$ such that the requirements of the lemma are met. ∎

Suppose we have already obtained a quasi-isometry q_* such that for every vertex $v' \in E_i$, vertices of $K' \setminus \mathrm{St}(v')$ are in the same v'-tier. Suppose $v_{i+1}' = E_{i+1} \setminus E_i$, and let B' be the v_{i+1}'-branch that contains all points in $E_i \setminus \mathrm{St}(v_{i+1}')$ (if $E_i \setminus \mathrm{St}(v_{i+1}') = \emptyset$, we pick an arbitrary v_{i+1}'-branch). Let $\{B_j\}_{j=1}^k$ be the collection of v_{i+1}-branches that have non-trivial intersection with K. Then $\{B_j\}_{j=1}^k$ are in the same v_{i+1}-tier (as they have the same height as x with respect to v_{i+1}). By Corollary 3.12, $\{B_j\}_{j=1}^k$ are in 1-1 correspondence with connected components of $K \setminus \mathrm{St}(v_{i+1}) \cong F(\Gamma) \setminus \mathrm{St}(\overline{v}_{i+1})$, and the correspondence is given by sending B_j to $\Pi(B_j)$. Let $B_j' = q_*(B_j)$. Since both $G(\Gamma)$ and $G(\Gamma')$ are prime, the stretch factor of q_* at v_{i+1} is 1, then there exists $g_* : \mathcal{P}(\Gamma') \to \mathcal{P}(\Gamma')$ such that

(1) g_* is a composition of finitely many elementary permutations of v_{i+1}'-branches, hence g_* fixes every point in $\mathrm{St}(v_{i+1}')$.
(2) g_* fixes every point in B'.
(3) $\{g_*(B_j')\}_{j=1}^k$ are in the same v_{i+1}'-tier.

By Lemma 6.17, we can assume additionally that

(4) For any $v' \in E_i \cap \mathrm{St}(v_{i+1}')$ and any v'-branch D such that $D \cap K' \neq \emptyset$, $g_*(D)$ and D are in the same v'-tier.

By (1) and (2), g_* fixes every point in E_{i+1}. We claim that vertices of $g_*(K') \setminus \mathrm{St}(v')$ are in the same v'-tier for any $v' \in g_*(E_{i+1})$. There are three cases to consider as follows:

- The case $v' = v_{i+1}'$ follows from property (3) of g_*.
- The case $v' \in E_i \cap \mathrm{St}(v_{i+1}')$ follows from the inductive assumption and property (4) of g_*.

- Let $v' \in E_i \setminus \mathrm{St}(v'_{i+1})$ and D be a v'-branch. If $v'_{i+1} \notin D$, then $g_*(D) = D$ by (2) and Lemma 6.14; if $v'_{i+1} \in D$, $g_*(D) = D$ is still true since g_* fixes v' and v'_{i+1}. Thus g_* does not permute the v'-branches and the claim follows.

Let $q'_* = g_* \circ q_*$, $K'' = g_*(K') = q'_*(K)$, $E'_i = g_*(E_i)$ and $v''_i = g_*(v'_i)$. Then $\{v''_i\}_{i=1}^n$ are exactly the vertices in $\mathcal{P}(\Gamma')$ such that $K'' \setminus \mathrm{St}(v''_i)$ is disconnected. Note that

$$E'_1 \subsetneq E'_2 \subsetneq \cdots \subsetneq E'_n$$

still satisfies Lemma 6.16. Moreover, vertices of $K'' \setminus \mathrm{St}(v'')$ are in the same v''-tier for any $v'' \in E'_{i+1}$. So we can repeat the previous process to deal with E'_{i+2}.

After finitely many steps, we can assume for every point $v' \in \{v'_i\}_{i=1}^n$, vertices of $K' \setminus \mathrm{St}(v')$ are in the same v'-tier, thus K' satisfies the assumption of Lemma 6.11 and $\pi \circ q_*$ induces a simplicial embedding $s : F(\Gamma) \to F(\Gamma')$. This finishes the proof of Lemma 6.12 modulo Lemma 6.17.

Lemma 6.17. *Take $F(\Gamma)$ which is of type II. Let $w \in \mathcal{P}(\Gamma)$ be a vertex. Let $\{B_i\}_{i=1}^n$ be a collection such that each B_i is a v_i-branch for some vertex $v_i \in \mathcal{P}(\Gamma)$ satisfying $d(v_i, w) = 1$. We assume $B_i \neq B_j$ for $i \neq j$ (however, $v_i = v_j$ is allowed for $i \neq j$). Let $q : X(\Gamma) \to X(\Gamma)$ be a quasi-isometry such that q_* fixes every point in* $\mathrm{St}(w)$. *Then there exists a quasi-isometry $q' : X(\Gamma) \to X(\Gamma)$ such that q'_* satisfies the following:*

(1) *q'_* fixes every point in* $\mathrm{St}(w)$.

(2) *$q'_*(B) = q_*(B)$ for any w-branch B.*

(3) *B_i and $q'_*(B_i)$ are in the same v_i-tier.*

(4) *If q_* fixes every point in a w-branch B, then q'_* also fixes every point in B.*

Proof. We start by introducing an auxiliary notion. Take vertices $w, v \in \mathcal{P}(\Gamma)$ and a v-peripheral complex $K \subset \mathcal{P}(\Gamma)$. The pair (v, K) is *w-non-crossing* if $d(v, w) = 1$ and $w \notin K$. In this case, $B \cap \mathrm{St}(w) = \emptyset$ for any v-branch B such that $\partial B = K$. Moreover, for any other v-branch B' with $\partial B' = K$, B' and B are in the same w-branch. To see this, note that

$$K = \partial B \not\subseteq \mathrm{lk}(v) \cap \mathrm{lk}(w),$$

otherwise B and v will be in different connected components of

$$\mathcal{P}(\Gamma) \setminus (\mathrm{lk}(v) \cap \mathrm{lk}(w)),$$

which contradicts Corollary 3.13 (5). On the other hand, $K \subset \mathrm{lk}(v)$. So K contains a vertex $w' \in \mathrm{lk}(v) \setminus \mathrm{St}(w)$ such that B' can be connected with B outside $\mathrm{St}(w)$ via w'. We refer to Remark 6.18 for a comment on the naming of "*w-non-crossing*".

If B_i and $q_*(B_i)$ are not in the same v_i-tier, we wish to post-composing q_* with elementary permutations to arrange Lemma 6.17 (3). Suppose in Step 1 we already arranged B_1 and $q'_*(B_1)$ to be in the same v_1-tier. Then in Step 2 when arranging the position of $q'_*(B_2)$, we want to maintain the outcome of Step 1. One ideal situation is that all the v_2-branches upon which we want to perform elementary permutations are contained in the same v_1-branch. Then whatever happens in Step 2 only takes place within one particular v_1-branch, and each v_1-branch is mapped to itself. This leads us to define the following binary relation, which will guide us on the order of treating elements in $\{B_i\}_{i=1}^n$.

We define a binary relation $\leq$ on the set of w-non-crossing pairs by $(v_1, K_1) \leq (v_2, K_2)$ if there exist v_1-branch B_1 with $\partial B_1 = K_1$ and v_2-branch B_2 with $\partial B_2 = K_2$ such that $B_1 \subset B_2$. If $(v_1, K_1) < (v_2, K_2)$, then $d(v_1, v_2) = 1$. To see this, note that if $v_1 = v_2$, we must have $B_1 = B_2$ and $K_1 = K_2$. Suppose $d(v_1, v_2) = 2$. Since $v_2 \notin B_1$, B_1 must belong to the v_2-branch that contains v_1 by Lemma 6.14. Hence $v_1 \in B_2$ and $w \in \partial B_2 = K_2$, which yields a contradiction.

Now we show the relation $<$ is a partial order. Suppose $(v_1, K_1) \leq (v_2, K_2)$. Since $B_1 \subset B_2$, we know $B_1 \cap \mathrm{St}(v_2) = \emptyset$, thus

$$(B_1 \cup \partial B_1) \cap \{v_2\} = \emptyset,$$

in particular

$$v_2 \notin \partial B_1 = K_1.$$

Thus (v_1, K_1) is v_2-non-crossing, and we deduce as before that $B_1' \subset B_2$ for any v_1-branch B_1' with $\partial B_1' = K_1$. Thus the relation $\leq$ is transitive. If $(v_1, K_1) \leq (v_2, K_2)$, $(v_2, K_2) \leq (v_1, K_1)$ and $(v_1, K_1) \neq (v_2, K_2)$, then it follows from previous discussion that all v_1-branches with boundary $=K_1$ stay inside one particular v_2-branch, and all v_2-branches with boundary $=K_2$ stay inside one particular v_1-branch. This implies all v_1-branches with boundary $=K_1$ stay inside one particular v_1-branch, which is impossible. So $\leq$ is antisymmetric.

Now we begin to arrange all the requirements of Lemma 6.17. We only need to consider the case when $B_i \subset \mathcal{P}(\Gamma) \setminus \mathrm{St}(w)$ for all i, otherwise B_i will contain a vertex fixed by q_* and (3) is automatic. Let $K_i = \partial B_i$. Then (v_i, K_i) is a w-non-crossing pair. Suppose (v_1, K_1) is a maximal element in $\{(v_i, K_i)\}_{i=1}^n$ with respect to the order defined above and suppose $(v_1, K_1) = (v_i, K_i)$ if and only if $1 \leq i \leq m$. Let $K_1' = q_*(K_1)$, and let $\{A_i\}_{i\in\mathbb{Z}}$ (or $\{A_i'\}_{i\in\mathbb{Z}}$) be the collection of v_1-branches with boundary K_1 (or K_1'). Then q_* induces a bijection between $\{A_i\}_{i\in\mathbb{Z}}$ and $\{A_i'\}_{i\in\mathbb{Z}}$. Since q_* fixes v_1, the stretch factor of q_* at v_1 is 1 by Lemma 6.9, so we can post-compose q_* with a finite sequence of elementary permutations of elements in $\{A_i'\}_{i\in\mathbb{Z}}$ such that (3) is true for $1 \leq i \leq m$. Note that (v_1, K_1') is also w-non-crossing, so $\{A_i'\}_{i\in\mathbb{Z}}$ are in the same w-branch, and each of the elementary permutations we post-compose before is supported on this particular w-branch (and is identity outside this w-branch), hence (1) and (2) still hold.

Pick $i_0 > m$, and let D_1 and D_2 be two QII v_{i_0}-branches such that

$$\partial D_1 = \partial D_2 = q_*(K_{i_0}).$$

Let f_* be an elementary permutation of D_1 and D_2. We claim

$$f_*(q_*(B_i)) = q_*(B_i)$$

for $1 \le i \le m$, then the lemma follows by induction on the number of B_i. To see the claim, note that

$$(v_1, K_1') \not\leq (v_{i_0}, q_*(K_{i_0}))$$

(since $(v_1, K_1) \not\leq (v_{i_0}, K_{i_0})$). Then for any v_1-branch E such that $\partial E = K_1'$, we know E contains a vertex $u \in \mathcal{P}(\Gamma) \setminus (D_1 \cup D_2)$, otherwise we would have $E \subset D_1$ or $E \subset D_2$. Recall that $f_*(u) = u$, so $f_*(E) = E$, in particular $f_*(q_*(B_i)) = q_*(B_i)$ for $1 \le i \le m$.

Property (4) is true since we only need to consider those B_i's that are not contained in w-branches which are fixed by q_* pointwise. ∎

Remark 6.18. The naming of "w-non-crossing" comes from the fact that if (v, K) is w-non-crossing, then for any connected component L of $X(\Gamma) \setminus P_v$ such that $\Delta(\partial L) = K$ (cf. Section 4.2), L does not cross any hyperplanes of $X(\Gamma)$ whose dual edge can extend to a standard line ℓ with $\Delta(\ell) = w$. In other words, all components L of $X(\Gamma) \setminus P_v$ with $\Delta(\partial L) = K$ are in the same height with respect to ℓ. However, we will not need this fact for the later part of the paper.

Remark 6.19. We record a direct consequence of the construction of Lemma 6.17 that will be used later. The map q' in Lemma 6.17 is obtained by post-composing q with a finite sequence of elementary permutations such that each of these elementary permutation induces identity map on $\mathcal{P}(\Gamma) \setminus q_*(\mathcal{B})$, where $\mathcal{B}$ is the union of all w-branches that contains at least one element from $\{B_i\}_{i=1}^n$.

6.3. Prime partition, sub-tiers, prime factors and dual cube complexes

Given RAAG $G(\Gamma)$ of type II (not necessarily prime), our goal in the next three subsections is to find a prime RAAG $G(\Gamma')$ which is quasi-isometric to $G(\Gamma)$. Such $G(\Gamma')$, if exists, must be unique by Theorem 6.10. In this subsection, we introduce a wall space structure on $F(\Gamma)$ and prove several basic properties of this wall space for later use.

Pick vertex $\overline{v} \in F(\Gamma)$, let $\{\mathcal{C}_i\}_{i=1}^k$ be the collection of QII classes in $F(\Gamma) \setminus \mathrm{St}(\overline{v})$ and let $(n_1, n_2, \ldots, n_k)$ be the associated tuple. Let $\{C_{ij}\}_{j=1}^{n_i}$ be the components in $\mathcal{C}_i$, and let

$$d = \gcd(n_1, n_2, \ldots, n_k).$$

For each i, we choose a map

$$f_i : \{C_{ij}\}_{j=1}^{n_i} \to \{1, 2, \ldots, d\}$$

such that for each $1 \le m \le d$, there are n_i/d elements in $f_i^{-1}(m)$. For $1 \le m \le d$, let

$$\mathfrak{C}_m = \bigcup_{i=1}^{k} f_i^{-1}(m).$$

This partition of components of $F(\Gamma) \setminus \mathrm{St}(\overline{v})$ into $\{\mathfrak{C}_m\}_{m=1}^d$ is called a *prime partition at* $\overline{v}$. Each $\mathfrak{C}_m$ is called a *prime factor* at $\overline{v}$. The prime partition comes together with an order, namely we define $\mathcal{C}_i \leq \mathcal{C}_j$ if $i \leq j$. Note that the prime partition is trivial if $\bar{v}$ is prime. Now we fix a prime partition for every non-prime vertex in $F(\Gamma)$.

Remark 6.20. Let $\alpha : F(\Gamma) \to F(\Gamma)$ be a simplicial automorphism. By considering the group automorphism of $G(\Gamma)$ induced by α, we deduce that the number of prime factors at $\bar{v}$ and the number of prime factors at $\alpha(\bar{v})$ are the same. However, α may not map prime factors at $\bar{v}$ to prime factors at $\alpha(\bar{v})$.

Let $v \in \mathcal{P}(\Gamma)$ be a vertex such that $\pi(v) = \bar{v}$, and let T be a v-tier. Recall that we have a map Π which maps v-branches to components of $F(\Gamma) \setminus \mathrm{St}(\bar{v})$. This gives rise to a partition

$$\{\Pi^{-1}(\mathfrak{C}_m) \cap T\}_{m=1}^{d}$$

of v-branches in T. Each element in the partition is called a *v-sub-tier*.

The following lemma follows directly from definition.

Lemma 6.21. *Pick vertex $x \in X(\Gamma)$, and let $i_x : F(\Gamma) \to \mathcal{P}(\Gamma)$ be the natural embedding. Then for vertices $\bar{u}, \bar{v}, \bar{w} \in F(\Gamma)$, $\bar{u}$ and $\bar{v}$ are in different prime factors at $\bar{w}$ if and only if $i_x(\bar{u})$ and $i_x(\bar{v})$ are in different $i_x(\bar{w})$-sub-tiers.*

Lemma 6.22. *Let S_1 and S_2 be two v-sub-tiers. Then there exists a quasi-isometry $q : X(\Gamma) \to X(\Gamma)$ such that the induced simplicial isomorphism $q_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ satisfies*

(1) *q_* fixes every vertex in $\mathcal{P}(\Gamma) \setminus (S_1 \cup S_2)$.*

(2) *$q_*(S_1) = S_2$ and $q_*(S_2) = S_1$.*

(3) *For every v-branch $B \subset S_1$, $q_*(B)$ and B are QII.*

Proof. To see this, note that there exist unique v-tiers T_1, T_2 and $1 \leq m_1, m_2 \leq d$ such that

$$S_i = T_i \cap \Pi^{-1}(\mathfrak{C}_{m_i})$$

for $i = 1, 2$. For each $1 \leq i \leq k$, pick a bijection between $f_i^{-1}(m_1)$ and $f_i^{-1}(m_2)$, and this induces a bijection $\bar{\Lambda}$ from components in $\mathfrak{C}_{m_1}$ to components in $\mathfrak{C}_{m_2}$. By Corollary 3.13 (1), $\bar{\Lambda}$ induces a bijection Λ from v-branches in S_1 to v-branches in S_2 such that B and $\Lambda(B)$ are QII. We define q as follows. Set $q(x) = x$ if $x \in P_v$. If $x \notin P_v$, let D be the component of $X(\Gamma) \setminus P_v$ with $x \in D$, and let B be the v-branch corresponding to D (see Proposition 4.6). If B is not inside $S_1 \cup S_2$, then set $q(x) = x$. Otherwise, we assume $B \subset S_1$. Let $B' = \Lambda(B)$, and let D' be the associated component of $X(\Gamma) \setminus P_v$. Let f be the elementary permutation (Definition 6.1) of B and B'. We can assume $f(D) = D'$ (Proposition 4.6) and f is an (L, A)-quasi-isometry with L and A independent of $B \subset S_1$ (see the discussion after Lemma 6.3). Set $q(x) = f(x)$ in this case. Then q is a quasi-isometry and satisfies all the requirements. ∎

The reader may check that the same proof of Lemma 6.17 works to give the following lemma.

Lemma 6.23. *Lemma 6.17 is still true if we replace v_i-tier by v_i-sub-tier in (3).*

In the rest of this subsection, we show the prime partitions on $F(\Gamma)$ give rise to a pocset structure on $F(\Gamma)$ and construct the dual cube complex.

Definition 6.24. Pick non-prime vertex $\bar{v} \in F(\Gamma)$, and let $\{\mathfrak{C}_j\}_{j=1}^d$ be the prime factors at $\bar{v}$. A *$\bar{v}$-halfspace* of $F(\Gamma)$ is a full subcomplex of form

$$\mathrm{St}(\bar{v}) \cup \Big(\bigcup_{j=1}^{m} \mathfrak{C}_j\Big)$$

or

$$\mathrm{St}(\bar{v}) \cup \Big(\bigcup_{j=m+1}^{d} \mathfrak{C}_j\Big)$$

with $1 \le m < d$. Let

$$H = \mathrm{St}(\bar{v}) \cup \Big(\bigcup_{j=1}^{m} \mathfrak{C}_j\Big) \quad \Big(\text{or } \mathrm{St}(\bar{v}) \cup \Big(\bigcup_{j=m+1}^{d} \mathfrak{C}_j\Big)\Big).$$

We define the *complement* of H, denoted by H^c, to be

$$\mathrm{St}(\bar{v}) \cup \Big(\bigcup_{j=m+1}^{d} \mathfrak{C}_j\Big) \quad \Big(\text{or } \mathrm{St}(\bar{v}) \cup \Big(\bigcup_{j=1}^{m} \mathfrak{C}_j\Big)\Big).$$

A *$\bar{v}$-wall* of $F(\Gamma)$ is a pair of halfspaces (H, H^c).

Let $\mathcal{H}(\Gamma)$ be the collection of pairs $(\bar{v}, H)$ such that $\bar{v}$ is non-prime and H is a $\bar{v}$-halfspace. If there is another pair $(\bar{v}', H') \in \mathcal{H}(\Gamma)$ such that $H = H'$ and $\bar{v} \neq \bar{v}'$, then $(\bar{v}', H')$ and $(\bar{v}, H)$ are viewed as different elements in $\mathcal{H}(\Gamma)$.

Let $\mathcal{W}(\Gamma)$ be the collection of triples $(\bar{v}, H, H^c)$ such that (H, H^c) is a $\bar{v}$-wall. Occasionally, we will omit $\bar{v}$ when there is no ambiguity.

Definition 6.25. We say two halfspaces $(\bar{v}_1, H_1), (\bar{v}_2, H_2) \in \mathcal{H}(\Gamma)$ are *compatible* if $d(\bar{v}_1, \bar{v}_2) = 1$ or $(H_1 \cap H_2) \nsubseteq \mathrm{St}(\bar{v}_1)$.

Lemma 6.26. *Suppose $d(\bar{v}_1, \bar{v}_2) \ge 2$. Let C_1 (or C_2) be the component of $F(\Gamma) \setminus \mathrm{St}(\bar{v}_1)$ (or $F(\Gamma) \setminus \mathrm{St}(\bar{v}_2)$) that contains $\bar{v}_2$ (or $\bar{v}_1$). Then the following hold:*

(1) *$(H_1 \cap H_2) \nsubseteq \mathrm{St}(\bar{v}_1)$ implies $(H_1 \cap H_2) \nsubseteq \mathrm{St}(\bar{v}_2)$ and vice versa.*

(2) *Assuming $(H_1 \cap H_2) \nsubseteq \mathrm{St}(\bar{v}_1)$, then exactly one of the following three possibilities is true:* (1) $\bar{v}_1 \in H_2$ *and* $\bar{v}_2 \in H_1$*;* (2) $H_2 \subsetneq H_1$*;* (3) $H_1 \subsetneq H_2$.

(3) *Assuming* $(H_1 \cap H_2) \not\subseteq \mathrm{St}(\overline{v}_1)$*, then*

- *Case* (1) *holds if and only if* $C_i \subset H_i$ *for* $i = 1, 2$.
- *Case* (2) *holds if and only if* $C_1 \subset H_1$ *and* $C_2 \cap H_2 = \emptyset$.
- *Case* (3) *holds if and only if* $C_2 \subset H_2$ *and* $C_1 \cap H_1 = \emptyset$.

(4) *In case* (1) *of assertion* (2)*, we have* $H_1 \not\subseteq H_2$ *and* $H_2 \not\subseteq H_1$.

Proof. We will first prove assertions (2) and (3).

We claim $C_1 \cap H_1 = C_2 \cap H_2 = \emptyset$ is impossible. Indeed, if $C_1 \cap H_1 = C_2 \cap H_2 = \emptyset$, then $\overline{v}_1 \notin H_2$. Lemma 6.14 implies that all the components of $F(\Gamma) \setminus \mathrm{St}(\overline{v}_2)$ that are in H_2, as well as $\overline{v}_2$, are contained in a single component of $F(\Gamma) \setminus \mathrm{St}(\overline{v}_1)$. Then $H_2 \setminus E \subset C_1$, where E is defined to be

$$\mathrm{St}(\overline{v}_1) \cap \mathrm{St}(\overline{v}_2) = \mathrm{lk}(\overline{v}_1) \cap \mathrm{lk}(\overline{v}_2).$$

Thus

$$H_2 \subset C_1 \cup \mathrm{St}(\overline{v}_2).$$

Hence

$$H_1 \cap H_2 \subset H_1 \cap (C_1 \cup \mathrm{St}(\overline{v}_2)) = H_1 \cap \mathrm{St}(\overline{v}_2).$$

As $C_1 \cap H_1 = \emptyset$, we know C_1 is disjoint from all components of $F(\Gamma) \setminus \mathrm{St}(\overline{v}_1)$ that are in H_1. Then

$$H_1 \cap \mathrm{St}(\overline{v}_2) = \mathrm{St}(\overline{v}_1) \cap \mathrm{St}(\overline{v}_2) \subset \mathrm{St}(\overline{v}_1),$$

which contradicts

$$(H_1 \cap H_2) \not\subseteq \mathrm{St}(\overline{v}_1).$$

Thus the claim is proved. Thus in order to prove assertion (2), it suffices to prove assertion (3).

If $C_i \cap H_i \neq \emptyset$ for $i = 1, 2$, then actually $C_i \subset H_i$ and case (1) holds. The converse is clear. If $C_1 \subset H_1$ and $C_2 \cap H_2 = \emptyset$, then $\overline{v}_1 \notin H_2$ and $H_2 \subset C_1 \cup \mathrm{St}(\overline{v}_2)$ as before. Note that $C_1 \subset H_1$ and $\overline{v}_2 \subset C_1$ imply that

$$(C_1 \cup \mathrm{St}(\overline{v}_2)) \subset H_1.$$

Also, $\overline{v}_1 \in H_1 \setminus H_2$, hence case (2) is true. Conversely, if $H_2 \subsetneq H_1$, then $\overline{v}_2 \in H_2 \subset H_1$. Thus $C_1 \subset H_1$. Now we prove $C_2 \cap H_2 = \emptyset$. If this is not true, then $C_2 \subset H_2$. Let D be a component of $F(\Gamma) \setminus \mathrm{St}(\overline{v}_1)$ such that $D \cap H_1 = \emptyset$. Then $D \cap C_1 = \emptyset$, hence $\overline{v}_2 \notin D$. Then Lemma 6.14 implies D and $\overline{v}_1$ are in the same connected component of $F(\Gamma) \setminus \mathrm{St}(\overline{v}_2)$. Thus $D \subset C_2 \subset H_2$. Then H_2 contains vertices which are not in H_1, which contradicts $H_2 \subset H_1$. Thus $C_2 \cap H_2 = \emptyset$. Similarly, we can prove $C_2 \subset H_2$ and $C_1 \cap H_1 = \emptyset$ iff case (3) holds. Thus assertion (3) holds. Assertion (1) follows, as in cases (1) and (3) of assertion (2), we have $\overline{v}_1 \in H_1 \cap H_2$ and $\overline{v}_1 \notin \mathrm{St}(\overline{v}_2)$. In case (2), $H_2 = H_1 \cap H_2$, thus $(H_1 \cap H_2) \not\subseteq \mathrm{St}(\overline{v}_2)$ is clear. Assertion (4) also follows, as we already showed that if $C_i \subset H_i$ for $i = 1, 2$, then H_2 contains vertices that are not in H_1. Similarly, H_1 contains vertices that are not in H_2. ■

Lemma 6.27. *We define $(\bar{v}_1, H_1) \le (\bar{v}_2, H_2)$ if $d(\bar{v}_1, \bar{v}_2) \neq 1$ and $H_1 \subset H_2$. Then $\le$ gives a partial order on $\mathcal{H}(\Gamma)$. Moreover, $(\mathcal{H}(\Gamma), \le)$ with the complement operation defined before form a pocset.*

Proof. If $(\bar{v}_1, H_1) \le (\bar{v}_2, H_2)$ and $d(\bar{v}_1, \bar{v}_2) \ge 2$, then $(H_1 \cap H_2) \not\subseteq \mathrm{St}(\bar{v}_1)$. Thus Lemma 6.26 that implies that $H_1 \subsetneq H_2$, which means $(\bar{v}_1, H_1) \ge (\bar{v}_2, H_2)$ is impossible. Thus if $(\bar{v}_1, H_1) \le (\bar{v}_2, H_2)$ and $(\bar{v}_1, H_1) \ge (\bar{v}_2, H_2)$, then $\bar{v}_1 = \bar{v}_2$. Thus the relation $\le$ is antisymmetric. Now we show transitivity. Pick $(\bar{v}_3, H_3) \in \mathcal{H}(\Gamma)$ such that $(\bar{v}_2, H_2) \le (\bar{v}_3, H_3)$; if two of $\bar{v}_1, \bar{v}_2, \bar{v}_3$ are the same, then

$$(\bar{v}_1, H_1) \le (\bar{v}_3, H_3)$$

by definition. If $\bar{v}_1, \bar{v}_2, \bar{v}_3$ are pairwise distinct, let C_1 and C_2 be as in Lemma 6.26, and let C_2' be the component of $F(\Gamma) \setminus \mathrm{St}(\bar{v}_2)$ that contains $\bar{v}_3$. Since $H_1 \subsetneq H_2$ and $H_2 \subsetneq H_3$, then $C_1 \cap H_1 = \emptyset$, $C_2 \subset H_2$ and $C_2' \cap H_2 = \emptyset$ by Lemma 6.26. Thus $\bar{v}_1$ and $\bar{v}_3$ are in different components of $F(\Gamma) \setminus \mathrm{St}(\bar{v}_2)$ and $d(\bar{v}_1, \bar{v}_3) \ge 2$, which implies $(\bar{v}_1, H_1) \le (\bar{v}_3, H_3)$. It follows that $\le$ is a partial order.

It remains to show $(\bar{v}_1, H_1) \le (\bar{v}_2, H_2)$ implies $(\bar{v}_2, H_2^c) \le (\bar{v}_1, H_1^c)$. The case $\bar{v}_1 = \bar{v}_2$ is clear. If $d(\bar{v}_1, \bar{v}_2) \ge 2$, then $H_1 \cap C_1 = \emptyset$ and $C_2 \subset H_2$ by Lemma 6.26, hence $C_1 \subset H_1^c$ and $C_2 \cap H_2^c = \emptyset$, which implies $(\bar{v}_2, H_2^c) \le (\bar{v}_1, H_1^c)$ by Lemma 6.26. ■

Lemma 6.28. *A subset $U \subset \mathcal{H}(\Gamma)$ is an ultrafilter in the sense of Definition 2.7 if and only if U satisfies both of the following conditions:*

1. *for each pair $(\bar{v}, H)$ and $(\bar{v}, H^c)$, U contains exactly one of them;*
2. *every pair of halfspaces in U is compatible.*

Proof. We first claim the following are equivalent:

1. $(\bar{v}_1, H_1)$ and $(\bar{v}_2, H_2)$ are not compatible.
2. $d(\bar{v}_1, \bar{v}_2) \neq 1$ and $(\bar{v}_1, H_1) \le (\bar{v}_2, H_2^c)$.
3. $d(\bar{v}_1, \bar{v}_2) \neq 1$ and $(\bar{v}_2, H_2) \le (\bar{v}_1, H_1^c)$.

To see the claim, let us assume $d(\bar{v}_1, \bar{v}_2) \ge 2$. Let C_1 and C_2 be as in Lemma 6.26. Then $(\bar{v}_1, H_1)$ and $(\bar{v}_2, H_2)$ are not compatible $\Leftrightarrow C_1 \cap H_1 = C_2 \cap H_2 = \emptyset \Leftrightarrow C_1 \cap H_1 = \emptyset$ and $C_2 \subset H_2^c \Leftrightarrow (\bar{v}_1, H_1) \le (\bar{v}_2, H_2^c)$, where the first step and the last step follow from Lemma 6.26. Similarly, we can establish the equivalence of (1) and (3) in the claim.

To prove the "if" direction of the lemma, we need to show if $(\bar{v}_1, H_1) \le (\bar{v}_2, H_2)$ and $(\bar{v}_1, H_1) \in U$, then $(\bar{v}_2, H_2) \in U$. Indeed, if $(\bar{v}_2, H_2) \notin U$, then $(\bar{v}_2, H_2^c) \in U$. Then

$$H_1 \cap H_2^c \subset H_2 \cap H_2^c \subset \mathrm{St}(\bar{v}_2),$$

which contradicts that $(\bar{v}_1, H_1)$ and $(\bar{v}_2, H_2^c)$ are compatible. Now we prove the only if direction. Suppose U is an ultrafilter. Then Lemma 6.28 (1) is clear. If U contains a pair of non-compatible halfspaces $(\overline{v}_1, H_1)$ and $(\overline{v}_2, H_2)$, then the claim in the previous paragraph implies that

$$(\overline{v}_1, H_1) \leq (\overline{v}_2, H_2^c).$$

Now Definition 2.7 (2) implies that $(\overline{v}_2, H_2^c) \in U$, which contradicts Definition 2.7 (1). This proves the only if direction of the lemma. ∎

Let X be the CAT(0) cube complex obtained from the pocset $\mathcal{H}(\Gamma)$ as in Theorem 2.8. Let Φ be the pocset isomorphism from the collection of halfspaces in X to $\mathcal{H}(\Gamma)$ as in Theorem 2.8. Then Φ induces a bijective map from hyperplanes of X to $\mathcal{W}(\Gamma)$ (cf. Definition 6.24), which is also denoted by Φ.

Denote the collection of vertices in X by $\{x_i\}_{i=1}^r$, and let $\{U(x_i)\}_{i=1}^r$ be the corresponding ultrafilters.

Let $\Phi(x_i)$ be the intersection of halfspaces in $U(x_i)$. For each subcomplex $A \subset X$, we define $\Phi(A) = \bigcup_{x \in A} \Phi(x)$, where x ranges over vertices in A.

Lemma 6.29. *The following are true:*

1. *For any vertex $\overline{u} \in F(\Gamma)$, $\Phi(x_i) \setminus \mathrm{St}(\overline{u})$ is contained in a prime factor at $\overline{u}$.*
2. *For arbitrary simplex $g \subset F(\Gamma)$, there exists an ultrafilter U such that the intersection of halfspaces in U contains g. In particular, $\bigcup_{i=1}^r \Phi(x_i) = F(\Gamma)$.*
3. *$\Phi(x_i) \neq \emptyset$ for all i.*
4. *If A is convex, then $\Phi(A)$ is a full subcomplex.*

Proof. Assertion (1) is true as each $\overline{u}$ wall has a $\overline{u}$-halfspace containing $\Phi(x_i)$, and all these $\overline{u}$-halfspaces are compatible. Now we prove assertion (2). Let E' be the collection of non-prime vertices in $F(\Gamma)$, and let G be the collection of vertices in g. Let

$$E = E' \cup G = \{\overline{u}_1, \overline{u}_2, \dots, \overline{u}_n\}.$$

For $1 \leq i \leq n$, define $E_i = \{\overline{u}_1, \dots, \overline{u}_i\}$. We order the elements in E such that for each $1 \leq i \leq n$ and any $\overline{u}_j \in E \setminus E_i^c$, we have $E_i \setminus \mathrm{St}(\overline{u}_j)$ inside a single connected component of $F(\Gamma) \setminus \mathrm{St}(\overline{u}_j)$. This can be arranged in the same way as in Lemma 6.16. We can assume in addition that $\overline{u}_i \in G$ if and only if $i \leq n_1$ and $\overline{u}_i \in E'$ if and only if $i \geq n_2$. For $i \geq n_2$, if $E_{i-1} \setminus \mathrm{St}(\overline{u}_i) \neq \emptyset$, let C_i be the component of $F(\Gamma) \setminus \mathrm{St}(\overline{u}_i)$ that contains $E_{i-1} \setminus \mathrm{St}(\overline{u}_i)$ (this is possible by our choice of E_i). If $E_{i-1} \setminus \mathrm{St}(\overline{u}_i) = \emptyset$, let C_i be an arbitrary component. We define U by choosing the unique halfspace that contains C_i in each $\overline{u}_i$-wall for $i \geq n_2$. It is clear that the intersection of halfspaces in U contains g. It remains to show two halfspaces

$$(\overline{u}_i, H_1), (\overline{u}_j, H_2) \in U$$

are compatible. The case $d(\overline{v}_1, \overline{v}_2) \leq 1$ is trivial. We assume $d(\overline{u}_i, \overline{u}_j) \geq 2$. Suppose $i < j$, then $\overline{u}_i \subset C_j \subset H_2$, hence

$$\overline{u}_i \in (H_1 \cap H_2) \setminus \mathrm{St}(\overline{u}_j).$$

It follows that U is an ultrafilter. This justifies (2).

We now prove assertion (3). Let $U = \{(\bar{u}_\lambda, H_\lambda)\}_{\lambda\in\Lambda}$ be an ultrafilter, and let $A = \bigcap_{\lambda\in\Lambda} H_\lambda$. To prove assertion (3), it suffices to justify that if $(\bar{u}_\lambda, H_\lambda)$ is minimal in U, then $\bar{u}_\lambda \in A$. Suppose the contrary is true, then there exists $(\bar{u}_{\lambda'}, H_{\lambda'}) \in U$ such that $\bar{u}_\lambda \notin H_{\lambda'}$, in particular $d(\bar{u}_{\lambda'}, \bar{u}_\lambda) \geq 2$. By Lemma 6.28, H_λ and $H_{\lambda'}$ are compatible. Now Lemma 6.26 (2) and (3) imply that we must have $H_{\lambda'} \subsetneq H_\lambda$, which contradicts the minimality of $(\bar{u}_\lambda, H_\lambda)$.

It remains to prove (4). Let $\{\mathfrak{h}_i\}_{i=1}^t$ be the collection of halfspaces in X with $A \subset \mathfrak{h}_i$, and let $\Phi(\mathfrak{h}_i) = (\bar{w}_i, \mathfrak{h}'_i)$. Suppose $K = \bigcap_{i=1}^t \mathfrak{h}'_i$. Since each $\mathfrak{h}'_i$ is a full subcomplex, so is K. It suffices to show $\Phi(A) = K$. The inclusion $\Phi(A) \subset K$ is clear. Let $\mathcal{W}'(\Gamma)$ be the Φ-image of hyperplanes in X that intersect A, and let $\mathcal{H}'(\Gamma)$ be the corresponding collection of halfspaces. Then $\mathcal{H}'(\Gamma)$ is a sub-pocset of $\mathcal{H}(\Gamma)$. We claim $U' \subset \mathcal{H}'(\Gamma)$ is an ultrafilter of $\mathcal{H}'(\Gamma)$ if and only if $U' \cup \{\mathfrak{h}'_i\}_{i=1}^t$ is an ultrafilter of $\mathcal{H}(\Gamma)$. To see this, we can use the pocset isomorphism Φ between the halfspaces of X and $\mathcal{H}(\Gamma)$ to translate this statement to a statement about halfspaces of X, which becomes obvious. We also deduce that $U' \cup \{\mathfrak{h}'_i\}_{i=1}^t$ corresponds to a vertex in A. Thus there is an isometric embedding from the CAT(0) cube complex associated with $\mathcal{H}'(\Gamma)$ to X, whose image is exactly A. Let $\{U'_i\}_{i=1}^\ell$ be the collection of ultrafilters on $\mathcal{H}'(\Gamma)$, and let K_i be the intersection of halfspaces in U'_i. Then we can prove

$$\bigcup_{i=1}^{\ell} K_i = F(\Gamma)$$

as in assertion (2). It follows that

$$K = K \cap \left(\bigcup_{i=1}^{\ell} K_i\right) = \bigcup_{i=1}^{\ell} (K \cap K_i),$$

but $K \cap K_i = U(x)$ for some vertex $x \in A$, so $K \subset \Phi(A)$. ∎

Recall that two distinct walls $(\bar{v}_1, H_1, H_1^c), (\bar{v}_2, H_2, H_2^c) \in \mathcal{W}(\Gamma)$ are *transverse* if none of $(\bar{v}_1, H_1) < (\bar{v}_2, H_2)$, $(\bar{v}_1, H_1) < (\bar{v}_2, H_2^c)$, $(\bar{v}_2, H_2) < (\bar{v}_1, H_1)$ and $(\bar{v}_2, H_2) < (\bar{v}_1, H_1^c)$ is true. Thus two such walls are transverse if and only if $d(\bar{v}_1, \bar{v}_2) = 1$ (note that when $d(\bar{v}_1, \bar{v}_2) = 1$, even if $H_1 \subset H_2$, we still have $(\bar{v}_1, H_1) \nleq (\bar{v}_2, H_2)$ by our definition). It follows that if h'_1 and h'_2 is a pair of crossing hyperplanes in X and $\Phi(h'_i)$ is a $\bar{v}_i$-wall for $i = 1, 2$, then $d(\bar{v}_1, \bar{v}_2) = 1$.

6.4. A filtration for X and $F(\Gamma)$

In this subsection, our goal is to understand the relationship between different $\Phi(x)$, with x ranging over the X, and how $F(\Gamma)$ is assembled from these $\Phi(x)$. The main result of this subsection is Lemma 6.30. As the material in this subsection is comparably technical, the reader might want to assume Lemma 6.30 and go ahead to Section 6.5 to see how it finishes the proof before coming back to this subsection.

We now define a filtration for X as well as for $F(\Gamma)$. Such filtration is motivated by the generalized star extension introduced in [21, Section 6.3].

We define a chain of convex subcomplexes in X by induction. Pick a vertex $x \in X$, and set $L_1 = \{x\}$. Suppose we have already defined L_i. If $L_i = X$, then we stop; if $L_i \subsetneq X$, pick an edge e_i such that $e_i \cap L_i$ is a vertex, and let L_{i+1} be the convex hull of $L_i \cup e_i$. Let $\{L_i\}_{i=1}^{s}$ be the resulting collection of convex subcomplexes. Here is an alternative way of describing L_{i+1}. Suppose h_i is the hyperplane dual to e_i and N_i is the carrier of h_i. Then $h_i \cap L_i = \emptyset$ by the convexity of L_i. Let M_i be the copy of $(L_i \cap N_i) \times [0, 1]$ inside N_i. Then $L_{i+1} = L_i \cup M_i$.

Now we look at the relation between $\Phi(L_i)$ and $\Phi(L_{i+1})$. For $j = 1, 2$, let M_{ij} be the subcomplex of M_i of form

$$(L_i \cap N_i) \times \{j - 1\}.$$

We assume $M_{i1} = L_i \cap N_i$, and let $p : M_{i1} \to M_{i2}$ be the map induced by parallelism. Suppose $(\overline{v}, H_i) \in \mathcal{H}(\Gamma)$ is the element corresponding to the halfspace of h_i that contains L_i. Then

$$\Phi(M_{i1}) \subset \Phi(K) \subset H_i$$

and $\Phi(M_{i2}) \subset H_i^c$. For any vertex $x \in M_{i1}$, $(\overline{v}, H_i)$ is a minimal element in $U(x)$, so $\overline{v} \in \Phi(x) \subset \Phi(M_{i1})$. Similarly, $\overline{v} \in \Phi(M_{i2})$.

Lemma 6.30. *The following are true:*

(1) *There is a simplicial isomorphism $\overline{h}_* : \Phi(M_{i1}) \to \Phi(M_{i2})$ with $\overline{h}_*(\Phi(x)) = \Phi(p(x))$ for any vertex $x \in \Phi(M_{i1})$ and $(\overline{h}_*)^{-1}(\Phi(x)) = \Phi(p^{-1}(x))$ for any vertex $x \in \Phi(M_{i2})$.*

(2) $\Phi(L_i) \cap \Phi(M_{i1}) = \Phi(L_i) \cap \Phi(M_{i2}) = \mathrm{St}(\overline{v}, \Phi(M_{i1}))$.

Thus $\Phi(L_{i+1})$ can be obtained by taking $\Phi(L_i)$ and $\Phi(M_{i1}) \cong \Phi(M_{i2})$ and gluing them along $\mathrm{St}(\overline{v}, \Phi(M_{i1}))$.

We extracted a technical part of the proof as Lemma 6.32. Now we prove Lemma 6.30, assuming Lemma 6.32.

Proof. We claim there exist $\mathfrak{C}_1$ and $\mathfrak{C}_2$ which are prime factors at $\overline{v}$ such that $\mathfrak{C}_1 \subset H_i$, $\mathfrak{C}_2 \subset H_i^c$ and $\Phi(M_{ij}) \setminus \mathrm{St}(\overline{v}) \subset \mathfrak{C}_j$ for $j = 1, 2$. Pick adjacent vertices $x_1, x_2 \in M_{i1}$, then there exists $(\overline{v}, H_i') \in U(x_1)$ such that $(H_i \cap H_i') \setminus \mathrm{St}(\overline{v})$ is a prime factor at $\overline{v}$. Denote this prime factor by $\mathfrak{C}_1$, then

$$\Phi(x_1) \setminus \mathrm{St}(\overline{v}) \subset \mathfrak{C}_1.$$

Let h be the hyperplane dual to the edge joining x_1 and x_2, and let $\Phi(h) = (\overline{w}, H, H^c)$. Then

$$U(x_2) = (U(x_1) \setminus \{H\}) \cup \{H^c\}.$$

Since h crosses h_i, $d(\overline{w}, \overline{v}) = 1$. Thus

$$(\overline{v}, H_i), (\overline{v}, H_i') \in U(x_2),$$

which implies $\Phi(x_2) \setminus \mathrm{St}(\overline{v}) \subset \mathfrak{C}_1$. Now $\Phi(M_{i1}) \setminus \mathrm{St}(\overline{v}) \subset \mathfrak{C}_1$ follows from the connectedness of M_{i1}. We can choose $\mathfrak{C}_2$ in a similar way.

The above argument also implies for any vertex $\overline{u}$ such that $d(\overline{u}, \overline{v}) \neq 1$, $\Phi(M_{i1}) \setminus \mathrm{St}(\overline{u})$ is contained in a prime factor at $\overline{u}$. Note that if $M_{i1} \subset \mathrm{St}(\overline{v})$, then $\Phi(x) = \Phi(p(x))$ for any vertex $x \in M_{i1}$, hence $M_{i2} = M_{i1}$. Now we assume $M_{i1} \not\subseteq \mathrm{St}(\overline{v})$, then we can set up as in Lemma 6.32 with respect to $K = M_{i1}$, $\overline{v} \in K$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$. Note that K is a full subcomplex of $F(\Gamma)$ by Lemma 6.29. Let h_* and $\overline{h}_*$ be the maps in Lemma 6.32. We claim $\overline{h}_*(M_{i1}) = M_{i2}$.

Pick vertex $x \in M_{i1}$. We claim for any vertex $\overline{u}$ with $d(\overline{u}, \overline{v}) \geq 1$, $\Phi(x) \setminus \mathrm{St}(\overline{u}) \neq \emptyset$ if and only if $\overline{h}_*(\Phi(x)) \setminus \mathrm{St}(\overline{u}) \neq \emptyset$, and in this case $\Phi(x) \setminus \mathrm{St}(\overline{u})$ and $\overline{h}_*(\Phi(x)) \setminus \mathrm{St}(\overline{u})$ are in the same prime factor at $\overline{u}$. The claim is a consequence of Lemma 6.32 (3b) when $d(\overline{u}, \overline{v}) = 1$. When $d(\overline{u}, \overline{v}) > 1$, note that $\overline{v} \in \Phi(x)$ and $\overline{v} \in \overline{h}_*(\Phi(x))$, then the claim follows from Lemma 6.32 (3a). Thus for any $(\overline{u}, H) \in U(x)$ with $d(\overline{u}, \overline{v}) \geq 1$, $\overline{h}_*(\Phi(x)) \subset H$. Moreover, (1) of Lemma 6.32 implies

$$\overline{h}_*(\Phi(x)) \setminus \mathrm{St}(\overline{v}) \subset \mathfrak{C}_2,$$

so

$$\overline{h}_*(\Phi(x)) \subset \Phi(p(x))$$

by our choice of $\mathfrak{C}_2$ (recall that $p: M_{i1} \to M_{i2}$ is the parallelism map). Denote the number of vertices in $\Phi(x)$ by $|\Phi(x)|$, then

$$|\Phi(x)| \leq |\Phi(p(x))|.$$

By reversing the role of M_{i1} and M_{i2} and applying Lemma 6.32 with $K = M_{i2}$, we have $|\Phi(p(x))| \leq |\Phi(x)|$, hence

$$|\Phi(x)| = |\Phi(p(x))|.$$

But $\overline{h}_*(\Phi(x))$ and $\Phi(p(x))$ are both full subcomplexes, so

$$\overline{h}_*(\Phi(x)) = \Phi(p(x)).$$

Thus $\overline{h}_*(\Phi(M_{i1})) = \Phi(M_{i2})$. This also implies $(\overline{h}_*)^{-1}(\Phi(x)) = \Phi(p^{-1}(x))$, which finishes the proof of assertion (1).

Since $\Phi(M_{ij})$ is a full subcomplex for $j = 1, 2$ (Lemma 6.29), we have

$$\overline{h}_*(\mathrm{St}(\overline{v}) \cap \Phi(M_{i1})) = \overline{h}_*(\mathrm{St}(\overline{v}, \Phi(M_{i1}))) = \mathrm{St}(\overline{v}, \Phi(M_{i2})) = \mathrm{St}(\overline{v}) \cap \Phi(M_{i2}).$$

However, $\overline{h}_*$ fixes every point in $\mathrm{St}(\overline{v}) \cap \Phi(M_{i1})$, so

$$\mathrm{St}(\overline{v}) \cap \Phi(M_{i1}) = \mathrm{St}(\overline{v}) \cap \Phi(M_{i2}).$$

Recall that

$$\Phi(L_i) \cap \Phi(M_{i2}) \subset \mathrm{St}(\overline{v}),$$

so

$$\begin{aligned}\Phi(M_{i1}) \cap \mathrm{St}(\overline{v}) \subset \Phi(M_{i1}) \cap \Phi(M_{i2}) \subset \Phi(L_i) \cap \Phi(M_{i2}) \\ \subset \Phi(M_{i2}) \cap \mathrm{St}(\overline{v}) = \Phi(M_{i1}) \cap \mathrm{St}(\overline{v}),\end{aligned}$$

and all these sets are equal. ∎

It remains to prove Lemma 6.32. We start with an auxiliary result needed in the proof of Lemma 6.32.

Lemma 6.31. *If $G(\Gamma)$ is of type II, then given any two vertices $v_1, v_2 \in \mathcal{P}(\Gamma)$, there only exist finitely many vertices w such that $v_1|_w v_2$.*

Proof. We first prove a preparatory claim as follows: pick vertex $x \in X(\Gamma)$ and $v \in \mathcal{P}(\Gamma) \setminus (F(\Gamma))_x$, let $\overline{w} \in \Gamma$ and let $S_{\overline{w}} \subset (F(\Gamma))_x$ be the lift of $\mathrm{St}(\overline{w}) \subset F(\Gamma)$. Then $S_{\overline{w}} \setminus \mathrm{St}(v) \neq \emptyset$. Now we prove this claim. Suppose the contrary is true. Put $\overline{v} = \pi(v)$. Then $\mathrm{St}(\overline{w}) \subset \mathrm{St}(\overline{v})$, hence $\overline{v} \in \mathrm{St}(\overline{w})$. Let $v' \in (F(\Gamma))_x$ be the lift of $\overline{v}$. Then $v' \in S_{\overline{w}} \subset \mathrm{St}(v)$. Note that $d(v', v) = 1$ is impossible since $\pi(v') = \pi(v)$, so $v' = v$, which is contradictory to $v \notin (F(\Gamma))_x$. Thus the claim is proved.

Now we prove the lemma. Pick an edge path ω which connects a vertex in P_{v_1} to a vertex in P_{v_2}. Let $\{x_i\}_{i=0}^n$ be consecutive vertices in ω, and let ℓ_i be the standard geodesic containing x_{i-1} and x_i. Let $K_i = (F(\Gamma))_{x_i}$ and $K = \bigcup_{i=0}^n K_i$. Then $v_1 \in K_0$ and $v_2 \in K_n$. It suffices to show that for any vertex $v \notin K$, v_1 and v_2 are in the same v-branch. To see this, note that

$$\pi(K_{i-1} \cap K_i) = \mathrm{St}(\pi(\Delta(\ell_i))),$$

so for $1 \le i \le n$, there exists vertex w_i such that

$$w_i \in (K_{i-1} \cap K_i) \setminus \mathrm{St}(v)$$

by the auxiliary result above. By Lemma 3.7, $w_i w_{i+1}|_v$ for $1 \le i \le n-1$, $v_1 w_1|_v$ and $w_n v_2|_v$, so $v_1 v_2|_v$. ∎

Lemma 6.32. *Let $\overline{v} \in F(\Gamma)$ be a non-prime vertex, and let $K \subset F(\Gamma)$ be a full subcomplex containing $\overline{v}$ such that for any vertex $\overline{u} \in F(\Gamma) \setminus \mathrm{lk}(\overline{v})$, $K \setminus \mathrm{St}(\overline{u})$ is inside a prime factor at $\overline{u}$. Suppose in addition that $K \setminus \mathrm{St}(\overline{v}) \neq \emptyset$, and let $\mathfrak{C}_1$ be the prime factor at $\overline{v}$ that contains $K \setminus \mathrm{St}(\overline{v})$. Let $\mathfrak{C}_2$ be a different prime factor at $\overline{v}$. Pick vertex $x \in X(\Gamma)$, and let $K', v, \mathfrak{C}'_1$ and $\mathfrak{C}'_2$ be the lift of $K, \overline{v}, \mathfrak{C}_1$ and $\mathfrak{C}_2$ in $(F(\Gamma))_x \subset \mathcal{P}(\Gamma)$, respectively. For $i = 1, 2$, let S_i be the v-sub-tier that contains $\mathfrak{C}'_i$. Let $q : X(\Gamma) \to X(\Gamma)$ be a quasi-isometry such that q_* permutes S_1 and S_2 and fixes every point in $\mathcal{P}(\Gamma) \setminus (S_1 \cup S_2)$ (Lemma 6.22).*

There exists a quasi-isometry $h : X(\Gamma) \to X(\Gamma)$ such that

(1) *h_* permutes S_1 and S_2 and fixes every vertex in $\mathcal{P}(\Gamma) \setminus (S_1 \cup S_2)$.*

(2) *The projection map $\pi : \mathcal{P}(\Gamma) \to F(\Gamma)$ restricted on $K' \cup h_*(K')$ is injective.*

(3) *Let $M = \pi(K' \cup h_*(K'))$. Let $\overline{h}_* : K \to \pi(h_*(K'))$ be the simplicial isomorphism induced by h_*. Pick vertex $\overline{u} \in F(\Gamma)$. Then*

  (a) *If $d(\overline{u}, \overline{v}) \geq 2$, then $M \setminus \mathrm{St}(\overline{u})$ is contained in one prime factor at $\overline{u}$.*

  (b) *If $d(\overline{u}, \overline{v}) = 1$, then $\overline{r} \in K \setminus \mathrm{St}(\overline{u})$ if and only if $\overline{h}_*(\overline{r}) \in \overline{h}_*(K) \setminus \mathrm{St}(\overline{u})$. In this case, $\overline{r}$ and $\overline{h}_*(\overline{r})$ are in the same prime factor at $\overline{u}$.*

  (c) *The subcomplex $\overline{h}_*(K)$ is full.*

Proof. We assume K is a full subcomplex. The general case follows from this special case by considering the full subcomplex spanned by K.

Let $L = K' \cup q_*(K')$. By Lemma 6.31, there are only finitely many vertices $w \in \mathcal{P}(\Gamma)$ such that $\mathrm{St}(w)$ separates two vertices in L. Denote these vertices by $\{w_i\}_{i=1}^n$. We claim if $\mathrm{St}(w)$ separates two vertices in K' and $d(w, v) \geq 2$, then vertices of $q_*(K') \setminus \mathrm{St}(w)$ are in the same w-branch. To see this, suppose $v_1|_w v_2$ for $v_1, v_2 \in K'$, then either $v_1|_w v$ or $v_2|_w v$, so $v_1 w|_v$ or $v_2 w|_v$ by Lemma 6.15. Then the claim follows from Lemma 6.14. Note that the claim is also true if we switch the role of K' and $q_*(K')$.

By the above claim, we can reorder and divide $\{w_i\}_{i=1}^n$ into the following four groups:

(1) $w_1 = v$.

(2) $w_i \in \mathrm{lk}(v)$ for $2 \leq i \leq n_1$.

(3) $\mathrm{St}(w_i)$ separates v from some vertex in K' for $n_1 + 1 \leq i \leq n_2$.

(4) $\mathrm{St}(w_i)$ separates v from some vertex in $q_*(K')$ for $n_2 + 1 \leq i \leq n$.

Note that q_* induces a bijection between $\{w_i\}_{i=n_1+1}^{n_2}$ and $\{w_i\}_{n_2+1}^{n}$. Let

$$k = n_2 - n_1 = n - n_2.$$

We also assume $q_*(w_i) = w_{i+k}$ for $n_1 + 1 \leq i \leq n_2$.

Let $D = \{w_i\}_{i=1}^{n_2}$. We claim $D \setminus \mathrm{St}(w_i)$ stays inside a w_i-branch for $i > n_2$. To see this, let B be the w_i-branch that contains v, and let $w_{i_0} \in D \setminus \mathrm{St}(w_i)$. It is clear that $w_{i_0} \in B$ if $i_0 \leq n_1$. If $n_1 < i_0 \leq n_2$, by above discussion, there exists $u \in K'$ such that $w_{i_0} u|_v$; similarly, there exists $u' \in q_*(K')$ such that $w_i u'|_v$. But $u|_v u'$, so $w_{i_0}|_v w_i$, and by Lemma 6.15, we have $w_{i_0} v|_{w_i}$ and $w_{i_0} \in B$. This discussion also implies $\{w_i\}_{i=n_1+1}^{n_2} \subset S_1$ and $\{w_i\}_{i=n_2+1}^{n} \subset S_2$.

Let $\{B_\lambda\}_{\lambda \in \Lambda}$ be the collection of w_i-branches that contain vertices of K', where i ranges over all values between 2 and n_1. By Lemma 6.17, we can post-compose q_* with a simplicial isomorphism f_* to obtain a map $q'_* = f_* \circ q_*$ that satisfies the conclusions of Lemma 6.17. By Lemma 6.23, we can assume that for vertex $u \in K' \setminus \mathrm{St}(w_i)$ $(2 \leq i \leq n_1)$, u and $q'_*(u)$ are in the same w_i-sub-tier. Let

$$L' = K' \cup q'_*(K').$$

Note that if $B_\lambda \subset \mathcal{P}(\Gamma) \setminus \mathrm{St}(v)$, then $B_\lambda \subset S_1$. So by Remark 6.19, f_* is a composition of elementary permutations that happen inside S_2, hence f_* fixes every point in S_1, in particular, $f_*(K') = K'$ and $f_*(L) = L'$. Let $w'_i = f_*(w_i)$. Then $\{w'_i\}_{i=1}^n$ is the collection of vertices such that $\mathrm{St}(w)$ separates two vertices in L'. We divide $\{w'_i\}_{i=1}^n$ into four groups as before, and this partition coincides with the partition induced by f_*. Since $w_i \in S_1$ for $n_1 + 1 \le i \le n_2$, so $w'_i = w_i$ for $i \le n_2$. Moreover, the claim in the previous paragraph also holds for $\{w'_i\}_{i=1}^n$.

We claim that for $n_1 < i \le n_2$, vertices of $L' \setminus \mathrm{St}(w'_i)$ are in the same w'_i-sub-tier. Actually, by Lemma 3.7, $w'_i \in (F(\Gamma))_x$ for $n_1 < i \le n_2$. Recall that $K \setminus \mathrm{St}(\pi(w'_i))$ is inside a prime factor at $\pi(w'_i)$, so vertices of $K' \setminus \mathrm{St}(w'_i)$ are inside a w'_i-sub-tier. We know S_2 contains vertices of $q'_*(K') \setminus \mathrm{St}(v)$, but $w'_i \in S_1$, so vertices of $q'_*(K') \setminus \mathrm{St}(v)$ and v are in the same w'_i-branch by Lemma 6.14. The claim follows.

Let $E_0 = \{w'_i\}_{i=1}^{n_2}$ and $E = \{w'_i\}_1^n$. Then E_0 is tight in E by previous discussion. Let $E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_k = E$ be a filtration such that each E_i is tight in E (this can be arranged as in Lemma 6.16). Up to reordering, we assume $w'_{i+n_2} = E_i \setminus E_{i-1}$ for $1 \le i \le k$ and $q'_*(w'_i) = w'_{i+k}$ for $n_1 < i \le n_2$. Suppose there is an integer m $(0 \le m \le k)$ such that

(1) q'_* permutes S_1 and S_2 and fixes every vertex in $\mathcal{P}(\Gamma) \setminus (S_1 \cup S_2)$.

(2) For $2 \le i \le n_1$ and vertex $u \in K' \setminus \mathrm{St}(w'_i)$, u and $q'_*(u)$ are in the same w'_i-sub-tier.

(3) For $n_1 < i \le n_2 + m$, vertices of $L' \setminus \mathrm{St}(w'_i)$ are contained in one w'_i-sub-tier.

Such m always exists, since (1) and (2) are always true and (3) is true when $m = 0$. Our goal is to modify the map q'_* such that $m = k$. Now we assume $m < k$ and argue by induction.

Let $a = n_1 + m + 1$ and $b = n_2 + m + 1$. Since vertices of $K' \setminus \mathrm{St}(w'_a)$ stay inside a w'_a-sub-tier and $w'_b = q'_*(w'_a)$, there is a simplicial isomorphism g_* which is a composition of finitely many elementary permutations of w'_b-branches such that

(a) Vertices in $g_*(q'_*(K')) \setminus \mathrm{St}(w'_b)$ are in the same w'_b-sub-tier.

(b) Let B' be the w'_b-branch that contains v, then g_* fixes every point in B'.

Lemma 6.14 implies vertices of $K' \setminus \mathrm{St}(w'_b)$ are in B', so g_* fixes every point in K'. Moreover, the tightness of E_m implies $g_*(w'_i) = w'_i$ for $1 \le i \le b$. By Lemmas 6.17 and 6.23, we can assume in addition that g_* satisfies

(c) For any vertex $t \in \mathrm{St}(w'_b) \cap E_m$ and any t-branch A with $A \cap L' \ne \emptyset$, $g_*(A)$ and A are in the same t-sub-tier.

Let

$$L'' = K' \cup g_*(q'_*(K')).$$

Then $g_*(L') = L''$. Let $w''_i = g_*(w'_i)$. Then $\{w''_i\}_{i=1}^n$ is the collection of vertices such that $\mathrm{St}(w''_i)$ separates two vertices in L''. Moreover, $g_*(E_0) \subsetneq g_*(E_1) \subsetneq \cdots \subsetneq g_*(E_k) = g_*(E)$ satisfies that each $g_*(E_i)$ is tight in $g_*(E)$. Note that $g_*(E_i) = E_i$ for $i \le m+1$. We claim

(i) $g_* \circ q'_*$ permutes S_1 and S_2 and fixes every vertex in $\mathcal{P}(\Gamma) \setminus (S_1 \cup S_2)$.

(ii) For $2 \le i \le n_1$ and vertex $u \in K' \setminus \mathrm{St}(w''_i)$, u and $g_*(q'_*(u))$ are in the same w''_i-sub-tier.

(iii) For $n_1 < i \le b$, vertices of $L'' \setminus \mathrm{St}(w''_i)$ are contained in one w''_i-sub-tier.

Part (i) follows from property (b) of g_* and Lemma 6.14.

We now verify (ii). Assume $2 \le i \le n_1$, then $w''_i = w'_i$. First we consider the case $d(w'_i, w'_b) \ge 2$. As g_* fixes every point in B', we know g_* induces trivial permutation of w'_i-branches (indeed, for a w'_i-branch D, if $w'_b \notin D$, then $g_*(D) = D$ by Lemma 6.14 and $g_*|_{B'} = \mathrm{Id}$; if $w'_b \in D$, then $g_*(D) = D$ is still true since g_* fixes w'_b and w'_i). Now (2) follows from the inductive assumption. If $d(w'_i, w'_b) = 1$, since $q'_*(u) \in L'$, $q'_*(u)$ and $g_*(q'_*(u))$ are in the same w'_i-sub-tier by property (c) of g_*. But u and $q'_*(u)$ are in the same w'_i-sub-tier by induction, thus (ii) follows.

It remains to verify (iii). Suppose $n_1 < i \le b$. Then $w''_i = w'_i$ and $w''_b = w'_b$. Since $g_*(L') = L''$, the case $i < b$ and $d(w'_i, w'_b) = 1$ follows from inductive assumption and property (c) of g_*. If $i < b$ and $d(w'_i, w'_b) = 2$, then we argue as before to see that g_* induces trivial permutation of w'_i-branches. Then (iii) follows from the inductive assumption. If $i = b$, by Lemma 6.14, vertices of $K' \setminus \mathrm{St}(w'_b)$ and v are in the same w'_b-branch, then (iii) follows from property (a) of g_*.

After applying the above induction process for finitely many times, we obtain a simplicial isomorphism $h_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ which satisfies (1) in Lemma 6.32. Moreover, let

$$\widetilde{L} = K' \cup h_*(K'),$$

and let $\{\widetilde{w}_i\}_{i=1}^n$ be the collection of vertices such that $\mathrm{St}(\widetilde{w}_i)$ separates two vertices of $\widetilde{L}$. Then

(1) For $2 \le i \le n_1$ and vertex $u \in K' \setminus \mathrm{St}(\widetilde{w}_i)$, u and $h_*(u)$ are in the same $\widetilde{w}_i$-sub-tier.

(2) For $n_1 < i \le n$, vertices of $\widetilde{L} \setminus \mathrm{St}(\widetilde{w}_i)$ are contained in one $\widetilde{w}_i$-sub-tier.

Let $2 \le i \le n_1$. Since $h_*(\widetilde{w}_i) = \widetilde{w}_i$ and vertices of $K' \setminus \mathrm{St}(\widetilde{w}_i)$ are contained in one $\widetilde{w}_i$-tier, (1) implies actually that vertices of $\widetilde{L} \setminus \mathrm{St}(\widetilde{w}_i)$ are contained in one $\widetilde{w}_i$-tier. Thus $\widetilde{L}$ satisfies the assumption of Lemma 6.11, and (2) of Lemma 6.32 follows. Note that K' is a full subcomplex, so is $h_*(K')$, then

$$\bar{h}_*(K) = \pi(h_*(K'))$$

is a full subcomplex.

Pick vertex $x_0 \in \bigcap_{w \in \widetilde{L}} P_w$ (x_0 may not be equal to x), then $\widetilde{L} \subset (F(\Gamma))_{x_0}$. Let

$$i_{x_0} : F(\Gamma) \to (F(\Gamma))_{x_0} \subset \mathcal{P}(\Gamma)$$

be the natural embedding. Then $i_{x_0}(M) = \widetilde{L}$, and (3a) follows from property (2) of h_* and Lemma 6.21. Now we look at (3b). For vertex $\bar{k} \in K$,

$$\begin{aligned} d(\bar{k}, \bar{u}) &= d(i_{x_0}(\bar{k}), i_{x_0}(\bar{u})) = d(h_* \circ i_{x_0}(\bar{k}), i_{x_0}(\bar{u})) \\ &= d(\pi \circ h_* \circ i_{x_0}(\bar{k}), \pi \circ i_{x_0}(\bar{u})) = d(\bar{h}_*(\bar{k}), \bar{u}). \end{aligned}$$

The first and the third equalities follow from Lemma 2.14, and the second equality holds since h_* fixes $i_{x_0}(\overline{u})$. Thus the first part of (3b) is true. The rest of (3b) follows from property (1) of h_*. ■

6.5. Recognizing special subgroups

Let $L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_s = X$ be the filtration of X defined in Section 6.4. Let Γ' be the 1-skeleton of $\Phi(L_1)$. Note that L_1 is a single vertex, and Lemma 6.30 implies that the isomorphism type of Γ' does not depend on the choice of L_1. The main goal of this subsection is the following.

Proposition 6.33. *The group $G(\Gamma')$ is a special subgroup of $G(\Gamma)$, and $G(\Gamma')$ is prime.*

The following is an immediate consequence of this proposition.

Theorem 6.34. *Let $G(\Gamma)$ be a RAAG of type II. Then there exists a prime RAAG $G(\Gamma')$ such that $G(\Gamma)$ is isomorphic to a special subgroup of finite index in $G(\Gamma')$.*

Now we prove Proposition 6.33.

Proof. For convex subcomplex $E \subset X(\Gamma')$, let $\{\ell_\lambda\}_{\lambda\in\Lambda}$ be the collection of standard geodesics in $X(\Gamma')$ such that $\ell_\lambda \cap E \neq \emptyset$. Denote the full subcomplex in $\mathcal{P}(\Gamma')$ spanned by $\{\Delta(\ell_\lambda)\}_{\lambda\in\Lambda}$ by $\widehat{E}$. An edge $e \subset X(\Gamma')$ is called a *v-edge* for $v \in \mathcal{P}(\Gamma)$ if $\Delta(\ell_e) = v$ (ℓ_e is the standard geodesic containing e). An edge $e \subset X$ is called a *$\overline{v}$-edge* for $\overline{v} \in F(\Gamma)$ if the ultrafilters corresponding to two vertices of e differ by a $\overline{v}$-halfspace.

We are going to define a sequence of simplicial embeddings $f_i : \Phi(L_i) \to \mathcal{P}(\Gamma')$ and cubical embeddings $g_i : L_i \to X(\Gamma')$ with

$$f_{i+1}|_{\Phi(L_i)} = f_i \quad \text{and} \quad g_{i+1}|_{L_i} = g_i$$

which satisfy the following compatibility conditions:

(1) $g_i(L_i)$ is a compact convex subcomplex of $X(\Gamma')$.

(2) For any vertex $x \in L_i$, $f_i(\Phi(x)) = \widehat{g_i(x)}$. In particular, $f_i(\Phi(L_i)) = \widehat{g_i(L_i)}$.

(3) g_i sends a $\overline{v}$-edge to an $f_i(\overline{v})$-edge.

Note that the existence of the embedding when $i = s$ will imply $G(\Gamma')$ is a special subgroup of $G(\Gamma)$.

We need several observations before the construction of g_i and f_i. Pick vertex $v \in \mathcal{P}(\Gamma')$, then the vertices of P_v are exactly those vertices $x \in X(\Gamma')$ with $v \in \hat{x}$. Let ℓ_v be a standard geodesic such that $\Delta(\ell_v) = v$, and let h_v be a hyperplane dual to ℓ_v. We identify P_v with $\ell_v \times h_v$. Then $e \subset X(\Gamma')$ is a v-edge if and only if $e \in P_v$ and e has trivial projection to the h_v-factor. Actually, these statements have their analogues in X.

Let $\mathcal{W}_{\overline{v}}(\Gamma)$ be the collection of $\overline{v}'$-walls with $d(\overline{v}, \overline{v}') \leq 1$, and let $\mathcal{H}_{\overline{v}}(\Gamma)$ be corresponding collection of halfspaces. Denote the corresponding CAT(0) cube complex by $X_{\overline{v}}$. Let $\Sigma \subset \mathcal{H}(\Gamma)$ be the subset made of elements $(\overline{w}, R)$ such that $d(\overline{w}, \overline{v}) \geq 2$ and $\overline{v} \in R$. Pick an ultrafilter $U_{\overline{v}}$ of $\mathcal{H}_{\overline{v}}(\Gamma)$; it is easy to see every pair of halfspaces in $\Sigma \cup U_{\overline{v}}$ is compatible, thus $\Sigma \cup U_{\overline{v}}$ is an ultrafilter of $\mathcal{H}(\Gamma)$, and this induces a cubical embedding

$$i_{\overline{v}} : X_{\overline{v}} \to X.$$

Note that $i_{\overline{v}}(X_{\overline{v}})$ is convex in X since two walls in $\mathcal{W}_{\overline{v}}(\Gamma)$ are transverse in $\mathcal{W}_{\overline{v}}(\Gamma)$ if and only if they are transverse in $\mathcal{W}(\Gamma)$. Since every $\overline{v}$-wall is transverse to all $\overline{w}$-walls with $d(\overline{w}, \overline{v}) = 1$, $X_{\overline{v}}$ admits a canonical splitting

$$X_{\overline{v}} = h_{\overline{v}} \times [0, d_{\overline{v}} - 1],$$

where $h_{\overline{v}}$ is isomorphic to the hyperplane in X corresponding to a $\overline{v}$-wall, and $d_{\overline{v}}$ is the number of prime factors at $\overline{v}$. We will view $X_{\overline{v}}$ as a convex subcomplex of X. Note that vertices of $X_{\overline{v}}$ are exactly those vertices x with $\overline{v} \subset \Phi(x)$. Moreover, $e \subset X$ is a $\overline{v}$-edge if and only if $e \in X_{\overline{v}}$ and e has trivial projection to the $h_{\overline{v}}$-factor.

Suppose we have already constructed g_i and f_i. Let e_i, h_i, N_i and $(\overline{v}, H_i, H_i^c) = \Phi(h_i)$ be as in Step 2, and let $v = f_i(\overline{v})$. Pick vertex $x \in L_i$. Then

$$x \in X_{\overline{v}} \Leftrightarrow \overline{v} \in \Phi(x) \Leftrightarrow v \in f_i(\Phi(x)) \Leftrightarrow v \in \widehat{g_i(x)} \Leftrightarrow g_i(x) \in P_v.$$

Thus g_i induces an isomorphism between $X_{\overline{v}} \cap L_i$ and $P_v \cap g_i(L_i)$. Let

$$X_{\overline{v}} \cap L_i = \overline{K}_i \times \overline{I}_i,$$

and

$$P_v \cap g_i(L_i) = K_i \times I_i$$

be the splitting induced from the splitting of $X_{\overline{v}}$ and P_v as above ($\overline{K}_i \subset h_{\overline{v}}$, $K_i \subset h_v$, $\overline{I}_i \subset [0, d_{\overline{v}} - 1]$ and $I_i \subset \ell_v$). By (3), $g_i|_{X_{\overline{v}} \cap L_i} = g_{i1} \times g_{i2}$ with $g_{i1} : \overline{K}_i \to K_i$ and $g_{i2} : \overline{I}_i \to I_i$. Suppose $\overline{I}_i = [0, a]$, we identify I_i with $[0, a]$ via g_{i2} and consistently identify ℓ_v with $\mathbb{R}$.

Since e_i is a $\overline{v}$-edge, $e_i \subset X_{\overline{v}}$. We assume without loss of generality that

$$x_i = e_i \cap L_i \in \overline{K}_i \times \{a\}.$$

Then

$$M_{i1} = L_i \cap N_i = \overline{K}_i \times \{a\}$$

and

$$N_i = \overline{K}_i \times [a, a+1].$$

Similarly, $g_i(M_{i1}) = K_i \times \{a\}$. Note that g_{i1} induces an isomorphism from $\overline{K}_i \times [a, a+1]$ to $K_i \times [a, a+1]$; this defines

$$g_{i+1} : L_{i+1} = L_i \cup N_i \to g_i(L_i) \cup (K_i \times [a, a+1]).$$

Moreover, $h_v \times [a, a+1]$, which is the carrier of the hyperplane $h_v \times \{a + 1/2\}$, satisfies

$$(h_v \times [a, a+1]) \cap g_i(L_i) = K_i \times \{a\},$$

so

$$g_i(L_i) \cup (K_i \times [a, a+1])$$

is a compact convex subcomplex in $X(\Gamma')$.

We consider the left action $G(\Gamma') \curvearrowright X(\Gamma')$, and let $\alpha \in G(\Gamma')$ be the translation along ℓ_v such that

$$\alpha(K_i \times \{a\}) = K_i \times \{a+1\}.$$

Then α induces an isomorphism

$$\alpha_* : \widehat{K_i \times \{a\}} \to \widehat{K_i \times \{a+1\}}.$$

It is clear that $\alpha_*(\hat{x}) = \widehat{\alpha(x)}$ for vertex $x \in K_i \times \{a\}$ and α sends v-edge to $\alpha_*(v)$-edge. We define f_{i+1} by

$$f_{i+1}(z) = \begin{cases} f_i(z) & \text{if } z \in \Phi(L_i), \\ (\alpha_* \circ f_i \circ (\bar{h}_*)^{-1})(z) & \text{if } z \in \Phi(M_{i2}). \end{cases}$$

Note that

$$f_i(z) = (\alpha_* \circ f_i \circ (\bar{h}_*)^{-1})(z)$$

for

$$z \in \Phi(L_i) \cap \Phi(M_{i2}) = \mathrm{St}(\bar{v}, \Phi(M_{i1}))$$

(cf. Lemma 6.30 (2)), so f_{i+1} is well defined. Now we show f_{i+1} and g_{i+1} satisfy the compatibility conditions (2) and (3). Since

$$g_{i+1}|_{M_{i2}} = \alpha \circ g_i \circ p^{-1},$$

(where $p^{-1} : M_{i2} \to M_{i1}$ is the parallelism map), it suffices to check p^{-1} and $(\bar{h}_*)^{-1}$ satisfy the corresponding compatibility conditions. However,

$$(\bar{h}_*)^{-1}(\Phi(x)) = \Phi(p^{-1}(x))$$

for vertex $x \in M_{i2}$ by Lemma 6.30 (1). Let $e \subset M_{i2}$ be a $\bar{w}$-edge. Then $p^{-1}(e)$ is also a $\bar{w}$-edge. We also deduce that $d(\bar{w}, \bar{v}) = 1$, hence $(\bar{h}_*)^{-1}(\bar{w}) = \bar{w}$. It follows that p^{-1} sends $\bar{w}$-edge to $(\bar{h}_*)^{-1}(\bar{w})$-edge.

Let $f : F(\Gamma) \to \mathcal{P}(\Gamma')$ and $g : X \to X(\Gamma')$ be the simplicial embedding and the cubical embedding obtained by the above induction. Then $E = g(X)$ is a compact convex subcomplex of $X(\Gamma')$ and

$$\hat{E} = f(F(\Gamma)) \cong F(\Gamma).$$

Thus $G(\Gamma)$ is isomorphic to a special subgroup of $G(\Gamma')$. In particular, $G(\Gamma')$ and $G(\Gamma)$ are quasi-isometric, so Γ' is also of type II by Corollary 3.18. Next we show $G(\Gamma')$ is prime.

Take a vertex in $\bar{u} \in F(\Gamma')$. Let $x \in E$ be a vertex, and let $v \in \hat{x}$ be the lift of $\bar{u}$. Put $\bar{v} = f^{-1}(v)$. Let $r : X(\Gamma') \to X(\Gamma)$ be the map in Theorem 2.16, and let $r_* : \mathcal{P}(\Gamma') \to \mathcal{P}(\Gamma)$ be the induced simplicial isomorphism. We claim the stretch factor of r_* at v is $d_{\bar{v}}$. Note that this claim and Lemma 6.9 imply $\bar{u}$ is prime.

It remains to prove the claim. Lemma 6.9 implies that this stretch factor is upper bounded by the number of prime factors at $\pi \circ r_*(v)$, which is equal to $d_{\bar{v}}$ by Remark 6.20 (note that the composition

$$F(\Gamma) \xrightarrow{f} \mathcal{P}(\Gamma') \xrightarrow{r_*} \mathcal{P}(\Gamma) \xrightarrow{\pi} F(\Gamma)$$

is a simplicial isomorphism). Now we produce the lower bound.

By considering the g-image of $X_{\bar{v}} \cong h_{\bar{v}} \times [0, d_{\bar{v}} - 1]$, we deduce that there is a segment $I_v \subset E$ of length $= d_{\bar{v}} - 1$ such that it is made of v-edges and it contains x. Let $\{x_1, \dots, x_{d_{\bar{v}}}\}$ be vertices of I_v. Then each component C of $F(\Gamma') \setminus \mathrm{St}(\bar{u})$ gives rise to a component C_i of $\hat{x}_i \setminus \mathrm{St}(v)$ via

$$F(\Gamma') \setminus \mathrm{St}(\bar{u}) \cong \hat{x} \setminus \mathrm{St}(v).$$

Let B_{C_i} be the v-branch containing C_i. It follows from Corollary 3.13 that $B_{C_i} \neq B_{C_j}$ unless $i = j$; moreover, for a component C' of $F(\Gamma') \setminus \mathrm{St}(\bar{u})$ with $C' \neq C$, we have

$$B_{C_i} \neq B_{C'_j}$$

for $1 \le i, j \le d_{\bar{v}}$. Let Π be the map defined before Definition 6.4. Then

$$\Pi(B_{C_i}) = C$$

for $1 \le i \le d_{\bar{v}}$. However, as each B_{C_i} contains a component of

$$\hat{I}_v \setminus \mathrm{St}(v) \subset \hat{E} \setminus \mathrm{St}(v),$$

we know that

$$\Pi(r_*(B_{C_i})) \neq \Pi(r_*(B_{C_j}))$$

for $i \neq j$ and

$$\Pi(r_*(B_{C_i})) \neq \Pi(r_*(B_{C'_j}))$$

when $C' \neq C$ (note that $r(E)$ is a vertex in $X(\Gamma)$, and $r_*(\hat{E}) = \widehat{r(E)}$ by (1) and (2) of Theorem 2.16). This means stretch factor of r_* at v (Lemma 6.9) is $\ge d_{\bar{v}}$. Thus the claim is proved. ■

Remark 6.35. Suppose Γ is of type II. For each vertex $v \in \mathcal{P}(\Gamma)$, we pick an identification f_v between the collection of v-sub-tiers and a copy of integers $\mathbb{Z}_v$. A v-halfspace is a subcomplex of $\mathcal{P}(\Gamma)$ of form $\mathrm{St}(v) \cup f_v^{-1}([a,\infty))$ or $\mathrm{St}(v) \cup f_v^{-1}((-\infty,a])$, where $a \in \mathbb{Z}_v$. We can put a pocset structure on the collection of all these halfspaces in a similar way as before. Then the CAT(0) cube complex associated with this pocset is isomorphic to $X(\Gamma')$.

We will not use this fact, so we will not give the detailed argument. However, it is instructive to think about the case when $\mathrm{Out}(G(\Gamma))$ is finite. Then the cube complex associated with the above pocset is actually isomorphic to $X(\Gamma)$. So the quasi-isometry rigidity/flexibility of $G(\Gamma)$ is reflected in how hard it is to reconstruct $X(\Gamma)$ from $\mathcal{P}(\Gamma)$ via cubulation.

The following is a consequence of Corollary 3.18 and Theorems 6.10 and 6.34.

Theorem 6.36. *If $G(\Gamma_1)$ is a RAAG of type II, then $G(\Gamma_2)$ is quasi-isometric to $G(\Gamma_1)$ if and only if $G(\Gamma_2)$ is commensurable with $G(\Gamma_1)$. Moreover, there exists a unique prime RAAG $G(\Gamma)$ such that $G(\Gamma_1)$ and $G(\Gamma_2)$ are isomorphic to special subgroups of finite index in $G(\Gamma)$.*

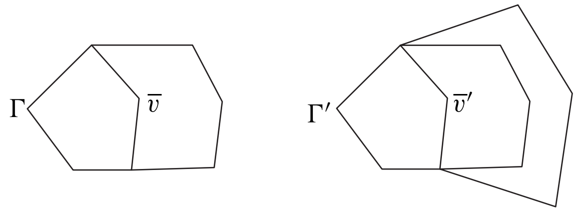
Now we discuss several related examples.

Example 6.37. Let Γ_1 be a pentagon, and let Γ_2 be a hexagon. We glue Γ_1 and Γ_2 along a vertex star to form Γ and claim Γ is prime. Let $\bar{v}$ be the only vertex of Γ such that $\mathrm{St}(\bar{v})$ separates Γ. For $i = 1, 2$, let $C_i = \Gamma_i \setminus \mathrm{St}(\bar{v})$. Pick $v \in \mathcal{P}(\Gamma)$ such that $\pi(v) = \bar{v}$, and let B_i be a v-branch such that $\Pi(B_i) = C_i$. It suffices to show B_1 and B_2 are not QII.

Suppose the contrary is true, and let q be the quasi-isometry such that $q_*(B_1) = B_2$. By Corollary 3.13, there exist vertices $x_1, x_2 \in P_v$ such that $(\Gamma_i)_{x_i} \setminus \mathrm{St}(v) \subset B_i$ for $i = 1, 2$ (where $(\Gamma_i)_{x_i} \subset (\Gamma)_{x_i}$ is the lift of Γ_i). Note that for any $v \in \mathcal{P}(\Gamma)$, $(\Gamma_1)_{x_1} \setminus \mathrm{St}(v)$ is contained in a single v-branch – this follows from Lemma 3.7. Thus for any $v \in \mathcal{P}(\Gamma)$, $q_*((\Gamma_1)_{x_1}) \setminus \mathrm{St}(v)$ is contained in a single v-branch. Then Lemma 6.11 implies that $\bigcap P_{v_i} \neq \emptyset$ with v_i ranges over vertices in $q_*((\Gamma_1)_{x_1})$. As $v \in q_*((\Gamma_1)_{x_1})$, we know $q_*((\Gamma_1)_{x_1}) = (\Gamma_1)_{x_3}$ for some vertex $x_3 \in P_v$. Take a vertex u in $(\Gamma_1)_{x_1} \setminus \mathrm{St}(v)$. Then $P_{q_*(u)} \cap P_v \neq \emptyset$ and $\pi(q_*(u)) \subset \pi((\Gamma_1)_{x_3}) = \Gamma_1$. On the other hand, $(\Gamma_2)_{x_2} \setminus \mathrm{St}(v) \subset B_2$. Then Lemma 3.8 implies that $q_*(B_1) \neq B_2$, which is a contradiction.

Theorem 6.36 implies that any $G(\Gamma')$ quasi-isometric to $G(\Gamma)$ is isomorphic to a finite index subgroup of $G(\Gamma)$. Note that by the same proof, this statement is true in the case when Γ is obtained by gluing two distinct graphs Γ_1 and Γ_2 ($\mathrm{Out}(G(\Gamma_i))$ is finite for $i = 1, 2$) along an isomorphic vertex star.

Example 6.38. Let Γ be a pentagon and a hexagon glued together over the star of a vertex. Let Γ' be a pentagon and two hexagons glued together over the star of a vertex. See the picture below. We claim $\mathcal{P}(\Gamma)$ and $\mathcal{P}(\Gamma')$ are isomorphic, but $G(\Gamma)$ and $G(\Gamma')$ are not quasi-isometric.



First we show $G(\Gamma)$ and $G(\Gamma')$ are not quasi-isometric. Suppose the contrary is true, and let $q: G(\Gamma) \to G(\Gamma')$ be a quasi-isometry. Let $\pi: \mathcal{P}(\Gamma) \to F(\Gamma)$ and $\pi': \mathcal{P}(\Gamma') \to F(\Gamma')$ be the canonical projection map defined after Lemma 2.12.

Pick vertex $v \in \mathcal{P}(\Gamma)$ such that $\pi(v) = \overline{v}$, and let $v' = q_*(v)$. Then $\pi'(v') = \overline{v}'$. This follows from the fact that $\pi(v) = \overline{v}$ (or $\pi'(v') = \overline{v}'$) if and only if there are at least two QII classes among all the v-branches (or v'-branches). This fact follows from the discussion in Example 6.37 and Corollary 3.13.

Now we compute the stretch factor of q_* at v. Corollary 3.13 implies that a v-tier contains two v-branches B_1 and B_2 such that $\Pi(B_1)$ and $\Pi(B_2)$ give the two connected components of $\Gamma \setminus \mathrm{St}(\overline{v})$. Suppose $\Pi(B_1)$ is contained in the pentagon subgraph of Γ, and $\Pi(B_2)$ is contained in the hexagon subgraph of Γ. Corollary 3.13 also implies that a v'-tier contains three v'-branches B'_1, B'_2 and B'_3 such that $\Pi(B'_1)$, $\Pi(B'_2)$ and $\Pi(B'_3)$ give the three connected components of $\Gamma' \setminus \mathrm{St}(\overline{v}')$. Suppose $\Pi(B'_2)$ and $\Pi(B'_3)$ are inside the two hexagons in Γ'. Then using the symmetry of Γ' that exchanges the two hexagons, we know B'_2 and B'_3 are QII. Thus the two hexagons give components of $\Gamma' \setminus \mathrm{St}(\overline{v}')$ that are QII.

We claim $\Pi(q_*(B_1)) = \Pi(B'_1)$. Let x_1 and $(\Gamma_1)_{x_1}$ be as in Example 6.37. We argue as in the second paragraph of Example 6.37 to see that there exists $x'_1 \in P_{v'}$ such that $q_*((\Gamma_1)_{x_1}) = (\Gamma'_1)_{x'_1}$, where Γ'_1 is the pentagon subgraph of Γ' and $(\Gamma'_1)_{x'_1}$ are defined in the same way as $(\Gamma_1)_{x_1}$. Thus $\Pi(q_*(B_1)) = \Pi(B'_1)$. Similarly, we can show $\Pi(q_*(B_2)) = \Pi(B'_2)$ or $\Pi(B'_3)$, $\Pi(q_*^{-1}(B'_1)) = \Pi(B_1)$ and $\Pi(q_*^{-1}(B'_2)) = \Pi(q_*^{-1}(B'_3)) = \Pi(B_2)$. Thus if we use B_1 to compute the stretch factor as in Definition 6.8, we will conclude that the factor is 1. If we use B_2 to compute the stretch factor as in Definition 6.8, we obtain that the factor is 2. This contradicts Lemma 6.9. Thus the quasi-isometry q does not exist.

It remains to show $\mathcal{P}(\Gamma)$ and $\mathcal{P}(\Gamma')$ are isomorphic. Let f_1 and f_2 be two simplicial embeddings from Γ to Γ' such that (1) they cover different hexagons in Γ'; (2) $f_1 = f_2$ when restricted to the pentagon in Γ. We also use f_i to denote the group monomorphism induced by f_i. Let $\omega \in G(\Gamma)$ be a geodesic word and write $\omega = \omega_1 a_1 \cdots \omega_n a_n \omega_{n+1}$, where a_i is a product of powers of elements in $\mathrm{St}(\overline{v})$ for all i, but ω_i does not contain any powers of elements in $\mathrm{St}(\overline{v})$ (ω_1 or ω_{n+1} may be trivial). By permuting letters in a_i, we have $a_i = \overline{v}^{k_i} b_i$, where b_i does not contain any power of $\overline{v}$.

Define a map $h: G(\Gamma) \to G(\Gamma')$ by mapping ω to $\omega'_1 a'_1 \cdots \omega'_n a'_n \omega'_{n+1}$ such that (1) $\omega'_i = f_1(\omega_i)$ if and only if $k_{i-1}/2$ is an integer, otherwise $\omega'_i = f_2(\omega_i)$; (2) $a'_i = f_1(a_i)$ if and only if the first letter of w_{i+1} is inside the pentagon, otherwise $a'_i = \overline{v}'^{\lfloor k_i/2 \rfloor} \cdot f_1(b_i)$. Given a different geodesic word $\omega_1 = \omega$, we can obtain ω_1 from ω by using the commutator relations to permute the letters in ω; moreover, each word in the middle is also a geodesic word. Now it is easy to check that h is well defined, and for each S-geodesic $\ell \subset G(\Gamma)$, there exists a unique S'-geodesic $\ell' \subset G(\Gamma')$ such that $h(\ell) = \ell'$ up to finitely many points; moreover, if two S-geodesic are parallel (or orthogonal), then the corresponding h-images are parallel (or orthogonal), thus h induces a simplicial map $h_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$. We can define a map $h' : G(\Gamma') \to G(\Gamma)$ in a similar fashion which serves as the inverse of h, which would imply that h_* is actually a simplicial isomorphism.

Acknowledgments. The author expresses gratitude to Ruth Charney, Bruce Kleiner and Thomas Koberda for helpful discussions and Do-Gyeom Kim for valuable comments. The author is very grateful to the anonymous referees for providing detailed comments, which greatly improved the writing of this paper.

References

[1] J. Behrstock, B. Kleiner, Y. Minsky, and L. Mosher, Geometry and rigidity of mapping class groups. *Geom. Topol.* **16** (2012), no. 2, 781–888 Zbl 1281.20045 MR 2928983

[2] J. A. Behrstock, T. Januszkiewicz, and W. D. Neumann, Quasi-isometric classification of some high dimensional right-angled Artin groups. *Groups Geom. Dyn.* **4** (2010), no. 4, 681–692 Zbl 1226.20033 MR 2727658

[3] J. A. Behrstock and W. D. Neumann, Quasi-isometric classification of graph manifold groups. *Duke Math. J.* **141** (2008), no. 2, 217–240 Zbl 1194.20045 MR 2376814

[4] M. Bestvina, B. Kleiner, and M. Sageev, The asymptotic geometry of right-angled Artin groups, I. *Geom. Topol.* **12** (2008), no. 3, 1653–1699 Zbl 1203.20038 MR 2421136

[5] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*. Grundlehren Math. Wiss. 319, Springer Berlin, Heidelberg, 1999 Zbl 0988.53001 MR 1744486

[6] P.-E. Caprace and M. Sageev, Rank rigidity for CAT(0) cube complexes. *Geom. Funct. Anal.* **21** (2011), no. 4, 851–891 Zbl 1266.20054 MR 2827012

[7] R. Charney, An introduction to right-angled Artin groups. *Geom. Dedicata* **125** (2007), no. 1, 141–158 Zbl 1152.20031 MR 2322545

[8] R. Charney, K. Ruane, N. Stambaugh, and A. Vijayan, The automorphism group of a graph product with no SIL. *Illinois J. Math.* **54** (2010), no. 1, 249–262 Zbl 1243.20047 MR 2776995

[9] I. Chatterji and G. Niblo, From wall spaces to CAT(0) cube complexes. *Internat. J. Algebra Comput.* **15** (2005), no. 5–6, 875–885 Zbl 1107.20027 MR 2197811

[10] C. Cunningham, A. Eisenberg, A. Piggott, and K. Ruane, CAT(0) extensions of right-angled Coxeter groups. *Topology Proc.* **48** (2016), 277–287 Zbl 1358.20040 MR 3431824

[11] C. Cunningham, A. Eisenberg, A. Piggott, and K. Ruane, Recognizing right-angled Coxeter groups using involutions. *Pacific J. Math.* **284** (2016), no. 1, 41–77 Zbl 1397.20046 MR 3530862

[12] M. G. Durham and S. J. Taylor, Convex cocompactness and stability in mapping class groups. *Algebr. Geom. Topol.* **15** (2015), no. 5, 2837–2857 Zbl 1364.20027 MR 3426695

[13] A. Eskin and B. Farb, Quasi-flats and rigidity in higher rank symmetric spaces. *J. Amer. Math. Soc.* **10** (1997), no. 3, 653–692 Zbl 0893.22004 MR 1434399

[14] V. Gerasimov, Fixed-point-free actions on cubings. *Siberian Adv. Math.* **8** (1998), no. 3, 36–58 Zbl 0912.20028 MR 1663779

[15] M. F. Hagen, Weak hyperbolicity of cube complexes and quasi-arboreal groups. *J. Topol.* **7** (2014), no. 2, 385–418 Zbl 1348.20044 MR 3217625
[16] F. Haglund, Finite index subgroups of graph products. *Geom. Dedicata* **135** (2008), no. 1, 167–209 Zbl 1195.20047 MR 2413337
[17] F. Haglund and F. Paulin, Simplicité de groupes d'automorphismes d'espaces à courbure négative. *Geom. Topol. Monogr.* **1** (1998), 181–248 Zbl 0916.51019 MR 1668359
[18] F. Haglund and D. T. Wise, Special cube complexes. *Geom. Funct. Anal.* **17** (2008), no. 5, 1551–1620 Zbl 1155.53025 MR 2377497
[19] U. Hamenstaedt, Geometry of the mapping class groups III: quasi-isometric rigidity. [v1] 2005, [v2] 2007, arXiv:math/0512429v2
[20] G. C. Hruska and D. T. Wise, Finiteness properties of cubulated groups. *Compos. Math.* **150** (2014), no. 3, 453–506 Zbl 1335.20043 MR 3187627
[21] J. Huang, Quasi-isometric classification of right-angled Artin groups, I: the finite out case. *Geom. Topol.* **21** (2017), no. 6, 3467–3537 Zbl 1404.20033 MR 3692971
[22] J. Huang, Top-dimensional quasiflats in CAT(0) cube complexes. *Geom. Topol.* **21** (2017), no. 4, 2281–2352 Zbl 1439.20045 MR 3654109
[23] S.-H. Kim and T. Koberda, Embedability between right-angled Artin groups. *Geom. Topol.* **17** (2013), no. 1, 493–530 Zbl 1278.20049 MR 3039768
[24] S.-H. Kim and T. Koberda, The geometry of the curve graph of a right-angled Artin group. *Internat. J. Algebra Comput.* **24** (2014), no. 2, 121–169 Zbl 1342.20042 MR 3192368
[25] B. Kleiner and B. Leeb, Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. *C. R. Acad. Sci. Ser. I Math.* **324** (1997), no. 6, 639–643 Zbl 0882.53037 MR 1447034
[26] M. R. Laurence, A generating set for the automorphism group of a graph group. *J. Lond. Math. Soc. (2)* **52** (1995), no. 2, 318–334 Zbl 0836.20036 MR 1356145
[27] B. Nica, Cubulating spaces with walls. *Algebr. Geom. Topol.* **4** (2004), no. 1, 297–309 Zbl 1131.20030 MR 2059193
[28] M. Roller, Poc sets, median algebras and group actions. An extended study of Dunwoody's construction and Sageev's theorem. 2016, arXiv:1607.07747v1
[29] M. Sageev, Ends of group pairs and non-positively curved cube complexes. *Proc. Lond. Math. Soc. (3)* **71** (1995), no. 3, 585–617 Zbl 0861.20041 MR 1347406
[30] M. Sageev, CAT(0) cube complexes and groups. In M. Bestvina, M. Sageev, and K. Vogtmann (eds.), *Geometric group theory*, pp. 7–54, IAS/Park City Math. Ser. 21, American Mathematical Society, Providence, RI, 2014 Zbl 1440.20015 MR 3329724
[31] H. Servatius, Automorphisms of graph groups. *J. Algebra* **126** (1989), no. 1, 34–60 Zbl 0682.20022 MR 1023285

Received 10 May 2016.

Jingyin Huang
Department of Mathematics, The Ohio State University, 231 W. 18th Ave, Columbus, OH 43210, USA; huang.929@osu.edu