Groups Geom. Dyn. 19 (2025), 1445–1478
DOI 10.4171/GGD/845

Isometric embeddings of surfaces for scl

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Abstract. Let $\varphi : F_1 \to F_2$ be an injective morphism of free groups. If φ is geometric (i.e., induced by an inclusion of oriented compact connected surfaces with nonempty boundary), then we show that φ is an isometric embedding for stable commutator length. More generally, we show that if T is a subsurface of an oriented compact (possibly closed) connected surface S, and c is an integral 1-chain on $\pi_1 T$, then there is an isometric embedding $H_2(T, c) \to H_2(S, c)$ for the relative Gromov seminorm. Those statements are proved by finding an appropriate standard form for admissible surfaces and showing that, under the right homology vanishing conditions, such an admissible surface in S for a chain in T is in fact an admissible surface in T.

1. Introduction

Stable commutator length is a function that measures the homological complexity of elements in a group. In a topological space X, an element $w \in \pi_1 X$ being homologically trivial means that a loop representing w bounds a surface in X. Stable commutator length measures the minimal complexity of such a surface, in a stable sense. More precisely, if (a power of) w is homologically trivial in $\pi_1 X$, the *stable commutator length* of w is

$$\mathrm{scl}_{\pi_1 X}(w) = \inf_{f,\Sigma} \frac{-\chi^-(\Sigma)}{2n(\Sigma)},$$

where the infimum is over all maps $f : \Sigma \to X$ from surfaces which send $\partial\Sigma$ to $w^{n(\Sigma)}$ for some $n(\Sigma) \in \mathbb{N}_{\geq 1}$, and $\chi^-(\Sigma)$ denotes the *reduced Euler characteristic* of Σ (i.e., the Euler characteristic after discarding disc and sphere components). Such maps f are called *admissible surfaces*. We also set $\mathrm{scl}_{\pi_1 X}(w) = \infty$ if no power of w is homologically trivial. This function $\mathrm{scl}_{\pi_1 X}$ is an invariant of the fundamental group $\pi_1 X$. We give more detailed definitions in Section 2.

Computations of scl remain elusive, and there are very few groups in which we can obtain exact values. A major exception is the case of free groups, in which Calegari [5] proved that scl is computable and has rational values. This was later generalised by Chen [10] to certain graphs of groups (encompassing previous results of Walker [26],

Mathematics Subject Classification 2020: 20F65 (primary); 20J05, 57M07 (secondary).
Keywords: stable commutator length, relative Gromov seminorm, surfaces, isometric embeddings, rationality.

Calegari [7], Chen [9], Susse [24] and Clay–Forester–Louwsma [11]). However, these examples are all, in some sense, 1-dimensional. A major open question is whether or not scl is computable and rational in closed surface groups.

Isometries for scl

The present paper aims to make progress towards understanding scl in free and surface groups by focusing on isometries. A group homomorphism $\varphi : G \to H$ is always scl-nonincreasing in the sense that $\mathrm{scl}_G(w) \geq \mathrm{scl}_H(\varphi(w))$ for all $w \in G$. We are interested in (injective) morphisms that preserve scl, that is, that satisfy $\mathrm{scl}_G(w) = \mathrm{scl}_H(\varphi(w))$ for all $w \in G$ – those morphisms are called *isometric embeddings*[1] for scl.

Previous isometry results include those of Calegari and Walker [8], who proved that a generic morphism between free groups preserves scl, as well as Chen [10], who proved that certain morphisms of graphs of groups are isometric. We give more precise statements of their results in Theorem 2.6.

There are several reasons why one might be interested in isometries of scl. One of them is that this can lead to a better understanding of the structure of the scl norm on the space $B_1(G;\mathbb{R})$ of real 1-boundaries on G. Calegari's rationality theorem [5] implies that, if G is a free group, then the unit ball of the scl norm is a rational polyhedron, and Calegari [4] also proved that certain top-dimensional faces of the scl norm ball are connected to dynamics via the rotation quasimorphism. Isometric endomorphisms of scl in a group G capture the symmetries of the scl norm ball in $B_1(G;\mathbb{R})$ and could reveal more information about scl in G.

Moreover, given an isometric embedding $\varphi : G \to H$ from a group G in which we can compute scl to another group H where scl is more mysterious, one can use knowledge about scl in G to learn about scl in H. For instance, Chen [10, Corollary 3.12] uses his isometric embedding theorem to show that the scl-spectrum of the Baumslag–Solitar group $\mathrm{BS}(m,\ell)$ contains the scl-spectrum of the free product $\mathbb{Z}/m * \mathbb{Z}/\ell$, which is better understood. Isometric embeddings from surfaces with boundary to closed surfaces are of particular interest to us, as they could potentially allow us to use what we know about scl in free groups to gain some ground in closed surface groups.

We first consider $\varphi : F_1 \to F_2$ a morphism of free groups. There is a condition that one might want to impose on φ in order to prove that it is an isometric embedding: If $\mathrm{scl}_{F_1}(w) = \infty$, then it is natural to ask that $\mathrm{scl}_{F_2}(\varphi(w)) = \infty$. This is equivalent to saying that the induced morphism $\varphi_* : H_1(F_1) \to H_1(F_2)$ on abelianisations should be injective. An isometric embedding that satisfies this condition will be called a *strong isometric embedding* (see Proposition 2.5). Our first result is that, if φ is *geometric* – in the sense that it is induced by an inclusion of surfaces with boundary – then this condition is sufficient.

[1] There are slight variations as to what different authors mean by isometries for scl, and we try to clarify the terminology in Definition 2.3.

Theorem A (Isometric embedding for scl). *Let S be an oriented, compact, connected surface with nonempty boundary and let $T \subseteq S$ be a subsurface. Consider the inclusion-induced map:*

$$\iota : \pi_1 T \to \pi_1 S.$$

If $\iota_* : H_1(\pi_1 T) \hookrightarrow H_1(\pi_1 S)$ is injective, then ι is a strong isometric embedding for scl.

In fact, it follows from our proof that extremal surfaces are preserved by the isometric embedding of Theorem A. This is also the case for the rotation quasimorphism when it is extremal. See Section 6 for a more detailed discussion.

Generalisation to closed surfaces

One might wonder if Theorem A generalises to closed surfaces. The answer is negative, as a simple example shows.

Example 1.1. Consider the inclusion of surfaces $T \hookrightarrow S$ of Figure 1. Then the induced map $H_1(T) \to H_1(S)$ is injective. However, let $w \in \pi_1 T \hookrightarrow \pi_1 S$ be the class of the boundary loop of T. It is a general fact that an oriented compact surface is extremal for its boundary (see [5, Lemma 4.62]), and therefore

$$\mathrm{scl}_{\pi_1 T}(w) = \frac{-\chi^-(T)}{2} = \frac{3}{2}.$$

On the other hand, there is a surface Σ of genus 1 with one boundary component bounding w in S, so that

$$\mathrm{scl}_{\pi_1 S}(w) \leq \frac{-\chi^-(\Sigma)}{2} = \frac{1}{2}.$$

Therefore, the inclusion-induced map $\pi_1 T \hookrightarrow \pi_1 S$ does not preserve scl.

However, this is not the end of the story. Given an admissible surface $\Sigma \to S$ bounding a power of a given loop $w \in \pi_1 T \hookrightarrow \pi_1 S$, one can consider the relative homology class represented by Σ in $H_2(S, w)$. In Example 1.1, the two admissible surfaces $T \to S$ and $\Sigma \to S$ represent classes in $H_2(S, w)$ that differ by the fundamental class $[S] \in H_2(S) \hookrightarrow H_2(S, w)$. Note that this is a phenomenon that cannot occur if S has nonempty boundary, because then $H_2(S) = 0$ and all admissible surfaces bounding w

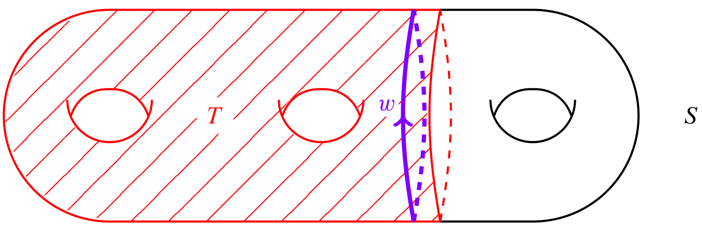


Figure 1. An inclusion of surfaces that is H_1-injective but not isometric for scl.

projectively represent the same class in $H_2(S, w)$. One might therefore ask whether we still get an isometric embedding when the relative homology class is fixed.

Isometries for the relative Gromov seminorm. We can make this more precise by introducing the Gromov seminorm on the degree-2 relative homology $H_2(S, w; \mathbb{Q})$. For a topological space X, the Gromov seminorm is defined on $H_n(X; \mathbb{Q})$ as the quotient seminorm of the ℓ^1-norm on the space of n-cycles. In degree 2, the Gromov seminorm of a class $\alpha \in H_2(X; \mathbb{Q})$ can also be interpreted as the infimum of $-2\chi^-(\Sigma)/n(\Sigma)$ over maps $\Sigma \to X$ from closed surfaces representing $n(\Sigma)\alpha$ for some $n(\Sigma) \in \mathbb{N}_{\geq 1}$.

Analogously, given $w \in \pi_1 X$, we define the *relative Gromov seminorm* of a class $\alpha \in H_2(X, w; \mathbb{Q})$ by

$$\|\alpha\|_{\mathrm{Grom}} = \inf_{f,\Sigma} \frac{-2\chi^-(\Sigma)}{n(\Sigma)},$$

where the infimum is over all maps $f : \Sigma \to X$ from surfaces with boundary representing $n(\Sigma)\alpha$ for some $n(\Sigma) \in \mathbb{N}_{\geq 1}$. Hence, $\mathrm{scl}(w)$ can be reinterpreted as the infimum of the relative Gromov seminorm on an affine subspace in $H_2(X, w; \mathbb{Q})$:

$$\mathrm{scl}(w) = \frac{1}{4} \inf_{\substack{\alpha \in H_2(X,w;\mathbb{Q}) \\ \partial\alpha = [S^1]}} \|\alpha\|_{\mathrm{Grom}}.$$

See Section 2.5 for more details on the relative Gromov seminorm and its relation to scl.

Asking whether an embedding of surfaces $T \hookrightarrow S$ is isometric when a relative homology class is fixed amounts to asking, given $w \in \pi_1 T$, whether or not the inclusion-induced map $H_2(T, w) \to H_2(S, w)$ is isometric for the relative Gromov seminorm.

We answer this question in the affirmative.

Theorem B (Isometric embedding for the relative Gromov seminorm). *Let S be an oriented, compact, connected surface, let $T \subseteq S$ be a π_1-injective subsurface, and let $c \in C_1(\pi_1 T; \mathbb{Z})$ be an integral chain in T. Then the inclusion-induced map*

$$\iota : H_2(T, c; \mathbb{Q}) \hookrightarrow H_2(S, c; \mathbb{Q})$$

is an injective isometric embedding for $\|\cdot\|_{\mathrm{Grom}}$.

Note that, if the surface S has nonempty boundary and w is homologically trivial, then $\mathrm{scl}_{\pi_1 S}(w) = \frac{1}{4}\|\alpha\|_{\mathrm{Grom}}$, where α is a generator of $H_2(S, w) \cong H_1(S^1)$. Therefore, in this context, Theorem B is really a statement about scl. But it has no assumption on H_1-injectivity, so it implies a stronger version of Theorem A.

Corollary C. *Let S be an oriented, compact, connected surface with nonempty boundary and let $T \subseteq S$ be a π_1-injective subsurface. Then the inclusion-induced map*

$$\iota : \pi_1 T \hookrightarrow \pi_1 S$$

is an isometric embedding for scl.

One of the points of this paper is to promote the study of the relative Gromov seminorm on $H_2(X, w)$ as an intermediate step in the computation of $\mathrm{scl}(w)$ when X has nontrivial second homology. This approach separates the problem of computing scl into two steps: One can try to understand the relative Gromov seminorm first, and then investigate the infimum in $H_2(X, w)$. Hence, some known results about scl in free groups might generalise to the relative Gromov seminorm in closed surface groups, for example, giving partial information about scl there.

Another instance of this phenomenon is that, even though extremal surfaces are not known to exist for arbitrary elements of closed surface groups, Calegari [4, Remark 3.18] proved that, if S is a closed surface and $w \in \pi_1 S$, then w rationally bounds a positive immersed surface in S, and this immersed surface is extremal in its relative homology class. In particular, there is a class $\alpha \in H_2(S, w)$ such that $\|\alpha\|_{\mathrm{Grom}}$ is rational.

The hope is that ideas that were successfully applied to the study of scl in free groups could be used to understand the relative Gromov seminorm in surface groups and wider classes of groups.

Strategy of proof

Let $T \subseteq S$ be a subsurface and let $w \in \pi_1 T$. The general idea to prove Theorems A and B is the following: Let $f : (\Sigma, \partial\Sigma) \to (S, w)$ be an admissible surface for w in S. The goal is to modify f to an admissible surface for w in T, as this will show that $\mathrm{scl}_{\pi_1 T}(w) \leq \mathrm{scl}_{\pi_1 S}(w)$, and the reverse inequality always holds. Note that the assumption of Theorem A that $H_1(\pi_1 T) \to H_1(\pi_1 S)$ be injective is equivalent (in the case where $\partial S \neq \emptyset$) to $H_2(S, T) = 0$, which means that every 2-chain in S with boundary in T does in fact lie in T. In particular, if $b \in C_2(\Sigma)$ is a 2-chain representing the fundamental class $[\Sigma] \in H_2(\Sigma, \partial\Sigma)$, then this implies that $f_* b \in C_2(T)$. If f is an embedding, then we can conclude that $f(\Sigma) \subseteq T$. However, this does not follow in general since it might be, for example, that a 2-cell σ in $S \setminus T$ is not visible in the 2-chain $f_* b$ because it appears once positively and once negatively, but still $\sigma \subseteq f(\Sigma)$.

We cannot, in general, assume that admissible surfaces are embedded. Our strategy is therefore to find a standard form for admissible surfaces that is general enough to allow one to compute scl or the relative Gromov seminorm, but nice enough to make the above argument work. Our standard form is described in Section 4 and, in particular, in Proposition 4.9; we expect it to be a helpful foundation for further study of scl in surface groups. With this standard form in hand, we can adapt the above argument to show that, under appropriate homology vanishing conditions, any admissible surface in S for $w \in \pi_1 T$ is in fact contained in T. This is the content of Theorem 5.1, which implies Theorems A and B.

Outline of the paper

In Section 2, we recall the algebraic and topological definitions of scl, and introduce the relative Gromov seminorm. We then go on to introduce 2-complexes and discuss some of their topological properties that are relevant for our proof in Section 3. In Section 4, we show how to reduce admissible surfaces for computations of scl and $\|\cdot\|_{\mathrm{Grom}}$ in surface groups to a certain standard form. Our main results are proved in Section 5. Finally, Section 6 is a discussion of how extremal surfaces and quasimorphisms behave with respect to our isometric embeddings.

2. scl and the relative Gromov seminorm

2.1. Stable commutator length algebraically

We give the definition of commutator length and stable commutator length via products of commutators. We will work with 1-chains throughout this paper, but the reader can harmlessly forget about chains and think about elements of a group.

Let G be a group. A *commutator* in G is an expression of the form $[a,b] = aba^{-1}b^{-1}$, for some $a, b \in G$. The *commutator length* $\mathrm{cl}_G(w)$ of an element $w \in G$ is its word length with respect to the set of all commutators:

$$\mathrm{cl}_G(w) = \inf\{g \geq 1 \mid \exists a_1, b_1, \dots, a_g, b_g \in G,\ w = [a_1,b_1]\cdots[a_g,b_g]\} \\ \in \mathbb{N}_{\geq 0} \cup \{\infty\},$$

where we agree that $\inf \emptyset = \infty$.

More generally, given finitely many elements $w_1, \dots, w_k \in G$, we set

$$\mathrm{cl}_G(w_1 + \cdots + w_k) = \inf_{t_i \in G} \mathrm{cl}_G(w_1(t_1 w_2 t_1^{-1}) \cdots (t_{k-1} w_k t_{k-1}^{-1})).$$

Given $R = \mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$, we will denote by $C_n(G;R)$ the group of n-chains over R, that is, the free R-module with basis G^n. These form a chain complex $C_*(G;R)$, called the *bar complex*. We will write $Z_n(G;R)$ for the group of n-cycles and $B_n(G;R)$ for the group of n-boundaries. See [27, Chapter 6] for more details. Note that there are natural inclusions $C_n(G;\mathbb{Z}) \hookrightarrow C_n(G;\mathbb{Q}) \hookrightarrow C_n(G;\mathbb{R})$. A chain in $C_n(G;\mathbb{R})$ will be called *real*, a chain in $C_n(G;\mathbb{Q})$ will be called *rational* and a chain in $C_n(G;\mathbb{Z})$ will be called *integral*.

Definition 2.1. Given an integral 1-chain $c = \sum_i n_i w_i \in C_1(G;\mathbb{Z})$ (with $n_i \in \mathbb{Z}$, $w_i \in G$), the *stable commutator length* of c is defined by

$$\mathrm{scl}_G(c) = \lim_{m\to\infty} \frac{\mathrm{cl}_G\big(\sum_i (w_i^{n_i})^m\big)}{m}.$$

The map $\mathrm{scl}_G : C_1(G;\mathbb{Z}) \to [0,\infty]$ is then extended to $C_1(G;\mathbb{Q})$ by linearity on rays, and to $C_1(G;\mathbb{R})$ by continuity.

For more details on why these definitions make sense, we refer the reader to Calegari's book [5, Section 2.6].

Observe that, given $c \in C_1(G;\mathbb{R})$, we have $\mathrm{scl}_G(c) < \infty$ if and only if $c \in B_1(G;\mathbb{R})$. If in addition $H_1(G)$ is torsion-free, then it is also true that, given $c \in C_1(G;\mathbb{Z})$, we have $\mathrm{scl}_G(c) < \infty$ if and only if $c \in B_1(G;\mathbb{Z})$.

2.2. Isometric embeddings

The following is immediate from the definition.

Proposition 2.2 (Monotonicity of scl). *Let $\varphi : G \to H$ be a group homomorphism. Then:*

(i) *For any $w_1, \dots, w_k \in G$, the following inequality holds:*

$$\mathrm{cl}_G(w_1 + \cdots + w_k) \geq \mathrm{cl}_H(\varphi(w_1) + \cdots + \varphi(w_k)).$$

(ii) *For any $c \in C_1(G; \mathbb{R})$, the following inequality holds:*

$$\mathrm{scl}_G(c) \geq \mathrm{scl}_H(\varphi(c)).$$

Hence, a group homomorphism is always scl-nonincreasing, and we would like to understand when a group homomorphism preserves scl.

Definition 2.3. Let $\varphi : G \to H$ be a group homomorphism.

- We say that φ is scl-*preserving* if for every 1-boundary $c \in B_1(G; \mathbb{R})$, the following equality holds:
$$\mathrm{scl}_G(c) = \mathrm{scl}_H(\varphi(c)). \tag{2.1}$$
- We say that φ is an *isometric embedding* for scl if it is injective and scl-preserving.
- We say that φ is a *strong isometric embedding* for scl if it is injective and (2.1) holds for every 1-chain $c \in C_1(G; \mathbb{R})$.

Remark 2.4. In Definition 2.3, replacing $\mathbb{R}$ with $\mathbb{Q}$ or $\mathbb{Z}$ does not change what it means for a group homomorphism to be scl-preserving or a (strong) isometric embedding for scl.

It is clear that a strong isometric embedding for scl is also an isometric embedding since $B_1(G; \mathbb{R}) \subseteq C_1(G; \mathbb{R})$. The following clarifies the relation between isometries and strong isometries.

Proposition 2.5. *Let $\varphi : G \to H$ be an isometric embedding for* scl. *Then φ is a strong isometric embedding if and only if the induced map*

$$\varphi_* : H_1(G; \mathbb{Q}) \to H_1(H; \mathbb{Q})$$

is injective.

Proof. Note that φ being a strong isometric embedding means that it preserves the stable commutator length of all chains in $C_1(G; \mathbb{Q})$, not just those in $B_1(G; \mathbb{Q})$. Equivalently, for each $c \in C_1(G; \mathbb{Q})$ such that $\varphi(c)$ is a 1-boundary, c itself is a 1-boundary. But the boundary map $C_1(G; \mathbb{Q}) \to C_0(G; \mathbb{Q})$ is the zero map, so $C_1(G; \mathbb{Q}) = Z_1(G; \mathbb{Q})$. Hence, φ is a strong isometric embedding if and only if the preimage under $\varphi : Z_1(G;\mathbb{Q}) \to Z_1(H;\mathbb{Q})$ of $B_1(H;\mathbb{Q})$ is precisely $B_1(G;\mathbb{Q})$, which means that the induced map

$$\varphi_* : H_1(G;\mathbb{Q}) \to H_1(H;\mathbb{Q})$$

is injective. ∎

We summarise here some known results on isometries of scl, focusing on free groups.

Theorem 2.6. *The following are examples of isometric embeddings for* scl.

(i) *Any left-invertible map $\varphi : G \to H$ is a strong isometric embedding for* scl. *(This follows from Proposition 2.2.)*

(ii) (Calegari [7, Corollary 3.16]) *Let F_m and F_n be free groups with respective free bases $(a_1, \dots, a_m)$ and $(b_1, \dots, b_n)$, with $m \leq n$, and let $\varphi : F_m \to F_n$ be given by*

$$\varphi : a_i \mapsto b_i^{k_i},$$

with $k_i \in \mathbb{Z} \setminus \{0\}$. Then φ is a strong isometric embedding for scl.

(iii) (Chen [10, Theorem 3.8]) *Let $\mathbb{X}, \mathbb{Y}$ be graphs of groups such that* scl *vanishes on all vertex groups. Assume that we are given an edge-injective morphism $h : X \to Y$ between the underlying graphs, monomorphisms $h_v : X_v \hookrightarrow Y_{h(v)}$ between the vertex groups and isomorphisms $h_e : X_e \xrightarrow{\cong} Y_{h(e)}$ between the edge groups, that commute with the inclusions of the edge groups in the vertex groups, and such that each map h_v induces a morphism that is injective in homology on the sum of the images of the incident edge groups. Then there is an induced map $\underline{h} : \pi_1\mathbb{X} \to \pi_1\mathbb{Y}$, and $\underline{h}$ is an isometric embedding for* scl.

(iv) (Calegari–Walker [8, Theorem 3.16]) *Let F_m and F_n be free groups of respective ranks m and n. Then there is a constant $C > 1$ such that a random homomorphism $\varphi : F_m \to F_n$ of length k is* scl*-preserving with probability $1 - O(C^{-k})$.*

2.3. Stable commutator length topologically

Stable commutator length can be given a topological interpretation, and we will use this interpretation as a working definition throughout this paper.

Fix a topological space X with $\pi_1 X = G$, and let $c = \sum_i n_i w_i \in C_1(G;\mathbb{Z})$ be an integral chain (with $n_i \in \mathbb{Z}$, $w_i \in G$). We assume that the w_i's are pairwise distinct, so that this decomposition of c is unique. For each i, we can pick a loop $\gamma_i : S^1 \to X$ representing the conjugacy class of $w_i^{n_i}$ in X, where S^1 is the (oriented) circle. Putting those together, we get a map $\gamma : \coprod_i S^1 \to X$. Note that γ is uniquely defined up to homotopy.

An *admissible surface*[2] for c in X is the data of an oriented compact (possibly disconnected) surface Σ, and of maps $f : \Sigma \to X$ and $\partial f : \partial\Sigma \to \coprod_i S^1$ making the following diagram commute:

$$\begin{array}{ccc} \partial\Sigma & \xrightarrow{\ \iota\ } & \Sigma \\ {\scriptstyle \partial f}\downarrow & & \downarrow{\scriptstyle f} \\ \coprod_i S^1 & \xrightarrow{\ \gamma\ } & X \end{array} \tag{2.2}$$

where $\iota : \partial\Sigma \hookrightarrow \Sigma$ is the inclusion. Such an admissible surface will be denoted by $f : (\Sigma, \partial\Sigma) \to (X, c)$.

The complexity of a compact connected surface Σ is measured by its *reduced Euler characteristic* $\chi^-(\Sigma) = \min\{0, \chi(\Sigma)\}$. If Σ is disconnected, we set $\chi^-(\Sigma) = \sum_K \chi^-(K)$, where the sum ranges over all connected components K of Σ.

Proposition 2.7 (Calegari [5, Proposition 2.74]). *If X is a space with $\pi_1 X = G$, and $c \in C_1(G;\mathbb{Z})$ is an integral chain, then there is an equality*

$$\mathrm{scl}_G(c) = \inf_{f,\Sigma} \frac{-\chi^-(\Sigma)}{2n(\Sigma)},$$

where the infimum is taken over all admissible surfaces $f : (\Sigma, \partial\Sigma) \to (X, c)$ such that $\partial f_*[\partial\Sigma] = n(\Sigma)\big[\coprod_i S^1\big]$ in $H_1\big(\coprod_i S^1;\mathbb{Z}\big)$ for some $n(\Sigma) \in \mathbb{N}_{\geq 1}$.

Such an admissible surface will be called an admissible surface for $\mathrm{scl}_G(c)$.

An admissible surface is called *extremal* if it realises the infimum in Proposition 2.7. Calegari [6] proved that extremal surfaces exist for all $c \in B_1(G;\mathbb{Z})$ if G is a free group. It follows that scl_G has rational values in this case.

2.4. The Gromov seminorm on homology

We recall here the definition of the Gromov seminorm, which was introduced in [16].

We will work with rational coefficients throughout.

Let X be a topological space and let $C_*(X;\mathbb{Q})$ denote its singular chain complex over $\mathbb{Q}$. Each module $C_n(X;\mathbb{Q})$ can be equipped with the ℓ^1-norm $\|\cdot\|_1$ defined by $\|\sum_i \lambda_i\sigma_i\|_1 = \sum_i |\lambda_i|$ (with $\lambda_i \in \mathbb{Q}$ and $\sigma_i : \Delta^n \to X$ a singular n-simplex). The *Gromov seminorm* (or *ℓ^1-seminorm*) on $H_n(X;\mathbb{Q})$ is defined to be the quotient seminorm:

$$\|\alpha\|_{\mathrm{Grom}} = \inf_{[a]=\alpha} \|a\|_1,$$

[2] Note that, contrary to the standard definition [5, Section 2.6.1], we impose no condition on $\partial f_*[\partial\Sigma]$ at this point. The reason why we are doing this should become clear in Section 2.5: we will want to consider admissible surfaces for all classes in $H_2(X, c)$, not just those mapping to $\big[\coprod_i S^1\big]$ under $\partial : H_2(X, c) \to H_1\big(\coprod_i S^1\big)$.

where the infimum is taken over all n-cycles $a \in Z_n(X;\mathbb{Q})$ representing the class $\alpha \in H_n(X;\mathbb{Q})$.

The Gromov seminorm on $H_2(X;\mathbb{Q})$ has the following geometric interpretation (see [20, Proposition 2.7] or [5, Section 1.2.5] for a proof).

Proposition 2.8. *If X is a topological space, then the Gromov seminorm of $\alpha \in H_2(X;\mathbb{Q})$ is given by*

$$\|\alpha\|_{\mathrm{Grom}} = \inf_{f,\Sigma} \frac{-2\chi^-(\Sigma)}{n(\Sigma)},$$

where the infimum is taken over all maps $f : \Sigma \to X$ from oriented closed surfaces Σ such that $f_*[\Sigma] = n(\Sigma)\alpha$ for some $n(\Sigma) \in \mathbb{N}_{\geq 1}$.

2.5. The relative Gromov seminorm

The Gromov seminorm is related to scl via the filling norm as explained in [5, Section 2.6], but it is another connection that we would like to discuss here. We will introduce an analogue of $\|\cdot\|_{\mathrm{Grom}}$ on $H_2(X,c)$, where X is a space and c is a chain, and whose calculation is an intermediate step in the calculation of scl.

We first explain what we mean by the homology of a space relative to a chain. Consider a topological space X with $\pi_1 X = G$ and an integral chain $c \in C_1(G;\mathbb{Z})$; this yields a map $\gamma : \coprod_i S^1 \to X$ as explained in Section 2.3. Now let X_γ denote the mapping cylinder of γ:

$$X_\gamma = \left(X \amalg \left(\coprod_i S^1 \times [0,1] \right) \right) / \sim,$$

where $\sim$ is the equivalence relation generated by $(u,0) \sim \gamma(u)$ for $u \in \coprod_i S^1$.

There is a natural embedding $\coprod_i S^1 \hookrightarrow X_\gamma$ via $u \mapsto (u,1)$, and the homology of the pair (X,c) is defined by

$$H_*(X,c;\mathbb{Q}) = H_*\left(X_\gamma, \coprod_i S^1; \mathbb{Q} \right).$$

Note that our choice of topological representative γ for c was unique up to homotopy (see Section 2.3), and homotopic choices of γ will yield homotopic pairs $\left(X_\gamma, \coprod_i S^1\right)$; hence, the homology $H_*(X,c)$ only depends on c.

We will sometimes omit $\mathbb{Q}$ from the notation, but the relative homology $H_*(X,c)$ should always be understood to be with rational coefficients throughout this paper.

Proposition 2.9. *Let X be a topological space and let $c \in C_1(\pi_1 X;\mathbb{Z})$ be an integral chain.*

(i) *There is a long exact sequence*

$$\cdots \to H_n\left(\coprod_i S^1 \right) \xrightarrow{\gamma_*} H_n(X) \to H_n(X,c) \xrightarrow{\partial} H_{n-1}\left(\coprod_i S^1 \right) \to \cdots .$$

(ii) *If* $c \in B_1(\pi_1 X; \mathbb{Q})$*, then* $\gamma_*\left[\coprod_i S^1\right] = 0$.
If in addition $c \in \pi_1 X$ *(i.e.,* $\coprod_i S^1$ *consists of a single circle), then* $\gamma_* = 0$ *and there is a short exact sequence:*

$$0 \to H_2(X) \to H_2(X,c) \xrightarrow{\partial} H_1(S^1) \to 0.$$

(iii) *If* $Y \subseteq X$ *and* $c \in C_1(\pi_1 Y; \mathbb{Z})$*, then there is a long exact sequence:*

$$\cdots \to H_n(Y,c) \to H_n(X,c) \to H_n(X,Y) \xrightarrow{\partial} H_{n-1}(Y,c) \to \cdots.$$

All the above exact sequences are with omitted rational coefficients.

Proof. This follows from the long exact sequences of pairs and triples in homology [17, p. 118], together with the fact that X_γ deformation retracts to X [17, p. 2]. ∎

Observe that, given an admissible surface $f : (\Sigma, \partial\Sigma) \to (X, c)$, the commutative square (2.2) gives an induced map

$$f_* : H_*(\Sigma, \partial\Sigma) \to H_*(X, c).$$

In particular, we have a class $f_*[\Sigma] \in H_2(X,c)$, where $[\Sigma] \in H_2(\Sigma, \partial\Sigma)$ is the (rational) fundamental class of Σ. Proposition 2.7 expresses scl as an infimum over all admissible surfaces with a condition on $\partial f_*[\partial\Sigma] = \partial(f_*[\Sigma])$; we can instead focus on admissible surfaces for which we impose a condition on $f_*[\Sigma]$.

Definition 2.10. Let X be a topological space and $c \in C_1(\pi_1 X; \mathbb{Z})$. The *relative Gromov seminorm* is defined on $H_2(X, c; \mathbb{Q})$ by

$$\|\alpha\|_{\mathrm{Grom}} = \inf_{f,\Sigma} \frac{-2\chi^-(\Sigma)}{n(\Sigma)},$$

where the infimum is taken over all admissible surfaces $f : (\Sigma, \partial\Sigma) \to (X, c)$ such that $f_*[\Sigma] = n(\Sigma)\alpha$ for some $n(\Sigma) \in \mathbb{N}_{\geq 1}$.

Such an admissible surface will be called an *admissible surface for* $\|\alpha\|_{\mathrm{Grom}}$.

The relative Gromov seminorm as we define it is indeed a seminorm: Homogeneity is obtained by replacing an admissible surface with a finite cover, and subadditivity by taking disjoint unions of admissible surfaces (after ensuring that they have the same degree by possibly taking finite covers).

Remark 2.11. If $c = 0$, then $H_2(X,c) = H_2(X)$, and Proposition 2.8 says that $\|\cdot\|_{\mathrm{Grom}}$ coincides with the usual definition of the Gromov seminorm. In the general case, the relative Gromov seminorm can also be defined as an ℓ^1-seminorm – see [20, Section 2] for more details.

The connection between scl and the relative Gromov seminorm is as follows.

Proposition 2.12. *Given an integral chain $c \in C_1(\pi_1 X; \mathbb{Z})$, we have*

$$\mathrm{scl}(c) = \frac{1}{4} \inf\left\{ \|\alpha\|_{\mathrm{Grom}} \mid \alpha \in H_2(X, c),\ \partial\alpha = \left[\coprod_i S^1\right]\right\},$$

where $\partial : H_2(X, c) \to H_1\big(\coprod_i S^1\big)$ is the boundary map in the long exact sequence of Proposition 2.9 (i).

Proof. This is a restatement of Proposition 2.7. ∎

Proposition 2.12 suggests that computations of scl could be tackled in two successive steps: First fix a relative class $\alpha \in H_2(X, c)$ with $\partial\alpha = \big[\coprod_i S^1\big]$ and estimate $\|\alpha\|_{\mathrm{Grom}}$, and then find the infimum over all such classes α. Note that, if $H_2(X) = 0$ (which happens, for instance, if G is free and X is a $K(G, 1)$), then the long exact sequence of Proposition 2.9 tells us that $\partial : H_2(X, c) \to H_1\big(\coprod_i S^1\big)$ is injective. If in addition c is a boundary, then there is a unique $\alpha \in H_2(X, c)$ such that $\partial\alpha = \big[\coprod_i S^1\big]$, and in this case, $\mathrm{scl}(c) = \frac{1}{4}\|\alpha\|_{\mathrm{Grom}}$ by Proposition 2.12.

However, our point is that in some cases, computations of scl can be made difficult by the presence of nonzero classes in $H_2(X)$, and in those cases, one may hope to obtain information on the relative Gromov seminorm as a stepping stone towards scl.

3. Links and orientability in 2-complexes

We start with an analysis of some topological properties of 2-complexes. In particular, we will need a notion of orientability for 2-complexes that are not surfaces. The right setting to make this work will be 2-complexes with small links. The goal of this section is to introduce those notions.

3.1. 2-complexes

We first specify the category of topological spaces we will be working with throughout this paper.

Following the terminology of [13, Chapter 2], we say that a continuous map $f : X \to Y$ between CW-complexes is *cellular* if, for each $n \in \mathbb{N}_{\geq 0}$, $f(X^{(n)}) \subseteq Y^{(n)}$, where $X^{(n)}$ and $Y^{(n)}$ denote the n-skeleta of X and Y, respectively. We say that f is *combinatorial* if it maps each cell of X homeomorphically onto a cell of Y.

A 2-*complex* is a 2-dimensional CW-complex X such that the attaching map $S^1_\sigma \to X^{(1)}$ of each 2-cell σ of X is combinatorial for a suitable subdivision of the circle S^1_σ. An edge in this subdivision of S^1_σ is called a *side* of σ, and the *degree* of σ is its number of sides. This follows Gersten's terminology [15].

We will assume that all 2-complexes are locally finite, but we allow noncompact 2-complexes.

Each vertex v in a 2-complex X has a *link* – denoted by $\mathrm{Lk}_X(v)$ – which can be defined as the boundary of a regular neighbourhood of v in X (see [15]). The link has the structure of a graph, with vertices of $\mathrm{Lk}_X(v)$ corresponding to oriented half-edges e of X starting at v, with an edge between e_1 and e_2 in $\mathrm{Lk}_X(v)$ corresponding to each 2-cell σ of X whose boundary traverses e_1^{-1} and e_2 successively.

A *cellulated surface* is a 2-complex that is also a topological surface, possibly with boundary.

3.2. A surface criterion for 2-complexes

To decide whether or not a given 2-complex is a surface, it suffices to examine the topology of links of vertices; this is the content of the following lemma.

Lemma 3.1 (Surface criterion). *A 2-complex X is a cellulated surface if and only if the link of every vertex in X is a circle or a nondegenerate arc. In this case, a vertex v of X lies on the boundary if and only if its link is homeomorphic to an arc.*

Proof. The direct implication ($\Rightarrow$) is clear from the definition of the link $\mathrm{Lk}_X(v)$ as the boundary of a regular neighbourhood of v. For ($\Leftarrow$), the key point is that every vertex v has a neighbourhood homeomorphic to a cone over $\mathrm{Lk}_X(v)$. If $\mathrm{Lk}_X(v)$ is a circle, then a cone over $\mathrm{Lk}_X(v)$ is homeomorphic to a (2-dimensional) disc; if $\mathrm{Lk}_X(v)$ is a nondegenerate arc, then a cone over $\mathrm{Lk}_X(v)$ is homeomorphic to a half-disc. It is also clear that points in the interior of 2-cells have neighbourhoods homeomorphic to $\mathbb{R}^2$. It remains to consider points in the interior of edges. Given an edge e, let v be one of its endpoints; then, $\mathrm{Lk}_X(v)$ has a vertex $\hat{e}$ corresponding to e, and by assumption, $\hat{e}$ has one or two neighbours in $\mathrm{Lk}_X(v)$. Since our 2-complexes are assumed to be combinatorial, this means that e is incident to one or two 2-cells of X; in both cases, points in the interior of e have neighbourhoods homeomorphic to $\mathbb{R}^2$ or $\mathbb{R} \times \mathbb{R}_{\geq 0}$. ∎

This motivates the following.

Definition 3.2. A 2-complex X has *small links* if one of the following equivalent conditions holds:

(i) The link of every vertex of X is homeomorphic to a circle or a union of (possibly degenerate) arcs.

(ii) Every edge e of X is incident to at most two 2-cells, counted with multiplicity (i.e., a 2-cell is counted as many times as it has sides that are glued to e).

In other words, Lemma 3.1 says that a 2-complex X is a surface if and only if it has nondegenerate small connected links.

3.3. Orientability of 2-complexes

We will need a notion of orientation for 2-complexes. To define it, we will work with *locally finite homology*, denoted by H_*^{lf}. For a CW-complex X, this is defined as the homology of the chain complex $C_*^{\mathrm{lf,cell}}(X)$, where $C_n^{\mathrm{lf,cell}}(X)$ consists of infinite formal sums of oriented n-cells of X with locally finite support. Note that, if X is compact, then $H_*^{\mathrm{lf}}(X) = H_*(X)$. See [14, Chapter 11] or [19, Section 5.1.1] for more details on locally finite homology.

Definition 3.3. Given a 2-complex X, we define the *boundary* ∂X of X to be the 1-dimensional subcomplex consisting of all the edges of X (and their endpoints) that are incident to a single 2-cell, and along only one side of this 2-cell. In other words, these are the edges e for which each point in the interior of e has a neighbourhood in X that is homeomorphic to a half-disc.

Note that, in a 2-complex, $C_3^{\mathrm{lf,cell}}(X) = 0$, so $H_2^{\mathrm{lf}}(X) = Z_2^{\mathrm{lf,cell}}(X)$. In particular, it makes sense to speak of the *support* of a 2-class: This is just the support of the corresponding 2-cycle.

Definition 3.4. Let X be a 2-complex and let A be an abelian group. We say that X is *A-orientable* if there is a class $\beta \in H_2^{\mathrm{lf}}(X, \partial X; A)$ whose support contains every 2-cell of X.

Note that, if $X = S$ is a cellulated surface, then our definition of boundary coincides with the usual one, and $\mathbb{Z}$-orientability of S is equivalent to orientability of S in the usual sense (see, e.g., [25, Corollary 6.7] in the closed case). Our definition of orientation applies to any 2-complex, but in the context of surfaces, it is less intrinsic and flexible than the usual one because it requires one to fix a cellular structure first.

We will use orientability via the following lemma.

Lemma 3.5. *Let X be an A-orientable 2-complex. Consider a subcomplex Y of X such that $\partial X \subseteq Y \subseteq X$ and $H_2^{\mathrm{lf}}(X, Y; A) = 0$. Then Y contains every 2-cell of X.*

Proof. The long exact sequence of the triple $(X, Y, \partial X)$ shows that the inclusion induces a surjective morphism:

$$H_2^{\mathrm{lf}}(Y, \partial X; A) \twoheadrightarrow H_2^{\mathrm{lf}}(X, \partial X; A).$$

Since X is A-orientable relative to ∂X, there is a class $\beta \in H_2^{\mathrm{lf}}(X, \partial X; A)$ with support containing every 2-cell of X. Let $\beta_0 \in H_2^{\mathrm{lf}}(Y, \partial X; A)$ be a preimage of β. Then the support of β_0 is contained in Y and must contain every 2-cell of X. ■

Corollary 3.6. *Let S be an orientable closed surface and let $T \subseteq S$ be a subsurface such that $H_2(S, T) = 0$. Then $S = T$.* ■

3.4. Subcomplex stability

For our proof, we will need to pass to a subcomplex, and it will be necessary to check that the relevant properties of the original 2-complex are inherited by the subcomplex. We start with the following easy observation.

Lemma 3.7 (Subcomplex stability of small links). *Any subcomplex X_0 of a 2-complex X with small links also has small links.*

Proof. For each vertex $v \in X_0$, there is an embedding $\mathrm{Lk}_{X_0}(v) \hookrightarrow \mathrm{Lk}_X(v)$, and any subgraph of a circle or a union of arcs is again a circle or a union of arcs. ∎

We also need to check that orientability, as well as vanishing of relative homology, descend to subcomplexes.

Lemma 3.8 (Subcomplex stability of orientability). *Let A be an abelian group and let X be an A-orientable 2-complex with small links. Then any subcomplex $X_0 \subseteq X$ is A-orientable.*

Proof. Orientability of X means that there exists a relative cellular 2-cycle $p = \sum_\sigma \lambda_\sigma \sigma \in Z_2^{\mathrm{lf,cell}}(X, \partial X; A)$ (with $\lambda_\sigma \in A$ for each 2-cell σ of X) whose support contains all 2-cells of X. Set

$$p_0 = \sum_{\sigma \subseteq X_0} \lambda_\sigma \sigma.$$

Since $dp \in C_1^{\mathrm{cell}}(\partial X; A)$, the support of dp_0 consists of 1-cells of X that lie in ∂X or are incident to at least one 2-cell in $X \setminus X_0$. In both cases, they are incident to at most one 2-cell of X_0; moreover, they are incident to at least one 2-cell of X_0 as they lie in the support of dp_0. Therefore, the support of dp_0 is contained in ∂X_0, showing that p_0 is a relative 2-cycle in $Z_2^{\mathrm{lf,cell}}(X_0, \partial X_0; A)$ whose support contains all the 2-cells of X_0. Hence, X_0 is A-orientable. ∎

Lemma 3.9 (Injectivity of relative homology). *Let X be a 2-complex, and let $Y, X_0 \subseteq X$ be two subcomplexes. Set $Y_0 = Y \cap X_0$. Then for any abelian group A, the inclusion-induced map*

$$H_2(X_0, Y_0; A) \to H_2(X, Y; A)$$

is injective.

Proof. We follow an argument of Howie [18, Lemma 3.2]. Applying excision to the triple $(X_0 \cup Y, Y, Y \setminus Y_0)$ shows that the inclusion induces an isomorphism:

$$H_2(X_0, Y_0; A) \cong H_2(X_0 \cup Y, Y; A).$$

But since X is a 2-complex, $H_3(X, X_0 \cup Y; A) = 0$, so the long exact sequence of the triple $(X, X_0 \cup Y, Y)$ shows that the inclusion induces an embedding:

$$H_2(X_0 \cup Y, Y; A) \hookrightarrow H_2(X, Y; A).$$

This proves that the inclusion-induced map $H_2(X_0, Y_0; A) \to H_2(X, Y; A)$ is injective. ■

4. Standard form for admissible surfaces

The aim of this section is to reduce admissible surfaces to a certain standard form for the purpose of computing scl or the relative Gromov seminorm in surface groups. This standard form can be thought of as an analogue of Culler's fatgraphs [12].

4.1. Incompressibility and monotonicity

Let X be a topological space and let $c \in C_1(\pi_1 X; \mathbb{Z})$. Suppose that we want to compute $\mathrm{scl}(c)$ or $\|\alpha\|_{\mathrm{Grom}}$ for some $\alpha \in H_2(X, c)$, and consider an admissible surface $f : (\Sigma, \partial\Sigma) \to (X, c)$ with $f_*[\Sigma] = n(\Sigma)\alpha$. Recall that we have a commutative diagram:

$$\begin{array}{ccc} \partial\Sigma & \xrightarrow{\iota} & \Sigma \\ {\scriptstyle \partial f}\downarrow & & \downarrow{\scriptstyle f} \\ \coprod_i S^1 & \xrightarrow{\gamma} & X \end{array}$$

Observe first that we can harmlessly remove any disc- or sphere-component of Σ, since this does not change $\chi^-(\Sigma)$. We then say that Σ is *disc- and sphere-free*.

Now assume that there is a simple closed curve β in Σ with null-homotopic image in X. Then we may cut Σ along β and glue two discs on the resulting boundary components. This makes $-\chi^-(\Sigma)$ decrease without changing $f_*[\Sigma]$, so it improves our estimate of $\mathrm{scl}(c)$ or $\|\alpha\|_{\mathrm{Grom}}$. We can therefore always assume that f is *incompressible*: Every noncontractible simple closed curve in Σ has noncontractible image in X.

Let ∂_j be a boundary component of Σ; hence, ∂f sends ∂_j to a component S^1_i of $\coprod_i S^1$. We say that $f : (\Sigma, \partial\Sigma) \to (X, c)$ is *monotone*[3] if the sign of the degree of the restriction

$$\partial f_{|\partial_j} : \partial_j \to S^1_i$$

[3] This definition is in general different from the usual definition of a monotone admissible surface (see [5, Definition 2.12]), but the two definitions coincide if f is an admissible surface for $\mathrm{scl}(c)$, or more generally if the coordinates of $\partial\alpha$ in the basis $([S^1_i])_i$ of $H_1\big(\coprod_i S^1\big)$ all have the same sign. Recall from Section 2.3 that each circle in $\coprod_i S^1$ comes with an orientation, and that α might not necessarily map to all circles with positive orientation under ∂. In particular, if $\partial\alpha$ has two components of opposite orientations, then there is no monotone admissible surface for $\|\alpha\|_{\mathrm{Grom}}$ in the usual sense.

only depends on i. In other words, two boundary components of Σ mapping to the same component of $\coprod_i S^1$ do so with the same orientation.

In the context of scl, it is a classical fact [5, Proposition 2.13] that one can work with monotone admissible surfaces only. Our definition of monotonicity allows us to adapt this to the relative Gromov seminorm.[4] Calegari's proof [5, Proposition 2.13] works in our context, where one should deal with boundary components of Σ mapping to distinct components of $\coprod_i S^1$ separately. See [21, Lemma II.1.4] for more details.

Lemma 4.1 (Monotone admissible surfaces). *Fix a class $\alpha \in H_2(X, c)$. Given an admissible surface $f : (\Sigma, \partial\Sigma) \to (X, c)$ with $f_*[\Sigma] = n(\Sigma)\alpha$, there is a monotone admissible surface $f' : (\Sigma', \partial\Sigma') \to (X, c)$ with $f'_*[\Sigma'] = n(\Sigma')\alpha$ such that*

$$\frac{-\chi^-(\Sigma')}{n(\Sigma')} \leq \frac{-\chi^-(\Sigma)}{n(\Sigma)}.$$

4.2. Transversality

Similarly to Brady, Clay and Forester's proof of the rationality theorem [2], we will use the notion of transversality from [3, Section VII.2] to obtain a nice decomposition of admissible surfaces.

Let X be a 2-complex, and let $c \in C_1(\pi_1 X; \mathbb{Z})$ be an integral chain. Recall from Section 2.3 that c can be represented by a map $\gamma : \coprod_i S^1 \to X$. We are considering an admissible surface $f : (\Sigma, \partial\Sigma) \to (X, c)$ with $f_*[\Sigma] = n(\Sigma)\alpha$. We can apply the transversality theorem from [3, Section VII.2] to ensure that $\gamma : \coprod_i S^1 \to X$ and $f : \Sigma \to X$ are *transverse*: This means that Σ decomposes into subsurfaces mapping to vertices of X, 1-handles (i.e., trivial I-bundles over edges of X), and discs mapping homeomorphically onto 2-cells of X.

We have seen in Section 4.1 that f may be assumed to be incompressible. This implies that each subsurface of Σ mapping to a vertex of X is in fact a disc. Hence, Σ decomposes into the following pieces:

- Discs mapping to vertices of X – called *vertex discs*,
- 1-*handles*, that is, trivial I-bundles over edges of X and
- Discs mapping homeomorphically onto 2-cells of X – called *cellular discs*.

We then say that $f : (\Sigma, \partial\Sigma) \to (X, c)$ is a *transverse incompressible admissible surface* (see Figure 2).

4.3. Connectedness of links

We henceforth assume that the 2-complex X is a cellulated, oriented, compact, connected surface, and we will denote it by S.

[4] This would be false with the usual definition – see Footnote 3.

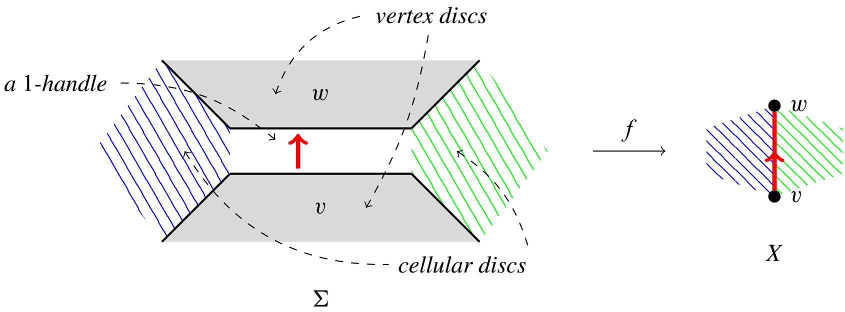


Figure 2. Pieces of a transverse incompressible admissible surface.

Consider an admissible surface $f : (\Sigma, \partial\Sigma) \to (S, c)$ with $f_*[\Sigma] = n(\Sigma)\alpha$ in $H_2(S, c)$. We may assume that f is transverse, incompressible, monotone, and disc- and sphere-free, as explained in Sections 4.1 and 4.2.

We now want to use the fact that S is a surface to ensure that Σ is 'thick enough', in the sense that its vertex discs have connected links.

More precisely, consider the 2-complex $\bar{\Sigma}$ obtained from Σ by collapsing all vertex discs to vertices and all 1-handles to edges – hence, f induces a combinatorial map $\bar{f} : \bar{\Sigma} \to S$, but $\bar{\Sigma}$ may not be a surface. (See, for instance, Figure 3: The vertex disc at the centre of Σ becomes a disconnecting vertex in $\bar{\Sigma}$.)

Definition 4.2. We say that a transverse incompressible admissible surface $f : (\Sigma, \partial\Sigma) \to (S, c)$ has *connected links* if the 2-complex $\bar{\Sigma}$ has connected vertex links.

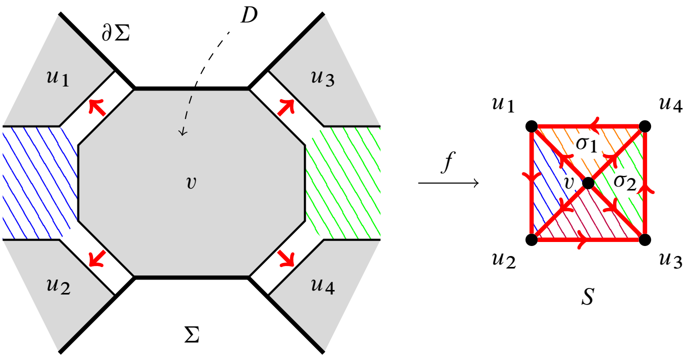


Figure 3. A vertex disc with disconnected link mapping to a vertex in the interior of S (note that the map f is orientation-preserving on the blue cellular disc but orientation-reversing on the green one).

If f does not have connected links, let D be a vertex disc in Σ whose corresponding vertex in $\bar{\Sigma}$ has disconnected link. Let v be the image of D under f; so v is a vertex of S. There are two cases: either v lies in the interior of S or on the boundary.

Assume first that v lies in the interior of S; an example is depicted in Figure 3 (all the 2-cells on the picture are triangles for simplicity, but the general case is similar). The main point is that there is a 2-cell in S between any two consecutive edges around v; in particular, there is a sequence of 2-cells $\sigma_1, \dots, \sigma_\ell$ lying between two successive edges whose preimages are 1-handles with one end each on $\partial\Sigma$, as in Figure 3. Now we perform a homotopy that moves the image of $\partial\Sigma$ across the 2-cells $\sigma_1, \dots, \sigma_\ell$ (see Figure 4). The new map $f : \Sigma \to S$ defines an admissible surface $(\Sigma, \partial\Sigma) \to (S, c)$. Note that f has been modified by a homotopy, so we still have $f_*[\Sigma] = n(\Sigma)\alpha$.

This operation decreases the number of connected components in the link of D (or more precisely, of its image in $\bar{\Sigma}$). Note that new vertex discs may have been created (such as the one mapping to u_4 in Figure 4), but they all have connected link. As for preexisting vertex discs of Σ, the operation does not impact the number of connected components of their links. Hence, if $\{D_i\}_i$ is the set of vertex discs of Σ mapping to the interior of S, and k_i denotes the number of connected components of the link of D_i, then we have made the quantity $\sum_i (k_i - 1)$ decrease strictly. We may therefore iterate to ensure that all vertex discs mapping to the interior of S have connected link.

We deal with the case where v lies on the boundary in the following way. We thicken S by gluing a cellulated annulus to each of its boundary components. This modifies the cellular structure of S but not its homeomorphism type (in other words, S has been replaced with another cell complex S' with more cells, but with S' homeomorphic to S), and this preserves all the properties of the map f – in particular, f is transverse for the new cellulation of S. Now all of the vertex discs with disconnected link map to the interior of S. Therefore, we can apply the operation described above to make all their links connected. This may create some new vertex discs mapping to the boundary, but they will have connected link. Hence, after both operations, all vertex discs have connected link.

We therefore obtain the following.

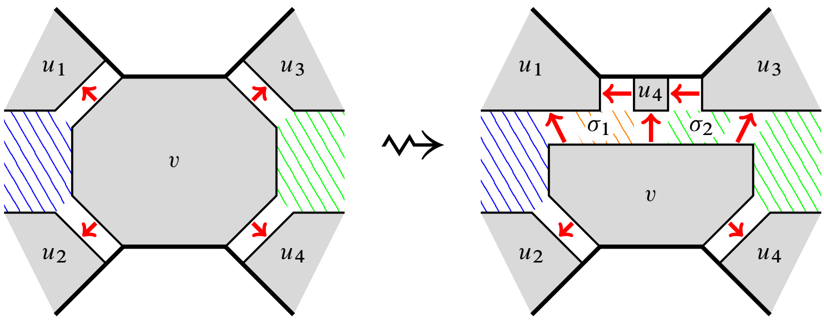


Figure 4. Making links of vertex discs connected (interior case).

Lemma 4.3 (Admissible surfaces with connected links). *Fix $\alpha \in H_2(S,c)$, and let $f : (\Sigma, \partial\Sigma) \to (S,c)$ be a transverse incompressible admissible surface with $f_*[\Sigma] = n(\Sigma)\alpha$. Then, after possibly changing the cellular structure on S, the map f may be homotoped to a transverse incompressible admissible surface with connected links.* ■

Note that since our admissible surface was modified by a homotopy, the properties of being incompressible, monotone, and disc- and sphere-free are preserved.

4.4. Folding

The key properties of admissible surfaces that we will need in the surface group case are related to orientation. Indeed, both S and the admissible surface Σ are oriented. Since f is transverse and cellular discs map homeomorphically into S, they can be of two types: Either they preserve the orientation or they reverse it. Having cellular discs of opposite orientations is undesirable, and we are now going to modify Σ to avoid this situation as much as possible.

We assume that $f : (\Sigma, \partial\Sigma) \to (S,c)$ is transverse, incompressible, monotone, disc- and sphere-free, and with connected links.

Suppose first that there is a connected component of Σ containing cellular discs of two different types – that is, one is orientation-preserving with respect to f and the other is orientation-reversing. Since $f : (\Sigma, \partial\Sigma) \to (S,c)$ has connected links, any two cellular discs in Σ mapping to cells in S with a common vertex on their boundaries must be connected by a path of cellular discs and 1-handles in Σ. It follows that any two cellular discs in the same connected component of Σ are connected by a path of cellular discs and 1-handles.

Therefore, Σ must contain two cellular discs of opposite types that are adjacent via a 1-handle. Since S is a surface, those two cellular discs must map to the same 2-cell of S, and we are in the situation of Figure 5 (pictures are given for the case of a 2-cell of degree 3, but the general case is similar) – in other words, f *folds* those two cellular discs onto one another.

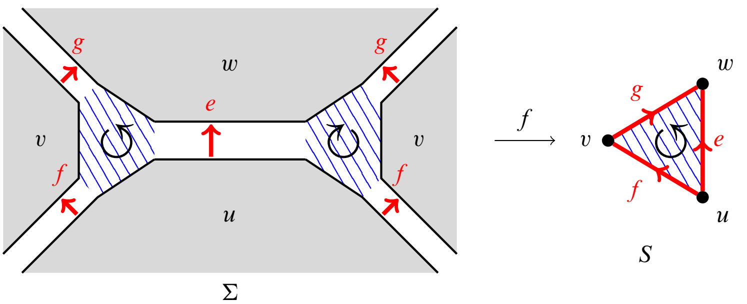


Figure 5. Adjacent cellular discs of opposite orientations.

In this case, we can delete the two adjacent cellular discs as illustrated in Figure 6. This operation does not change the homotopy type of Σ nor the boundary map $\partial f : \partial\Sigma \to \coprod_i S^1$. It also preserves the class $f_*[\Sigma] = n(\Sigma)\alpha$, as well as transversality, incompressibility, monotonicity of f and the property of being disc- and sphere-free. Moreover, it makes the number of cellular discs of Σ decrease strictly. Hence, after repeating a finite number of times, we may assume that each connected component of Σ contains cellular discs of only one type: either orientation-preserving or orientation-reversing.

In other words, we have reduced to the case where Σ has the following property.

Definition 4.4. Let S be a cellulated, oriented, compact, connected surface. A transverse incompressible admissible surface $f : (\Sigma, \partial\Sigma) \to (S, c)$ is said to be *non-folded* if each connected component of Σ contains cellular discs that are either all orientation-preserving or all orientation-reversing.

However, the operation just described could have created some vertex discs with disconnected link in Σ (as removing cellular discs amounts to deleting edges in the links of vertex discs).

To fix this, we would like to successively apply the operation described above and the one of Section 4.3. This relies on the following crucial observation: In the process described in Section 4.3, we can always choose the orientation of the cellular discs that we add. Indeed, the added cellular discs correspond to a path in the link of v in S, and this link is a circle, so there are two possible paths, one corresponding to adding cellular discs of positive orientation to Σ, and the other corresponding to adding cellular discs of negative orientation. For example, Figure 7 shows two possible choices that make the link of the vertex disc of Figure 3 connected. In particular, when applying the operation of Section 4.3, we can assume that we are only adding cellular discs of *positive* orientation.

We can now apply the following procedure: We alternately apply the operation of Section 4.3 to make links connected – and with the choice of only adding cellular discs of positive orientation – and then the operation described above to remove folding. Each iteration of the latter removes one cellular disc of each orientation, and each iteration of the former does not increase the number of cellular discs of negative orientation. Hence, the total number of discs of negative orientation decreases strictly at each pair of iterations,

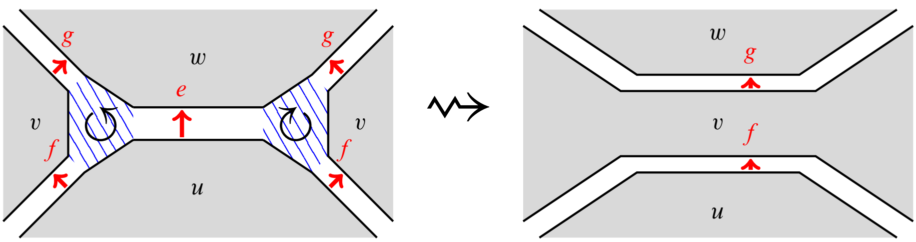


Figure 6. Eliminating cellular discs of opposite orientations.

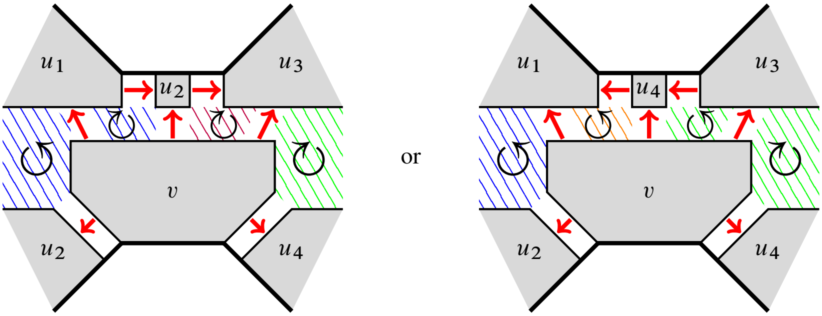


Figure 7. Two possible choices for making the link of the vertex disc of Figure 3 connected.

ensuring that the process terminates in an admissible surface that is both non-folded and with connected links.

Note that the operations just described do not impact the boundary map (up to homotopy), and hence the monotonicity, of f. Hence, we can first apply Lemma 4.1 and ensure that the resulting surface is monotone. We can also readily remove any disc- or sphere-component.

This proves the following.

Lemma 4.5 (Non-folded admissible surfaces). *Fix a class $\alpha \in H_2(S, c)$. Given a transverse incompressible admissible surface $f : (\Sigma, \partial\Sigma) \to (S, c)$ with $f_*[\Sigma] = n(\Sigma)\alpha$, there is a non-folded admissible surface $f' : (\Sigma', \partial\Sigma') \to (S, c)$ with connected links, with $f'_*[\Sigma'] = n(\Sigma')\alpha$, and such that*

$$\frac{-\chi^-(\Sigma')}{n(\Sigma')} \leq \frac{-\chi^-(\Sigma)}{n(\Sigma)}.$$

Moreover, f' can be assumed to be monotone and disc- and sphere-free. ■

4.5. Asymptotic promotion to orientation-perfect surfaces

In order to obtain isometric embedding results for surface groups in Section 5, we will need admissible surfaces to satisfy the following orientation property.

Definition 4.6. Let S be a cellulated, oriented, compact, connected surface. A transverse incompressible admissible surface $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ is *orientation-perfect* if there are no two cellular discs in Σ that map to the same 2-cell of S with opposite orientations.

There is an operation which one may be tempted to perform to obtain an orientation-perfect surface: Given two cellular discs of Σ mapping to the same 2-cell σ of S with opposite orientations, one obtains a new admissible surface by removing the two cellular discs and gluing the two resulting boundary components to one another. This changes neither ∂f nor $f_*[\Sigma]$; however, $-\chi^-(\Sigma)$ increases by 2. If we do this carelessly, then we get a worse estimate of scl or $\|\cdot\|_{\mathrm{Grom}}$.

Instead, we will perform this operation in an asymptotic way that is inspired by Chen's *asymptotic promotion* [10] – albeit in a much simpler case. This comes at a cost: we will not be able to obtain extremal surfaces anymore; however, we will still be able to compute scl or $\|\cdot\|_{\mathrm{Grom}}$.

We start with a transverse, incompressible, monotone, and disc- and sphere-free, non-folded admissible surface with connected links $f : (\Sigma, \partial\Sigma) \to (S, c)$ with $f_*[\Sigma] = n(\Sigma)\alpha$, and we assume that f is not orientation-perfect. Fix a small $\varepsilon > 0$, and pick a large $N \in \mathbb{N}_{\geq 1}$ such that $\frac{1}{N} \leq \varepsilon$. Let $\Sigma_0 \to \Sigma$ be a degree-N covering under which the preimage of every connected component of Σ is connected. The composite map $\Sigma_0 \to \Sigma \to S$ is also a transverse, incompressible, monotone, disc- and sphere-free, non-folded admissible surface with connected links, with $\chi^-(\Sigma_0) = N\chi^-(\Sigma)$ and $n(\Sigma_0) = Nn(\Sigma)$. Since f is non-folded but not orientation-perfect, there are two cellular discs in distinct components of Σ_0 that map to the same 2-cell of S with opposite orientations. We remove those two discs and glue the resulting boundary components to one another in a way that is compatible with the map f. There is an admissible surface $f' : (\Sigma'_0, \partial\Sigma'_0) \to (S, c)$ resulting from this operation, which is still transverse, incompressible, monotone, and disc- and sphere-free; it satisfies $f'_*[\Sigma'_0] = Nn(\Sigma)\alpha$, and

$$-\chi^-(\Sigma'_0) = -\chi^-(\Sigma_0) + 2 = -N\chi^-(\Sigma) + 2.$$

Therefore,

$$\frac{-\chi^-(\Sigma'_0)}{n(\Sigma'_0)} \leq \frac{-\chi^-(\Sigma)}{n(\Sigma)} + 2\varepsilon.$$

We can then perform the process of Section 4.4 again to ensure that Σ'_0 is non-folded and with connected links.

After the complete operation, the number of connected components of Σ has decreased by 1, while the quantity $\frac{-\chi^-(\Sigma)}{n(\Sigma)}$ has not increased more than a controlled arbitrarily small amount. Since Σ has a finite number of connected components, we may iterate until we obtain an orientation-perfect surface. We obtain the following.

Lemma 4.7 (Orientation-perfect admissible surfaces). *Fix $\alpha \in H_2(S, c)$. Given an $\varepsilon > 0$ and a transverse incompressible admissible surface $f : (\Sigma, \partial\Sigma) \to (S, c)$ with $f_*[\Sigma] = n(\Sigma)\alpha$, there is an orientation-perfect admissible surface $f' : (\Sigma', \partial\Sigma') \to (S, c)$, with $f'_*[\Sigma'] = n(\Sigma')\alpha$, and such that*

$$\frac{-\chi^-(\Sigma')}{n(\Sigma')} \leq \frac{-\chi^-(\Sigma)}{n(\Sigma)} + \varepsilon.$$

Moreover, f' can be assumed to be monotone, disc- and sphere-free, non-folded and with connected links. ∎

Remark 4.8. If the 1-chain c consists simply of an element $w \in \pi_1 S$, and if $f : (\Sigma, \partial\Sigma) \to (S, w)$ is an admissible surface for $\mathrm{scl}(w)$ (not for $\|\alpha\|_{\mathrm{Grom}}$), then we can bypass the asymptotic promotion argument and in fact replace Σ with a connected admissible surface. Indeed, consider the connected components $\{\Sigma_i\}_i$ of Σ, and observe that the restriction of f to each Σ_i is an admissible surface for $\mathrm{scl}(w)$. (But note that distinct components may represent distinct classes in $H_2(S, w)$.) We have

$$\frac{-\chi^-(\Sigma)}{n(\Sigma)} = \frac{\sum_i(-\chi^-(\Sigma_i))}{\sum_i n(\Sigma_i)} \geq \min_i \frac{-\chi^-(\Sigma_i)}{n(\Sigma_i)}.$$

Hence, there is a component Σ_i of Σ for which $-\chi^-(\Sigma_i)/n(\Sigma_i) \leq -\chi^-(\Sigma)/n(\Sigma)$, and we may replace Σ with Σ_i. Now Σ is connected, so making it non-folded is enough to guarantee that it is orientation-perfect.

4.6. Standard form

We have shown the following.

Proposition 4.9 (Standard form). *Let S be an oriented, compact, connected surface, let $c \in C_1(\pi_1 S; \mathbb{Z})$ be an integral chain, and $\alpha \in H_2(S, c; \mathbb{Q})$. Then:*

(i) *The relative Gromov seminorm of α can be computed via*

$$\|\alpha\|_{\mathrm{Grom}} = \inf_{f,\Sigma} \frac{-2\chi^-(\Sigma)}{n(\Sigma)},$$

where the infimum is taken over all admissible surfaces $f : (\Sigma, \partial\Sigma) \to (S, c)$ that are transverse, incompressible, monotone, disc- and sphere-free and orientation-perfect, with connected links for some cellulation of S.
Such an admissible surface is said to be in perfect standard form.

(ii) *If there exists an extremal surface for $\|\alpha\|_{\mathrm{Grom}}$ (i.e., realising the infimum in Definition 2.10), then there exists one which is transverse, incompressible, monotone, disc- and sphere-free and non-folded, with connected links for some cellulation of S.*
Such an admissible surface is said to be in standard form.

Proof. This follows from Section 4.2 (for transversality) and Lemmas 4.1 (for monotonicity), 4.3 (for connected links), 4.5 (for the non-folding property) and 4.7 (for the orientation-perfect property). ∎

Remark 4.10. By the discussion of Section 4.4, an orientation-perfect admissible surface is automatically non-folded. It follows that an admissible surface in perfect standard form is also in standard form.

It follows from Proposition 2.12 that the obvious analogue of Proposition 4.9 holds for scl: The stable commutator length of c can be computed with surfaces in perfect standard form, and if there exists an extremal surface, then there exists one in standard form.

Moreover, if the 1-chain c consists of a single element $w \in \pi_1 S$, and if there exists an extremal surface for $\mathrm{scl}(w)$, then there exists one in perfect standard form (see Remark 4.8).

5. Isometries for scl and the relative Gromov seminorm

We now have all the tools we need to prove our isometric embedding theorems. We consider S an oriented, compact, connected surface, and $T \subseteq S$ a subsurface that is π_1*-injective*, in the sense that the induced morphism

$$\iota : \pi_1 T \hookrightarrow \pi_1 S$$

is injective. We would like to understand when ι is a (strong) isometric embedding for scl or the relative Gromov seminorm. We will identify $\pi_1 T$ with its image in $\pi_1 S$. Hence, a chain $c \in C_1(\pi_1 T; \mathbb{Z})$ can also be seen as a chain in $C_1(\pi_1 S; \mathbb{Z})$, and admissible surfaces for c can be considered either in T or in S.

5.1. Main theorem

Our main technical result is the following, which says that, with our standard form and with appropriate homology vanishing conditions, an admissible surface in S for a chain in T is in fact entirely contained in T.

Theorem 5.1. *Let S be a cellulated, oriented, compact, connected surface, let $T \subseteq S$ be a π_1-injective subcomplex, let $c \in C_1(\pi_1 T; \mathbb{Z})$ be an integral chain and let $\alpha \in H_2(S, c; \mathbb{Q})$. Consider an admissible surface $f : (\Sigma, \partial\Sigma) \to (S, c)$ in S with $f_*[\Sigma] = n(\Sigma)\alpha$ for some $n(\Sigma) \in \mathbb{N}_{\geq 1}$, and with $f(\partial\Sigma) \subseteq T$.*

Let $R = \mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$ and assume that one of the following holds:

(i) *f is in standard form and $H_2(S, T; R) = 0$, or*

(ii) *f is in perfect standard form and $f_*[\Sigma] = 0$ in $H_2(S, T; R)$.*

Then $f(\Sigma) \subseteq T$.

Proof. Consider $S_0 = \mathrm{Im}\, f \subseteq S$, and let $T_0 = S_0 \cap T$. Hence, S_0 is a subcomplex of S, and T_0 is a (π_1-injective) subcomplex of S_0. The map f induces $f_0 : \Sigma \to S_0$. Our subcomplex stability lemmas imply that

- S_0 is an R-orientable 2-complex with small links by Lemmas 3.7 and 3.8.
- With assumption (i) (i.e., $H_2(S, T; R) = 0$), we also have $H_2(S_0, T_0; R) = 0$ by Lemma 3.9.
- With assumption (ii) (i.e., $f_*[\Sigma] = 0$ in $H_2(S, T; R)$), then we have $f_{0*}[\Sigma] = 0$ in $H_2(S_0, T_0; R)$ by Lemma 3.9.

Claim 1. We have an inclusion $\partial S_0 \subseteq f_0(\partial\Sigma)$ (where ∂S_0 should be understood in the sense of Definition 3.3).

Proof. Let e be an edge of ∂S_0. By definition, $e \subseteq S_0 = \mathrm{Im}\, f$, so there is a 1-handle H in Σ that maps to e. Now each end of the 1-handle H can be either incident to a cellular disc or to $\partial\Sigma$. If each end is incident to a cellular disc, then those two cellular discs must map to the same 2-cell σ of S_0, because e is incident to only one 2-cell as it lies on ∂S_0. In this case, the two cellular discs map to σ with opposite orientations (as in Figure 5), which contradicts the non-folding property – which f has since it is in standard form. Therefore, at least one end of H must be incident to $\partial\Sigma$. This implies that $e \subseteq f_0(\partial\Sigma)$. ∎

By assumption, $f_0(\partial\Sigma) \subseteq T_0$. Hence, we have

$$\partial S_0 \subseteq f_0(\partial\Sigma) \subseteq T_0 \subseteq S_0.$$

With assumption (i), we have $H_2(S_0, T_0; R) = 0$, so it follows immediately from Lemma 3.5 that every 2-cell of S_0 is contained in T_0.

With assumption (ii), we have $f_{0*}[\Sigma] = 0$ in $H_2(S_0, T_0; R)$. Let σ be a 2-cell in $S_0 = \mathrm{Im}\, f$. Recall that f_0 is in perfect standard form, so it is orientation-perfect. This means that all cellular discs of Σ mapping to σ do so with the same orientation. Hence, the image $f_{0*}[\Sigma]$ in $H_2(S_0, T_0; R)$ has a term in σ with nonzero coefficient. But $f_{0*}[\Sigma] = 0$ in $H_2(S_0, T_0; R)$, so we must have $\sigma \subseteq T_0$. This shows that every 2-cell in S_0 is contained in T_0.

Claim 2. Every 0- or 1-cell of S_0 is incident to a 2-cell of S_0.

Proof. Note first that there is no isolated 0-cell in S_0 since f_0 is incompressible. Now assume for contradiction that there is a 1-cell $e \subseteq S_0$ without any incident 2-cell. Let H be a 1-handle of Σ mapping to e. Then both ends of H lie on $\partial\Sigma$. Hence, any vertex disc in Σ incident to H meets $\partial\Sigma$ on both sides of H. But links of vertex discs are connected since f is in standard form, so neither of the vertex discs incident to H is incident to any other 1-handle (see Figure 8). Hence, Σ has a disc component consisting of H and its two incident vertex discs (note that these two discs are distinct by connectedness of their link). This is impossible since admissible surfaces in standard form are assumed to be disc- and sphere-free. ∎

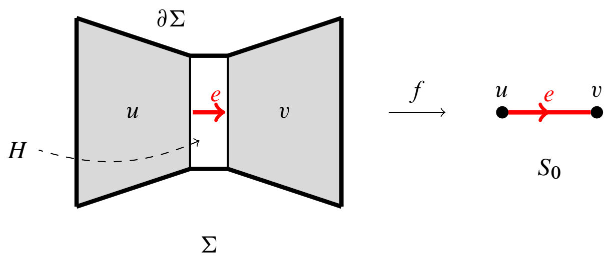


Figure 8. A 1-handle mapping to an edge with no incident 2-cell.

Since every 2-cell of S_0 is contained in T_0, it follows from Claim 2 that $S_0 = T_0$, and therefore $\operatorname{Im} f = S_0 \subseteq T$ as wanted. ■

5.2. Isometric embeddings

We now discuss applications of Theorem 5.1 to isometric embeddings for scl and $\|\cdot\|_{\mathrm{Grom}}$.

If S is a surface, we say that a subsurface $T \subseteq S$ is *H_1-injective* if the induced map $H_1(T) \to H_1(S)$ is injective.

Theorem A (Isometric embedding for scl). *Let S be an oriented, compact, connected surface with nonempty boundary, and let $T \subseteq S$ be an H_1-injective subsurface. Then the inclusion-induced map*

$$\iota : \pi_1 T \hookrightarrow \pi_1 S$$

is a strong isometric embedding for scl.

Proof. Note first that $\pi_1 T$ and $\pi_1 S$ are free groups because $\partial S \neq \emptyset$. Since $\iota_* : H_1(\pi_1 T) \to H_1(\pi_1 S)$ is injective, we have $\operatorname{rk}(\pi_1 T) = \operatorname{rk}(\operatorname{Im} \iota)$, and it follows from the Hopf property for free groups that $\iota : \pi_1 T \to \pi_1 S$ is injective.

Moreover, by the universal coefficient theorem, the inclusion-induced map

$$H_1(T; \mathbb{Q}) \to H_1(S; \mathbb{Q})$$

is injective, or equivalently, the map

$$\iota_* : H_1(\pi_1 T; \mathbb{Q}) \to H_1(\pi_1 S; \mathbb{Q})$$

is injective.

It remains to show that ι preserves the stable commutator length of integral boundaries (strong isometry will follow from Proposition 2.5).

Let $c \in B_1(\pi_1 T; \mathbb{Z})$. By Proposition 4.9, $\operatorname{scl}_{\pi_1 S}(c)$ can be computed as an infimum over all admissible surfaces $f : (\Sigma, \partial\Sigma) \to (S, c)$ in standard form. But since S has nonempty boundary, $H_2(S) = 0$, so injectivity of $H_1(T) \to H_1(S)$ implies that $H_2(S, T) = 0$ by the long exact sequence of (S, T). Therefore, by Theorem 5.1, every admissible surface in standard form in S is in fact contained in T. It follows that

$$\operatorname{scl}_{\pi_1 T}(c) \leq \operatorname{scl}_{\pi_1 S}(c),$$

and the reverse inequality always holds by monotonicity of scl (see Proposition 2.2). ■

In Theorem A, observe that H_1-injectivity of T could be replaced with the equivalent assumption that $H_2(S,T) = 0$. With that assumption, the theorem would also hold when S is closed since in that case, $H_2(S,T) = 0$ implies that $S = T$ by Corollary 3.6. However, this would only add a trivial statement and therefore the generality of the theorem would not be increased.

Our second isometric embedding theorem is the following, which also applies – and gives a nontrivial result – in the closed case.

Theorem B (Isometric embedding for the relative Gromov seminorm). *Let S be an oriented, compact, connected surface, let $T \subseteq S$ be a π_1-injective subsurface and let $c \in C_1(\pi_1 T; \mathbb{Z})$ be an integral chain in T. Then the inclusion-induced map*

$$\iota : H_2(T, c; \mathbb{Q}) \hookrightarrow H_2(S, c; \mathbb{Q})$$

is an injective isometric embedding for $\|\cdot\|_{\mathrm{Grom}}$.

Proof. Note that injectivity follows from the long exact sequence of the triple (S, T, c) (see Proposition 2.9) since $H_3(S, T) = 0$.

Let $\alpha \in H_2(T, c; \mathbb{Q})$. Then Proposition 4.9 says that $\|\iota\alpha\|_{\mathrm{Grom}}$ can be computed as an infimum over all admissible surfaces $f : (\Sigma, \partial\Sigma) \to (S, c)$ in perfect standard form. Let f be such an admissible surface, with $f_*[\Sigma] = n(\Sigma)\iota\alpha$ in $H_2(S, c; \mathbb{Q})$. By the long exact sequence of the triple (S, T, c) (see Proposition 2.9), $\iota\alpha$ maps to zero in $H_2(S, T; \mathbb{Q})$, and so the image of $f_*[\Sigma]$ in $H_2(S, T; \mathbb{Q})$ is also zero. Therefore, Theorem 5.1 applies and f can be homotoped to an admissible surface in T. This proves that $\|\alpha\|_{\mathrm{Grom}} \leq \|\iota\alpha\|_{\mathrm{Grom}}$, and the reverse inequality always holds since $\|\cdot\|_{\mathrm{Grom}}$ is monotone with respect to continuous maps. ■

In fact, applying Theorem B to the context of surfaces with nonempty boundary yields a stronger version of Theorem A.

Corollary C. *Let S be an oriented, compact, connected surface with nonempty boundary and let $T \subseteq S$ be a π_1-injective subsurface. Then the inclusion-induced map*

$$\iota : \pi_1 T \hookrightarrow \pi_1 S$$

is an isometric embedding for scl.

Proof. The morphism ι is assumed to be injective, so it suffices to prove that it preserves the scl of homologically trivial 1-chains. Let $c \in B_1(\pi_1 T; \mathbb{Z})$. Since $H_2(T) = 0$, the long exact sequence of the pair (T, c) (see Proposition 2.9) shows that there is a unique class $\alpha \in H_2(T, c; \mathbb{Q})$ such that $\partial\alpha = \left[\coprod_i S^1\right]$. By Proposition 2.12, we have $4\,\mathrm{scl}_{\pi_1 T}(c) = \|\alpha\|_{\mathrm{Grom}}$. Similarly, $H_2(S) = 0$ and $4\,\mathrm{scl}_{\pi_1 S}(c) = \|\iota\alpha\|_{\mathrm{Grom}}$. But Theorem B implies that $\|\alpha\|_{\mathrm{Grom}} = \|\iota\alpha\|_{\mathrm{Grom}}$, so $\mathrm{scl}_{\pi_1 T}(c) = \mathrm{scl}_{\pi_1 S}(c)$. ■

6. Extremal surfaces and quasimorphisms

We conclude with a discussion of how our isometric embeddings of surfaces behave with respect to extremal surfaces and quasimorphisms. To be more precise, consider $\iota : G \hookrightarrow H$ an isometric embedding for scl. Note, in particular, that ι is injective and induces an embedding $K(G, 1) \hookrightarrow K(H, 1)$. Given a 1-chain $c \in C_1(G; \mathbb{Z}) \hookrightarrow C_1(H; \mathbb{Z})$, our aim is to find conditions under which

- there is an extremal surface $(\Sigma, \partial\Sigma) \to (K(G,1), c)$ for $\mathrm{scl}_G(c)$ that is also extremal for $\mathrm{scl}_H(c)$, or
- there is an extremal quasimorphism $\varphi \in Q(H)$ for $\mathrm{scl}_H(c)$ that restricts to an extremal quasimorphism $\varphi_{|G} \in Q(G)$ for $\mathrm{scl}_G(c)$.

We address these problems for the isometric embedding $\iota : \pi_1 T \hookrightarrow \pi_1 S$ of Theorem A.

6.1. Extremal surfaces

Recall that an *extremal surface* for $\mathrm{scl}_G(c)$ is one that realises the infimum in Proposition 2.7. A major result, due to Calegari [6], is that extremal surfaces exist for all $c \in B_1(G;\mathbb{Z})$ if G is a free group.

Proposition 4.9 says that, for the purpose of finding extremal surfaces, we can assume that admissible surfaces are in standard form – but not necessarily in perfect standard form. In the context of Theorem A, this is sufficient: H_1-injectivity implies that $H_2(S,T) = 0$ since $H_2(S) = 0$, so Theorem 5.1 with assumption (i) says that any admissible surface $(\Sigma, \partial\Sigma) \to (S, c)$ in standard form is in fact contained in T. Note also that $\pi_1 S$ and $\pi_1 T$ are free groups, so extremal surfaces exist by Calegari's theorem [6]. This gives the following.

Corollary 6.1. *Let S be an oriented, compact, connected surface with nonempty boundary, and let $T \subseteq S$ be an H_1-injective subsurface. Let $c \in B_1(\pi_1 T;\mathbb{Z})$. Then there exists an admissible surface $f : (\Sigma, \partial\Sigma) \to (T, c)$ that is extremal for both $\mathrm{scl}_{\pi_1 T}(c)$ and $\mathrm{scl}_{\pi_1 S}(c)$.* ∎

Note, however, that even if extremal surfaces were known to exist for the relative Gromov seminorm, we would not obtain an analogue of Corollary 6.1 in that setting. Indeed, to prove Theorem B, we needed to apply Theorem 5.1 with assumption (ii) and work with admissible surfaces in perfect standard form. But an asymptotic promotion argument was necessary to obtain the perfect standard form (see Section 4.5), and this does not preserve extremal surfaces.

In the case where S has nonempty boundary and the subsurface $T \subseteq S$ is only π_1-injective rather than H_1-injective, then Corollary C says that $\pi_1 T \hookrightarrow \pi_1 S$ is still an isometric embedding for scl. In general, as Corollary C relies on Theorem B, this isometric embedding might not preserve extremal surfaces, but in the special case where the 1-chain c consists of a single element w of $\pi_1 T$, then Remark 4.8 says that the asymptotic promotion argument can be bypassed, and therefore extremal surfaces can be assumed to be in perfect standard form. We obtain the following.

Corollary 6.2. *Let S be an oriented, compact, connected surface with nonempty boundary, and let $T \subseteq S$ be a π_1-injective subsurface. Let $w \in [\pi_1 T, \pi_1 T]$. Then there exists an admissible surface $f : (\Sigma, \partial\Sigma) \to (T, w)$ that is extremal for both $\mathrm{scl}_{\pi_1 T}(w)$ and $\mathrm{scl}_{\pi_1 S}(w)$.* ∎

6.2. Extremal quasimorphisms

Recall that a *quasimorphism* on a group G is a map $\phi : G \to \mathbb{R}$ such that

$$\sup_{a,b \in G} |\phi(ab) - \phi(a) - \phi(b)| < \infty.$$

This supremum is called the *defect* of ϕ and denoted by $D(\phi)$. We say that ϕ is *homogeneous* if $\phi(w^n) = n\phi(w)$ for all $w \in G$ and $n \in \mathbb{Z}$. We denote by $Q(G)$ the space of homogeneous quasimorphisms $G \to \mathbb{R}$.

Given a quasimorphism $\phi : G \to \mathbb{R}$, we can naturally extend ϕ to a map $C_1(G;\mathbb{R}) \to \mathbb{R}$ by linearity.

The connection between quasimorphisms and scl is given by the following result, which says essentially that $(Q(G), D(\cdot))$ is the dual space of $(B_1(G;\mathbb{R}), \mathrm{scl}_G)$ (after quotienting by the kernels of the respective seminorms).

Proposition 6.3 (Bavard duality [1]). *Let G be a group and $c \in C_1(G;\mathbb{R})$ be a 1-chain. Then*

$$\mathrm{scl}_G(c) = \sup_{\substack{\phi \in Q(G) \\ D(\phi) \neq 0}} \frac{\phi(c)}{2D(\phi)}.$$

A quasimorphism is called *extremal* if it realises the supremum in Proposition 6.3. As opposed to extremal surfaces, extremal quasimorphisms exist for all 1-boundaries [5, Proposition 2.88], but finding an explicit extremal quasimorphism for a given element is usually a hard problem. There are, however, some results of this form; the following will be of particular interest to us.

Proposition 6.4 (Calegari [4]). *Let S be a hyperbolic, compact, connected surface with $\partial S \neq \emptyset$. Let $c \in B_1(\pi_1 S;\mathbb{Z})$. Then the following are equivalent:*

(i) *There exists an admissible surface $f : (\Sigma, \partial\Sigma) \to (S, c)$ for $\mathrm{scl}_{\pi_1 S}(c)$ which is immersed and orientation-preserving – we say that c* rationally bounds a positive immersed surface.

(ii) *The rotation quasimorphism rot_S is extremal for c.*

The *rotation quasimorphism* is an object that encodes the dynamics of the action of $\pi_1 S$ on the boundary of the hyperbolic plane given by the choice of a hyperbolic structure. The defect of rot_S is always 1. See [4] or [5, Section 2.3.3] for more details. Note that Proposition 6.4 applies, in particular, to the 1-chain c given by the (oriented) boundary of S.

We make the following observation.

Proposition 6.5. *Let S be a hyperbolic, compact, connected surface with $\partial S \neq \emptyset$, and let $T \subseteq S$ be an H_1-injective convex subsurface. Given $c \in B_1(\pi_1 T;\mathbb{Z})$, there is an equality*

$$\mathrm{rot}_T(c) = \mathrm{rot}_S(c).$$

Proof. There is an interpretation of the rotation number in terms of the area enclosed by a chain: $\mathrm{rot}_S(c) = \frac{1}{2\pi}\,\mathrm{area}_S(c)$. We refer to [5, Lemma 4.68] for full details, but it suffices for our purpose to say that $\mathrm{area}_S(c) = \sum_i n_i\,\mathrm{area}(\sigma_i)$, where $b = \sum_i n_i \sigma_i$ is a cellular 2-chain in S with $db = c$. Now H_1-injectivity of T implies that $H_2(S,T) = 0$ since $H_2(S) = 0$, so the differential

$$d : C_2^{\mathrm{cell}}(S,T) \to C_1^{\mathrm{cell}}(S,T)$$

is injective. Hence, the fact that $db = c \in C_1^{\mathrm{cell}}(T)$ implies that $b \in C_2^{\mathrm{cell}}(T)$, and therefore $\mathrm{area}_T(c) = \sum_i n_i\,\mathrm{area}(\sigma_i) = \mathrm{area}_S(c)$. ∎

It follows that, if T is H_1-injective, then for any chain $c \in B_1(\pi_1 T;\mathbb{Z})$ that rationally bounds a positive immersed surface in T, we have

$$\mathrm{scl}_{\pi_1 T}(c) = \frac{1}{2}\,\mathrm{rot}_T(c) = \frac{1}{2}\,\mathrm{rot}_S(c) \le \mathrm{scl}_{\pi_1 S}(c) \le \mathrm{scl}_{\pi_1 T}(c),$$

so rot_S is an extremal quasimorphism for c in S and restricts to rot_T, which is an extremal quasimorphism for c in T.

It is natural at this point to ask whether it is equivalent for a chain $c \in B_1(\pi_1 T;\mathbb{Z})$ to rationally bound a positive immersed surface in T or in S. Using the same kind of argument as in the proof of Proposition 6.5, it is easy to see that this is true if the word ‘immersed’ is replaced with ‘embedded’. In general, using Scott’s theorem on subgroup separability of surface groups [22, 23], we can lift an immersed surface to an embedded one, and thus obtain an affirmative answer.

Proposition 6.6. *Let S be a hyperbolic, compact, connected surface with $\partial S \neq \emptyset$, and let $T \subseteq S$ be an H_1-injective convex subsurface. Given $c \in B_1(\pi_1 T;\mathbb{Z})$, the following are equivalent:*

(i) rot_S *is an extremal quasimorphism for* $\mathrm{scl}_{\pi_1 S}(c)$.

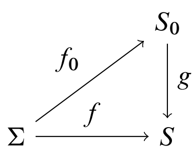
(ii) *c rationally bounds a positive immersed surface in S.*

(iii) rot_T *is an extremal quasimorphism for* $\mathrm{scl}_{\pi_1 T}(c)$.

(iv) *c rationally bounds a positive immersed surface in T.*

Proof. The equivalences (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) follow from Proposition 6.4. It is clear that (iv) $\Rightarrow$ (ii) since $T \subseteq S$, so it remains to prove that (ii) $\Rightarrow$ (iv).

Assume that (ii) holds: There is an admissible surface $f : (\Sigma, \partial\Sigma) \to (S, c)$ for $\mathrm{scl}_{\pi_1 S}(c)$ that is immersed and orientation-preserving. It follows from Scott’s theorem [22,23] that there is a finite covering $g : S_0 \to S$ over which f lifts as an embedding $f_0 : \Sigma \hookrightarrow S_0$:



Consider $T_0 = g^{-1}(T) \subseteq S_0$. Since T is H_1-injective, it follows that $H_2(S, T) = 0$. This means that every connected component of $\overline{S \setminus T}$ contains at least one boundary component of S. This property lifts to a finite cover, so $H_2(S_0, T_0) = 0$.

Now we have an embedded surface $f_0 : \Sigma \hookrightarrow S_0$. For an appropriate cellular structure on S_0, this embedding gives rise to a cellular 2-chain $b \in C_2^{\text{cell}}(S_0)$ with $db \in C_1^{\text{cell}}(T_0)$. The same argument as in the proof of Proposition 6.5 then gives $b \in C_2^{\text{cell}}(T_0)$. As f_0 is an embedding, this implies that $f_0(\Sigma) \subseteq T_0$, so $f(\Sigma) \subseteq T$ and c rationally bounds a positive immersed surface in T. ∎

Remark 6.7. We could also have used Theorem A to prove that (i) ⇒ (iii) in Proposition 6.6. Indeed, assume that rot_S is extremal for $\text{scl}_{\pi_1 S}(c)$. Recall that $\pi_1 T \hookrightarrow \pi_1 S$ is isometric and that rot_S and rot_T agree on $\pi_1 T$ (by Proposition 6.5). Therefore,

$$\text{scl}_{\pi_1 T}(c) = \text{scl}_{\pi_1 S}(c) = \frac{\text{rot}_S(c)}{2} = \frac{\text{rot}_T(c)}{2},$$

so that rot_T is extremal for $\text{scl}_{\pi_1 T}(c)$.

However, the previous proof, using Scott's theorem, has the advantage of being independent of Theorem A.

This does not say whether the isometric embedding of Theorem A respects extremal quasimorphisms for all 1-chains c, but it does for some of them.

Corollary 6.8. *Let S be a hyperbolic, compact, connected surface with $\partial S \neq \emptyset$, and let $T \subseteq S$ be an H_1-injective convex subsurface. Let $c \in B_1(\pi_1 T; \mathbb{Z})$.*

If c rationally bounds a positive immersed surface in S, then the rotation quasimorphism $\text{rot}_S \in Q(\pi_1 S)$ is extremal for $\text{scl}_{\pi_1 S}(c)$, and restricts to $\text{rot}_T \in Q(\pi_1 T)$ which is extremal for $\text{scl}_{\pi_1 T}(c)$. ∎

This gives an alternative proof that the embedding $\pi_1 T \hookrightarrow \pi_1 S$ preserves the stable commutator length of every $c \in B_1(\pi_1 T; \mathbb{Z})$ which rationally bounds a positive immersed surface in S, independently of Theorem A.

Acknowledgements. First and foremost, I would like to thank my supervisor Henry Wilton for our weekly discussions and his constant support. Thanks are also due to Lvzhou Chen, Francesco Fournier-Facio and Kevin Li for helpful conversations, as well as to the anonymous referee for careful reading and detailed comments. Financial support from an EPSRC PhD studentship is gratefully acknowledged.

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Received 8 March 2023.

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