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A Dirichlet-to-Neumann map for the Allen–Cahn equation on manifolds with boundary

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Abstract. We study the asymptotic behavior of Dirichlet minimizers to the Allen–Cahn equation on manifolds with boundary, and we relate the Neumann data to the geometry of the boundary. We show that Dirichlet minimizers are asymptotically local in orders of ε and compute expansions of the solution to high order. A key tool is showing that the linearized Allen–Cahn operator is invertible at the heteroclinic solution, on functions with 0 boundary condition. We apply our results to separating hypersurfaces in closed Riemannian manifolds. This gives a projection theorem about Allen–Cahn solutions near minimal surfaces, as constructed by Pacard–Ritoré.

1. Introduction

Consider (M^n, g), a closed, smooth Riemannian manifold with boundary $Y = \partial M$ (see Figure 3). We assume that Y is at least $C^{2,\alpha}$ and will state higher regularity when needed.

For any $\varepsilon > 0$, there exists a non-negative minimizer of the Allen–Cahn energy [2]

$$E_\varepsilon(u) = \int_M \varepsilon \frac{|\nabla_g u|^2}{2} + \frac{1}{\varepsilon} W(u) \tag{1.1}$$

such that $u|_Y \equiv 0$, where $W(u) = \frac{1}{4}(1-u^2)^2$ is taken to be the standard double-well potential. Minimizers of this energy functional satisfy the Allen–Cahn equation on the interior of M

$$\varepsilon^2 \Delta_g u = W'(u) = u(u^2 - 1). \tag{1.2}$$

On closed Riemannian manifolds, there is a well-known correspondence between zero sets of solutions to the Allen–Cahn equation and minimal surfaces: Modica and Mortola [12] showed that the Allen–Cahn energy functional Γ-converges to the perimeter. Under certain geometric constraints, Wang and Wei [17, Thm 1.1] showed that the level sets of a sequence of stable solutions to (1.2), $\{u_{\varepsilon_i}\}$, converge to a minimal surface with good regularity. On the other hand, given a minimal surface $Y \subseteq M$, Pacard and Ritoré [14, Thm 4.1] showed that one can construct solutions to Allen–Cahn with zero sets converging to Y as $\varepsilon \to 0$ (see Figure 1).

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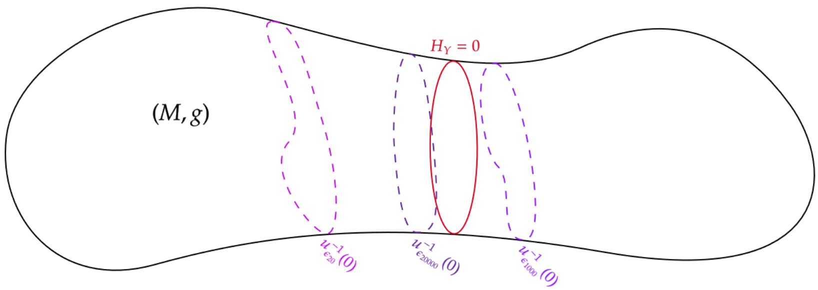

Figure 1. Illustration of level set convergence to a minimal hypersurface

In this paper, we are concerned with solutions to equation (1.2) on manifolds with boundary. Given $Y = \partial M$, how can one see the geometry of Y in a solution, $u_{\varepsilon,Y}$, to (1.2) that vanishes on Y? We take $u_{\varepsilon,Y} : M \to \mathbb{R}$ to be the non-negative minimizer of (1.1) with zero Dirichlet data on Y. We then show an asymptotic expansion of the Neumann data, $\nu_Y(u_{\varepsilon,Y})$, in powers of ε, with the coefficients depending on the curvatures of Y. Finally, we apply our results to the setting of (M^n, g) closed with $Y^{n-1} \subseteq M^n$, a separating hypersurface so that $M = M^+ \sqcup_Y M^-$.

1.1. Background

For Y as above, consider the non-negative energy minimizer of (1.1), $u_{\varepsilon,Y}$, with Dirichlet conditions on Y. By standard methods in calculus of variations, this minimizer exists. By work of Brezis–Oswald [3, Thm 1], there is at most one such solution to (1.2) on M with this Dirichlet condition, and such a solution minimizes (1.1) among all such functions. We ask, what is $\partial_\nu u_{\varepsilon,Y}$? We describe normal derivative, as well as an expansion of $u_{\varepsilon,Y}$ itself, *asymptotically in* ε, by mimicking the techniques of Wang–Wei [17] and also Mantoulidis [9, Section 4].

Our main application is the closed setting of this problem. Let (M^n, g) be a smooth Riemannian manifold, and $Y^{n-1} \subseteq M^n$ a separating, two-sided hypersurface (see Figure 2) such that $M = M^+ \sqcup_Y M^-$. One can consider non-negative (resp. non-positive) minimizers of (1.1) on $M^\pm$, referred to as $u^\pm_{\varepsilon,Y}$. For $\nu = \nu_+$ a normal pointing inward to M^+, the condition $\partial_\nu u^+_{\varepsilon,Y} = \partial_\nu u^-_{\varepsilon,Y}$ means that $u^\pm_{\varepsilon,Y}$ can be pasted together to form a smooth solution to equation (1.2) with level set on Y. In particular, we are motivated by the following theorem of Pacard and Ritoré:

Theorem 1.1 (Pacard–Ritoré [14, Thm 1.1]). *Assume that* (M, g) *is an* n*-dimensional closed Riemannian manifold and* $Y^{n-1} \subset M$ *is a two-sided, smooth, nondegenerate min-*

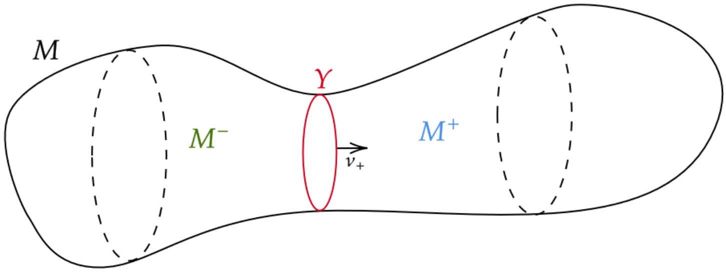

Figure 2. Closed Setting

imal hypersurface. Then there exists $\varepsilon_0 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0)$ there exists $\{u_\varepsilon\}$, solutions to the Allen–Cahn equation, such that u_ε converges to $+1$ (resp. -1) on compact subsets of $(M^+)^o$ (resp. $(M^-)^o$). Furthermore,

$$\mathcal{E}_\varepsilon(u_\varepsilon) \xrightarrow{\varepsilon \to 0} \frac{1}{\sqrt{2}} A(Y)$$

where $A(Y)$ is the $(n-1)$-dimensional area of Y.

Our Theorem 1.13 describes the projection of the solutions constructed by Pacard and Ritoré [14, Thm 4.1] onto a specific kernel.

While Pacard–Ritoré showed that one can construct solutions with zero sets converging to a prescribed minimal surface, Wang–Wei consider a stable sequence of solutions to (1.2) and produce curvature bounds on the level sets [17, Thm 1.1]. Recall that $B(u_\varepsilon)$ denotes the extended second fundamental form on graphical functions (see [17, eqn 1.4]).

Theorem 1.2 (Wang–Wei [17, Thm 1.1]). *For any $\theta \in (0,1)$, $0 < b_1 \le b_2 < 1$, and $\Lambda > 0$, there exist two constants $C = C(\theta, b_1, b_2, \Lambda)$ and $\varepsilon_* = \varepsilon(\theta, b_1, b_2, \Lambda)$ so that the following holds: suppose u_ε is a stable solution of equation (1.2) in $B_1(0) \subseteq \mathbb{R}^n$ satisfying*

$$|\nabla u_\varepsilon| \neq 0 \quad \text{and} \quad |B(u_\varepsilon)| \le \Lambda \quad \text{in} \quad \{|u_\varepsilon| \le 1 - b_2\} \cap B_1(0)$$

If $n \le 10$ and $\varepsilon \le \varepsilon_*$, then for any $t \in [-1 + b_1, 1 - b_1]$, $\{u_\varepsilon = t\}$ are smooth hypersurfaces and

$$[H(u_\varepsilon)]_\theta \le C, \qquad \|H(u_\varepsilon)\|_{C^0} \le C\varepsilon(\log|\log \varepsilon|)^2$$

where $H(u_\varepsilon)$ denotes the mean curvature of $\{u_\varepsilon = t\}$.

We are heavily inspired by the techniques used in their paper, though our results have a different theme.

1.2. Motivating example

As mentioned in the previous section, one can construct solutions of (1.2) by matching Dirichlet and Neumann conditions along a hypersurface. We are motivated by the following example from [7, Ex. 19]:

Example 1.3. Let $M = S^n \subseteq \mathbb{R}^{n+1}$. Define regions $A_\tau = S^n \cap \{|x_{n+1}| < \tau\}$ and $S^n \backslash A_\tau = D_\tau^+ \cup D_\tau^-$, where $D_\tau^\pm$ are the discs forming the complement of the annulus A_τ. Consider $u_{\varepsilon,\tau}^\pm$ the *non-negative* energy minimizers of the Allen–Cahn energy on $D_\tau^\pm$. Let $v_{\varepsilon,\tau}$ denote the *nonpositive* energy minimizer on A_τ and define

$$\tilde{u}_{\tau,\varepsilon}(p) := \begin{cases} u_{\varepsilon,\tau}^+(p) & p \in D_\tau^+, \\ u_{\varepsilon,\tau}^-(p) & p \in D_\tau^-, \\ v_{\varepsilon,\tau}(p) & p \in A_\tau; \end{cases}$$

see Figure 4. The function $\tilde{u}_{\tau,\varepsilon} \in C^0(S^n)$ is a solution to equation (1.2) on $S^n \backslash \partial A_\tau$. We aim to find $0 < \tau < 1$ such that $\tilde{u}_\tau$ is C^1 across ∂A_τ, that is, the Neumann data matches on $x_{n+1} = \pm\tau$.

Sketch of the proof. One can show that

$$C_{\varepsilon,\tau}^\pm := \frac{\partial u_{\varepsilon,\tau}^\pm}{\partial x_{n+1}} - \frac{\partial v_{\varepsilon,\tau}}{\partial x_{n+1}}|_{x_{n+1}=\pm\tau}$$

varies continuously with τ and is only dependent on ε and τ (i.e., these solutions are one dimensional). Note by symmetry that $C_{\varepsilon,\tau}^- = -C_{\varepsilon,\tau}^+$. In particular, for ε fixed and τ sufficiently close to 1, $u_{\varepsilon,\tau}^\pm \equiv 0$ while $v_{\varepsilon,\tau} < 0$ on $\mathrm{Int}(A_\tau)$. Similarly, for τ sufficiently close to 0, $u_{\varepsilon,\tau}^\pm > 0$ and $v_{\varepsilon,\tau} \equiv 0$, that is, for some $\delta > 0$,

$$\begin{aligned} |1-\tau| < \delta &\implies C_{\varepsilon,\tau}^+ > 0, \\ |1-\tau| > 1-\delta &\implies C_{\varepsilon,\tau}^+ < 0. \end{aligned}$$

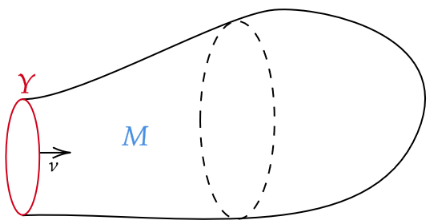

Figure 3. Image of our setup

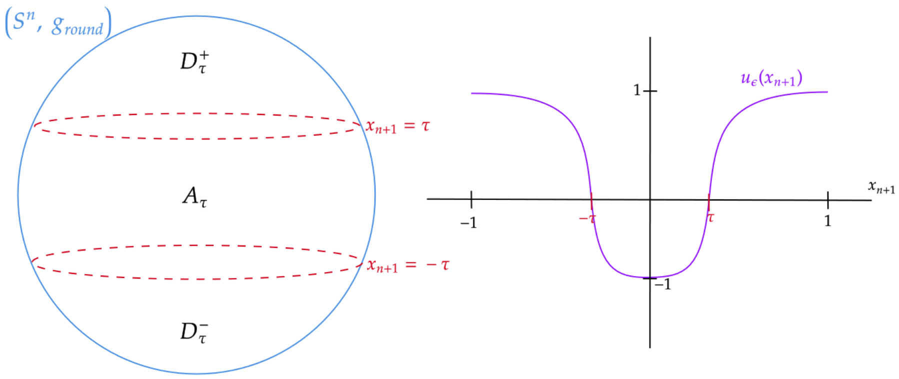

Figure 4. Example of matching Neumann data on the sphere

By continuity of $C^{+}_{\varepsilon,\tau}$, there exists τ such that $C^{\pm}_{\varepsilon,\tau} = 0$, and so that $\tilde{u}_{\tau,\varepsilon}$ is a C^1 and hence, by elliptic regularity, a smooth solution to equation (1.2) on S^n.

1.3. Results

We recall the classical Dirichlet-to-Neumann map: consider an elliptic operator, L, arising from an energy functional. Given (M, g) a Riemannian manifold and ∂M smooth, consider $f : \partial M \to \mathbb{R}$. Suppose there exists a unique $\tilde{u}$ such that

$$\tilde{u} : M \to \mathbb{R}, \quad L\tilde{u} = 0, \quad \tilde{u}|_{\partial M} = f.$$

Then one can formulate a map

$$\mathcal{D} : H^1(\partial M) \to L^2(\partial M), \quad \mathcal{D}(f) = \partial_\nu(\tilde{u});$$

see [11, Section 7] for details. We investigate the following Dirichlet-to-Neumann-type map, where the Dirichlet data is fixed at 0, but the manifold and its boundary $(M, Y = \partial M)$, are variable. This is pictured in Figure 5 where we think of M as a subset of a larger closed manifold. Suppose we take the unique energy minimizer on M such that

$$\begin{cases} \tilde{u} : M \to \mathbb{R}, \\ \varepsilon^2 \Delta_g(\tilde{u}) - W'(\tilde{u}) = 0, \\ \tilde{u}|_M > 0, \\ \tilde{u}|_Y = 0. \end{cases}$$

Let ν be the positive normal to Y and define

$$\mathcal{N}(Y) := \partial_\nu(\tilde{u})|_Y.$$

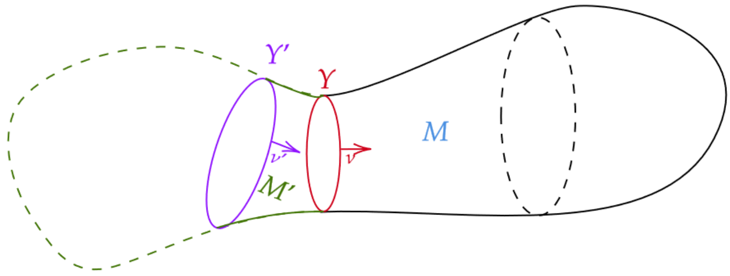

Figure 5. Variable boundary visualization

The function $\mathcal{N}(Y)$ is a global term and depends on the geometry of M, as opposed to just the geometry of Y. We prove the following result, which expands the Neumann data asymptotically in ε:

Theorem 1.4. *For* $0 < \alpha < 1$, $Y = \partial M$ *a* $C^{3,\alpha}$ *hypersurface, and* $u_{\varepsilon,Y}$ *the positive minimizer of* E_ε *on* M *with* 0 *Dirichlet condition on* Y*, we have that*

$$\partial_\nu u_{\varepsilon,Y} = \frac{1}{\varepsilon\sqrt{2}} - \frac{2}{3}H_Y + O(\varepsilon^{1-\alpha})$$

with error in $C^\alpha(Y)$.

Remark 1.5. We can posit this more formally as a Dirichlet-to-Neumann operator when M is closed and $Y \subseteq M$ is a separating hypersurface. Let $\mathcal{M}^{2,\alpha}(M)$ denote the space of $C^{2,\alpha}$ two-sided, closed hypersurfaces with bounded geometry (cf. equation (2.1)). For $Y \subseteq \mathcal{M}^{2,\alpha}(M)$, let $U(Y)$ denote a normal neighborhood such that each $Y' \in U$ can be represented as

$$Y' = Y_\eta = F(Y,\eta) := \exp_Y(\eta(p)\nu(p))$$

with $\eta \in C^{2,\alpha}(Y)$ and ν a normal to Y. Define

$$\begin{aligned} F &: Y \times (-\delta,\delta) \to M, \\ F(p,t) &= \exp_p(t\nu(p)), \\ F_\eta(p,t) &= \exp_p((t+\eta(p))\nu(p)) \end{aligned} \tag{1.3}$$

so that

$$\begin{aligned} &\mathcal{N} : U(Y) \to C^{1,\alpha}(Y), \\ &Y_\eta \mapsto F_\eta^*(\nu_{Y_\eta} u^+_{\varepsilon,Y_\eta})|_{t=0} = (F_\eta^{-1})_*(\nu_\eta) F_\eta^*(u^+_{\varepsilon,Y_\eta})|_{t=0}. \end{aligned} \tag{1.4}$$

Now noting that $U(Y) =: V \subseteq C^{2,\alpha}(Y)$, we can frame $\mathcal{N} : V \to C^\alpha(Y)$ as a map between functions on Y. In this sense, the variable initial level set, Y_η, is the Dirichlet data, and the normal derivative is the Neumann data, which forms a Dirichlet-to-Neumann-type map on Y after pulling back. Our results do not just apply to perturbations of a fixed Y, as all of our theorems (with the exception of Theorem 1.13) apply to *any* separating hypersurface of the appropriate regularity and bounded geometry. For this, we use the term “Dirichlet-to-Neumann-*type* map” to describe $\mathcal{N}$.

As noted in [7, Ex. 8], for Y fixed, a positive minimizer on $M^{\pm}$ will exist for ε sufficiently small.

We begin with a key estimate on $L_\varepsilon = \varepsilon^2\Delta_g - W''(\overline{\mathbb{H}}_\varepsilon)$ with respect to the weighted holder space, $C_\varepsilon^{k,\alpha}$ (see the definition in (2.8)).

Theorem 1.6. *Let $Y = \partial M^+$ be a $C^{2,\alpha}$ surface, and suppose $f : M^+ \to \mathbb{R}$ in $C_\varepsilon^{2,\alpha}(M^+)$ with $f(s,0) \equiv 0$. Then there exists an $\varepsilon_0 > 0$ sufficiently small, independent of f, such that for all $\varepsilon < \varepsilon_0$, we have*

$$\|f\|_{C_\varepsilon^{2,\alpha}(M^+)} \le K\|L_\varepsilon f\|_{C_\varepsilon^{\alpha}(M)}$$

for K independent of ε.

Immediately, we see that the linearized Allen–Cahn operator is invertible as a map from $C_\varepsilon^{2,\alpha}(M^+) \cap C_0(M^+)$ (i.e., with zero boundary conditions) to $C_\varepsilon^{\alpha}(M^+)$ (see [6, Thm 6.15]). After establishing this, Theorem 1.4 is then proved in the following manner:

(1) Let t denote the signed distance from Y and s be a Fermi coordinate on Y. We decompose
$$u_{\varepsilon,Y}(s,t) = \overline{\mathbb{H}}\left(\frac{t}{\varepsilon}\right) + \phi(s,t)$$
where $\overline{\mathbb{H}}(t)$ is a modification of the heteroclinic solution. We then rewrite the Allen–Cahn equation in terms of ϕ.

(2) We prove a modified Schauder estimate,
$$\|\phi\|_{C_\varepsilon^{2,\alpha}(M^+)} \le \|L_\varepsilon \phi\|_{C_\varepsilon^{\alpha}(M^+)},$$
reminiscent of [13, Prop. 3.21].

(3) We integrate the Allen–Cahn equation, showing that
$$\partial_t\phi(s,0)|_{t=0} = H_Y(s)\sigma_0 + \int_0^{-\omega\varepsilon\ln(\varepsilon)} (\Delta_t\phi)\dot{\overline{\mathbb{H}}}_\varepsilon + O(\varepsilon^2).$$

(4) We show that the quantity $\int_0^{-\omega\varepsilon\ln(\varepsilon)}(\Delta_t\phi)\dot{\overline{\mathbb{H}}}_\varepsilon$ is small by proving better $C_\varepsilon^{2,\alpha}$ estimates for $\partial_{s_i}\phi$. This mimics [17, Section 7].

We can improve our analysis of the Neumann data when $H_Y = 0$:

Theorem 1.7. *When $Y = \partial M$ is minimal, we have that*

$$\nu^+(u_{\varepsilon,Y}^+) = \frac{1}{\varepsilon\sqrt{2}} + \sigma_0^{-1}\kappa_0\varepsilon[Ric_Y(\nu,\nu) + |A_Y|^2] + O(\varepsilon^{2-\alpha})$$

with error in $C^\alpha(Y)$.

In the manifold with boundary setting, we see that as Y is more regular, we can capture more terms in the expansion of $\nu(u_{\varepsilon,Y})$. This culminates in our main theorem (Theorem 1.8). Recall the (s,t) coordinates for a tubular neighborhood of Y, where t denotes the signed distance from Y. Let H_t denote the mean curvature of the set of points a distance t from Y. Then

Theorem 1.8. *For $Y = \partial M$ a $C^{\bar{k}+3,\alpha}$ hypersurface, the minimizer of* (1.1) *can be expanded as*

$$\begin{aligned} u_\varepsilon(s,t) &= \overline{\mathbb{H}}_\varepsilon(t) \\ &\quad + \sum_{i=1}^{\bar{k}} \varepsilon^i \cdot \Big(\sum_{j=0}^{m_i} a_{i,j}\big(\{\partial_s^\beta \partial_t^j H_t(s)|_{t=0}\}_{j+|\beta|\le i}\big)\overline{w}_{i,j,\varepsilon}(t)\Big) + \phi, \quad (1.5) \\ \|\phi\|_{C^{2,\alpha}_\varepsilon(M)} &= O(\varepsilon^{k+1}), \end{aligned}$$

where $a_{i,j}(s)$ are polynomials of derivatives of H_t up to a certain order and $\overline{w}_{i,j,\varepsilon}(t) = \overline{w}_{i,j}(t/\varepsilon)$ are modifications of functions $w_{i,j,\varepsilon}(t) = w_{i,j}(t/\varepsilon)$ for $w_{i,j} : [0,+\infty) \to \mathbb{R}$ smooth and exponentially decaying in t (see Definition 2.1).

When $\bar{k} \geq 1$, this yields

Corollary 1.9. *For u_ε a solution to equation* (1.2) *with Dirichlet data on a $C^{4,\alpha}$ hypersurface Y, we have that*

$$\begin{aligned} u^+_{\varepsilon,Y}(s,t) &= \overline{\mathbb{H}}_\varepsilon(t) + \varepsilon H_Y(s)\overline{w}_\varepsilon(t) + \phi(s,t), \\ \|\phi\|_{C^{2,\alpha}_\varepsilon(M)} &= O(\varepsilon^2). \end{aligned}$$

When Y is minimal (and has no singular set, by assumption at the beginning of Section 1), we have the following corollary:

Corollary 1.10. *For $u_{\varepsilon,Y}$ a solution to equation* (1.2) *with Dirichlet data on Y, a minimal surface, the expansion in* (1.5) *exists to any order.*

Such surfaces appear when the minimal surface exhibits symmetry inside of the ambient manifold. See, for example, [7, Section 6] and [4, Section 5] among other sources.

Remark 1.11. In general, we see that both the expansion of $u_{\varepsilon,Y}$ and its Neumann derivative are *asymptotically local* in terms of a series expansion in powers of ε, despite being determined by the *global* geometry of M.

Remark 1.12. Similar expansions have been done by Wang and Wei [16], Chodosh and Mantoulidis [5], and Mantoulidis [9], among other authors. These works begin with $u : M \to \mathbb{R}$ a smooth solution to (1.2) and then expand u about its zero set. This approach actually gives the zero set better regularity by a Simons-type equation (see [16, Lem 8.6]), allowing for more terms in the expansion. By contrast, we start with Y, a prescribed zero set with limited regularity, and one-sided solutions, $u^\pm_{\varepsilon,Y} : M^\pm \to \mathbb{R}$, for which the Simons-type equation does not apply.

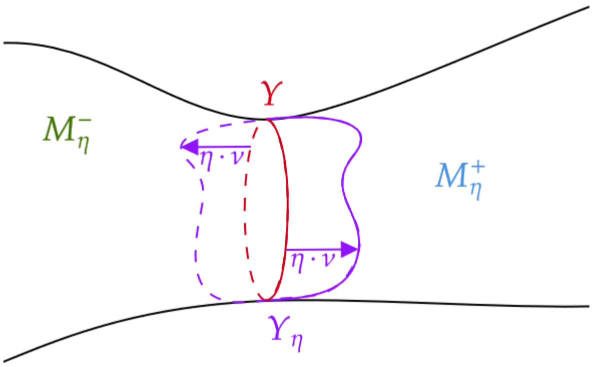

Figure 6. Perturbation of Y and the corresponding splitting of $M = M_\eta^+ \sqcup_{Y_\eta} M_\eta^-$

Returning to the setting of M closed and $Y \subseteq M$ a separating hypersurface: take $\eta \in C^{2,\alpha}(Y)$, along with equation (1.3):

$$F_\eta : Y \to U(Y) \subseteq M, \quad F_\eta(p,0) := \exp_p(\eta(p)\nu(p)), \quad Y_\eta = \{F_\eta(p) \mid p \in Y\},$$

where $U(Y)$ is some open neighborhood of Y in M. Decompose $M = M_\eta^+ \sqcup_{Y_\eta} M_\eta^-$ (see Figure 6), and consider the (positive) energy minimizers $u_{\varepsilon,\eta}^\pm$ on $M_\eta^\pm$. By Brezis–Oswald [3, Thm 1], these are the unique solutions to (1.2) on $M_\eta^\pm$ and we can paste them together to form:

$$u_{\varepsilon,\eta} : M \to \mathbb{R}, \tag{1.6}$$
$$u_{\varepsilon,\eta} = \begin{cases} u_{\varepsilon,\eta}^+(p) & p \in M_\eta^+, \\ -u_{\varepsilon,\eta}^-(p) & p \in M_\eta^-. \end{cases}$$

We now use the map in equation (1.4)

$$\mathcal{N}^\pm : U(Y) \subseteq C^{2,\alpha}(Y) \to C^{1,\alpha}(Y), \quad \mathcal{N}^\pm(\eta) = F_\eta^*(\partial_{\nu_{Y_\eta}^\pm} u_{\varepsilon,\eta}^\pm)(p).$$

The Neumann data of $u_{\varepsilon,\eta}$ match along Y_η if and only if

$$\mathcal{N}^+(\eta) - \mathcal{N}^-(\eta) = 0.$$

When this is the case, $u_{\varepsilon,\eta}$ is a smooth solution to (1.2) and we can characterize the projection of $u_{\varepsilon,\eta}$ onto $\dot{\overline{\mathbb{H}}}_\varepsilon$, the kernel of $L_\varepsilon := \varepsilon^2\partial_t^2 - W''(\mathbb{H}_\varepsilon)$.

Theorem 1.13. *Let $Y \subseteq M$ be a minimal separating hypersurface in a closed, smooth Riemannian manifold. For $u_{\varepsilon,\eta} : M \to \mathbb{R}$ as in (1.6), suppose that $u_{\varepsilon,\eta}$ is C^1 across Y_η and $\|\eta\|_{C^{2,\alpha}} = O(\varepsilon^{1+\beta})$ for some $\beta \geq \alpha$ fixed. Then*

$$\int_{\mathbb{R}} \Delta_Y(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon(t)dt = 2\sigma_0 J_Y(\eta) + \tilde{O}(\varepsilon^{1+2\beta})$$

with error holding in $C^{\alpha}(Y)$. If we further have that Y is nondegenerate, then

$$\int_{\omega\varepsilon\ln(\varepsilon)}^{-\omega\varepsilon\ln(\varepsilon)} u_{\varepsilon,\eta}(s,t)\dot{\overline{\mathbb{H}}}_{\varepsilon}(t)dt = \frac{\sqrt{2}}{3}\eta(s) + \tilde{O}(\varepsilon^2,\eta^2)$$

with error in $C^{2,\alpha}(Y)$.

The above theorem tells us that when we perturb Y to Y_η to find a solution to (1.2) with zero set Y_η, then we can detect η via the projection of our solution onto $\dot{\overline{\mathbb{H}}}_{\varepsilon}$. Also note that for $u_{\varepsilon,\eta}(s,t) = \overline{\mathbb{H}}_{\varepsilon}(t) + \phi(s,t)$,

$$\int_{\omega\varepsilon\ln(\varepsilon)}^{-\omega\varepsilon\ln(\varepsilon)} u_{\varepsilon,\eta}(s,t)\dot{\overline{\mathbb{H}}}_{\varepsilon}(t)dt = \int_{\omega\varepsilon\ln(\varepsilon)}^{-\omega\varepsilon\ln(\varepsilon)} \phi(s,t)\dot{\overline{\mathbb{H}}}_{\varepsilon}(t)dt + O(\varepsilon^k),$$

so Theorem 1.13 is equivalent to computing the projection of ϕ onto $\dot{\overline{\mathbb{H}}}_{\varepsilon}$. Further note that Theorem 1.13 differs from Corollary 1.9 in that we compute a two-sided integral for Theorem 1.13.

Following the completion of this paper, the expansions in Theorem 1.8 and the linearized operator bound in Theorem 1.6 have been used in a fundamental way in [10, 15] in order to construct an Allen–Cahn like energy on hypersurfaces and establish a strong min-max property for it.

1.4. Paper organization

This paper is organized as follows:

- In Section 2, we define our notation and recall some known geometric equations and quantities.
- In Section 3, we pose the initial decomposition of $u^{+}_{\varepsilon,Y} = \overline{\mathbb{H}}(t/\varepsilon) + \phi$ where $\overline{\mathbb{H}}$ is a modification of the heteroclinic solution. We prove Theorem 1.6, which is used to estimate $\|\phi\|_{C^{2,\alpha}_{\varepsilon}}$ and $\|\partial_{s_i}\phi\|_{C^{2,\alpha}_{\varepsilon}}$. We then prove Theorem 1.4.
- In Section 4, we prove higher order expansions of the Neumann data, Theorem 1.7, and our main result, Theorem 1.8. This theorem says that given Y a $C^{k+3,\alpha}$ surface with bounded geometry, we can expand $u^{\pm}_{\varepsilon,Y}$ to order ε^k.
- In Section 5, we prove Theorem 1.13.

2. Setup

We first describe the manifold with boundary setting. Let (M^n, g) be a Riemannian manifold with $Y = \partial M$ a $C^{k,\alpha}$ surface for some $k \geq 2$. Throughout this paper, we will assume uniform bounds on the geometry of Y, that is, we fix a $C > 0$ (independent of ε) such that

$$\|A_Y\|_{C^{k-2,\alpha}} \leq C. \tag{2.1}$$

We define $u_{\varepsilon,Y}$ (also denoted as u_ε) to be the energy minimizer of (1.1) on M with Dirichlet conditions on Y. Let $p \in Y$ a base point, $\{E_i\}$ an orthonormal frame for TY at p, and ν a normal on Y with respect to g. We coordinatize a tubular neighborhood of Y via the following maps:

$$\begin{aligned} &G : B_1(0)^{n-1} \to Y, \quad G(s) = \exp_p^Y(s^i E_i), \\ &F : B_1(0)^{n-1} \times [0, \delta_0) \to M, \quad F(s,t) := \exp^M_{G(s)}(\nu(G(s))t), \end{aligned} \tag{2.2}$$

for any $\omega > 5$ fixed. Here, $\exp_p^Y : B_1(0) \to Y$ denotes the exponential map into Y, and $\exp^M : Y \times [0, \delta_0) \to M$ is the M-exponential map. We will often suppress the $F(s,t)$ notation and identify $F(s,t)$ with (s,t) notationally. Using equation (2.2), we will also denote $u = u(s,t) = u(F(s,t))$ in an ε-neighborhood of Y via the above. In this neighborhood, we can expand the metric and second fundamental form, $g_{ij}(s,t)$ and $A_{ij}(s,t)$, in coordinates, smoothly in t from the following equations:

$$\begin{aligned} g_{ij}(s,t) = g_{ij}(s,0) &- 2t \sum_{k=1}^{n} A_i^k(s,0) g_{jk}(s,0) \\ &+ t^2 \sum_{i,j=1}^{n} A^2(\partial_i, \partial_j) - \mathrm{Rm}_g(\partial_i, \partial_t, \partial_t, \partial_j) + O(t^3), \end{aligned} \tag{2.3}$$

$$A(s,t) = A(s,0) + t\Big[A^2|_{(s,t)} - \mathrm{Rm}(\cdot, \partial_t, \partial_t, \cdot)|_{(s,t)}\Big] + O(t^2), \tag{2.4}$$

$$H(s,t) = H(s,0) + t[-|A|^2|_{(s,t)} - \mathrm{Ric}_g(\partial_t, \partial_t)|_{(s,t)}] + O(t^2). \tag{2.5}$$

Here, $g(s,0)$, $A(s,0)$, and $H(s,0)$ denote the corresponding geometric quantities on Y. Moreover, A^2 denotes a single trace of $A \otimes A$ (see [5, A.1–A.2]). We can decompose the Laplacian on M in a neighborhood of Y via

$$\Delta_g = \Delta_t - H_t \partial_t + \partial_t^2 \tag{2.6}$$

where $\Delta_t = \Delta_{Y+t\nu}$ is the Laplacian on the surface $Y + t\nu = \{p \mid d_{\text{signed}}(p, Y) = t\}$ and H_t denotes the mean curvature of $Y + t\nu$. See [17, Section 3] for details. In light of this notation, $H_t|_{t=0} = H_0 = H_Y$ and we will use H_0 and H_Y interchangeably. Similarly, $\Delta_t|_{t=0} = \Delta_0 = \Delta_Y$ and we will use these two interchangeably as well. While the above expansions hold on $t \in [0, \delta_0)$, we will often restrict u to $t \in [0, -\omega\varepsilon\ln(\varepsilon))$ as $\||u| - 1| = O(\varepsilon^\omega)$ in $C_\varepsilon^{k,\alpha}$ for $t > -\omega\varepsilon\ln(\varepsilon)$ [7, Exercise 10 and Remark]. In the closed setting for which $Y \subseteq M$ is separating, we decompose $M = M^+ \sqcup_Y M^-$ and use the above framework for $(M^+, Y = \partial M^+)$ and $(M^-, Y = \partial M^-)$ respectively.

We also define the rescaled metric

$$g_\varepsilon := \varepsilon^{-2} g \tag{2.7}$$

along with the following geometric Hölder spaces:

$$\begin{aligned}
\|f\|_{C^{0,\alpha}_\varepsilon} &:= \|f\|_{C^0} + [f]_{0,\alpha,\varepsilon}, \\
[f]_{\alpha,M} &= \sup_{p_1 \neq p_2 \in M} \frac{|f(p_1) - f(p_2)|}{|\mathrm{dist}_g(p_1,p_2)|^\alpha}, \\
[f]_{k,\alpha,\varepsilon} &:= [f]_{k,\alpha,\varepsilon,M} = \varepsilon^k \sup_{\beta \text{ s.t. } |\beta| = k} \sup_{p_1 \neq p_2 \in M} \frac{|D^\beta f(p_1) - D^\beta f(p_2)|}{\mathrm{dist}_{g_\varepsilon}(p_1,p_2)^\alpha}, \\
\|f\|_{C^{k,\alpha}_\varepsilon} &:= \sum_{j=0}^k \varepsilon^j \|D^j f\|_{C^0} + \varepsilon^k \|D^k f\|_{C^{0,\alpha}_\varepsilon}, \\
C^{k,\alpha}_{\varepsilon,0}(M) &= C^{k,\alpha}_\varepsilon \bigcap \{f : M \to \mathbb{R} \mid f|_{\partial M} \equiv 0\}.
\end{aligned} \tag{2.8}$$

Note that with respect to the rescaled metric, equation (1.2) becomes

$$\Delta_{g_\varepsilon} u = W'(u).$$

In accordance with this, we can define the following blow-up maps:

$$\begin{aligned}
&G_\varepsilon : B_{\varepsilon^{-1}}(0)^{n-1} \to Y, \quad G_\varepsilon(\sigma) := \exp^Y_{p,g_\varepsilon}(\sigma^i E_i), \\
&F_\varepsilon : B_{\varepsilon^{-1}}(0)^{n-1} \times [0, -\omega \ln(\varepsilon)) \to M, \quad F_\varepsilon(\sigma,\tau) := \exp^{NY}_{G(\sigma),g_\varepsilon}(\nu(G(\sigma))\tau).
\end{aligned} \tag{2.9}$$

We may refer to (σ, τ) as "scaled" Fermi coordinates, as opposed to (s, t), the actual Fermi coordinates. For $\eta \in C^{2,\alpha}(Y)$ non-negative and $\|\eta\|_{C^0} \leq \delta_0$, define the perturbed graph

$$Y_\eta := \{p = F(s, \eta(s)) \mid s \in Y\} \tag{2.10}$$

where $F(s, \eta(s))$ (see equation (2.2)) is the point that is a distance of $\eta(s)$ away from $s \in Y$. We also define

$$M_\eta := \{p = F(s,t) \in M \mid \eta(s) \leq t < \delta_0\} \cup \{p \in M \mid \mathrm{dist}(p, Y) \geq \delta_0\}$$

As with Y, we define $Y_\eta := \partial M_\eta$. We then define $u_{\eta,\varepsilon}$, the minimizer of E_ε on M_η with 0 Dirichlet condition on Y_η.

For the closed setting, we have (M^n, g) a closed Riemannian manifold, and $Y^{n-1} \subseteq M^n$, a separating, two-sided hypersurface. In this case, η in (2.10) is real valued. Moreover, Y_η divides M into M^+_η and M^-_η. We then define $u^\pm_{\eta,\varepsilon}$, the non-negative (resp. nonpositive minimizers) of E_ε on $M^\pm_\eta$ with 0 Dirichlet condition on $Y_\eta = \partial M^\pm_\eta$ (see Figure 6).

2.1. Constants and definitions

We recall the 1-dimensional solution to equation (1.2), the *heteroclinic solution*, denoted by

$$\mathbb{H}(t) := \tanh\left(\frac{t}{\sqrt{2}}\right),$$

following the convention of [5]. For any $\varepsilon > 0$, we define

$$\overline{\mathbb{H}}(t) := \Big[1 - \chi\Big(\frac{t}{-\omega\ln(\varepsilon)}\Big)\Big]\mathbb{H}(t) + \chi\Big(\frac{t}{-\omega\ln(\varepsilon)}\Big),$$

where $\chi(t)$ is a smooth function that is 0 for $t < 1$, goes from $0 \to 1$ on $[1, 2]$, and is 1 for $t \geq 2$. For reference, we will also use $\chi_\delta(t) := \chi(t/\delta)$. Note that

$$\partial_t^2\overline{\mathbb{H}} - W'(\overline{\mathbb{H}}) = O(\varepsilon^\omega)$$

in a $C^{k,\alpha}(\mathbb{R}^+)$ sense and $\overline{\mathbb{H}}$ is supported on $[-\omega\ln(\varepsilon), -2\omega\ln(\varepsilon)]$. Now let $\dot{\mathbb{H}}$, $\ddot{\mathbb{H}}$ denote the first and second derivatives of $\mathbb{H}(t)$. We further denote

$$\mathbb{H}_\varepsilon(t) := \mathbb{H}\Big(\frac{t}{\varepsilon}\Big), \quad \dot{\mathbb{H}}_\varepsilon := \dot{\mathbb{H}}\Big(\frac{t}{\varepsilon}\Big), \quad \ddot{\mathbb{H}}_\varepsilon(t) := \ddot{\mathbb{H}}\Big(\frac{t}{\varepsilon}\Big)$$

to be the rescaled versions of $\mathbb{H}$ and its derivatives and analogously for $\overline{\mathbb{H}}_\varepsilon, \dot{\overline{\mathbb{H}}}_\varepsilon, \ddot{\overline{\mathbb{H}}}_\varepsilon$. Furthermore, let

$$R_{\omega,\varepsilon} = \varepsilon^2\partial_t^2\overline{\mathbb{H}}_\varepsilon - W'(\overline{\mathbb{H}}_\varepsilon),$$

which is $O(\varepsilon^\omega)$ in $C_\varepsilon^{k,\alpha}(\mathbb{R})$ and supported on $[-\omega\varepsilon\ln(\varepsilon), -2\omega\varepsilon\ln(\varepsilon)]$.

Define the constants

$$\sigma_0 := \int_0^\infty \dot{\overline{\mathbb{H}}}^2\, dt = \frac{\sqrt{2}}{3}, \quad \kappa_0 := \int_0^\infty t\dot{\overline{\mathbb{H}}}^2\, dt = \frac{1}{6}[4\ln(2) - 1], \quad \sigma = \dot{\mathbb{H}}(0) = \frac{1}{\sqrt{2}}.$$

Similarly, consider $w : [0, \infty) \to \mathbb{R}$, the solution to

$$w''(t) - W''(\overline{\mathbb{H}})w(t) = \mathbb{H}(t), \tag{2.11}$$

$$w(0) = 0, \tag{2.12}$$

$$\lim_{t\to\infty} w(t) = 0, \tag{2.13}$$

which exists and is unique by Section A.5 (see also [1, Lemma B.1 and Remark B.3]). We note that $\dot{w}(0) < 0$. As with $\overline{\mathbb{H}}$ and $\overline{\mathbb{H}}_\varepsilon$, let

$$\overline{w}(t) := \Big[1 - \chi\Big(\frac{t}{-\omega\ln(\varepsilon)}\Big)\Big]w(t)$$

(i.e., smooth cut off to 0). Also let $w_\varepsilon(t) := w(t/\varepsilon)$ and similar for $\dot{w}_\varepsilon, \ddot{w}_\varepsilon, \dot{\overline{w}}_\varepsilon, \ddot{\overline{w}}_\varepsilon$. In general, for any exponentially decaying function satisfying an ODE similar to equation (2.11), we will adopt the same notation of

$$f \to \bar{f}(t) = \Big(1 - \chi\Big(\frac{t}{-\omega\ln(\varepsilon)}\Big)\Big)f(t).$$

With this, we define the linearized Allen–Cahn operator about $\overline{\mathbb{H}}_\varepsilon$:

$$L_\varepsilon := \varepsilon^2\Delta_g - W''(\overline{\mathbb{H}}_\varepsilon) : C^{2,\alpha}(M^+) \to C^\alpha(M^+).$$

We also define big O and $\widetilde{O}$ notation to capture the size of error terms. We write $E = O(\varepsilon^m)$ or $\|E\|_{C^{k,\alpha}_{\varepsilon}} = O(\varepsilon^m)$ to denote

$$\|E\|_{C^{k,\alpha}_{\varepsilon}} \leq C\varepsilon^m$$

for some C independent of ε. Similarly, $E = \widetilde{O}(f_1, f_2, \dots)$ denotes error depending on the collection of functions

$$E = \widetilde{O}(f_1, f_2, \dots) \implies \|E\|_{C^{k,\alpha}_{\varepsilon}} \leq C \sum_i \|f_i\|_{C^{k,\alpha}_{\varepsilon}}.$$

for some C independent of ε and the $\{f_i\}$.

Finally, we establish a definition for exponentially decaying functions.

Definition 2.1. A function $f : [0, \infty) \to \mathbb{R}$ is exponentially decaying if there exists $\gamma > 0$, $C > 0$, $t_0 > 0$ such that

$$\forall t > t_0, \quad |f(t)| \leq Ce^{-\gamma t}.$$

Moreover, f is exponentially decaying in C^k if such a bound holds for all $p \leq k$ derivatives of f with constants γ_p, C_p, t_p for each p. The function f is exponentially decaying in C^∞ if such coefficients exist and the bound holds for each $p \in \mathbb{Z}^+$.

3. Normal derivative for Y

In this section we work in the manifold with boundary setting. The goal is to prove Theorem 1.4.

3.1. Initial decomposition

Decompose

$$u_\varepsilon(s,t) = \overline{\mathbb{H}}_\varepsilon(t) + \phi(s,t) \tag{3.1}$$

with the above holding on a tubular neighborhood of Y in normal coordinates, that is, $Y \times [0, \delta_0)$. We recall the following initial bound that $\|\phi\|_{C^{k,\alpha}_{\varepsilon}} = o(1)$ as $\varepsilon \to 0$:

Lemma 3.1. *Let Y be a $C^{k+1,\alpha}$ surface. For $\phi(s,t) : M^+ \to \mathbb{R}$ as in (3.1), we have that for all $\mu > 0$, there exists an $\varepsilon_0(\mu)$ such that for all $\varepsilon < \varepsilon_0$,*

$$\|\phi\|_{C^{k,\alpha}_{\varepsilon}(M^+)} \leq \mu.$$

Proof. Let $R > 0$ to be determined. Note that $\phi : B_1(0)^{n-1} \times [0, \varepsilon R) \to \mathbb{R}$ is smooth away from the boundary and $C^{k+1,\alpha}$ near the boundary (i.e., about $t = 0$; see [[6, Lem 6.18]). On this subdomain, we have

$$\phi = o(1) \in C^{k,\alpha}_{\varepsilon}(Y \times [0, \varepsilon R]) \tag{3.2}$$

as $\varepsilon \to 0$. This follows by

- Blowing up our sequence of u_ε on (M, g_ε) to get a C^{k+1} solution $u_\infty : \mathbb{R}^{n-1} \times \mathbb{R}^+ \to \mathbb{R}$ to (1.2) with $\varepsilon = 1$.
- Considering the odd reflection of u_∞, to get $\tilde{u}_\infty : \mathbb{R}^n \to \mathbb{R}$ a smooth solution on the whole space (the Dirichlet and Neumann data match at $t = 0$!).
- Using a classification of solutions to equation (1.2) on $\mathbb{R}^n$ with $u^{-1}(0) = \{x_n = 0\}$ [8, Thm 3].

This gives us C^{k+1}_{loc} convergence on $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, which when we restrict to $t \in [0, R]$ gives uniform convergence in t for any $B_r(p) \subseteq \mathbb{R}^{n-1}$ fixed. Bounding $C^{k,\alpha}$ norms by C^{k+1} norms, translating this back to the unscaled setting of $Y \times [0, \varepsilon R]$, and noting that Y is closed, we see that (3.2) holds. For $t > R\varepsilon$, we recall that for u_ε a solution to (1.2) with $u_\varepsilon^{-1}(0) = Y$, we have the following decay estimate from [7, Exercise 10] for any $\ell \in \mathbb{Z}^+$, $\alpha > 0$,

$$\|\phi\|_{C^{\ell,\alpha}} = \|u_\varepsilon(s,t) - \mathbb{H}\left(\frac{t}{\varepsilon}\right)\|_{C^{\ell,\alpha}} \leq Ce^{-\sigma t/\varepsilon}$$

where $C = C(\ell, \alpha)$ and $\sigma > 0$ independent of ε, ℓ, α. This holds for all t, and on $I_2 = (\varepsilon R, \delta_0)$ for R sufficiently large,

$$\|\phi\|_{C^{k,\alpha}(\{t > R\varepsilon\})} \leq Ce^{-\sigma R} \leq \mu$$

for some ε sufficiently small. Since $\|f\|_{C^{k,\alpha}_\varepsilon} \leq \|f\|_{C^{k,\alpha}}$, the conclusion follows. ∎

Having given an initial bound on ϕ, we write a PDE describing it.

Lemma 3.2. *For $\phi(s,t)$ as in (3.1), we have*

$$L_\varepsilon(\phi) = \varepsilon H_t \dot{\overline{\mathbb{H}}}_\varepsilon + Q_0(\phi), \tag{3.3}$$

where $Q_0(\phi) = \tilde{O}(\phi^2, \varepsilon^\omega)$ holds in $C^{k,\alpha}_\varepsilon$ for any $k > 0, 0 < \alpha < 1$.

Proof. In this decomposition, the Allen–Cahn equation for $t \in [0, \delta)$ is

$$\begin{aligned}
\varepsilon^2 \Delta_g u &= -\varepsilon H_t \dot{\overline{\mathbb{H}}}_\varepsilon + \ddot{\overline{\mathbb{H}}}_\varepsilon + \varepsilon^2[\Delta_t \phi - H_t \phi_t + \phi_{tt}], \\
W'(u) &= W'(\overline{\mathbb{H}}_\varepsilon) + W''(\overline{\mathbb{H}}_\varepsilon)\phi + Q_0(\phi^2) \\
\implies 0 &= \varepsilon^2 \Delta_g u - W'(u) \\
&= -\varepsilon H_t \dot{\overline{\mathbb{H}}}_\varepsilon + L_\varepsilon(\phi) + Q_0(\phi^2) + R_{\omega,\varepsilon} \\
&= -\varepsilon H_t \dot{\overline{\mathbb{H}}}_\varepsilon + L_\varepsilon(\phi) + \tilde{Q}_0.
\end{aligned}$$

For

$$\begin{aligned}
Q_0(\phi) &= -\frac{1}{2} W'''(\overline{\mathbb{H}}_\varepsilon)\phi^2 - \phi^3 = \tilde{O}(\phi^2), \\
\tilde{Q}_0(\phi) &= Q_0(\phi) + R_{\omega,\varepsilon} = \tilde{O}(\phi^2, \varepsilon^\omega, \varepsilon^\omega \phi) \\
&= \tilde{O}(\phi^2, \varepsilon^\omega).
\end{aligned}$$

∎

3.2. Proof of Theorem 1.6

We now demonstrate the invertibility of the linearized operator, L_ε, proving Theorem 1.6.

Proof of Theorem 1.6. Rescale the metric to g_ε as in (2.7) so that

$$L_\varepsilon = \varepsilon^2 \Delta_g - W''(\overline{\mathbb{H}}_\varepsilon) = \Delta_{g_\varepsilon} - W''(\overline{\mathbb{H}}_\varepsilon).$$

For any U in the interior of (M, g_ε), that is, $\mathrm{dist}(U, Y) > \delta$ fixed, we have that

$$\|f\|_{C^{2,\alpha}_\varepsilon(U)} \leq K[\|L_\varepsilon \phi\|_{C^\alpha_\varepsilon(U)} + \|\phi\|_{C^0(U)}]$$

by Schauder theory, with K independent of ε. For points in $Y \times [0,\delta)$, we consider scaled Fermi coordinates $\sigma = \varepsilon^{-1}s$, $\tau = \varepsilon^{-1}t$ (along with (2.9)). This gives

$$\|f(\sigma,\tau)\|_{C^{2,\alpha}(Y\times[0,\delta\varepsilon^{-1}))} \leq K(\|L_\varepsilon f(\sigma,\tau)\|_{C^\alpha(Y\times[0,\delta\varepsilon^{-1}))} + \|f\|_{C^0(Y\times[0,\delta\varepsilon^{-1}))}).$$

Note that we have changed $C^{k,\alpha}_\varepsilon \to C^{k,\alpha}$ by parameterizing by (σ,τ) instead of (s,t). Moreover, K is independent of ε by the expansion of $\varepsilon^2\Delta_g = \Delta_{g_\varepsilon}$ in the scaled coordinates, (σ,τ). Undoing the scaling and using compactness of Y and M^+, our two bounds give

$$\|f\|_{C^{2,\alpha}_\varepsilon(M)} \leq K[\|L_\varepsilon f\|_{C^\alpha_\varepsilon(M)} + \|f\|_{C^0(M)}].$$

It suffices to prove

$$\|f\|_{C^0(M)} \leq \widetilde{K}\|L_\varepsilon f\|_{C^\alpha_\varepsilon(M)}$$

for some $\widetilde{K} > 0$ also independent of ε. To this end, suppose this is not the case. Then there exists a sequence of $\{f_j\}$ and $\{\varepsilon_j\}$ such that

$$\|f_j\|_{C^0(M)} \geq j\,\|L_\varepsilon f_j\|_{C^\alpha_\varepsilon(M)}.$$

Normalize each f_j by $\|f_j\|_{C^0}$ so that

$$j^{-1} \geq \|L_\varepsilon f_j\|_{C^\alpha_\varepsilon(M)}.$$

Choose $p_j \in M$ so that $|f_j(p_j)| = 1$. By doing a maximum principle comparison with $f \equiv 1$, we see that $\mathrm{dist}(Y, p_j) < \kappa\varepsilon_j$ for some $\kappa > 0$ independent of ε_j (by using Section A.2). Thus, $p_j = (s_j, t_j)$ with $t_j < \kappa\varepsilon_j$. Define the blowups of f_j around s_j as

$$\widetilde{f_j} : B_{\varepsilon^{-1}}(0) \times [0,\delta\varepsilon^{-1}) \to \mathbb{R}, \quad \widetilde{f_j}(\sigma,\tau) := f_j(\varepsilon\sigma + s_j, \varepsilon\tau).$$

In (σ,τ) coordinates, we have the local estimate $\|L_\varepsilon \widetilde{f_j}\|_{C^\alpha(B_{\varepsilon^{-1}}(0)\times[0,\delta\varepsilon^{-1})} \to 0$, since

$$\|L_\varepsilon f_j\|_{C^\alpha_\varepsilon(M)} > \|L_\varepsilon \widetilde{f_j}\|_{C^\alpha(B_{\varepsilon^{-1}}(0)\times[0,\delta\varepsilon^{-1})}.$$

Having normalized by $\|f_j\|_{C^0}$, we have uniform $C^{2,\alpha}_{\varepsilon_j}$ bounds. Moreover, $g_\varepsilon \to g_{\mathbb{R}^n}$ uniformly in ε after pulling back to (σ,τ) coordinates. Thus, we have uniform $C^{2,\alpha}$ estimates on $\widetilde{f_j}(\sigma,\tau)$. Use the Arzelà–Ascoli theorem to pass to a subsequence that converges in C^2 to $f_\infty : \mathbb{R}^n \times \mathbb{R}^+$ on compact sets. The subsequence comes with $\{\tau_j\} \to \tau^*$ with $0 < \tau^* < \infty$, so that

$$|f_\infty(0, \tau^*)| = 1.$$

We also have convergence at the boundary, that is,

$$\forall \sigma \in \mathbb{R}^n, \quad f_\infty(\sigma, 0) \equiv 0.$$

This follows because for $C > \tau > 0$, all of the $\widetilde{f}_j$'s satisfy

$$|\widetilde{f}_j(\sigma, \tau)| \leq 2K\tau$$

by the $C^{2,\alpha}$ bounds and $\widetilde{f}_j(\sigma, 0) \equiv 0$. Thus, we get the same interior bound for f_∞, which forces the same Dirichlet data. Since $j \to \infty \implies \varepsilon_j \to 0$, we get

$$\begin{aligned} L_{\varepsilon_j} = \Delta_{g_\varepsilon} - W''(\overline{\mathbb{H}}(\tau)) &\xrightarrow{j\to\infty} \Delta_{\mathbb{R}^n} + \partial_\tau^2 - W''(\overline{\mathbb{H}}) =: \widetilde{L} \\ \implies \widetilde{L} f_\infty = 0. \end{aligned}$$

This tells us that $f_\infty(\sigma, \tau) \equiv 0$ by Lemma A.1 for $\widetilde{L} = \Delta_{\mathbb{R}^n} + \partial_\tau^2 - W''(\overline{\mathbb{H}})$ on the half space. But we have picked our sequence of points so that $f_\infty(0, \tau^*) \neq 0$, which is a contradiction. ■

From the lemma, we immediately have

Corollary 3.3. *For all $\varepsilon < \varepsilon_0$ sufficiently small, the operator*

$$L_\varepsilon : C^{2,\alpha}_{\varepsilon,0}(M) \to C^\alpha_\varepsilon(M)$$

is invertible.

In the exact same way, we prove the corresponding $C^{1,\alpha}_\varepsilon$ estimate.

Lemma 3.4. *Let Y be $C^{2,\alpha}$ and suppose $f : M \to \mathbb{R}$ in $C^{2,\alpha}_\varepsilon(M)$ satisfies $f(s, 0) \equiv 0$. There exists $\varepsilon_0 > 0$ such that $\forall \varepsilon < \varepsilon_0$,*

$$\|f\|_{C^{1,\alpha}_\varepsilon(M)} \leq K\|L_\varepsilon f\|_{C^0(M)}.$$

Remark 3.5. Of course, our results apply to $u^\pm_{\varepsilon,Y}$ in the closed setting, recreating the lemmas on each of $M^\pm$ in the decomposition of $M = M^+ \sqcup_Y M^-$. We compare the above bounds with the analogous bound in Pacard–Ritoré [14]. In [14, Prop 8.6], $\phi : Y \times \mathbb{R} \to \mathbb{R}$ and the bound

$$\|\phi\|_{C^{2,\alpha}_\varepsilon(Y\times\mathbb{R})} \leq K\|L_\varepsilon \phi\|_{C^\alpha_\varepsilon(Y\times\mathbb{R})}$$

holds when the authors enforce the orthogonality condition of $\int_\mathbb{R} \phi \dot{\mathbb{H}} = 0$. In this case, ϕ has been projected away from the kernel of L_ε, which allows the authors to exclude $\|\cdot\|_{C^0}$ terms in the Schauder estimate. In our setting, we use the Dirichlet condition of $u(s, 0) \equiv 0 \implies \phi(s, 0) \equiv 0$ (instead of an orthogonality condition) to obtain the estimate of Lemma 3.4.

Corollary 3.6. *For ϕ satisfying the same conditions as in Theorem 1.6, we have*

$$\|\phi\|_{C^{2,\alpha}_\varepsilon(M)} = O(\varepsilon). \tag{3.4}$$

Proof. Theorem 1.6 and equation (3.3) let us conclude that for $u_\varepsilon(s,t) = \overline{\mathbb{H}}_\varepsilon(t) + \phi(s,t)$,

$$\begin{aligned}
\|\phi\|_{C^{2,\alpha}_\varepsilon(M)} &\leq \|\varepsilon H_t \dot{\overline{\mathbb{H}}}_\varepsilon\|_{C^{\alpha}_\varepsilon(M)} + \|\tilde{Q}_0(\phi)\|_{C^{\alpha}_\varepsilon(M)}, \\
\|\tilde{Q}_0(\phi)\|_{C^{\alpha}_\varepsilon(M)} &\leq \|\phi\|^2_{C^{\alpha}_\varepsilon(M)} + K\varepsilon^\omega \\
\implies \|\phi\|_{C^{2,\alpha}_\varepsilon(M)} &\leq O(\varepsilon) + \mu\|\phi\|_{C^{\alpha}_\varepsilon(M)}, \\
\|\phi\|_{C^{2,\alpha}_\varepsilon(M)} &\leq O(\varepsilon)
\end{aligned}$$

where μ can be made arbitrarily small as $\varepsilon \to 0$ by Lemma 3.1. ∎

3.3. Better tangential behavior

In this section, we get improved horizontal estimates when Y is $C^{3,\alpha}$. Let ∇^Y denote the gradient on Y, extended as an operator on functions on $Y \times [0, \delta_0)$ in (s,t) coordinates via

$$\nabla^Y f(s,t) = g^{ij}(s,0)\partial_{s_i}(f)\partial_{s_j}|_{(s,t)},$$

where ∂_{s_i} is identified with $F_*(\partial_{s_i})|_{s,t}$ using equation (2.2). Then we have

Lemma 3.7. *Let Y be $C^{3,\alpha}$. For $\phi(s,t)$ as in $u(s,t) = \mathbb{H}_\varepsilon(t) + \phi(s,t)$ from equation (3.1) and any $\delta > 0$, there exists $K = K(\delta)$ such that*

$$\|\nabla_Y \phi\|_{C^{2,\alpha}_\varepsilon(Y\times[0,\delta))} \leq K\varepsilon. \tag{3.5}$$

Remark 3.8. The reader may ask how this estimate is "improved" since ϕ satisfies the same $C^{2,\alpha}_\varepsilon$ bound. The point is that because of the weighting in the $\|\phi\|_{C^{2,\alpha}_\varepsilon}$ bound, a priori we have

$$\begin{aligned}
\|\phi_{s_i}\|_{C^{1,\alpha}_\varepsilon(Y\times[0,\delta))} &\leq \varepsilon^{-1}\|\phi\|_{C^{2,\alpha}_\varepsilon(M)} = O(1), \\
\|\phi_t\|_{C^{1,\alpha}_\varepsilon(Y\times[0,\delta))} &\leq \varepsilon^{-1}\|\phi\|_{C^{2,\alpha}_\varepsilon(M)} = O(1)
\end{aligned}$$

by definition of the $C^{k,\alpha}_\varepsilon$ norms and (3.4). By contrast, the above Lemma 3.7 gives an $O(\varepsilon)$ bound for ϕ_{s_i} near the boundary, that is, one order in ε better. This method does not work for ϕ_t, since $[\partial_t, L_\varepsilon](\phi)$ is large a priori. However, we do note that for $t > -\omega\varepsilon\ln(\varepsilon)$, the same proof in Lemma 3.1 gives

$$\|\phi\|_{C^{k,\alpha}_\varepsilon(t>-\omega\varepsilon\ln(\varepsilon))} \leq C(k)\varepsilon^\omega \tag{3.6}$$

for $\varepsilon < \varepsilon_0(k)$ and $C(k)$ independent of ε. This will be used below.

Proof. Starting with

$$\varepsilon H_t \dot{\overline{\mathbb{H}}}_\varepsilon(t) = L_\varepsilon(\phi) + Q_0(\phi) + R_{\omega,\varepsilon},$$

move into Fermi coordinates and apply $\chi_\delta(t)\partial_{s_i}$ to each side:

$$\begin{aligned}\varepsilon(\chi_\delta \partial_{s_i} H_t)\dot{\overline{\mathbb{H}}}_\varepsilon &= L_\varepsilon(\chi_\delta \phi_{s_i}) + T_0(\phi)\chi_\delta\phi_{s_i} + [L_\varepsilon, \chi_\delta\partial_{s_i}](\phi) + \chi_\delta R_{\omega,\varepsilon}\\ &= L_\varepsilon(\chi_\delta\phi_{s_i}) + E,\end{aligned}$$

where

$$\begin{aligned}E &= T_0(\phi)\chi_\delta\phi_{s_i} + [L_\varepsilon, \chi_\delta\partial_{s_i}](\phi) + \chi_\delta R_{\omega,\varepsilon},\\ T_0(\phi) &= -\frac{1}{2}W'''(\overline{\mathbb{H}}_\varepsilon)\phi - 3\phi^2\end{aligned}$$

and we can bound

$$\|\varepsilon(\chi_\delta\partial_{s_i}(H_t)\dot{\overline{\mathbb{H}}}_\varepsilon)\|_{C^\alpha_\varepsilon(M)} \leq K\varepsilon.$$

For the error term, we have

$$\begin{aligned}\|E\|_{C^\alpha_\varepsilon(M)} &\leq \|\chi_\delta\phi_{s_i}\|_{C^\alpha_\varepsilon(M)} \cdot \|T_0(\phi)\|_{C^\alpha_\varepsilon(M)} + \|[L_\varepsilon, \chi_\delta\partial_{s_i}](\phi)\|_{C^\alpha_\varepsilon(M)}\\ &\qquad + \|\chi_\delta R_{\omega,\varepsilon}\|_{C^\alpha_\varepsilon(M)},\\ T_0(\phi) &= \widetilde{O}(\phi) \leq K\varepsilon,\\ \|\chi_\delta R_{\omega,\varepsilon}\|_{C^\alpha_\varepsilon(M)} &= o(\varepsilon),\\ \implies \|E\|_{C^\alpha_\varepsilon(M)} &\leq K\varepsilon\|\phi_{s_i}\|_{C^\alpha_\varepsilon(M)} + \|[L_\varepsilon, \chi_\delta\partial_{s_i}](\phi)\|_{C^\alpha_\varepsilon(M)} + o(\varepsilon)\\ &\leq K\varepsilon + \|[L_\varepsilon, \chi_\delta\partial_{s_i}](\phi)\|_{C^\alpha_\varepsilon(M)}.\end{aligned}$$

Here, we noted that $T_0(\phi) = \widetilde{O}(\phi)$ and that $\|\phi_{s_i}\|_{C^\alpha_\varepsilon(M)} = O(1)$ a priori. We now compute the commutator

$$\begin{aligned}[L_\varepsilon, \chi_\delta\partial_{s_i}] &= \varepsilon^2[\Delta_t - H_t\partial_t + \partial_t^2 - W''(\overline{\mathbb{H}}_\varepsilon), \chi_\delta(t)\partial_{s_i}]\\ &= [\varepsilon^2\Delta_t, \chi_\delta(t)\partial_{s_i}] + [-\varepsilon^2 H_t\partial_t, \chi_\delta(t)\partial_{s_i}] + [\varepsilon^2\partial_t^2, \chi_\delta(t)\partial_{s_i}]\\ &\qquad - [\varepsilon^2 W''(\overline{\mathbb{H}}_\varepsilon), \chi_\delta(t)\partial_{s_i}]\\ &= A + B + C + D.\end{aligned}$$

We bound each summand in the commutator as follows:

$$\begin{aligned}\|A\|_{C^\alpha_\varepsilon(M)} &= \|[\varepsilon^2\Delta_t, \chi_\delta(t)\partial_{s_i}]\phi\|_{C^\alpha_\varepsilon(M)} \leq \varepsilon^2\|[\Delta_t, \chi_\delta(t)\nabla^Y]\phi\|_{C^\alpha_\varepsilon(M)}\\ &\leq \varepsilon^2\|[\Delta_Y, \nabla^Y]\chi_\delta(t)\phi\|_{C^\alpha_\varepsilon(M)} + \varepsilon^2\|[(\Delta_t - \Delta_Y), \nabla^Y]\chi_\delta(t)\phi\|_{C^\alpha_\varepsilon(M)}\\ &\leq \varepsilon^2\|\chi_\delta(t)\mathrm{Ric}(\nabla^Y\phi, \cdot)\|_{C^\alpha_\varepsilon(M)} + \varepsilon^2\|[(\Delta_t - \Delta_Y), \nabla^Y]\chi_\delta(t)\phi\|_{C^\alpha_\varepsilon(M)}\\ &\leq \varepsilon^2\|\chi_\delta(t)\mathrm{Ric}(\nabla^Y\phi, \cdot)\|_{C^\alpha_\varepsilon(M)} + \varepsilon^2\|[(\Delta_t - \Delta_Y), \nabla^Y]\chi_\delta(t)\phi\|_{C^\alpha_\varepsilon(Y\times[0,2\delta))}\end{aligned}$$

$$\leq K\varepsilon\|\phi\|_{C_\varepsilon^{1,\alpha}(M)} + K\delta\|\phi\|_{C_\varepsilon^{2,\alpha}(M)} + O(\varepsilon^\omega) \leq K\delta\varepsilon,$$

having used (3.6) and for K independent of δ. We also compute

$$\begin{aligned}\|B\|_{C_\varepsilon^\alpha(M)} &= \|[-\varepsilon^2 H_t\partial_t, \chi_\delta(t)\partial_{s_i}]\phi\|_{C_\varepsilon^\alpha(M)} \\ &\leq \varepsilon^2[\|\chi_\delta\partial_{s_i}(H_t)\phi_t\|_{C_\varepsilon^\alpha(M)} + \|H_t\chi_\delta'(t)\phi_s\|_{C_\varepsilon^\alpha(M)}] \\ &\leq K\varepsilon^2\|\phi_t\|_{C_\varepsilon^\alpha(M)} + K\varepsilon^2\delta^{-1}\|\phi_s\|_{C_\varepsilon^\alpha(M)} \leq K\delta^{-1}\varepsilon\|\phi\|_{C_\varepsilon^{1,\alpha}(M)} \leq K\delta^{-1}\varepsilon^2.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|C\|_{C_\varepsilon^\alpha(M)} &= \|[\varepsilon^2\partial_t^2, \chi_\delta\partial_{s_i}]\phi\|_{C_\varepsilon^\alpha(M)} = \varepsilon^2\|\chi_\delta'\phi_{st} + \chi_\delta''\phi_s\|_{C_\varepsilon^\alpha(M)} \leq \delta^{-2}\|\phi\|_{C_\varepsilon^{2,\alpha}(t>\delta)} \\ &\leq K\varepsilon^\omega, \\ \|D\|_{C_\varepsilon^\alpha(M)} &= \|[\chi_\delta\partial_{s_i}, \varepsilon^2 W''(\overline{\mathbb{H}}_\varepsilon)]\phi\| = 0.\end{aligned}$$

So in conclusion, we have

$$\|E\|_{C_\varepsilon^\alpha(M)} \leq K\varepsilon,$$

and so

$$\begin{aligned}L_\varepsilon(\chi_\delta\phi_{s_i}) &= \varepsilon(\chi_\delta\partial_{s_i}(H_t)\dot{\overline{\mathbb{H}}}_\varepsilon) - E, \\ \|\phi_{s_i}\|_{C_\varepsilon^{2,\alpha}(Y\times[0,\delta))} &\leq \|\chi_\delta\phi_{s_i}\|_{C_\varepsilon^{2,\alpha}(M)} \leq K\|L_\varepsilon(\chi_\delta\phi_{s_i})\|_{C_\varepsilon^\alpha(M)} \leq K\varepsilon.\end{aligned}$$

Here, we have used Theorem 1.6 (note that $\phi_{s_i}(s,0) \equiv 0$). The bound on $\nabla^Y\phi$ now follows. ∎

3.4. Proof of Theorem 1.4

Referencing the decomposition in equation (3.3), we multiply by $\dot{\overline{\mathbb{H}}}_\varepsilon$ (recall the definition in Section 2.1) and integrate from $t = 0$ to $t = -\omega\varepsilon\ln(\varepsilon)$:

$$\tilde{O}(\phi^2) = -\varepsilon H_t\dot{\overline{\mathbb{H}}}_\varepsilon + L_\varepsilon(\phi), \tag{3.7}$$

which implies

$$\begin{aligned}\int_0^{-\omega\varepsilon\ln(\varepsilon)} \tilde{O}(\phi^2)\dot{\overline{\mathbb{H}}}_\varepsilon = -\varepsilon\int_0^{-\omega\varepsilon\ln(\varepsilon)} H_t\dot{\overline{\mathbb{H}}}_\varepsilon^2 + \varepsilon^2\int_0^{-\omega\varepsilon\ln(\varepsilon)} (\Delta_t\phi)\dot{\overline{\mathbb{H}}}_\varepsilon \\ - \varepsilon^2\int_0^{-\omega\varepsilon\ln(\varepsilon)} H_t\phi_t\dot{\overline{\mathbb{H}}}_\varepsilon \\ + \varepsilon^2\int_0^{-\omega\varepsilon\ln(\varepsilon)} [\phi_{tt} - W''(\overline{\mathbb{H}}_\varepsilon)\phi]\dot{\overline{\mathbb{H}}}_\varepsilon.\end{aligned} \tag{3.8}$$

Note that the left-hand side of (3.8) can be bounded:

$$\left\|\int_0^{-\omega\varepsilon\ln(\varepsilon)} \tilde{O}(\phi^2)\dot{\overline{\mathbb{H}}}_\varepsilon dt\right\|_{C^\alpha(Y)} \leq K\varepsilon^{2-\alpha}\int_0^{-\omega\varepsilon\ln(\varepsilon)} \dot{\overline{\mathbb{H}}}_\varepsilon dt \leq K\varepsilon^{3-\alpha} + O(\varepsilon^\omega) \leq K\varepsilon^{3-\alpha},$$

since $\omega > 5$. For the right-hand side of (3.8), we note that

$$|H_t - H_0| \leq t \|\dot{H}_t\|_{C^0(Y\times[0,\delta))} \leq Kt,$$

which follows by the mean value theorem and (2.4). Moreover, recall from Section 2.1 that

$$\int_0^{-\omega\varepsilon\ln(\varepsilon)} \dot{\overline{\mathbb{H}}}_\varepsilon^2 = \varepsilon\sigma_0 + C\varepsilon^\omega.$$

We further note from (3.5) that

$$\begin{aligned}\Big\| \int_0^{-\omega\varepsilon\ln(\varepsilon)} \varepsilon^2(\Delta_t\phi)\dot{\overline{\mathbb{H}}}_\varepsilon \Big\|_{C^\alpha(Y)} &= \Big\| \int_0^{-\omega\varepsilon\ln(\varepsilon)} \varepsilon^2(\Delta_0\phi)\dot{\overline{\mathbb{H}}}_\varepsilon \Big\|_{C^\alpha_Y} \\ &\quad + \Big\| \int_0^{-\omega\varepsilon\ln(\varepsilon)} \varepsilon^2([\Delta_t - \Delta_0]\phi)\dot{\overline{\mathbb{H}}}_\varepsilon \Big\|_{C^\alpha_Y} \\ &= O(\varepsilon^{3-\alpha}) + O(\varepsilon^{4-\alpha}) = O(\varepsilon^{3-\alpha}).\end{aligned}$$

Here, we again used that $\Delta_t - \Delta_0 = tL$ where L is a second-order linear differential operator with bounded coefficients. In both cases, we use equation (3.7) to get the final bounds. Similarly, from equation (3.4), we have

$$\Big\| \int_0^{-\omega\varepsilon\ln(\varepsilon)} \varepsilon^2 H_t\phi_t\dot{\overline{\mathbb{H}}}_\varepsilon \Big\|_{C^\alpha(Y)} = O(\varepsilon^{3-\alpha}).$$

We also compute

$$\begin{aligned}\int_0^{-\omega\varepsilon\ln(\varepsilon)} \dot{\overline{\mathbb{H}}}_\varepsilon &= \varepsilon(1 + O(\varepsilon^\omega)), \\ \int_0^{-\omega\varepsilon\ln(\varepsilon)} \big(\varepsilon^2\phi_{tt}\dot{\overline{\mathbb{H}}}_\varepsilon - W''(\overline{\mathbb{H}}_\varepsilon)\phi\dot{\overline{\mathbb{H}}}_\varepsilon\big) &= -\varepsilon^2\sigma\phi_t(s,0) + O(\varepsilon^k), \\ \int_0^{-\omega\varepsilon\ln(\varepsilon)} \varepsilon H_t\dot{\overline{\mathbb{H}}}_\varepsilon^2 &= \varepsilon^2 H_0\sigma_0 + O(\varepsilon^3)\end{aligned}$$

with all estimates holding in $C^\alpha(Y)$. Combining and noting that $\sigma_0\sigma^{-1} = \frac{2}{3}$, we have

$$\varepsilon^2\phi_t(s,0) = -\varepsilon^2\frac{2}{3}H_0 + O(\varepsilon^{3-\alpha})$$

for any $k \geq 3$. We summarize this as

$$\phi_t(s,0) = -\frac{2}{3}H_0 + O(\varepsilon^{1-\alpha}),$$

so that

$$\partial_\nu u(p) = \partial_t u = \varepsilon^{-1}\dot{\mathbb{H}}(0) + \phi_t(p,0) = \frac{1}{\varepsilon\sqrt{2}} - \frac{2}{3}H_Y(p) + O(\varepsilon^{1-\alpha})$$

holds in $C^\alpha(Y)$. This proves Theorem 1.4. ∎

4. Higher order expansions

4.1. Proof of Theorem 1.7

In this section, we give a next order expansion of the normal derivative when $H_Y = 0$.

Proof of Theorem 1.7. When Y is minimal (and hence smooth, since we assumed all our hypersurfaces are at least $C^{2,\alpha}$), (3.3) becomes

$$L_\varepsilon(\phi) = \varepsilon H_t \dot{\overline{\mathbb{H}}}_\varepsilon + Q_0(\phi^2).$$

If $H_Y = H_0 = 0$, then

$$H_t = \int_0^t \dot{H}_r dr.$$

We expand this as

$$L_\varepsilon(\phi) = \varepsilon^2 \dot{H}_0(s)\Big(\frac{t}{\varepsilon}\Big)\dot{\overline{\mathbb{H}}}_\varepsilon(t) + \varepsilon\Big(\int_0^t \int_0^r \ddot{H}_w(s)dwdr\Big)\dot{\overline{\mathbb{H}}}_\varepsilon(t) + Q_0(\phi^2) \tag{4.1}$$

with the goal of showing

$$\Big\|\varepsilon\Big(\int_0^t \int_0^r \ddot{H}_w(s)dwdr\Big)\dot{\overline{\mathbb{H}}}_\varepsilon(t)\Big\|_{C^\alpha_\varepsilon(M)} = o(\varepsilon^2).$$

The C^0 bound holds clearly, as

$$\begin{aligned}\Big|\varepsilon\Big(\int_0^t \int_0^r \ddot{H}_w dwdr\Big)\dot{\overline{\mathbb{H}}}_\varepsilon(t)\Big| &\leq \varepsilon\Big(\int_0^t \int_0^r \sup_{\substack{s\in Y\\ w\in[0,-\omega\varepsilon\ln(\varepsilon))}} |\ddot{H}_w(s)|dwdr\Big)\dot{\overline{\mathbb{H}}}_\varepsilon \\ &\leq K\varepsilon t^2 \dot{\overline{\mathbb{H}}}_\varepsilon \leq K\varepsilon^3 \sup_{t\in[0,-\omega\varepsilon\ln(\varepsilon))}\Big|\Big(\frac{t}{\varepsilon}\Big)^2\dot{\overline{\mathbb{H}}}_\varepsilon(t)\Big| \leq K\varepsilon^3.\end{aligned}$$

For the $[\cdot]_\alpha$ bound, we have

$$\begin{aligned} f(s,t) &:= \varepsilon\Big(\int_0^t \int_0^r \ddot{H}_w(s)dwdr\Big)\dot{\overline{\mathbb{H}}}_\varepsilon(t), \\ [f]_{\alpha,Y\times[0,-2\omega\varepsilon\ln(\varepsilon))} &\leq \varepsilon\Big(\Big\|\int_0^t \int_0^r \ddot{H}_w(s)dwdr\Big\|_{C^0(Y\times[0,-2\omega\varepsilon\ln(\varepsilon))}[\dot{\overline{\mathbb{H}}}_\varepsilon]_{\alpha,Y\times[0,-2\omega\varepsilon\ln(\varepsilon)} \\ &\quad + \Big[\int_0^t \int_0^r \ddot{H}_w(s)dwdr\Big]_{\alpha,Y\times[0,-2\omega\varepsilon\ln(\varepsilon)} \\ &\quad \cdot\|\dot{\overline{\mathbb{H}}}_\varepsilon\|_{C^0(Y\times[0,-2\omega\varepsilon\ln(\varepsilon))}\Big),\end{aligned}$$

having noted that $\dot{\overline{\mathbb{H}}}_\varepsilon \equiv 0$ for $t > -2\omega\varepsilon\ln(\varepsilon)$. On $Y\times[0,-2\omega\varepsilon\ln(\varepsilon))$, these norms are bounded by

$$\Big\|\int_0^t \int_0^r \ddot{H}_w(s)dwdr\Big\|_{C^0(Y\times[0,-2\omega\varepsilon\ln(\varepsilon))} \leq Kt^2 = O(\varepsilon^2\ln(\varepsilon)^2),$$

$$\Big[\int_0^t\int_0^r \ddot{H}_w(s)dwdr\Big]_{\alpha,Y\times[0,-2\omega\varepsilon\ln(\varepsilon)} \le Kt^{2-\alpha} = O(\varepsilon^{2-\alpha}\ln(\varepsilon)^{2-\alpha}),$$
$$\|\dot{\overline{\mathbb{H}}}_\varepsilon\|_{C^0(Y\times[0,-2\omega\varepsilon\ln(\varepsilon))} = O(1),$$
$$[\dot{\overline{\mathbb{H}}}_\varepsilon]_{\alpha,Y\times[0,-2\omega\varepsilon\ln(\varepsilon)} = O(\varepsilon^{-\alpha}),$$

so that

$$[f]_{\alpha,\varepsilon} \le O(\varepsilon^{3-2\alpha}\ln(\varepsilon)^{2-\alpha}) = o(\varepsilon^2).$$

Further noting that

$$\|Q_0(\phi)\|_{C^\alpha_\varepsilon(M)} \le K\|\phi\|_{C^\alpha(M)} + K\varepsilon^\omega \le K\varepsilon^2,$$

we then have to leading order

$$L_\varepsilon(\phi) = O(\varepsilon^2)$$

in $C^\alpha_\varepsilon(M)$. From Theorem 1.6,

$$\|\phi\|_{C^{2,\alpha}_\varepsilon(M)} \le K\varepsilon^2.$$

If we differentiate (4.1) with respect to s_i again, we get by the same bounding techniques

$$L_\varepsilon(\partial_{s_i}\phi) = \varepsilon\Big(\int_0^t \partial_{s_i}(\dot{H}_r)dr\Big)\dot{\overline{\mathbb{H}}}_\varepsilon + \bar{Q}_0(\phi\phi_{s_i}, \varepsilon^2 D^2\phi, \varepsilon^2 D\phi) + o(\varepsilon^2).$$

Using our bound on $\|\phi\|_{C^{2,\alpha}_\varepsilon}$ and Theorem 1.6 composed with χ_δ as in Lemma 3.7, we get

$$\|\phi_{s_i}\|_{C^{2,\alpha}_\varepsilon(Y\times[0,\delta))} = O(\varepsilon^2)$$

so that

$$\sup_{t\in[0,-\omega\varepsilon\ln(\varepsilon))}\|\Delta_t\phi(s,t)\|_{C^\alpha(Y)} \le \|\Delta_t\phi\|_{C^\alpha(Y\times[0,\delta))} = O(\varepsilon^{1-\alpha}).$$

Now we multiply (4.1) by $\dot{\overline{\mathbb{H}}}_\varepsilon$ and integrate:

$$\begin{aligned}\int_0^{-\omega\varepsilon\ln(\varepsilon)} L_\varepsilon(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon(t)dt &= \int_0^{-\omega\varepsilon\ln(\varepsilon)} (\varepsilon^2[\Delta_t(\phi) - H_t\phi_t + \phi_{tt}] \\ &\qquad - W''(\overline{\mathbb{H}}_\varepsilon)\phi)\dot{\overline{\mathbb{H}}}_\varepsilon(t)dt \\ &= -\varepsilon^2\sigma\phi_t(s,0) + O(\varepsilon^{4-\alpha}),\end{aligned}$$

$$\varepsilon^2\int_0^{-\omega\varepsilon\ln(\varepsilon)} \dot{H}_0(s)\Big(\frac{t}{\varepsilon}\Big)\dot{\overline{\mathbb{H}}}^2_\varepsilon(t) = \kappa_0\varepsilon^3\dot{H}_0,$$

$$\begin{aligned}\Big|\varepsilon\int_0^{-\omega\varepsilon\ln(\varepsilon)}\Big(\int_0^t\int_0^r \ddot{H}_w(s)dwdr\Big)\dot{\overline{\mathbb{H}}}_\varepsilon(t)^2dt\Big| &\le C\varepsilon\int_0^{-\omega\varepsilon\ln(\varepsilon)} t^2\dot{\overline{\mathbb{H}}}_\varepsilon(t)^2dt \\ &= O(\varepsilon^4),\end{aligned}$$

$$\int_0^{-\omega\varepsilon\ln(\varepsilon)} Q_0(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon dt = O(\varepsilon^4),$$

where these error terms hold in $C^\alpha(Y)$. Now note that

$$\dot{H}_0 = [\mathrm{Ric}(\nu,\nu) + |A_Y|^2]$$

so that

$$\phi_t(s,0) = \sigma^{-1}\kappa_0\varepsilon[\mathrm{Ric}(\nu,\nu) + |A_Y|^2] + O(\varepsilon^{2-\alpha}). \qquad \blacksquare$$

4.2. Full characterization of Neumann data

One can compare Theorems 1.4 and 1.7 and note that more terms can be gleaned. In fact, if Y is a $C^{k+3,\alpha}$ surface, we can find an expansion for $u_\varepsilon(s,t)$ (and hence, $\partial_t u_\varepsilon|_{t=0}$) up to order k ($k-1$, respectively). Let

$$a_{i,j}(s) := a_{i,j}\big(\{\partial_s^\beta \partial_t^j H_t|_{t=0}\}_{j+|\beta|\le i}\big)(s)$$

denote a polynomial in derivatives of $H_t(s)$ at $t=0$. Define

$$i \in \mathbb{Z}^{\ge 0}, \quad \sigma(i) := \max(0, 2\lceil i/2\rceil - 2) = \begin{cases} 0 & i = 0, \\ \text{largest even integer less than } i & i > 0. \end{cases}$$

We now prove our main result, Theorem 1.8.

Proof of Theorem 1.8. Actually, we prove the following by induction: for any $k < \bar{k}$, we have

$$\begin{aligned} u_\varepsilon^+(s,t) &= \overline{\mathbb{H}}_\varepsilon(t) + \sum_{i=1}^{k} \varepsilon^i \cdot \Big(\sum_{j=0}^{m_i} a_{i,j}(s)\bar{w}_{i,j}\Big(\frac{t}{\varepsilon}\Big)\Big) + \phi, \\ L_\varepsilon(\phi) &= R_{k+1}(s,t) + F_k(\phi), \\ \|a_{i,j}\|_{C^\alpha(Y)} &= O(1), \\ \|w_{i,j}\|_{C^{k,\alpha}([0,\infty))} &= O(1) \end{aligned}$$

where $\{w_{i,j}(t)\}$ are all exponentially decaying (again see Definition 2.1). Moreover, we require that R_{k+1} can be expanded in powers of ε to arbitrary order less than ω (which we can choose $\omega > \bar{k} + 2$) as follows:

$$\begin{aligned} &\forall \ell \ge 0, \quad \exists\{b_{k+1,i,j}(s)\}, \{f_{k+1,i,j}(t)\} \quad \text{s.t.} \\ &R_{k+1}(s,t) = \varepsilon^{k+1}\sum_{j=0}^{N_{k+1}} b_{k+1,0,j}(s)\bar{f}_{k+1,0,j}\Big(\frac{t}{\varepsilon}\Big) \\ &\qquad + \sum_{i=1}^{\ell}\varepsilon^{k+1+i}\sum_{j=0}^{N_{k+1,i}} b_{k+1,i,j}(s)\bar{f}_{k+1,i,j}\Big(\frac{t}{\varepsilon}\Big) + O(\varepsilon^{\ell+k+2}), \\ &\|f_{k+1,i,j}\|_{C^\alpha([0,\infty))} = O(1), \end{aligned}$$

where the expansion of R_{k+1} holds in $C^\alpha_\varepsilon(M)$ for $\ell + k + 2 \le \bar{k} + 1$ assuming $\bar{k} - k - 1 \ge 0$. In this sense, we see that there is a partial expansion of the remainder up to any order. Here, we require that

- $b_{k+1,0,j}(s) = b_{k+1,0,j}(\{\partial_t^p \partial_s^\beta H_t|_{t=0})$ depends on **at most** $\sigma(k+1)$ **tangential derivatives** of $\{\partial_t^p H_t|_{t=0}(s)\}$.
- For $i \geq 1$, $b_{k+1,i,j}(s) = b_{k+1,i,j}(\{\partial_t^p \partial_s^\beta H_t|_{t=0})$ is a polynomial in **at most** $\sigma(k+2)$ **tangential derivatives** of $\partial_t^p H_t|_{t=0}(s)$.
- Each $f_{k+1,i,j}(t)$ is exponentially decaying in C^∞ and $\overline{f}_{k+1,i,j}$ is the modification with a smooth cutoff. This allows us to solve

$$\begin{aligned} w_{k+1,i,j} &: [0,\infty) \to \mathbb{R} \\ \ddot{w}_{k+1,i,j}(t) - W''(\mathbb{H}(t))w_{k+1,i,j}(t) &= f_{k+1,i,j}(t) \\ w_{k+1,i,j}(0) &= 0 \\ \lim_{t\to\infty} w_{k+1,i,j}(t) &= 0 \end{aligned} \tag{4.2}$$

  by Section A.5.

We also require that $F_k(\phi)$ is an error term that has at most cubic dependency on ϕ in the following form:

$$F_k(\phi) = \varepsilon\Big[\sum_{i=1}^{n_k} c_{k,i}(s)\overline{h}_{k,i}\Big(\frac{t}{\varepsilon}\Big)\Big]\phi + \Big[\sum_{i=1}^{m_k} d_{k,i}(s)\overline{p}_{k,i}\Big(\frac{t}{\varepsilon}\Big)\Big]\phi^2 - \phi^3,$$

$$\|h_{k,i}(t)\|_{C^\alpha([0,\infty))} = O(1),$$

$$\|p_{k,i}(t)\|_{C^\alpha([0,\infty))} = O(1).$$

Moreover,

- $\{h_{k,i}\}$ and $\{p_{k,i}\}$ are expontentially decaying in C^∞.
- $\{c_{k,i}\}$, $\{d_{k,i}\}$ depend on **at most** $\sigma(k)$ **tangential derivatives** of $\partial_t^p H_t|_{t=0}(s)$.

Note that $L_\varepsilon(\phi) = R_{k+1} + F_k$ and Theorem 1.6 automatically gives the conclusion of

$$\|\phi\|_{C^{2,\alpha}_\varepsilon(M)} = O(\varepsilon^{k+1}).$$

From hereon in the proof, we assume that $\omega > \overline{k} + 2$. **Base Case** $k = 0$. This is the content of Corollary 3.6:

$$u_\varepsilon^+(s,t) = \overline{\mathbb{H}}_\varepsilon(t) + \phi(s,t),$$

$$L_\varepsilon(\phi) = (\varepsilon H_t \dot{\overline{\mathbb{H}}}_\varepsilon - R_{\omega,\varepsilon}) - \Big[\frac{1}{2}W'''(\overline{\mathbb{H}}_\varepsilon)\phi^2 + \phi^3\Big] = R_1(s,t) + F_0(\phi)$$

from (3.3). At this level of expansion, $\{a_{1,i,j} = 0\}, \{w_{1,i,j} = 0\}$. We see that $R_1(s,t) = \varepsilon H_t \dot{\overline{\mathbb{H}}}_\varepsilon - R_{\omega,\varepsilon}$ satisfies our inductive assumptions simply by expanding

$$H_t = H_0 + t\dot{H} + \frac{t^2}{2}\ddot{H} + \cdots + \frac{t^{\ell+1}}{(\ell+1)!}\partial_t^{\ell+1}H_t|_{t=0} + O(t^{\ell+2})$$

$$\implies \varepsilon H_t \dot{\overline{\mathbb{H}}}_\varepsilon = \varepsilon(H_0)\dot{\overline{\mathbb{H}}}_\varepsilon + \sum_{i=1}^{\ell} \varepsilon^{i+1}\Big(\frac{1}{(i+1)!}\partial_t^{\ell+1}H_t|_{t=0}\Big)\Big[\Big(\frac{t}{\varepsilon}\Big)^{i+1}\dot{\overline{\mathbb{H}}}_\varepsilon(t)\Big]$$

$$+ O(\varepsilon^{\ell+2}),$$
$$R_{\omega,\varepsilon} = \ddot{\overline{\mathbb{H}}}_\varepsilon - W'(\overline{\mathbb{H}}_\varepsilon) = O(\varepsilon^\omega) = O(\varepsilon),$$

since $\omega > \overline{k} + 2$. Here, we noted that $(\frac{t}{\varepsilon})^i \dot{\overline{\mathbb{H}}}_\varepsilon$ is bounded in $C^{k,\ell}_\varepsilon$ for all ℓ, α, and i. Thus, $R_1(s,t)$ satisfies our inductive assumptions. Note that computing $\partial_t^i H_t$ does not require extra regularity of Y - simply expand (2.4) in t. In particular,

$$N_1 = N_{1,i} = 0,$$
$$b_{1,i,0}(s) = \Big(\frac{1}{(i+1)!}\partial_t^{\ell+1} H_t|_{t=0}\Big), \quad f_{1,i,0}(s) = \Big(\frac{t}{\varepsilon}\Big)^{i+1} \dot{\overline{\mathbb{H}}}_\varepsilon(t).$$

Moreover, each b_{i+1} depends on $0 \le \sigma(1), \sigma(2)$ tangential derivatives of $\{\partial_t^p H_t|_{t=0}\}$. And finally, each $f_{1,i,0}(t)$ is exponentially decaying. Similarly, it is clear that $F_0(\phi)$ satisfies our inductive assumptions as it only has quadratic and cubic terms with bounded coefficients in $C^{k,\alpha}_\varepsilon$ that are also exponentially decaying:

$$d_{0,1}(s) = 1, \quad p_{0,1}\Big(\frac{t}{\varepsilon}\Big) = \frac{1}{2} W'''(\overline{\mathbb{H}}_\varepsilon) = 3\overline{\mathbb{H}}_\varepsilon.$$

Induction. Now assume that we have an expansion up to order $k-1$ for $k \le \overline{k}$:

$$u_\varepsilon^+(s,t) = \overline{\mathbb{H}}_\varepsilon(t) + \sum_{i=1}^{k-1} \varepsilon^i \sum_{j=0}^{m_i} a_{i,j}(s) \overline{w}_{i,j}\Big(\frac{t}{\varepsilon}\Big) + \phi,$$
$$L_\varepsilon(\phi) = R_k(s,t) + F_{k-1}(\phi).$$

We expand (for any $\ell \ge 1$)

$$R_k(s,t) = \varepsilon^k \sum_{j=0}^{N_k} b_{k,0,j}(s) \overline{f}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big) + \Big[\sum_{i=1}^{\ell} \varepsilon^{i+k} \sum_{j=0}^{N_{k,i}} b_{k,i,j}(s) \overline{f}_{k,i,j}\Big(\frac{t}{\varepsilon}\Big) + O(\varepsilon^{\ell+1+k})\Big] \tag{4.3}$$

where we know $\{b_{k,0,j}\}$ depend on at most $\sigma(k-1+1) = \sigma(k) = \max(0, 2\lceil k/2\rceil - 2) \le k-1$ derivatives of $\partial_t^p H_t|_{t=0}(s)$. We can compute two more tangential derivatives of $b_{k,0,j}$ when Y is $C^{k+3,\alpha}$. With this, we use (4.3) and write

$$\phi(s,t) = \varepsilon^k \sum_{j=0}^{N_k} b_{k,0,j}(s) w_{k,0,j}\Big(\frac{t}{\varepsilon}\Big) + \tilde{\phi}$$

such that each $w_{k,0,j}$ solves

$$\ddot{w}_{k,0,j}(t) - W''(\mathbb{H}_\varepsilon) w_{k,0,j}(t) = f_{k,0,j}(t)$$

and has bounded $C^\alpha(\mathbb{R}^+)$ norm and is exponentially decaying. This follows again by Section A.5. Multiplying these functions by cutoffs, we get

$$\|\ddot{\overline{w}}_{k,0,j}(t) - W''(\overline{\mathbb{H}}_\varepsilon)\overline{w}_{k,0,j}(t) - \overline{f}_{k,0,j}(t)\|_{C^\alpha(\mathbb{R}^+)} \leq C_{k,0,j}\varepsilon^\omega \leq C_{k,0,j}\varepsilon^{k+2}$$

for some constants $C_{k,0,j}$ independent of ε. With this expansion, we have

$$\begin{aligned} L_\varepsilon(\phi) = L_\varepsilon(\widetilde{\phi}) &+ \varepsilon^{k+2}\sum_{j=0}^{N_k}\Delta_t(b_{k,0,j})(s)\overline{w}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big) \\ &- \varepsilon^{k+1}\sum_{j=0}^{N_k}H_t(s)b_{k,0,j}(s)\dot{\overline{w}}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big) \\ &+ \varepsilon^k\sum_{j=0}^{N_k}b_{k,0,j}(s)[\ddot{\overline{w}}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big) - W''(\overline{\mathbb{H}}_\varepsilon)\overline{w}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big)]. \end{aligned}$$

Because $\{b_{k,0,j}\}$ depend on $\sigma(k) \leq k-1$ derivatives of $\partial_t^p H_t|_{t=0}(s)$, we know that $\Delta_t b_{k,0,j}$ is at least in $C^\alpha(Y)$, since $k \leq \overline{k}$ and Y is $C^{\overline{k}+3,\alpha}$. Using (4.2), we see that the last line cancels with the first term in (4.3) at the cost of an $O(\varepsilon^{k+2})$ error. We also expand

$$\begin{aligned} \Delta_t(b_{k,0,j})\overline{w}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big) &= \Delta_0(b_{k,0,j})(s)\overline{w}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big) \\ &\quad + \varepsilon\Big(\frac{\Delta_t - \Delta_0}{t}\Big)(b_{k,0,j})\cdot\Big(\frac{t}{\varepsilon}\Big)\overline{w}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big), \\ H_t(s)b_{k,0,j}(s)\dot{w}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big) &= H_0(s)b_{k,0,j}(s)\dot{\overline{w}}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big) \\ &\quad + \varepsilon\Big(\frac{H_t(s) - H_0(s)}{t}\Big)\Big(\frac{t}{\varepsilon}\Big)b_{k,0,j}(s)\dot{\overline{w}}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big), \end{aligned}$$

where we can write

$$\begin{aligned} &(\Delta_t b_{k,0,j}) = \sum_{i=0}^m \frac{t^i}{i!}(\partial_t^i\Delta_t|_{t=0})(b_{k,0,j})(s) + O(t^{m+1}), \\ &(\partial_t^m\Delta_t)|_{t=0} : C^{l+2}(Y) \to C^l = (\partial_t^m g^{ij}(s,t)|_{t=0})\partial_{s_i}\partial_{s_j} - (\partial_t^m b^k(s,t)|_{t=0})\partial_{s_k}. \end{aligned}$$

These expansions in t do not require higher regularity of $H_t(s)$, as can be seen from the expansion of the metric, $g(s,t)$, and the second fundamental form, $A(s,t)$, in equations (2.3) and (2.4). This allows us to make sense of $\frac{\Delta_t - \Delta_0}{t}$. Similarly,

$$H_t(s)b_{k,0,j}(s)\dot{\overline{w}}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big) = \sum_{i=0}^m \varepsilon^i\frac{1}{i!}(\partial_t^i H_t|_{t=0}(s))\Big[\Big(\frac{t}{\varepsilon}\Big)^i\dot{\overline{w}}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big)\Big] + O(\varepsilon^{m+1})$$

for any m. Moreover, we have

$$F_{k-1}(\phi) = \varepsilon\Big[\sum_{i=1}^{n_{k-1}} c_{k-1,i}(s)\overline{h}_{k-1,i}\Big(\frac{t}{\varepsilon}\Big)\Big]\phi + \Big[\sum_{i=1}^{m_{k-1}} d_{k-1,i}(s)\overline{p}_{k-1,i}\Big(\frac{t}{\varepsilon}\Big)\Big]\phi^2 - \phi^3$$

$$
\begin{aligned}
&= \varepsilon\Big[\sum_{i=1}^{n_{k-1}} c_{k-1,i}(s)\overline{h}_{k-1,i}\Big(\frac{t}{\varepsilon}\Big)\Big]\Big(\varepsilon^k \sum_{j=0}^{N_k} b_{k,0,j}(s)\,\overline{f}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big) + \widetilde{\phi}\Big) \\
&\quad + \Big[\sum_{i=1}^{m_{k-1}} d_{k-1,i}(s)\overline{p}_{k-1,i}\Big(\frac{t}{\varepsilon}\Big)\Big]\Big(\varepsilon^k \sum_{j=0}^{N_k} b_{k,0,j}(s)\,\overline{f}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big) + \widetilde{\phi}\Big)^2 \\
&\quad - \Big(\varepsilon^k \sum_{j=0}^{N_k} b_{k,0,j}(s)\,\overline{f}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big) + \widetilde{\phi}\Big)^3
\end{aligned}
$$

with $\{c_{k-1,i}\}$, $\{d_{k-1,i}\}$ depending on at most $\sigma(k-1)$ derivatives of $\partial_t^p H_t|_{t=0}(s)$. If we expand and relabel, noting that the product of exponentially decaying functions are themselves exponentially decaying, we get

$$
\begin{aligned}
F_{k-1}(\phi) = \varepsilon^{k+1}\Big[\sum_{i=1}^{\widetilde{n}_k} C_{k,i}(s)\overline{h}^*_{k,i}\Big(\frac{t}{\varepsilon}\Big)\Big] + \varepsilon\Big[\sum_{i=1}^{n_k} c_{k,i}(s)\overline{h}_{k,i}\Big(\frac{t}{\varepsilon}\Big)\Big]\widetilde{\phi} \\
+ \Big[\sum_{i=1}^{m_k} d_{k,i}(s)\overline{p}_{k,i}\Big(\frac{t}{\varepsilon}\Big)\Big]\widetilde{\phi}^2 - \widetilde{\phi}^3
\end{aligned}
$$

for some $n_k, m_k, \widetilde{n}_k$. Here

- $\{h^*_{k,i}, h_{k,i}, p_{k,i}\}$ are all exponentially decaying and $O(1)$ in C^α norm.
- $\{C_{k,i}\}$, $\{c_{k,i}\}$, $\{d_{k,i}\}$ depend on at most $\sigma(k)$ derivatives of $\partial_t^p H_t|_{t=0}(s)$.

We define

$$
\begin{aligned}
R_{k+1}(s,t) &:= \Big[R_k(s,t) - \varepsilon^k \sum_{j=0}^{N_k} b_{k,0,j}(s)\,\overline{f}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big)\Big] \\
&\quad + \varepsilon^{k+1}\Big[\sum_{i=1}^{\widetilde{n}_k} C_{k,i}(s)\overline{h}^*_{k,i}\Big(\frac{t}{\varepsilon}\Big)\Big] \\
&\quad - \varepsilon^{k+2}\sum_{j=0}^{N_k} \Delta_t(b_{k,0,j})(s)\overline{w}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big) \\
&\quad + \varepsilon^{k+1}\sum_{j=0}^{N_k} H_t(s)b_{k,0,j}(s)\dot{\overline{w}}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big) \\
\implies R_{k+1}(s,t) &= \varepsilon^{k+1}\Big[\sum_{j=0}^{N_{k,1}} b_{k,1,j}(s)\,\overline{f}_{k,1,j}\Big(\frac{t}{\varepsilon}\Big) + \sum_{i=1}^{\widetilde{n}_k} C_{k,i}(s)\overline{h}^*_{k,i}\Big(\frac{t}{\varepsilon}\Big) \\
&\quad + \sum_{j=0}^{N_k} H_0(s)b_{k,0,j}(s)\dot{\overline{w}}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big)\Big]
\end{aligned}
$$

$$+\,\varepsilon^{k+2}\sum_{j=0}^{N_k}\Big(\frac{H_t(s)-H_0(s)}{t}\Big)\Big(\frac{t}{\varepsilon}\Big)b_{k,0,j}(s)\dot{\bar{w}}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big)$$
$$-\,\varepsilon^{k+2}\sum_{j=0}^{N_k}\Delta_t(b_{k,0,j})(s)\bar{w}_{k,0,j}\Big(\frac{t}{\varepsilon}\Big)$$
$$+\sum_{i=2}^{\ell}\varepsilon^{i+k}\sum_{j=0}^{N_{k,i}}b_{k,i,j}(s)\bar{f}_{k,i,j}\Big(\frac{t}{\varepsilon}\Big)+O(\varepsilon^{\ell+1+k}).$$

Moreover, note that $\{b_{k+1,i,j}\}$ depends on at most $\sigma(k+1)$ derivatives for $i\geq 1$, while $C_{k,i}(s)$ and $b_{k,0,j}(s)$ depend on at most $\sigma(k)$ derivatives. Also recalling that ℓ is any value such that $\ell+1+k\leq\bar{k}+1$, we can rewrite the above as

$$R_{k+1}(s,t)=\varepsilon^{k+1}\sum_{j=0}^{N_{k+1}}b_{k+1,0,j}(s)\bar{f}_{k+1,0,j}\Big(\frac{t}{\varepsilon}\Big)$$
$$+\sum_{i=1}^{\ell}\varepsilon^{i+k+1}\sum_{j=0}^{N_{k+1,i}}b_{k+1,i,j}(s)\bar{f}_{k+1,i,j}\Big(\frac{t}{\varepsilon}\Big)+O(\varepsilon^{\ell+k+2}),$$

adjusting the expansion depending on the value of $\bar{k}-k-1$. If $k=\bar{k}$, then no such expansion is needed, as we have reached the maximal value of k in the induction. Furthermore, we define

$$F_k(\tilde{\phi}):=\varepsilon\Big[\sum_{i=1}^{n_k}c_{k,i}(s)\bar{h}_{k,i}\Big(\frac{t}{\varepsilon}\Big)\Big]\tilde{\phi}+\Big[\sum_{i=1}^{m_k}d_{k,i}(s)\bar{p}_{k,i}\Big(\frac{t}{\varepsilon}\Big)\Big]\tilde{\phi}^2-\tilde{\phi}^3$$

so that

$$L_\varepsilon(\tilde{\phi})=R_{k+1}(s,t)+F_k(\tilde{\phi})$$

with the correct decomposition and regularity of coefficients. Now with the decomposition of R_{k+1}, we use Theorem 1.6 and get

$$\|\tilde{\phi}\|_{C^{2,\alpha}_\varepsilon(M)}=O(\varepsilon^{k+1}).$$

This finishes the induction. ∎

As a result, we have the following corollaries for $k=1$:

Corollary 4.1. *For u_ε a solution to equation (1.2) with Dirichlet data on $Y=\partial M$ a $C^{4,\alpha}$ hypersurface, we have that*

$$u_\varepsilon(s,t)=\mathbb{H}_\varepsilon(t)+\varepsilon H_Y(s)w_\varepsilon(t)+\phi(s,t),$$
$$\|\phi\|_{C^{2,\alpha}_\varepsilon(M)}=O(\varepsilon^2).$$

Similarly, for $k=2$, we have

Corollary 4.2. *For u_ε a solution to equation* (1.2) *with Dirichlet data on $Y = \partial M$ a $C^{5,\alpha}$ hypersurface, we have that*

$$u_\varepsilon^+(s,t) = \mathbb{H}_\varepsilon(t) + \varepsilon H_Y(s) w_\varepsilon(t) + \varepsilon^2\Big[\dot{H}_0(s)\tau_\varepsilon(t) + H_Y^2\rho_\varepsilon(t) + \frac{1}{2}H_Y^2(s)\kappa_\varepsilon(t)\Big] + \phi,$$
$$\|\phi\|_{C^{2,\alpha}_\varepsilon(M)} = O(\varepsilon^3).$$

5. Proof of Theorem 1.13

In this section, we work in the closed setting. Consider Y a minimal hypersurface, and perturbations $\eta : Y \to \mathbb{R}$, with Y_η defined as in Section 1.3. Recall the definition of $u_{\varepsilon,\eta}$ from (1.6):

$$u_{\varepsilon,\eta} = \begin{cases} u_{\varepsilon,\eta}^+(p) & p \in M^+, \\ -u_{\varepsilon,\eta}^-(p) & p \in M^-. \end{cases}$$

We aim to prove Theorem 1.13. As a corollary, we can describe the horizontal variation of the solutions constructed in Pacard–Ritoré ([14, Thm 4.1] and [13, Thm 1.1]). We recall their notation

$$u_\varepsilon(t) := \tanh\Big(\frac{t}{\varepsilon\sqrt{2}}\Big),$$
$$\overline{u}(y,t) = u_\varepsilon(t - \zeta(y)) + v(y,t).$$

In our notation, $v(y,t) \leftrightarrow \phi(s,t)$ and $\zeta(y) \leftrightarrow \eta(s)$. With this, we have

Corollary 5.1. *Suppose ζ is the perturbation constructed in* [13, Thm 3.33] *and v the solution to* (1.2) *with $v^{-1}(0) = Y_\zeta$ and Y nondegenerate, minimal, and separating. Then*

$$\int_{\mathbb{R}} v(s,t)\dot{\overline{\mathbb{H}}}_\varepsilon(t)dt = \frac{\sqrt{2}}{3}\zeta(s) + O(\varepsilon^{1+2\beta})$$

with error in $C^{2,\alpha}(Y)$. In particular,

$$\int_{\mathbb{R}} \Delta_Y v(s,t)\dot{\overline{\mathbb{H}}}_\varepsilon(t) = \frac{\sqrt{2}}{3}J_Y(\zeta) + O(\varepsilon^{1+2\beta}).$$

Remark 5.2. Note that

$$J_Y = \Delta_Y + (|A_Y|^2 + \mathrm{Ric}_g(\nu,\nu))$$

and we show that

$$\int_{\mathbb{R}} (|A_Y|^2 + \mathrm{Ric}_g(\nu,\nu))v\dot{\overline{\mathbb{H}}}_\varepsilon = o(\varepsilon^{1+2\beta})$$

in $C^\alpha(Y)$. Thus, we could replace $\Delta_Y v$ with $J_Y v$ on the left-hand side of 5.1.

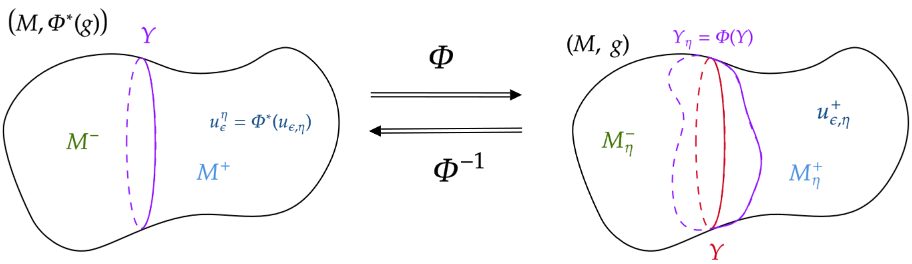

Figure 7. Φ map describing our setup

Remark 5.3. The corollary tells us that the Pacard–Ritoré solutions *have horizontal variation as large as the perturbation*, $\zeta(s)$, off of the initial Y minimal.

The proof is essentially the same as for Theorem 1.7, but we have to confront the low regularity of Y_η given that $\eta \in C^{2,\alpha}(Y)$. We do this by pulling back $u_{\varepsilon,\eta}^+$ to $Y \times [0, -\omega\varepsilon \ln(\varepsilon))$ and then showing

$$\nu^+(u_{\varepsilon,\eta}^+) = \frac{1}{\varepsilon\sqrt{2}} + \kappa_0\varepsilon[\operatorname{Ric}(\nu,\nu) + |A_Y|^2] + \left[\sigma_0 J_Y(\eta) + \int_0^{-\omega\varepsilon\ln(\varepsilon)} \Delta_Y(u_{\varepsilon,\eta})\dot{\overline{\mathbb{H}}}_\varepsilon\right] + \widetilde{O}(\eta^2, \varepsilon^2).$$

5.1. Setup

For Y_η as in (2.10), we consider the decomposition of $M = M_\eta^+ \cup_{Y_\eta} M_\eta^-$ and $u_{\varepsilon,\eta}^\pm$ the minimizers on $M_\eta^\pm$. With Fermi coordinates about Y, define

$$\Phi : Y \times (\omega\varepsilon\ln(\varepsilon)^2, -\omega\varepsilon\ln(\varepsilon)^2) \to Y \times (\omega\varepsilon\ln(\varepsilon)^2, -\omega\varepsilon\ln(\varepsilon)^2),$$
$$\Phi(s,t) := F\Big(s, t + \eta(s)\zeta\Big(\frac{-4t}{\omega\varepsilon\ln(\varepsilon)^2}\Big)\Big)$$

for F as in (2.2) and where $\zeta(t)$ is the standard bump function, which is 1 on $(-1, 1)$ and goes to zero outside of $[-2, 2]$ (see Figure 7). Note that we choose a factor of $\ln(\varepsilon)^2$ so that Φ restricted to $Y \times (\omega\varepsilon\ln(\varepsilon), -\omega\varepsilon\ln(\varepsilon))$ is a diffeomorphism onto its image.

In fact, on this subdomain

$$|t| < -\omega\varepsilon\ln(\varepsilon) \implies \Phi(s,t) = (s, t + \eta(s)).$$

We pull back $u^+_{\varepsilon,\eta}$ by this function and compute the Allen–Cahn equation under this pull-back. We define (dropping the $\pm$ notation)

$$\begin{aligned} u^\eta_\varepsilon &: Y \times [0, -\omega\varepsilon \ln(\varepsilon)) \to \mathbb{R}, \\ &:= \Phi^*(u_{\varepsilon,\eta})(s,t) \end{aligned}$$

so that

$$\begin{aligned} \Phi^*(W'(u_{\varepsilon,\eta})) &= W'(u^\eta_\varepsilon), \\ u^\eta_\varepsilon(s,0) &\equiv 0 \end{aligned}$$

and

$$\Phi^*(\Delta_g u^+_{\varepsilon,\eta}) = \Delta_{\Phi^*(g)} u^\eta_\varepsilon$$

using diffeomorphism invariance of the Laplacian. Instead of computing $\Phi^*(g)$ in coordinates, we push forward Δ_g to $(M, \Phi^*(g))$, as a differential operator, by Φ^{-1}, that is,

$$\Delta_{\Phi^*(g)} = (\Phi^{-1})_*(\Delta_g).$$

We first expand Δ_g on (M, g), recalling (2.6):

$$\Delta_g = \Delta_t - H_t \partial_t + \partial_t^2,$$

where Δ_{t_0} denotes the Laplacian on $Y_{t_0} := Y + t_0 \nu$, that is, the set at signed distance t_0 from Y. We now compute $\Delta_{\Phi^*(g)} = (\Phi^{-1})_*(\Delta_g)$ by pushing forward each summand:

$$\begin{aligned} (\Phi^{-1})_*(H_t \partial_t) &= H_{t+\eta} \partial_t, \\ (\Phi^{-1})_*(\partial_t^2) &= \partial_t^2, \\ (\Phi^{-1})_*(\Delta_t) &= \Delta_{t+\eta} + E_\eta \\ \implies (\Phi^{-1})_*(\Delta_g)|_t &= \Delta_g|_{t+\eta} + E_\eta \end{aligned}$$

where in the last line, $\Delta_g|_{t+\eta}$ denotes the ambient Laplacian on M but with metric coefficients evaluated at the point $(s, t + \eta(s))$. We also define

$$\begin{aligned} E_\eta &:= g^{ij}(s, t + \eta(s))[-\eta_i(s)\partial_t \partial_{s_j} - \eta_j(s)\partial_t \partial_{s_i} - \eta_{ij}(s)\partial_t + \eta_i(s)\eta_j(s)\partial_t^2] \\ &= -\Delta_{t+\eta}(\eta)\partial_t - 2\nabla^{t+\eta}(\eta)\partial_t + |\nabla^{t+\eta}\eta|^2 \partial_t^2. \end{aligned}$$

From hereon, we only consider u^η_ε restricted to $Y \times [0, -\omega\varepsilon \ln(\varepsilon))$, and we rewrite the pulled back Allen–Cahn equation as

$$\varepsilon^2[\Delta_{t+\eta} - H_{t+\eta}\partial_t + \partial_t^2 + E_\eta](u^\eta_\varepsilon) = W'(u^\eta_\varepsilon).$$

Now we decompose (using that Y is minimal and inspired by 4.2)

$$u^\eta_\varepsilon = \overline{\mathbb{H}}_\varepsilon(t) + \varepsilon^2 \dot{H}_0(s)\overline{\tau}_\varepsilon(t) + \phi^\eta(s,t) \tag{5.1}$$

where

$$\ddot{\tau}(t) - W''(\mathbb{H}(t))\tau(t) = t\dot{\mathbb{H}}(t)$$

on $\mathbb{R}^+$. From hereon, we label $\phi^\eta =: \phi$, and our pulled back Allen–Cahn equation becomes

$$\begin{aligned}
L_{\varepsilon,t+\eta}(\phi) = {} & \varepsilon[\Delta_{t+\eta}(\eta) + \dot{H}_0\eta]\dot{\overline{\mathbb{H}}}_\varepsilon + |\nabla^{t+\eta}\eta|^2\ddot{\overline{\mathbb{H}}}_\varepsilon \\
& + \varepsilon^3\Big(\frac{H_{t+\eta}(s) - \dot{H}_0(s)(t+\eta)}{(t+\eta)^2}\Big)\Big(\frac{t+\eta}{\varepsilon}\Big)^2\dot{\overline{\mathbb{H}}}_\varepsilon \\
& - \varepsilon^4\Delta_{t+\eta}(\dot{H}_0)\overline{\tau}_\varepsilon + \varepsilon^3 H_{t+\eta}\dot{H}_0\dot{\overline{\tau}}_\varepsilon + \varepsilon^3\Delta_{t+\eta}(\eta)\dot{H}_0\dot{\overline{\tau}}_\varepsilon + 2\varepsilon^3\nabla^{t+\eta}(\eta)(\dot{H}_0)\dot{\overline{\tau}}_\varepsilon \\
& - \varepsilon^2|\nabla^{t+\eta}\eta|^2\dot{H}_0\ddot{\overline{\tau}}_\varepsilon + \frac{1}{2}W'''(\overline{\mathbb{H}}_\varepsilon)\varepsilon^4\dot{H}_0^2\overline{\tau}_\varepsilon^2 + \varepsilon^6\dot{H}_0^3\overline{\tau}_\varepsilon^3 \\
& + R(\phi) + O(\varepsilon^\omega)
\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}
L_{\varepsilon,r} &:= \varepsilon^2(\Delta_r - H_r\partial_t + \partial_t^2 - W''(\overline{\mathbb{H}}_\varepsilon(t))), \\
R(\phi) &:= \varepsilon^2 E_\eta(\phi) - F_0(\phi), \qquad (5.3)\\
F_0(\phi) &:= W'''(\overline{\mathbb{H}}_\varepsilon)\varepsilon^2\dot{H}_0\overline{\tau}_\varepsilon\phi + \Big[3\varepsilon^4\dot{H}_0^2\tau_\varepsilon^2 + \frac{1}{2}W'''(\overline{\mathbb{H}}_\varepsilon)\Big]\phi^2 + \phi^3 \qquad (5.4)
\end{aligned}$$

where $F_0(\phi)$ is the error term from expanding $W'(u)$. Here, all of the $O(\varepsilon^\omega)$ terms come from replacing $\mathbb{H}_\varepsilon \to \overline{\mathbb{H}}_\varepsilon$ and the like. We abbreviate the right-hand side of equation (5.2) as $G(\phi)$, and we will often use $L_{\varepsilon,t}$ and L_ε interchangeably when it is clear from context.

5.2. Estimates on ϕ

Again using Theorem 1.6 and equation (3.6), we have

$$\begin{aligned}
\|\phi\|_{C^{2,\alpha}_\varepsilon(M)} &\le K\|L_\varepsilon\phi\|_{C^\alpha_\varepsilon(M)} \\
&\le \Big(\frac{2}{\delta}\Big)^\alpha[\|L_\varepsilon\phi\|_{C^\alpha_\varepsilon(t<\delta)} + \|L_\varepsilon\phi\|_{C^\alpha_\varepsilon(t>\delta/2)}] \\
&\le K\|L_\varepsilon\phi\|_{C^\alpha_\varepsilon(t<\delta)} + O(\varepsilon^\omega) \\
&\le K\|L_{\varepsilon,t+\eta}\phi\|_{C^\alpha_\varepsilon(t<\delta)} + \|\varepsilon^2(\Delta_{t+\eta} - \Delta_t)\phi\|_{C^\alpha_\varepsilon(t<\delta)} \\
&\qquad + \|\varepsilon^2(H_{t+\eta} - H_t)\phi_t\|_{C^\alpha_\varepsilon(t<\delta)} + O(\varepsilon^\omega),
\end{aligned}$$

having used (3.6) to bound $\|L_\varepsilon\phi\|_{C^\alpha_\varepsilon(t>\delta/2)}$. With $\|\eta\|_{C^{2,\alpha}(Y)} \le K\varepsilon^{1+\beta}$, this implies

$$\begin{aligned}
& \|\phi\|_{C^{2,\alpha}_\varepsilon(M)} \le K\|L_{\varepsilon,t+\eta}\phi\|_{C^\alpha_\varepsilon(t<\delta)} + \varepsilon^{1+\beta}\|\phi\|_{C^{2,\alpha}_\varepsilon(M)} + O(\varepsilon^\omega) \\
\implies & \|\phi\|_{C^{2,\alpha}_\varepsilon(M)} \le K[\|\varepsilon[\Delta_{t+\eta} + \dot{H}_0](\eta)\dot{\overline{\mathbb{H}}}_\varepsilon\|_{C^\alpha_\varepsilon(t<\delta)} \\
& \qquad + \|\,|\nabla^{t+\eta}\eta|^2\ddot{\overline{\mathbb{H}}}_\varepsilon\|_{C^\alpha_\varepsilon(t>\delta)}] + O(\varepsilon^3) \\
& \quad \le O(\varepsilon^{2+\beta}).
\end{aligned} \tag{5.5}$$

5.3. Proof of Theorem 1.13

Now as in Section 3.4, we again decompose $L_{\varepsilon,t+\eta}$, multiply by $\dot{\overline{\mathbb{H}}}_\varepsilon$, integrate, and extract the normal derivative:

$$\int_0^{-\omega\varepsilon\ln(\varepsilon)} L_{\varepsilon,t+\eta}(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon = -\varepsilon^2\sigma\phi_t(s,0) + \varepsilon^2\int_0^{-\omega\varepsilon\ln(\varepsilon)} \Delta_Y(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon + O(\varepsilon^{4+\beta-\alpha}),$$

where the above holds in $C^\alpha(Y)$, having used (5.5). Similarly,

$$\int_0^{-\omega\varepsilon\ln(\varepsilon)} G(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon = \varepsilon^2\dot{H}_0\eta\sigma_0 + \varepsilon^2\Delta_0(\eta)\sigma_0 + O(\varepsilon^{3+2\beta})$$

with error terms holding in $C^\alpha(Y)$. The details are sketched in Section A.4. Because $\|\eta\|_{C^{2,\alpha}(Y)} \leq K\varepsilon^{1+\beta}$, we see that in terms of order of ε,

$$\phi_t(s,0) = \underbrace{\sqrt{2}\int_0^{-\omega\varepsilon\ln(\varepsilon)} \Delta_Y(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon(t)dt}_{O(\varepsilon^{1+\beta-\alpha})} - \underbrace{\frac{2}{3}[\Delta_0 + \dot{H}_0](\eta)}_{O(\varepsilon^{1+\beta})} + O(\varepsilon^{1+2\beta})$$

where the above asymptotics hold in $C^\alpha(Y)$. We frame this as

$$\left\|\phi_t(s,0) - \sqrt{2}\int \Delta_Y(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon + \frac{2}{3}J_Y(\eta)\right\|_{C^\alpha(Y)} = O(\varepsilon^{1+2\beta}).$$

We now note that ∂_t is comparable to $(\Phi^{-1})_*(\nu_\eta)$ (see Section A.3), that is, the normal vector for Y and that of Y_η (translated to Y) are comparable since η is small:

$$\begin{aligned}
\nu_\eta &= (1 + A(s))\partial_t + B^i(s)\partial_{s_i}|_{t=\eta(s)}, \\
\|A(s)\|_{C^\alpha(Y)} &\leq C\|\eta\|^2_{C^{1,\alpha}(Y)}, \\
\|B^i(s)\|_{C^\alpha(Y)} &\leq C\|\eta\|_{C^{1,\alpha}(Y)} \\
\implies (\Phi^{-1})_*(\nu_\eta) &= (1 + \tilde{A}(s))\partial_t + \tilde{B}^i(s)\partial_{s_i}|_{t=0}, \\
\|\tilde{A}(s)\|_{C^\alpha(Y)} &\leq C\|\eta\|^2_{C^{1,\alpha}(Y)}, \\
\|\tilde{B}^i(s)\|_{C^\alpha(Y)} &\leq C\|\eta\|_{C^{1,\alpha}(Y)}.
\end{aligned}$$

As such,

$$\|(\Phi^{-1})_*(\nu_\eta)(\phi) - \phi_t(s,0)\|_{C^\alpha(Y)} = O(\varepsilon^{3+3\beta-\alpha})$$

(recall that $\phi_{s_i}(s,0) \equiv 0$) and so

$$\begin{aligned}
(\Phi^{-1})_*(\nu_\eta)(\phi) &= \phi_t(s,0) + O(\varepsilon^{3+3\beta-\alpha}) \\
&= \sqrt{2}\int_0^{-\omega\varepsilon\ln(\varepsilon)} \Delta_Y(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon(t)dt - \frac{2}{3}J_Y(\eta) + O(\varepsilon^{1+2\beta}).
\end{aligned}$$

To prove Theorem 1.13, we note that if $u_{\varepsilon,\eta}$ is C^1 across $Y_\eta = u_{\varepsilon,\eta}^{-1}(0)$, then the Neumann data match. If we take

$$\begin{aligned} u_{\varepsilon,\eta} &: Y \times (\omega\varepsilon \ln(\varepsilon)^2, -\omega\varepsilon \ln(\varepsilon)^2) \to \mathbb{R}, \\ \overline{u}_\varepsilon^\eta &:= \Phi^*(u_{\varepsilon,\eta}) : Y \times (\omega\varepsilon \ln(\varepsilon), -\omega\varepsilon \ln(\varepsilon)) \to \mathbb{R}, \\ \overline{u}_\varepsilon^\eta &= \overline{\mathbb{H}}_\varepsilon(t) + \varepsilon^2 \dot{H}_0(s)\overline{\tau}_\varepsilon(t) + \overline{\phi}(s,t), \\ \overline{\tau}_\varepsilon(t) &:= \begin{cases} \tau_\varepsilon(t) & t \geq 0, \\ -\tau_\varepsilon(-t) & t < 0, \end{cases} \\ \overline{\phi}(s,t) &:= \begin{cases} \phi^+(s,t) & t \geq 0, \\ \phi^-(s,t) & t < 0 \end{cases} \end{aligned}$$

where $\phi^\pm$ are the same functions as in (5.1) with the $\pm$ referring to $M^\pm$ (i.e., $t > 0$ or $t < 0$). With the above, $\nu_\eta(u_{\varepsilon,\eta}) = (\Phi_\eta)_*^{-1}(\nu_\eta)(u_\varepsilon^\eta)$ is well defined, and the Neumann data matches, that is,

$$0 = \nu_\eta(u_{\varepsilon,\eta})|_{t=\eta(s)^+} - \nu_\eta(u_{\varepsilon,\eta})|_{t=\eta(s)^-}$$

if and only if the following holds:

$$\begin{aligned} 0 &= (\Phi_\eta)_*^{-1}(\nu_\eta)(u_\varepsilon^\eta)|_{t=0^+} - (\Phi_\eta)_*^{-1}(\nu_\eta)(u_\varepsilon^\eta)|_{t=0^-} \\ &= -\frac{4}{3} J_Y(\eta) + \sqrt{2} \int_{\omega\varepsilon \ln(\varepsilon)}^{-\omega\varepsilon \ln(\varepsilon)} \Delta_Y(\overline{\phi})\dot{\overline{\mathbb{H}}}_\varepsilon(t)dt + O(\varepsilon^{1+2\beta}) \\ \implies \int_{\omega\varepsilon \ln(\varepsilon)}^{-\omega\varepsilon \ln(\varepsilon)} \Delta_Y(\overline{\phi})\dot{\overline{\mathbb{H}}}_\varepsilon(t)dt &= \frac{2\sqrt{2}}{3}\sigma_0 J_Y(\eta) + \tilde{O}(\varepsilon^{1+2\beta}), \end{aligned}$$

where the error holds in $C^\alpha(Y)$. Now note that

$$\left\| \int_{\omega\varepsilon \ln(\varepsilon)}^{-\omega\varepsilon \ln(\varepsilon)} \dot{H}_0 \overline{\phi}\dot{\overline{\mathbb{H}}}_\varepsilon dt \right\|_{C^\alpha(Y)} = O(\varepsilon^{3+\beta-\alpha})$$

so that we can write the above as

$$J_Y\Big(\int_{\omega\varepsilon \ln(\varepsilon)}^{-\omega\varepsilon \ln(\varepsilon)} \overline{\phi}\dot{\overline{\mathbb{H}}}_\varepsilon(t)dt\Big) = 2\sigma_0 J_Y(\eta) + O(\varepsilon^{1+2\beta})$$

with error in $C^\alpha(Y)$. We now substitute $\int u_{\varepsilon,\eta}\dot{\overline{\mathbb{H}}}_\varepsilon$ for $\int \phi\dot{\overline{\mathbb{H}}}_\varepsilon$ at the cost of negligible error, since

$$\begin{aligned} \left| \int_{\omega\varepsilon \ln(\varepsilon)}^{-\omega\varepsilon \ln(\varepsilon)} \mathbb{H}_\varepsilon(t)\dot{\mathbb{H}}_\varepsilon(t)dt \right| &= O(\varepsilon^\omega), \\ \left\| \int_{\omega\varepsilon \ln(\varepsilon)}^{-\omega\varepsilon \ln(\varepsilon)} \varepsilon^2 \dot{H}_0(s)\tau_\varepsilon(t)\dot{\mathbb{H}}_\varepsilon(t) \right\|_{C^\alpha(Y)} &= O(\varepsilon^3) \end{aligned}$$

and the same order of bound holds if we replace $\mathbb{H}_\varepsilon \to \overline{\mathbb{H}}_\varepsilon$, $\tau_\varepsilon \to \overline{\tau}_\varepsilon$. Furthermore, if Y is nondegenerate, we invert both sides by J_Y,

$$\int_{\omega\varepsilon\ln(\varepsilon)}^{-\omega\varepsilon\ln(\varepsilon)} u_{\varepsilon,\eta}(s,t)\dot{\overline{\mathbb{H}}}_\varepsilon(t)dt = 2\sigma_0\eta(s) + O(\varepsilon^{1+2\beta}),$$

where the error holds in $C^{2,\alpha}(Y)$. This concludes the proof of the theorem. ∎

A. Miscellaneous lemmas and computations

A.1. Lemma on $L^* = \Delta_{\mathbb{R}^n} + \partial_t^2 - W''(\overline{\mathbb{H}})$ for $\mathbb{R}^n \times \mathbb{R}^+$

Lemma A.1. *For $\phi \in C^1(\mathbb{R}^n \times \mathbb{R}^+)$, suppose $\phi(s,0) \equiv 0$ and $L^*(\phi) = 0$ on $\mathbb{R}^n \times \mathbb{R}^+$. Then $\phi \equiv 0$.*

Proof. The proof is a slight extension of the well-known classification of $\ker(L)$ on $\mathbb{R}^n \times \mathbb{R}$. See [13, Lemma 3.7] for reference. Because of the Dirichlet condition at $t = 0$, consider the odd reflection

$$\widetilde{\phi}(s,t) = \begin{cases} \phi(s,t) & t \geq 0, \\ -\phi(s,-t) & t < 0. \end{cases}$$

Then $\widetilde{\phi}(s,t)$ is a C^1 solution to L^*. By the maximum principle, $\widetilde{\phi}$ converges to 0 exponentially and uniformly as $t \to \pm\infty$. Thus, it is in L^2 and via an energy argument (again [13, Lemma 3.7]), we see that

$$\widetilde{\phi}(s,t) = c\dot{\mathbb{H}}(t),$$

but $\phi(s,0) = 0$, so $c = 0$. ∎

A.2. Boundedness of τ in Theorem 1.6

Recall that we have a sequence $\{f_j\} : M \to \mathbb{R}$ and p_j such that $|f_j(p_j)| = \|f_j\|_{C^0(M)} = 1$ and $\|L_\varepsilon f_j\|_{C^\alpha_\varepsilon(M)} \leq j^{-1}$. We show that $\operatorname{dist}(Y, p_j) < \kappa\varepsilon_j$ for some κ independent of ε_j. In terms of scaled Fermi coordinates, (σ, τ), we have

$$L_\varepsilon = \Delta_{g_\varepsilon} - W''(\overline{\mathbb{H}}(\tau)).$$

Consider $\phi = 1$ for which

$$L(1) = -W''(\overline{\mathbb{H}}) = 1 - 3\overline{\mathbb{H}}^2.$$

We see that this $L(1) < -1$ for all $\tau > \operatorname{arctanh}(\sqrt{\tfrac{2}{3}}) =: c_0$. Moreover, $1 \geq |\phi_j|$ by the normalization. We now apply the maximum principle to L and $(1 \pm \phi)$ on the open set $U = \{p \mid \operatorname{dist}(p, Y) > c_0\varepsilon\}$. This tells us that for $\tau > c_0$ (i.e., $t > c_0\varepsilon$), $1 \pm \phi_j$ achieves its minimum on the boundary of $\{\tau > c_0\}$. Immediately, this tells us that we can choose $p_j = (q_j, t_j)$ for some $0 \leq t_j < c_0\varepsilon$. Similarly, we can show that $\tau_j = \varepsilon^{-1} t_j \geq \tau_0 > 0$. Recentering ϕ at $(q_j, 0)$ and using (σ, τ) coordinates, we have

$$|\partial_\tau \phi_j(\sigma, \tau)| \leq \|\phi_j\|_{C^{2,\alpha}_\varepsilon(M))} \leq K(\|L\phi_j\|_{C^\alpha_\varepsilon(M)} + \|\phi_j\|_{C^0(M)}) \leq K(o(1) + 1) \leq 2K,$$

so that because $\phi_j(\sigma, 0) \equiv 0$ for all σ, we have

$$|\phi_j(\sigma_j, \tau_j)| \geq 1/2 \implies \tau_j \geq \frac{1}{4K}$$

where K is the Schauder constant and independent of j and ε. This tells us that

$$0 < \frac{1}{4K} \leq \tau_j \leq c_0,$$

so there exists a convergent subsequence of $\{\tau_j\}$ that converges to $0 < \tau < \infty$.

A.3. Normal for Y_η

In this section, we show that for η a perturbation with $\|\eta\|_{C^{2,\alpha}}\| \leq K\varepsilon^{1+\beta}$, we have

$$\begin{aligned} \nu_{Y_\eta} &= a^t(s)\partial_t + a^i(s)\partial_{s_i}, \\ \|a^t - 1\|_{C^{2,\alpha}(Y)} &\leq C\varepsilon^{2+2\beta}, \\ \|a^i\|_{C^{2,\alpha}(Y)} &\leq C\varepsilon^{1+\beta}. \end{aligned}$$

Lemma A.2. *For any $\eta \in C^{2,\alpha}(Y)$ and $\|\eta\|_{C^{2,\alpha}(Y)} \leq K\varepsilon^{1+\beta}$, there exists $C > 0$ so that the normal derivative to Y_η expands as*

$$\begin{aligned} \nu_\eta &= a^t(s)\partial_t + a^i(s)\partial_{s_i}, \\ \|a^t(s) - 1\|_{C^{1,\alpha}(Y)} &\leq C\|\eta\|^2_{C^{2,\alpha}(Y)}, \\ \|a^i(s)\|_{C^{1,\alpha}(Y)} &\leq C\|\eta\|_{C^{2,\alpha}(Y)}. \end{aligned}$$

Proof. In coordinates, we compute the tangent basis for Y_η as

$$v_i = \partial_{s_i} + \eta_i \partial_t|_{(s,t=\eta)}.$$

Let $g(\eta)_{ij} := g(v_i, v_j)$ and $g(\eta)^{ij}$ be the corresponding inverse. Then

$$g(\eta)_{ij} = g_{ij}|_{(s,t=\eta)} + \eta_i\eta_j = \delta_{ij} + \eta A_{ij} + O((D\eta)^2),$$

so that for

$$\begin{aligned} w_\eta := \partial_t - \Pi_{TY}(\partial_t) &= \partial_t - g^{ij} g(\partial_t, v_i) v_j \\ &= \partial_t - (\delta_{ij} - \eta A_{ij} + \widetilde{O}((D\eta)^2))\eta_i(\partial_{s_j} + \eta_j\partial_t) \\ &= (1 + \widetilde{O}(\eta\eta_i\eta_j, (D\eta)^4)\partial_t - (\eta_i\delta_{ij} + \widetilde{O}(\eta D\eta, (D\eta)^3))\partial_j, \end{aligned}$$

we compute

$$\|w_\eta\|^2 = 1 + \widetilde{O}((D\eta)^2) \implies \|w_\eta\|^{-1} = 1 + \widetilde{O}((D\eta)^2)$$

and so

$$\begin{aligned} \nu_\eta &= \frac{w_\eta}{\|w_\eta\|} = a^t(s)\partial_t + B^i(s)\partial_{s_i}, \\ \|a^t - 1\|_{C^{1,\alpha}(Y)} &\leq C\|\eta\|^2_{C^{2,\alpha}(Y)}, \\ \|B^i\|_{C^{1,\alpha}(Y)} &\leq C\|\eta\|_{C^{2,\alpha}(Y)}. \end{aligned}$$

■

We now note that for the diffeomorphism

$$\begin{aligned} \Phi &: Y \times (\omega\varepsilon\ln(\varepsilon), -\omega\varepsilon\ln(\varepsilon)) \to Y \times (-\omega\varepsilon\ln(\varepsilon)^2, \omega\varepsilon\ln(\varepsilon)^2), \\ \Phi(s,t) &= (s, t+\eta) \\ \implies (\Phi^{-1})_*(\nu_\eta) &= (1 + A(s))(\Phi^{-1})_*(\partial_t) + B^i(s)(\Phi^{-1})_*(\partial_{s_i}) \\ &= (1 + A(s))\partial_t + B^i(s)(\partial_{s_i} + \eta_i\partial_t) \\ &= (1 + \widetilde{A}(s))\partial_t + B^i(s)\partial_{s_i}, \\ \|\widetilde{A}(s)\|_{C^{1,\alpha}(Y)} &\leq C\|\eta\|^3_{C^{2,\alpha}(Y)} \leq C\|\eta\|^2_{C^{2,\alpha}(Y)} \end{aligned}$$

using that $\|\eta\|_{C^{2,\alpha}(Y)} = O(\varepsilon^{1+\beta})$.

A.4. Integrating the η equation for the normal derivative

In this section, we keep track of all the terms in equation (5.2) when integrating against $\dot{\overline{\mathbb{H}}}_\varepsilon$ to extract $\phi_t^\eta(s,0)$.

Lemma A.3. *The function $\phi_t^\eta(s,0)$ decomposes as*

$$\phi_t^\eta(s,0) = \sqrt{2}\int_0^{-\omega\varepsilon\ln(\varepsilon)} \Delta_Y(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon(t)dt - \frac{2}{3}J_Y(\eta) + O(\varepsilon^{1+2\beta})$$

where the error bound holds in $C^\alpha(Y)$.

Proof. Recall from (5.2) that we have

$$\begin{aligned} L_{\varepsilon,t+\eta}(\phi) &= \varepsilon[\Delta_{t+\eta}(\eta) + \dot{H}_0\eta]\dot{\overline{\mathbb{H}}}_\varepsilon + |\nabla^{t+\eta}\eta|^2\ddot{\overline{\mathbb{H}}}_\varepsilon \\ &\quad + \varepsilon^3\Big(\frac{H_{t+\eta}(s) - \dot{H}_0(s)(t+\eta)}{(t+\eta)^2}\Big)\Big(\frac{t+\eta}{\varepsilon}\Big)^2\dot{\overline{\mathbb{H}}}_\varepsilon \\ &\quad - \varepsilon^4\Delta_{t+\eta}(\dot{H}_0)\overline{\tau}_\varepsilon + \varepsilon^3 H_{t+\eta}\dot{H}_0\dot{\overline{\tau}}_\varepsilon + \varepsilon^3\Delta_{t+\eta}(\eta)\dot{H}_0\dot{\overline{\tau}}_\varepsilon + 2\varepsilon^3\nabla^{t+\eta}(\eta)(\dot{H}_0)\dot{\overline{\tau}}_\varepsilon \\ &\quad - \varepsilon^2|\nabla^{t+\eta}\eta|^2\dot{H}_0\ddot{\overline{\tau}}_\varepsilon + \frac{1}{2}W'''(\overline{\mathbb{H}}_\varepsilon)\varepsilon^4\dot{H}_0^2\overline{\tau}_\varepsilon^2 + \varepsilon^6\dot{H}_0^3\overline{\tau}_\varepsilon^3 + R(\phi) + O(\varepsilon^\omega) \\ &=: G(\phi). \end{aligned} \tag{A.1}$$

Starting with the left-hand side, we first recall

$$L_{\varepsilon,t+\eta}(\phi) = \varepsilon^2(\Delta_{t+\eta}(\phi^\eta) - H_{t+\eta}\phi_t^\eta + \phi_{tt}^\eta) - W''(\overline{\mathbb{H}}_\varepsilon)\phi^\eta.$$

We multiply by $\dot{\overline{\mathbb{H}}}_\varepsilon$ and integrate from $t = 0 \to t = -\omega\varepsilon\ln(\varepsilon)$. From hereon, all integrals will be from $[0, -\omega\varepsilon\ln(\varepsilon))$. We get

$$\begin{aligned}\int_0^{-\omega\varepsilon\ln(\varepsilon)} L_{\varepsilon,t+\eta}(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon &= -\varepsilon^2\sigma\phi_t^\eta(s,0) + \varepsilon^2\int \Delta_{t+\eta}(\phi^\eta)\dot{\overline{\mathbb{H}}}_\varepsilon - \varepsilon^2\int H_{t+\eta}\phi_t^\eta\dot{\overline{\mathbb{H}}}_\varepsilon \\ &= -\varepsilon^2\sigma\phi_t^\eta(s,0) + \varepsilon^2\int \Delta_Y(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon + O(\varepsilon^{4-\alpha}).\end{aligned}$$

Here we used equation (5.5), that is,

$$\|\phi\|_{C^{2,\alpha}_\varepsilon(M))} = O(\varepsilon^{2+\beta})$$

and $\|\int \dot{\overline{\mathbb{H}}}_\varepsilon H_{t+\eta}\|_{C^\alpha(Y)} = O(\varepsilon^{1+\beta})$, since Y is minimal and $\|\eta\|_{C^{2,\alpha}} = O(\varepsilon^{1+\beta})$. On the right-hand side of (A.1), we have

$$\begin{aligned}\int_0^{-\omega\varepsilon\ln(\varepsilon)} G(\phi^\eta)\dot{\overline{\mathbb{H}}}_\varepsilon dt &= \varepsilon\int[\Delta_{t+\eta}(\eta) + \dot{H}_0\eta]\dot{\overline{\mathbb{H}}}_\varepsilon^2 dt + \int |\nabla^{t+\eta}\eta|^2\dot{\overline{\mathbb{H}}}_\varepsilon\ddot{\overline{\mathbb{H}}}_\varepsilon dt \\ &+ \varepsilon^3\int\Big(\frac{H_{t+\eta}(s) - \dot{H}_0(s)(t+\eta)}{(t+\eta)^2}\Big)\Big(\frac{t+\eta}{\varepsilon}\Big)^2\dot{\overline{\mathbb{H}}}_\varepsilon^2 dt \\ &- \varepsilon^4\int \Delta_{t+\eta}(\dot{H}_0)\overline{\tau}_\varepsilon\dot{\overline{\mathbb{H}}}_\varepsilon dt + \varepsilon^3\int H_{t+\eta}\dot{H}_0\dot{\overline{\tau}}_\varepsilon\dot{\overline{\mathbb{H}}}_\varepsilon dt \\ &+ \varepsilon^3\int \Delta_{t+\eta}(\eta)\dot{H}_0\dot{\overline{\tau}}_\varepsilon\dot{\overline{\mathbb{H}}}_\varepsilon + 2\varepsilon^3\int \nabla^{t+\eta}(\eta)(\dot{H}_0)\dot{\overline{\tau}}_\varepsilon\dot{\overline{\mathbb{H}}}_\varepsilon \\ &- \varepsilon^2\int |\nabla^{t+\eta}\eta|^2\dot{H}_0\ddot{\overline{\tau}}_\varepsilon\dot{\overline{\mathbb{H}}}_\varepsilon + \frac{1}{2}\varepsilon^4\int W'''(\overline{\mathbb{H}}_\varepsilon)\dot{H}_0^2\overline{\tau}_\varepsilon^2\dot{\overline{\mathbb{H}}}_\varepsilon \\ &+ \varepsilon^6\int \dot{H}_0^3\overline{\tau}_\varepsilon^3\dot{\overline{\mathbb{H}}}_\varepsilon + \int R(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon + O(\varepsilon^{\omega+1}).\end{aligned}$$

We write this as

$$\begin{aligned}\int_0^{-\omega\varepsilon\ln(\varepsilon)} G(\phi^\eta)\dot{\overline{\mathbb{H}}}_\varepsilon dt = A_1 &+ A_2 \\ &+ B_1 \\ &+ C_1 + C_2 \\ &+ C_3 + C_4 \\ &+ D_1 + D_2 \\ &+ D_3 + E + O(\varepsilon^{\omega+1}),\end{aligned}$$

with the aim of extracting the leading terms and an appropriate error bounded in $C^\alpha(Y)$. We have

$$A_1 = \varepsilon(\Delta_0(\eta) + \dot{H}_0\eta)\int \dot{\overline{\mathbb{H}}}_\varepsilon^2 dt + \varepsilon\int[\Delta_{t+\eta} - \Delta_0](\eta)\dot{\overline{\mathbb{H}}}_\varepsilon^2 dt$$

$$
\begin{aligned}
&= \sigma_0 \varepsilon^2 J_Y(\eta) + \varepsilon \int [\Delta_{t+\eta} - \Delta_0](\eta) \dot{\overline{\mathbb{H}}}_\varepsilon^2 dt \\
&= \sigma_0 \varepsilon^2 J_Y(\eta) + O(\varepsilon^3 \|\eta\|_{C^{2,\alpha}(Y)}, \varepsilon^2 \|\eta\|_{C^\alpha(Y)} \|\eta\|_{C^{2,\alpha}(Y)}) \\
&= \sigma_0 \varepsilon^2 J_Y(\eta) + O(\varepsilon^{4+\beta}),
\end{aligned}
$$

which comes from expanding $\Delta_{t+\eta} - \Delta_0$ in powers of $(t + \eta)$. Similarly,

$$
A_2 = \int |\nabla^Y \eta|^2 \dot{\overline{\mathbb{H}}}_\varepsilon \ddot{\overline{\mathbb{H}}}_\varepsilon + \int [|\nabla^{t+\eta} \eta|^2 - |\nabla^Y \eta|^2] \dot{\overline{\mathbb{H}}}_\varepsilon \ddot{\overline{\mathbb{H}}}_\varepsilon dt = O(\varepsilon^{3+2\beta}),
$$

which comes from expanding $g^{ij}(s, t + \eta)$ in powers of $(t + \eta)$ and

$$
|\nabla^{t+\eta} \eta|^2 = g^{ij}(s, t + \eta) \eta_i \eta_j = g^{ij}(s, 0) \eta_i \eta_j + [g^{ij}(s, t + \eta) - g^{ij}(s, 0)] \eta_i \eta_j.
$$

For B_1, we see that

$$
\|B_1\|_{C^0} \leq \varepsilon^3 \int K\Big(\frac{|t| + |\eta|}{\varepsilon}\Big) \dot{\overline{\mathbb{H}}}_\varepsilon^2 dt \leq O(\varepsilon^4).
$$

To see the $[\cdot]_\alpha$ bound, we write

$$
\begin{aligned}
B_1 &= \varepsilon^3 \int_0^{-\omega \varepsilon \ln(\varepsilon)} \Big(\int_0^{t+\eta} [\dot{H}_r(s) - \dot{H}_0(s)] dr\Big) \dot{\overline{\mathbb{H}}}_\varepsilon^2 dt \\
&= \varepsilon^3 \int_0^{-\omega \varepsilon \ln(\varepsilon)} \Big(\int_0^{t+\eta} \int_0^r \ddot{H}_w(s) dw dr\Big) \dot{\overline{\mathbb{H}}}_\varepsilon^2 dt \\
\implies [B_1]_{C^\alpha(Y)} &= O(\varepsilon^4).
\end{aligned}
$$

For $\{C_i\}$, we compute in a straightforward manner, using that $Y \in C^{4,\alpha}$ and satisfies (2.1),

$$
\begin{aligned}
C_1 &= \varepsilon^4 \int \Delta_{t+\eta}(\dot{H}_0) \dot{\overline{\tau}}_\varepsilon \dot{\overline{\mathbb{H}}}_\varepsilon dt = O(\varepsilon^5), \\
C_2 &= \varepsilon^3 \int H_{t+\eta} \dot{H}_0 \dot{\overline{\tau}}_\varepsilon \dot{\overline{\mathbb{H}}}_\varepsilon dt = O(\varepsilon^4), \\
C_3 &= \varepsilon^3 \int \Delta_{t+\eta}(\eta) \dot{H}_0 \dot{\overline{\tau}}_\varepsilon \dot{\overline{\mathbb{H}}}_\varepsilon dt = O(\varepsilon^{5+\beta}), \\
C_4 &= 2\varepsilon^3 \int \nabla^{t+\eta}(\eta)(\dot{H}_0) \dot{\overline{\tau}}_\varepsilon \dot{\overline{\mathbb{H}}}_\varepsilon dt = O(\varepsilon^{5+\beta}),
\end{aligned}
$$

which is seen from making a change of variables $t \to t/\varepsilon$ to gain another factor of ε, and then noting that the integrals converge and are bounded in $C^\alpha(Y)$. For the $\{D_i\}$ terms, we similarly have

$$
\begin{aligned}
D_1 &= -\varepsilon^2 \int |\nabla^{t+\eta} \eta|^2 \dot{H}_0 \ddot{\overline{\tau}}_\varepsilon \dot{\overline{\mathbb{H}}}_\varepsilon dt = O(\varepsilon^{5+2\beta}), \\
D_2 &= \frac{1}{2} \varepsilon^4 \int W'''(\overline{\mathbb{H}}_\varepsilon) \dot{H}_0^2 \overline{\tau}_\varepsilon^2 \dot{\overline{\mathbb{H}}}_\varepsilon dt = O(\varepsilon^5),
\end{aligned}
$$

$$D_3 = \varepsilon^6 \int \dot{H}_0^3 \overline{\tau}_\varepsilon^3 \dot{\overline{\mathbb{H}}}_\varepsilon = O(\varepsilon^7).$$

Finally, recall (5.3) to decompose the E term

$$E_1 = \int R(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon = \int \varepsilon^2 E_\eta(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon - \int F_0(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon,$$

and we have

$$\begin{aligned} E_\eta &= -\Delta_{t+\eta}(\eta)\partial_t - 2\nabla^{t+\eta}(\eta)\partial_t + |\nabla^{t+\eta}\eta|^2\partial_t^2, \\ \varepsilon^2 \int (-\Delta_{t+\eta}(\eta))\phi_t \dot{\overline{\mathbb{H}}}_\varepsilon &= O(\varepsilon^{5+2\beta-\alpha}), \\ -2\varepsilon^2 \int \nabla^{t+\eta}(\eta)(\phi_t)\dot{\overline{\mathbb{H}}}_\varepsilon &= O(\varepsilon^{4+2\beta-\alpha}), \\ \varepsilon^2 \int |\nabla^{t+\eta}\eta|^2 \phi_{tt}\dot{\overline{\mathbb{H}}}_\varepsilon &= O(\varepsilon^{5+3\beta-\alpha}), \\ \implies \varepsilon^2 \int E_\eta(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon &= O(\varepsilon^{4+2\beta-\alpha}) \end{aligned}$$

with bounds holding in $C^\alpha(Y)$. Similarly, using the definition of F_0 in (5.4),

$$\begin{aligned} F_0(\phi) &:= W'''(\overline{\mathbb{H}}_\varepsilon)\varepsilon^2 \dot{H}_0 \overline{\tau}_\varepsilon \phi + \left[3\varepsilon^4 \dot{H}_0^2 \overline{\tau}_\varepsilon^2 + \frac{1}{2}W'''(\overline{\mathbb{H}}_\varepsilon)\right]\phi^2 + \phi^3, \\ \varepsilon^2 \int W'''(\overline{\mathbb{H}}_\varepsilon)\dot{H}_0 \overline{\tau}_\varepsilon \phi \dot{\overline{\mathbb{H}}}_\varepsilon &= O(\varepsilon^{5+\beta-\alpha}), \\ 3\varepsilon^4 \int \dot{H}_0^2 \overline{\tau}_\varepsilon^2 \phi^2 \dot{\overline{\mathbb{H}}}_\varepsilon &= O(\varepsilon^{9+2\beta-\alpha}), \\ \frac{1}{2}\int W'''(\overline{\mathbb{H}}_\varepsilon)\phi^2 \dot{\overline{\mathbb{H}}}_\varepsilon &= O(\varepsilon^{5+2\beta-\alpha}), \\ \int \phi^3 \dot{\overline{\mathbb{H}}}_\varepsilon &= O(\varepsilon^{7+3\beta-\alpha}) \\ \implies \int F_0(\phi)\dot{\overline{\mathbb{H}}}_\varepsilon &= O(\varepsilon^{5+\beta-\alpha}). \end{aligned}$$

With this, we have shown that

$$\int G(\phi^\eta)\dot{\overline{\mathbb{H}}}_\varepsilon dt = \varepsilon^2 \sigma_0 J_Y(\eta) + O(\varepsilon^{3+2\beta})$$

with error in $C^\alpha(Y)$. This finishes the proof. ∎

A.5. Existence of solutions to $\ddot{F} - W''(\overline{\mathbb{H}})F = \varphi$

Given $\varphi : [0,\infty) \to \mathbb{R}$ smooth and asymptotically exponentially decaying, consider

$$\begin{aligned} \partial_t^2 F(t) - W''(\mathbb{H}(t))F(t) &= \varphi(t), \\ F(0) &= 0, \\ \lim_{t\to\infty} F(t) &= 0. \end{aligned} \tag{A.2}$$

We reprove the following lemma seen in [9] and proven in [1, Lemma B.1, Remark B.3]:

Lemma A.4. *Given* $\varphi \in C^{\infty}([0,\infty))$ *such that*

$$\exists t_0 > 0,\ K > 0,\ \gamma > 0 \quad s.t. \quad \forall t > t_0, \quad |\varphi(t)| \leq K e^{-\gamma t}$$

then there exists a smooth solution to system (A.2) *with exponential decay.*

Proof. Consider the a priori solution of the form

$$F(t) = v(t)\dot{\mathbb{H}}(t)$$

where $v(t)$ is to be constructed with $v(0) = 0$. We plug this into (A.2), multiply by $\dot{\mathbb{H}}$, and integrate twice to get a general solution of

$$v(t) = b_0 + \int_0^t \dot{\mathbb{H}}(s)^{-2}\Big[a_0 + \int_0^s \varphi(r)\dot{\mathbb{H}}(r)dr\Big]ds.$$

Using the condition of $v(0) = 0$, we have $b_0 = 0$. Moreover, we can set

$$a_0 = -\int_0^{\infty} \varphi(r)\dot{\mathbb{H}}(r).$$

We now show that $v(t)$ is bounded so that $\lim_{t\to\infty} F(t) = 0$. First, we know that for t large,

$$\dot{\mathbb{H}}(t)^{-2} \sim e^{\sqrt{2}t}$$

and from this, we can enforce

$$|\dot{v}(t)| = |a_0 + \int_0^t \varphi(r)\dot{\mathbb{H}}(r)| \leq K e^{-(\beta+\sqrt{2})t}$$

for some $\beta > -\sqrt{2}$ and $K > 0$ by using the exponential decay of $\varphi(s)$ in L^{∞}. Thus, $\dot{v}$ is exponentially decaying, so that $v(t)$ is bounded and, hence,

$$\lim_{t\to\infty} F(t) = \lim_{t\to\infty} v(t)\dot{\mathbb{H}}(t) = 0.$$

Moreover, since $v(t)$ is bounded and $\dot{\mathbb{H}}(t)$ is exponentially decaying, $F(t)$ is also exponentially decaying by differentiating the equation for F. ■

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