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Well-posedness and stability for the two-phase periodic quasistationary Stokes flow

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Abstract. The two-phase horizontally periodic quasistationary Stokes flow in $\mathbb{R}^2$, describing the motion of two immiscible fluids with equal viscosities that are separated by a sharp interface, parameterized as the graph of a function $f = f(t)$, is considered in the general case when both gravity and surface tension effects are included. Using potential theory, the moving boundary problem is formulated as a fully nonlinear and nonlocal parabolic problem for the function f. Based on abstract parabolic theory, it is shown that the problem is well-posed in all subcritical spaces $\mathrm{H}^r(\mathbb{S})$, with $r \in (3/2, 2)$. Moreover, the stability properties of the flat equilibria are analyzed in dependence on the physical properties of the fluids.

1. Introduction

We consider the two-phase horizontally periodic quasistationary Stokes flow driven by surface tension and gravity effects, which is modeled by the system

$$\left.\begin{aligned} \mu\Delta v^\pm - \nabla q^\pm &= 0 && \text{in } \Omega^\pm(t),\\ \operatorname{div} v^\pm &= 0 && \text{in } \Omega^\pm(t),\\ [v] &= 0 && \text{on } \Gamma(t),\\ [T_\mu(v,q)]\tilde{\nu} &= (\Theta x_2 - \sigma\tilde{\kappa})\tilde{\nu} && \text{on } \Gamma(t),\\ (v^\pm, q^\pm)(x) &\to (\pm c_{1,\Gamma}, 0, \pm c_{2,\Gamma}) && \text{for } x_2 \to \pm\infty,\\ V_n &= v\cdot\tilde{\nu} && \text{on } \Gamma(t) \end{aligned}\right\} \tag{1.1a}$$

for $t > 0$. We assume that $\Gamma(t)$ is the graph of a function $f(t)$ that separates the two fluid domains

$$\Omega^\pm(t) := \{x = (x_1, x_2) \in \mathbb{S}\times\mathbb{R} : x_2 \gtrless f(t, x_1)\}, \quad t > 0.$$

We denote by $\mathbb{S} := \mathbb{R}/2\pi\mathbb{Z}$ the unit circle, functions that depend on the real variable $\xi \in \mathbb{S}$ being 2π-periodic with respect to it. In particular, the unknown $(f, v^\pm, q^\pm)$ is assumed to be 2π-periodic with respect to the horizontal variable x_1. At time $t = 0$, we impose the initial condition

$$f(0) = f_0. \tag{1.1b}$$

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Moreover, the constants $c_{1,\Gamma}$ and $c_{2,\Gamma}$ evolve over time and are identified by $f = f(t)$ and the other constants in (1.1a) according to

$$c_{1,\Gamma} := -\frac{\sigma}{2\mu}\Big\langle \frac{f'}{(1+f'^2)^{1/2}}\Big\rangle, \quad c_{2,\Gamma} := -\frac{\Theta}{2}\langle f\rangle, \quad t>0, \tag{1.1c}$$

where

$$\langle h\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} h(s)\,\mathrm{d}s$$

denotes the integral mean of an integrable function $h : \mathbb{S} \to \mathbb{R}$ and

$$\Theta := g(\rho^- - \rho^+) \in \mathbb{R}. \tag{1.2}$$

The equation set (1.1) describes the dynamics of two incompressible and immiscible Newtonian fluids with equal viscosities, the positive constants σ and μ representing the surface tension coefficient at the interface and the viscosity of the fluids, respectively. The constant $g \geq 0$ is the Earth's gravity and $\rho^\pm$ stands for the density of the fluid occupying $\Omega^\pm$. Moreover, $\widetilde{\nu}$ is the unit exterior normal to $\partial\Omega^-$ and $\widetilde{\kappa}$ is the curvature of the moving interface. We further denote by $v^\pm = v^\pm(t) : \Omega^\pm(t) \to \mathbb{R}^2$ the velocity field in the fluid domain $\Omega^\pm(t)$ and $q^\pm = q^\pm(t) : \Omega^\pm(t) \to \mathbb{R}$ is defined by

$$q^\pm(t,x) = p^\pm(t,x) + g\rho^\pm x_2, \quad x = (x_1,x_2) \in \Omega^\pm(t),$$

where $p^\pm = p^\pm(t)$ is the pressure in $\Omega^\pm(t)$. The stress tensor $T_\mu(v^\pm, q^\pm)$ is given by

$$T_\mu(v^\pm,q^\pm) := -q^\pm I_2 + \mu\big(\nabla v^\pm + (\nabla v^\pm)^\top\big), \quad (\nabla v^\pm)_{ij} := \partial_j v_i^\pm, \quad i,j = 1,2, \tag{1.3}$$

while $[v]$ and $[T_\mu(v,q)]$ denote the jump of the velocity and the stress tensor across the moving interface, respectively, as defined in (2.3) below. Finally, V_n is the normal velocity of the interface $\Gamma(t)$, $x \cdot y$ denotes the Euclidean scalar product of two vectors $x, y \in \mathbb{R}^n$, and $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix.

First studies of the quasistationary Stokes flow investigated the setting of a single fluid phase which occupies a sufficiently smooth domain $\Omega(t) \subset \mathbb{R}^d$, $d \geq 2$. In [20], the authors have established the well-posedness of the problem for initial data which are close to a smooth and strictly star-shaped domain, together with the exponential stability of balls. Subsequently, the exponential stability of balls has been proven in [11, 12] in space dimension $d \in \{2,3\}$ by a different power series argument. Moreover, in the two-phase setting in $\mathbb{R}^2$, where one bounded fluid phase is surrounded by the other and the system is driven by the surface tension at the interface, it was recently shown in [7] that balls are exponentially stable. Furthermore, in the confined setting, the quasistationary Stokes flow is the singular limit of the Navier–Stokes problem when the Reynolds number vanishes, cf. [35, 36].

The two-phase quasistationary Stokes flow in a bounded geometry, with one of the fluid phases surrounding the other one and with possible phase transitions, has been considered in arbitrary space dimension $d \geq 2$ in the monograph [33] on the basis of maximal regularity in weighted L^p-spaces. We also refer to [8] for a study concerning a feedback stabilization issue in this setting.

In the absence of gravity effects, the nonperiodic version of the quasistationary Stokes flow (1.1)—with equal and general viscosities—has been investigated recently in [27,28], the local well-posedness property being provided in $H^r(\mathbb{R})$, with $r \in (3/2, 2)$ arbitrarily close to the critical exponent $r = 3/2$, cf. [28, Remark 1.2]. Moreover, as shown in [29], the unconfined one-phase flow is the singular limit of the two-phase problem when the viscosity of one of the fluids vanishes.

In the absence of surface tension effects, that is when $\sigma = 0$, the problem (1.1) has been analyzed in [14, 15]. If the interface between the fluids is parameterized as an arbitrary curve, the problem can be reformulated as an ODE and local well-posedness is established by using Picard's theorem [15]. In [14], the authors showed that the problem is actually globally well-posed. Global existence results of solutions in the graph case are provided in [15] for initial data that are small in $H^3(\mathbb{S})$ or in certain Wiener algebras in the stable regime when $\Theta > 0$. In fact, the solutions to (1.1) in the case $\sigma = 0$ also solve the transport Stokes system, see [19,21,22,30,31], which is a model for the settling process of a cluster of rigid particles in a viscous fluid.

Lastly, we mention the related Peskin problem which models the evolution of an elastic string (or membrane) immersed in a viscous fluid. In this context, the equations governing the motion in the fluid match those in equation (1.1) (with $g = 0$), while the dynamics of the elastic string are described using Lagrangian coordinates, see the very recent research in [5, 6, 13, 16–18, 23, 32].

In this paper, we consider the horizontally periodic quasistationary Stokes flow (1.1) with both gravity (neglected if $g = 0$) and surface tension effects incorporated—which could not be achieved in the nonperiodic case [27,28] and was neither investigated in [14, 15]—in Sobolev spaces $H^r(\mathbb{S})$ with $r \in (3/2, 2)$ arbitrary (again $r = 3/2$ is the critical exponent). A striking difference to the nonperiodic case [27,28] is the far field boundary condition $(1.1a)_5$ for (v, q). While in the nonperiodic case (v, q) vanishes at infinity, under the periodicity assumption (v, q) converges, at each fixed time t, towards a constant vector explicitly determined by $f(t)$, cf. (1.1c). In particular, for $x_2 \to \pm\infty$, the velocity is asymptotically horizontal, but the asymptotic profiles at $\pm\infty$ are opposite to each other. Moreover, for $x_2 \to \pm\infty$, the pressure may deviate from the hydrostatic pressure by some constant which is determined by $f(t)$ and which has opposite sign at $\pm\infty$.

Our strategy to solve (1.1) is to prove that $(v^\pm, q^\pm)$ is determined at each time instant $t > 0$ by $f(t)$, provided $f(t) \in H^3(\mathbb{S})$, see Remark 2.1 and Theorem 2.2. In this way, we may reformulate (1.1) as a fully nonlinear and nonlocal problem for f, see (3.10), with nonlinearities expressed by (singular) integral operators involving f, which are well-defined when merely $f \in H^r(\mathbb{S})$ with $r > 3/2$. The fully nonlinear character of (3.10) is due to the fact that the phase space $H^r(\mathbb{S})$ can be chosen arbitrarily close to the critical space, meaning that the curvature has to be interpreted as a distribution. The situation is different in [11, 12, 20, 33, 35, 36] where the interface is at least of class C^2 and due to quasilinearity of the curvature operator, the Stokes problem (1.1) may be formulated as a quasilinear evolution problem. Using results on certain families of singular integral operators from Appendices A and B—which might be of interest also in the context of other evolution problems (such as the Hele–Shaw, Muskat, or Mullins–Sekerka problems)—we then prove that the new formulation (3.10) is of parabolic type and provide its well-posedness by using abstract parabolic theory from [24], cf. Theorem 1.1 below. Additionally, we show a parabolic smoothing property for (3.10), which justifies the assumption $f(t) \in \mathrm{H}^3(\mathbb{S})$ in Theorem 2.2.

Our first main result is the following theorem.

Theorem 1.1. *Let $\Theta \in \mathbb{R}$, $\mu, \sigma \in (0, \infty)$, $r \in (3/2, 2)$, and $f_0 \in \mathrm{H}^r(\mathbb{S})$.*

(i) (Well-posedness) *There exists a unique maximal solution $(f, v^\pm, q^\pm)$ to (1.1) such that*

$$f := f(\cdot, f_0) \in \mathrm{C}([0, T_+), \mathrm{H}^r(\mathbb{S})) \cap \mathrm{C}^1([0, T_+), \mathrm{H}^{r-1}(\mathbb{S}))$$

and

$$\left.\begin{aligned} &f(t) \in \mathrm{H}^3(\mathbb{S}),\\ &v^\pm(t) \in \mathrm{C}^2(\Omega^\pm(t), \mathbb{R}^2) \cap \mathrm{C}^1(\overline{\Omega^\pm(t)}, \mathbb{R}^2),\\ &q^\pm(t) \in \mathrm{C}^1(\Omega^\pm(t)) \cap \mathrm{C}(\overline{\Omega^\pm(t)}) \end{aligned}\right\} \quad \text{for all } t \in (0, T_+),$$

where $T_+ = T_+(f_0) \in (0, \infty]$.

(ii) (Parabolic smoothing) *We have $[(t, \xi) \mapsto f(t, \xi)] \in \mathrm{C}^\infty((0, T_+) \times \mathbb{S}, \mathbb{R})$.*

(iii) (Global existence) *The solution is global, that is $T_+(f_0) = \infty$, if for each $T > 0$, we have*

$$\sup_{[0,T]\cap[0,T_+(f_0))} \|f(t)\|_{\mathrm{H}^r} < \infty.$$

Our second main objective is to study the stability properties of the solutions with a flat interface, which are all equilibria to (1.1). Indeed, if $(f, v^\pm, q^\pm)$ is a solution to (1.1), then, for each constant $c \in \mathbb{R}$, the tuple $(f + c, \tilde{v}^\pm, \tilde{q}^\pm)$ defined by

$$\tilde{v}^\pm(t, x) = v^\pm(t, x - (0, c)), \quad \tilde{q}^\pm(t, x) = q^\pm(t, x - (0, c)) \mp \frac{c\Theta}{2}, \quad x_2 \neq f(t, x_1) + c,$$

is again a solution to (1.1) (with initial data $f_0 + c$ and the same maximal existence time). Moreover, the integral mean $\langle f \rangle$ is preserved by the flow since, by $(1.1a)_2$ and $(1.1a)_{5-6}$,

$$\frac{\mathrm{d}\langle f \rangle}{\mathrm{d}t}(t) = \int_{\Gamma(t)} v(t) \cdot \tilde{\nu}(t)\, \mathrm{d}\sigma = \int_{\Omega^\pm(t)} \operatorname{div} v^\pm(t)\, \mathrm{d}x = 0, \quad t \in [0, T_+). \tag{1.4}$$

Since it is straightforward to verify that $(f, v, q) = 0$ is a stationary solution to (1.1), it follows that $(c, 0, \mp c\Theta/2)$ is a stationary solution to (1.1) for each $c \in \mathbb{R}$. This observation together with our second main result in Theorem 1.2 shows, on the one hand, that if

$$\sigma + \Theta > 0, \tag{1.5}$$

solutions corresponding to initial data $f_0 \in \mathrm{H}^r(\mathbb{S})$ which are close to a constant exist globally and $f(t)$ converges exponentially fast towards the integral mean of f_0. On the other hand, if (1.5) holds with reverse inequality, that is $\sigma + \Theta < 0$, then the constant solutions are (nonlinearly) unstable.

Theorem 1.2 (Exponential stability/instability).

(i) *Assume (1.5) and let ϑ_0 denote the positive constant*

$$\vartheta_0 := \frac{\sigma + \Theta}{4\mu} \mathbf{1}_{[0,\infty)}(\sigma - \Theta) + \frac{\sqrt{\sigma\Theta}}{2\mu} \mathbf{1}_{(0,\infty)}(\Theta - \sigma). \tag{1.6}$$

Then, given $\vartheta \in [0, \vartheta_0)$, there exist constants $\delta > 0$ and $M > 0$ such that for any $f_0 \in \mathrm{H}^r(\mathbb{S})$ satisfying

$$\|f_0\|_{\mathrm{H}^r} < \delta \quad \textit{and} \quad \langle f_0 \rangle = 0,$$

we have $T_+(f_0) = \infty$ and

$$\|f(t)\|_{\mathrm{H}^r} + \left\| \frac{\mathrm{d}f}{\mathrm{d}t}(t) \right\|_{\mathrm{H}^{r-1}} \le M e^{-\vartheta t} \|f_0\|_{\mathrm{H}^r} \quad \textit{for all } t \ge 0.$$

(ii) *If $\sigma + \Theta < 0$, then the zero solution is unstable.*

Outline. In Section 2 and Appendix C, we solve the fixed time problem $(1.1a)_1$–$(1.1a)_5$ with a general right-hand side in $(1.1a)_4$. Then, in Section 3.1, we introduce two classes of (singular) integral operators, studied in Appendices A and B, which enable us to reformulate in Section 3.2 the flow (1.1) as a nonlinear and nonlocal evolution problem for f. In Sections 3.3 and 3.4, it is shown that the problem is of parabolic type, the main results being established in Section 3.5.

2. The fixed time problem

In this section, we address the solvability of the boundary value problem $(1.1a)_1$–$(1.1a)_5$ at a fixed time instant $t > 0$, under the assumption that $f = f(t)$ is sufficiently regular and with a general right-hand side in $(1.1a)_4$. More precisely, we fix $f \in \mathrm{H}^3(\mathbb{S})$ and consider the boundary value problem

$$\left.\begin{aligned} \mu\Delta v^\pm - \nabla q^\pm &= 0 && \text{in } \Omega^\pm,\\ \operatorname{div} v^\pm &= 0 && \text{in } \Omega^\pm,\\ [v] &= 0 && \text{on } \Gamma,\\ [T_\mu(v,q)]\tilde{\nu} &= (\omega^{-1}G)\circ \Xi^{-1} && \text{on } \Gamma,\\ (v^\pm, q^\pm)(x) &\to (\pm c_1, 0, \pm c_2) && \text{for } x_2 \to \pm\infty, \end{aligned}\right\} \tag{2.1}$$

where $G := (G_1, G_2) \in \mathrm{H}^1(\mathbb{S})^2$ satisfies $\langle G_1 \rangle = 0$ and the constants $c_1, c_2 \in \mathbb{R}$ are arbitrary. The domains $\Omega^\pm$ and their common boundary Γ are defined by

$$\Omega^\pm := \big\{x = (x_1, x_2) \in \mathbb{S} \times \mathbb{R} : x_2 \gtrless f(x_1)\big\}, \quad \Gamma := \big\{(\xi, f(\xi)) \in \mathbb{S} \times \mathbb{R} : \xi \in \mathbb{S}\big\}.$$

Note that $\Xi := \Xi_f := (\mathrm{id}_{\mathbb{S}}, f)$ is a diffeomorphism that maps the x_1-axis onto Γ. Further, ν and τ are the componentwise pull-back under Ξ of the unit normal $\widetilde{\nu}$ on Γ exterior to Ω^- and of the tangent $\widetilde{\tau}$, that is,

$$\nu := \nu(f) := \omega^{-1}(-f', 1)^\top, \quad \tau := \tau(f) := \omega^{-1}(1, f')^\top, \quad \omega := \omega(f) := (1 + f'^2)^{1/2}. \tag{2.2}$$

For any functions $z^\pm$ defined on $\Omega^\pm$, respectively, and having limits at some $(\xi, f(\xi)) \in \Gamma$, we will write

$$[z](\xi, f(\xi)) := \lim_{\Omega^+ \ni x \to (\xi, f(\xi))} z^+(x) - \lim_{\Omega^- \ni x \to (\xi, f(\xi))} z^-(x). \tag{2.3}$$

Remark 2.1. We note that, in the particular case, when

$$G := G(f) := \Theta(-ff', f) - \sigma(\omega^{-1} - 1, \omega^{-1} f')', \tag{2.4}$$

we have $G \in \mathrm{H}^1(\mathbb{S})^2$ and

$$(\omega^{-1} G) \circ \Xi^{-1} = \big(\Theta x_2 - \sigma \widetilde{\kappa}\big)\widetilde{\nu}.$$

Consequently, the right-hand sides of (2.1)$_4$ and of (1.1a)$_4$ coincide in this case.

Before stating our result on the solvability of (2.1), we introduce some further notation. To start, we define for $0 \neq x = (x_1, x_2) \in \mathbb{S} \times \mathbb{R}$

$$G_\pi(x) := -\frac{1}{4\pi} \ln\left(\frac{t_{[x_1]}^2 + T_{[x_2]}^2}{(1 + t_{[x_1]}^2)(1 - T_{[x_2]}^2)}\right) = -\frac{1}{4\pi} \ln\left(\sin^2\left(\frac{x_1}{2}\right) + \sinh^2\left(\frac{x_2}{2}\right)\right), \tag{2.5}$$

which is the fundamental solution to the x_1-periodic Laplace equation, that is, G_π solves the equation $-\Delta G_\pi = \delta_0$ in $\mathcal{D}'(\mathbb{S} \times \mathbb{R})$. We use the shorthand notation

$$t_{[x_1]} := \tan\left(\frac{x_1}{2}\right), \quad x_1 \in \mathbb{R} \setminus (\pi + 2\pi\mathbb{Z}), \quad T_{[x_2]} := \tanh\left(\frac{x_2}{2}\right), \quad x_2 \in \mathbb{R}. \tag{2.6}$$

The x_1-periodic Stokeslet $(\mathcal{U}, \mathcal{P})$, with $\mathcal{U} := (\mathcal{U}^1, \mathcal{U}^2)$ and $\mathcal{P} := (\mathcal{P}^1, \mathcal{P}^2)$, is defined by

$$\begin{aligned} \mathcal{U}^1(x) &= -\frac{1}{2\pi}\big(G_\pi(x) + x_2 \partial_2 G_\pi(x), -x_2 \partial_1 G_\pi(x)\big), \quad && \mathcal{P}^1(x) = \partial_1 G_\pi(x), \\ \mathcal{U}^2(x) &= -\frac{1}{2\pi}\big(-x_2 \partial_1 G_\pi(x), G_\pi(x) - x_2 \partial_2 G_\pi(x)\big), \quad && \mathcal{P}^2(x) = \partial_2 G_\pi(x) \end{aligned} \tag{2.7}$$

for $x = (x_1, x_2) \in (\mathbb{S} \times \mathbb{R}) \setminus \{0\}$, and we may reexpress (2.7) as follows:

$$\begin{aligned} \mathcal{U}(x) &= \frac{1}{8\pi}\left(\ln\left(\frac{t_{[x_1]}^2 + T_{[x_2]}^2}{(1 + t_{[x_1]}^2)(1 - T_{[x_2]}^2)}\right) I_2 - x_2 \begin{pmatrix} -\frac{(1+t_{[x_1]}^2)T_{[x_2]}}{t_{[x_1]}^2 + T_{[x_2]}^2} & \frac{t_{[x_1]}(1-T_{[x_2]}^2)}{t_{[x_1]}^2 + T_{[x_2]}^2} \\ \frac{t_{[x_1]}(1-T_{[x_2]}^2)}{t_{[x_1]}^2 + T_{[x_2]}^2} & \frac{(1+t_{[x_1]}^2)T_{[x_2]}}{t_{[x_1]}^2 + T_{[x_2]}^2} \end{pmatrix}\right), \\ \mathcal{P}^1(x) &= -\frac{1}{4\pi} \frac{t_{[x_1]}(1 - T_{[x_2]}^2)}{t_{[x_1]}^2 + T_{[x_2]}^2}, \quad \mathcal{P}^2(x) = -\frac{1}{4\pi} \frac{(1 + t_{[x_1]}^2)T_{[x_2]}}{t_{[x_1]}^2 + T_{[x_2]}^2}. \end{aligned} \tag{2.8}$$

Using (2.7), it is straightforward to prove that $(\mathcal{U}^k, \mathcal{P}^k)$, $k = 1, 2$, are fundamental solutions to the Stokes equations in the sense that

$$\left.\begin{aligned} \Delta \mathcal{U}^k - \nabla \mathcal{P}^k &= \delta_0 e_k, \\ \operatorname{div} \mathcal{U}^k &= 0 \end{aligned}\right\} \quad \text{in } \mathcal{D}'(\mathbb{S} \times \mathbb{R}), \tag{2.9}$$

where $e_1 := (1, 0)$ and $e_2 := (0, 1)$. In particular, they solve the Stokes equations (2.9) pointwise in $(\mathbb{S} \times \mathbb{R}) \setminus \{0\}$. For the derivation of the x_1-periodic Stokeslet $(\mathcal{U}, \mathcal{P})$, we refer to the recent paper [15] (see also [3] for an alternative derivation).

In Theorem 2.2 below and further on we sum over indices appearing twice in a product.

Theorem 2.2. *Given $f \in \mathrm{H}^3(\mathbb{S})$ and $G \in \mathrm{H}^1(\mathbb{S})^2$ with $\langle G_1 \rangle = 0$, the boundary value problem (2.1) has a solution $(v^\pm, q^\pm)$ such that*

$$v^\pm \in \mathrm{C}^2(\Omega^\pm, \mathbb{R}^2) \cap \mathrm{C}^1(\overline{\Omega^\pm}, \mathbb{R}^2) \quad \textit{and} \quad q^\pm \in \mathrm{C}^1(\Omega^\pm) \cap \mathrm{C}(\overline{\Omega^\pm}) \tag{2.10}$$

if and only if the constants c_1, c_2 in $(2.1)_5$ are given by

$$c_1 = -\frac{\langle f G_1 \rangle}{2\mu} \quad \textit{and} \quad c_2 = -\frac{\langle G_2 \rangle}{2}. \tag{2.11}$$

If c_1, c_2 are defined by (2.11), then the solution $(v^\pm, q^\pm)$ is unique and is given by the formula

$$v^\pm := v_G^\pm + \Big(0, \frac{\langle G_2 \rangle \ln 4}{4\mu}\Big) \quad \textit{and} \quad q^\pm := q_G^\pm, \tag{2.12}$$

where, given $x \in \Omega^\pm$,

$$\begin{aligned} v_G^\pm(x) &:= \frac{1}{\mu} \int_{-\pi}^{\pi} \mathcal{U}^k(x - (s, f(s))) G_k(s)\, \mathrm{d}s, \\ q_G^\pm(x) &:= \int_{-\pi}^{\pi} \mathcal{P}^k(x - (s, f(s))) G_k(s)\, \mathrm{d}s. \end{aligned} \tag{2.13}$$

Proof. We devise the proof into several parts.

Uniqueness. For the uniqueness statement, we need to show that if $(\mathsf{v}^\pm, \mathsf{q}^\pm)$ satisfies (2.10) and solves the boundary value problem

$$\left.\begin{aligned} \mu \Delta \mathsf{v}^\pm - \nabla \mathsf{q}^\pm &= 0 && \text{in } \Omega^\pm, \\ \operatorname{div} \mathsf{v}^\pm &= 0 && \text{in } \Omega^\pm, \\ [\mathsf{v}] &= 0 && \text{on } \Gamma, \\ [T_\mu(\mathsf{v}, \mathsf{q})]\tilde{\nu} &= 0 && \text{on } \Gamma, \\ (\mathsf{v}^\pm, \mathsf{q}^\pm)(x) &\to (\pm \mathsf{c}_1, 0, \pm \mathsf{c}_2) && \text{for } x_2 \to \pm\infty \end{aligned}\right\} \tag{2.14}$$

for some $(\mathsf{c}_1, \mathsf{c}_2) \in \mathbb{R}^2$, then actually $(\mathsf{v}^\pm, \mathsf{q}^\pm) = (0,0)$ in $\Omega^\pm$ and $\mathsf{c}_1 = \mathsf{c}_2 = 0$. We first note, in view of (2.14)$_2$, that

$$T_\mu(\mathsf{v}^\pm, \mathsf{q}^\pm)\widetilde{\nu} = -\mathsf{q}^\pm\widetilde{\nu} + \mu \begin{pmatrix} \partial_{\widetilde{\nu}}\mathsf{v}_1^\pm + \partial_{\widetilde{\tau}}\mathsf{v}_2^\pm \\ \partial_{\widetilde{\nu}}\mathsf{v}_2^\pm - \partial_{\widetilde{\tau}}\mathsf{v}_1^\pm \end{pmatrix},$$

and, since $[\partial_{\widetilde{\tau}}\mathsf{v}] = 0$ as a consequence of (2.10) and (2.14)$_3$, we arrive together with (2.14)$_4$ at

$$\mu[\partial_{\widetilde{\nu}}\mathsf{v}] - [\mathsf{q}]\widetilde{\nu} = [T_\mu(\mathsf{v}, \mathsf{q})]\widetilde{\nu} = 0. \tag{2.15}$$

Set $(\mathsf{v}, \mathsf{q}) := \mathbf{1}_{\Omega^+}(\mathsf{v}^+, \mathsf{q}^+) + \mathbf{1}_{\Omega^-}(\mathsf{v}^-, \mathsf{q}^-) \in \mathrm{L}^\infty(\mathbb{S} \times \mathbb{R}, \mathbb{R}^2 \times \mathbb{R})$. We then compute, in light of (2.10), (2.14)$_1$–(2.14)$_3$, and (2.15), that

$$\left.\begin{aligned} \mu\Delta\mathsf{v} - \nabla\mathsf{q} &= 0, \\ \operatorname{div}\mathsf{v} &= 0 \end{aligned}\right\} \quad \text{in } \mathcal{D}'(\mathbb{S} \times \mathbb{R}).$$

In particular, taking the divergence of the first equation yields $\Delta\mathsf{q} = 0$, hence, q is a harmonic function in $\mathbb{S} \times \mathbb{R}$. Since q is bounded, Liouville's theorem and (2.14)$_5$ now yield $\mathsf{q} = 0$ in $\mathbb{R}^2$. This in turn means that v_1 and v_2 are harmonic in $\mathbb{S} \times \mathbb{R}$, and, since v is bounded, we conclude together with (2.14)$_5$ that $\mathsf{v} = 0$ and $\mathsf{c}_1 = \mathsf{c}_2 = 0$, which proves the uniqueness claim.

Solution of the Stokes equations. To prove that $(v^\pm, q^\pm)$ actually solves the homogeneous Stokes equation, we fix $x_0 \in \Omega^\pm$ and choose $\varepsilon > 0$ such that the closed ball $\overline{B}_\varepsilon(x_0)$ is contained in $\Omega^\pm$. Since $(\mathcal{U}, \mathcal{P})(\cdot - (s, f(s))) \in \mathrm{C}^\infty(\Omega^\pm, \mathbb{R}^{2\times 2} \times \mathbb{R}^2)$, cf. (2.5) and (2.7), for each fixed $s \in \mathbb{S}$, the partial derivatives $\partial_x^\alpha \mathcal{U}_j^k(\cdot - (s, f(s)))$, $\partial_x^\alpha \mathcal{P}^k(\cdot - (s, f(s)))$, $\alpha \in \mathbb{N}^2$, are bounded in $\overline{B}_\varepsilon(x_0)$ uniformly in $s \in \mathbb{S}$. Therefore, the function $(v^\pm, q^\pm)$ is well-defined in (2.12)–(2.13) and we may interchange differentiation with respect to x and the integral sign in these formulas. Recalling that $(\mathcal{U}^k, \mathcal{P}^k)$, $k = 1, 2$, solve the Stokes equations (2.9) pointwise in $(\mathbb{S} \times \mathbb{R}) \setminus \{0\}$, it follows immediately that $(v^\pm, q^\pm)$ solve (2.1)$_1$-(2.1)$_2$ in $\Omega^\pm$.

Boundary conditions. The boundary conditions (2.1)$_3$–(2.1)$_4$ together with the far-field boundary condition (2.1)$_5$ for $(v^\pm, q^\pm)$ follow by combining the results of Lemmas C.4 and C.6 below. ∎

3. The evolution problem

In this section, we combine the results from Section 2, Appendices A and B to reformulate the moving boundary problem (1.1) as a fully nonlinear and nonlocal evolution problem for the function f, see (3.10) below, with nonlinearities defined in terms of (singular) integral operators. Then, exploiting estimates from Appendix A and the localization result in Lemma B.2, we prove that the evolution problem (3.10) is of parabolic type. This property together with the abstract parabolic theory from [24] is then used to establish our main results in Theorems 1.1 and 1.2.

3.1. Two classes of (singular) integral operators

In this section, we introduce two classes of (singular) integral operators $B_{n,m}^{p,q}$ and $C_{n,m}$, the operators $B_{n,m}^{p,q}$ (together with the integral operator B_0) constituting via (3.9) the main building blocks of the evolution operator in (3.10), while the operators $C_{n,m}$ (which in a suitable sense retain the singular part of the operators $B_{n,m}^{p,q}$ with $p = 0$) are important in the analysis of (3.10).

To start, we define for integers $m, n, p, q \in \mathbb{N}_0$ satisfying $p \le n + q + 1$, and Lipschitz continuous mappings

$$\mathbf{a} = (a_1, \ldots, a_m) : \mathbb{R} \to \mathbb{R}^m, \quad \mathbf{b} = (b_1, \ldots, b_n) : \mathbb{R} \to \mathbb{R}^n,$$
$$\mathbf{c} = (c_1, \ldots, c_q) : \mathbb{R} \to \mathbb{R}^q$$

the integral operators

$$B_{n,m}^{p,q}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi](\xi) := \frac{1}{2\pi}\mathrm{PV}\int_{-\pi}^{\pi} \frac{\prod_{i=1}^{n} \frac{T_{[\xi,s]}b_i}{t_{[s]}} \prod_{i=1}^{q} \frac{\delta_{[\xi,s]}c_i/2}{t_{[s]}}}{\prod_{i=1}^{m}\big[1 + \big(\frac{T_{[\xi,s]}a_i}{t_{[s]}}\big)^2\big]} \frac{\varphi(\xi - s)}{t_{[s]}} t_{[s]}^p \, \mathrm{d}s \tag{3.1}$$

and

$$C_{n,m}(\mathbf{a})[\mathbf{b}, \varphi](\xi) := \frac{1}{\pi}\mathrm{PV}\int_{-\pi}^{\pi} \frac{\prod_{i=1}^{n} \frac{\delta_{[\xi,s]}b_i}{s}}{\prod_{i=1}^{m}\big[1 + \big(\frac{\delta_{[\xi,s]}a_i}{s}\big)^2\big]} \frac{\varphi(\xi - s)}{s} \, \mathrm{d}s, \tag{3.2}$$

where $\varphi \in \mathrm{L}^2(\mathbb{S})$ and $\xi \in \mathbb{R}$. We use the notation introduced in (2.6) together with the shorthand

$$\delta_{[\xi,s]}f := f(\xi) - f(\xi - s), \quad T_{[\xi,s]}f := \tanh\Big(\frac{\delta_{[\xi,s]}f}{2}\Big), \qquad \xi, s \in \mathbb{R}. \tag{3.3}$$

As shown in Lemma A.2 below, the PV symbol is not needed in (3.1) if $p \ge 1$.

Moreover, we point out that if the functions $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are 2π-periodic, then so are also the mappings $B_{n,m}^{p,q}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi]$ and $C_{n,m}(\mathbf{a})[\mathbf{b}, \varphi]$. In particular,

$$B_{0,0}^{0,0}[\varphi](\xi) = \frac{1}{2\pi}\mathrm{PV}\int_{-\pi}^{\pi} \frac{\varphi(\xi - s)}{t_{[s]}} \, \mathrm{d}s = H[\varphi](\xi), \quad \xi \in \mathbb{R}, \tag{3.4}$$

where H is the periodic Hilbert transform, see, e.g., [4, 37]. Mapping properties for the operators $B_{n,m}^{p,q}$ and $C_{n,m}$ are established in Appendix A.

If all coordinate functions of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are identical to a given function $f \in \mathrm{W}^{1,\infty}(\mathbb{S})$, we set

$$B_{n,m}^{p,q}(f) := B_{n,m}^{p,q}(f, \ldots, f | f, \ldots, f)[f, \ldots, f, \cdot], \quad 0 \le p \le n + q + 1, \tag{3.5}$$

respectively,

$$C_{n,m}^{0}(f) := C_{n,m}(f, \ldots, f)[f, \ldots, f, \cdot]. \tag{3.6}$$

The operators $B_{n,m}^{p,q}(f)$ appear in the reformulation (3.10) of the Stokes problem and the operators $C_{n,m}^{0}(f)$ are used in its analysis.

Finally, we introduce a further integral operator B_0 by setting

$$B_0(f)[\varphi](\xi) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left(\frac{t_{[s]}^2 + (T_{[\xi,s]}f)^2}{(1+t_{[s]}^2)(1-(T_{[\xi,s]}f)^2)}\right)\varphi(\xi - s)\,\mathrm{d}s, \quad \xi \in \mathbb{S}, \tag{3.7}$$

where again $f \in \mathrm{W}^{1,\infty}(\mathbb{S})$ and $\varphi \in \mathrm{L}^2(\mathbb{S})$.

As shown in Corollary A.8, given $r \in (3/2, 2)$, the mappings

$$\begin{aligned} &\big[f \mapsto B_{n,m}^{0,q}(f)\big] : \mathrm{H}^r(\mathbb{S}) \to \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S})), \\ &\big[f \mapsto B_0(f)\big], \big[f \mapsto B_{n,m}^{p,q}(f)\big] : \mathrm{H}^r(\mathbb{S}) \to \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}), \mathrm{H}^r(\mathbb{S})), \quad 1 \le p \le n+q+1, \end{aligned} \tag{3.8}$$

are smooth. These properties are essential in the study of (3.10).

Finally, we introduce the operators

$$\begin{aligned} B_1(f) &:= B_{0,1}^{0,0}(f) - B_{2,1}^{2,0}(f), \\ B_2(f) &:= B_{1,1}^{0,0}(f) + B_{1,1}^{2,0}(f), \\ B_3(f) &:= B_{0,2}^{0,1}(f) + B_{0,2}^{2,1}(f) - B_{2,2}^{0,1}(f) - 2B_{2,2}^{2,1}(f) \\ &\quad - B_{2,2}^{4,1}(f) + B_{4,2}^{2,1}(f) + B_{4,2}^{4,1}(f), \\ B_4(f) &:= B_{1,2}^{0,1}(f) + B_{1,2}^{2,1}(f) - B_{3,2}^{2,1}(f) - B_{3,2}^{4,1}(f), \\ B_5(f) &:= 2\big(B_{0,1}^{1,1}(f) - B_{2,1}^{3,1}(f)\big), \\ B_6(f) &:= 2\big(B_{1,1}^{1,1}(f) + B_{1,1}^{3,1}(f)\big), \end{aligned} \tag{3.9}$$

which appear in a natural way in the analysis, see (C.19) and (C.21).

3.2. The reformulation of the Stokes problem (1.1)

Let $(f, v^\pm, q^\pm)$ be a solution to (1.1) enjoying the regularity properties in Theorem 1.1 (i). Since $f(t) \in \mathrm{H}^3(\mathbb{S})$, and consequently $G(f(t)) \in \mathrm{H}^1(\mathbb{S})^2$, see (2.4), for all $t > 0$, we infer from Remark 2.1 and Theorem 2.2 that the function $(v^\pm(t), q^\pm(t))$ is identified by the system $(1.1a)_1$–$(1.1a)_5$ and (1.1c) according to (2.12)–(2.13). Together with the kinematic boundary condition $(1.1a)_6$ and the formulas (C.19), and (C.21) for the trace of the velocities v_G, we deduce that f solves the evolution problem

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t) = \Psi(f(t)), \quad t > 0, \quad f(0) = f_0, \tag{3.10}$$

where the operator Ψ is defined by

$$\Psi(f) := \frac{\sigma}{4\mu} f'\Psi_1(f) + \frac{\Theta}{4\mu} f'\Psi_3(f) - \frac{\sigma}{4\mu}\Psi_2(f) + \frac{\Theta}{4\mu}\Psi_4(f) + \frac{\Theta \ln(4)}{4\mu}\langle f\rangle, \tag{3.11}$$

with

$$
\begin{aligned}
\Psi_1(f) &:= (B_1 - 2B_4)(f)[\phi_1(f) - f'\phi_2(f)] \\
&\qquad + (2B_2 + B_3)(f)[f'\phi_1(f)] + B_3(f)[\phi_2(f)], \\
\Psi_2(f) &:= B_1(f)[\phi_2(f) - f'\phi_1(f)] \\
&\qquad + B_3(f)[\phi_1(f) - f'\phi_2(f)] + 2B_4(f)[f'\phi_1(f) + \phi_2(f)], \\
\Psi_3(f) &:= (B_0(f) + B_6(f))[ff'] + B_5(f)[f], \\
\Psi_4(f) &:= (B_0(f) - B_6(f))[f] + B_5(f)[ff'],
\end{aligned}
\tag{3.12}
$$

where $\phi = \phi(f)$ is given by

$$\phi(f) := (\phi_1(f), \phi_2(f)) := (\omega(f)^{-1} - 1, f'\omega(f)^{-1}). \tag{3.13}$$

The function $\phi = \phi(f)$ appears in the definition of $G = G(f)$ in (2.4).

Let $r \in (3/2, 2)$ be fixed in the following. As an important observation, we note that the right-hand side of (3.10) is well-defined for all functions f which belong to $\mathrm{H}^r(\mathbb{S})$. In order to study (3.10), we first establish the following result.

Lemma 3.1. *Given $r \in (3/2, 2)$, we have $\phi \in \mathrm{C}^\infty(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})^2)$ and the Fréchet derivative $\partial\phi(f_0) = (\partial\phi_1(f_0), \partial\phi_2(f_0))$, $f_0 \in \mathrm{H}^r(\mathbb{S})$, satisfies*

$$\partial\phi_i(f_0) = a_i(f_0)\frac{\mathrm{d}}{\mathrm{d}\xi} \in \mathcal{L}(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})), \quad i = 1, 2, \tag{3.14}$$

where $a_i(f_0) \in \mathrm{H}^{r-1}(\mathbb{S})$ are given by

$$a_1(f_0) := -\frac{f_0'}{(1 + f_0'^2)^{3/2}} \quad \text{and} \quad a_2(f_0) := \frac{1}{(1 + f_0'^2)^{3/2}}. \tag{3.15}$$

Proof. The proof is similar to that of [27, Lemma 3.5]. ∎

Combining (3.9), (3.11), (3.12), Lemma 3.1 and Corollary A.8, we conclude that

$$\Psi \in \mathrm{C}^\infty(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})). \tag{3.16}$$

3.3. The Fréchet derivative

In order to apply the abstract parabolic theory from [24, Chapter 8] in the context of (3.10), which we now view as an evolution equation in the ambient space $\mathrm{H}^{r-1}(\mathbb{S})$, it remains to show that the Fréchet derivative $\partial\Psi(f_0) \in \mathcal{L}(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S}))$ generates a strongly continuous analytic semigroup in $\mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}))$. This is the content of the next result (where we use notation from [2]).

Proposition 3.2. *Given $f_0 \in \mathrm{H}^r(\mathbb{S})$, we have*

$$-\partial\Psi(f_0) \in \mathcal{H}(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})). \tag{3.17}$$

For the remainder of this section and in Section 3.4, we fix $f_0 \in \mathrm{H}^r(\mathbb{S})$, $r \in (3/2, 2)$. The proof of Proposition 3.2 requires some preparation. To start, we infer from (3.11) that

$$\begin{aligned}\partial\Psi(f_0)[f] &= \frac{1}{4\mu}(\sigma\Psi_1(f_0) + \Theta\Psi_3(f_0))f' + \frac{\sigma}{4\mu}(f_0'\partial\Psi_1(f_0) - \partial\Psi_2(f_0))[f] \\ &\quad + \frac{\Theta}{4\mu}(\partial\Psi_4(f_0) + f_0'\partial\Psi_3(f_0))[f] + \frac{\Theta\ln(4)}{4\mu}\langle f\rangle, \qquad f \in \mathrm{H}^r(\mathbb{S}).\end{aligned} \tag{3.18}$$

The terms on the second line of (3.18) are lower order perturbations. To quantify this, let $r' \in (3/2, r)$ be fixed in the following. Recalling (3.9) and (3.12), Corollary A.8 (with r replaced by r') yields

$$\partial B_i(f_0) \in \mathcal{L}(\mathrm{H}^{r'}(\mathbb{S}), \mathcal{L}(\mathrm{H}^{r'-1}(\mathbb{S}), \mathrm{H}^{r'}(\mathbb{S}))), \quad i \in \{0, 5, 6\},$$

and therefore,

$$\|\partial\Psi_i(f_0)[f]\|_{\mathrm{H}^{r-1}} \le C\|f\|_{\mathrm{H}^{r'}}, \quad f \in \mathrm{H}^r(\mathbb{S}), \quad i \in \{3, 4\}. \tag{3.19}$$

Moreover, we clearly have

$$\|\langle f\rangle\|_{\mathrm{H}^{r-1}} \le C\|f\|_1 \le C\|f\|_{\mathrm{H}^{r'}}, \quad f \in \mathrm{H}^r(\mathbb{S}). \tag{3.20}$$

The first two terms on the right-hand side of (3.18) are easy to handle as they are first order differential operators, however, the next two terms are more intricate. To analyze them we first compute, for some fixed $\varphi_0 \in \mathrm{H}^{r-1}(\mathbb{S})$, by combining (3.9), (A.2), (A.10), (A.32), Lemma A.2, and Corollary A.8 that

$$\begin{aligned}\partial B_1(f_0)[f][\varphi_0] &= -2\varphi_0 C^0_{1,2}(f_0)[f'] + R_1[f], \\ \partial B_2(f_0)[f][\varphi_0] &= \varphi_0\big(C^0_{0,1} - 2C^0_{2,2}\big)(f_0)[f'] + R_2[f], \\ \partial B_3(f_0)[f][\varphi_0] &= \varphi_0\big(C^0_{0,2} - 3C^0_{2,2} - 4C^0_{2,3} + 4C^0_{4,3}\big)(f_0)[f'] + R_3[f], \\ \partial B_4(f_0)[f][\varphi_0] &= \varphi_0\big(2C^0_{1,2} - 4C^0_{3,3}\big)(f_0)[f'] + R_4[f]\end{aligned} \tag{3.21}$$

for all $f \in \mathrm{H}^r(\mathbb{S})$, where

$$\|R_i[f]\|_{\mathrm{H}^{r-1}} \le C\|f\|_{\mathrm{H}^{r'}}, \quad f \in \mathrm{H}^r(\mathbb{S}), \quad 1 \le i \le 4. \tag{3.22}$$

Moreover, (3.9), (A.2), Lemma A.2, and Corollary A.8 entail that

$$\begin{aligned}B_1(f_0)[\varphi] &= C^0_{0,1}(f_0)[\varphi] + \widetilde{R}_1[\varphi], \\ B_2(f_0)[\varphi] &= C^0_{1,1}(f_0)[\varphi] + \widetilde{R}_2[\varphi], \\ B_3(f_0)[\varphi] &= \big(C^0_{1,2} - C^0_{3,2}\big)(f_0)[\varphi] + \widetilde{R}_3[\varphi], \\ B_4(f_0)[\varphi] &= C^0_{2,2}(f_0)[\varphi] + \widetilde{R}_4[\varphi]\end{aligned} \tag{3.23}$$

for all $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, where

$$\|\widetilde{R}_i[\varphi]\|_{\mathrm{H}^{r-1}} \le C\|\varphi\|_{\mathrm{H}^{r'-1}}, \quad \varphi \in \mathrm{H}^{r-1}(\mathbb{S}), \quad 1 \le i \le 4. \tag{3.24}$$

Setting

$$a_i := a_i(f_0), \quad \phi_i := \phi_i(f_0), \quad i = 1, 2, \tag{3.25}$$

see (3.13) and (3.15), we infer from (3.12), (3.14), (3.21)-(3.25), and the algebraic relation

$$C^0_{n,m}(f_0) + C^0_{n+2,m}(f_0) = C^0_{n,m-1}(f_0), \quad m \geq 1,$$

that

$$\partial\Psi_i(f_0)[f] = T_i(f_0)[f] + T_{i,\mathrm{lot}}(f_0)[f], \quad i = 1, 2, \tag{3.26}$$

where

$$\begin{aligned}
T_1(f_0)[f] :=& \big(C^0_{0,2} - C^0_{2,2}\big)(f_0)[(a_1 - \phi_2 - f_0'a_2)f'] \\
&+ C^0_{1,2}(f_0)[(3(\phi_1 + f_0'a_1) + a_2)f'] \\
&+ C^0_{3,2}(f_0)[(\phi_1 + f_0'a_1 - a_2)f'] \\
&+ \phi_1\big(3f_0'C^0_{0,3} - 6C^0_{1,3} - 6f_0'C^0_{2,3} + 2C^0_{3,3} - f_0'C^0_{4,3}\big)(f_0)[f'] \\
&+ \phi_2\big(C^0_{0,3} + 6f_0'C^0_{1,3} - 6C^0_{2,3} - 2f_0'C^0_{3,3} + C^0_{4,3}\big)(f_0)[f'], \\
T_2(f_0)[f] :=& \big(C^0_{1,2} - C^0_{3,2}\big)(f_0)[(a_1 + \phi_2 - f_0'a_2)f'] - C^0_{0,2}(f_0)[(\phi_1 + f_0'a_1 - a_2)f'] \\
&+ C^0_{2,2}(f_0)[(\phi_1 + f_0'a_1 + 3a_2)f'] \\
&+ \phi_1\big(C^0_{0,3} + 6f_0'C^0_{1,3} - 6C^0_{2,3} - 2f_0'C^0_{3,3} + C^0_{4,3}\big)(f_0)[f'] \\
&+ \phi_2\big(-f_0'C^0_{0,3} + 2C^0_{1,3} + 6f_0'C^0_{2,3} - 6C^0_{3,3} - f_0'C^0_{4,3}\big)(f_0)[f']
\end{aligned} \tag{3.27}$$

and

$$\|T_{i,\mathrm{lot}}(f_0)[f]\|_{\mathrm{H}^{r-1}} \leq C\|f\|_{\mathrm{H}^{r'}}, \quad f \in \mathrm{H}^r(\mathbb{S}), \quad i = 1, 2. \tag{3.28}$$

3.4. Localization of the Fréchet derivative

Using the formulas for $\partial\Psi(f_0)$ provided in Section 3.3 and inspired by the papers [9, 10, 25], we prove in this section that the Fréchet derivative $\partial\Psi(f_0)$ can be locally approximated by certain Fourier multipliers which are themselves generators of strongly continuous analytic semigroups, see Proposition 3.3 and (3.40)–(3.41). The proof of Proposition 3.2 relies heavily on these properties and concludes this section.

To start, we choose for each $\varepsilon \in (0, 1)$ a set of smooth functions $\{\pi^\varepsilon_j : 1 \leq j \leq N\}$ in $\mathrm{C}^\infty(\mathbb{S}, [0, 1])$, where the integer $N = N(\varepsilon)$ is sufficiently large, such that

$$\begin{aligned}
&\operatorname{supp}\pi^\varepsilon_j = I^\varepsilon_j + 2\pi\mathbb{Z} \quad \text{with } I^\varepsilon_j := [x^\varepsilon_j - \varepsilon, x^\varepsilon_j + \varepsilon],\ x^\varepsilon_j := j\varepsilon; \\
&\sum_{j=1}^N \pi^\varepsilon_j = 1 \quad \text{in } \mathrm{C}^\infty(\mathbb{S}).
\end{aligned} \tag{3.29}$$

We call $\{\pi^\varepsilon_j : 1 \leq j \leq N\}$ an ε-partition of unity. Moreover, associated to each ε-partition of unity, we choose a further set $\{\chi^\varepsilon_j : 1 \leq j \leq N\} \subset \mathrm{C}^\infty(\mathbb{S}, [0, 1])$ with

$$\begin{aligned}
&\operatorname{supp}\chi^\varepsilon_j = J^\varepsilon_j + 2\pi\mathbb{Z} \quad \text{with } J^\varepsilon_j = [x^\varepsilon_j - 2\varepsilon, x^\varepsilon_j + 2\varepsilon]; \\
&\chi^\varepsilon_j = 1 \quad \text{on } \operatorname{supp}\pi^\varepsilon_j.
\end{aligned} \tag{3.30}$$

Each ε-partition of unity allows us to define a new norm on $\mathrm{H}^s(\mathbb{S})$, $s \geq 0$, via the mapping

$$\Big[f \mapsto \sum_{j=1}^{N} \|\pi_j^\varepsilon f\|_{\mathrm{H}^s}\Big] : \mathrm{H}^s(\mathbb{S}) \to \mathbb{R},$$

which is equivalent to the standard norm. Indeed, it is straightforward to show there exists a constant $c = c(\varepsilon, s) \in (0, 1)$ such that

$$c\|f\|_{\mathrm{H}^s} \leq \sum_{j=1}^{N} \|\pi_j^\varepsilon f\|_{\mathrm{H}^s} \leq c^{-1}\|f\|_{\mathrm{H}^s}, \quad f \in \mathrm{H}^s(\mathbb{S}). \tag{3.31}$$

Following [27], we introduce the continuous path $\Phi : [0, 1] \to \mathcal{L}(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S}))$ defined by

$$\begin{aligned}\Phi(\tau) := {} & \frac{\tau}{4\mu}(\sigma\Psi_1(f_0) + \Theta\Psi_3(f_0))\frac{\mathrm{d}}{\mathrm{d}\xi} + \frac{\sigma}{4\mu}(\tau f_0'\partial\Psi_1(\tau f_0) - \partial\Psi_2(\tau f_0)) \\ & + \frac{\tau\Theta}{4\mu}(\partial\Psi_4(f_0) + f_0'\partial\Psi_3(f_0)) + \frac{\tau\Theta\ln(4)}{4\mu}\langle\cdot\rangle, \quad \tau \in [0, 1],\end{aligned} \tag{3.32}$$

which connects the Fréchet derivative $\partial\Psi(f_0) = \Phi(1)$, see (3.18), to the Fourier multiplier

$$\Phi(0) = -\frac{\sigma}{4\mu}H \circ \frac{\mathrm{d}}{\mathrm{d}\xi} = -\frac{\sigma}{4\mu}\Big(-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\Big)^{1/2}, \tag{3.33}$$

see (3.4). In (3.33), we use that H is a Fourier multiplier with symbol $(-i\,\mathrm{sign}(k))_{k\in\mathbb{Z}}$.

The homotopy Φ will be used to conclude invertibility of $\lambda - \Phi(1)$ from $\lambda - \Phi(0)$ for $\mathrm{Re}\,\lambda$ large enough. In the arguments below, we use the estimate

$$\|fg\|_{\mathrm{H}^s} \leq C(\|f\|_\infty\|g\|_{\mathrm{H}^s} + \|g\|_\infty\|f\|_{\mathrm{H}^s}), \quad \text{with } s \in (1/2, 1), \tag{3.34}$$

which holds for all $f, g \in \mathrm{H}^s(\mathbb{S})$. The following proposition shows that the operator $\Phi(\tau)$ can be locally approximated by certain Fourier multipliers for all $\tau \in [0, 1]$.

Proposition 3.3. *Let $\gamma > 0$ and $3/2 < r' < r < 2$. Then, there exists $\varepsilon \in (0, 1)$ together with an ε-partition of unity $\{\pi_j^\varepsilon : 1 \leq j \leq N\}$, a constant $K = K(\varepsilon)$, and bounded operators*

$$\mathbb{A}_{j,\tau} \in \mathcal{L}(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})), \quad j \in \{1, \ldots, N\}, \quad \tau \in [0, 1],$$

such that

$$\|\pi_j^\varepsilon\Phi(\tau)[f] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon f]\|_{\mathrm{H}^{r-1}} \leq \gamma\|\pi_j^\varepsilon f\|_{\mathrm{H}^r} + K\|f\|_{\mathrm{H}^{r'}} \tag{3.35}$$

for all $j \in \{1, \ldots, N\}$, $f \in \mathrm{H}^r(\mathbb{S})$, and $\tau \in [0, 1]$. The operators $\mathbb{A}_{j,\tau}$ are defined by

$$\mathbb{A}_{j,\tau} := -\alpha_\tau(x_j^\varepsilon)\Big(-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\Big)^{1/2} + \beta_\tau(x_j^\varepsilon)\frac{\mathrm{d}}{\mathrm{d}\xi}, \quad j \in \{1, \ldots, N\}, \quad \tau \in [0, 1],$$

with the functions α_τ, β_τ given by (see (2.2))

$$\alpha_\tau := \frac{\sigma}{4\mu}\omega^{-1}(\tau f_0) \quad \text{and} \quad \beta_\tau := \frac{\tau}{4\mu}(\sigma\Psi_1(f_0) + \Theta\Psi_3(f_0)). \tag{3.36}$$

Proof. Let $\varepsilon \in (0,1)$ and let $\{\pi_j^\varepsilon : 1 \le j \le N\}$ be an ε-partition of unity with the associated set $\{\chi_j^\varepsilon : 1 \le j \le N\}$ satisfying (3.30). In the following, we use the symbol C for constants that are independent of ε and denote constants that depend on ε by K. Recalling (3.19) and (3.20), the algebra property of $\mathrm{H}^{r-1}(\mathbb{S})$ leads us to

$$\Big\| \pi_j^\varepsilon \Big(\frac{\tau\Theta}{4\mu}\big(\partial\Psi_4(f_0) + f_0'\partial\Psi_3(f_0)\big)[f] + \frac{\tau\Theta\ln(4)}{4\mu}\langle f\rangle \Big) \Big\|_{\mathrm{H}^{r-1}} \le K\|f\|_{\mathrm{H}^{r'}} \tag{3.37}$$

for all $j \in \{1,\ldots,N\}$, $f \in \mathrm{H}^r(\mathbb{S})$, and $\tau \in [0,1]$.

Moreover, since $\Psi_k(f_0) \in \mathrm{H}^{r-1}(\mathbb{S}) \hookrightarrow \mathrm{C}^{r-3/2}(\mathbb{S})$, $k \in \{1,3\}$, we use $(3.30)_2$ and (3.34) to derive that

$$\begin{aligned} \|\pi_j^\varepsilon \beta_\tau f' - \beta_\tau(x_j^\varepsilon)(\pi_j^\varepsilon f)'\|_{\mathrm{H}^{r-1}} \le{}& C\|(\Psi_1(f_0) - \Psi_1(f_0)(x_j^\varepsilon))(\pi_j^\varepsilon f)'\|_{\mathrm{H}^{r-1}} \\ &+ C\|(\Psi_3(f_0) - \Psi_3(f_0)(x_j^\varepsilon))(\pi_j^\varepsilon f)'\|_{\mathrm{H}^{r-1}} \\ &+ K\|f\|_{\mathrm{H}^{r-1}}, \end{aligned}$$

where

$$\begin{aligned} &C\|(\Psi_k(f_0) - \Psi_k(f_0)(x_j^\varepsilon))(\pi_j^\varepsilon f)'\|_{\mathrm{H}^{r-1}} \\ &\quad \le C\|\chi_j^\varepsilon(\Psi_k(f_0) - \Psi_k(f_0)(x_j^\varepsilon))\|_\infty \|(\pi_j^\varepsilon f)'\|_{\mathrm{H}^{r-1}} + K\|f\|_{\mathrm{H}^{r'}} \\ &\quad \le \frac{\gamma}{4}\|\pi_j^\varepsilon f\|_{\mathrm{H}^r} + K\|f\|_{\mathrm{H}^{r'}}, \quad k \in \{1,3\}, \end{aligned}$$

for all $j \in \{1,\ldots,N\}$, $f \in \mathrm{H}^r(\mathbb{S})$, and $\tau \in [0,1]$, provided that ε is sufficiently small, and therefore,

$$\Big\| \pi_j^\varepsilon \Big(\frac{\tau\sigma}{4\mu}\Psi_1(f_0) + \frac{\tau\Theta}{4\mu}\Psi_3(f_0) \Big) f' - \beta_\tau(x_j^\varepsilon)(\pi_j^\varepsilon f)' \Big\|_{\mathrm{H}^{r-1}} \le \frac{\gamma}{2}\|\pi_j^\varepsilon f\|_{\mathrm{H}^r} + K\|f\|_{\mathrm{H}^{r'}}. \tag{3.38}$$

Finally, (3.26)–(3.28) and repeated use of Lemma B.2 lead us to

$$\begin{aligned} &\Big\| \pi_j^\varepsilon \frac{\sigma}{4\mu}\big(\tau f_0'\partial\Psi_1(\tau f_0) - \partial\Psi_2(\tau f_0)\big)[f] - \alpha_\tau(x_j^\varepsilon)H[(\pi_j^\varepsilon f)'] \Big\|_{\mathrm{H}^{r-1}} \\ &\quad \le \frac{\gamma}{2}\|\pi_j^\varepsilon f\|_{\mathrm{H}^r} + K\|f\|_{\mathrm{H}^{r'}} \end{aligned} \tag{3.39}$$

for all $j \in \{1,\ldots,N\}$, $f \in \mathrm{H}^r(\mathbb{S})$, and $\tau \in [0,1]$, provided that ε is sufficiently small.

Gathering (3.37)–(3.39), the claim follows in view of (3.32). ■

Since $\Psi_k(f_0) \in \mathrm{H}^{r-1}(\mathbb{S})$, $k \in \{1,3\}$, there exists a constant $\eta \in (0,1)$ such that the functions α_τ and β_τ defined in (3.36) satisfy

$$\eta \le \alpha_\tau \le \eta^{-1}, \quad |\beta_\tau| \le \eta^{-1}, \quad \tau \in [0,1].$$

Introducing the Fourier multiplier

$$\mathbb{A}_{\alpha,\beta} := -\alpha\Big(-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\Big)^{1/2} + \beta\frac{\mathrm{d}}{\mathrm{d}\xi} \in \mathcal{L}(\mathrm{H}^s(\mathbb{S}), \mathrm{H}^{s-1}(\mathbb{S})), \quad \alpha \in [\eta,\eta^{-1}],\ \beta \in [-\eta^{-1},\eta^{-1}],$$

it is straightforward to show, by using Fourier analysis techniques, that for all $\alpha \in [\eta, \eta^{-1}]$ and $\beta \in [-\eta^{-1}, \eta^{-1}]$,

$$\lambda - \mathbb{A}_{\alpha,\beta} : \mathrm{H}^r(\mathbb{S}) \to \mathrm{H}^{r-1}(\mathbb{S}) \text{ is an isomorphism for all } \operatorname{Re}\lambda \geq 1. \tag{3.40}$$

Moreover, there exists a constant $\kappa_0 = \kappa_0(\eta) \geq 1$ with the property that for all $\alpha \in [\eta, \eta^{-1}]$ and $\beta \in [-\eta^{-1}, \eta^{-1}]$,

$$\kappa_0 \|(\lambda - \mathbb{A}_{\alpha,\beta})[f]\|_{\mathrm{H}^{r-1}} \geq |\lambda| \|f\|_{\mathrm{H}^{r-1}} + \|f\|_{\mathrm{H}^r}, \quad f \in \mathrm{H}^r(\mathbb{S}), \quad \operatorname{Re}\lambda \geq 1. \tag{3.41}$$

The relations (3.40)–(3.41) imply, in particular, that the operator $\mathbb{A}_{\alpha,\beta}$ generates a strongly continuous analytic semigroup, cf. [2, Section I.1.2]. Moreover, together with Proposition 3.3 and the interpolation property

$$[\mathrm{H}^{s_0}(\mathbb{S}), \mathrm{H}^{s_1}(\mathbb{S})]_\theta = \mathrm{H}^{(1-\theta)s_0+\theta s_1}(\mathbb{S}), \quad \theta \in (0,1), \quad -\infty < s_0 \leq s_1 < \infty, \tag{3.42}$$

where $[\cdot,\cdot]_\theta$ is the complex interpolation functor, they enable us to prove Proposition 3.2.

Proof of Proposition 3.2. Let $r' \in (3/2, r)$ and $\kappa_0 \geq 1$ be the constant in (3.41). We may use Proposition 3.3 with $\gamma := 1/2\kappa_0$ to find $\varepsilon \in (0,1)$, an ε-partition of unity $\{\pi_j^\varepsilon : 1 \leq j \leq N\}$, a constant $K = K(\varepsilon) > 0$, operators $\mathbb{A}_{j,\tau} \in \mathcal{L}(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S}))$, $1 \leq j \leq N$ and $\tau \in [0,1]$, satisfying

$$2\kappa_0 \|\pi_j^\varepsilon \Phi(\tau)[f] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon f]\|_{\mathrm{H}^{r-1}} \leq \|\pi_j^\varepsilon f\|_{\mathrm{H}^r} + 2\kappa_0 K \|f\|_{\mathrm{H}^{r'}}, \quad f \in \mathrm{H}^r(\mathbb{S}).$$

Furthermore, (3.41) yields for all $1 \leq j \leq N$, $\tau \in [0,1]$, and $\operatorname{Re}\lambda \geq 1$

$$2\kappa_0 \|(\lambda - \mathbb{A}_{j,\tau})[\pi_j^\varepsilon f]\|_{\mathrm{H}^{r-1}} \geq 2|\lambda| \|\pi_j^\varepsilon f\|_{\mathrm{H}^{r-1}} + 2\|\pi_j^\varepsilon f\|_{\mathrm{H}^r}, \quad f \in \mathrm{H}^r(\mathbb{S}).$$

Combining the above inequalities, we get

$$\begin{aligned} &2\kappa_0 \left\|\pi_j^\varepsilon(\lambda - \Phi(\tau))[f]\right\|_{\mathrm{H}^{r-1}} \\ &\quad \geq 2\kappa_0 \left\|(\lambda - \mathbb{A}_{j,\tau})[\pi_j^\varepsilon f]\right\|_{\mathrm{H}^{r-1}} - 2\kappa_0 \left\|\pi_j^\varepsilon \Phi(\tau)[f] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon f]\right\|_{\mathrm{H}^{r-1}} \\ &\quad \geq 2|\lambda| \left\|\pi_j^\varepsilon f\right\|_{\mathrm{H}^{r-1}} + \left\|\pi_j^\varepsilon f\right\|_{\mathrm{H}^r} - 2\kappa_0 K \|f\|_{\mathrm{H}^{r'}}. \end{aligned}$$

Using (3.31), (3.42), and Young's inequality we conclude that there exist constants $\kappa \geq 1$ and $\omega > 1$ such that

$$\kappa \|(\lambda - \Phi(\tau))[f]\|_{\mathrm{H}^{r-1}} \geq |\lambda| \, \|f\|_{\mathrm{H}^{r-1}} + \|f\|_{\mathrm{H}^r} \tag{3.43}$$

for all $\tau \in [0,1]$, $\operatorname{Re}\lambda \geq \omega$, and $f \in \mathrm{H}^r(\mathbb{S})$.

Since $\omega - \Phi(0) = \omega - \mathbb{A}_{\sigma/4\mu,0}$ is an isomorphism, see (3.33) and (3.40), the method of continuity, cf. [2, Proposition I.1.1.1] and (3.43) imply that $\omega - \Phi(1) = \omega - \partial\Psi(f_0)$ is also an isomorphism. This property combined with (3.43) (with $\tau = 1$) leads us to the desired conclusion, see [2, Section I.1.2]. ∎

3.5. Proof of the main results

This section is devoted to the proof of the main results in Theorems 1.1 and 1.2.

Proof of Theorem 1.1. The proof follows from (3.16) and Proposition 3.2, by using the abstract parabolic theory in [24, Chapter 8]. Given the substantial resemblance of the arguments to those in the non-periodic case, see the proof of [27, Theorem 3.2], we refrain from presenting them herein. ∎

It remains to establish Theorem 1.2. For this we define

$$\widehat{\mathrm{H}}^s(\mathbb{S}) := \big\{f \in \mathrm{H}^s(\mathbb{S}) : \langle f \rangle = 0\big\}, \quad s \geq 0,$$

and infer from (1.4) and (3.10) that $\Psi(f) \in \widehat{\mathrm{H}}^{r-1}(\mathbb{S})$ for all $f \in \mathrm{H}^r(\mathbb{S})$, $r \in (3/2, 2)$. Therefore, the mapping

$$\widehat{\Psi} := \Psi\big|_{\widehat{\mathrm{H}}^r(\mathbb{S})} : \widehat{\mathrm{H}}^r(\mathbb{S}) \to \widehat{\mathrm{H}}^{r-1}(\mathbb{S})$$

is well-defined and smooth, see (3.16). Moreover, for initial data $f_0 \in \widehat{\mathrm{H}}^r(\mathbb{S})$, the evolution problem (3.10) is equivalent to

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t) = \widehat{\Psi}(f(t)), \quad t > 0, \quad f(0) = f_0, \tag{3.44}$$

which is also of parabolic type. Indeed, given $f_0 \in \widehat{\mathrm{H}}^r(\mathbb{S})$, the Fréchet derivative $\partial\widehat{\Psi}(f_0)$ is the generator of a strongly continuous analytic semigroup in $\mathcal{L}(\widehat{\mathrm{H}}^{r-1}(\mathbb{S}))$. This is a consequence of [2, Corollary I.1.6.3] since, observing that $\widehat{\mathrm{H}}^s(\mathbb{S})$ is the orthogonal complement of the set of constant functions in $\mathrm{H}^s(\mathbb{S})$, $s \geq 0$, we may interpret $\partial\Psi(f_0)$ as the matrix operator

$$\partial\Psi(f_0) = \begin{bmatrix} \partial\widehat{\Psi}(f_0) & 0 \\ 0 & 0 \end{bmatrix} : \widehat{\mathrm{H}}^r(\mathbb{S}) \oplus \mathbb{R} \to \widehat{\mathrm{H}}^{r-1}(\mathbb{S}) \oplus \mathbb{R}.$$

It thus remains to study the stability properties of the zero solution to (3.44). This is advantageous because in contrast to $\partial\Psi(0)$, the Fréchet derivative $\partial\widehat{\Psi}(0)$ does not have zero as an eigenvalue, when assuming (1.5), as the next lemma shows.

Lemma 3.4. *The spectrum* $\sigma(\partial\widehat{\Psi}(0))$ *of* $\partial\widehat{\Psi}(0) \in \mathcal{L}(\widehat{\mathrm{H}}^r(\mathbb{S}), \widehat{\mathrm{H}}^{r-1}(\mathbb{S}))$ *is given by*

$$\sigma\big(\partial\widehat{\Psi}(0)\big) = \Big\{-\frac{\sigma k^2 + \Theta}{4\mu|k|} : k \in \mathbb{N}\Big\}. \tag{3.45}$$

Proof. In view of (3.4), (3.7), (3.9), (3.12), (3.13), (3.18), (3.33), and Lemma 3.1, we have

$$\partial\Psi(0) = \frac{\Theta}{4\mu}B_0(0) - \frac{\sigma}{4\mu}\Big(-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\Big)^{1/2}.$$

The operator $B_0(0)$ is actually also a Fourier multiplier. Indeed, letting $S \in \mathcal{L}(\widehat{\mathrm{H}}^{r-1}(\mathbb{S}))$ denote the operator which associates to each function $f \in \widehat{\mathrm{H}}^{r-1}(\mathbb{S})$ its antiderivative, that is,

$$S[f](\xi) := \int_0^{\xi} f(s)\,\mathrm{d}s + \frac{1}{2\pi}\int_0^{2\pi} sf(s)\,\mathrm{d}s, \quad \xi \in \mathbb{S},$$

integration by parts leads to

$$B_0(0)[f](\xi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln(\sin^2(s/2))f(\xi - s)\,\mathrm{d}s = H[S[f]](\xi), \quad \xi \in \mathbb{S}.$$

The relation (3.45) is now an immediate consequence of the latter relation. ■

We are now in a position to prove Theorem 1.2, which is based on asymptotic theory for abstract parabolic problems from [24, Chapter 9].

Proof of Theorem 1.2. In order to establish (i), let (1.5) be satisfied. Then, in view of Lemma 3.4, we have

$$\sup\big\{\mathrm{Re}\,\lambda : \lambda \in \sigma(\partial\widehat{\Psi}(0))\big\} \le -\vartheta_0 < 0.$$

Therefore, the assumptions of [24, Theorem 9.1.2] are fulfilled in the context of the evolution problem (3.44) and, together with Theorem 1.1, we conclude Theorem 1.2 (i).

Concerning (ii), assume now that $\sigma + \Theta < 0$. Then,

$$\begin{cases} -\dfrac{\sigma+\Theta}{4\mu} \in \sigma_+(\partial\widehat{\Psi}(0)) := \sigma(\partial\widehat{\Psi}(0)) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > 0\}; \\ \inf\big\{\mathrm{Re}\,\lambda : \lambda \in \sigma_+(\partial\widehat{\Psi}(0))\big\} > 0. \end{cases}$$

A direct application of [24, Theorem 9.1.3] provides the desired instability result. ■

A. Some classes of (singular) integral operators

In this section, we establish several important mapping properties for the (singular) integral operators $B^{p,q}_{n,m}$, $C_{n,m}$, and B_0 introduced in (3.1), (3.2), and (3.7).

We start by relating the two families of singular integral operators $B^{0,q}_{n,m}$ and $C_{n,m}$. To this end, we define for integers $m, n, q \in \mathbb{N}_0$, $\ell \in \{1,2\}$, and Lipschitz continuous mappings $\mathbf{a} = (a_1,\ldots,a_m) : \mathbb{R} \to \mathbb{R}^m$, $\mathbf{b} = (b_1,\ldots,b_n) : \mathbb{R} \to \mathbb{R}^n$, $\mathbf{c} = (c_1,\ldots,c_q) : \mathbb{R} \to \mathbb{R}^q$ the integral operator

$$A^{\ell,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi](\xi) := \frac{1}{2\pi}\int_{-\pi}^{\pi}\Bigg[\frac{\prod_{i=1}^{n}\frac{T_{[\xi,s]}b_i}{t_{[s]}}\prod_{i=1}^{q}\frac{\delta_{[\xi,s]}c_i/2}{t_{[s]}}}{\prod_{i=1}^{m}\big[1+\big(\frac{T_{[\xi,s]}a_i}{t_{[s]}}\big)^2\big]}\frac{1}{t_{[s]}^{\ell}} - \frac{\prod_{i=1}^{n}\frac{\delta_{[\xi,s]}b_i/2}{s/2}\prod_{i=1}^{q}\frac{\delta_{[\xi,s]}c_i/2}{s/2}}{\prod_{i=1}^{m}\big[1+\big(\frac{\delta_{[\xi,s]}a_i/2}{s/2}\big)^2\big]}\frac{1}{(s/2)^{\ell}}\Bigg]\varphi(\xi-s)\,\mathrm{d}s, \tag{A.1}$$

where $\varphi \in \mathrm{L}^2(\mathbb{S})$ and $\xi \in \mathbb{R}$ (see (2.6) and (3.3)). The following relation:

$$B^{0,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi] = A^{1,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi] + C_{n+q,m}(\mathbf{a})[(\mathbf{b},\mathbf{c}),\varphi], \quad m,n,q \in \mathbb{N}_0, \tag{A.2}$$

and the fact that the operators $A^{\ell,q}_{n,m}$ are regularizing, see Lemma A.2, will enable us to transfer mapping properties obtained for the operators $C_{n,m}$, see Section A.1, to the operators $B^{0,q}_{n,m}$ (which have kernels with a higher degree of nonlinearity than the former). We note that $A^{\ell,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]$ is 2π-periodic if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ have this property.

Before establishing mapping properties for these operators, we collect below some useful elementary inequalities:

$$|\tanh(x)| \le |x| \quad \text{and} \quad |\tanh(x) - x| \le |x\tanh^2(x)|, \tag{A.3}$$

$$|y| \le |\tan(y)| \quad \text{and} \quad |\tan(y) - y| \le |y^2\tan(y)|, \tag{A.4}$$

$$\left|\frac{T_{[x,s]}d}{t_{[s]}}\right| \le \left|\frac{\delta_{[x,s]}d/2}{t_{[s]}}\right| \le \left|\frac{\delta_{[x,s]}d}{s}\right| \cdot \left|\frac{s/2}{t_{[s]}}\right| \le \|d'\|_\infty \left|\frac{s/2}{t_{[s]}}\right| \le \|d'\|_\infty, \quad d \in \mathrm{W}^{1,\infty}(\mathbb{S}), \tag{A.5}$$

for $x \in \mathbb{R}$, $y \in (-\pi/2,\pi/2)$ and $0 \ne s \in (-\pi,\pi)$.

When estimating these operators, the standard norm on $\mathrm{H}^r(\mathbb{S})$, defined by means of the Fourier transform, is not so practical. Instead of using this norm, we recall the classical identity $\mathrm{H}^r(\mathbb{S}) = \mathrm{W}^{r,2}(\mathbb{S})$ for all $r \ge 0$, cf., e.g., [34]. For non-integer $r > 0$ it holds that

$$\mathrm{W}^{r,2}(\mathbb{S}) := \{v \in \mathrm{W}^{[r],2}(\mathbb{S}) : [v]_{\mathrm{W}^{r,2}} < \infty\}, \quad r = [r] + \{r\},\ [r] \in \mathbb{N}_0,\ \{r\} \in (0,1),$$

where

$$[v]^2_{\mathrm{W}^{r,2}} := \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|v^{([r])}(\xi+y) - v^{([r])}(\xi)|^2}{|y|^{1+2\{r\}}}\,\mathrm{d}\xi\,\mathrm{d}y = \int_{-\pi}^{\pi} \frac{\|\tau_y v^{([r])} - v^{([r])}\|_2^2}{|y|^{1+2\{r\}}}\,\mathrm{d}y$$

and $\tau_y v := v(\cdot + y)$ is the left shift operator. The space $\mathrm{W}^{r,2}(\mathbb{S})$ is equipped with the norm

$$\|v\|^2_{\mathrm{W}^{r,2}} := \|v\|^2_{\mathrm{W}^{[r],2}} + [v]^2_{\mathrm{W}^{r,2}}.$$

Using Plancherel's identity, it is easy to verify the norms $\|\cdot\|_{\mathrm{H}^r}$ and $\|\cdot\|_{\mathrm{W}^{r,2}}$ are equivalent.

A.1. Estimates for the operators $C_{n,m}$

In Lemma A.1, we gather some useful mapping properties of the singular integral operators $C_{n,m}$.

Lemma A.1. *Let $n, m \in \mathbb{N}_0$, $\mathbf{a} = (a_1,\ldots,a_m) : \mathbb{R} \to \mathbb{R}^m$, and $\mathbf{b} = (b_1,\ldots,b_n) : \mathbb{R} \to \mathbb{R}^n$.*

(i) *Given $\mathbf{a} \in \mathrm{W}^{1,\infty}(\mathbb{R})^m$, there exists a constant $C > 0$ depending only on n, m and $\|\mathbf{a}'\|_\infty$ such that for all $\mathbf{b} \in \mathrm{W}^{1,\infty}(\mathbb{R})^n$ and $\theta \in \mathbb{R}$ we have*

$$\|C_{n,m}(\mathbf{a})[\mathbf{b},\cdot]\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{S}),\mathrm{L}^2((\theta-\pi,\theta+\pi)))} \le C\prod_{i=1}^{n}\|b_i'\|_\infty. \tag{A.6}$$

Moreover, $C_{n,m} \in \mathrm{C}^{1-}(\mathrm{W}^{1,\infty}(\mathbb{S})^m, \mathcal{L}^n_{\mathrm{sym}}(\mathrm{W}^{1,\infty}(\mathbb{S}), \mathcal{L}(\mathrm{L}^2(\mathbb{S}))))$.

(ii) *Given* $n \in \mathbb{N}$, $r \in (3/2, 2)$, $\tau \in (5/2 - r, 1)$, *and* $\mathbf{a} \in \mathrm{H}^r(\mathbb{S})^n$, *there exists a constant* $C > 0$ *that depends only on* n, m, r, *and* $\|\mathbf{a}\|_{\mathrm{H}^r}$ *(and on* τ *in* (A.8)*), such that for all* $\mathbf{b} \in \mathrm{H}^r(\mathbb{S})^n$ *and* $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, *we have*

$$\|C_{n,m}(\mathbf{a})[\mathbf{b}, \varphi]\|_2 \leq C\|b_1'\|_2\|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=2}^{n} \|b_i'\|_{\mathrm{H}^{r-1}} \tag{A.7}$$

and

$$\begin{aligned} &\|C_{n,m}(\mathbf{a})[\mathbf{b}, \varphi] - \varphi C_{n-1,m}(\mathbf{a})[b_2, \ldots, b_n, b_1']\|_2 \\ &\leq C\|b_1\|_{\mathrm{H}^\tau}\|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=2}^{n} \|b_i'\|_{\mathrm{H}^{r-1}}. \end{aligned} \tag{A.8}$$

(iii) *Given* $r \in (3/2, 2)$ *and* $\mathbf{a} \in \mathrm{H}^r(\mathbb{S})^m$, *there exists a constant* $C > 0$ *that depends only on* n, m, r *and* $\|\mathbf{a}\|_{\mathrm{H}^r}$ *such that for all* $\mathbf{b} \in \mathrm{H}^r(\mathbb{S})^n$ *and* $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, *we have*

$$\|C_{n,m}(\mathbf{a})[\mathbf{b}, \varphi]\|_{\mathrm{H}^{r-1}} \leq C\|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=1}^{n} \|b_i'\|_{\mathrm{H}^{r-1}}. \tag{A.9}$$

(iv) *Given* $n \in \mathbb{N}$, $3/2 < r' < r < 2$, *and* $\mathbf{a} \in \mathrm{H}^r(\mathbb{S})^m$, *there exists a constant* $C > 0$ *that depends only on* n, m, r, r', *and* $\|\mathbf{a}\|_{\mathrm{H}^r}$ *such that for all* $\mathbf{b} \in \mathrm{H}^r(\mathbb{S})^n$ *and* $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, *we have*

$$\begin{aligned} &\|C_{n,m}(\mathbf{a})[\mathbf{b}, \varphi] - \varphi C_{n-1,m}(\mathbf{a})[b_2, \ldots, b_n, b_1']\|_{\mathrm{H}^{r-1}} \\ &\leq C\|b_1\|_{\mathrm{H}^{r'}}\|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=2}^{n} \|b_i\|_{\mathrm{H}^r}. \end{aligned} \tag{A.10}$$

Proof. The claim (i) is established in [26, Lemma A.1] in the case $\theta = 0$. The result for $\theta \neq 0$ is obtained from the result for $\theta = 0$ via the identity

$$C_{n,m}(\mathbf{a})[\mathbf{b}, \varphi](\xi) = C_{n,m}(\tau_\theta \mathbf{a})[\tau_\theta \mathbf{b}, \tau_\theta \varphi](\xi - \theta), \quad \xi, \theta \in \mathbb{R}.$$

Moreover, the proof of (ii) is similar to that of [1, Lemma 4] and we therefore omit it. Finally, (iii) and (iv) can be established by arguing analogously as in the nonperiodic version of these results, cf. [1, Lemmas 5 and 6]. ∎

A.2. Estimates for the operators $A_{n,m}^{\ell,q}$ and $B_{n,m}^{p,q}$

We study the integral operators $B_{n,m}^{p,q}$ and $A_{n,m}^{\ell,q}$ and show first in Lemma A.2 that $A_{n,m}^{\ell,q}$ regularizes, the same being true for $B_{n,m}^{p,q}$ provided that $p \geq 1$, see Lemma A.6.

Lemma A.2. *Let* $n, m, p, q \in \mathbb{N}_0$ *with* $1 \leq p \leq n + q + 1$, $\ell \in \{1, 2\}$, $r \in (3/2, 2)$, *and let* $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathrm{W}^{1,\infty}(\mathbb{S})^{m+n+q}$ *be given. Then,*

$$A_{n,m}^{\ell,q}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \cdot],\ B_{n,m}^{p,q}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \cdot] \in \mathcal{L}(\mathrm{L}^1(\mathbb{S}), \mathrm{C}(\mathbb{S})), \tag{A.11}$$

and there exists a constant $C > 0$ *that depends only on* n, m, p, q, *and* $\|(\mathbf{a}', \mathbf{b}')\|_\infty$ *such that for all* $\varphi \in \mathrm{L}^1(\mathbb{S})$, *we have*

$$\big\| A^{\ell,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi]\big\|_\infty + \big\| B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi]\big\|_\infty \le C\|\varphi\|_1 \prod_{i=1}^{q} \|c_i'\|_\infty. \tag{A.12}$$

Moreover, if $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathrm{C}^1(\mathbb{S})^{m+n+q}$, *there exists a constant* $C > 0$ *depending only on* n, m, q, *and* $\|(\mathbf{a}', \mathbf{b}')\|_\infty$ *such that for all* $\varphi \in \mathrm{C}(\mathbb{S})$, *we have*

$$\big\| A^{1,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi]\big\|_{\mathrm{C}^1} \le C\|\varphi\|_\infty \prod_{i=1}^{q} \|c_i\|_{\mathrm{C}^1}. \tag{A.13}$$

Proof. To start, we denote the kernels of the integral operators $A^{\ell,q}_{n,m}$ and $B^{p,q}_{n,m}$ by K_A and K_B, respectively, that is,

$$\begin{aligned} A^{\ell,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi](\xi) &= \int_{-\pi}^{\pi} K_A(\xi, s)\varphi(\xi - s)\,\mathrm{d}s, \\ B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi](\xi) &= \int_{-\pi}^{\pi} K_B(\xi, s)\varphi(\xi - s)\,\mathrm{d}s, \quad \xi \in \mathbb{R}. \end{aligned} \tag{A.14}$$

We begin by establishing (A.12) for $B^{p,q}_{n,m}$. Since $p \ge 1$ and $n + q + 1 \ge p$, we infer from (A.5) that

$$|K_B(\xi, s)| \le \bigg(\prod_{i=1}^{n} \|b_i'\|_\infty\bigg)\bigg(\prod_{i=1}^{q} \|c_i'\|_\infty\bigg)|s|^{p-1}\left|\frac{s/2}{t_{[s]}}\right|^{n+q+1-p} \le C\prod_{i=1}^{q} \|c_i'\|_\infty$$

for $\xi \in \mathbb{R}$ and $0 \ne s \in (-\pi, \pi)$, which proves (A.12) for $B^{p,q}_{n,m}$.

Since $A^{\ell,q}_{n,m}$ is linear in c_i, $1 \le i \le q$, it suffices to establish the estimate (A.12) for $A^{\ell,q}_{n,m}$ under the assumption that $\|\mathbf{c}'\|_\infty \le 1$. Let $F : \mathbb{R}^{n+q+m} \to \mathbb{R}$ be the locally Lipschitz continuous function defined by

$$F(x, y, z) = \frac{1}{2\pi}\frac{\big(\prod_{i=1}^{n} x_i\big)\big(\prod_{i=1}^{q} y_i\big)}{\prod_{i=1}^{m}(1 + z_i^2)} \quad \text{for } (x, y, z) \in \mathbb{R}^{n+q+m}. \tag{A.15}$$

Given $\xi \in \mathbb{R}$, $s \ne 0$, and $\mathbf{d} = (d_1, \ldots, d_l) \in \mathrm{W}^{1,\infty}(\mathbb{S})^l$, $l \in \mathbb{N}$, we introduce the shorthand notation

$$\frac{T_{[\xi,s]}\mathbf{d}}{t_{[s]}} := \bigg(\frac{T_{[\xi,s]}d_1}{t_{[s]}}, \ldots, \frac{T_{[\xi,s]}d_l}{t_{[s]}}\bigg). \tag{A.16}$$

Together with (A.5), we may now estimate for $\xi \in \mathbb{R}$ and $0 \ne s \in (-\pi, \pi)$:

$$\begin{aligned} |K_A(\xi, s)| \le C&\left|\frac{1}{t^\ell_{[s]}} - \frac{1}{(s/2)^\ell}\right| \\ &+ \left|\frac{1}{(s/2)^\ell}\right|\left|F\bigg(\frac{T_{[\xi,s]}\mathbf{b}}{t_{[s]}}, \frac{\delta_{[\xi,s]}\mathbf{c}/2}{t_{[s]}}, \frac{T_{[\xi,s]}\mathbf{a}}{t_{[s]}}\bigg) - F\bigg(\frac{\delta_{[\xi,s]}\mathbf{b}/2}{s/2}, \frac{\delta_{[\xi,s]}\mathbf{c}/2}{s/2}, \frac{\delta_{[\xi,s]}\mathbf{a}/2}{s/2}\bigg)\right|. \end{aligned} \tag{A.17}$$

In view of (A.3) and (A.4), we have

$$\left|\frac{1}{t_{[s]}^{\ell}}-\frac{1}{(s/2)^{\ell}}\right|\leq 2|s|^{2-\ell},\quad 0\neq s\in(-\pi,\pi),\quad \ell\in\{1,2\},$$
$$\left|\frac{T_{[\xi,s]}\mathbf{d}}{t_{[s]}}-\frac{\delta_{[\xi,s]}\mathbf{d}/2}{s/2}\right|+\left|\frac{\delta_{[\xi,s]}\mathbf{d}/2}{t_{[s]}}-\frac{\delta_{[\xi,s]}\mathbf{d}/2}{s/2}\right|\leq C|s|^2,\quad 0\neq s\in(-\pi,\pi),\quad \xi\in\mathbb{R},$$

with C depending only on $\|\mathbf{d}'\|_\infty$. These estimates together with (A.17) immediately imply that

$$|K_A(\xi,s)|\leq C|s|^{2-\ell},\quad 0\neq s\in(-\pi,\pi),\quad \xi\in\mathbb{R},\quad \ell\in\{1,2\},\tag{A.18}$$

and the desired estimate (A.12) for $A^{\ell,q}_{n,m}$ follows.

Since $\varphi\in \mathrm{C}(\mathbb{S})$, the continuity of parameter integrals implies that both $A^{\ell,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]$ and $B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]$ belong to $\mathrm{C}(\mathbb{S})$, and thus, the density of $\mathrm{C}(\mathbb{S})$ in $\mathrm{L}^1(\mathbb{S})$ leads us to (A.11).

It remains to establish (A.13). To this end, we first assume that $\varphi\in\mathrm{C}^1(\mathbb{S})$. Since

$$K_A(\cdot,s)\varphi(\cdot-s)\in\mathrm{C}^1(\mathbb{S}),\quad 0\neq s\in(-\pi,\pi),$$
$$K_A(\xi,\cdot)\varphi(\xi-\cdot)\in\mathrm{C}^1([-\pi,\pi]),\quad \xi\in\mathbb{R},$$

Fubini's theorem and integration by parts imply that $A^{1,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]$ is weakly differentiable with

$$\begin{aligned}&\big(A^{1,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\big)'(\xi)\\&\quad=(K_A(\xi,-\pi)-K_A(\xi,\pi))\varphi(\xi-\pi)+\int_{-\pi}^{\pi}(\partial_\xi K_A+\partial_s K_A)(\xi,s)\varphi(\xi-s)\,\mathrm{d}s\end{aligned}$$

for $\xi\in\mathbb{R}$, hence, we have

$$\begin{aligned}&\big(A^{1,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\big)'\\&=2\frac{1-(-1)^{n+q+1}}{2\pi}\frac{\prod_{i=1}^n(\delta_{[\cdot,\pi]}b_i/\pi)\prod_{i=1}^q(\delta_{[\cdot,\pi]}c_i/\pi)}{\prod_{i=1}^m\big[1+\big(\delta_{[\cdot,\pi]}a_i/\pi\big)^2\big]}\frac{\varphi(\cdot-\pi)}{\pi}\\&\quad+\sum_{j=1}^n\frac{b_j'}{2}\big(A^{2,q}_{n-1,m}(\mathbf{a}|\mathbf{b}_j)[\mathbf{c},\varphi]-B^{1,q}_{n+1,m}(\mathbf{a}|\mathbf{b},b_j)[\mathbf{c},\varphi]\big)\\&\quad+\sum_{j=1}^q\frac{c_j'}{2}A^{2,q-1}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}_j,\varphi]\\&\quad-\frac{n+q+1}{2}\big(A^{2,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]+B^{1,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\big)\\&\quad+\sum_{j=1}^m\Big[A^{2,q}_{n+2,m+1}(\mathbf{a},a_j|\mathbf{b},a_j,a_j)[\mathbf{c},\varphi]+B^{1,q}_{n+2,m+1}(\mathbf{a},a_j|\mathbf{b},a_j,a_j)[\mathbf{c},\varphi]\\&\qquad-a_j'A^{2,q}_{n+1,m+1}(\mathbf{a},a_j|\mathbf{b},a_j)[\mathbf{c},\varphi]+a_j'B^{1,q}_{n+3,m+1}(\mathbf{a},a_j|\mathbf{b},a_j,a_j,a_j)[\mathbf{c},\varphi]\Big],\end{aligned}\tag{A.19}$$

where we use the notation

$$\begin{aligned} \mathbf{b}_j &:= (b_1, \dots, b_{j-1}, b_{j+1}, \dots, b_n), \quad 1 \le j \le n, \\ \mathbf{c}_j &:= (c_1, \dots, c_{j-1}, c_{j+1}, \dots, c_q), \quad 1 \le j \le q. \end{aligned} \tag{A.20}$$

The remaining claim (A.13) follows now from the previous relation and (A.11) in view of the density of $\mathrm{C}^1(\mathbb{S})$ in $\mathrm{C}(\mathbb{S})$. ■

We next study the singular integral operator $B^{0,q}_{n,m}$.

Lemma A.3. *Let $n, m, q \in \mathbb{N}_0$ and let $(\mathbf{a}, \mathbf{b}) \in \mathrm{W}^{1,\infty}(\mathbb{S})^{m+n}$ be given. Then, there exists a constant $C > 0$ that depends only on n, m, q, and $\|(\mathbf{a}', \mathbf{b}')\|_\infty$ such that for all $\mathbf{c} \in \mathrm{W}^{1,\infty}(\mathbb{S})^q$ and $\varphi \in \mathrm{L}^2(\mathbb{S})$, we have*

$$\left\| B^{0,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi] \right\|_2 \le C \|\varphi\|_2 \prod_{i=1}^{q} \|c_i'\|_\infty. \tag{A.21}$$

Proof. The claim follows directly from (A.2), Lemma A.1 (i), and Lemma A.2. ■

Lemma A.4. *Let $n, m, q \in \mathbb{N}_0$, $r \in (3/2, 2)$, and $(\mathbf{a}, \mathbf{b}) \in \mathrm{H}^r(\mathbb{S})^{m+n}$ be given. Then, there exists a constant $C > 0$ that depends only on n, m, q, r, and $\|(\mathbf{a}, \mathbf{b})\|_{\mathrm{H}^r}$ such that for all $\mathbf{c} \in \mathrm{H}^r(\mathbb{S})^q$ and $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, we have*

$$\left\| B^{0,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi] \right\|_{\mathrm{H}^{r-1}} \le C \|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=1}^{q} \|c_i\|_{\mathrm{H}^r}. \tag{A.22}$$

Proof. The claim follows immediately from equation (A.2), Lemma A.1 (iii), (A.12), and (A.19). ■

The next result shows that the (singular) integral operators $B^{p,q}_{n,m}$ are locally Lipschitz continuous with respect to $(\mathbf{a}, \mathbf{b}, \mathbf{c})$.

Lemma A.5. *Given $n, m, p, q \in \mathbb{N}_0$ with $p \le n + q + 1$, we have*

$$\big[(\mathbf{a}, \mathbf{b}, \mathbf{c}) \mapsto B^{0,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \cdot]\big] \in \mathrm{C}^{1-}(\mathrm{W}^{1,\infty}(\mathbb{S})^{m+n+q}, \mathcal{L}(\mathrm{L}^2(\mathbb{S}))), \tag{A.23}$$

$$\big[(\mathbf{a}, \mathbf{b}, \mathbf{c}) \mapsto B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \cdot]\big] \in \mathrm{C}^{1-}(\mathrm{W}^{1,\infty}(\mathbb{S})^{m+n+q}, \mathcal{L}(\mathrm{L}^1(\mathbb{S}), \mathrm{C}(\mathbb{S}))), \quad p \ge 1. \tag{A.24}$$

Proof. Given $(\mathbf{a}, \mathbf{b}, \mathbf{c}), (\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}) \in \mathrm{W}^{1,\infty}(\mathbb{S})^{m+n+q}$, $\varphi \in \mathrm{C}^\infty(\mathbb{S})$, and $p \in \mathbb{N}_0$, we have

$$\begin{aligned} &B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi] - B^{p,q}_{n,m}(\tilde{\mathbf{a}}|\tilde{\mathbf{b}})[\tilde{\mathbf{c}}, \varphi] \\ &= \sum_{i=1}^{q} B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\tilde{c}_1, \dots, \tilde{c}_{i-1}, c_i - \tilde{c}_i, c_{i+1}, \dots, c_q, \varphi] \\ &\quad + \sum_{i=1}^{n} \big(B^{p,q}_{n,m}(\mathbf{a}|\tilde{b}_1, \dots, \tilde{b}_{i-1}, b_i, \dots, b_n) - B^{p,q}_{n,m}(\mathbf{a}|\tilde{b}_1, \dots, \tilde{b}_i, b_{i+1}, \dots, b_n)\big)[\tilde{\mathbf{c}}, \varphi] \end{aligned}$$

$$+\sum_{i=1}^{m}\big(B^{p,q}_{n+2,m+1}(\tilde a_1,\ldots,\tilde a_i,a_i,\ldots,a_m|\tilde{\mathbf{b}},\tilde a_i,\tilde a_i)$$
$$-B^{p,q}_{n+2,m+1}(\tilde a_1,\ldots,\tilde a_i,a_i,\ldots,a_m|\tilde{\mathbf{b}},\tilde a_i,a_i)\big)[\tilde{\mathbf{c}},\varphi]$$
$$+\sum_{i=1}^{m}\big(B^{p,q}_{n+2,m+1}(\tilde a_1,\ldots,\tilde a_i,a_i,\ldots,a_m|\tilde{\mathbf{b}},\tilde a_i,a_i)$$
$$-B^{p,q}_{n+2,m+1}(\tilde a_1,\ldots,\tilde a_i,a_i,\ldots,a_m|\tilde{\mathbf{b}},a_i,a_i)\big)[\tilde{\mathbf{c}},\varphi].$$

The first term on the right-hand side may be estimated by using Lemma A.3 if $p=0$ and Lemma A.2 whenever $p\geq 1$. For the remaining terms, it thus suffices to show that given $d,\tilde d\in \mathrm{W}^{1,\infty}(\mathbb{S})$, we have

$$\big\|B^{0,q}_{n+1,m}(\mathbf{a}|\mathbf{b},d)[\mathbf{c},\varphi]-B^{0,q}_{n+1,m}(\mathbf{a}|\mathbf{b},\tilde d)[\mathbf{c},\varphi]\big\|_2\leq C\|d-\tilde d\|_{\mathrm{W}^{1,\infty}}\|\varphi\|_2 \tag{A.25}$$

for (A.23), respectively,

$$\big\|B^{p,q}_{n+1,m}(\mathbf{a}|\mathbf{b},d)[\mathbf{c},\varphi]-B^{p,q}_{n+1,m}(\mathbf{a}|\mathbf{b},\tilde d)[\mathbf{c},\varphi]\big\|_\infty\leq C\|d-\tilde d\|_{\mathrm{W}^{1,\infty}}\|\varphi\|_1,\quad p\geq 1, \tag{A.26}$$

for (A.24), with a constant C that depends only on $\|(\mathbf{a},\mathbf{b},\mathbf{c},d,\tilde d)\|_{\mathrm{W}^{1,\infty}}$ and n,m,p,q.

To show (A.25)-(A.26), we infer from the fundamental theorem of calculus that

$$|(x-\tanh(x))-(y-\tanh(y))|\leq (x^2+y^2)|x-y|,\quad x,y\in\mathbb{R}. \tag{A.27}$$

With $F:\mathbb{R}^{n+q+m}\to\mathbb{R}$ denoting the smooth function defined in (A.15) we then compute, by using also the notation (A.16), that

$$\big(B^{p,q}_{n+1,m}(\mathbf{a}|\mathbf{b},d)[\mathbf{c},\varphi]-B^{p,q}_{n+1,m}(\mathbf{a}|\mathbf{b},\tilde d)[\mathbf{c},\varphi]\big)(\xi)$$
$$=\mathrm{PV}\int_{-\pi}^{\pi}F\Big(\frac{T_{[\xi,s]}\mathbf{b}}{t_{[s]}},\frac{\delta_{[\xi,s]}\mathbf{c}/2}{t_{[s]}},\frac{T_{[\xi,s]}\mathbf{a}}{t_{[s]}}\Big)\frac{T_{[\xi,s]}d-T_{[\xi,s]}\tilde d}{t_{[s]}}\frac{\varphi(\xi-s)}{t_{[s]}^{1-p}}\,\mathrm{d}s$$
$$=B^{p,q+1}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},d-\tilde d,\varphi](\xi)-\int_{-\pi}^{\pi}K(\xi,s)\varphi(\xi-s)\,\mathrm{d}s$$

for $\xi\in\mathbb{R}$ and $p\in\mathbb{N}_0$, where, given $\xi\in\mathbb{R}$ and $0\neq s\in(-\pi,\pi)$, we set

$$K(\xi,s):=F\Big(\frac{T_{[\xi,s]}\mathbf{b}}{t_{[s]}},\frac{\delta_{[\xi,s]}\mathbf{c}/2}{t_{[s]}},\frac{T_{[\xi,s]}\mathbf{a}}{t_{[s]}}\Big)\times\frac{\big(\delta_{[\xi,s]}d/2-T_{[\xi,s]}d\big)-\big(\delta_{[\xi,s]}\tilde d/2-T_{[\xi,s]}\tilde d\big)}{t_{[s]}^{2-p}}.$$

The function $B^{p,q+1}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},d-\tilde d,\varphi]$ may be estimated by using Lemma A.3 if $p=0$ and Lemma A.2 for $p\geq 1$, and we are left to estimate the integral term. To this end we rely on (A.27) and (A.5) to obtain that $|K(\xi,s)|\leq C\|d-\tilde d\|_\infty$ for all $\xi\in\mathbb{R}$ and $0\neq s\in(-\pi,\pi)$, and therefore,

$$\Big|\int_{-\pi}^{\pi}K(\xi,s)\varphi(\xi-s)\,\mathrm{d}s\Big|\leq C\|d-\tilde d\|_\infty\|\varphi\|_1\leq C\|d-\tilde d\|_\infty\|\varphi\|_2,\quad \xi\in\mathbb{R}.$$

This completes the proof. ∎

Using Lemma A.5, we next prove that the operators $B^{p,q}_{n,m}$, with $p \geq 1$, have a regularizing effect.

Lemma A.6. *Let $n, m, p, q \in \mathbb{N}_0$, $1 \leq p \leq n+q+1$, and $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathrm{C}^1(\mathbb{S})^{m+n+q}$. Then, there exists a constant $C > 0$ that depends only on n, m, p, q, and $\|(\mathbf{a}', \mathbf{b}')\|_\infty$ such that for all $\varphi \in \mathrm{L}^2(\mathbb{S})$, we have*

$$\big\| B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi] \big\|_{\mathrm{H}^1} \leq C \|\varphi\|_2 \prod_{i=1}^{q} \|c_i'\|_\infty. \tag{A.28}$$

Furthermore, given $r \in (3/2, 2)$ and $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathrm{H}^r(\mathbb{S})^{m+n+q}$, there exists a constant $C > 0$ that depends only on n, m, p, q, and $\|(\mathbf{a}, \mathbf{b})\|_{\mathrm{H}^r}$ such that for all $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, we have

$$\big\| B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi] \big\|_{\mathrm{H}^r} \leq C \|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=1}^{q} \|c_i\|_{\mathrm{H}^r}. \tag{A.29}$$

Proof. We first assume that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \varphi) \in \mathrm{C}^\infty(\mathbb{S})^{m+n+q+1}$. Recalling the notation (A.14), the theorem on the differentiation of parameter integrals ensures that $B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi]$ is continuously differentiable with

$$\big(B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi]\big)'(\xi) = \int_{-\pi}^{\pi} \partial_\xi K_B(\xi, s)\varphi(\xi - s) - K_B(\xi, s)\partial_s(\varphi(\xi - s))\, ds, \quad \xi \in \mathbb{R}.$$

Using integration by parts, we then get

$$\begin{aligned}
&(B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi])' \\
&= \frac{1}{2}\sum_{j=1}^{n} b_j'\big(B^{p-1,q}_{n-1,m}(\mathbf{a}|\mathbf{b}_j) - B^{p+1,q}_{n+1,m}(\mathbf{a}|\mathbf{b}, b_j)\big)[\mathbf{c}, \varphi] + \frac{1}{2}\sum_{j=1}^{q} c_j' B^{p-1,q-1}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}_j, \varphi] \\
&\quad + \sum_{j=1}^{m} a_j'\big(B^{p+1,q}_{n+3,m+1}(\mathbf{a}, a_j|\mathbf{b}, a_j, a_j, a_j) - B^{p-1,q}_{n+1,m+1}(\mathbf{a}, a_j|\mathbf{b}, a_j)\big)[\mathbf{c}, \varphi] \\
&\quad + \sum_{j=1}^{m} \big(B^{p-1,q}_{n+2,m+1}(\mathbf{a}, a_j|\mathbf{b}, a_j, a_j) + B^{p+1,q}_{n+2,m+1}(\mathbf{a}, a_j|\mathbf{b}, a_j, a_j)\big)[\mathbf{c}, \varphi] \\
&\quad + \frac{p-n-q-1}{2}\big(B^{p-1,q}_{n,m}(\mathbf{a}|\mathbf{b}) + B^{p+1,q}_{n,m}(\mathbf{a}|\mathbf{b})\big)[\mathbf{c}, \varphi],
\end{aligned}$$

with the observation that the last term is meaningful only if $1 \leq p \leq n+q$, otherwise, it is not present in the formula above. The functions $\mathbf{b}_j$, $1 \leq j \leq n$, and $\mathbf{c}_j$, $1 \leq j \leq q$, are as defined in (A.20). A standard density argument together with Lemma A.5 ensures now that $B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi] \in \mathrm{H}^1(\mathbb{S})$ for all $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathrm{C}^1(\mathbb{S})^{m+n+q}$ and $\varphi \in \mathrm{L}^2(\mathbb{S})$, the estimate (A.28) being a direct consequence of Lemmas A.2 and A.3 while the estimate (A.29) follows from Lemmas A.2, A.4, and (A.28). ∎

The next result shows that the operator B_0 defined in (3.7) has similar regularity properties as $B^{p,q}_{n,m}$ with $1 \leq p \leq n+q+1$, see (A.29).

Lemma A.7. *Let $r \in (3/2, 2)$. Given $f \in \mathrm{H}^r(\mathbb{S})$, there exists a constant $C > 0$ that depends only on $\|f\|_{\mathrm{H}^r}$ such that for all $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$ we have*

$$\|B_0(f)[\varphi]\|_{\mathrm{H}^r} \leq C\|\varphi\|_{\mathrm{H}^{r-1}}. \tag{A.30}$$

Proof. We first prove that $B_0(f) \in \mathcal{L}(\mathrm{L}^2(\mathbb{S}), \mathrm{L}^\infty(\mathbb{S}))$ if $f \in \mathrm{W}^{1,\infty}(\mathbb{S})$. Indeed, using (2.5) and the fact that $\ln(\sin(\cdot/2)^2) \in \mathrm{L}^2(\mathbb{S})$, we deduce, in view of the inequality

$$\begin{aligned}&|\ln(\sin^2(s/2) + \sinh^2(\delta_{[\xi,s]}f/2))|\\ &\quad \leq |\ln(\sin^2(s/2))| + \ln(1 + \sinh^2(\pi\|f'\|_\infty)), \quad \xi, s \in \mathbb{S},\end{aligned}$$

that for $\varphi \in \mathrm{L}^2(\mathbb{S})$ we have

$$|B_0(f)[\varphi](\xi)| \leq \int_{-\pi}^{\pi} \big[|\ln(\sin^2(s/2))| + \ln(1 + \sinh^2(\pi\|f'\|_\infty))\big]|\varphi(\xi - s)|\,\mathrm{d}s \leq C\|\varphi\|_2.$$

We now assume that $f \in \mathrm{H}^r(\mathbb{S})$ and $\varphi \in \mathrm{C}^\infty(\mathbb{S})$. Using the theorem on the differentiation of parameter integrals and subsequently integration by parts, we find that $B_0(f)[\varphi]$ is continuously differentiable and its derivative is given by

$$(B_0(f)[\varphi])' = f'B_2(f)[\varphi] + B_1(f)[\varphi] \in \mathrm{H}^{r-1}(\mathbb{S}),$$

cf. Lemmas A.4 and A.6. The claim follows now by a standard density argument in view of Lemmas A.4 and A.6. ∎

A.3. Fréchet differentiability

This section is devoted to establishing the following result.

Corollary A.8. *Given $r \in (3/2, 2)$, the mappings*

$$\begin{aligned}&\big[f \mapsto B^{0,q}_{n,m}(f)\big] : \mathrm{H}^r(\mathbb{S}) \to \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S})),\\ &\big[f \mapsto B_0(f)\big], \big[f \mapsto B^{p,q}_{n,m}(f)\big] : \mathrm{H}^r(\mathbb{S}) \to \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}), \mathrm{H}^r(\mathbb{S})), \quad 1 \leq p \leq n + q + 1,\end{aligned}$$

are smooth.

The proof of Corollary A.8 is presented at the end of this section, as it requires some preparation. Let us first note that Lemmas A.4 and A.6 ensure that the mappings defined above are well-defined. In order to establish the smoothness of these mappings, we further introduce the operators

$$\begin{aligned}&B^{0,q,k}_{n,m} : \mathrm{H}^r(\mathbb{S}) \to \mathcal{L}^k_{\mathrm{sym}}(\mathrm{H}^r(\mathbb{S}), \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}))),\\ &B^{p,q,k}_{n,m} : \mathrm{H}^r(\mathbb{S}) \to \mathcal{L}^k_{\mathrm{sym}}(\mathrm{H}^r(\mathbb{S}), \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}), \mathrm{H}^r(\mathbb{S}))), \quad 1 \leq p \leq n + q + k + 1,\end{aligned} \tag{A.31}$$

by

$$B^{p,q,k}_{n,m}(f)[f_1, \ldots, f_k][\cdot] := B^{p,q+k}_{n,m}(f, \ldots, f\,|\,f, \ldots, f)[f, \ldots, f, f_1, \ldots, f_k, \cdot].$$

Let us note that $B^{p,q}_{n,m}(f) = B^{p,q,0}_{n,m}(f)$. The next lemma is the main step towards proving the smoothness property for the operators $B^{p,q}_{n,m}$.

Lemma A.9. *The mappings* (A.31) *are Fréchet differentiable. Furthermore, the Fréchet derivative* $\partial B^{p,q,k}_{n,m}(f_0)$ *is given by*

$$\begin{aligned}\partial B^{p,q,k}_{n,m}(f_0)[f][f_1,\dots,f_k] &= n\big(B^{p,q,k+1}_{n-1,m}(f_0) - B^{p+2,q,k+1}_{n+1,m}(f_0)\big)[f_1,\dots,f_k,f]\\ &\quad + 2m\big(B^{p+2,q,k+1}_{n+3,m+1}(f_0) - B^{p,q,k+1}_{n+1,m+1}(f_0)\big)[f_1,\dots,f_k,f]\\ &\quad + qB^{p,q-1,k+1}_{n,m}(f_0)[f_1,\dots,f_k,f] \qquad \text{(A.32)}\end{aligned}$$

for $f_0, f, f_1, \dots, f_k \in \mathrm{H}^r(\mathbb{S})$*, where terms with negative indices are to be neglected.*

Proof. Defining $\phi := \phi^{p,q}_{n,m}$ by the formula

$$\phi(\eta, s) := \frac{1}{2\pi}\frac{\big(\frac{\tanh(\eta)}{t_{[s]}}\big)^n\big(\frac{\eta}{t_{[s]}}\big)^q}{\big[1+\big(\frac{\tanh(\eta)}{t_{[s]}}\big)^2\big]^m}t^p_{[s]}, \qquad \eta\in\mathbb{R}, \quad 0\neq s\in(-\pi,\pi),$$

we have for $\xi \in \mathbb{R}$, $f, f_1, \dots, f_k \in \mathrm{H}^r(\mathbb{S})$, and $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$

$$B^{p,q,k}_{n,m}(f)[f_1,\dots,f_k][\varphi](\xi) = \mathrm{PV}\int_{-\pi}^{\pi}\Big(\prod_{i=1}^{k}\frac{\delta_{[\xi,s]}f_i/2}{t_{[s]}}\Big)\phi\big(\delta_{[\xi,s]}f/2, s\big)\frac{\varphi(\xi-s)}{t_{[s]}}\,\mathrm{d}s,$$

the PV being needed only when $p = 0$. Our goal is to prove that

$$\begin{aligned}&\partial B^{p,q,k}_{n,m}(f_0)[f][f_1,\dots,f_k][\varphi](\xi)\\ &= \mathrm{PV}\int_{-\pi}^{\pi}\Big(\prod_{i=1}^{k}\frac{\delta_{[\xi,s]}f_i/2}{t_{[s]}}\Big)(\delta_{[\xi,s]}f/2)\partial_\eta\phi(\delta_{[\xi,s]}f_0/2, s)\frac{\varphi(\xi-s)}{t_{[s]}}\,\mathrm{d}s \qquad \text{(A.33)}\end{aligned}$$

for $\xi \in \mathbb{R}$, $f_0, f, f_1, \dots, f_k \in \mathrm{H}^r(\mathbb{S})$, and $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, as straightforward computations show that the formulas (A.32) and (A.33) are equivalent.

Using Taylor's formula, we compute

$$\begin{aligned}&\big(B^{p,q,k}_{n,m}(f_0+f) - B^{p,q,k}_{n,m}(f_0) - \partial B^{p,q,k}_{n,m}(f_0)[f]\big)[f_1,\dots,f_k][\varphi](\xi)\\ &= \mathrm{PV}\int_{-\pi}^{\pi}\Big(\prod_{i=1}^{k}\frac{\delta_{[\xi,s]}f_i/2}{t_{[s]}}\Big)\big(\delta_{[\xi,s]}f/2\big)^2\int_0^1(1-\tau)\partial_\eta^2\phi\big(\delta_{[\xi,s]}f_\tau/2, s\big)\,\mathrm{d}\tau\,\frac{\varphi(\xi-s)}{t_{[s]}}\,\mathrm{d}s, \qquad \text{(A.34)}\end{aligned}$$

where $f_\tau := f_0 + \tau f$ for $\tau \in [0, 1]$, and $\partial_\eta^2\phi = \partial_\eta^2\phi^{p,q}_{n,m}$ is given by

$$\begin{aligned}&\partial_\eta^2\phi^{p,q}_{n,m}\\ &= \frac{1}{t^2_{[s]}}\Big\{n(n-1)\phi^{p,q}_{n-2,m} + 2nq\phi^{p,q-1}_{n-1,m} + q(q-1)\phi^{p,q-2}_{n,m} - 2m(2n+1)\phi^{p,q}_{n,m+1}\\ &\quad - 2nq\phi^{p+2,q-1}_{n+1,m} - 2n^2\phi^{p+2,q}_{n,m} + 8m(n+1)\phi^{p+2,q}_{n+2,m+1} + n(n+1)\phi^{p+4,q}_{n+2,m}\\ &\quad - 2m(2n+3)\phi^{p+4,q}_{n+4,m+1} + 4mq\phi^{p+2,q-1}_{n+3,m+1} - 4mq\phi^{p,q-1}_{n+1,m+1}\\ &\quad + 4m(m+1)\phi^{p+4,q}_{n+6,m+2} - 8m(m+1)\phi^{p+2,q}_{n+4,m+2} + 4m(m+1)\phi^{p,q}_{n+2,m+2}\Big\} \qquad \text{(A.35)}\end{aligned}$$

in $\mathbb{R} \times ((-\pi, \pi) \setminus \{0\})$ and for all $0 \leq p \leq n + q + k + 1$. Recalling (A.5), in all the terms on the right-hand side of (A.34) where $\phi^{p,q}_{n,m}$ with $p \geq 1$ appear, the PV is not needed and we may interchange the order of integration by using Fubini's theorem.

Assume first that $p \geq 1$. We then infer from (A.34) and (A.35), after interchanging the order of integration in the last line of (A.34), that

$$\begin{aligned}
&\big(B^{p,q,k}_{n,m}(f_0 + f) - B^{p,q,k}_{n,m}(f_0) - \partial B^{p,q,k}_{n,m}(f_0)[f]\big)[f_1, \dots, f_k][\varphi] \\
&= \int_0^1 (1-\tau)\Big\{n(n-1)B^{p,q,k+2}_{n-2,m} + 2nqB^{p,q-1,k+2}_{n-1,m} \\
&\qquad + q(q-1)B^{p,q-2,k+2}_{n,m} - 2m(2n+1)B^{p,q,k+2}_{n,m+1} \\
&\qquad - 2nqB^{p+2,q-1,k+2}_{n+1,m} - 2n^2 B^{p+2,q,k+2}_{n,m} \\
&\qquad + 8m(n+1)B^{p+2,q,k+2}_{n+2,m+1} + n(n+1)B^{p+4,q,k+2}_{n+2,m} \\
&\qquad - 2m(2n+3)B^{p+4,q,k+2}_{n+4,m+1} \\
&\qquad + 4mqB^{p+2,q-1,k+2}_{n+3,m+1} - 4mqB^{p,q-1,k+2}_{n+1,m+1} \\
&\qquad + 4m(m+1)B^{p+4,q,k+2}_{n+6,m+2} - 8m(m+1)B^{p+2,q,k+2}_{n+4,m+2} \\
&\qquad + 4m(m+1)B^{p,q,k+2}_{n+2,m+2}\Big\}(f_\tau)[f_1, \dots, f_k, f, f][\varphi]\, d\tau.
\end{aligned} \tag{A.36}$$

Moreover, Lemma A.6 implies there exists a constant $C > 0$ such that for all $\|f\|_{\mathrm{H}^r} \leq 1$ we have

$$\begin{aligned}
&\big\|\big(B^{p,q,k}_{n,m}(f_0 + f) - B^{p,q,k}_{n,m}(f_0) - \partial B^{p,q,k}_{n,m}(f_0)[f]\big)[f_1, \dots, f_k]\big\|_{\mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}), \mathrm{H}^r(\mathbb{S}))} \\
&\leq C\|f\|^2_{\mathrm{H}^r} \prod_{i=1}^k \|f_i\|_{\mathrm{H}^r},
\end{aligned}$$

which proves (A.32) for $p \geq 1$.

Let now $p = 0$. In this case, the formula (A.36) is still valid (and defines a function in $\mathrm{H}^{r-1}(\mathbb{S})$). This formula is obtained again by interchanging the order of integration in (A.34) via (A.35), but slightly more subtle arguments are needed when considering the terms of (A.35) with $p = 0$ as the PV symbol appears in front of the first integral in (A.34). More precisely, letting

$$I(\xi, s, \tau) := \left(\prod_{i=1}^k \frac{\delta_{[\xi,s]} f_i/2}{t_{[s]}}\right)(\delta_{[\xi,s]} f/2)^2 (1-\tau)\partial^2_\eta \phi\big(\delta_{[\xi,s]} f_\tau/2, s\big)\frac{\varphi(\xi - s)}{t_{[s]}}$$

denote the integrand in (A.34), it holds that

$$\begin{aligned}
&\mathrm{PV}\int_{-\pi}^{\pi}\left(\int_0^1 I(\xi, s, \tau)\, d\tau\right) ds = \int_0^{\pi}\left(\int_0^1 I(\xi, s, \tau) + I(\xi, -s, \tau)\, d\tau\right) ds \\
&= \int_0^1\left(\int_0^{\pi} I(\xi, s, \tau) + I(\xi, -s, \tau)\, ds\right) d\tau = \int_0^1\left(\mathrm{PV}\int_{-\pi}^{\pi} I(\xi, s, \tau)\, ds\right) d\tau,
\end{aligned}$$

by Fubini's theorem and in view of the estimate

$$|I(\xi,s,\tau)+I(\xi,-s,\tau)|\leq \frac{C}{|s|^{5/2-r}},\quad \xi\in\mathbb{R},\quad 0\neq s\in(-\pi,\pi),\quad \tau\in[0,1].$$

Applying Lemmas A.4 and A.6, we conclude from (A.36) that there exists a constant $C>0$ such that for all $\|f\|_{\mathrm{H}^r}\leq 1$, we have

$$\begin{aligned}&\big\|\big(B^{p,q,k}_{n,m}(f_0+f)-B^{p,q,k}_{n,m}(f_0)-\partial B^{p,q,k}_{n,m}(f_0)[f]\big)[f_1,\dots,f_k]\big\|_{\mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}))}\\&\quad\leq C\|f\|^2_{\mathrm{H}^r}\prod_{i=1}^k\|f_i\|_{\mathrm{H}^r},\end{aligned}$$

which proves the claim for $p=0$. ■

We now show the Fréchet differentiability of the operator B_0 defined in (3.7).

Lemma A.10. *Given $r\in(3/2,2)$, the map $B_0:\mathrm{H}^r(\mathbb{S})\to\mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}),\mathrm{H}^r(\mathbb{S}))$ is Fréchet differentiable and the Fréchet derivative $\partial B_0(f_0)$ is given by*

$$\partial B_0(f_0)[f]=2B^{1,0,1}_{1,1}(f_0)[f]+2B^{3,0,1}_{1,1}(f_0)[f],\quad f_0,f\in\mathrm{H}^r(\mathbb{S}).\tag{A.37}$$

Proof. We apply the same strategy as in the proof of Lemma A.9. Defining ϕ by

$$\phi(\eta,s):=\frac{1}{2\pi}\ln\left(\frac{t^2_{[s]}+\tanh^2(\eta)}{(1+t^2_{[s]})(1-\tanh^2(\eta))}\right),\quad 0\neq\eta\in\mathbb{R},\quad s\in(-\pi,\pi),$$

we have

$$\begin{aligned}\partial_\eta\phi(\eta,s)&=\frac{1}{\pi}\frac{(1+t^2_{[s]})\tanh(\eta)}{t^2_{[s]}+\tanh^2(\eta)},\\ \partial^2_\eta\phi(\eta,s)&=\frac{1}{\pi}\frac{(1+t^2_{[s]})(1-\tanh^2(\eta))(t^2_{[s]}-\tanh^2(\eta))}{(t^2_{[s]}+\tanh^2(\eta))^2}.\end{aligned}\tag{A.38}$$

We prove that

$$\partial B_0(f_0)[f][\varphi](\xi)=\int_{-\pi}^{\pi}\big(\delta_{[\xi,s]}f/2\big)\partial_\eta\phi\big(\delta_{[\xi,s]}f_0/2,s\big)\varphi(\xi-s)\,\mathrm{d}s,\tag{A.39}$$

since straightforward calculations show that (A.37) and (A.39) coincide. Using Taylor's formula, Fubini's theorem, (A.38), and (A.39), we compute for $\xi\in\mathbb{R}$, $f_0,f\in\mathrm{H}^r(\mathbb{S})$, and $\varphi\in\mathrm{H}^{r-1}(\mathbb{S})$ that

$$\begin{aligned}&B_0(f_0+f)[\varphi](\xi)-B_0(f_0)[\varphi](\xi)-\partial B_0(f_0)[f][\varphi](\xi)\\&=\int_{-\pi}^{\pi}\big(\delta_{[\xi,s]}f/2\big)^2\int_0^1(1-\tau)\partial^2_\eta\phi\big(\delta_{[\xi,s]}f_\tau/2,s\big)\,\mathrm{d}\tau\,\varphi(\xi-s)\,\mathrm{d}s\end{aligned}$$

$$= \int_0^1 (1-\tau) \int_{-\pi}^{\pi} \big(\delta_{[\xi,s]} f/2\big)^2 \partial_\eta^2 \phi\big(\delta_{[\xi,s]} f_\tau/2, s\big) \varphi(\xi - s)\, ds\, d\tau$$
$$= 2 \int_0^1 (1-\tau) \Big\{ B_{0,2}^{1,0,2} + B_{0,2}^{3,0,2} - B_{2,2}^{1,0,2} - 2B_{2,2}^{3,0,2} - B_{2,2}^{5,0,2} + B_{4,2}^{3,0,2} + B_{4,2}^{5,0,2} \Big\}(f_\tau)[f, f][\varphi]\, d\tau,$$

where $f_\tau = f_0 + \tau f$. Using (A.29), we thus find a constant $C > 0$ such that for all $f \in \mathrm{H}^r(\mathbb{S})$ with $\|f\|_{\mathrm{H}^r} \le 1$, we have

$$\|B_0(f_0 + f) - B_0(f_0) - \partial B_0(f_0)[f]\|_{\mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}), \mathrm{H}^r(\mathbb{S}))} \le C\|f\|_{\mathrm{H}^r}^2,$$

which proves the claim. ∎

We are now in a position to establish Corollary A.8.

Proof of Corollary A.8. Recalling that $B_{n,m}^{p,q}(f) = B_{n,m}^{p,q,0}(f)$ for $f \in \mathrm{H}^r(\mathbb{S})$, the assertion is a direct consequence of Lemmas A.9 and A.10. ∎

B. Localization of the singular integral operators $C_{n,m}$

In this section, we show that the singular integral operators $C_{n,m}^0$ defined in (3.6) can be locally approximated by Fourier multipliers, see Lemma B.2 for the precise statement. As a starting point, we infer from (3.2) the following algebraic relations:

$$\big(C_{n,m}(\tilde{\mathbf{a}}) - C_{n,m}(\mathbf{a})\big)[\mathbf{b}, \varphi] = \sum_{i=1}^{m} C_{n+2,m+1}(a_1, \ldots, a_i, \tilde{a}_i, \ldots, \tilde{a}_m)[\mathbf{b}, a_i + \tilde{a}_i, a_i - \tilde{a}_i, \varphi], \quad n \in \mathbb{N}_0, \quad m \in \mathbb{N}, \tag{B.1}$$

and

$$d C_{n,m}(\mathbf{a})[\mathbf{b}, \varphi] - C_{n,m}(\mathbf{a})[\mathbf{b}, d\varphi] = b_1 C_{n,m}(\mathbf{a})[b_2, \ldots, b_n, d, \varphi] - C_{n,m}(\mathbf{a})[b_2, \ldots, b_n, d, b_1\varphi], \quad n \in \mathbb{N}, \quad m \in \mathbb{N}_0, \tag{B.2}$$

which hold for all $\mathbf{a}, \tilde{\mathbf{a}} \in \mathrm{W}^{1,\infty}(\mathbb{S})^m$, $\mathbf{b} \in \mathrm{W}^{1,\infty}(\mathbb{S})^n$, $d \in \mathrm{W}^{1,\infty}(\mathbb{S})$, and $\varphi \in \mathrm{L}^2(\mathbb{S})$.

The following commutator property, see [1, Lemma 12] for a similar result in a nonperiodic setting, is an important tool in the analysis that follows.

Lemma B.1. *Given $n, m \in \mathbb{N}_0$ and $a, f \in \mathrm{C}^1(\mathbb{S})$, there exists a constant $C > 0$ that depends only on n, m, r, and $\|(a, f)\|_{\mathrm{C}^1}$ such that for all $\varphi \in \mathrm{L}^2(\mathbb{S})$ we have*

$$\|a C_{n,m}^0(f)[\varphi] - C_{n,m}^0(f)[a\varphi]\|_{\mathrm{H}^1} \le C\|\varphi\|_2. \tag{B.3}$$

Proof. The proof is similar to that of [1, Lemma 12], and therefore we omit it. ∎

Let us now recall the definition of an ε-partition of unity from Section 3.4. The central result in this section is the following lemma.

Lemma B.2. *Let $n, m \in \mathbb{N}_0$, $3/2 < r' < r < 2$, $f \in \mathrm{H}^r(\mathbb{S})$, $a, b \in \mathrm{H}^{r-1}(\mathbb{S})$, and $\eta > 0$ be given. Then, for any sufficiently small $\varepsilon \in (0,1)$, there exists a constant $K > 0$ that depends on $\varepsilon, n, m, \|f\|_{\mathrm{H}^r}$, and $\|(a,b)\|_{\mathrm{H}^{r-1}}$ such that for all $1 \le j \le N$ and $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, we have*

$$\left\| \pi_j^\varepsilon a C_{n,m}^0(f)[b\varphi] - \frac{a(x_j^\varepsilon)b(x_j^\varepsilon)(f'(x_j^\varepsilon))^n}{\left[1 + (f'(x_j^\varepsilon))^2\right]^m} H[\pi_j^\varepsilon \varphi] \right\|_{\mathrm{H}^{r-1}} \le \eta \|\pi_j^\varepsilon \varphi\|_{\mathrm{H}^{r-1}} + K \|\varphi\|_{\mathrm{H}^{r'-1}}. \tag{B.4}$$

The proof of Lemma B.2 relies heavily on the result provided by the next lemma.

Lemma B.3. *Given $n, m \in \mathbb{N}_0$, $3/2 < r < 2$, $\eta \in (0, \infty)$, and $f \in \mathrm{H}^r(\mathbb{S})$, for sufficiently small $\varepsilon \in (0,1)$ and all $1 \le j \le N$, $|y| \le \varepsilon$, and $\varphi \in \mathrm{L}^2(\mathbb{S})$, we have*

$$\left\| T_j^\varepsilon(f)[\tau_y(\pi_j^\varepsilon \varphi) - \pi_j^\varepsilon \varphi] \right\|_2 \le \eta \|\tau_y(\pi_j^\varepsilon \varphi) - \pi_j^\varepsilon \varphi\|_2, \tag{B.5}$$

where $T_j^\varepsilon(f) := \chi_j^\varepsilon C_{n+1,m}(f, \ldots, f)[f, \ldots, f, f - f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}}, \cdot]$.

Proof. Let $\varepsilon \in (0,1)$. Since

$$T_j^\varepsilon(f)[\tau_y(\pi_j^\varepsilon \varphi) - \pi_j^\varepsilon \varphi] = \chi_j^\varepsilon \big(C_{n+1,m}^0(f) - f'(x_j^\varepsilon) C_{n,m}^0(f)\big)[\tau_y(\pi_j^\varepsilon \varphi) - \pi_j^\varepsilon \varphi] \in \mathrm{L}^2(\mathbb{S}),$$

we have by Lemma A.1 (i) that

$$\|T_j^\varepsilon(f)[\tau_y(\pi_j^\varepsilon \varphi) - \pi_j^\varepsilon \varphi]\|_2 = \|T_j^\varepsilon(f)[\tau_y(\pi_j^\varepsilon \varphi) - \pi_j^\varepsilon \varphi]\|_{\mathrm{L}^2((x_j^\varepsilon - \pi, x_j^\varepsilon + \pi))}. \tag{B.6}$$

We now introduce the Lipschitz continuous function $F_j : \mathbb{R} \to \mathbb{R}$ that satisfies $F_j = f$ on J_j^ε and $F_j' = f'(x_j^\varepsilon)$ on $\mathbb{R} \setminus J_j^\varepsilon$. Given $\xi \in (x_j^\varepsilon - \pi, x_j^\varepsilon + \pi)$, we then have

$$\begin{aligned}
&T_j^\varepsilon(f)[\tau_y(\pi_j^\varepsilon \varphi) - \pi_j^\varepsilon \varphi](\xi) \\
&= \chi_j^\varepsilon(\xi) \frac{1}{\pi} \mathrm{PV} \int_{-\pi}^{\pi} \phi\left(\frac{\delta_{[\xi,s]} f}{s}\right) \frac{\delta_{[\xi,s]}(f - f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}})}{s} \frac{(\tau_y(\pi_j^\varepsilon \varphi) - \pi_j^\varepsilon \varphi)(\xi - s)}{s} \, ds \\
&= \chi_j^\varepsilon(\xi) \frac{1}{\pi} \mathrm{PV} \int_{-\pi}^{\pi} \phi\left(\frac{\delta_{[\xi,s]} f}{s}\right) \frac{\delta_{[\xi,s]}(F_j - f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}})}{s} \frac{\big(\tau_y(\pi_j^\varepsilon \varphi) - \pi_j^\varepsilon \varphi\big)(\xi - s)}{s} \, ds \\
&= \big(\chi_j^\varepsilon C_{n+1,m}(f, \ldots, f)[f, \ldots, f, F_j - f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}}, \tau_y(\pi_j^\varepsilon \varphi) - \pi_j^\varepsilon \varphi]\big)(\xi),
\end{aligned} \tag{B.7}$$

where $\phi(x) = x^n(1 + x^2)^{-m}$, $x \in \mathbb{R}$. Indeed, if on the one hand $\xi \in (x_j^\varepsilon - \pi, x_j^\varepsilon + \pi) \setminus J_j^\varepsilon$, this is a consequence of $\chi_j^\varepsilon(\xi) = 0$. If on the other hand $\xi \in J_j^\varepsilon$, then $f(\xi) = F_j(\xi)$ by the definition of F_j. Since $|s| < \pi$, we have $\xi - s \in (x_j^\varepsilon - 3\pi/2, x_j^\varepsilon + 3\pi/2)$ for sufficiently small ε, while $\operatorname{supp} \pi_j^\varepsilon \cap (x_j^\varepsilon - 3\pi/2, x_j^\varepsilon + 3\pi/2) = I_j^\varepsilon$. Therefore, in the case $\xi - s \notin J_j^\varepsilon$ it holds that $\xi - s + y \notin I_j^\varepsilon$ for all $|y| \le \varepsilon$, hence, $\pi_j^\varepsilon \varphi(\xi - s) = \tau_y(\pi_j^\varepsilon \varphi)(\xi - s) = 0$. As a direct consequence, the integrand is not zero at most when $\xi - s \in J_j^\varepsilon$, and in this case, we also have $f(\xi - s) = F_j(\xi - s)$. This proves (B.7).

Lemma A.1 (i) together with (B.6), (B.7), and the definition of F_j enables us to deduce that there exists a constant $C>0$ such that for all $1\le j\le N$, $|y|\le\varepsilon$, and $\varphi\in L^2(\mathbb{S})$, we have

$$\|T_j^\varepsilon(f)[\tau_y(\pi_j^\varepsilon\varphi)-\pi_j^\varepsilon\varphi]\|_2\le C\|f'-f'(x_j^\varepsilon)\|_{L^\infty(J_j^\varepsilon)}\|\tau_y(\pi_j^\varepsilon\varphi)-\pi_j^\varepsilon\varphi\|_2.$$

The estimate (B.5) follows by choosing $\varepsilon\in(0,1)$ sufficiently small since $f'\in C^{r-3/2}(\mathbb{S})$. ■

We are now in a position to establish Lemma B.2.

Proof of Lemma B.2. In the following, we denote constants that do not depend on ε by C and constants that depend on ε by K.

Recalling that $H=B_{0,0}^{0,0}$, cf. (3.4), the relation $H=A_{0,0}^{1,0}+C_{0,0}$, cf. (A.2), together with Lemma A.2 yields

$$\|(H-C_{0,0})[\pi_j^\varepsilon\varphi]\|_{H^{r-1}}\le C\|A_{0,0}^{1,0}[\pi_j^\varepsilon\varphi]\|_{C^1}\le C\|\pi_j^\varepsilon\varphi\|_\infty\le K\|\varphi\|_{H^{r'-1}},$$

and therefore,

$$\begin{aligned}&\Big\|\pi_j^\varepsilon aC_{n,m}^0(f)[b\varphi]-\frac{a(x_j^\varepsilon)b(x_j^\varepsilon)(f'(x_j^\varepsilon))^n}{[1+(f'(x_j^\varepsilon))^2]^m}H[\pi_j^\varepsilon\varphi]\Big\|_{H^{r-1}}\\&\quad\le\Big\|\pi_j^\varepsilon aC_{n,m}^0(f)[b\varphi]-\frac{a(x_j^\varepsilon)b(x_j^\varepsilon)(f'(x_j^\varepsilon))^n}{[1+(f'(x_j^\varepsilon))^2]^m}C_{0,0}[\pi_j^\varepsilon\varphi]\Big\|_{H^{r-1}}+K\|\varphi\|_{H^{r'-1}}.\end{aligned}$$

To estimate the first term on the left-hand side of the latter inequality, we write

$$\begin{aligned}&\pi_j^\varepsilon aC_{n,m}^0(f)[b\varphi]-\frac{a(x_j^\varepsilon)b(x_j^\varepsilon)(f'(x_j^\varepsilon))^n}{\big[1+(f'(x_j^\varepsilon))^2\big]^m}C_{0,0}[\pi_j^\varepsilon\varphi]\\&\quad=a(T_1+T_2)+b(x_j^\varepsilon)(T_3+a(x_j^\varepsilon)T_4),\end{aligned}$$

where

$$\begin{aligned}T_1&:=\pi_j^\varepsilon C_{n,m}^0(f)[(b-b(x_j^\varepsilon))\varphi]-C_{n,m}^0(f)[\pi_j^\varepsilon(b-b(x_j^\varepsilon))\varphi],\\T_2&:=C_{n,m}^0(f)[\pi_j^\varepsilon(b-b(x_j^\varepsilon))\varphi],\\T_3&:=\pi_j^\varepsilon aC_{n,m}^0(f)[\varphi]-a(x_j^\varepsilon)C_{n,m}^0(f)[\pi_j^\varepsilon\varphi],\\T_4&:=C_{n,m}^0(f)[\pi_j^\varepsilon\varphi]-\frac{(f'(x_j^\varepsilon))^n}{\big[1+(f'(x_j^\varepsilon))^2\big]^m}C_{0,0}[\pi_j^\varepsilon\varphi].\end{aligned}$$

We consider these terms successively.

The term aT_1. In view of Lemma B.1 and of the algebra property of $H^{r-1}(\mathbb{S})$, we have

$$\|aT_1\|_{H^{r-1}}\le K\|(b-b(x_j^\varepsilon))\varphi\|_2\le K\|\varphi\|_{H^{r'-1}}.\tag{B.8}$$

The term aT_2. We use Lemma A.1 (iii), (3.34), the identity $\chi_j^\varepsilon \pi_j^\varepsilon = \pi_j^\varepsilon$, and the algebra property of $\mathrm{H}^{r-1}(\mathbb{S})$ to obtain, in view of $b \in \mathrm{C}^{r-3/2}(\mathbb{S})$, that is,

$$\begin{aligned}\|aT_2\|_{\mathrm{H}^{r-1}} &\leq C\|\pi_j^\varepsilon(b-b(x_j^\varepsilon))\varphi\|_{\mathrm{H}^{r-1}}\\ &\leq C\|\chi_j^\varepsilon(b-b(x_j^\varepsilon))\|_\infty\|\pi_j^\varepsilon\varphi\|_{\mathrm{H}^{r-1}}+K\|\varphi\|_{\mathrm{H}^{r'-1}}\\ &\leq (\eta/3)\|\pi_j^\varepsilon\varphi\|_{\mathrm{H}^{r-1}}+K\|\varphi\|_{\mathrm{H}^{r'-1}},\end{aligned} \tag{B.9}$$

provided that $\varepsilon \in (0,1)$ is sufficiently small.

The term $b(x_j^\varepsilon)T_3$. Since $\chi_j^\varepsilon \pi_j^\varepsilon = \pi_j^\varepsilon$, we have $T_3 = T_{3,1} + T_{3,2} + T_{3,3}$, where

$$\begin{aligned}T_{3,1} &:= (\chi_j^\varepsilon a)\big(\pi_j^\varepsilon C_{n,m}^0(f)[\varphi] - C_{n,m}^0(f)[\pi_j^\varepsilon\varphi]\big),\\ T_{3,2} &:= \chi_j^\varepsilon(a-a(x_j^\varepsilon))C_{n,m}^0(f)[\pi_j^\varepsilon\varphi],\\ T_{3,3} &:= a(x_j^\varepsilon)\big(\chi_j^\varepsilon C_{n,m}^0(f)[\pi_j^\varepsilon\varphi] - C_{n,m}^0(f)[\chi_j^\varepsilon(\pi_j^\varepsilon\varphi)]\big),\end{aligned}$$

and Lemma B.1 yields

$$\|b(x_j^\varepsilon)T_{3,1}\|_{\mathrm{H}^{r-1}} + \|b(x_j^\varepsilon)T_{3,3}\|_{\mathrm{H}^{r-1}} \leq K\|\varphi\|_{\mathrm{H}^{r'-1}}.$$

Moreover, (3.34), Lemma A.1 (iii), and the property $a \in \mathrm{C}^{r-3/2}(\mathbb{S})$ lead us to

$$\begin{aligned}\|b(x_j^\varepsilon)T_{3,2}\|_{\mathrm{H}^{r-1}} &\leq C\|\chi_j^\varepsilon(a-a(x_j^\varepsilon))\|_\infty\|C_{n,m}^0(f)[\pi_j^\varepsilon\varphi]\|_{\mathrm{H}^{r-1}} + K\|C_{n,m}^0(f)[\pi_j^\varepsilon\varphi]\|_{\mathrm{H}^{r'-1}}\\ &\leq (\eta/3)\|\pi_j^\varepsilon\varphi\|_{\mathrm{H}^{r-1}} + K\|\varphi\|_{\mathrm{H}^{r'-1}},\end{aligned}$$

provided that $\varepsilon \in (0,1)$ is small enough, and therefore,

$$\|b(x_j^\varepsilon)T_3\|_{\mathrm{H}^{r-1}} \leq (\eta/3)\|\pi_j^\varepsilon\varphi\|_{\mathrm{H}^{r-1}} + K\|\varphi\|_{\mathrm{H}^{r'-1}}. \tag{B.10}$$

The term $(ab)(x_j^\varepsilon)T_4$. Using again the relation $\chi_j^\varepsilon \pi_j^\varepsilon = \pi_j^\varepsilon$, we have $T_4 = T_{4,1} + T_{4,2}$, where

$$\begin{aligned}T_{4,1} &:= \frac{(f'(x_j^\varepsilon))^n}{[1+(f'(x_j^\varepsilon))^2]^m}\big(\chi_j^\varepsilon C_{0,0}[\pi_j^\varepsilon\varphi] - C_{0,0}[\chi_j^\varepsilon(\pi_j^\varepsilon\varphi)]\big)\\ &\qquad - \big(\chi_j^\varepsilon C_{n,m}^0(f)[\pi_j^\varepsilon\varphi] - C_{n,m}^0(f)[\chi_j^\varepsilon(\pi_j^\varepsilon\varphi)]\big),\\ T_{4,2} &:= \chi_j^\varepsilon\bigg(C_{n,m}^0(f)[\pi_j^\varepsilon\varphi] - \frac{(f'(x_j^\varepsilon))^n}{[1+(f'(x_j^\varepsilon))^2]^m}C_{0,0}[\pi_j^\varepsilon\varphi]\bigg),\end{aligned}$$

and, by Lemma B.1,

$$\|T_{4,1}\|_{\mathrm{H}^{r-1}} \leq K\|\varphi\|_2. \tag{B.11}$$

It remains to estimate the term $T_{4,2}$ for which we first use Lemma A.1 (i) to deduce that

$$\|T_{4,2}\|_2 \leq K\|\varphi\|_2. \tag{B.12}$$

In order to estimate the seminorm $[T_{4,2}]_{W^{r-1,2}}$, we note, by using (B.1) together with the identity $f'(x_j^\varepsilon) = \delta_{[\xi,s]}(f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}})/s$, that

$$T_{4,2} = \sum_{k=0}^{n-1} (f'(x_j^\varepsilon))^{n-k-1} \chi_j^\varepsilon C_{k+1,m}(f,\dots,f)[f,\dots,f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon\varphi]$$
$$-\sum_{k=0}^{m-1} \frac{(f'(x_j^\varepsilon))^n}{\big[1+(f'(x_j^\varepsilon))^2\big]^{m-k}} \chi_j^\varepsilon C_{2,k+1}(f,\dots,f)[f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon\varphi]$$
$$-\sum_{k=0}^{m-1} \frac{(f'(x_j^\varepsilon))^{n+1}}{\big[1+(f'(x_j^\varepsilon))^2\big]^{m-k}} \chi_j^\varepsilon C_{1,k+1}(f,\dots,f)[f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon\varphi].$$

Consequently,

$$[T_{4,2}]_{W^{r-1,2}} \le C_0\Bigg(\sum_{k=0}^{n-1}\big[\chi_j^\varepsilon C_{k+1,m}(f,\dots,f)[f,\dots,f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon\varphi]\big]_{W^{r-1,2}}$$
$$+\sum_{k=0}^{m-1}\big[\chi_j^\varepsilon C_{2,k+1}(f,\dots,f)[f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon\varphi]\big]_{W^{r-1,2}}$$
$$+\sum_{k=0}^{m-1}\big[\chi_j^\varepsilon C_{1,k+1}(f,\dots,f)[f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon\varphi]\big]_{W^{r-1,2}}\Bigg). \tag{B.13}$$

Set

$$S_k := \chi_j^\varepsilon C_{k+1,m}(f,\dots,f)[f,\dots,f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon\varphi], \quad 0\le k\le n-1.$$

In order to estimate the $W^{r-1,2}$-seminorm of S_k, we write for $y \in (-\pi,\pi)$

$$\tau_y S_k - S_k = S_{k,1} + S_{k,2} + \chi_j^\varepsilon S_{k,3},$$

where, using again (B.1), we have

$$S_{k,1} := (\tau_y\chi_j^\varepsilon - \chi_j^\varepsilon)\tau_y C_{k+1,m}(f,\dots,f)[f,\dots,f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon\varphi],$$
$$S_{k,2} := \chi_j^\varepsilon C_{k+1,m}(f,\dots,f)[f,\dots,f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\tau_y(\pi_j^\varepsilon\varphi)-\pi_j^\varepsilon\varphi],$$
$$S_{k,3} := \sum_{i=1}^{k} C_{k+1,m}(f,\dots,f)[\underbrace{f,\dots,f}_{i-1},\tau_y f - f,\tau_y f,\dots,\tau_y f, f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\tau_y(\pi_j^\varepsilon\varphi)]$$
$$+ C_{k+1,m}(f,\dots,f)[\tau_y f,\dots,\tau_y f,\tau_y f - f,\tau_y(\pi_j^\varepsilon\varphi)]$$
$$-\sum_{i=1}^{m} C_{k+3,m+1}^i[\tau_y f,\dots,\tau_y f,\tau_y f - f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\tau_y f + f,\tau_y f - f,\tau_y(\pi_j^\varepsilon\varphi)]$$

and

$$C_{k+3,m+1}^i := C_{k+3,m+1}(\underbrace{f,\dots,f}_{i},\tau_y f,\dots,\tau_y f).$$

Lemma A.1 (ii) (with $r = r'$) yields

$$\|S_{k,1}\|_2 \leq K\|\tau_y \chi_j^\varepsilon - \chi_j^\varepsilon\|_2\|\varphi\|_{\mathrm{H}^{r'-1}}.$$

To estimate $S_{k,2}$, we consider two cases. If $|y| > \varepsilon$, we use Lemma A.1 (i) and obtain

$$\|S_{k,2}\|_2 \leq K\|\varphi\|_2.$$

If $|y| \leq \varepsilon$, we use (B.5), which gives

$$\|S_{k,2}\|_2 \leq (\eta/C_1)\|\tau_y(\pi_j^\varepsilon\varphi) - \pi_j^\varepsilon\varphi\|_2,$$

provided that $\varepsilon \in (0, 1)$ is small enough, with a positive constant C_1 which we fix below. Finally, Lemma A.1 (ii) (with $r = r'$) produces

$$\|\chi_j^\varepsilon S_{k,3}\|_2 \leq K\|\tau_y f' - f'\|_2\|\varphi\|_{\mathrm{H}^{r'-1}}.$$

Combining the above estimates, we have

$$[S_k]_{\mathrm{W}^{r-1,2}} \leq (\eta/C_1)\|\pi_j^\varepsilon\varphi\|_{\mathrm{H}^{r-1}} + K\|\varphi\|_{\mathrm{H}^{r'-1}}. \tag{B.14}$$

It is now obvious that all the terms on the right-hand side of (B.13) can be estimated by the right-hand side of (B.14), provided that $\varepsilon \in (0, 1)$ is sufficiently small. From (B.12)–(B.14), we then deduce, after choosing $C_1 := 3CC_0(n + 2m)(1 + \|ab\|_\infty)$, that is,

$$\begin{aligned}\|T_{4,2}\|_{\mathrm{H}^{r-1}} &\leq C(\|T_{4,2}\|_2 + [T_{4,2}]_{\mathrm{W}^{r-1,2}})\\ &\leq \frac{CC_0(n+2m)\eta}{C_1}\|\pi_j^\varepsilon\varphi\|_{\mathrm{H}^{r-1}} + K\|\varphi\|_{\mathrm{H}^{r'-1}}\\ &\leq \frac{\eta}{3(1+\|ab\|_\infty)}\|\pi_j^\varepsilon\varphi\|_{\mathrm{H}^{r-1}} + K\|\varphi\|_{\mathrm{H}^{r'-1}},\end{aligned}$$

and together with (B.11), we get

$$\|(ab)(x_j^\varepsilon)T_4\|_{\mathrm{H}^{r-1}} \leq (\eta/3)\|\pi_j^\varepsilon\varphi\|_{\mathrm{H}^{r-1}} + K\|\varphi\|_{\mathrm{H}^{r'-1}}. \tag{B.15}$$

Gathering (B.8)-(B.10) and (B.15), we obtain (B.4), and the proof is complete. ■

C. The behavior of the pressure and velocity near the interface and in the far-field

In this section we consider the function $(v^\pm, q^\pm)$ defined in (2.12)–(2.13) and prove, under the assumptions in Theorem 2.2, that $(v^\pm, q^\pm)$ satisfies the boundary conditions $(2.1)_{3-4}$, as well as the far field boundary condition $(2.1)_5$, see Lemmas C.6 and C.4 below.

Thus, in this section, we fix $f \in \mathrm{H}^3(\mathbb{S})$ and use the notation introduced in Section 2. Some additional notation is also needed. Given a function $w : (\mathbb{S}\times\mathbb{R})\setminus\Gamma \to \mathbb{R}$, we set $w^\pm := w\big|_{\Omega^\pm}$ and denote by

$$\{w\}^\pm \circ \Xi(\xi) := \lim_{\Omega^\pm \ni x \to (\xi, f(\xi))} w(x), \quad \xi \in \mathbb{S},$$

the one-sided limits of w in $\Xi(\xi)$, whenever these limits exist. Conversely, given functions $w^\pm : \Omega^\pm \to \mathbb{R}$, we set

$$w := \mathbf{1}_{\Omega^+} w^+ + \mathbf{1}_{\Omega^-} w^-,$$

which is viewed as a function defined almost everywhere in $\mathbb{S}\times\mathbb{R}$. Moreover, since the gradient $\nabla v^\pm$ is determined by simply differentiating under the integral sign in (2.13) (see the proof of Theorem 2.2), we need to calculate the first order partial derivatives of $\mathcal{U}$. From formula (2.8) we infer, that, for given $x \in (\mathbb{S}\times\mathbb{R})\setminus\{0\}$, they are given by the following expressions:

$$\begin{aligned}
\partial_1 \mathcal{U}^{1\top}(x) &= \frac{1}{8\pi}\begin{pmatrix} \frac{t_{[x_1]}(1-T^2_{[x_2]})}{t^2_{[x_1]}+T^2_{[x_2]}} - x_2 \frac{t_{[x_1]}T_{[x_2]}(1+t^2_{[x_1]})(1-T^2_{[x_2]})}{(t^2_{[x_1]}+T^2_{[x_2]})^2} \\ \frac{x_2}{2}\frac{(1+t^2_{[x_1]})(1-T^2_{[x_2]})(t^2_{[x_1]}-T^2_{[x_2]})}{(t^2_{[x_1]}+T^2_{[x_2]})^2} \end{pmatrix},\\
\partial_2 \mathcal{U}^{1\top}(x) &= \frac{1}{8\pi}\begin{pmatrix} 2\frac{T_{[x_2]}(1+t^2_{[x_1]})}{t^2_{[x_1]}+T^2_{[x_2]}} + \frac{x_2}{2}\frac{(1+t^2_{[x_1]})(1-T^2_{[x_2]})(t^2_{[x_1]}-T^2_{[x_2]})}{(t^2_{[x_1]}+T^2_{[x_2]})^2} \\ -\frac{t_{[x_1]}(1-T^2_{[x_2]})}{t^2_{[x_1]}+T^2_{[x_2]}} + x_2 \frac{t_{[x_1]}T_{[x_2]}(1+t^2_{[x_1]})(1-T^2_{[x_2]})}{(t^2_{[x_1]}+T^2_{[x_2]})^2} \end{pmatrix},\\
\partial_1 \mathcal{U}^{2\top}(x) &= \frac{1}{8\pi}\begin{pmatrix} \frac{x_2}{2}\frac{(1+t^2_{[x_1]})(1-T^2_{[x_2]})(t^2_{[x_1]}-T^2_{[x_2]})}{(t^2_{[x_1]}+T^2_{[x_2]})^2} \\ \frac{t_{[x_1]}(1-T^2_{[x_2]})}{t^2_{[x_1]}+T^2_{[x_2]}} + x_2 \frac{t_{[x_1]}T_{[x_2]}(1+t^2_{[x_1]})(1-T^2_{[x_2]})}{(t^2_{[x_1]}+T^2_{[x_2]})^2} \end{pmatrix},\\
\partial_2 \mathcal{U}^{2\top}(x) &= \frac{1}{8\pi}\begin{pmatrix} -\frac{t_{[x_1]}(1-T^2_{[x_2]})}{t^2_{[x_1]}+T^2_{[x_2]}} + x_2 \frac{t_{[x_1]}T_{[x_2]}(1+t^2_{[x_1]})(1-T^2_{[x_2]})}{(t^2_{[x_1]}+T^2_{[x_2]})^2} \\ -\frac{x_2}{2}\frac{(1+t^2_{[x_1]})(1-T^2_{[x_2]})(t^2_{[x_1]}-T^2_{[x_2]})}{(t^2_{[x_1]}+T^2_{[x_2]})^2} \end{pmatrix}.
\end{aligned} \tag{C.1}$$

This motivates us to establish first the following preparatory result.

Lemma C.1. *Given $\varphi \in \mathrm{L}^2(\mathbb{S})$, let $Z_n(f)[\varphi] : (\mathbb{S}\times\mathbb{R})\setminus\Gamma \to \mathbb{R}$, $1 \le n \le 6$, be defined by*

$$Z_1(f)[\varphi](x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{t_{[r_1]}\big(1-T^2_{[r_2]}\big)}{t^2_{[r_1]}+T^2_{[r_2]}}\varphi(s)\,\mathrm{d}s,$$

$$
\begin{aligned}
Z_2(f)[\varphi](x) &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{T_{[r_2]}\big(1+t_{[r_1]}^2\big)}{t_{[r_1]}^2+T_{[r_2]}^2}\varphi(s)\,\mathrm{d}s,\\
Z_3(f)[\varphi](x) &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r_2}{2}\frac{\big(1+t_{[r_1]}^2\big)\big(1-T_{[r_2]}^2\big)\big(t_{[r_1]}^2-T_{[r_2]}^2\big)}{\big(t_{[r_1]}^2+T_{[r_2]}^2\big)^2}\varphi(s)\,\mathrm{d}s,\\
Z_4(f)[\varphi](x) &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r_2}{2}\frac{t_{[r_1]}T_{[r_2]}\big(1+t_{[r_1]}^2\big)\big(1-T_{[r_2]}^2\big)}{\big(t_{[r_1]}^2+T_{[r_2]}^2\big)^2}\varphi(s)\,\mathrm{d}s,\\
Z_5(f)[\varphi](x) &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} r_2\frac{t_{[r_1]}\big(1-T_{[r_2]}^2\big)}{t_{[r_1]}^2+T_{[r_2]}^2}\varphi(s)\,\mathrm{d}s,\\
Z_6(f)[\varphi](x) &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} r_2\frac{T_{[r_2]}\big(1+t_{[r_1]}^2\big)}{t_{[r_1]}^2+T_{[r_2]}^2}\varphi(s)\,\mathrm{d}s,
\end{aligned}
\tag{C.2}
$$

where $r=(r_1,r_2)$ is defined by

$$
r:=r(x,s):=x-(s,f(s)),\quad x\in\Omega^\pm,\quad s\in\mathbb{R}. \tag{C.3}
$$

We set

$$
B_n(f)[\varphi](\xi):=\mathrm{PV}\big(Z_n(f)[\varphi](\Xi(\xi))\big),\quad 1\le n\le 6,\quad \xi\in\mathbb{S}. \tag{C.4}
$$

Then, $Z_n(f)[\varphi]^\pm\in \mathrm{C}^\infty(\Omega^\pm)$, $1\le n\le 6$, and $Z_n(f)[\varphi]\in \mathrm{C}(\mathbb{S}\times\mathbb{R})$, $n=5,6$, with

$$
\{Z_n(f)[\varphi]\}^\pm\circ\Xi=B_n(f)[\varphi],\quad n=5,6. \tag{C.5}
$$

Moreover, if additionally $\varphi\in \mathrm{H}^1(\mathbb{S})$, then $Z_n(f)[\varphi]^\pm\in \mathrm{C}(\overline{\Omega^\pm})$, $1\le n\le 4$, and

$$
\left\{\begin{pmatrix} Z_1(f)[\varphi]\\ Z_2(f)[\varphi]\\ Z_3(f)[\varphi]\\ Z_4(f)[\varphi]\end{pmatrix}\right\}^\pm\circ\Xi=\begin{pmatrix} B_1(f)[\varphi]\\ B_2(f)[\varphi]\\ B_3(f)[\varphi]\\ B_4(f)[\varphi]\end{pmatrix}\pm\frac{1}{\omega^2}\begin{pmatrix}-f'\\ 1\\ -\frac{2f'^2}{\omega^2}\\ \frac{f'-f'^3}{2\omega^2}\end{pmatrix}\varphi. \tag{C.6}
$$

Related to the definition (C.4), we observe that we may evaluate the integrals (C.2) at $\Xi(\xi)$ with $\xi\in\mathbb{S}$, provided that we interpret some of the integrals as being singular, see Lemmas A.2 and A.3, since the operators $B_n(f)$, $1\le n\le 6$, can be represented as linear combinations of the operators $B^{p,q}_{n,m}(f)$, $n,m,p,q\in\mathbb{N}_0$, $1\le p\le n+q+1$, defined in (3.5), see (3.9). In fact, Lemma A.2 ensures that $B_n(f)[\varphi]\in \mathrm{C}(\mathbb{S})$, $n=5,6$, while, for $\varphi\in \mathrm{H}^1(\mathbb{S})$, we also have $B_n(f)[\varphi]\in \mathrm{C}(\mathbb{S})$, $1\le n\le 4$, cf. Lemmas A.2 and A.4.

Proof of Lemma C.1. Arguing as in the proof of Theorem 2.2, it immediately follows that the function $Z_n(f)[\varphi]^\pm$ belongs to $\mathrm{C}^\infty(\Omega^\pm)$ for $1\le n\le 6$. Moreover, Lebesgue's dominated convergence theorem leads to

$$
\{Z_n(f)[\varphi]\}^\pm\circ\Xi=B_n(f)[\varphi]\in \mathrm{C}(\mathbb{S}),\quad n=5,6,
$$

so that $Z_n(f)[\varphi]\in \mathrm{C}(\mathbb{S}\times\mathbb{R})$ for $n=5,6$. This proves (C.5).

In the remaining, we assume that $\varphi \in \mathrm{H}^1(\mathbb{S})$. Since $B_n(f) \in \mathrm{C}(\mathbb{S})$, $n = 1, 2$, together with [26, Lemma 2.2], we conclude that $Z_n(f)[\varphi]^\pm \in \mathrm{C}(\overline{\Omega^\pm})$ for $n = 1, 2$, with

$$\begin{aligned} \{Z_1(f)[\varphi]\}^\pm \circ \Xi &= B_1(f)[\varphi] \mp \frac{f'}{\omega^2}\varphi, \\ \{Z_2(f)[\varphi]\}^\pm \circ \Xi &= B_2(f)[\varphi] \pm \frac{1}{\omega^2}\varphi. \end{aligned} \tag{C.7}$$

In order to derive similar properties for $Z_n(f)[\varphi]$, $n = 3, 4$, we use integration by parts to deduce that

$$\left.\begin{aligned} Z_5(f)[\varphi'] &= Z_1(f)[f'\varphi] - Z_3(f)[\varphi] - 2Z_4(f)[f'\varphi], \\ Z_6(f)[\varphi'] &= Z_2(f)[f'\varphi] + Z_3(f)[f'\varphi] - 2Z_4(f)[\varphi] \end{aligned}\right\} \quad \text{in } (\mathbb{S}\times\mathbb{R})\setminus\Gamma,$$

respectively,

$$\left.\begin{aligned} B_5(f)[\varphi'] &= B_1(f)[f'\varphi] - B_3(f)[\varphi] - 2B_4(f)[f'\varphi], \\ B_6(f)[\varphi'] &= B_2(f)[f'\varphi] + B_3(f)[f'\varphi] - 2B_4(f)[\varphi] \end{aligned}\right\} \quad \text{in } \mathrm{C}(\mathbb{S}).$$

Since $Z_n(f)[\varphi'] \in \mathrm{C}(\mathbb{S}\times\mathbb{R})$, $n = 5, 6$, the latter formulas combined with (C.5) and (C.7) (with φ replaced by $f'\varphi$) yield

$$\begin{aligned} \{Z_3(f)[\varphi] + 2Z_4(f)[f'\varphi]\}^\pm \circ \Xi &= B_3(f)[\varphi] + 2B_4(f)[f'\varphi] \mp \frac{f'^2}{\omega^2}\varphi, \\ \{Z_3(f)[f'\varphi] - 2Z_4(f)[\varphi]\}^\pm \circ \Xi &= B_3(f)[f'\varphi] - 2B_4(f)[\varphi] \mp \frac{f'}{\omega^2}\varphi. \end{aligned} \tag{C.8}$$

We now replace φ by φ/ω^2 in $(\mathrm{C.8})_1$ and by $(f'\varphi)/\omega^2$ in $(\mathrm{C.8})_2$ to obtain, after taking the sum of the two relations, that

$$\begin{aligned} \{Z_3(f)[\varphi]\}^\pm \circ \Xi &= B_3(f)[\varphi] \mp \frac{2f'^2}{\omega^4}\varphi, \\ \{Z_4(f)[\varphi]\}^\pm \circ \Xi &= B_4(f)[\varphi] \pm \frac{f' - f'^3}{2\omega^4}\varphi, \end{aligned} \tag{C.9}$$

with $(\mathrm{C.9})_2$ being a direct consequence of $(\mathrm{C.9})_1$ and $(\mathrm{C.8})_2$. This proves (C.6) and completes the proof. ∎

As a further preparatory result, we establish the following lemma which is related to the logarithmic term in $\mathcal{U}$, see (2.5) and (2.8).

Lemma C.2. *Given $\varphi \in \mathrm{L}^2(\mathbb{S})$, let $Z_0(f)[\varphi] : (\mathbb{S}\times\mathbb{R})\setminus\Gamma \to \mathbb{R}$ be given by*

$$Z_0(f)[\varphi](x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\Big(\sin^2\Big(\frac{r_1}{2}\Big) + \sinh^2\Big(\frac{r_2}{2}\Big)\Big)\varphi(s)\,\mathrm{d}s. \tag{C.10}$$

Then, $Z_0(f)[\varphi] \in \mathrm{C}^\infty((\mathbb{S}\times\mathbb{R})\setminus\Gamma)$ and $\nabla\big(Z_0(f)[\varphi]\big) = (Z_1(f)[\varphi], Z_2(f)[\varphi])$. Additionally, if $\varphi \in \mathrm{H}^1(\mathbb{S})$, we have $Z_0(f)[\varphi] \in \mathrm{C}(\mathbb{S}\times\mathbb{R})$ and $Z_0(f)[\varphi]^\pm \in \mathrm{C}^1(\overline{\Omega^\pm}, \mathbb{R}^2)$, with

$$\{Z_0(f)[\varphi]\}^\pm \circ \Xi = B_0(f)[\varphi], \tag{C.11}$$

where

$$B_0(f)[\varphi](\xi) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\big(\sin^2(s/2)+\sinh^2(\delta_{[\xi,s]}f/2)\big)\varphi(\xi-s)\,\mathrm{d}s, \qquad \xi\in\mathbb{S}. \quad \text{(C.12)}$$

Proof. Arguing as in the proof of Theorem 2.2, we obtain that $Z_0(f)[\varphi]^\pm \in \mathrm{C}^\infty(\Omega^\pm)$, with gradient

$$\nabla\big(Z_0(f)[\varphi]\big) = (Z_1(f)[\varphi], Z_2(f)[\varphi]).$$

Since $Z_n(f)[\varphi]^\pm \in \mathrm{C}^1(\overline{\Omega^\pm})$, $n=1,2$, for $\varphi\in \mathrm{H}^1(\mathbb{S})$, cf. Lemma C.1, we deduce that $Z_0(f)[\varphi]^\pm \in \mathrm{C}^1(\overline{\Omega^\pm},\mathbb{R}^2)$. Additionally, Lebesgue's dominated convergence theorem ensures that both one-sided limits of $Z_0(f)[\varphi]$ in $\Xi(\xi)$ exist for all $\xi\in\mathbb{S}$ and coincide with $B_0(f)[\varphi](\xi)$ (which exists as an improper integral). This proves (C.11) and the continuity property $Z_0(f)[\varphi]\in \mathrm{C}(\mathbb{S}\times\mathbb{R})$. ∎

Related to the asymptotic behavior of the operators defined above, we establish the following lemma.

Lemma C.3. *Given $\varphi\in \mathrm{L}^2(\mathbb{S})$ for $x_2\to\pm\infty$, we have*

$$Z_5(f)[\varphi]^\pm \to 0, \tag{C.13}$$

$$Z_6(f)[\varphi]^\pm \mp x_2\langle\varphi\rangle \to \mp\langle f\varphi\rangle, \tag{C.14}$$

$$Z_0(f)[\varphi]^\pm \mp x_2\langle\varphi\rangle \to \mp\langle f\varphi\rangle - \langle\varphi\rangle\ln 4. \tag{C.15}$$

Proof. The property (C.13) is a simple consequence of Lebesgue's dominated convergence theorem, which implies, via

$$Z_6(f)[\varphi]^\pm(x) \mp x_2\langle\varphi\rangle \pm \langle f\varphi\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} r_2(1\mp T_{[r_2]})\frac{T_{[r_2]}-t_{[r_1]}^2}{t_{[r_1]}^2+T_{[r_2]}^2}\varphi(s)\,\mathrm{d}s, \qquad x\in\Omega^\pm,$$

also (C.14). Finally, with respect to (C.15), we note that since

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln(4^{\mp1}e^{r_2})\varphi\,\mathrm{d}s = \langle f\varphi\rangle + (\pm\ln 4 - x_2)\langle\varphi\rangle, \qquad x\in\Omega^\pm,$$

Lebesgue's dominated convergence theorem yields

$$\begin{aligned}
&Z_0(f)[\varphi]^\pm(x) \pm \big[\langle f\varphi\rangle + (\pm\ln 4 - x_2)\langle\varphi\rangle\big]\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\ln\left(\sin^2\left(\frac{r_1}{2}\right)+\frac{e^{r_2}-2+e^{-r_2}}{4}\right)\mp\ln(4^{\mp1}e^{r_2})\right]\varphi(s)\,\mathrm{d}s\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left(4e^{\mp r_2}\sin^2\left(\frac{r_1}{2}\right)+e^{\mp2r_2}-2e^{\mp r_2}+1\right)\varphi(s)\,\mathrm{d}s \xrightarrow[x_2\to\pm\infty]{} 0.
\end{aligned}$$ ∎

We are now in a position to study the behavior of the velocity v defined in (2.12)–(2.13) close to the interface and in the far field (under the assumptions of Theorem 2.2).

Lemma C.4. *We have* $v \in \mathrm{C}(\mathbb{S}\times\mathbb{R},\mathbb{R}^2)$, $v^\pm \in \mathrm{C}^1(\overline{\Omega^\pm},\mathbb{R}^2)$, *and*

$$\mu\big([\nabla v + (\nabla v)^\top]\tilde{\nu}\big)\circ \Xi = \omega^{-1}(G\cdot\tau)\tau \quad \text{on } \mathbb{S}, \tag{C.16}$$

$$v^\pm(x) \to \Big(\mp\frac{\langle fG_1\rangle}{2\mu}, 0\Big) \quad \text{for } x_2 \to \pm\infty. \tag{C.17}$$

Proof. Recalling (2.13), we write

$$v_G = \frac{1}{4\mu}\begin{pmatrix}(Z_0(f)+Z_6(f))[G_1] - Z_5(f)[G_2]\\ (Z_0(f)-Z_6(f))[G_2] - Z_5(f)[G_1]\end{pmatrix}^\top \quad \text{in } (\mathbb{S}\times\mathbb{R})\setminus\Gamma, \tag{C.18}$$

and Lemmas C.1 and C.2 ensure that indeed $v_G \in \mathrm{C}(\mathbb{S}\times\mathbb{R},\mathbb{R}^2)$ and

$$\{v_G\}^\pm \circ \Xi = \frac{1}{4\mu}\begin{pmatrix}(B_0(f)+B_6(f))[G_1] - B_5(f)[G_2]\\ (B_0(f)-B_6(f))[G_2] - B_5(f)[G_1]\end{pmatrix}^\top. \tag{C.19}$$

Noticing also that

$$\left.\begin{aligned}\nabla(Z_5(f)[\varphi]) &= (-Z_3(f)[\varphi], Z_1(f)[\varphi] - 2Z_4(f)[\varphi])\\ \nabla(Z_6(f)[\varphi]) &= (-2Z_4(f)[\varphi], Z_2(f)[\varphi] + Z_3(f)[\varphi])\end{aligned}\right\} \quad \text{in } \mathbb{S}\times\mathbb{R}\setminus\Gamma,$$

we infer from Lemmas C.1 and C.2 that $v_G^\pm \in \mathrm{C}^1(\overline{\Omega^\pm},\mathbb{R}^2)$ and the formula (C.6) leads us to

$$[\nabla v_G]\circ\Xi = \begin{bmatrix}\partial_1 v_{G,1} & \partial_2 v_{G,1}\\ \partial_1 v_{G,2} & \partial_2 v_{G,2}\end{bmatrix}\circ\Xi = \frac{G\cdot\tau}{\mu\omega^3}\begin{pmatrix}-f' & 1\\ -f'^2 & f'\end{pmatrix},$$

hence,

$$\mu\big([\nabla v_G + (\nabla v_G)^\top]\tilde{\nu}\big)\circ\Xi = \omega^{-1}(G\cdot\tau)\tau,$$

and (C.16) follows.

Moreover, in view of Lemma C.3, we have

$$v_G^\pm(x) \to \Big(\mp\frac{\langle fG_1\rangle}{2\mu}, -\frac{\langle G_2\rangle \ln 4}{4\mu}\Big) \quad \text{for } x_2\to\pm\infty,$$

which proves (C.17). ∎

The following observation, together with (C.19), is used when formulating the Stokes problem (1.1) as an evolution problem for f, as it provides an expression for the trace of v_G on Γ, in the particular case when $G = F'$ for some function $F = (F_1, F_2)$, which involves the function F (and not its derivative), see (C.21) below.

Remark C.5. Assume that $G = F'$ for some function $F = (F_1, F_2) \in \mathrm{H}^2(\mathbb{S})$. Then, observing that $[s \mapsto \mathcal{U}(x-(s,f(s)))] : \mathbb{S} \to \mathbb{R}^{2\times 2}$ is continuously differentiable, integration by parts in (2.13) leads to the following representation:

$$v_G^\pm(x) = \frac{1}{\mu}\int_{-\pi}^{\pi} F(s)\Big(\partial_1\begin{pmatrix}\mathcal{U}^1\\ \mathcal{U}^2\end{pmatrix}(r) + f'(s)\partial_2\begin{pmatrix}\mathcal{U}^1\\ \mathcal{U}^2\end{pmatrix}(r)\Big)\,ds, \quad x \in \Omega^\pm. \tag{C.20}$$

In view of (C.1) and (C.2), we conclude from Lemma C.1 that

$$\{v_G\}^{\pm} \circ \Xi = \frac{1}{4\mu} \begin{pmatrix} (B_1 - 2B_4)(f)[F_1 - f'F_2] + (2B_2 + B_3)(f)[f'F_1] + B_3(f)[F_2] \\ B_1(f)[F_2 - f'F_1] + B_3(f)[F_1 - f'F_2] + 2B_4(f)[f'F_1 + F_2], \end{pmatrix}^{\top}. \tag{C.21}$$

Finally, we consider the pressure q.

Lemma C.6. *We have* $q^{\pm} \in \mathrm{C}(\overline{\Omega^{\pm}})$ *and*

$$[q] \circ \Xi = -\omega^{-1} G \cdot \nu \quad \text{on } \mathbb{S},$$
$$q^{\pm}(x) \to \mp \frac{\langle G_2 \rangle}{2} \quad \text{for } x_2 \to \pm\infty.$$

Proof. Since

$$q_G = -\frac{Z_1(f)[G_1] + Z_2(f)[G_2]}{2},$$

Lemma C.1 yields $q^{\pm} \in \mathrm{C}^1(\overline{\Omega^{\pm}})$ together with $[q_G] \circ \Xi = -\omega^{-1} G \cdot \nu$. Moreover, a simple application of Lebesgue's dominated convergence theorem shows that

$$q^{\pm}(x) \to \mp \frac{\langle G_2 \rangle}{2} \quad \text{for } x_2 \to \pm\infty,$$

which completes the proof. ∎

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