Doc. Math. (Online first)
DOI 10.4171/DM/1037

Second-countable spaces and Reverse Mathematics

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Abstract. Reverse Mathematics (often abbreviated RM) is a program in the foundations of mathematics that seeks to identify the minimal axioms needed to prove theorems of ordinary mathematics. The development of RM generally is based on the rather frugal language of second-order arithmetic, essentially including only variables for natural and real numbers. As a result, higher-order notions have to be 'coded' or 'represented' indirectly and it is a natural question – at the very heart of RM – whether this coding practise has any influence on the minimal axioms needed to prove certain theorems. In this paper, we investigate this question for basic properties of second-countable spaces, including the *Ginsburg–Sands theorem*, recently studied in RM. We show that the coding of *countable* second-countable spaces already has a significant influence, namely shifting the classification of basic statements from 'provable using arithmetical comprehension' to 'inhabits the range of hyperarithmetical analysis'. We also show that basic statements about (countable) second-countable spaces can imply – or are equivalent to – strong axioms, including countable choice, the enumeration principle, and full second-order arithmetic in various guises.

1. Introduction and preliminaries

1.1. Short summary

The study of certain classes of topological spaces in mathematical logic has been developed via 'representations' or 'codes' in the rather frugal language of second-order arithmetic, where the latter only has variables for natural numbers and real numbers (see [18, §10.8]). In this paper, we establish the following observations (O1)–(O3) concerning this coding practise for the subject of second-countable spaces that are at most the size of the continuum.

- (O1) *Hyperarithmetical shift:* coding *countable* second-countable spaces in the language of second-order arithmetic can change the logical strength of basic statements about such spaces from 'inhabits the range of hyperarithmetical analysis' to 'is provable from arithmetical comprehension'.
- (O2) *New equivalences*: certain basic third-order statements about countable second-countable spaces are equivalent to the *enumeration principle*, which simply states that certain countable sets of reals can be enumerated.

Mathematics Subject Classification 2020: 03B30 (primary); 03F35 (secondary).
Keywords: higher-order arithmetic, second-order arithmetic, coding, representations, topology, second-countable spaces, Reverse Mathematics.

(O3) *Strong Axioms*: other basic statements about second-countable spaces imply – or are equivalent to – strong axioms, including countable choice and full second-order arithmetic in various guises.

When formulated with codes/representations, the basic theorems covered by (O1)–(O3) can be proved in rather weak fragments of second-order arithmetic, usually arithmetical comprehension. Our conception of 'basic' statement includes the supremum principle for (strongly) continuous functions on compact spaces and similar theorems. We also study the *Ginsburg–Sands theorem* as the latter has recently received some attention in mathematical logic [4]. The former theorem is a prime example of (O1) by Corollary 2.12.

We believe that every mathematician should be aware of (O1)–(O3), as major foundational topics in mathematics, like *Turing computability* [3, 19, 20, 54], *computable analysis* [8–10, 56] and *Reverse Mathematics* [18, 53, 55] are based at their core on second-order codes/representations. The associated classification apparently depends greatly on this coding practise and 'less coding-heavy' alternatives should be studied, in our humble opinion. A less foundational result in this paper is the observation that basic statements about second-countable spaces, like that a continuous function is bounded or the existence of discontinuous functions, are equivalent to a known[1] fragment of the Axiom of *countable* Choice, namely by Theorems 2.9 and Corollary 2.11.

An important point regarding the above centred slogan is that the results in this paper are *robust*, i.e., we obtain the same results for many variations of the theorems at hand. Robustness is studied as a property of logical systems in mathematical logic as follows in [35, p. 432].

> *[...] one distinction that I think is worth making is the one between robust systems and non-robust systems. A system is robust if it is equivalent to small perturbations of itself. This is not a precise notion yet, but we can still recognize some robust systems.*

Finally, we provide a more detailed introduction in Section 1.2 while Section 1.3 lists required definitions and axioms. Our main results are in Sections 2.2–2.4.

1.2. Detailed summary

It is a commonplace that the abstract and general nature of topology entails a deep connection to set theory, the usual foundations of most of mathematics (see e.g. [36, Preface]). Nonetheless, topology has also been studied in much more frugal logical systems, including *second-order arithmetic* Z_2 [53, I–II]. A central topic here is to identify the minimal axioms needed to prove a given theorem of ordinary mathematics, i.e., which subsystem of Z_2 suffices for a proof. This constitutes the aim of *Reverse Mathematics* [18, 53, 55] where a central result is that the majority of theorems of ordinary mathematics are provable in rather weak fragments of Z_2, often carrying foundational import.

[1] The fragment at hand is called $\mathsf{QF\text{-}AC}^{0,1}$ and not provable in ZF, i.e., the usual foundations of mathematics without the Axiom of Choice (see [31] and Section 1.3.1).

The previous paragraph describes a theme that permeates the exact sciences: *classification*. In particular, a central topic of mathematical logic is the classification of theorems from mathematics in hierarchies according to their logical strength. It seems self-evident that such a classification should not be an artifact of the representation used for mathematical objects. However, the language of Z_2, called L_2, only has variables for natural numbers $n \in \mathbb{N}$ and sets $X \subset \mathbb{N}$. Thus, higher-order objects have to be 'represented' or 'coded' via second-order objects in L_2. Well-known third-order examples that are studied via codes are continuous functions [53, II.6.1], metric spaces [53, II.5.1], and topological spaces [18, §10.8].

The goal of this paper is to show that the coding of *second-countable* topological spaces can have a profound influence on the logical strength of the associated theorems. Our starting point is the notion of *countable second-countable space* (abbreviated 'CSC-space'), introduced by Dorais in [13, 15], and studied in [4, 18, 52]. We introduce the *third-order* version of CSC-space, called 'RSC-space', in Section 1.3.2; we stress that RSC-spaces have at most the size of the continuum. It is then a natural question, in the very spirit of RM, whether there is any difference between CSC-spaces and 'RSC-spaces that are countable', i.e., where the latter come with an injection (or bijection) to $\mathbb{N}$. The answer to this question is three-fold, mirroring the observations (O1)–(O3) from Section 1.1.

Now, on one hand, arithmetical comprehension seems to suffice to establish the known theorems about CSC-spaces in second-order RM. On the other hand, a number of basic theorems about countable RSC-spaces are equivalent to a new 'Big' system of RM, namely the *enumeration principle*, which expresses that countable sets can be enumerated. The latter is central in the RM of the Jordan decomposition theorem [42, 47, 49], as discussed in detail in Section 2.2. By 'basic theorem', we mean e.g. the supremum principle for continuous functions or the fact that compact RSC-spaces are separable. This establishes observation (O2) from Section 1.1.

The previous paragraph does not tell the *entire* story: we also identify theorems about (countable or uncountable) RSC-spaces that *inhabit the range of hyperarithmetical analysis*, where the latter italicised text is explained at the end of Section 1.3.1. This includes the *Ginsburg–Sands theorem*, recently studied for CSC-spaces in [4] and formulated as follows in [24, p. 574].

Principle 1.1 (GS). *An infinite topological space has a sub-space homeomorphic to exactly one of the following topologies over $\mathbb{N}$:*

- *The discrete topology: all sets are open.*
- *The indiscrete topology: only $\emptyset$ and $\mathbb{N}$ are open.*
- *The co-finite topology: the open sets are $\emptyset$, $\mathbb{N}$, and any sub-set of $\mathbb{N}$ with finite complement.*
- *The initial segment topology: the open sets are $\emptyset$, $\mathbb{N}$, and any set of the form $[0, n] = \{k \in \mathbb{N} : k \leq n\}$.*
- *The final segment topology: the open sets are $\emptyset$, $\mathbb{N}$, and any set of the form $[n, +\infty) = \{k \in \mathbb{N} : n \leq k\}$.*

The results on GS and related topics may be found in Section 2.3 and establish the observation (O1). We stress that on one hand the Ginsburg–Sands theorem for CSC-spaces is provable in ACA_0 (see [4]), while for (countable) RSC-spaces it is stronger, namely GS inhabits the range of hyperarithmetical analysis over ACA_0^ω.

Finally, we also identify basic third-order theorems about RSC-spaces that imply strong axioms, including countable choice and second-order arithmetic in various incarnations. In particular, we study the following statements for RSC-spaces.

(a) A (compact) RSC-space is separable.

(b) A (compact) RSC-space is Lindelöf.

(c) A continuous function on a compact RSC-space has a supremum.

(d) A continuous function on a compact RSC-space attains a maximum.

When formulated for CSC-spaces, the associated second-order versions of (a)–(d) are provable in rather weak fragments of second-order arithmetic, i.e., arithmetical comprehension seems sufficient. By contrast, the third-order theorems in items (a)–(d) imply at least Feferman's highly non-constructive *projection principle* (see [21] and Section 2.4) and even full second-order arithmetic or countable choice in some cases, as established in Section 2.4 as a contribution to (O3) from Section 1.1.

For our background framework, we make use of Kohlenbach's base theory RCA_0^ω of higher-order Reverse Mathematics (often abbreviated RM), introduced in [31]. We shall assume familiarity with RCA_0^ω and the associated RM of real analysis as in e.g. [31, §2] or [43]. To be absolutely clear, RCA_0^ω is a weak logical system that we assume as a 'background theory'. In the latter, we prove that the above statements (a)–(d) imply or are equivalent to strong axioms, including even second-order arithmetic Z_2 and countable choice $\mathsf{QF\text{-}AC}^{0,1}$ (see Section 1.3.1 for the latter). Along the way, we will obtain a number of elegant equivalences for Feferman's projection principle and countable choice $\mathsf{QF\text{-}AC}^{0,1}$ in higher-order RM.

In conclusion, the use of codes or representations in general can have a tremendous influence on the logical strength of basic theorems of topology. The results in this paper are novel since we show that this logical strength can vary as much as Z_2 itself, or even require countable choice. Moreover, items (a)–(d) are rather elementary. We believe that many variations on these results are possible and look forward to the associated exploration.

1.3. Preliminaries

We introduce some (mostly standard) topology definitions in higher-order arithmetic (Section 1.3.2) and some required axioms (Section 1.3.1). We assume basic familiarity with the formalisation of the real numbers, which is the same in second- and higher-order RM (see [53, II.5] or [31, §3]).

1.3.1. Some axioms of higher-order arithmetic. We introduce some higher-order axioms needed in the below. We assume basic familiarity with Kohlenbach's base theory RCA_0^ω of higher-order RM (see [31, §2]).

First of all, the functional E in ($\exists^2$) is also called *Kleene's quantifier* $\exists^2$:

$$\big(\exists E\colon \mathbb{N}^{\mathbb{N}} \to \{0,1\}\big)(\forall f \in \mathbb{N}^{\mathbb{N}})\big[(\exists n \in \mathbb{N})\big(f(n)=0\big) \leftrightarrow E(f)=0\big]. \tag{$\exists^2$}$$

Related to ($\exists^2$), the functional μ^2 in (μ^2) is called *Feferman's* μ (see [1]) and may be found – with this symbol – in Hilbert–Bernays' Grundlagen [25, Supplement IV]:

$$\mu(f) := \begin{cases} n & \text{if } n \text{ is the least natural number such that } f(n)=0,\\ 0 & \text{if } f(n)>0 \text{ for all } n \in \mathbb{N}. \end{cases} \tag{μ^2}$$

We have ($\exists^2$)$\leftrightarrow$(μ^2) over RCA_0^ω (see [31, §3]) while $\mathsf{ACA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\exists^2)$ proves the same sentences as ACA_0 by [28, Theorem 2.5].

Secondly, the functional S^2 in (S^2) is called *the Suslin functional* [31]:

$$\big(\exists \mathsf{S}^2\colon \mathbb{N}^{\mathbb{N}} \to \{0,1\}\big)(\forall f \in \mathbb{N}^{\mathbb{N}})\big[(\exists g \in \mathbb{N}^{\mathbb{N}})(\forall n \in \mathbb{N})\big(f(\overline{g}n)=0\big) \leftrightarrow \mathsf{S}(f)=0\big]. \tag{S^2}$$

The system $\Pi_1^1\text{-}\mathsf{CA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\mathsf{S}^2)$ proves the same Π_3^1-sentences as $\Pi_1^1\text{-}\mathsf{CA}_0$ by [46, Theorem 2.2]. By definition, the Suslin functional S^2 can decide whether a Σ_1^1-formula as in the left-hand side of (S^2) is true or false. We similarly define the functional S_k^2 which decides the truth or falsity of Σ_k^1-formulas from L_2; we also define the system $\Pi_k^1\text{-}\mathsf{CA}_0^\omega$ as $\mathsf{RCA}_0^\omega + (\mathsf{S}_k^2)$, where ($\mathsf{S}_k^2$) expresses that S_k^2 exists. We note that the operators ν_n from [11, p. 129] are essentially S_n^2 strengthened to return a witness (if existent) to the Σ_n^1-formula at hand.

Thirdly, full second-order arithmetic Z_2 is readily derived from $\cup_k \Pi_k^1\text{-}\mathsf{CA}_0^\omega$, or from:

$$\big(\exists E\colon (\mathbb{N}^{\mathbb{N}} \to \mathbb{N})\big)(\forall Y\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N})\big[(\exists f \in \mathbb{N}^{\mathbb{N}})\big(Y(f)=0\big) \leftrightarrow E(Y)=0\big], \tag{$\exists^3$}$$

and we therefore define $\mathsf{Z}_2^\Omega \equiv \mathsf{RCA}_0^\omega + (\exists^3)$ and $\mathsf{Z}_2^\omega \equiv \cup_k \Pi_k^1\text{-}\mathsf{CA}_0^\omega$, which are conservative over Z_2 by [28, Cor. 2.6]. The functional from ($\exists^3$) is also called 'Kleene's quantifier $\exists^3$', and we use the same convention for other functionals.

Fourth, many results in higher-order RM are established in RCA_0^ω plus the following special case of the Axiom of countable Choice [31,40].

Principle 1.2 ($\mathsf{QF\text{-}AC}^{0,1}$). *Let φ be quantifier-free and such that*

$$(\forall n \in \mathbb{N})(\exists f \in \mathbb{N}^{\mathbb{N}})\varphi(f,n).$$

Then there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{N}^{\mathbb{N}}$ with $(\forall n \in \mathbb{N})\varphi(f_n,n)$.

The local equivalence between sequential and 'epsilon-delta' continuity cannot be proved in ZF, but can be established in $\mathsf{RCA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$ (see [31]). Thus, it should not be a surprise that the latter system is often used as a base theory too. Below, we show that $\mathsf{QF\text{-}AC}^{0,1}$ is equivalent to various basic statements about second-countable spaces. We shall sometimes use a 'bounded' version of countable choice, which follows from the induction axiom.

Principle 1.3 ($\textsf{B-QF-AC}^{0,1}$). *Let φ be quantifier-free and such that*

$$(\forall n \in \mathbb{N})(\exists f \in \mathbb{N}^{\mathbb{N}})\varphi(f, n).$$

For every $k \in \mathbb{N}$, there exists $(f_n)_{n\in\mathbb{N}}$ with $(\forall n \leq k)\varphi(f_n, n)$.

The (necessary) use of extra induction axioms is not unheard of in RM [38]. We let $\textsf{B-QF-AC}_!^{0,1}$ be the previous principle with the 'unique existence' restriction $(\forall n \in \mathbb{N})(\exists! f \in \mathbb{N}^{\mathbb{N}})\varphi(f, n)$ in place.

Finally, we introduce the notion of *hyperarithmetical analysis* as it is essential for (O1) from Section 1.1.

Now, hyperarithmetical analysis refers to a cluster of logical systems just above $\textsf{ACA}_0$. In particular, going back to Kreisel [32], the notion of *hyperarithmetical set* (see e.g. [53, VIII.3]) gives rise to the second-order definition of *theory/theorem of hyperarithmetical analysis* (THA for brevity, see e.g. [2]). Well-known THAs are Σ_1^1-$\textsf{CA}_0$ and weak-Σ_1^1-$\textsf{CA}_0$ (see [53, VII.6.1 and VIII.4.12]), where the latter is the former with the antecedent restricted to unique existence.

Moreover, any system *between* two THAs is *also* a THA, which is a convenient way of establishing that a given system is a THA. At the higher-order level, $\textsf{ACA}_0^\omega + \textsf{QF-AC}^{0,1}$ is a conservative extension of Σ_1^1-$\textsf{CA}_0$ by [28, Cor. 2.7]. We therefore arrive at the following definition, pioneered in [50].

Definition 1.4. We say that a system of higher-order arithmetic T *exists in the range of hyperarithmetical analysis* in case $\textsf{ACA}_0^\omega + \textsf{QF-AC}^{0,1} \rightarrow T \rightarrow \textsf{ACA}_0^\omega + \text{weak-}\Sigma_1^1\text{-}\textsf{AC}_0$.

Many theorems from real analysis exist in the latter range (see [50]) while natural (second-order) THAs are considered to be relatively rare.

1.3.2. Definitions. We introduce some required definitions, stemming from mainstream mathematics. We note that subsets of $\mathbb{R}$ are given by their characteristic functions as in Definition 1.5, well-known from measure and probability theory. We shall generally work over $\textsf{ACA}_0^\omega$ as some definitions make little sense over $\textsf{RCA}_0^\omega$. The notion of 'CSC-space' exclusively refers to the second-order definition [13, 18].

First of all, we make use the usual definition of (open) set, where $B(x, r)$ is the open ball with radius $r > 0$ centred at $x \in \mathbb{R}$.

Definition 1.5. Sets of reals are defined as follows.

- A function $F : \mathbb{R} \rightarrow \mathbb{R}$ is represented by $\Phi : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ satisfying

$$(\forall x, y \in \mathbb{R})\big(x =_{\mathbb{R}} y \rightarrow F(x) =_{\mathbb{R}} F(y)\big), \tag{1.1}$$

  which is an instance of the axiom of function extensionality.
- A subset $A \subset \mathbb{R}$ is given by its characteristic function $\mathbb{1}_A : \mathbb{R} \rightarrow \{0, 1\}$, i.e., we write $x \in A$ for $\mathbb{1}_A(x) = 1$, for any $x \in \mathbb{R}$.
- A subset $O \subset \mathbb{R}$ is *open* in case $x \in O$ implies that there is $k \in \mathbb{N}$ such that $B(x, \frac{1}{2^k}) \subset O$.

- A subset $O \subset \mathbb{R}$ is *RM-open* in case there are sequences of reals $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ such that $O = \cup_{n\in\mathbb{N}}(a_n, b_n)$.
- A subset $C \subset \mathbb{R}$ is *closed* if the complement $\mathbb{R} \setminus C$ is open.
- A subset $C \subset \mathbb{R}$ is *RM-closed* if the complement $\mathbb{R} \setminus C$ is RM-open.
- A set $A \subset \mathbb{R}$ is *enumerable* if there is a sequence of reals that includes all elements of A.
- A set $A \subset \mathbb{R}$ is *countable* if there is $Y: \mathbb{R} \to \mathbb{N}$ that is injective on A, i.e.,
$$(\forall x, y \in A)\big(Y(x) =_0 Y(y) \to x =_{\mathbb{R}} y\big).$$
- A set $A \subset \mathbb{R}$ is *strongly countable* if there is $Y: \mathbb{R} \to \mathbb{N}$ that is injective and surjective on A; the latter means that $(\forall n \in \mathbb{N})(\exists x \in A)(Y(x) = n)$.

Secondly, our definition of second-countable spaces is given by Definition 1.6. To be absolutely clear, we include the mapping k in the definition as the latter is generally used in second-order RM [4, 13, 18]. Below, we only study second-countable spaces with size at most the continuum, which we call RSC-spaces.

Definition 1.6. Second-countable spaces are defined as follows.

- A *basis/base* for a topology on X is a collection $(U_i)_{i\in I}$ and a mapping $k: (X \times I^2) \to I$ satisfying the following.
  - For every $x \in X$, there is $i \in I$ with $x \in U_i$.
  - For $x \in X$ and $i, j \in I$, we have $x \in U_i \cap U_j \to x \in U_{k(x,i,j)} \subseteq U_i \cap U_j$.
- A *real second-countable space* consists of a set $X \subset \mathbb{R}$ and a basis with index set $I = \mathbb{N}$. We abbreviate this by 'RSC-space X'.
- An RSC-space X is (*strongly*) *countable* if X is (strongly) countable.

As promised, the definition of 'countable RSC space' amounts to the second-order one with 'enumerable set $X \subset \mathbb{N}$' replaced by 'countable set $X \subset \mathbb{R}$'. For an RSC-space X, a sub-set $Z \subset X$ is defined via its characteristic function $\mathbb{1}_Z: \mathbb{R} \to \{0, 1\}$, keeping in mind (1.1). We now have the following definition, again mirroring CSC-spaces as close as possible.

Definition 1.7. Let X be an RSC-space with basis $(U_i)_{i\in\mathbb{N}}$ and $k: (X \times \mathbb{N}^2) \to \mathbb{N}$.

- A set $O \subset X$ is *open* if for $x \in O$, there exists $i \in \mathbb{N}$ with $x \in U_i \subset O$.
- [13, 18] A set $O \subset X$ is *uniformly open* if there is a function $\eta: X \to \mathbb{N}$ such that for all $x \in O$, $x \in U_{\eta(x)} \subset O$.
- A set $C \subset X$ is *closed* if the complement $X \setminus C$ is open.
- A set $C \subset X$ is *uniformly closed* if $X \setminus C$ is uniformly open.

The notion of uniform openness is similar to having an R2-representation from [39]. In the latter, a set $O \subset \mathbb{R}$ is called *R2-open* if there is $Y: \mathbb{R} \to \mathbb{R}$ such that $x \in O \leftrightarrow Y(x) >_{\mathbb{R}} 0$ and $x \in O \to B(x, Y(x)) \subset O$ for all $x \in \mathbb{R}$.

Thirdly, the following definitions are mostly standard, where we note that a different nomenclature is sometimes used in the logical literature.

Definition 1.8 (Compactness and around). For an RSC-space X with basis $(U_i)_{i\in\mathbb{N}}$ and $k\colon (X\times\mathbb{N}^2)\to\mathbb{N}$, we say that

- X is *countably-compact* if for any sequence $(O_n)_{n\in\mathbb{N}}$ of open sets in X such that $X\subset\cup_{n\in\mathbb{N}}O_n$, there is $m\in\mathbb{N}$ such that $X\subset\cup_{n\leq m}O_n$,
- X is (open-cover) *compact* in case for any covering[2] generated by $\Psi\colon X\to\mathbb{N}$, there are $x_0,\dots,x_k\in X$ such that $\cup_{i\leq k}U_{\Psi(x_i)}$ covers X,
- X is *Lindelöf* in case for any covering[2] generated by $\Psi\colon X\to\mathbb{N}$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\cup_{i\in\mathbb{N}}U_{\Psi(x_i)}$ covers X,
- a sequence $(x_n)_{n\in\mathbb{N}}$ in X *converges to* $y\in X$ in case for every open U containing y, we have $(\exists m\in\mathbb{N})(\forall n\geq m)(x_n\in U)$.
- X is *sequentially compact* if any sequence in X has a limit point[3],
- a set $Z\subset X$ is *finite* if there is $N\in\mathbb{N}$ such that for any pairwise different $x_0,\dots,x_N\in Z$, there is $i\leq N$ with $x_i\notin Z$; we then write $|Z|\leq N$,
- X is *limit point compact* if any infinite set in X has a limit point[3],
- X is *separable* if there is a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that for any non-empty open $O\subset X$, there is $n\in\mathbb{N}$ with $x_n\in O$.

It is well-known that for second-countable spaces, various compactness notions are equivalent and that such spaces are separable. We argue in Section 1.4 that our notion of 'finite set' from Definition 1.8 is the right one for developing higher-order RM. By Theorem 2.14, this choice does not greatly influence our classification anyway.

We now introduce the (usual) definition of continuity, alongside its effective notion from [4, 13, 18]. We let $(V_j)_{j\in\mathbb{N}}$ be an enumeration of all balls with rational centre and radius. Since we shall work over ACA_0^ω, we need not address partiality.

Definition 1.9 (Continuity). For an RSC-space X, a function $f\colon X\to\mathbb{R}$ is

- *continuous* if for any open $V\subset\mathbb{R}$, $f^{-1}(V)=\{x\in X : f(x)\in V\}$ is open,
- *continuous at* $x\in X$ if for any open $V\subset\mathbb{R}$ containing $f(x)$, there is an open $U\subset X$ containing x and $f(U)\subset V$,
- *effectively continuous* if there exists $\varphi\colon (X\times\mathbb{N})\to\mathbb{N}$ such that if $f(x)\in V_j$, then $x\in U_{\varphi(x,j)}\subset f^{-1}(V_j)$,

[2]A mapping $\Psi\colon X\to\mathbb{N}$ generates a covering of X in case $x\in U_{\Psi(x)}$ for any $x\in X$. Both Cousin and Lindelöf (only) study this kind of uncountable coverings in [12, 34], which are the papers in which the Cousin and Lindelöf lemmas first appeared.

[3]We say that $y\in X$ is a *limit point* of the sequence $(x_n)_{n\in\mathbb{N}}$ if there exists $g\in\mathbb{N}^{\mathbb{N}}$ such that $(x_{g(n)})_{n\in\mathbb{N}}$ converges to y. Similarly, $y\in X$ is a *limit point* of the (infinite) set $Z\subset X$ if there is a sequence in $Z\setminus\{y\}$ that converges to y.

- *sequentially continuous* if for any sequence $(x_n)_{n\in\mathbb{N}}$ in X with limit $y \in X$, the limit of $(f(x_n))_{n\in\mathbb{N}}$ is $f(y)$,
- *strongly continuous* if for any $V \subset \mathbb{R}$, $f^{-1}(V)$ is open (see [33, 37]).

We could also study notions intermediate between strong and normal continuity, with the same results.

Finally, separation axioms in topology are well-known.

Definition 1.10. The T_i-separation axioms ($i = 1, 2$) are defined as follows.

- A space is T_1 in case any two distinct points are separated, i.e., each lies in a neighbourhood not including the other.
- A space is Hausdorff or T_2 in case for any two distinct points x, y, there is a neighbourhood U of x and a neighbourhood V of y such that $U \cap V = \emptyset$.

1.4. On the choice of definitions

Most mathematical notions have a number of equivalent definitions. In this section, we provide motivation for our choice of 'finite set of reals' as in Definition 1.8, as it is (slightly) different from the mainstream definitions like *Dedekind finite*.

In particular, the following three items show that our definition of finite set naturally comes to the fore in basic real analysis.

- Limit point compactness goes back to Weierstrass, according to Jordan [29, p. 73]. A basic result regarding limit points is that

  $$\textit{for } X \subset \mathbb{R} \textit{ without limit points},\ \ X \cap [-n, n] \textit{ is } \text{finite} \textit{ for all } n \in \mathbb{N}.$$

  A standard compactness argument establishes the centred statement in $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$, assuming 'finite' refers to Definition 1.8. By contrast, finding an injection or bijection from $X \cap [-n, n]$ to some $\{0, 1, \ldots, k\}$ seems much harder (see [42, 49] for details).
- A function $f : [0, 1] \to \mathbb{R}$ is *regulated* if the left and right limits $f(x-)$ and $f(x+)$ exist. Now define the following set for regulated f:

  $$D_k := \big\{x \in [0, 1] : \big|f(x) - f(x-)\big| > \tfrac{1}{2^k} \vee \big|f(x) - f(x+)\big| > \tfrac{1}{2^k}\big\}.$$

  All this can be done in ACA_0^ω. The set D_k is finite, which again follows from a standard compactness argument in $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$, assuming 'finite' refers to Definition 1.8. By contrast, finding an injection or bijection from D_k to some $\{0, 1, \ldots, n\}$ seems much harder [42, 49].
- Borel studies *height functions* in [5–7]. The latter generalises the notion of 'injection to $\mathbb{N}$' by allowing finitely many elements to map to the same natural number. In light of [47, 49], height functions allow for a smooth development of the higher-order RM of the uncountability of $\mathbb{R}$, assuming 'finite' refers to Definition 1.8. By contrast, the notion of 'injection to $\mathbb{N}$' or other notions do not seem to yield such a development.

By the previous items, Definition 1.8 provides a smooth development of higher-order RM. By contrast, a development based on the set-theoretic definition, involving injections and bijections, seems to require a stronger base theory. *Nonetheless*, Theorem 2.14 implies that we obtain the same classification for the Ginsburg–Sands theorem, independent of whether we use Definition 1.8 or the set-theoretic definition. At the same time, Theorem 2.14 suggests that using finiteness notions other than Definition 1.8, one seems to need more induction than with the latter.

2. Main results

2.1. Introduction

We establish the results sketched in Section 1, i.e., that basic theorems concerning RCS-spaces imply or are equivalent to relatively strong axioms. We study basic properties like the supremum principle for continuous functions (see Principle 2.1 right below), but also more advanced theorems like the Ginsburg–Sands theorem (see Principle 1.1 in Section 1.2).

- In Section 2.2, we establish (O2) from Section 1.1, i.e., we show that various properties of countable RSC-spaces are equivalent to *enumeration principles* which express that (strongly) countable sets can be enumerated.
- In Section 2.3, we establish (O1) and (O3) from Section 1.1. In particular, we show that basic statements about *un*countable RSC-spaces are equivalent to countable choice as in $\mathsf{QF\text{-}AC}^{0,1}$. As a result, related statements are classified in the range of hyperarithmetical analysis.
- In Section 2.4, we establish (O3) from Section 1.1. In particular, we show that basic statements about *un*countable RSC-spaces imply -or are equivalent to- various incarnations of second-order arithmetic, including Kleene's quantifier $(\exists^3)$ and Feferman's *projection principle*.

Next, the supremum principle studied in the below is defined as usual.

Principle 2.1 (SUP). *Let X be a compact RSC-space. For a continuous function $f\colon X\to\mathbb{R}$ and a decreasing sequence $(C_n)_{n\in\mathbb{N}}$ of closed sets, there is a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n=\sup_{x\in C_n} f(x)$.*

We also study the sequential version of the supremum and maximum principles. Sequential versions are studied in RM in e.g. [14, 16, 18, 22, 23, 26, 27, 30, 53, 57].

Principle 2.2 (SUP'). *Let X be a compact RSC-space. For a sequence of continuous $X\to\mathbb{R}$-functions $(f_n)_{n\in\mathbb{N}}$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n=\sup_{x\in X} f_n(x)$.*

Principle 2.3 (MAX). *For a compact RSC-space X and a sequence of continuous $X\to\mathbb{R}$-functions $(f_n)_{n\in\mathbb{N}}$, there is $(x_n)_{n\in\mathbb{N}}$ with $(\forall x\in X,\ n\in\mathbb{N})(f_n(x)\le f_n(x_n))$.*

These principles seem rather basic compared to the other principles in this paper.

Finally, we shall work over ACA_0^ω for convenience, but could in principle obtain most results over RCA_0^ω or $\mathsf{RCA}_0^\omega + \mathsf{WKL}_0$ using the following trick.

Remark 2.4 (On the law of excluded middle). Our starting point is Kleene's arithmetical quantifier $(\exists^2)$ from Section 1.3.1. By [31, Prop. 3.12], $(\exists^2)$ is equivalent over RCA_0^ω to the statement

There exists an $\mathbb{R} \to \mathbb{R}$-function that is not continuous.

Clearly, $\neg(\exists^2)$ is then equivalent to *Brouwer's theorem*, i.e., the statement that all $\mathbb{R} \to \mathbb{R}$-functions are continuous. Now, if we wish to prove a given statement T of real analysis about possibly discontinuous functions in $\mathsf{RCA}_0^\omega + \mathsf{WKL}_0$, we may invoke the law of excluded middle as in $(\exists^2) \vee \neg(\exists^2)$. We can then split the proof of T in two cases: one assuming $\neg(\exists^2)$ and one assuming $(\exists^2)$. In the latter case, since $(\exists^2) \to \mathsf{ACA}_0$, we have access to much more powerful tools (than just WKL_0). In the former case, since $\neg(\exists^2)$ implies that all functions are continuous, we only need to establish T restricted to the special case of continuous functions. Moreover, we can use WKL_0 to provide codes for all (continuous) functions (see [43, §2]). After that, we can use the second-order RM literature to establish T restricted to codes for continuous functions, and hence T. To be absolutely clear, the 'law of excluded middle trick' is the above splitting of proofs based on $(\exists^2) \vee \neg(\exists^2)$.

2.2. Countable spaces

In this section, we establish observation (O2) from Section 1.1, i.e., we show that various properties of countable RSC-spaces are equivalent to *enumeration principles* which express that (strongly) countable sets can be enumerated. This includes basic properties like the supremum principle for continuous functions (see Theorem 2.7), but also more advanced theorems, like the Ginsburg–Sands theorem (Theorem 2.8).

First of all, we introduce the aforementioned enumeration principles. The principle cocode_0 is a new 'Big' system in higher-order RM, boasting many equivalences involving basic properties of functions of bounded variation, including the Jordan decomposition theorem and approximation theorems [42, 47].

Principle 2.5 (cocode_0). *Any countable set $A \subset \mathbb{R}$ is enumerable.*

As to logical properties (see [41, 42]), $\mathsf{RCA}_0^\omega + \mathsf{cocode}_0$ is conservative over RCA_0 while Z_2^Ω (and hence ZF) proves cocode_0. Moreover, cocode_0 is 'explosive' in that $\mathsf{ACA}_0^\omega + \mathsf{cocode}_0$ proves ATR_0 while $\Pi_1^1\text{-}\mathsf{CA}_0^\omega + \mathsf{cocode}_0$ proves $\Pi_2^1\text{-}\mathsf{CA}_0$.

Next, the principle cocode_1 boasts some equivalences (see [42, 50]).

Principle 2.6 (cocode_1). *Any strongly countable set $A \subset \mathbb{R}$ is enumerable.*

As explored in detail in [50], cocode_1 and (many) related third-order principles populate the range of *hyperarithmetical analysis*. In particular, $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$ is conservative over $\Sigma_1^1\text{-}\mathsf{AC}_0$ [28, Ch. 2], while $\mathsf{ACA}_0^\omega + \mathsf{cocode}_1$ implies weak-$\Sigma_1^1\text{-}\mathsf{AC}_0$. We refer to [50, 53] for the exact definitions of these second-order systems and a detailed discussion of hyperarithmetical analysis.

Secondly, we have the following theorem where item (a) essentially expresses that countable RSC-spaces have a representation as in second-order RM. The results seem rather robust, following Montalbàn's criterion from Section 1.

Theorem 2.7 (ACA_0^ω). *The following are equivalent.*

(a) *A countable RSC-space is separable.*

(b) *The previous item restricted to compact spaces.*

(c) *A countable RSC-space is Lindelöf.*

(d) *The supremum principle* SUP *for countable RSC-spaces.*

(e) *The previous item restricted to effectively or strongly continuous functions.*

(f) *The previous item restricted to sequences* $(C_n)_{n\in\mathbb{N}}$ *of* uniformly *closed sets.*

(g) *The enumeration principle* cocode_0.

The equivalence holds for 'sequential', 'countable', and 'limit point' compactness, the latter additionally assuming $\mathsf{QF\text{-}AC}^{0,1}$.

Proof. First of all, that cocode_0 implies all other items is relatively straightforward. Indeed, the former provides an enumeration of the set X, after which the 'usual' second-order proofs go through (or are trivial). These second-order proofs can be found in [13, 15, 18, 52].

Secondly, let $A \subset \mathbb{R}$ and $Y: \mathbb{R} \to \mathbb{N}$ be such that the latter is injective on the former. Without loss of generality, we may assume that $0 \notin A$ as we can always append one real to an enumeration, even in RCA_0^ω. Define the sequence of sets $(U_i)_{i\in\mathbb{N}}$ as follows:

$$U_{2n} = \big\{x \in A : Y(x) = n\big\} \quad \text{and} \quad U_{2n+1} = \{0\} \cup \big\{x \in A : Y(x) > n\big\}. \tag{2.1}$$

Use ($\exists^2$) to define $k: (A \times \mathbb{N}^2) \to \mathbb{N}$ as follows for $x \in A$ and $i \leq j$ in $\mathbb{N}$

$$k(x,i,j) := \begin{cases} j & \text{if } i \text{ and } j \text{ are odd,} \\ j & \text{if } i \text{ is odd and } j \text{ is even,} \\ i & \text{otherwise,} \end{cases} \tag{2.2}$$

and observe that this mapping has the required properties for forming a base of $X = A \cup \{0\}$. Hence, X is an RSC-space, which can be seen to be compact as follows: let $\Psi: X \to \mathbb{N}$ be given and note that $n_0 = \Psi(0)$ must be odd to guarantee $0 \in U_{n_0}$ following (2.1). Clearly, X is covered by $\cup_{i<n_0} U_{2i} \cup U_{n_0}$. In exactly the same way, one proves that X is countably compact.

To prove that X is sequentially compact, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the former. In case $(\forall k \in \mathbb{N})(\exists n \in \mathbb{N})(x_n \in A \wedge Y(x_n) > k)$, then 0 is an accumulation point of the sequence. Note that $\mathsf{QF\text{-}AC}^{0,0}$, included in RCA_0^ω, suffices to obtain the required subsequence. Otherwise, 0 must occur infinitely often in $(x_n)_{n\in\mathbb{N}}$, i.e., the former is again an accumulation point. In exactly the same way, one proves that X is limit point compact, using $\mathsf{QF\text{-}AC}^{0,1}$ to obtain a sequence in a given infinite set.

Next, either item (a) or (b) provide a sequence $(x_n)_{n\in\mathbb{N}}$ such that for any open set $O \subset X$, there is $m \in \mathbb{N}$ with $x_m \in O$. Clearly, this sequence provides an enumeration of A as $(\exists a \in A)(Y(a) = n) \leftrightarrow (\exists x \in \mathbb{R})(x \in U_{2n}) \leftrightarrow (\exists m \in \mathbb{N})(x_m \in U_{2n})$ for any $n \in \mathbb{N}$. Thus, the aforementioned items imply cocode_0. For item (c), repeat the above while omitting 0 from X and omitting U_{2n+1} from the base. The function $\Psi(x) := 2Y(x)$ generates the required covering.

Now assume item (d) and consider the RSC-space $X = A \cup \{0\}$ defined above. Define $f: X \to \mathbb{R}$ as $f(0) := 0$ and $f(x) := \frac{1}{2^{Y(x)}}$ otherwise. To establish effective and strong continuity, define $\varphi(x, j) := 2Y(x)$ and note that indeed $x \in U_{\varphi(x,j)} \subset f^{-1}(V_j)$ in case $f(x) \in V_j$. Now use item (d) or item (e) and note that for $n \geq 1$:

$$(\exists x \in A)\big(Y(x) = n\big) \leftrightarrow \big[\tfrac{1}{2^n} = \sup_{x\in C_n} f(x)\big],$$

where $C_n := X \setminus \cup_{i<n} U_{2i}$; the complement of C_n is uniformly open as witnessed by $\eta(x) := 2Y(x)$. Hence, we can define the range of Y restricted to A, which yields cocode_0 by [42, Theorem 2.11]. ∎

The equivalences in the theorem still go through if we require certain separation axioms T_i. The reason is of course that A is countable and that U_{2n} in (2.1) are therefore at most singletons.

We have studied Borel's notion of *height countability* in [47, 49]. The equivalences in Theorem 2.7 go through for this notion *mutatis mutandis*. We could similarly study Weierstrass' extreme value theorem and obtain equivalences for the latter.

Finally, the previous results pertain to countable sets and -with some effort- similar equivalences go through for *strongly* countable sets and the associated enumeration principle cocode_1. We study one example in detail in Theorem 2.8. We sometimes write 'w^{1^*}' to indicate that w is a finite sequence of type 1 objects with length $|w|$. We assume a fixed coding of such objects, e.g. via type $0 \to 1$.

Theorem 2.8 (ACA_0^ω). *The following are equivalent.*

- *The enumeration principle* cocode_1.
- *The combination of the following.*
  - *The principle* $\mathsf{B\text{-}QF\text{-}AC}_!^{0,1}$.
  - *The Ginsburg–Sands theorem* GS *for strongly countable RSC-spaces.*

Proof. First of all, cocode_1 allows us to enumerate strongly countable sets, i.e., a strongly countable RSC-space is homeomorphic (in the obvious way) to a CSC-space. Since ACA_0 proves GS for CSC-spaces [4, Theorem 4.6], the second item from the theorem follows from cocode_1. To show that cocode_1 implies $\mathsf{B\text{-}QF\text{-}AC}_!^{0,1}$, let φ be such that $(\forall n \in \mathbb{N})(\exists! f \in \mathbb{N}^\mathbb{N})\varphi(n, f)$. Define the set $A = \{f \in \mathbb{N}^\mathbb{N} : (\exists n \in \mathbb{N})\varphi(n, f)\}$ and the function $Y: \mathbb{R} \to \mathbb{N}$ via $Y(f) := (\mu n)\varphi(n, f)$. By assumption, Y is injective and surjective on A and cocode_1 provides an enumeration of A, say $(f_n)_{n\in\mathbb{N}}$. We thus have $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})\varphi(n, f_m)$ as required.

Secondly, fix $A \subset \mathbb{R}$ and let Y be injective and surjective on A. Fix a standard coding of finite sequences of reals where we write $w^{1^*} = \langle x_0, \ldots, x_k \rangle$ for $x_i \in \mathbb{R}$ and $|w| = k+1$ is called the 'length' of w. Now define X as

$$\{w^{1^*} : (\forall i < |w|)(w(i) \in A \wedge Y(w(i)) = i)\}.$$

Then consider the base defined by $U_n = \{w \in X : |w| = n\}$ and where the associated function k is obvious. Note that the RSC-space X is infinite thanks to $\mathsf{B\text{-}QF\text{-}AC}^{0,1}_!$. Clearly, all sub-sets of X are open and the same for any sub-space of X. By GS as in the final item, X has a sub-space that is homeomorphic to the indiscrete topology on $\mathbb{N}$. Let $f : X \to \mathbb{N}$ and $f^{-1} : \mathbb{N} \to X$ be the associated continuous bijection and its inverse. Now consider the sequence $(f^{-1}(n))_{n\in\mathbb{N}}$ in X. Since $f : X \to \mathbb{N}$ is a bijection, this sequence includes $w \in X$ of arbitrary length $|w|$. Thus, we obtain an enumeration of A, and we are done ■

The Ginsburg–Sands theorem for CSC-spaces is provable in ACA_0 by [4, Theorem 4.6]. By Theorem 2.8, replacing 'enumerated sets' by 'strongly countable sets', the Ginsburg–Sands theorem is equivalent to cocode_1 and therefore inhabits the range of hyperarithmetical analysis. Following Theorem 2.11, the Ginsburg–Sands theorem for (possibly uncountable) RSC-spaces is equivalent to $\mathsf{QF\text{-}AC}^{0,1}$, i.e., this version is in the range of hyperarithmetical analysis.

In conclusion, we have established that various properties of countable second-countable spaces are equivalent to the enumeration principles cocode_i for $i = 0, 1$.

2.3. Uncountable spaces and countable choice

In this section, we establish observations (O1) and (O3) from Section 1.1. In particular, we show that basic statements about *un*countable RSC-spaces are equivalent to countable choice as in $\mathsf{QF\text{-}AC}^{0,1}$. As a result, related statements are classified in the range of hyperarithmetical analysis, including the Ginsburg–Sands theorem.

First of all, we obtain some elegant equivalences involving countable choice, in line with observation (O3) from Section 1.1. Moreover, countable choice is now fundamental in light of item (c), given how central continuity is to topology.

Theorem 2.9 (ACA_0^ω). *The following are equivalent.*

(a) *The principle* $\mathsf{QF\text{-}AC}^{0,1}$.

(b) *The conjunction of the following.*

- $\mathsf{B\text{-}QF\text{-}AC}^{0,1}$ (*see Principle* 1.3).
- *For a sequentially* (*or: open cover, or: countably*) *compact RSC-space X, a* (*sequentially*) *continuous function $f : X \to \mathbb{R}$ is bounded.*

(c) *The conjunction of the following.*

- $\mathsf{B\text{-}QF\text{-}AC}^{0,1}$ (*see Principle* 1.3).
- *For a sequentially compact RSC-space X, if X is infinite, there is a discontinuous $f : X \to \mathbb{R}$.*

(d) *The previous item restricted to 'not effectively continuous'.*

(e) *The conjunction of the following.*

- $\mathsf{B\text{-}QF\text{-}AC}^{0,1}$ (*see Principle* 1.3).
- *For any RSC-space X, sequential compactness implies countable compactness.*

(f) *The conjunction of the following.*

- $\mathsf{B\text{-}QF\text{-}AC}^{0,1}$ (*see Principle* 1.3).
- *For a sequentially compact RSC-space X and a continuous function $f : X \to \mathbb{R}$ with $y = \sup_{x \in X} f(x)$, there is $x_0 \in X$ with $f(x_0) = y$.*

The second bullet in item (b) with all instances of 'sequentially' omitted, is provable.

Proof. For the final sentence, fix continuous $f : X \to \mathbb{R}$ and define $O_n := \{x \in X : f(x) > n\} = f^{-1}((+\infty, n))$, which is open since f is continuous. Clearly, $\cup_{n\in\mathbb{N}} O_n$ covers X and if $\cup_{n\leq n_0} O_n$ is a finite sub-covering, then f is bounded above by n_0 on X. Hence, the final sentence of the theorem follows.

To establish item (b) using $\mathsf{QF\text{-}AC}^{0,1}$, let $f : X \to \mathbb{R}$ be (sequentially) continuous. In case $(\forall n \in \mathbb{N})(\exists x \in X)(|f(x)| > n)$, apply $\mathsf{QF\text{-}AC}^{0,1}$ and let $(x_n)_{n\in\mathbb{N}}$ be the associated sequence. If $y \in X$ is an accumulation point of this sequence, let $(y_n)_{n\in\mathbb{N}}$ be the associated sub-sequence converging to y. Clearly, $(f(y_n))_{n\in\mathbb{N}}$ does not converge to $f(y)$, nor is f continuous at y, a contradiction. As a result, f must be bounded and item (b) follows.

To derive $\mathsf{QF\text{-}AC}^{0,1}$ from item (b), fix $F : \mathbb{R} \to \mathbb{N}$ such that $(\forall n \in \mathbb{N})(\exists x \in \mathbb{R})(F(x) = n)$ but there is no sequence $(x_n)_{n\in\mathbb{N}}$ such that $F(x_n) = n$ for all $n \in \mathbb{N}$. Like in the previous, fix a standard coding of finite sequences of reals where we write $w^{1^*} = \langle x_0, \ldots, x_k \rangle$ for $x_i \in \mathbb{R}$ and $|w| = k + 1$ is called the 'length' of w. Up to coding, define the set $X \subset \mathbb{R}$ such that $w \in X$ in case w equals a finite sequence $y_0, \ldots, y_k \in \mathbb{R}$ with $F(y_i) = i$ for $i \leq k = |w|$. The set X also includes a new symbol 0_X different by fiat from all reals. Now define

$$U_{2n} = \big\{w \in X \setminus \{0_X\} : |w| = n\big\}, \quad U_{2n+1} = \{0_X\} \cup \big\{w \in X \setminus \{0_X\} : |w| > n\big\}. \tag{2.3}$$

To obtain a basis, consider the function from (2.2). To establish sequential compactness, let $(y_n)_{n\in\mathbb{N}}$ be a sequence in X. We must have $(\exists k \in \mathbb{N})(\forall n \in \mathbb{N})(y_n = 0_X \vee |y_n| \leq k)$ as otherwise we would obtain a sequence $(x_n)_{n\in\mathbb{N}}$ such that $F(x_n) = n$ for all $n \in \mathbb{N}$. As a result, $(y_n)_{n\in\mathbb{N}}$ has a constant sub-sequence and hence trivially a limit point. For (countable) compactness, note that any open neighbourhood of 0_X covers most of X.

Now define the function $f : X \to \mathbb{R}$ by $f(0_X) = 0$ and $f(x) = |x| + 1$ for $0_X \neq x \in X$. Then f is sequentially continuous at any fixed $x \neq 0_X$, as any sequence converging to x must eventually be in $U_{2|x|}$. Moreover, any sequence converging to 0_X must be eventually 0_X as otherwise we would obtain a sequence $(x_n)_{n\in\mathbb{N}}$ such that $F(x_n) = n$ for all $n \in \mathbb{N}$. Hence, f is also sequentially continuous at 0_X. However, f is not bounded, for which we seem to need $\mathsf{B\text{-}QF\text{-}AC}^{0,1}$, contradicting item (b). Hence, we have obtained a version of $\mathsf{QF\text{-}AC}^{0,1}$ for the real numbers; one readily shows that this implies $\mathsf{QF\text{-}AC}^{0,1}$ itself.

To prove item (c) using $\mathsf{QF\text{-}AC}^{0,1}$, apply the latter to '$X$ is infinite' to obtain a sequence of distinct points in X, say $(x_n)_{n\in\mathbb{N}}$. Now define $g: X \to \mathbb{R}$ as $g(x) = 0$ if $x \neq y_n$ for any $n \in \mathbb{N}$ and $g(y_n) = n + 1$. We proved above that $\mathsf{QF\text{-}AC}^{0,1}$ implies item (b), i.e., g must by discontinuous. To derive $\mathsf{QF\text{-}AC}^{0,1}$ from item (c), suppose the former is again false and consider the RSC-space $X_0 := X \setminus \{0_X\}$ with base $(U_{2n})_{n\in\mathbb{N}}$ from (2.3) and the function from (2.2). The RSC-space X_0 is sequentially compact in the same way as for the RSC-space X (but the former is not (countably) compact). Let $f: X_0 \to \mathbb{R}$ be any function and fix open $V \subset \mathbb{R}$. In case $x \in f^{-1}(V)$, then $x \in U_{2|x|} \subset f^{-1}(V)$, i.e., $f^{-1}(V)$ is (uniformly) open. Hence, all functions on X_0 are (effectively) continuous, contradicting item (c). Hence $\mathsf{QF\text{-}AC}^{0,1}$ follows from the latter, and the same for item (d).

That item (e) implies $\mathsf{QF\text{-}AC}^{0,1}$ follows from the previous. To establish the former using the latter, let X be a sequentially compact RSC-space and suppose there is an open covering $(O_n)_{n\in\mathbb{N}}$ with $(\forall n \in \mathbb{N})(\exists x \in X)(x \notin \cup_{m\leq n} O_m)$. Apply $\mathsf{QF\text{-}AC}^{0,1}$ to the latter and let $(x_n)_{n\in\mathbb{N}}$ be the associated sequence. By sequential compactness, there is a convergent sub-sequence $(y_n)_{n\in\mathbb{N}}$, say with limit $y \in X$. However, y is in some O_{n_0} and therefore so is the tail of $(y_n)_{n\in\mathbb{N}}$, a contradiction, and item (e) follows.

To derive item (f) using $\mathsf{QF\text{-}AC}^{0,1}$, let y be as in the former and apply $\mathsf{QF\text{-}AC}^{0,1}$ to $(\forall n \in \mathbb{N})(\exists x \in X)(|f(x) - y|)$. The resulting sequence has a convergent sub-sequence, say with limit $x_0 \in X$, and by continuity $f(x_0) = y$. To derive $\mathsf{QF\text{-}AC}^{0,1}$ from (f), suppose the former is false and consider the RSC-space X_0 defined above (on which all functions are continuous). Define $f: X_0 \to \mathbb{R}$ as $f(x) := 1 - \frac{1}{2^{|x|}}$ for which $\sup_{x\in X_0} f(x) = 1$ by $\mathsf{B\text{-}QF\text{-}AC}^{0,1}$. However, there clearly is no $x_0 \in X_0$ such that $f(x_0) = 1$, contradicting item (f). Hence, the latter implies $\mathsf{QF\text{-}AC}^{0,1}$. ■

Secondly, an interesting corollary is that the implication *sequential implies countable compactness* for countable RSC-spaces, inhabits the range of hyperarithmetical analysis. By contrast, the second-order version of this implication is equivalent to ACA_0 by [18, §10.8.9]. This is our first example of the *hyperarithmetical shift* from (O1) in Section 1.1.

Corollary 2.10 (ACA_0^ω). *The higher items imply the lower ones.*

- $\mathsf{QF\text{-}AC}^{0,1}$.
- *For a RSC-space X, sequential compactness implies countable compactness.*
- *The previous item restricted to countable spaces.*
- *The previous item restricted to strongly countable spaces.*
- cocode_1.
- $\mathsf{weak\text{-}}\Sigma_1^1\mathsf{\text{-}AC}_0$.

Proof. Most implications are immediate or follow from the theorem. To show that the fifth item implies cocode_1, consider the strongly countable RSC-space X from the proof of Theorem 2.8. Assuming there is no enumeration of A, this space is sequentially compact as any sequence $(x_n)_{n\in\mathbb{N}}$ in X is such that $(\exists m \in \mathbb{N})(\forall n \in \mathbb{N})(|x_n| \leq m)$. Hence, the former sequence must have a constant sub-sequence, which of course has an accumulation point. By contrast, the covering $\cup_{n\in\mathbb{N}} X_n$ of the space X has no finite sub-covering. ■

Regarding similar results, item (f) of Theorem 2.9 for CSC-spaces is provable in ACA_0 while the restriction to (strongly) countable sets implies cocode_1. Thus, item (f) and these restrictions to (strongly) provide more examples of the hyperarithmetical shift as in (O1) from Section 1.1.

Thirdly, we connect the Ginsburg–Sands theorem and countable choice.

Theorem 2.11 (ACA_0^ω). *The following are equivalent.*

- *The combination of the following.*
  - *The principle* $\mathsf{B\text{-}QF\text{-}AC}^{0,1}$.
  - *The Ginsburg–Sands theorem* GS *for RSC-spaces* $X \subset \mathbb{R}$.
- *Countable choice as in* $\mathsf{QF\text{-}AC}^{0,1}$.

Proof. First of all, we show that GS for RSC-spaces implies $\mathsf{QF\text{-}AC}^{0,1}$, assuming $\mathsf{B\text{-}QF\text{-}AC}^{0,1}$. To this end, let φ be quantifier-free and such that $(\forall n \in \mathbb{N})(\exists x \in \mathbb{R})\varphi(n, x)$. We assume a fixed coding of finite sequences of reals as real numbers, which is readily defined in ACA_0^ω. Now define $w^{1^*} \in X$ if and only if $(\forall i < |w|)\varphi(i, w(i))$, i.e., X contains initial segments of the choice function we are after. Define $U_n := \{w \in X : |w| = n\}$ and note that we obtain a base in the obvious way. Thanks to $\mathsf{B\text{-}QF\text{-}AC}^{0,1}$, the RSC-space X is infinite. Clearly, all sets are open in X and the same for any sub-space of X. By GS as in the first item, X has a sub-space that is homeomorphic to the discrete topology on $\mathbb{N}$. Let $f : X \to \mathbb{N}$ and $f^{-1} : \mathbb{N} \to X$ be the associated continuous bijection and its inverse. Now consider the sequence $(f^{-1}(n))_{n\in\mathbb{N}}$ in X. By definition, $f^{-1}(n)$ is a finite sequence and since $f : X \to \mathbb{N}$ is a bijection, we have that $(\forall m \in \mathbb{N})(\exists n \in \mathbb{N})(\exists x \in f^{-1}(n))\varphi(m, x)$. Thus, $(f^{-1}(n))_{n\in\mathbb{N}}$ readily yields a choice function for $(\forall n \in \mathbb{N})(\exists x \in \mathbb{R})\varphi(n, x)$. Since ACA_0^ω is available, $\mathsf{QF\text{-}AC}^{0,1}$ also follows.

Secondly, to prove GS for an RSC-space X, apply $\mathsf{QF\text{-}AC}^{0,1}$ to the statement 'X is infinite', yielding a sequence of distinct elements of X. Identifying an element with its index, apply the second-order GS, provable in ACA_0 by [4, Theorem 4.6], to obtain the required result. ■

Finally, an interesting corollary is that the third-order Ginsburg–Sands theorem inhabits the range of hyperarithmetical analysis. This is our second explicit example of the hyperarithmetical shift from (O1) in Section 1.1. Note that we can also restrict the fifth item to *stable* spaces (see [4]).

Corollary 2.12 (ACA_0^ω). *The higher items imply the lower ones.*

- $\mathsf{QF\text{-}AC}^{0,1}$
- *The Ginsburg–Sands theorem* GS *for RSC-spaces.*
- *The Ginsburg–Sands theorem* GS *for countable RSC-spaces.*
- *The Ginsburg–Sands theorem* GS *for strongly countable RSC-spaces.*
- *The previous item restricted to* T_1 *or Hausdorff spaces.*
- cocode_1.
- $\mathsf{weak\text{-}}\Sigma_1^1\text{-}\mathsf{AC}_0$.

Proof. The implications are either trivial or follow by Theorems 2.8 and 2.11. Note that the RSC-space in the proof of the former theorem is T_1 and Hausdorff. ∎

Regarding the fifth item in Corollary 2.12, GS for T_1-CSC-spaces is exceptional from the point of view of the RM zoo [4, 17], but all second-order versions of GS do not go beyond ACA_0. By contrast, in higher-order RM, most variations of GS are classified in the range of hyperarithmetical analysis, in the sense of Corollary 2.12. Of course, it is then a natural question whether *our* classification of GS depends on our choice of representation, in particular the definition of 'finite set' from Definition 1.8. Theorem 2.14 right below suggests that the answer is negative, where we use the usual definitions from set theory as follows.

Definition 2.13. A set of reals $X \subset \mathbb{R}$ is called

- *set theory finite* if there is $k \in \mathbb{N}$ and $Y : \mathbb{R} \to \mathbb{N}$ such that Y is injective on X and $Y(x) \leq k$ for $x \in X$,
- *set theory infinite* if it is not set theory finite,
- *Dedekind infinite* if there is $Z \subsetneq X$ and $Y : \mathbb{R} \to \mathbb{R}$ with $(\forall z \in Z)(Y(z) \in X)$ and Y is injective on Z, i.e., $Y(z) = Y(z') \to z = z'$ for all $z, z' \in Z$.

We also assume $\Sigma\text{-}\mathsf{IND}$, the higher-order[4] version of Σ^1_1-induction.

Theorem 2.14. *The system* $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1} + \Sigma\text{-}\mathsf{IND}$ *exists in the range of hyperarithmetical analysis and proves the Ginsburg–Sands theorem* GS *for RSC-spaces with 'infinite' replaced by 'set theory infinite' or 'Dedekind infinite'.*

Proof. First of all, the induction axiom $\Sigma\text{-}\mathsf{IND}$ proves that a finite set $X \subset \mathbb{R}$ can be enumerated (via a finite sequence) by considering the formula

$$\varphi(n) \equiv (\exists w^{1^*})\big[|w| = n \wedge \big(\forall i < |w|\big)\big(w(i) \in X\big)\big].$$

Hence, a set theory infinite or Dedekind infinite set must be infinite, as the former notions do not hold for finite-sets-with-an-enumeration, by the pigeon-hole principle. Thus, assuming $\Sigma\text{-}\mathsf{IND}$, GS for set theory or Dedekind infinite sets, implies GS for infinite sets, as required.

Secondly, for the remaining claim, we recall that $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$ is conservative over $\Sigma^1_1\text{-}\mathsf{AC}_0$ by [28, Cor. 2.7]. This is established by a construction that extends any model $\mathcal{M}$ of $\Sigma^1_1\text{-}\mathsf{AC}_0$ to a model $\mathcal{N}$ of $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$ where the second-order part of $\mathcal{N}$ is isomorphic to $\mathcal{M}$. Via exactly the same construction, any model $\mathcal{M}'$ of $\Sigma^1_1\text{-}\mathsf{AC}_0 + \Sigma^1_1\text{-induction}$ extends to a model $\mathcal{N}'$ of $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1} + \Sigma\text{-}\mathsf{IND}$, where the second-order part of $\mathcal{N}'$ is isomorphic to $\mathcal{M}'$. Hence, $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1} + \Sigma\text{-}\mathsf{IND}$ is conservative over $\Sigma^1_1\text{-}\mathsf{AC}_0 + \Sigma^1_1\text{-induction}$. The latter is known to be a system of hyperarithmetical analysis, as observed between [2, Def. 3.4 and Thm. 3.6]. ∎

[4] The induction schema $\Sigma\text{-}\mathsf{IND}$ is $[\varphi(0) \wedge (\forall n \in \mathbb{N})(\varphi(n) \to \varphi(n+1))] \to (\forall n \in \mathbb{N})\varphi(n)$ for any $\varphi(n)$ of the form $(\exists f \in \mathbb{N}^{\mathbb{N}})\varphi_0(n, f)$ for quantifier-free φ_0 and any parameters.

In conclusion, we have obtained a number of equivalences involving countable choice and RSC-space, contributing to (O3) from Section 1.1. As an aside, we showed that the Ginsburg–Sands theorem for (countable) RSC-spaces inhabits the range of hyperarithmetical analysis, contributing to (O1). By contrast, arithmetical comprehension suffices to establish this theorem for CSC-spaces.

2.4. Uncountable spaces and comprehension

In this section, we develop observation (O3) from Section 1.1. In particular, we show that basic statements about *un*countable RSC-spaces imply – or are equivalent to – strong axioms, including Kleene's quantifier $(\exists^3)$, second-order arithmetic, and Feferman's projection principle, with the latter defined next.

First of all, Feferman introduces the 'projection principle' Proj_1 in [21, §5] as a third-order version of Kleene's quantifier $(\exists^3)$ from Section 1.3.1. Working over a base theory akin to ACA_0^ω, it is then shown that Proj_1 implies various well-known theorems of analysis, like the supremum principle. Moreover, Proj_1 also yields Z_2 when combined with (μ^2). Thus, $\mathsf{Proj1}$ can be said to be impredicative and highly non-constructive. Now, Feferman's language is slightly richer than that of ACA_0^ω and the following axiom constitutes the higher-order RM version of Proj_1:

$$(\forall Y : \mathbb{N}^\mathbb{N} \to \mathbb{N})(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})\big[n \in X \leftrightarrow (\exists f \in \mathbb{N}^\mathbb{N})\big(Y(f, n) = 0\big)\big]. \quad (\mathsf{BOOT})$$

The name refers to the verb 'to bootstrap' as Π_k^1-CA_0^ω + BOOT prove Π_{k+1}^1-CA_0. Convergence theorems for nets are equivalent to BOOT, as well as the supremum principle for certain weak continuity notions [48, 51].

Secondly, we have the following theorem, in line with (O3) from Section 1.1.

Theorem 2.15 (ACA_0^ω + $\mathsf{B\text{-}QF\text{-}AC}_!^{0,1}$). *The following are equivalent.*

(a) *An RSC-space is separable.*

(b) *An RSC-space is Lindelöf.*

(c) *A compact RSC-space is separable.*

(d) *The maximum principle* MAX *for sequentially compact RSC-spaces.*

(e) *The supremum principle* SUP' *for RSC-spaces.*

(f) *The supremum principle* SUP *for RSC-spaces.*

(g) *The previous item restricted to effectively continuous functions.*

(h) *The previous item restricted to sequences* $(C_n)_{n\in\mathbb{N}}$ *of* uniformly *closed sets.*

(i) $\mathsf{QF\text{-}AC}^{0,1}$ + BOOT.

The equivalence holds for 'sequential', 'limit point', and 'countable' compactness.

Proof. First of all, to derive the other items from $\mathsf{QF\text{-}AC}^{0,1}$ + BOOT, let $X \subset \mathbb{R}$ be an RSC-space with base $(U_i)_{i\in\mathbb{N}}$. Use BOOT to form the set $X_0 \subset \mathbb{N}$ such that $(i \in X_0 \leftrightarrow (\exists x \in \mathbb{R})(x \in U_i))$. Now apply $\mathsf{QF\text{-}AC}^{0,1}$ to $(\forall i \in \mathbb{N})(i \in X_0 \to (\exists x \in U_i))$ to obtain the sequence witnessing the separability of X. This sequence also establishes that X is Lindelöf. The supremum principle now follows via the usual interval-halving technique, replacing $(\exists x \in X)(f(x) > q)$ by $(\exists n \in \mathbb{N})(f(x_n) > q)$, where $(x_n)_{n\in\mathbb{N}}$ is the sequence provided by the separability of the space. Note that by Theorem 2.9, $\mathsf{QF\text{-}AC}^{0,1}$ suffices to show that continuous functions are bounded. The principle SUP' follows in the same way. To obtain MAX, apply $\mathsf{QF\text{-}AC}^{0,1}$ to

$$(\forall k, n \in \mathbb{N})(\exists y \in X)\big(\big|f_n(y) - \sup_{x\in X} f_n(x)\big| < \tfrac{1}{2^k}\big).$$

The resulting sequence has a convergent (relative to k) sub-sequence, say with limits $(y_n)_{n\in\mathbb{N}}$. The latter is as required for MAX.

Secondly, assume item (c) and note that BOOT is equivalent over ACA_0^ω to the statement that for any $F:\mathbb{R}\to\mathbb{N}$, there is $X_1 \subset \mathbb{N}$ such that

$$(\forall n \in \mathbb{N})\big(n \in X_1 \leftrightarrow (\exists x \in \mathbb{R})\big(F(x) = n\big)\big).$$

Now fix $F:\mathbb{R}\to\mathbb{N}$ and define $(U_i)_{i\in\mathbb{N}}$ as follows:

$$U_{2n} = \big\{x \in \mathbb{R} : F(x) = n\big\} \quad \text{and} \quad U_{2n+1} = \{0_X\} \cup \big\{x \in \mathbb{R} : F(x) \geq n\big\},$$

where $X := \mathbb{R} \cup \{0_X\}$ and 0_X is a new symbol different from all reals by fiat. Then $(U_n)_{n\in\mathbb{N}}$ is a base for X, as the associated function k is straightforward. Note that any open set containing 0_X covers all but finitely many points in X, i.e., the latter is (countably) compact. For sequential compactness (and similar for limit point compactness), the point 0_X is the required limit point for sequences that do not contain a constant sub-sequence. Now let $(x_n)_{n\in\mathbb{N}}$ be the sequence provided by item (c) and note that for any $n \in \mathbb{N}$, by definition:

$$(\exists m \in \mathbb{N})\big(F(x_m) = n\big) \leftrightarrow U_{2n} \neq \emptyset \leftrightarrow (\exists x \in \mathbb{R})\big(F(x) = n\big).$$

Since the left-most formula is arithmetical, BOOT follows. To derive $\mathsf{QF\text{-}AC}^{0,1}$, we assume $(\forall n \in \mathbb{N})(\exists x \in \mathbb{R})(F(x) = n)$ and proceed in the same way.

Thirdly, assume item (f) and note that $\mathsf{QF\text{-}AC}^{0,1}$ follows by Theorem 2.9. To obtain BOOT, fix $F:\mathbb{R}\to\mathbb{N}$ and define the compact RSC-space X as in the previous paragraph. Moreover, the function $f: X \to \mathbb{R}$ defined as $f(x) = \frac{1}{2^{F(x)}}$ for $x \neq 0_X$ and $f(0_X) = 0$ is (effectively) continuous. Now consider the following:

$$(\exists x \in \mathbb{R})\big(F(x) = n\big) \leftrightarrow \big[\tfrac{1}{2^n} = \sup_{x\in C_n} f(x)\big], \tag{2.4}$$

where $C_n := X \setminus \cup_{i<n} U_{2i}$; the complement of C_n is uniformly open as witnessed by $\eta(x) := 2F(x)$. Since the right-hand side of (2.4) is arithmetical, BOOT follows. The proof involving items (d) and (e) is similar. ■

Fourth, the following theorem is now relatively straightforward. We emphasise that the norm $\|f\|_\infty = \lambda f.\sup_{x\in X} f(x)$ is found throughout mathematical textbooks. The final item in the theorem merely expresses the existence of $\|f\|_\infty$ for continuous functions on certain spaces.

Theorem 2.16 (RCA_0^ω). *The following are equivalent.*

- *Kleene's quantifier* $(\exists^3)$.
- *The combination of the following.*
  - *Kleene's quantifier* $(\exists^2)$.
  - *For any compact RSC-space X, there exists $\xi\colon (X \to \mathbb{R}) \to \mathbb{R}$ such that for any continuous $f\colon X \to [0,1]$, we have $\xi(f) = \sup_{x\in X} f(x)$.*

Proof. First of all, Kleene's $(\exists^3)$ implies BOOT and can perform the usual interval-halving technique for finding the supremum of bounded functions, as quantifying over an RSC-space X amounts to quantifying over Baire space.

Secondly, fix $F\colon\mathbb{R}\to\mathbb{N}$ and consider the RSC-space X as in the proof of Theorem 2.15. Recall the function $f\colon X \to [0,1]$ define by $f(x) := 1 - \frac{1}{2^{F(x)}}$ and $f(0_X) = 0$. Now consider

$$(\exists x \in \mathbb{R})\big(F(x) = 0\big) \leftrightarrow \big[1 = \sup_{x\in X} f(x)\big].$$

Thus, $\xi(f)$ as in the final item of the theorem allows us to decide where F has a zero or not. Using $(\exists^2)$, this readily yields $(\exists^3)$. ∎

Clearly, we could restrict the final item in the theorem to *effectively* (or strongly) continuous functions. The associated function φ could then be an input to the modified ξ-functional. Many other variations seem possible.

Finally, the definition of RSC-space as in Definition 1.6 is readily adapted to sets $X \times Y \subset \mathbb{R}^2$. We now study the following supremum principle where we stress that the supremum-as-a-function $\lambda x.\sup_{y\in Y} f(x,y)$ is found in textbooks like [44, 45].

Principle 2.17 (SUP_2). *Let $X \times Y \subset \mathbb{R}^2$ be a compact RSC-space. For a continuous function $f\colon X \times Y \to [0,1]$, there exists an $X \to \mathbb{R}^2$-function Φ such that*

$$\Phi(x)(1) = \sup_{y\in Y} f(x,y) \quad \textit{and} \quad \Phi(x)(2) = \inf_{y\in Y} f(x,y) \quad \textit{for all } x \in X.$$

We now show that Principle 2.17 implies the following generalisation of BOOT to arbitrary quantifier alternations, where $k \geq 2$.

Principle 2.18 (BOOT_k). *For any $Y\colon (\mathbb{N}^\mathbb{N})^k \to \mathbb{N}$, there is $X \subset \mathbb{N}$ with*

$$(\forall n \in \mathbb{N})\Big[n \in X \leftrightarrow \underbrace{(\exists f_0 \in \mathbb{N}^\mathbb{N})(\forall f_1 \in \mathbb{N}^\mathbb{N})\cdots}_{k-1 \text{ quantifier alternations}}\big(Y(f_0,\ldots,f_k,n) = 0\big)\Big].$$

We note that $\mathsf{BOOT}_k \to \Pi_k^1\text{-}\mathsf{CA}_0$ over RCA_0^ω while $\mathsf{ACA}_0^\omega + \mathsf{BOOT}_k$ proves $\Pi_{k+1}^1\text{-}\mathsf{CA}_0$. We now have the following implication.

Theorem 2.19 (ACA_0^ω). *We have $\mathsf{SUP}_2 \to \mathsf{BOOT}_k$ for any $k \geq 2$. We can replace 'sequential' by 'limit point' or 'countable' or 'open cover' in SUP_2.*

Proof. We first establish $\mathsf{SUP}_2 \to \mathsf{BOOT}_2$ using Theorem 2.15. Like in the proof of the latter, BOOT_2 is equivalent over ACA_0^ω to the statement that for any $F : \mathbb{R}^2 \to \mathbb{N}$, there is $X_2 \subset \mathbb{N}$ such that

$$(\forall n \in \mathbb{N})\big(n \in X_2 \leftrightarrow (\exists x \in \mathbb{R})(\forall y \in \mathbb{R})\big(F(x,y) = n\big)\big). \tag{2.5}$$

Now fix $F\colon \mathbb{R}^2 \to \mathbb{N}$ and define $(U_i)_{i\in\mathbb{N}}$ as follows:

$$U_{2n} = \big\{(x,y) \in \mathbb{R}^2 : F(x,y) = n\big\} \quad \text{and} \quad U_{2n+1} = \{\overline{0}_X\} \cup \big\{(x,y) \in \mathbb{R}^2 : F(x,y) \geq n\big\},$$

where $\overline{0}_X$ is a new symbol different from all $(x,y) \in \mathbb{R}^2$ by fiat; also, $X := \mathbb{R}^2 \cup \{\overline{0}_X\}$. Clearly, the RSC-space X is (countably) compact as any open set containing $\overline{0}_X$ covers all but finitely many points in X. Now define $f\colon (X \times X) \to \mathbb{R}$ as $f(\overline{0}_X) = 0$ and $f((x,y)) = \frac{1}{2^{F(x,y)}}$ for $x, y \in \mathbb{R}$, which is readily seen to be (effectively) continuous. Hence, we have access to the functions $\lambda x.\sup_{y\in X} f((x,y))$ and $\lambda x.\inf_{z\in X} f((x,z))$. Now consider the following equivalence, for all $n \in \mathbb{N}$:

$$(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})\big(F(x,y) = n\big) \leftrightarrow (\exists x \in \mathbb{R})\big(\sup_{y\in X} f((x,y)) = \tfrac{1}{2^n} = \inf_{z\in X} f((x,z))\big).$$

The right-hand side only has one quantifier over $\mathbb{R}$, i.e., BOOT provides a set $X_1 \subset \mathbb{N}$ of exactly those $n \in \mathbb{N}$ satisfying $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(F(x,y) = n)$. Thus, we obtain (2.5) and hence BOOT_2 following Theorem 2.15.

Finally, we prove BOOT_3 from SUP_2; the general case is then straightforward. Fix $G\colon \mathbb{R}^3 \to \mathbb{N}$, let $Z\colon \mathbb{R} \to \mathbb{R}^2$ be a bijection and let $W\colon \mathbb{R}^2 \to \mathbb{R}$ be the inverse, readily defined in ACA_0^ω. Then define $F(x,y) := G(Z(x)(1), Z(x)(2), y)$ and consider the RSC-space X and function $f : X^2 \to \mathbb{R}$ defined in terms of F as in the previous paragraph. Now consider the following, for any $n \in \mathbb{N}$:

$$\begin{aligned}&(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(\forall z \in \mathbb{R})\big(G(x,y,z) = n\big)\\ &\quad\leftrightarrow (\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\big(\sup_{z_0\in X} f((W(x,y), z_0)) = \tfrac{1}{2^n} = \inf_{z_1\in X} f((W(x,y), z_1))\big),\end{aligned}$$

and the bottom formula only involves one quantifier alternation, i.e., BOOT_2 applies, yielding BOOT_3 as required. ∎

Funding. Our research was supported by the *Klaus Tschira Boost Fund* via the grant Projekt KT43. We express our gratitude towards the latter institution.

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Communicated by Michael Rathjen

Received 13 September 2024; revised 7 February 2025.

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