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Khovanov homology and rational unknotting

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Abstract. Building on the work by Alishahi–Dowlin, we extract a new knot invariant $\lambda \geq 0$ from universal Khovanov homology. While λ is a lower bound for the unknotting number, in fact more is true: λ is a lower bound for the proper rational unknotting number (the minimal number of rational tangle replacements preserving connectivity necessary to relate a knot to the unknot). Moreover, we show that, for all $n \geq 0$, there exists a knot K with $\lambda(K) = n$. Along the way, following Thompson, we compute the Bar-Natan complexes of rational tangles.

1. Introduction

The most famous geometric application of Khovanov's categorification of the Jones polynomial [18] comes from the Rasmussen invariant, a knot concordance homomorphism giving a strong lower bound for the slice genus [41]. The Rasmussen invariant may be read off the grading of the limit of Lee's spectral sequence [26], which starts at Khovanov homology. Recently, Alishahi and Dowlin [2] discovered that there is further geometric information contained in Lee's spectral sequence. Namely, the number of the page on which that sequence collapses is a lower bound for the unknotting number u. This new lower bound behaves rather differently from the Rasmussen invariant; e.g., it is not invariant under concordance and not additive under the connected sum of knots.

In this paper, we use Alishahi and Dowlin's methods on a different variation of Khovanov homology to define a new knot invariant λ taking non-negative integer values. This invariant is greater than or equal to all the previously known variations of Alishahi–Dowlin's bound appearing in [1, 2, 8, 13]. Before giving the definition of λ, let us state our two main results.

Theorem 1.1. *For all knots K, one has $\lambda(K) \leq u_q(K)$.*

Theorem 1.2. *For every $n \geq 0$, there exists a knot K such that $\lambda(K) = n$.*

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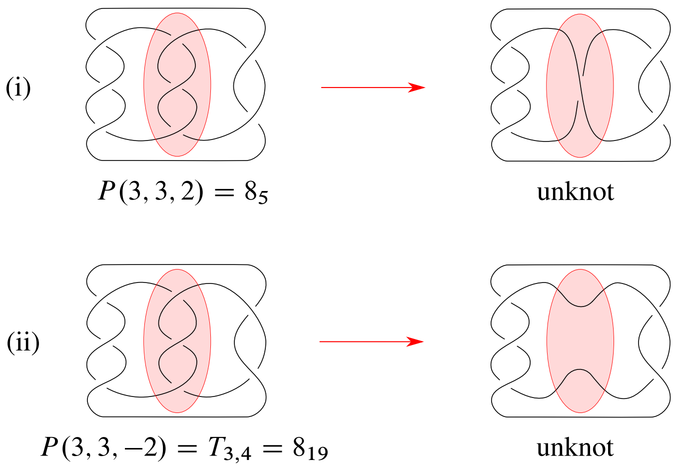

Figure 1. (i) An example of a proper rational replacement ($1/3$ by -1 in the language of Definition 5.14), showing that the $P(3,3,2)$ pretzel knot has proper rational unknotting number 1. (ii) An example of a non-proper rational replacement ($1/3$ by 0), showing that the $P(3,3,-2)$ pretzel knot, which is also the $T_{3,4}$ torus knot, has rational unknotting number 1. Since $\lambda(T_{3,4}) = 2$, it follows from Theorem 1.1 that there is no *proper* rational replacement transforming the $T_{3,4}$ pretzel knot into the unknot; i.e., $T_{3,4}$ has proper rational unknotting number at least 2 (and in fact equal to 2).

Here, the *proper rational unknotting number* $u_q(K)$ is defined as follows.

Definition 1.3. Two knots K, K' are related by a *rational replacement* if K' may be obtained from K by replacing a rational tangle T in K with another rational tangle T'. If the arcs of T and T' connect the same tangle end points, we say that the rational replacement is *proper*. Now, $u_q(K)$ is defined as the minimal number of proper rational replacements relating K to the unknot.

Figure 1 shows examples of rational replacements and an application of Theorem 1.1. Since a crossing change is merely a special case of a proper rational replacement, we find that $u_q(K) \leq u(K)$ holds for all knots K. So, Theorem 1.1 can be seen as a strengthening of the inequality $\mathfrak{u}_X(K) \leq u(K)$, where $\mathfrak{u}_X(K)$ denotes one of the bounds constructed by Alishahi–Dowlin [2]. Indeed, we have

$$\mathfrak{u}_X(K) \leq \lambda(K) \leq u_q(K) \leq u(K).$$

We will see that none of these inequalities are equalities, and in fact, the gaps between the four involved invariants can be arbitrarily large.

In contrast to Theorem 1.2, let us note that currently $\mathfrak{u}_X(K) \leq 3$ holds for all knots K for which $\mathfrak{u}_X$ has been computed. The essentially only known knot with

$\mathfrak{u}_X(K) = 3$ was recently found by Manolescu and Marengon [34]. Still, it seems a reasonable conjecture that $\mathfrak{u}_X$ is unbounded, though the proof of that conjecture might require more complicated knots and methods of computation than our proof of the unboundedness of λ in Theorem 1.2.

In the remainder of the introduction, let us provide details and background about Khovanov homologies, the invariant λ, and rational unknotting.

1.1. Rational replacements and rational unknotting

Rational unknotting has previously been considered by Lines [30] and McCoy [36], while proper rational unknotting was explored in a recent paper by McCoy and Zentner [37]. In those papers, rational unknotting is obstructed via the double branched cover, relying on the so-called Montesinos trick: if two knots K and J are related by a rational replacement, then their double branched covers M_K, M_J are related by a surgery. So, one may obstruct the existence of certain rational replacements by obstructing the existence of certain surgeries.

The arguably easiest upshot of that method is the following: the minimal number of generators of $H_1(M_K; \mathbb{Z})$ is a lower bound for the rational unknotting number of K. For example, this implies that the connected sum of n trefoil knots has (proper and non-proper) rational unknotting number equal to n. On the other hand, one may easily compute that λ of the connected sum of $n \geq 1$ trefoil knots equals 1. This may be taken as a first sign that our lower bound λ is quite different from the lower bounds for u_q obtainable from the double branched cover. In the instance of connected sums of trefoils, the λ bound is weaker. However, there are also knots for which λ is the strongest lower bound for u_q that we know of. Let us give an explicit example.

Question 1.4. In Example 3.18, we compute $\lambda(T_{5,6}) = 3$, from which it follows that $u_q(T_{5,6}) \geq 3$. Is there any other way of showing $u_q(T_{5,6}) \geq 3$?

Note that the gap between the proper rational unknotting number u_q and the (classical) unknotting number u may be arbitrarily high. For example, $u_q(K) = 1$ clearly holds for all two-bridge knots K; but $u(K)$ of two-bridge knots can take any value, which can, e.g., be shown using the signature bound $|\sigma(K)/2| \leq u(K)$. This also demonstrates that $|\sigma|/2$ is not a lower bound for the proper rational unknotting number.

In Section 2.1, we will fix our conventions regarding tangles and tangle diagrams. We will then revisit Definition 1.3 and give a more refined definition of rational replacement in Definition 5.14. As an aside, let us also remark that, in the definition of the proper rational unknotting number $u_q(K)$, the proper rational replacements relating K and the unknot are sequential: happening one after another. However, by a standard transversality argument (see, e.g., [43]), one can show that, for every knot K, there exist $u_q(K)$ many simultaneous rational replacements, i.e., rational replacements taking place in pairwise disjoint balls.

1.2. A simple universal Khovanov homology

An algebra A over a ring R equipped with an (A, A)-bilinear map $\Delta\colon A \to A \otimes A$ and an R-linear map $\varepsilon\colon A \to R$, such that $(\varepsilon \otimes \mathrm{Id}) \circ \Delta = \mathrm{Id}$, is called a *Frobenius algebra*, and the tuple $\mathcal{F} = (R, A, \Delta, \varepsilon)$ is called a *Frobenius system*. We will only consider the so-called *rank two Frobenius systems*. These are Frobenius systems $\mathcal{F}$ with an $X \in A$ such that A is freely generated by 1 and X as an R-module. Moreover, all our Frobenius algebras will be equipped with a filtration or a grading such that 1 and X are homogeneous elements of degree 0 and -2, respectively. We call this the *quantum grading*. Every such rank two Frobenius system $\mathcal{F}$ yields a variation of Khovanov homology theory, i.e., a way to associate with all diagrams D of a link L a chain complex $C_{\mathcal{F}}(D)$ of R-modules such that $C_{\mathcal{F}}(L)$ is well defined up to homotopy equivalence [20]. For links with a marked component, or for knots, there is an action of A on $C_{\mathcal{F}}(D)$, which is well defined up to homotopy, so we may consider $C_{\mathcal{F}}(D)$ as a chain complex of A-modules, which are free.

Khovanov's original homology theory corresponds to the Frobenius algebra $\mathbb{Z}[X]/(X^2)$ over $\mathbb{Z}$. On the other hand, the theory coming from the Frobenius algebra $A_{\mathrm{univ}} = R_{\mathrm{univ}}[X]/(X^2 - hX - t)$ over $R_{\mathrm{univ}} = \mathbb{Z}[h, t]$ is called *universal*, since for all rank two Frobenius algebras $\mathcal{F}$, the chain complex $C_{\mathcal{F}}(D)$ is determined by $C_{\mathrm{univ}}(D)$ [20]. Recently, Khovanov and Robert defined another theory called α-homology, which is also universal in the sense above [21]. To define λ, we will use a third universal theory, which we call $\mathbb{Z}[G]$-homology.[1] The universality of this theory is due to Naot [39, 40]. This $\mathbb{Z}[G]$-theory associates with a diagram D of a knot K the reduced Khovanov chain complex coming from the Frobenius algebra $R[X]/(X^2 + GX)$ with $R = \mathbb{Z}[G]$. We denote this chain complex by $[\![D]\!]$ (well defined up to isomorphism) or $[\![K]\!]$ (well defined up to homotopy equivalence). Our reason to use $\mathbb{Z}[G]$-homology is that it is the simplest of the three mentioned universal theories, in so far as the ground ring is a polynomial ring in only one, instead of two variables. Let us explicitly state how $\mathbb{Z}[G]$-homology determines $\mathcal{F}_{\mathrm{univ}}$-homology. (This is implicit in the work of Naot [39, 40].)

Theorem 1.5. *Endow $A_{\mathrm{univ}} = \mathbb{Z}[h, t][X]/(X^2 - hX - t)$ with the structure of a $\mathbb{Z}[G]$-module by letting G act as $2X - h$. Then, for every knot K,*

$$C_{\mathrm{univ}}(K) \simeq [\![K]\!] \otimes_{\mathbb{Z}[G]} A_{\mathrm{univ}}\{1\}.$$

[1] Here, G is simply a symbolic variable, and so, $\mathbb{Z}[G]$ is the one-variable polynomial ring over the integers (not some group ring).

Here, $\{\cdot\}$ denotes a shift in quantum degree.

Corollary 1.6. *For every knot K, $C_{\mathrm{univ}}(K)$ is homotopy equivalent to a chain complex of free shifted A_{univ}-modules, with differentials consisting only of integer multiples of powers of $2X - h$.* ∎

Theorem 1.5 and Corollary 1.6 can be understood to say that $\mathbb{Z}[G]$-homology encodes the same amount of information present in $\mathcal{F}_{\mathrm{univ}}$- and α-homology in a more compact way. In particular, the original reduced Khovanov homology over $\mathbb{Z}$ of K as defined in [19] may be obtained from $[\![K]\!]$ simply by setting $G = 0$, i.e., by tensoring with $\mathbb{Z}[G]/(G) \cong \mathbb{Z}$. The original unreduced Khovanov homology over $\mathbb{Z}$ is also determined by $\mathbb{Z}[G]$-homology; see Corollary 2.18.

Let us give three examples of $\mathbb{Z}[G]$-complexes $[\![\cdot]\!]$ of knots. For the unknot U, $[\![U]\!]$ is simply homotopy equivalent to one copy of $\mathbb{Z}[G]$ supported in homological degree 0. For the trefoil $T_{2,3}$, we have a homotopy equivalence

$$[\![T_{2,3}]\!] \simeq {}_0\mathbb{Z}[G]\{2\} \to 0 \to \mathbb{Z}[G]\{6\} \xrightarrow{G} \mathbb{Z}[G]\{8\},$$

where the subscript to the left denotes homological degree. Finally, for the $T_{3,4}$ torus knot, we have

$$[\![T_{3,4}]\!] \simeq {}_0\mathbb{Z}[G]\{6\} \to 0 \to \mathbb{Z}[G]\{10\} \xrightarrow{G} \mathbb{Z}[G]\{12\} \xrightarrow{0} \mathbb{Z}[G]\{12\} \xrightarrow{G^2} \mathbb{Z}[G]\{16\}.$$

Many more examples will be given in Sections 3 and 4.

We will give an introduction to Khovanov homologies and the proof of Theorem 1.5 in Section 2. The idea of the proof is to show that $\mathbb{Z}[G]$-homology of a knot K determines the *Bar-Natan complex* [5] of the 2-ended tangle obtained by cutting K open at some point. The Bar-Natan complex in turn is known to be universal and determine $\mathcal{F}_{\mathrm{univ}}$-homology. The Bar-Natan complex is discussed in Section 2, and also in Section 1.4 below.

1.3. The definition of λ

Let us now introduce this paper's protagonist.

Definition 1.7. For a knot K, let $\lambda(K)$ be the minimal integer $k \geq 0$ such that there exist ungraded chain maps (i.e., chain maps that do not need to respect the homological or the quantum degree, cf. Definition 3.1)

$$[\![K]\!] \underset{g}{\overset{f}{\rightleftarrows}} [\![U]\!]$$

and homotopies $g \circ f \simeq G^k \cdot \mathrm{id}_{[\![K]\!]}$, $f \circ g \simeq G^k \cdot \mathrm{id}_{[\![U]\!]}$.

It is not obvious that, for a given knot, f, g, k as in Definition 1.7 exist at all. So, for the time being, we scrupulously set $\lambda(K) = \infty$ if they do not. It will then be a consequence of Theorem 1.1 that this case does not occur, since $\lambda(K) \leq u_q(K) \leq u(K) < \infty$. To get acquainted with calculating λ, the reader is invited to convince themself that $\lambda(U) = 0$, $\lambda(T_{2,3}) = 1$, $\lambda(T_{3,4}) = 2$.

Our Definition 1.7 is based on the work of Alishahi and Dowlin, who use analogous maps f, g in the proof that their invariant $\mathfrak{u}_X$ is a lower bound for the unknotting number u. The invariant $\mathfrak{u}_X(K)$ is defined as the maximal X-torsion order of the homology of K coming from the Frobenius algebra $\mathcal{F}_{\text{Lee}} = \mathbb{Q}[X,t]/(X^2 - t)$ over $\mathbb{Q}[t]$, i.e., the minimal n such that $X^n H_{\text{Lee}}(K)$ is torsion-free. At first glance, the definition of $\mathfrak{u}_X$ and λ appear to be rather different, but on a closer look, one finds that $\mathfrak{u}_X(K) = \lambda_X(K)$, where $\lambda_X(K)$ is the minimal $k \geq 0$ such that there exist ungraded chain maps

$$C_{\text{Lee}}(K) \underset{g}{\overset{f}{\rightleftarrows}} C_{\text{Lee}}(U)$$

and homotopies $g \circ f \simeq (2X)^k \cdot \mathrm{id}_{C_{\text{Lee}(K)}}$, $f \circ g \simeq (2X)^k \cdot \mathrm{id}_{C_{\text{Lee}(U)}}$.

In this sense, λ is a direct generalization of $\mathfrak{u}_X$, obtained from the reduced homology coming from the Frobenius algebra $\mathcal{F}_{\mathbb{Z}[G]}$ instead of from the unreduced homology coming from $\mathcal{F}_{\text{Lee}}$. But why do not we instead of λ consider $\mathfrak{u}_G(K)$, defined (see Definition 3.21) as the maximal G-torsion order of $\mathbb{Z}[G]$-homology of K? There are two reasons. Firstly, λ is not equal to $\mathfrak{u}_G$ for all knots; the proof of the equality $\lambda_X = \mathfrak{u}_X$ does not carry over from $\mathbb{Q}[X,t]/(X^2 - t)$ to $\mathbb{Z}[G]$ because it relies on

$$\mathbb{Q}[X,t]/(X^2 - t) \cong \mathbb{Q}[X]$$

being a PID, which $\mathbb{Z}[G]$ is not. In fact, $\lambda(K) \geq \mathfrak{u}_G(K)$ holds for all knots K. Secondly, $\mathfrak{u}_G$ displays some unorthodox behavior; for example, the value of $\mathfrak{u}_G(-K)$ is not determined by the value of $\mathfrak{u}_G(K)$, where $-K$ denotes the mirror image of K. Again, the ring $\mathbb{Z}[G]$ not being a PID is to blame for this.

More details about the invariant λ and related invariants can be found in Section 1.5 below and in Section 3.

1.4. The Bar-Natan complex of rational tangles

The proof of Theorem 1.1, given at the end of Section 5, proceeds by relating the Bar-Natan complexes of different rational tangles. So, first, we need to compute the Bar-Natan complexes of rational tangles. For this, we heavily rely on Thompson's computation [50] of *dotted* Bar-Natan complexes of rational tangles. Thompson's proof adapts mutatis mutandis to the (more general) version of Bar-Natan complexes that we are using. We also use Kotelskiy–Watson–Zibrowius' theorem [22, Theorem 1.1] that Bar-Natan's category of 4-ended tangles and cobordisms is equivalent to a category coming from a quiver with two vertices and four edges, which yields a quite simple calculus for chain complexes of 4-ended tangles.

As a result, we obtain a recursive algorithm that takes as input a rational tangle T corresponding to a rational number $p/q \in \mathbb{Q} \cup \{\infty\}$ and returns as output a chain complex in the homotopy class of the Bar-Natan complex of T. That chain complex is relatively small, consisting of $|p| + |q|$ many objects and $|p| + |q| - 1$ non-trivial morphisms. We refer the reader to Section 5 and Theorem 5.6 for details.

1.5. Further properties and generalizations of λ

From Kronheimer–Mrowka's result that Khovanov homology detects the unknot [24], it follows that λ does, too.

Proposition 1.8. *The λ-invariant detects the unknot; i.e., $\lambda(K) = 0$ holds if and only if K is the unknot.*

We will see that the value of $\lambda(K\#J)$ for the connected sum $K\#J$ is not determined by the values of $\lambda(K)$ and $\lambda(J)$. However, we can say the following.

Proposition 1.9. (i) $\lambda(K\#J) \leq \lambda(K) + \lambda(J)$ *for all knots K, J.*

(ii) λ *does not change under taking mirror images, or orientation reversal.*

Let us call a knot K *thin* if its reduced integral Khovanov homology consists of free modules supported in a single δ-degree (see Section 3.5 for the definition of δ and further details).

Proposition 1.10. *For all non-trivial thin knots K, we have $\lambda(K) = 1$.*

In particular, $\lambda(K) = 1$ holds for all non-trivial quasi-alternating knots, since those knots are thin in the above sense [35]. This leads to applications, such as the following.

Example 1.11. In Example 3.18, we will compute $\lambda(T_{5,6}) = 3$. It follows that there is no proper rational replacement relating $T_{5,6}$ to a quasi-alternating knot (compare this to [8, Example 10]).

In the definition of λ, replacing U by an arbitrary second knot J yields the definition of a function $\lambda(K, J) \geq 0$ that is symmetric and obeys the triangle inequality: $\lambda(\cdot, \cdot)$ is a pseudometric on the set of isotopy classes of knots. In fact, we can even further extend the definition of λ and define λ for pairs of tangles. This leads to a pseudometric on the set of equivalence classes of tangles in a fixed ball, with fixed base point, connectivity, and number of components; see Proposition 3.13. Section 3 provides details and also the proofs of the above propositions.

1.6. A comparison of λ with previously known invariants

Alishahi and Dowlin's article [2] appeared at the same time as an article by Alishahi [1], in which, similarly to $\mathfrak{u}_X$, a lower bound $\mathfrak{u}_h$ for the unknotting number was constructed using the Frobenius algebra $\mathbb{F}_2[X,h]/(X^2+hX)$ over $\mathbb{F}_2[h]$. Then, further papers followed: Caprau–González–Lee–Lowrance–Sazdanović–Zhang generalized Alishahi and Dowlin's work for $\mathbb{Q}$ to the fields $\mathbb{F}_p$ for all odd primes p [8]. Gujral [13], using α-homology, defined an invariant ν which can be seen to equal our invariant $\mathfrak{u}_G$ and showed that it provides a lower bound for the ribbon distance between knots; this was a generalization of earlier work by Sarkar [42]. Here, the *ribbon distance* between two smoothly concordant knots K and J is the minimal k such that there is a sequence $K = K_1, \dots, K_n = J$ of knots such that each consecutive K_i, K_{i+1} are related by a ribbon concordance in either direction with at most k saddles. This leads to the following question (see Section 1.5 or Definition 3.2 for the definition of $\lambda(K,J)$).

Question 1.12. Is $\lambda(K,J)$ less than or equal to the ribbon distance of K and J for all pairs of knots K, J?

The previously defined invariants mentioned above will be discussed in more detail in Section 3.4. By construction, λ is greater than or equal to all of them. (The price to pay is that λ is generally slightly harder to compute.) Let us explicitly emphasize that this observation combined with Theorem 1.1 implies that all of those previously defined invariants are also lower bounds for the proper rational unknotting number.

The construction principle that underlies λ and the other above-mentioned invariants from Khovanov homology goes back to the work by Alishahi and Eftekhary, who applied it to knot Floer homology [3, 4]. They obtained a lower bound for the unknotting number, as well as lower bounds for other quantities, such as the minimal number of negative-to-positive crossing changes in any unknotting sequence of a knot. Further, knot Floer torsion order invariants were defined by Juhász, Miller, and Zemke [17], who find lower bounds for even more topological quantities, such as the bridge index, the band-unlinking number, etc., Still, the following question remains open.

Question 1.13. Is one of the knot Floer torsion order invariants a lower bound for the proper rational unknotting number? [2]

[2] After the article at hand first had appeared as a preprint, Eftekhary answered this question in the positive [10].

1.7. Computations

Computations of $\mathbb{Z}[G]$-homology are theoretically possible by hand using Bar-Natan's divide-and-conquer approach [6]. Nevertheless, to proceed efficiently, we use the program `khoca` [28] (originally written for [27]) to compute $\mathbb{Z}[G]$-complexes of knots.[3] As input, `khoca` accepts diagrams of a knot K, e.g., in PD notation. From `khoca`'s output, one may read off a chain complex of $\mathbb{Z}[G]$-modules in the homotopy class of $[\![K]\!]$. For further simplification, `khoca`'s output may be fed into the new program `homca` [14], which attempts to decompose $[\![K]\!]$ as a direct sum of simpler chain complexes. From these simpler pieces, one may typically calculate λ by hand. See Example 3.18 for an application of this strategy to the $T_{5,6}$ torus knot. For small knots, we find the following.

Proposition 1.14. *For all knots up to* 10 *crossings, we have* $\lambda = 1$*, except for the knots* 8_{19}, 10_{124}, 10_{128}, 10_{139}, 10_{152}, 10_{154}, 10_{161}*, where* $\lambda = 2$.

This proposition and further calculations for small knots are discussed in Section 4.3.

1.8. Structure of the paper

In Section 2, we introduce $\mathbb{Z}[G]$-homology and the other variations of Khovanov homology needed in this article and prove Theorem 1.5. (Parts of that proof have been moved to Appendix A.) Section 3 contains properties and generalizations of the λ-invariant and other closely related invariants (in particular, the proofs of the propositions mentioned in Section 1.5 above). In Section 4, we calculate the λ invariant for various families of knots, thereby proving Theorem 1.2. Section 5 includes a discussion of the Bar-Natan complexes of rational tangles and the proof of Theorem 1.1.

2. $\mathbb{Z}[G]$-homology and other variations of Khovanov homology

The aim of this section is to lay the foundations of $\mathbb{Z}[G]$-homology and discuss how it fits in the general picture of Khovanov homology. We will in particular show that the $\mathbb{Z}[G]$-theory is equivalent to Khovanov's universal theory described in [20], which establishes $\mathbb{Z}[G]$-homology as a simpler alternative of equal strength. Note that the

[3] Note that `javakh` [12], while being very fast, apparently only calculates Morrison's 'universal homology', which corresponds to $\mathbb{Q}[G]$-homology. Currently, the program `kht++` [53] also only simplifies complexes over fields, not over the integers.

$\mathbb{Z}[G]$-theory is not new; it was previously described by Naot [39, 40][4]. Other topics of discussion in this section include Frobenius systems and topological quantum field theories (TQFTs for short), as well as homology of reduced type. Throughout this section, we assume familiarity with Bar-Natan's theory for tangles and cobordisms [5].

2.1. Tangles and tangle diagrams

Let us start by giving precise definitions of tangles and tangle diagrams and discuss their relationship.

Definition 2.1. A *tangle* T is a proper smooth 1-submanifold of a closed oriented 3-ball B. The points in $T \cap \partial B$ are called *end points* of T. Every tangle is $2n$*-ended*; i.e., has $2n$ end points for some $n \geq 0$. Throughout this text, we will consider oriented tangles, unless explicitly mentioned otherwise. Two tangles in the same 3-ball B with the same set of $2n$ end points in ∂B are *equivalent* if there is an orientation-preserving homeomorphism of B, fixing the boundary pointwise, mapping one tangle to the other, and preserving the orientation of the tangles if they are oriented.

Note that a $2n$-ended tangle consists of n arcs and a finite number of circles.

Definition 2.2. The *connectivity* of a $2n$-ended tangle T with arcs $\alpha_1, \ldots, \alpha_n \subset T$ is the set $\{\partial\alpha_1, \ldots, \partial\alpha_n\}$.

For example, for tangles in a fixed ball B with a fixed set of $0, 2, 4, 6, \ldots$ end points on ∂B, there are $1, 1, 3, 15, \ldots$ possible connectivities[5]. Bleiler [7] called this notion 'string attachments', but for its brevity, we prefer the term connectivity, which is also used in [23, 49]. Note that proper rational replacements (see Definition 1.3) are precisely those rational replacements that preserve connectivity.

Definition 2.3. A *tangle diagram* D is an immersed proper smooth 1-submanifold of a closed 2-disk E such that all self-intersections are transverse double points, endowed with over-under information at each such double point. Similarly, as for tangles, every tangle diagram has an even number of *end points* $D \cap \partial E$, and we call two tangle diagrams in the same disk E with the same set of end points *equivalent* if there is an orientation-preserving homeomorphism of E, fixing the boundary pointwise, mapping one tangle diagram to the other, preserving over-under information, and preserving orientation if the tangle diagram is oriented.

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[4] Naot's notation is $\mathbb{Z}[H]$ instead of $\mathbb{Z}[G]$. H is for 'handle', while G is for 'genus' ….

[5] With $2n > 0$ end points, there are $(2n-1)!!$ connectivities, where !! denotes the double factorial.

Remark 2.4. All tangles in a ball that is embedded into the 3-sphere arise as intersections of that ball with a link that is transverse to the ball's boundary sphere. Similarly, all tangle diagrams in a disk embedded into the plane arise as intersections of that disk with link diagrams that are transverse to the disk's boundary circle.

How are tangle diagrams with $2n$ end points in two *different* disks E_1, E_2 related? Let us consider homeomorphisms $\varphi\colon E_1 \to E_2$ that are orientation-preserving and end point-preserving, i.e., sending end points to end points. If two such homeomorphisms φ, φ' are end point-preservingly isotopic (i.e., isotopic along end point-preserving maps), then they send a tangle diagram $D \subset E_1$ to two equivalent tangle diagrams $\varphi(D), \varphi'(D) \subset E_2$. By Alexander's trick, the isotopy class of a homeomorphism $E_1 \to E_2$ is determined by the isotopy class of its restriction to the boundary. So, there are $2n$ end point-preserving isotopy classes of homeomorphisms $E_1 \to E_2$, each giving a way to identify equivalence classes of $2n$-ended tangle diagrams in two different disks. If one considers tangle diagrams with *base points*, i.e., one distinguished end point, then requiring that φ sends base point to base point determines φ uniquely up to isotopy.

The situation is more complicated, however, for tangles in different balls B_1, B_2. As before, the end point-preserving isotopy classes of homeomorphisms $\varphi\colon B_1 \to B_2$ are determined by the end point-preserving isotopy classes of homeomorphisms $\partial B_1 \to \partial B_2$. Those are in (non-canonical) one-to-one correspondence with the elements of the mapping class group of the $2n$-punctured sphere (see, e.g., [11] for an introduction to mapping class groups). For $2n \geq 4$, there are non-trivial mapping classes fixing some boundary point; so, in contrast to the situation for tangle diagrams, base-pointed tangles with four or more end points in different balls cannot be identified in a canonical fashion.

This also has consequences for the tangle diagrams of a tangle, which one may obtain by projection.

Definition 2.5. Let $T \subset B$ be a $2n$-ended tangle. Let φ be a homeomorphism from B to the unit ball $B_0 \subset \mathbb{R}^3$, mapping the end points of T on ∂B to

$$\{(\cos(k\pi/n), \sin(k\pi/n), 0) \mid 0 \leq k < 2n\}.$$

If the projection $\mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x, y)$ sends $\varphi(T)$ to a tangle diagram D_T in the unit disk in the xy-plane, we call D_T a *tangle diagram of* T.

A fixed homeomorphism φ sends equivalent tangles T, T' to tangle diagrams $D_T, D_{T'}$ related by Reidemeister moves and tangle diagram equivalence. But if one does not specify φ, this is no longer true, and the equivalence class of T is no longer determined by D_T. We will revisit this in Section 3.2.

For now, let us focus on the special case $n = 1$. (Note that, for $n = 0$, tangles without end points are just links in a ball and of no interest beyond that.) Two-ended tangles and tangle diagrams will be the objects we use to construct $\mathbb{Z}[G]$-homology in this section. We have the following one-to-one correspondences:

isotopy classes of base-pointed oriented links $L \subset S^3$	$\overset{1:1}{\longleftrightarrow}$	equivalence classes of 2-ended tangles T in a fixed ball with fixed end points x, y, with the arc of T oriented from x to y.	(2.1)
isotopy classes of base-pointed oriented link diagrams	$\overset{1:1}{\longleftrightarrow}$	equivalence classes of 2-ended tangle diagrams D in a fixed disk with fixed end points x, y, with the arc of D oriented from x to y.	(2.2)

Let us describe how to get from L to T and vice versa in (2.1). The complement of an open ball neighborhood of the base point of L is a closed ball B containing a 2-ended tangle $B \cap L$. There are two non-isotopic homeomorphisms sending end points to end points between B and another fixed ball; these two correspond to the two elements of the mapping class group of the twice-punctured sphere. By specifying the orientation of the arc on the right-hand side of (2.1), we eliminate this ambiguity. In the other direction, a fixed ball containing a 2-ended tangle T may be embedded into S^3, and the two end points of T may be joined by an arc outside of the embedded ball, producing a link $L \subset S^3$. The correspondence (2.2) can be shown in a similar way.

So, from now on, we will work with the notions of base-pointed link (diagrams) and 2-ended tangle (diagrams) interchangeably. Moreover, we may assign tangle diagrams to given 2-ended tangles, without the ambiguities arising for tangles with more end points.

Note that this setup is also well suited to describe the connected sum operation: given two base-pointed oriented links L, L', glue their corresponding 2-ended tangles together in a way that is compatible with the orientation of the arcs. This produces another oriented 2-ended tangle, which corresponds to the *connected sum* $L\#L'$.

2.2. Categorical framework

Let us now fix some notions from category theory, following Bar-Natan [5]. Whenever we refer to "category" in this paper, we assume that the category is small; i.e., its classes of objects and morphisms are actually sets. A category $\mathcal{C}$ is called *pre-additive* if it has the following additional structure: for any two given objects $\mathcal{O}, \mathcal{O}' \in \mathrm{ob}(\mathcal{C})$, the set $\hom_{\mathcal{C}}(\mathcal{O}, \mathcal{O}')$ is an Abelian group and the composition of morphisms is bilinear. An arbitrary category $\mathcal{C}$ can be made pre-additive by allowing formal $\mathbb{Z}$-linear

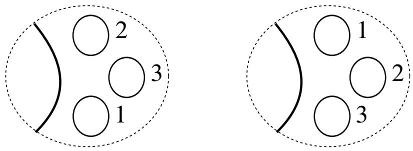


Figure 2. Two non-equal but isomorphic objects in $\mathrm{Cob}^3(2n)$.

combinations in every set of morphisms $\hom_{\mathcal{C}}(\mathcal{O}, \mathcal{O}')$ and by extending composition of morphisms in the natural bilinear way. Given a pre-additive category $\mathcal{C}$, we denote by $\mathrm{Mat}(\mathcal{C})$ the additive closure and by $\mathrm{Kom}(\mathcal{C})$ the category of complexes over $\mathcal{C}$ (in the sense of Bar-Natan [5]). We call a pre-additive category $\mathcal{C}$ *graded* if it carries the following additional structure.

(1) For any two objects $\mathcal{O}, \mathcal{O}' \in \mathrm{ob}(\mathcal{C})$, $\hom_{\mathcal{C}}(\mathcal{O}, \mathcal{O}')$ forms a graded Abelian group such that composition of morphisms respects the grading and such that all identity maps are of degree zero.

(2) There is a $\mathbb{Z}$-action $(m, \mathcal{O}) \mapsto \mathcal{O}\{m\}$ on the objects of $\mathcal{C}$, called *grading shift by* m. Note that this action changes gradings of morphisms, but not the set of morphisms itself (i.e., $\hom_{\mathcal{C}}(\mathcal{O}, \mathcal{O}') = \hom_{\mathcal{C}}(\mathcal{O}\{m\}, \mathcal{O}'\{n\})$, but if $f \in \hom_{\mathcal{C}}(\mathcal{O}, \mathcal{O}')$ has degree d, then $f \in \hom_{\mathcal{C}}(\mathcal{O}\{m\}, \mathcal{O}'\{n\})$ has degree $d - m + n$).

If a pre-additive category $\mathcal{C}$ only satisfies the first point above, we can extend the category to have the second point as well: simply add "artificial" objects $\mathcal{O}\{m\}$ for any $\mathcal{O} \in \mathrm{ob}(\mathcal{C})$ and $m \in \mathbb{Z}$ and define the $\mathbb{Z}$-action in the obvious way. Let us call this construction the *graded closure*. If $\mathcal{C}$ is a graded category, the additive closure $\mathrm{Mat}(\mathcal{C})$ and the category of complexes $\mathrm{Kom}(\mathcal{C})$ can be considered as graded categories as well.

Next, let us describe the categories we are going to work with to construct $\mathbb{Z}[G]$-homology.

- $\mathrm{Cob}^3(2n)$: the objects of $\mathrm{Cob}^3(2n)$ are crossingless unoriented $2n$-ended tangle diagrams D_T (possibly empty if $n = 0$) in some disk that is fixed throughout, together with an enumeration of every circle appearing in D_T (see Figure 2). Morphisms are 2-dimensional cobordisms (orientable, possibly disconnected surfaces) between two such diagrams D_T, $D_{T'}$, considered up to boundary-fixing isotopy. The identity is given by the product cobordism, and composition is done by concatenating cobordisms. We turn $\mathrm{Cob}^3(2n)$ into a pre-additive category as described above. For better readability, we will frequently keep the enumeration implicit and omit it in our diagrams.

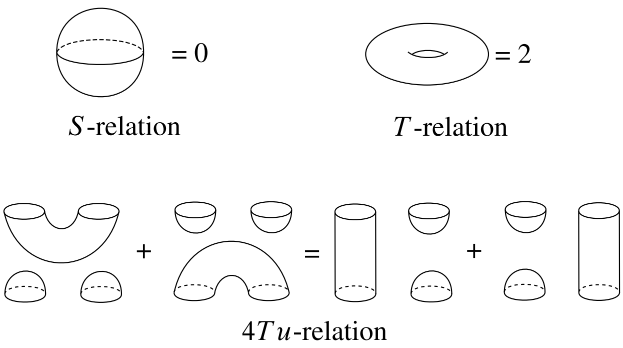


Figure 3. The defining relations for $\mathrm{Cob}^3_{/l}(2n)$.

- $\mathrm{Cob}^3_{/l}(2n)$: by modding out the *local relations* S, T, and $4Tu$ on the morphisms of $\mathrm{Cob}^3(2n)$, we obtain the category $\mathrm{Cob}^3_{/l}(2n)$ (see Figure 3).
- $\mathcal{E}$: there is only one object in $\mathcal{E}$, namely, the crossingless diagram D_{T_0} of the trivial 2-ended tangle T_0. Morphisms are connected cobordisms up to boundary-fixing isotopy. The identity is given by the product cobordism (a “curtain”, see Definition 2.7), and composition is done by concatenating cobordisms. We turn $\mathcal{E}$ into a pre-additive category as described above.
- $\mathcal{M}_R$: let R be a graded ring. We write $R\{m\}$ for R with grading shifted by $m \in \mathbb{Z}$, i.e., $R\{m\}_n = R_{n-m}$. Let $\mathcal{M}_R$ be the category whose objects are graded R-modules isomorphic to a direct sum $\bigoplus_{i=1}^n R\{m_i\}$, and whose morphisms are graded homomorphisms between R-modules. We turn $\mathcal{M}_R$ into a graded category by introducing the shift operation

$$\left(\bigoplus_{i=1}^{n} R\{m_i\}\right)\{n\} := \bigoplus_{i=1}^{n} R\{m_i + n\}, \quad n \in \mathbb{Z}.$$

Remark 2.6. Note that our definition of the category $\mathrm{Cob}^3(2n)$ (resp., $\mathrm{Cob}^3_{/l}(2n)$) differs from Bar-Natan’s definition in [5]: we require that the objects in $\mathrm{Cob}^3(2n)$, i.e., crossingless tangle diagrams, come with an enumeration of the circles in the diagram. This enumeration will be needed in subsequent sections in order to obtain well-defined TQFTs. It is worthwhile to note that while the enumeration enlarges the set of objects in $\mathrm{Cob}^3(2n)$, it does *not* introduce any new isomorphism classes of objects compared to Bar-Natan’s definition. Indeed, the morphisms of $\mathrm{Cob}^3(2n)$ are unaffected by the enumeration, so any two differently enumerated tangle diagrams with the same underlying un-enumerated tangle diagram are isomorphic via the product cobordism. In fact, the functor that forgets the enumeration of the circles is an equivalence of categories.

Connected cobordisms between the trivial 2-ended tangle diagram D_{T_0} and itself will have a special role throughout this paper, so let us give them a proper name.

Definition 2.7. A connected cobordism of genus k between the trivial 2-ended tangle diagram and itself will be called a *curtain of genus* k.

Figure 4 shows a curtain of genus one.

Let $C \in \hom_{\mathrm{Cob}^3(2n)}(D_{T_1}, D_{T_2})$ be a morphism between two tangle diagrams D_{T_1} and D_{T_2}. We can turn $\hom_{\mathrm{Cob}^3(2n)}(D_{T_1}, D_{T_2})$ into a graded Abelian group by setting

$$\deg C := \chi(C) - n,$$

where $\chi(C)$ is the Euler characteristic of C. Consequently, we can extend $\mathrm{Cob}^3(2n)$ to become a graded category (cf. [5]). Since the three local relations S, T, and $4Tu$ are degree-homogeneous, $\mathrm{Cob}^3_{/l}(2n)$ is graded too. Last but not least, we use the same notion of degree to turn $\mathcal{E}$ into a graded category as well. For the sake of simplicity, we will use the same notation for the graded versions of $\mathrm{Cob}^3(2n)$, $\mathrm{Cob}^3_{/l}(2n)$, and $\mathcal{E}$.

2.3. Definition of the $\mathbb{Z}[G]$-complex

Given a $2n$-ended tangle T with diagram D_T, Bar-Natan [5] showed how to obtain from the cube of resolutions of D_T a chain complex $[D_T]$ living in

$$\mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2n)))$$

and well defined up to isomorphism. For 2-ended tangles, this complex is an invariant up to homotopy equivalence for equivalent tangles. If T is obtained from a link L with base point by the correspondence (2.1), Bar-Natan showed that one can obtain the original Khovanov homology of L from the complex $[T]$ [5, Section 9].

There is an isomorphism in the category $\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))$ known as *delooping*[6], which can be used to reduce the complexity of the complex $[T]$. It is described in Figure 5.

It is a simple exercise to check that the morphisms depicted in Figure 5 yield two mutually inverse isomorphisms in $\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))$. In other words, we have an isomorphism of objects

 $\cong$  $\{-1\} \oplus$  $\{+1\}$.

[6]Delooping was first described by Bar-Natan [6], with a different version given later by Naot [39]. Our version is closest to Naot's, with the exception that we do not use dotted cobordisms.

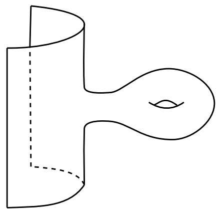


Figure 4. A curtain of genus one.

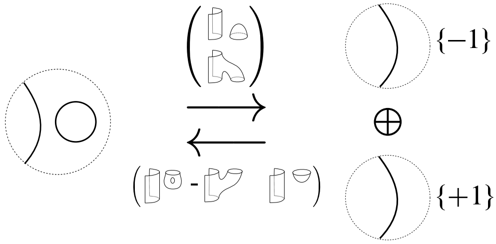


Figure 5. Delooping.

Given a 2-ended tangle T with diagram D_T, we can use delooping to successively resolve every circle appearing in the complex $[D_T]$. This yields an isomorphic complex whose chain objects consist solely of sums of grading shifted copies of D_{T_0}, giving us a connection between the categories $\mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2)))$ and $\mathrm{Kom}(\mathrm{Mat}(\mathcal{E}))$. In fact, we have the following.

Proposition 2.8. *The functor* $B\colon \mathrm{Mat}(\mathcal{E}) \to \mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))$ *given by inclusion is an equivalence of categories.*

The proof requires quite a bit of work and is postponed to Appendix A.

As B is an equivalence of categories, there is a functor $I\colon \mathrm{Mat}(\mathrm{Cob}^3_{/l}(2)) \to \mathrm{Mat}(\mathcal{E})$ such that $I \circ B$ and $B \circ I$ are naturally isomorphic to the identity functors $\mathrm{Id}_{\mathrm{Mat}(\mathcal{E})}$ and $\mathrm{Id}_{\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))}$, respectively. This functor can be constructed by using delooping, though not in a natural way: there is an ambiguity in the order of which circles get delooped.[7] Observe that if I is constructed in this way, then $I \circ B = \mathrm{Id}_{\mathrm{Mat}(\mathcal{E})}$ while $B \circ I$ is still only naturally isomorphic to $\mathrm{Id}_{\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))}$. The functor I induces an equivalence of categories

$$\hat{I}\colon \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))) \to \mathrm{Kom}(\mathrm{Mat}(\mathcal{E})).$$

[7] This problem can be resolved by introducing the convention of always delooping the *last* circle with respect to the enumeration. However, since we do not need a natural inverse, we are not going to introduce such a convention.

Now, let G be a formal variable and consider the ring $\mathbb{Z}[G]$. We equip $\mathbb{Z}[G]$ with a grading by setting $\deg 1 = 0$ and $\deg G = -2$, and we consider the category $\mathcal{M}_{\mathbb{Z}[G]}$. There is a functor $F\colon \mathcal{E} \to \mathcal{M}_{\mathbb{Z}[G]}$ sending the object $D_{T_0}\{m\} \in \mathrm{ob}(\mathcal{E})$ to the $\mathbb{Z}[G]$-module $\mathbb{Z}[G]\{m\}$ and a cobordism of genus k to the linear map given by multiplication with G^k. This functor extends by linearity to a functor $F\colon \mathrm{Mat}(\mathcal{E}) \to \mathcal{M}_{\mathbb{Z}[G]}$, which is in fact an isomorphism. Moreover, it induces yet another functor

$$\widehat{F}\colon \mathrm{Kom}(\mathrm{Mat}(\mathcal{E})) \to \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]}).$$

Let us make the following definition.

Definition 2.9. The $\mathbb{Z}[G]$-*complex* of a 2-ended tangle T, denoted by $\Omega(T)$, is defined as the chain complex

$$\Omega(T) := \widehat{F}(\widehat{I}([T])) \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]}),$$

where $[T]$ is the Bar-Natan complex of T. If L is a link with base point and T_L its corresponding 2-ended tangle, we define the $\mathbb{Z}[G]$-*complex of* L as $\Omega(L) := \widehat{F}(\widehat{I}([T_L]))$.

By construction, the homotopy class of $\Omega(T)$ is an invariant for 2-ended tangles. The construction is summarized in the following schematic below:

$$\begin{array}{c}
\text{2-ended tangle } T \\
\Big\downarrow \\
\text{cube of resolutions of } T \\
\Big\downarrow {\scriptstyle [5]} \\
[T] \in \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))) \\
\Big\downarrow {\scriptstyle \widehat{I},\ \text{Proposition 2.8}} \\
\widehat{I}([T]) \in \mathrm{Kom}(\mathrm{Mat}(\mathcal{E})) \\
\Big\downarrow {\scriptstyle \widehat{F}} \\
\Omega(T) := \widehat{F}(\widehat{I}([T])) \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]}).
\end{array}$$

2.4. Frobenius systems and TQFTs

In [20], Khovanov describes a rank 2 Frobenius system $\mathcal{F}_{\mathrm{univ}}$ (denoted by $\mathcal{F}_5$ in [20]) which is universal in the sense that any other rank two Frobenius system can be obtained from it by a base change and a twist. Consequently, the chain complex obtained by applying the $\mathcal{F}_{\mathrm{univ}}$-TQFT to the cube of resolutions is sometimes called *universal Khovanov complex*. Before we go into more details, let us first recall the definition of a Frobenius system (as in [20]).

Definition 2.10. A Frobenius system $\mathcal{F} = (R, A, \Delta, \varepsilon)$ is a 4-tuple consisting of a graded commutative unitary ring R and a graded or filtered free R-module A equipped with a commutative algebra structure (multiplication m, unit ι) and a cocommutative coalgebra structure (comultiplication Δ, counit ε) that are related by the so-called *Frobenius identity*:

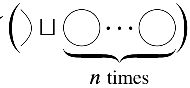
$$\Delta \circ m = (\mathrm{Id} \otimes m) \circ (\Delta \otimes \mathrm{Id}).$$

The maps defining the (co-)algebra structure are required to be homogeneous of a certain degree.

We will only be interested in rank 2 Frobenius systems, i.e., Frobenius systems where $A \cong R1 \oplus RX$ as R-modules for some $X \in A$, and for such systems, we will always use the grading $\deg 1 = 0$ and $\deg X = -2$.

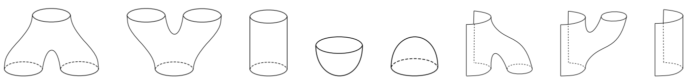
Given a rank 2 Frobenius system $\mathcal{F} = (R, A, \Delta, \varepsilon)$, we can define a functor (a TQFT) $\mathcal{F}\colon \mathrm{Cob}^3_{/l}(2) \to \mathcal{M}_A$ as follows.[8]

(1) On objects, $\mathcal{F}$ acts in the following way:

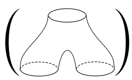
$\mathcal{F}$  $= \underline{A}\{1\} \otimes_R \underbrace{A\{1\} \otimes_R \cdots \otimes_R A\{1\}}_{n \text{ times}}.$

Here, the special strand corresponds to the first tensor factor while the other factors are ordered according to the enumeration of the circles. The underline indicates the action of A on the tensor product $\underline{A}\{1\} \otimes_R A\{1\}^{\otimes n}$, turning it into an A-module.

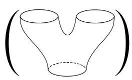
(2) The morphisms of $\mathrm{Cob}^3_{/l}(2)$ can be expressed as sums of compositions of disjoint unions of the following elementary cobordisms (details can be found in [18]):

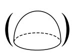


Hence, it is enough to define $\mathcal{F}$ on these elementary cobordisms:

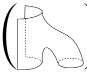
$\mathcal{F}$  $= m\colon A\{1\} \otimes A\{1\} \to A\{1\},$

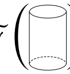
$\mathcal{F}$  $= \iota\colon R\{1\} \to A\{1\},$

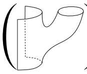
$\mathcal{F}$  $= \Delta\colon A\{1\} \to A\{1\} \otimes A\{1\},$

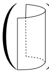
$\mathcal{F}$  $= \varepsilon\colon A\{1\} \to R\{1\},$

[8] We abuse notation and use $\mathcal{F}$ to denote the Frobenius system, the corresponding TQFT, and sometimes even the Frobenius algebra of the system.

$\mathcal{F}\Big($  $\Big) = m\colon \underline{A}\{1\} \otimes A\{1\} \to \underline{A}\{1\},$

$\mathcal{F}\Big($  $\Big) = \mathrm{Id}\colon A\{1\} \to A\{1\},$

$\mathcal{F}\Big($  $\Big) = \Delta\colon \underline{A}\{1\} \to \underline{A}\{1\} \otimes A\{1\},$

$\mathcal{F}\Big($  $\Big) = \mathrm{Id}\colon \underline{A}\{1\} \to \underline{A}\{1\}.$

Given a 2-ended tangle T with diagram D_T, we can apply $\mathcal{F}$ to its cube of resolutions and obtain a Khovanov-type chain complex in the usual way. The resulting complex is sometimes called *unreduced*. We will denote it by either $C_{\mathcal{F}}(T)$ or $C_{\mathcal{F}}(D_T)$, which is justified since different choices of the diagram D_T yield homotopy equivalent complexes.

Remark 2.11. (1) By our definition, a TQFT yields a complex over the category $\mathcal{M}_A$. However, this category is not Abelian, so in order to take homology, one has to move from $\mathrm{Kom}(\mathcal{M}_A)$ to $\mathrm{Kom}(A\text{-Mod})$ using inclusion, where A-Mod denotes the category of graded A-modules. We accept this inconvenience because $\mathcal{M}_A$ is a simpler category and complexes are our main working tool (in contrast to homology).

(2) If we forget about the base point, then the TQFT above can be viewed as $\mathcal{F}\colon \mathrm{Cob}^3_{/l}(0) \to \mathcal{M}_R$ which yields the usual Khovanov complex corresponding to the Frobenius system $\mathcal{F}$.

We will mainly be interested in the following two Frobenius systems.

Definition 2.12. The Frobenius system $\mathcal{F}_{\mathrm{univ}} = (R_{\mathrm{univ}}, A_{\mathrm{univ}}, \Delta, \varepsilon)$ is defined as follows:

$$\begin{aligned} R_{\mathrm{univ}} &= \mathbb{Z}[h,t], \quad A_{\mathrm{univ}} = R_{\mathrm{univ}}[X]/(X^2 - hX - t),\\ \varepsilon(1) &= 0, \quad \Delta(1) = 1 \otimes X + X \otimes 1 - h1 \otimes 1,\\ \varepsilon(X) &= 1, \quad \Delta(X) = X \otimes X + t1 \otimes 1. \end{aligned}$$

We equip $\mathcal{F}_{\mathrm{univ}}$ with a grading by setting $\deg X = \deg h = -2$ and $\deg t = -4$.

Definition 2.13. The Frobenius system $\mathcal{F}_{\mathbb{Z}[G]} = (R_{\mathbb{Z}[G]}, A_{\mathbb{Z}[G]}, \Delta, \varepsilon)$ is defined as follows:

$$\begin{aligned} R_{\mathbb{Z}[G]} &= \mathbb{Z}[G], \quad A_{\mathbb{Z}[G]} = R_{\mathbb{Z}[G]}[X]/(X^2 + GX),\\ \varepsilon(1) &= 0, \quad \Delta(1) = 1 \otimes X + X \otimes 1 + G1 \otimes 1,\\ \varepsilon(X) &= 1, \quad \Delta(X) = X \otimes X. \end{aligned}$$

We equip $\mathcal{F}_{\mathbb{Z}[G]}$ with a grading by setting $\deg X = \deg G = -2$.

It is easy to see that $\mathcal{F}_{\mathbb{Z}[G]}$ can be obtained from $\mathcal{F}_{\text{univ}}$ by a base change that sends $t \mapsto 0$ and $h \mapsto -G$. To simplify notation, we will write C_{univ} for the unreduced complex $C_{\mathcal{F}_{\text{univ}}}$ and $C_{\mathbb{Z}[G]}$ for $C_{\mathcal{F}_{\mathbb{Z}[G]}}$.

Using the Frobenius system $\mathcal{F}_{\mathbb{Z}[G]}$, we can define a second type of chain complex as follows.

Definition 2.14. Let T be a 2-ended tangle with diagram D_T and $C_{\mathbb{Z}[G]}(D_T)$ the corresponding $\mathcal{F}_{\mathbb{Z}[G]}$-complex. The *reduced* $\mathcal{F}_{\mathbb{Z}[G]}$-complex of T is defined as follows:

$$[\![T]\!] := C_{\mathbb{Z}[G]}(D_T) \otimes_{A_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}/(X)\{-1\} \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]}).$$

The notation $[\![T]\!]$ is justified since different choices of the diagram D_T yield homotopy equivalent complexes. (We will sometimes use $[\![D_T]\!]$ nevertheless.) Observe that reducing has the following effect on summands in $C_{\mathbb{Z}[G]}$:

$$\underline{A_{\mathbb{Z}[G]}}\{1\} \otimes_{R_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}\{1\}^{\otimes n} \xrightarrow{\text{reduce}} R_{\mathbb{Z}[G]} \otimes_{R_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}\{1\}^{\otimes n}.$$

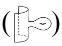
In particular, the reduced complex has no longer an $A_{\mathbb{Z}[G]}$-module structure. Also, note that the first factor is no longer affected by a shift in grading.[9] We will see in the next section that the reduced $\mathcal{F}_{\mathbb{Z}[G]}$-complex $[\![T]\!]$ is in fact isomorphic to the $\mathbb{Z}[G]$-complex $\Omega(T)$ defined in Section 2.3.

2.5. Equivalence of the $\mathcal{F}_{\text{univ}}$- and $\mathbb{Z}[G]$-theory

Let $\mathcal{F} = (R, A, \Delta, \varepsilon)$ be a rank 2 Frobenius system. The corresponding TQFT gives us a functor

$$\alpha = \alpha_{\mathcal{F}} \colon \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))) \to \mathrm{Kom}(\mathcal{M}_A).$$

Let us now show that the $\mathcal{F}_{\text{univ}}$- and the $\mathbb{Z}[G]$-theory are equivalent.

Lemma 2.15. *Let D_T be a 2-ended tangle diagram, and let $\mathcal{F} = (R, A, \Delta, \varepsilon)$ be a rank 2 Frobenius system. We can see A as a $\mathbb{Z}[G]$-module by letting G act as $\mathcal{F}($*  *$)$. Then, the functor $\gamma \colon \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]}) \to \mathrm{Kom}(\mathcal{M}_A)$ defined as*

$$\gamma(Y) := Y \otimes_{\mathbb{Z}[G]} A\{1\}, \quad Y \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$$

satisfies

$$\alpha([D_T]) \cong \gamma(\Omega(D_T))$$

(cf. Figure 6).

[9] This is needed in order for the dual of the reduced complex to correspond to the reduced complex of the mirror image. Here, we use the usual convention that the signs of the homological and quantum grading in the dual are switched.

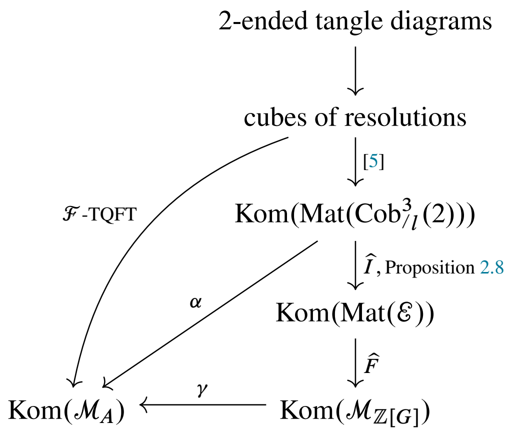


Figure 6. The functors and constructions figuring in the statement of Lemma 2.15.

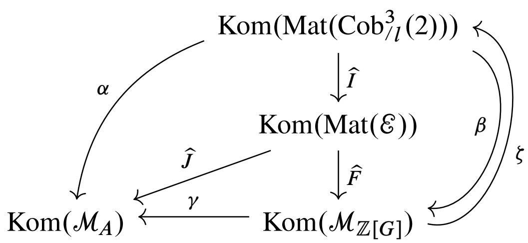
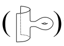


Figure 7. Functors used in the proof of Lemma 2.15.

Proof. We know that the functor $\widehat{I}$ is an equivalence of categories (with "inverse" $\widehat{B}$, cf. Proposition 2.8) and that $\widehat{F}$ is an isomorphism of categories; thus, if $\beta = \widehat{F} \circ \widehat{I}$ and $\zeta = \widehat{B} \circ \widehat{F}^{-1}$, we have that $\zeta(\beta(C)) \cong C$ for all $C \in \text{Kom}(\text{Mat}(\text{Cob}^3_{/l}(2)))$ (see Figure 7). In order to show that $\alpha([D_T]) \cong \gamma(\widehat{F}(\widehat{I}([D_T])))$, it is enough to prove that $\alpha(\zeta(Y)) \cong \gamma(Y)$ for all $Y \in \text{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$. This is done by introducing a new functor $J\colon \mathcal{E} \to \mathcal{M}_A$ sending the trivial 2-ended tangle diagram D_{T_0} to the A-module A and the curtain with genus $k \geq 0$ to the linear map given by $\mathcal{F}($  $)^k$. The functor J induces a functor

$$\widehat{J}\colon \text{Kom}(\text{Mat}(\mathcal{E})) \to \text{Kom}(\mathcal{M}_A).$$

It is easy to see that $\widehat{J} = \alpha \circ \widehat{B}$, so α is naturally isomorphic to $\widehat{J} \circ \widehat{I}$. Thus, it only remains to check that $\widehat{J} \cong \gamma \circ \widehat{F}$. This follows immediately from the definitions. ■

Before we move on, let us first show that the reduced $\mathcal{F}_{\mathbb{Z}[G]}$-complex $[\![\cdot]\!]$ is isomorphic to Ω.

Proposition 2.16. *The reduced $\mathcal{F}_{\mathbb{Z}[G]}$-complex $[\![D_T]\!]$ is isomorphic to the $\mathbb{Z}[G]$-complex $\Omega(D_T)$.*

Proof. Using Lemma 2.15 with $\mathcal{F} = \mathcal{F}_{\mathbb{Z}[G]}$ and that

$$A_{\mathbb{Z}[G]}/(X) \cong \mathbb{Z}[G],$$

we obtain

$$\begin{aligned} [\![D_T]\!] &= C_{\mathbb{Z}[G]}(D_T) \otimes_{A_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}/(X)\{-1\} \\ &= \alpha([D_T]) \otimes_{A_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}/(X)\{-1\} \\ &\cong \gamma(\Omega(D_T)) \otimes_{A_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}/(X)\{-1\} \\ &\cong \Omega(D_T) \otimes_{\mathbb{Z}[G]} A_{\mathbb{Z}[G]}/(X)\{-1\} \\ &\cong \Omega(D_T). \end{aligned}$$ ■

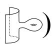
Proposition 2.16 tells us that the reduced $\mathcal{F}_{\mathbb{Z}[G]}$-complex and the $\mathbb{Z}[G]$-complex are isomorphic. We will from now on denote both complexes by $[\![\cdot]\!]$ and no longer distinguish between them.

Theorem 2.17. *Let D_T be a 2-ended tangle diagram, and let*

$$\mathcal{F} = (R, A, \Delta, \varepsilon)$$

be a rank 2 Frobenius system. The $\mathcal{F}$-complex $C_{\mathcal{F}}(D_T)$ is determined by the $\mathbb{Z}[G]$-complex $[\![D_T]\!]$ in the following way:

$$C_{\mathcal{F}}(D_T) \cong [\![D_T]\!] \otimes_{\mathbb{Z}[G]} A\{1\} \in \mathrm{Kom}(\mathcal{M}_A),$$

where A is a $\mathbb{Z}[G]$-module via G acting as $\mathcal{F}($$)$.

Proof. The statement of the theorem follows immediately from Lemma 2.15 and Proposition 2.16:

$$C_{\mathcal{F}}(D_T) = \alpha([D_T]) \cong \gamma(\Omega(D_T)) \cong [\![D_T]\!] \otimes_{\mathbb{Z}[G]} A\{1\}.$$ ■

Observe that Theorem 2.17 specializes to Theorem 1.5 from the introduction.

Theorem 1.5. *Endow $A_{\mathrm{univ}} = \mathbb{Z}[h,t][X]/(X^2 - hX - t)$ with the structure of a $\mathbb{Z}[G]$-module by letting G act as $2X - h$. Then, for every knot K,*

$$C_{\mathrm{univ}}(K) \simeq [\![K]\!] \otimes_{\mathbb{Z}[G]} A_{\mathrm{univ}}\{1\}.$$

Proof. Apply Theorem 2.17 with $\mathcal{F} = \mathcal{F}_{\mathrm{univ}}$. ■

Let us make explicit how $\mathbb{Z}[G]$-homology determines the original Khovanov homology. (Naot mentions this statement in [39, Section 6.6].)

Corollary 2.18. *For all knots K, the unreduced integral Khovanov chain complex may be obtained from $[\![K]\!]$ by tensoring with the $\mathbb{Z}[G]$-module $\mathbb{Z}\{-1\} \oplus \mathbb{Z}\{1\}$, where G acts as $\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right)$. More sloppily said, replace every copy of $\mathbb{Z}[G]\{m\}$ by $\mathbb{Z}\{m-1\} \oplus \mathbb{Z}\{m+1\}$, and every differential nG^k with $n, k \in \mathbb{Z}, k \geq 0$ by $\left(\begin{smallmatrix} n & 0 \\ 0 & n \end{smallmatrix}\right)$ for $k = 0$, by $\left(\begin{smallmatrix} 0 & 2n \\ 0 & 0 \end{smallmatrix}\right)$ for $k = 1$, and by the zero matrix for $k \geq 2$.*

Proof. Apply Theorem 2.17 to the Frobenius system $\mathbb{Z}[X]/(X^2)$ over $\mathbb{Z}$, and forget the action of the algebra. ∎

Theorem 2.17 shows us how to obtain the $\mathcal{F}$-complex $C_{\mathcal{F}}(D_T)$ from the $\mathbb{Z}[G]$-complex $[\![D_T]\!]$ for any rank 2 Frobenius system $\mathcal{F}$, which is in particular true for the universal system $\mathcal{F}_{\text{univ}}$. In order to show that the $\mathcal{F}_{\text{univ}}$- and the $\mathbb{Z}[G]$-theory are in fact equivalent, it remains to prove that $[\![D_T]\!]$ is also determined by $C_{\text{univ}}(D_T)$.

Theorem 2.19. *Let D_T be a 2-ended tangle diagram. The $\mathbb{Z}[G]$-complex $[\![D_T]\!]$ is determined by the $\mathcal{F}_{\text{univ}}$-complex $C_{\text{univ}}(D_T)$ in the following way:*

$$[\![D_T]\!] \cong C_{\text{univ}}(D_T) \otimes_{A_{\text{univ}}} \mathbb{Z}[G]\{-1\} \in \text{Kom}(\mathcal{M}_{\mathbb{Z}[G]}),$$

where $\mathbb{Z}[G]$ is an A_{univ}-module by X and t acting as 0 and h as $-G$.

Proof. By Theorem 2.17,

$$C_{\mathcal{F}_{\text{univ}}}(D_T) \cong [\![D_T]\!] \otimes_{\mathbb{Z}[G]} A_{\text{univ}}\{1\}. \tag{*}$$

Consider $A_{\mathbb{Z}[G]}$ as an A_{univ}-module by letting t act as 0 and h as $-G$. Tensoring (*) with $A_{\mathbb{Z}[G]}$ over A_{univ} yields

$$\begin{aligned} C_{\text{univ}}(D_T) \otimes_{A_{\text{univ}}} A_{\mathbb{Z}[G]} &\cong \big([\![D_T]\!] \otimes_{\mathbb{Z}[G]} A_{\text{univ}}\{1\}\big) \otimes_{A_{\text{univ}}} A_{\mathbb{Z}[G]} \\ &\cong [\![D_T]\!] \otimes_{\mathbb{Z}[G]} A_{\mathbb{Z}[G]}\{1\} \\ &\cong C_{\mathbb{Z}[G]}(D_T). \end{aligned}$$

Therefore,

$$\begin{aligned} [\![D_T]\!] &= C_{\mathbb{Z}[G]}(D_T) \otimes_{A_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}/(X)\{-1\} \\ &\cong (C_{\text{univ}}(D_T) \otimes_{A_{\text{univ}}} A_{\mathbb{Z}[G]}) \otimes_{A_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}/(X)\{-1\} \\ &\cong C_{\text{univ}}(D_T) \otimes_{A_{\text{univ}}} A_{\mathbb{Z}[G]}/(X)\{-1\} \\ &\cong C_{\text{univ}}(D_T) \otimes_{A_{\text{univ}}} \mathbb{Z}[G]\{-1\}. \end{aligned}$$ ∎

The discussion in this section can be summarized by the commutative diagram in Figure 8, where $\xi\colon \text{Kom}(\mathcal{M}_{A_{\text{univ}}}) \to \text{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$ is the functor given by

$$\xi(C) := C \otimes_{A_{\text{univ}}} \mathbb{Z}[G]\{-1\}$$

for $C \in \text{Kom}(\mathcal{M}_{A_{\text{univ}}})$.

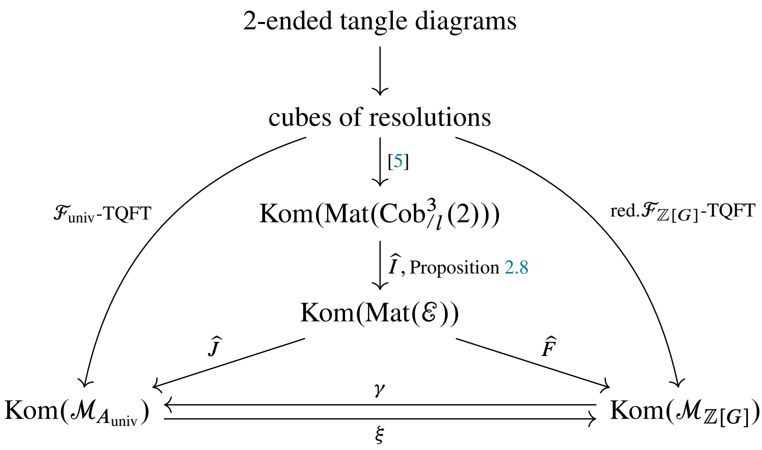


Figure 8. A summary of the relationships discussed in Section 2.5.

2.6. Reduced $\mathbb{Z}[G]$-homology

Let T be a 2-ended tangle. We have seen in the previous subsection that the reduced $\mathbb{Z}[G]$-complex $[\![T]\!]$ is determined by the $\mathcal{F}_{\mathrm{univ}}$-complex $C_{\mathrm{univ}}(T)$ and vice versa. One advantage of the $\mathbb{Z}[G]$-complex is that setting $G = 1$ yields a particularly simple homology theory.

Proposition 2.20. *Let T be a 2-ended tangle with a single component. Then,*

$$H([\![T]\!]_{G=1}) \cong \mathbb{Z},$$

where $[\![T]\!]_{G=1}$ is the complex $[\![T]\!]_{G=1} := [\![T]\!] \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/(G-1)$.

Proof. Let us first look at the unreduced situation over the rationals, i.e., the complex

$$C(T) := (C_{\mathbb{Z}[G]}(T) \otimes_{A_{\mathbb{Z}[G]}} A_{\mathbb{Q}[G]}) \otimes_{A_{\mathbb{Q}[G]}} \mathbb{Q}[G]/(G-1),$$

where $A_{\mathbb{Z}[G]} = \mathbb{Z}[X, G]/(X^2 + GX)$ and $A_{\mathbb{Q}[G]} = \mathbb{Q}[X, G]/(X^2 + GX)$. Note that $C(T)$ can be equivalently obtained from the Frobenius algebra $\mathbb{Q}[X]/(X^2 + X)$ in the usual way. By [33, Proposition 2.3], we know that

$$H(C(T)) \cong \mathbb{Q} \oplus \mathbb{Q}.$$

In fact, using Wehrli's edge-coloring technique [52, Section 2.1], one obtains a decomposition

$$C(T) = XC(T) \oplus (X+1)C(T),$$

where $XC(T)$ and $(X+1)C(T)$ are the subcomplexes generated by all elements having X and $X+1$ as the first tensor factor (i.e., at the base point), respectively. Similar to [52, Theorem 5], one can show that both $XC(T)$ and $(X+1)C(T)$ have homology of dimension one. Now, let us look at the reduced situation over the rationals, i.e., the complex

$$[\![T]\!]_{\mathbb{Q},G=1} := [\![T]\!]_{G=1} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

By construction, $[\![T]\!]_{\mathbb{Q},G=1}$ is equivalent to the complex $C(T)$ with X set to 0 in the first tensor factor of every summand in $C(T)$, which means that the summand $XC(T)$ becomes trivial after reducing. Hence,

$$H([\![T]\!]_{\mathbb{Q},G=1}) \cong \mathbb{Q}.$$

Switching back to the integers, the above tells us that $\dim_{\mathbb{Q}}([\![T]\!]_{G=1}) = 1$. Hence, it remains to show that $[\![T]\!]_{G=1}$ has no torsion. This is done in the same way as in the proof of [33, Proposition 2.4 (ii)]. ∎

Remark 2.21. Let T be a 2-ended tangle corresponding to a knot K. It is interesting to note that one can extract the Rasmussen $s_{\mathbb{F}}$-invariant of K over any field $\mathbb{F}$ from the $\mathbb{Z}[G]$-homology of K. Indeed, consider the $\mathbb{Z}[G]$-complex with coefficients switched to some field $\mathbb{F}$, i.e.,

$$[\![K]\!]_{\mathbb{F}[G]} = [\![K]\!] \otimes_{\mathbb{Z}[G]} \mathbb{F}[G].$$

This is a Khovanov-type complex over the PID $\mathbb{F}[G]$; hence, it decomposes into a single grading-shifted copy of the base ring $\mathbb{F}[G]\{n\}$ and some summands of the form $\mathbb{F}[G]\{m\} \xrightarrow{G^k} \mathbb{F}[G]\{2k+m\}$ for $k,m,n \in \mathbb{Z}$ and $k \geq 0$ (a so-called *pawn* and several G^k*-knights*, cf. Definition 3.16). Therefore, setting $G=1$ in $[\![K]\!]_{\mathbb{F}[G]}$ yields

$$[\![K]\!]_{\mathbb{F}[G]} \otimes_{\mathbb{F}[G]} \mathbb{F}[G]/(G-1) \simeq \mathbb{F}[G]\{n\} \otimes_{\mathbb{F}[G]} \mathbb{F}[G]/(G-1).$$

We claim that n, i.e., the filtered degree of the generator of $\mathbb{F}[G]\{n\}$ in homology, is equal to $s_{\mathbb{F}}(K)$. By [33], $s_{\mathbb{F}}(K)$ can be obtained from the homology of the unreduced complex $C_{\mathcal{F}_{\mathbb{F}[G]}}(K)$ corresponding to the Frobenius algebra

$$A_{\mathbb{F}[G]} = \mathbb{F}[G,X]/(X^2+GX)$$

after setting $G=1$. On the other hand, using Theorem 2.17 and the decomposition of $[\![K]\!]_{\mathbb{F}[G]}$ described above, we can write $C_{\mathcal{F}_{\mathbb{F}[G]}}(K)$ as

$$\begin{aligned} C_{\mathcal{F}_{\mathbb{F}[G]}}(K) &\cong [\![K]\!]_{\mathbb{F}[G]} \otimes_{\mathbb{F}[G]} A_{\mathbb{F}[G]}\{1\} \\ &\cong (\mathbb{F}[G]\{n\} \oplus \mathcal{R}) \otimes_{\mathbb{F}[G]} A_{\mathbb{F}[G]}\{1\} \\ &\cong A_{\mathbb{F}[G]}\{n+1\} \oplus (\mathcal{R} \otimes_{\mathbb{F}[G]} A_{\mathbb{F}[G]}\{1\}), \end{aligned}$$

where $\mathcal{R}$ consists solely of summands $\mathbb{F}[G]\{m\} \xrightarrow{G^k} \mathbb{F}[G]\{2k+m\}$. If we now set $G = 1$ and take homology, we obtain

$$H(C_{\mathcal{F}_{\mathbb{F}[G]}}(K) \otimes_{\mathbb{F}[G]} \mathbb{F}[G]/(G-1)) \cong \mathbb{F}[X]/(X^2+X)\{n+1\}.$$

Now, $\mathbb{F}[X]/(X^2+X)\{n+1\}$ is generated by 1 and X in filtered degrees $n+1$ and $n-1$, respectively; hence, $s_{\mathbb{F}}(K) = n$ by [41], as claimed.

2.7. The Bar-Natan complex of tangles with base point

Recall that the crossingless unoriented $2n$-ended tangle diagrams in $\mathrm{Cob}^3(2n)$ lie inside a fixed disk with fixed end points. Let us fix one of those end points as base point. Given a cobordism from the trivial 2-ended tangle diagram D_{T_0} to itself, and a cobordism in $\mathrm{Cob}^3(2n)$ between diagrams D and D', one may glue these two cobordisms together such that one of the end points of D_{T_0} gets attached to the base points of D and D'. This gives a bilinear map

$$\hom_{\mathrm{Cob}^3(2)}(D_{T_0}, D_{T_0}) \times \hom_{\mathrm{Cob}^3(2n)}(D, D') \to \hom_{\mathrm{Cob}^3(2n)}(D, D').$$

Quotienting by the relations l, and using that $\hom_{\mathrm{Cob}^3(2)}(D_{T_0}, D_{T_0})$ is isomorphic to $\mathbb{Z}[G]$, we obtain a $\mathbb{Z}[G]$-action on each of the morphism $\mathbb{Z}$-modules of $\mathrm{Cob}^3_{/l}(2n)$.

Definition 2.22. Denote by $\mathrm{Cob}^{3\bullet}_{/l}(2n)$ the $\mathbb{Z}[G]$-enriched category obtained from $\mathrm{Cob}^3_{/l}(2n)$ by fixing one of the tangle end points as base point and letting $\mathbb{Z}[G]$ act on the morphism groups as described above. For a $2n$-ended tangle diagram D with base point, denote by $[D]^\bullet$ the Bar-Natan chain complex of D over $\mathrm{Cob}^{3\bullet}_{/l}(2n)$. Here, we identify equivalence classes of tangle diagrams in the disk in which D lives with equivalence classes of tangle diagrams in the disk fixed for $\mathrm{Cob}^{3\bullet}_{/l}(2n)$, using a homeomorphism (which is unique up to isotopy) between these disks that sends end points to end points and base point to base point.

Note that one may recover $\mathrm{Cob}^3_{/l}(2n)$ from $\mathrm{Cob}^{3\bullet}_{/l}(2n)$ and $[D]$ from $[D]^\bullet$ by simply forgetting the $\mathbb{Z}[G]$-action and the base point. In other words, Definition 2.22 only introduces the action of G but does not introduce any new objects and morphisms.

Remark 2.23. For $n = 1$, the $\mathbb{Z}[G]$-action on $\mathrm{Cob}^3_{/l}(2)$ is by construction the same as the one obtained via the equivalence of $\mathrm{Cob}^3_{/l}(2)$ and $\mathcal{M}_{\mathbb{Z}[G]}$. In particular, for 2-ended tangle diagrams D, both choices of base point result in the same $\mathbb{Z}[G]$-action on $[D]^\bullet$.

Gluing constructions as the one above have been formalized by Bar-Natan [5] using the following tool.

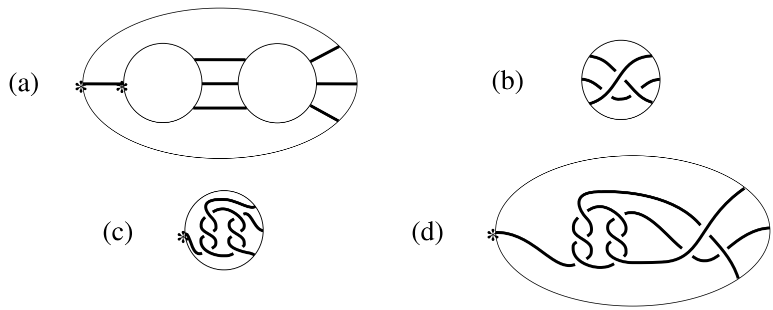


Figure 9. Examples of the concepts introduced in Definition 3.8. An asterisk marks the base point. (a) $\mathcal{D}_4$, (b) a braid-like 6-ended tangle diagram Q, (c) a 4-ended tangle diagram T, (d) $\mathcal{D}_4(T, Q)$, called a braiding of T using Q.

Definition 2.24. A d*-input planar arc diagram* $\mathcal{D}$ is a disk (called *output disk*), with d enumerated open so-called *input disks* removed from its interior, together with a proper smooth oriented 1-submanifold of $\mathcal{D}$, with *end points* on $\partial\mathcal{D}$. Here, $\partial\mathcal{D}$ consists of the union of the ∂E, with E ranging over the input disks and the output disk. The number of end points on each such ∂E is required to be even; if it is non-zero, then one of the end points is distinguished as base point of E. An example can be seen below and in Figure 9.

Let $\mathcal{D}$ be a d-input planar arc diagram with $2n_0$ end points on the output disk and $2n_i$ end points on the i-th input disk. By gluing tangle diagrams, $\mathcal{D}$ yields an operator that takes as input d base-pointed tangle diagrams $D_1, \dots, D_d$ that fit into the input disks, and that gives as output a base-pointed tangle diagram $\mathcal{D}(D_1, \dots, D_d)$. For what follows, recall that, for each $n \geq 1$, we fixed a base point on the boundary of the disk containing the crossingless tangles of $\mathrm{Cob}^3(2n)$. By gluing crossingless tangle diagrams and cobordisms, $\mathcal{D}$ then gives a functor

$$\prod_{i=1}^{d} \mathrm{Cob}^3(2n_i) \to \mathrm{Cob}^3(2n_0),$$

which is compatible with modding out the relations l. By taking tensor products, this functor extends to a functor

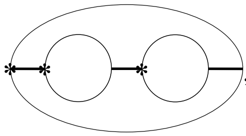
$$\prod_{i=1}^{d} \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2n_i))) \to \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2n_0))),$$

which is compatible with homotopy equivalence. Note that the orientations of the arcs and circles in $\mathcal{D}$ matter for the operator, but not for the functors. We have the following compatibility result:

$$\mathcal{D}([D_1],\dots,[D_d]) \cong [\mathcal{D}(D_1,\dots,D_d)]. \tag{2.3}$$

Equipped with this tool set, one could give a more formal definition of the $\mathbb{Z}[G]$ action on $[D]^\bullet$ given in Definition 2.22, using a 2-input planar arc diagram whose two input disks have 2 and $2n$ end points, respectively.

Moreover, for the following input diagram

$\mathcal{D} =$  ,

clearly, $\mathcal{D}(D_1, D_2)$ is a diagram of the connected sum $L_1\#L_2$, if D_i is a 2-ended tangle diagram corresponding do the base-pointed link L_i, in the sense of (2.1). From $[\mathcal{D}(D_1, D_2)] \cong \mathcal{D}([D_1],[D_2])$, it now follows that

$$[\![L_1\#L_2]\!] \cong [\![L_1]\!] \otimes [\![L_2]\!]. \tag{2.4}$$

To adapt to $\mathbb{Z}[G]$-complexes, we consider planar arc diagrams $\mathcal{D}$ satisfying the following condition: $\mathcal{D}$ contains an arc connecting the base point of the output disk to the base point of the first input disk. Then, $\mathcal{D}$ induces a functor

$$\mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^{3;\bullet}_{/l}(2n_1))) \times \prod_{i=2}^{d} \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^{3}_{/l}(2n_i))) \to \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^{3;\bullet}_{/l}(2n_0))).$$

For this functor, we have the following analog of (2.3):

$$\mathcal{D}([D_1]^\bullet,[D_2],\dots,[D_d]) \simeq [\mathcal{D}(D_1,\dots D_d)]^\bullet.$$

3. Properties of the λ-invariant

For the convenience of the reader, let us restate the definition of λ from the introduction.

Definition 1.7. For a knot K, let $\lambda(K)$ be the minimal integer $k \geq 0$ such that there exist ungraded chain maps (i.e., chain maps that do not need to respect the homological or the quantum degree, cf. Definition 3.1)

$$[\![K]\!] \underset{g}{\overset{f}{\rightleftarrows}} [\![U]\!]$$

and homotopies $g \circ f \simeq G^k \cdot \mathrm{id}_{[\![K]\!]}$, $f \circ g \simeq G^k \cdot \mathrm{id}_{[\![U]\!]}$.

Let us take this opportunity to clarify what we mean by ‘ungraded’.

Definition 3.1. For chain complexes (C, d), (C', d') in some additive category, an *ungraded* chain map $f: C \to C'$ is a morphism

$$f\colon \bigoplus_{i=-\infty}^{\infty} C_i \to \bigoplus_{i=-\infty}^{\infty} C'_i$$

that need *not* respect homological degree such that $d' \circ f = f \circ d$. Whenever we want to highlight the difference, we call a chain map in the usual sense *graded*. If the underlying category is Abelian (so that one may take homology), then the ungraded chain map f induces a morphism

$$f_*\colon H(C) = \bigoplus_{i=-\infty}^{\infty} H_i(C) \to \bigoplus_{i=-\infty}^{\infty} H_i(C') = H(C').$$

Some authors also call a chain complex without homological grading a *differential module*.

3.1. Basic properties and generalizations of λ

First, let us extend the above definition of λ.

Definition 3.2. Abusing notation, we denote by λ all of the following functions.

- $\lambda(A, B)$ for two chain complexes A, B over $\mathbb{Z}[G]$, or over $\mathrm{Cob}^{3;\bullet}_{/l}(2n)$, is defined as the minimal integer $k \geq 0$ such that there exist ungraded chain maps $f: A \to B$ and $g: B \to A$ and homotopies $g \circ f \simeq G^k \cdot \mathrm{id}_A$, $f \circ g \simeq G^k \cdot \mathrm{id}_B$, if such a k exists, and ∞ otherwise.
- $\lambda(A)$ for A a chain complex over $\mathbb{Z}[G]$ is an abbreviation for $\lambda(A, [\![U]\!])$.
- $\lambda(D, D')$ for D and D' two tangle diagrams in a fixed disk with the same end points and the same base point is defined as $\lambda([D]^\bullet, [D']^\bullet)$.
- $\lambda(K, J)$ for K and J two knots is defined as $\lambda([\![K]\!], [\![J]\!])$.

Note that, for all knots K, $\lambda(K)$ as in Definition 1.7 equals

$$\lambda(K, U) = \lambda([\![K]\!], [\![U]\!]) = \lambda([\![K]\!]),$$

as in Definition 3.2.

Remark 3.3. One can naturally extend the definition of λ from knots to links with base point by setting $\lambda(L, L')$ to be $\lambda(T, T')$, where T, T' are the 2-ended tangles corresponding to the links L, L' via (2.1). In this sense, most of this paper’s results will generalize from knots to links. For simplicity's sake, however, we are sticking with knots.

Next, let us prove some useful basic properties of λ.

Lemma 3.4. *For some $k \geq 1$, let $A_1, \ldots, A_k, B_1, \ldots, B_k$ be chain complexes over $\mathbb{Z}[G]$ or $\mathrm{Cob}^3_{/l}{}^{\bullet}(2n)$. Then, for $A = A_1 \oplus \cdots \oplus A_k$ and $B = B_1 \oplus \cdots \oplus B_k$, one has*

$$\lambda(A, B) \leq \max(\lambda(A_1, B_1), \ldots, \lambda(A_k, B_k)).$$

Proof. Without loss of generality, we can assume that $k = 2$. If either $\lambda(A_1, B_1)$ or $\lambda(A_2, B_2)$ are equal to ∞, the statement of the lemma is trivial, so let us assume that they are both finite. We pick chain maps f_1, g_1 such that $f_1 \circ g_1 \simeq G^{\lambda(A_1,B_1)} \cdot \mathrm{id}_{B_1}$ and $g_1 \circ f_1 \simeq G^{\lambda(A_1,B_1)} \cdot \mathrm{id}_{A_1}$, and choose maps f_2, g_2 similarly for $\lambda(A_2, B_2)$. Let $m = \max(\lambda(A_1, B_1), \lambda(A_2, B_2))$ and define

$$f\colon A_1 \oplus A_2 \to B_1 \oplus B_2, \quad g\colon B_1 \oplus B_2 \to A_1 \oplus A_2$$

as follows:

$$f = \begin{pmatrix} G^{m-\lambda(A_1,B_1)} \cdot f_1 & 0 \\ 0 & G^{m-\lambda(A_2,B_2)} \cdot f_2 \end{pmatrix}, \quad g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}.$$

We leave it to the reader to check that

$$f \circ g \simeq G^m \cdot \mathrm{id}_{B_1 \oplus B_2} \quad \text{and} \quad g \circ f \simeq G^m \cdot \mathrm{id}_{A_1 \oplus A_2}. \qquad \blacksquare$$

Taking one of the B_i as $[\![U]\!]$, and all the others as 0, we obtain the following special case of Lemma 3.4, which gives a useful upper bound for λ of a direct sum.

Corollary 3.5. *Let $C^1, \ldots, C^n$ be chain complexes of $\mathbb{Z}[G]$-modules, fix a $j \in \{1, \ldots, n\}$, and let $l_k = \lambda(C^k, 0)$, for all $k \neq j$, and $l_j = \lambda(C^j)$. Then,*

$$\lambda\Big(\bigoplus_i C^i\Big) \leq \max_i l_i. \qquad \blacksquare$$

Lemma 3.6. (i) $\lambda(A_1 \otimes A_2) \leq \lambda(A_1) + \lambda(A_2)$ *for A_1, A_2 chain complexes of $\mathbb{Z}[G]$-modules.*

(ii) $\lambda(\bar{A}) = \lambda(A)$, *where A is a $\mathbb{Z}[G]$-complex and $\bar{A}$ is its dual.*

Proof. For the first statement, let us assume that $\lambda(A_1), \lambda(A_2)$ are both finite (if either one is ∞ the statement is trivial). Let $f_i\colon A_i \to [\![U]\!]$, $g_i\colon [\![U]\!] \to A_i$ be chain maps such that $g_i \circ f_i \simeq G^{\lambda(A_i)} \cdot \mathrm{id}_{A_i}$ and $f_i \circ g_i \simeq G^{\lambda(A_i)} \cdot \mathrm{id}_{[\![U]\!]}$ for $i = 1, 2$. Define

$$f\colon A_1 \otimes A_2 \to [\![U]\!] \otimes [\![U]\!] \cong [\![U]\!], \quad g\colon [\![U]\!] \otimes [\![U]\!] \cong [\![U]\!] \to A_1 \otimes A_2$$

as

$$f = f_1 \otimes f_2, \quad g = g_1 \otimes g_2.$$

Then, $g \circ f \simeq G^{\lambda(A_1)+\lambda(A_2)} \cdot \mathrm{id}_{A_1 \otimes A_2}$ and $f \circ g \simeq G^{\lambda(A_1)+\lambda(A_2)} \cdot \mathrm{id}_{[\![U]\!]}$, so

$$\lambda(A_1 \otimes A_2) \leq \lambda(A_1) + \lambda(A_2),$$

as desired.

As for the second statement, it follows from the fact that if $f\colon A \to [\![U]\!]$, $g\colon [\![U]\!] \to A$ are chain maps such that $g \circ f \simeq G^k \cdot \mathrm{id}_A$ and $f \circ g \simeq G^k \cdot \mathrm{id}_{[\![U]\!]}$, then the induced dual chain maps $\bar{g}\colon \bar{A} \to [\![\bar{U}]\!] \cong [\![U]\!]$ and $\bar{f}\colon [\![\bar{U}]\!] \cong [\![U]\!] \to \bar{A}$ satisfy

$$\bar{f} \circ \bar{g} \simeq G^k \cdot \mathrm{id}_{\bar{A}} \quad \text{and} \quad \bar{g} \circ \bar{f} \simeq G^k \cdot \mathrm{id}_{[\![U]\!]}\,. \qquad \blacksquare$$

Proposition 1.9 now follows directly from Lemma 3.6, since $[\![K\#J]\!] \cong [\![K]\!] \otimes [\![J]\!]$ (see (2.4)) and $[\![-K]\!] \cong [\![\bar{K}]\!]$.

3.2. A closer look at λ for tangles

Here, we will again make use of planar arc diagrams, as introduced in Section 2.7.

Lemma 3.7. *Let $\mathcal{D}$ be a 2-input planar arc diagram containing an arc connecting the base points of the output disk and the first input disk. Let D_1 and D_1' be two tangle diagrams fitting into the first input disk, and let D_2 be a tangle diagram fitting into the second input disk. Then,*

$$\lambda(\mathcal{D}(D_1, D_2), \mathcal{D}(D_1', D_2)) \leq \lambda(D_1, D_1').$$

Proof. If $\lambda(D_1, D_1') = \infty$, the statement is clear. Suppose that

$$\lambda(D_1, D_1') = n \in \mathbb{N}$$

and consider chain maps $f\colon [D_1]^\bullet \to [D_1']^\bullet$ and $g\colon [D_1']^\bullet \to [D_1]^\bullet$ satisfying $f \circ g \simeq G^n \cdot \mathrm{id}_{[D_1']^\bullet}$ and $g \circ f \simeq G^n \cdot \mathrm{id}_{[D_1]^\bullet}$. Using the functor induced by $\mathcal{D}$, we may define maps $\tilde{f}$ and $\tilde{g}$ as

$$\mathcal{D}([D_1]^\bullet, [D_2]) \underset{\tilde{g}=\mathcal{D}(g,\mathrm{id}_{[D_2]})}{\overset{\tilde{f}=\mathcal{D}(f,\mathrm{id}_{[D_2]})}{\rightleftarrows}} \mathcal{D}([D_1']^\bullet, [D_2]).$$

These maps satisfy

$$\tilde{g} \circ \tilde{f} = \mathcal{D}(g \circ f, \mathrm{id}_{D_2}) \simeq \mathcal{D}(G^n \cdot \mathrm{id}_{D_1}, \mathrm{id}_{D_2}) = G^n \cdot \mathrm{id}_{\mathcal{D}([D_1']^\bullet, [D_2])},$$

and the analogous equality for $\tilde{g} \circ \tilde{f}$. This shows the desired statement. $\blacksquare$

See Figure 9 for examples of the following definitions.

Definition 3.8. Let $\mathcal{D}_{2n}$ be the following 2-input planar arc diagram: the two input disks are $2n$-ended and $(4n-2)$-ended, respectively; $\mathcal{D}_{2n}$ consists of one arc connecting the base point of the output disk to the base point of the first input disk, $2n-1$ arcs connecting end points of the two input disks, and $2n-1$ arcs connecting end points of the second input disk to end points of the output disk.

We say that a tangle diagram Q with $2m$ end points is *braid-like*, if it may be isotoped such that m end points are on the left, m end points are on the right, and Q consists of m arcs that at no point have a vertical tangent.

For $\mathcal{D}_{2n}$ as above, Q a $(4n-2)$-ended braid-like tangle diagram, and D a $2n$-ended tangle diagram, we say that $\mathcal{D}_{2n}(D, Q)$ is a *braiding* of D.

Recall from Definition 2.5 that, to obtain a tangle diagram of a given tangle in a ball B, one must choose a homeomorphism between B and the unit ball B_0. We will now show that two tangle diagrams of a fixed tangle are related by a finite sequence of Reidemeister moves and a braiding. In fact, the braiding only depends on the homeomorphisms between the balls, and not on the tangles. Let us make this precise.

Lemma 3.9. *Let B be a ball, and $P = \{p_1, \dots, p_{2n}\} \subset \partial B$ for some $n \geq 1$. Let φ_1, φ_2 be homeomorphisms from B to the unit ball B_0 with $\varphi_1(P) = \varphi_2(P)$ and $\varphi_1(p_1) = \varphi_2(p_1)$. Let $\mathcal{D}_{2n}$ be the 2-input planar arc diagram from Definition 3.8. Then, there is an unoriented braid-like $(4n-2)$-ended tangle diagram Q such that for all tangles T in B with end points P and base point p_1 the following holds: if D_1 and D_2 are the tangle diagrams of T coming from φ_1 and φ_2, respectively, then $\mathcal{D}_{2n}(D_1, Q)$ and D_2 are related by a finite sequence of Reidemeister moves and tangle diagram equivalences.*

Proof. Let $f\colon S^2 \to S^2$ be the restriction of $\varphi_2 \circ \varphi_1^{-1}$ to $S^2 = \partial B_0$. Let us write $\widetilde{P} = \varphi_1(P) = \varphi_2(P)$ and $\tilde{p}_i = \varphi_1(p_i)$. Note $f(\widetilde{P}) = \widetilde{P}$. In case that f is isotopic to id_{S^2} along homeomorphisms fixing $\widetilde{P}$ pointwise, it follows that $\varphi_1(T)$ and $\varphi_2(T)$ are equivalent tangles, and thus, D_1 and D_2 are related by a finite sequence of Reidemeister moves and tangle diagram equivalences. To deal with general f, let us consider the mapping class group of homeomorphisms $f\colon S^2 \to S^2$ with $f(\widetilde{P}) = \widetilde{P}$ and $f(\tilde{p}_1) = \tilde{p}_1$, up to isotopy along such maps. Every such f is isotopic to a homeomorphism fixing a neighborhood of $\tilde{p}_1$ pointwise, and so this mapping class group is isomorphic to the mapping class group of the $(2n-1)$-punctured disk, which is isomorphic to the braid group on $2n-1$ strands. More explicitly, it is generated by $\sigma_1, \dots, \sigma_{2n-2}$, where σ_i is a so-called *half-twist*, switching the positions of the punctures $\tilde{p}_{i+1}$ and $\tilde{p}_{i+2}$ [11, Section 9.1.3]. So, f is isotopic to a product β of the generators $\sigma_1, \dots, \sigma_{2n-2}$. Let Q be a braid-like $(4n-2)$-ended tangle diagram corresponding to β. Then, one sees that $\mathcal{D}_{2n}(D_1, Q)$ is a tangle diagram of T coming from the homeomorphism $(\varphi_2 \circ \varphi_1^{-1}) \circ \varphi_1 = \varphi_2$. Therefore, $\mathcal{D}_{2n}(D_1, Q)$ and D_2 are related by a finite sequence of Reidemeister moves and tangle diagram equivalences, as desired. ∎

Proposition 3.10. *Let S and T be tangles with the same end points and the same base point in a ball B. Let φ_1 and φ_2 be homeomorphisms from B to the unit ball B_0, leading to tangle diagrams D_{S1}, D_{S2} for S and D_{T1}, D_{T2} for T, respectively. Then,*

$$\lambda(D_{S1}, D_{T1}) = \lambda(D_{S2}, D_{T2}).$$

Proof. By Lemma 3.9, there is a 2-input planar arc diagram $\mathcal{D}$ and a tangle Q such that $\mathcal{D}(D_{S1}, Q)$ and D_{S2} are related by a finite sequence of Reidemeister moves, and so, are $\mathcal{D}(D_{T1}, Q)$ and D_{T2}. By Lemma 3.7, it follows that $\lambda(D_{S2}, D_{T2}) \leq \lambda(D_{S1}, D_{T1})$. Switching the roles of φ_1 and φ_2, the opposite inequality also follows. ∎

As a consequence, the following is well defined, since it does not depend on the choice of homeomorphism.

Definition 3.11. Let S and T be tangles with the same end points and the same base point in a ball B. Then, let $\lambda(S, T)$ be defined as $\lambda(D_S, D_T)$, where D_S and D_T are tangle diagrams of S and T, respectively, obtained via the same homeomorphism from B to the unit ball B_0.

Proposition 3.12. *Let S and T be two tangles in a ball B with the same connectivity, base point and end points. Let R be a tangle in another ball B' and $\varphi\colon \partial B \to \partial B'$ an orientation-reversing homeomorphism sending end points to end points such that $S \cup R$ and $T \cup R$ are knots in $B \cup_\varphi B' \cong S^3$. Then,*

$$\lambda(S \cup R, T \cup R) \leq \lambda(S, T).$$

Proof. One may pick tangle diagrams D_S, D_T, and D_R for S, T, and R, respectively, such that D_S and D_T come from the same homeomorphism from B to B_0; gluing D_S and D_R (using a 2-input planar arc diagram $\mathcal{D}$) results in a knot diagram of $S \cup R$; and similarly, $\mathcal{D}(D_T, D_R)$ is a diagram of $T \cup R$. Then, we have

$$\begin{aligned} \lambda(S \cup R, T \cup R) &= \lambda(\mathcal{D}(D_S, D_R), \mathcal{D}(D_T, D_R)) \\ &\leq \lambda(D_S, D_T) \\ &= \lambda(S, T) \end{aligned}$$

by the definition of λ for knots, Lemma 3.7, and Definition 3.11, respectively. ∎

Proposition 3.13. *Let us fix a ball and $2n$ end points on its boundary, and consider unoriented tangles T with a fixed number of components, a fixed connectivity, and* *a fixed base point in that ball. On the set of equivalence classes of such tangles T, λ is a pseudometric.*

Proof. It is straightforward to see that λ is symmetric, satisfies the triangle inequality, and $\lambda(T, T) = 0$. Since any two tangles S, T with same connectivity are related by crossing changes, Theorem 1.1 implies that $\lambda(S, T) < \infty$. ∎

Note that $\lambda(S, T) = 0$ if and only if $[S]^\bullet$ and $[T]^\bullet$ are ungradedly homotopy equivalent. So, the existence of non-equivalent tangles with homotopy equivalent Bar-Natan homology prevents λ from being a metric. Still, this pseudometric allows for a nice formulation of the main step of the proof of Theorem 1.1.

Proposition 3.14. *Fix a ball and four end points on its boundary, where one of them is distinguished as base point. On the set of equivalence classes of unoriented rational tangles in that ball with fixed connectivity, the pseudometric given by λ is in fact equal to the discrete metric. That is to say, $\lambda(S, T) = 1$ for inequivalent rational tangles S and T.*

The proof of Proposition 3.14 will be given in Section 5.

3.3. Decomposing $\mathbb{Z}[G]$-chain complexes into pieces

To analyze the $\mathbb{Z}[G]$-chain complex $[\![K]\!]$ of a knot K and compute $\lambda(K)$, one may follow a divide-and-conquer strategy and decompose $[\![K]\!]$ as a direct sum. This motivates the following definition.

Definition 3.15. For a graded ring R, a graded chain complex C of free shifted R-modules of finite total rank, i.e., $C \in \mathrm{Kom}(\mathcal{M}_R)$, is called a *piece* if it satisfies the following: C is not contractible (i.e., not homotopy equivalent to the trivial complex), and if C is homotopy equivalent to $C' \oplus C''$ with $C', C'' \in \mathrm{Kom}(\mathcal{M}_R)$, then either C' or C'' is contractible. In other words, a piece is an indecomposable object in the category $\mathrm{Kom}(\mathcal{M}_R)/h$ of chain complexes of finite total rank up to homotopy.

Let us now define the two most common kinds of pieces.

Definition 3.16. For a graded integral domain R, let the *pawn piece*, denoted by ♙, be the chain complex consisting of just one copy R in homological degree 0 and quantum degree 0. Given a non-trivial prime power $z \in R$, we define the *z-knight piece*, denoted by ♘(z), to be the chain complex

$${}_0R \xrightarrow{z} R\{-\deg z\},$$

where the left subscript denotes the homological degree.

The names of these pieces, coined by Bar-Natan [5], come from the fact that a ♙ and ♘(G) piece in $[\![K]\!]$ result in the patterns

$\mathbb{Q}$
$\mathbb{Q}$

and

	$\mathbb{Q}$
$\mathbb{Q}$	

,

respectively, in unreduced rational Khovanov homology. This can be seen using Corollary 2.18.

Remark 3.17. A complex P is a piece if and only if the ring of endomorphisms of P up to homotopy has precisely two distinct idempotents, namely, the zero map and the identity map. Let us use this to check that pawns and knights actually are pieces. For $P =$ ♙, the endomorphism ring of P is isomorphic to R, and there are no non-trivial homotopies. Since R is assumed to be an integral domain, the only idempotents are 0 and 1, and $0 \neq 1$. So, ♙ is indeed a piece.

Now, consider $P =$ ♘(z). Ignoring the chain complex structure, R-module endomorphisms $P \to P$ are given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Which among those maps are chain maps? To respect homological degree, we must have $b = c = 0$. To commute with the differential, we must have $az = dz$. Since $z \neq 0$ and R is an integral domain, this implies $a = d$. So, the endomorphism ring of P consists (as for ♙) just of multiples of id_P; i.e., this ring is isomorphic to R. All homotopies are multiples of $h\colon P_1 \to P_0$, $h(1) = 1$. We have $h \circ d + d \circ h = z \cdot \mathrm{id}_P$, and so, the endomorphism ring of P modulo homotopy is isomorphic to $R/(z)$. Since z is by assumption a non-trivial prime power, $R/(z)$ has no non-trivial idempotents. Thus, ♘(z) is indeed a piece.

In Example 3.18 and Section 4 below, we will claim that various chain complexes are pieces. This may be checked by similar arguments as above, but since we do not actually make use of the fact that those complexes are pieces, we omit these arguments from the text.

If R is a graded PID, then pawns and knights are the only pieces. This fact has been used previously to analyze homology theories coming from Frobenius algebras over fields, e.g., by Khovanov [20] or by Morrison [38][10]. In the introduction, we have seen $[\![K]\!]$ for $K = U, T_{2,3}, T_{3,4}$, and for those examples, $[\![K]\!]$ also decomposes into

[10]Morrison's "universal Khovanov homology" is equivalent to $[\![\cdot]\!] \otimes \mathbb{Q}$, i.e., the reduced theory coming from the Frobenius algebra $\mathbb{Q}[t, X]/(X^2 - tX)$ over $\mathbb{Q}[t]$. Since $\mathbb{Q}[t]$ is a PID, the chain complexes coming from that theory are homotopy equivalent to a sum of ♙ and ♘(t^n) pieces, which Morrison calls E and C_n (or $KhC[n]$), respectively. This homology theory can be calculated with `JavaKh` [12].

sum of pawns and knights. Let us consider a further example, which demonstrates that the pieces of $\mathbb{Z}[G]$-chain complexes can be significantly more complicated. (In fact, we do not know a classification of those pieces, cf. [39, Question 1].)

Example 3.18. As one may compute with `khoca` and `homca`, the chain complex $[\![T_{5,6}]\!]$ is homotopy equivalent to the sum of

$$_0\text{♙}\{20\} \oplus {}_2\text{♘}(G)\{24\} \oplus {}_4\text{♘}(G^2)\{26\}$$

and the following four more complicated pieces (where we write $R = \mathbb{Z}[G]$):

$$P_1 = \begin{array}{ccc} {}_6R\{28\} & \xrightarrow{G} & R\{30\} \\ \oplus & \overset{2}{\nearrow} & \oplus \\ R\{30\} & \xrightarrow[G]{} & R\{32\} \end{array}, \qquad P_2 = {}_8R\{30\} \begin{array}{c} \xrightarrow{2G^2} \\ \\ \xrightarrow[G^3]{} \end{array} \begin{array}{c} R\{34\} \\ \oplus \\ R\{36\} \end{array} \begin{array}{c} \xrightarrow{G} \\ \\ \xrightarrow[-2]{} \end{array} R\{36\},$$

$$P_3 = {}_{10}R\{34\} \begin{array}{c} \xrightarrow{5G} \\ \\ \xrightarrow[G^2]{} \end{array} \begin{array}{c} R\{36\} \\ \oplus \\ R\{38\} \end{array} \begin{array}{c} \xrightarrow{G^2} \\ \\ \xrightarrow[-5G]{} \end{array} R\{40\}, \qquad P_4 = {}_{12}R\{36\} \begin{array}{c} \xrightarrow{3G^2} \\ \\ \xrightarrow[G^3]{} \end{array} \begin{array}{c} R\{40\} \\ \oplus \\ R\{42\} \end{array} \begin{array}{c} \xrightarrow{G} \\ \\ \xrightarrow[-3]{} \end{array} R\{42\}.$$

Note that P_3 is isomorphic to ${}_{10}\text{♘}(G^2) \otimes \text{♘}(5G)\{34\}$. Let us now compute λ of $T_{5,6}$. We have $\lambda(\text{♘}(G^k)) = k$ for $k \in \{1, 2\}$ (in fact, for all $k \geq 1$) and leave it to the reader to check that $\lambda(P_i, 0) \leq 3$ for $i \in \{1, 2, 3, 4\}$. Using Corollary 3.5, this implies $\lambda(T_{5,6}) \leq 3$. To show $\lambda(T_{5,6}) \geq 3$, we rely on the maximal G-torsion order of homology, denoted by $\mathfrak{u}_G$. This invariant is discussed in detail in Section 3.4. It gives a lower bound $\mathfrak{u}_G \leq \lambda$ (see Lemma 3.27). Consider the homology of $\overline{P_4}$, the dual of P_4. The annihilator of the class of a generator of ${}_{-12}R\{-36\}$ is the ideal $(3G^2, G^3) \subset \mathbb{Z}[G]$, and so, the G-torsion order of that homology class is equal to 3. Hence, $\lambda(T_{5,6}) = \lambda(-T_{5,6}) \geq \mathfrak{u}_G(-T_{5,6}) \geq 3$, and thus, $\lambda(T_{5,6}) = 3$.

Remark 3.19. If R is Noetherian, then every chain complex in $\mathrm{Kom}(\mathcal{M}_R)/h$ can be written as a sum of finitely many pieces. If R is a graded PID, then this decomposition is essentially unique, i.e., unique up to the order of the summands. This is not true for $R = \mathbb{Z}[G]$, as the following example demonstrates. So, in this text, we will often decompose chain complexes $[\![K]\!]$ as sums of pieces, but we will never rely on this decomposition being unique.

Let us now give an example of a chain complex that admits two essentially different decompositions as sums of pieces. For any non-zero integer n, let Q_n be the complex

$${}_0R\{0\} \begin{array}{c} \xrightarrow{nG} \\ \\ \xrightarrow[G^2]{} \end{array} \begin{array}{c} R\{2\} \\ \oplus \\ R\{4\} \end{array} \begin{array}{c} \xrightarrow{-G} \\ \\ \xrightarrow[n]{} \end{array} R\{4\}.$$

One computes that the endomorphism ring of Q_n modulo homotopy is isomorphic to $R/(G^2, nG)$. This ring does not admit non-trivial idempotents, and so, Q_n is a piece. Now, the Smith normal form gives us invertible 2×2 integer matrices S, T such that $S\left(\begin{smallmatrix}2&0\\0&3\end{smallmatrix}\right)T = \left(\begin{smallmatrix}1&0\\0&6\end{smallmatrix}\right)$. This leads to the following change of basis, which demonstrates $Q_2 \oplus Q_3 \cong Q_1 \oplus Q_6$, giving us the desired example. Note that $Q_1 \simeq$ ♘(G):

$$
{}_0R\{0\}^{\oplus 2}
\begin{array}{c}
\xrightarrow{G\left(\begin{smallmatrix}2&0\\0&3\end{smallmatrix}\right)} \\ \\ \xrightarrow[G^2]{}
\end{array}
\begin{array}{c}
R\{2\}^{\oplus 2} \\ \oplus \\ R\{4\}^{\oplus 2}
\end{array}
\begin{array}{c}
\xrightarrow{-G} \\ \\ \xrightarrow[\left(\begin{smallmatrix}2&0\\0&3\end{smallmatrix}\right)]{}
\end{array}
R\{4\}^{\oplus 2}
\cong
{}_0R\{0\}^{\oplus 2}
\begin{array}{c}
\xrightarrow{GS\left(\begin{smallmatrix}2&0\\0&3\end{smallmatrix}\right)T} \\ \\ \xrightarrow[G^2]{}
\end{array}
\begin{array}{c}
R\{2\}^{\oplus 2} \\ \oplus \\ R\{4\}^{\oplus 2}
\end{array}
\begin{array}{c}
\xrightarrow{-G} \\ \\ \xrightarrow[S\left(\begin{smallmatrix}2&0\\0&3\end{smallmatrix}\right)T]{}
\end{array}
R\{4\}^{\oplus 2}
$$

3.4. Torsion orders

When computing λ of a knot K, it is fairly simple to find an upper bound $k \geq \lambda(K)$ by defining ungraded chain maps

$$f\colon [\![K]\!] \to [\![U]\!], \quad g\colon [\![U]\!] \to [\![K]\!]$$

such that $g \circ f$ and $f \circ g$ are homotopic to G^k. In order to compute the exact value of λ, however, one has to find the minimal such k, which can be a hard task. The invariants described in this subsection give lower bounds for λ in terms of the maximal torsion order in homology.

In 2017, Alishahi and Dowlin [1,2] introduced the following knot invariants which are lower bounds for the unknotting number: $\mathfrak{u}_h$ is defined as the maximal order of h-torsion in the unreduced homology with Frobenius algebra $\mathbb{F}_2[h, X]/(X^2 + hX)$, while $\mathfrak{u}_X$ is the maximal X-torsion order in the unreduced homology $H_{\mathbb{Q}[t]}$ with Frobenius algebra $\mathbb{Q}[t, X]/(X^2 - t)$ (a lift of Lee homology). Note that $H_{\mathbb{Q}[t]}$ is a module over $\mathbb{Q}[t, X]/(X^2 - t)$, but that ring is just equal to its subring $\mathbb{Q}[X]$, and so, we consider $H_{\mathbb{Q}[t]}$ as a module over $\mathbb{Q}[X]$. It was then remarked in [8] that for the latter invariant one can replace $\mathbb{Q}$ with $\mathbb{F}_p$ for any odd prime p in order to obtain new bounds $\mathfrak{u}_{(X,p)}$. Finally, Gujral [13] introduced a lower bound ν for the ribbon distance: ν is the maximal order of $(2X - (\alpha_1 + \alpha_2))$-torsion in the α-homology of a knot, which is the unreduced homology with Frobenius algebra $\mathbb{Z}[X, \alpha_1, \alpha_2]/((X - \alpha_1)(X - \alpha_2))$ over the ground ring $\mathbb{Z}[\alpha_1, \alpha_2]$, introduced in [21].

The following invariant is the analog of those bounds in the $\mathbb{Z}[G]$-setting.

Definition 3.20. Let K be a knot and consider its $\mathbb{Z}[G]$-complex $[\![K]\!] \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$. Since $\mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]}) \subset \mathrm{Kom}(\mathbb{Z}[G]\text{-Mod})$, we can take the homology $H([\![K]\!])$ of $[\![K]\!]$ in the latter category (see Remark 2.11). Let now $a \in H([\![K]\!])$. We say that a is *G-torsion* if there is an $n \in \mathbb{Z}_{\geq 0}$ such that $G^n \cdot a = 0$. Let the *order* of a G-torsion element a, $\mathrm{ord}_G(a)$, be the minimal such n and $T(H([\![K]\!]))$ the $\mathbb{Z}[G]$-module of G-torsion elements.

Definition 3.21. We define $\mathfrak{u}_G(K)$ to be the maximal order of a G-torsion element:

$$\mathfrak{u}_G(K) := \max_{a \in T(H(\llbracket K \rrbracket))} \operatorname{ord}_G(a).$$

Proposition 3.22. *For all knots K, we have*

(i) $\mathfrak{u}_G = \nu$,

(ii) $\mathfrak{u}_G \geq \mathfrak{u}_X$,

(iii) $\lambda \geq \mathfrak{u}_{(X,p)}$,

(iv) $\lambda \geq \mathfrak{u}_h$.

Proof. (i) By Theorem 1.5, the α-chain complex of K is homotopy equivalent to $\llbracket K \rrbracket \otimes_{\mathbb{Z}[G]} \mathbb{Z}[X, \alpha_1, \alpha_2]/((X - \alpha_1)(X - \alpha_2))$, where G acts as $2X - \alpha_1 - \alpha_2$ on the second tensor factor. Less formally, α-homology is just $\mathbb{Z}[G]$-homology with G disguised as $2X - \alpha_1 - \alpha_2$. The statement follows.

(ii) Again by Theorem 1.5, $C_{\mathbb{Q}[t]}(K) \simeq \llbracket K \rrbracket \otimes_{\mathbb{Z}[G]} \mathbb{Q}[X]$, where G acts as $2X$ on $\mathbb{Q}[X]$. So, X-torsion in $H_{\mathbb{Q}[t]}(K)$ corresponds to G-torsion in $H(\llbracket K \rrbracket \otimes \mathbb{Q})$. Thus, it suffices to show that the maximal G-torsion order in $\mathbb{Z}[G]$-homology is greater than or equal to the maximal G-torsion order in $\mathbb{Q}[G]$-homology. So, let a cycle x in the chain complex $\llbracket K \rrbracket \otimes \mathbb{Q}$ be given that represents a homology class of maximal G-torsion order in $\mathbb{Q}[G]$-homology, i.e., $G^{\mathfrak{u}_X(K)}[x] = 0$ and $G^{\mathfrak{u}_X(K)-1}[x] \neq 0$. Choose $n \in \mathbb{Z} \setminus \{0\}$ such that $nx \in \llbracket K \rrbracket$. Since $G^{\mathfrak{u}_X(K)}x$ is a boundary over $\mathbb{Q}$, one may choose $m \in \mathbb{Z} \setminus \{0\}$ such that $G^{\mathfrak{u}_X(K)}nmx$ is a boundary over $\mathbb{Z}$. So, $nm[x]$ is G-torsion in $H(\llbracket K \rrbracket)$. Moreover, if $G^{\mathfrak{u}_X(K)-1}nmx$ were a boundary dy in $\llbracket K \rrbracket$, then $G^{\mathfrak{u}_X(K)-1}x = d(y/(nm))$ and thus $G^{\mathfrak{u}_X(K)-1}[x] = 0 \in H(\llbracket K \rrbracket \otimes \mathbb{Q})$ would follow, which is a contradiction. So, the G-torsion order of $nm[x] \in H(\llbracket K \rrbracket)$ equals $\mathfrak{u}_X(K)$, which implies the claim $\mathfrak{u}_G(K) \geq \mathfrak{u}_X(K)$.

(iii) Let $f\colon \llbracket K \rrbracket \to \mathbb{Z}[G]$ and $g\colon \mathbb{Z}[G] \to \llbracket K \rrbracket$ be chain maps such that $f \circ g$ and $g \circ f$ are homotopic to $G^{\lambda(K)}$ times the identity of $\mathbb{Z}[G]$ and $\llbracket K \rrbracket$, respectively. Once again by Theorem 1.5, $C_{\mathbb{F}_p[t]}(K) \simeq \llbracket K \rrbracket \otimes_{\mathbb{Z}[G]} \mathbb{F}_p[X]$, where G acts as $2X$ on $\mathbb{F}_p[X]$. So, f and g induce maps $f_p\colon H_{\mathbb{F}_p[t]}(K) \to \mathbb{F}_p[X]$ and $g_p\colon \mathbb{F}_p[X] \to H_{\mathbb{F}_p[t]}(K)$ such that $f_p \circ g_p$ and $g_p \circ f_p$ are multiplication with $(2X)^{\lambda(K)}$ on $\mathbb{F}_p[X]$ and $H_{\mathbb{F}_p[t]}(K)$, respectively. The existence of such maps implies, in the usual way (see, e.g., the proof of Lemma 3.27 below), that X-torsion orders in $H_{\mathbb{F}_p[t]}(K)$ are at most $\lambda(K)$.

(iv) This inequality may be proven similarly as the previous one, using that by Theorem 1.5, $C_{\mathbb{F}_2[h]}(K) \simeq \llbracket K \rrbracket \otimes_{\mathbb{Z}[G]} \mathbb{F}_2[h]$, where G acts as h on $\mathbb{F}_2[h]$. ∎

Note that the inequalities $\mathfrak{u}_G \geq \mathfrak{u}_{(X,p)}$ and $\mathfrak{u}_G \geq \mathfrak{u}_h$ do not hold in general; counterexamples are given by the complexes P_4 (for $\mathfrak{u}_{(X,3)}$) and P_2 (for $\mathfrak{u}_h$) from Example 3.18.

As a side note, it is worthwhile to observe, although we will not make use of it, that $\mathfrak{u}_h$, $\mathfrak{u}_X$ are also linked to the convergence of the Bar-Natan [51] and the Lee [26] spectral sequences, respectively. For all knots K, these sequences start at Khovanov homology of K (with coefficients in $\mathbb{F}_2$ and $\mathbb{Q}$, respectively) and, letting n_{BN} and n_{Lee} be the pages at which they collapse, we have $\mathfrak{u}_h(K) = n_{\mathrm{BN}} - 1$ and $\lceil \mathfrak{u}_X(K)/2 \rceil = n_{\mathrm{Lee}} - 1$. As a consequence, the following interesting result holds.

Corollary 3.23 ([2]). *The Knight Move Conjecture is true for all knots K with $u(K) \leq 2$.*

In light of Theorem 1.1, we even have the following corollary.

Corollary 3.24. *The Knight Move Conjecture is true for all knots K with $u_q(K) \leq 2$.* ■

The connection discussed above between invariants related to λ and spectral sequences brings us to the following natural question.

Question 3.25. Is there a spectral sequence E_G such that, for any knot K, $E_G(K)$ starts at Khovanov homology (with coefficients in $\mathbb{Z}$) and collapses at a page whose number is determined by $\mathfrak{u}_G(K)$?

We suspect this question has a positive answer. Namely, consider the chain complex obtained from $[\![K]\!]$ by setting $G = 1$. The resulting complex is filtered and gives rise to a spectral sequence $E_G(K)$ starting at (reduced) Khovanov homology with integer coefficients. It seems likely that $E_G(K)$ collapses at page $\mathfrak{u}_G(K) - 1$.

Lemma 3.26. *The invariant $\mathfrak{u}_G$ detects the unknot, i.e., $\mathfrak{u}_G(K) = 0$ holds if and only if K is trivial.*

Proof. We start by noticing that $\mathfrak{u}_G(U) = 0$: this is clear since $H([\![U]\!]) = \mathbb{Z}[G]$ is torsion free. Note that

$$H_{\mathbb{Q}[t]}(K) \cong \mathbb{Q}[X] \oplus T(H_{\mathbb{Q}[t]}(K)),$$

where $T(H_{\mathbb{Q}[t]}(K))$ is the X-torsion part. Since Khovanov homology detects the unknot we have $T(H_{\mathbb{Q}[t]}(K)) = \{0\}$ only if $K = U$. This implies that if K is not the unknot, then $\mathfrak{u}_G(K) \geq \mathfrak{u}_X(K) > 0$. ■

Lemma 3.27. *Let K be a knot. Then, $\mathfrak{u}_G(K) \leq \lambda(K)$.*

Proof. Let $n = \lambda(K)$, and let $f\colon [\![K]\!] \to [\![U]\!]$, $g\colon [\![U]\!] \to [\![K]\!]$ be ungraded chain maps such that $g \circ f \simeq G^n \cdot \mathrm{id}_{[\![K]\!]}$ and $f \circ g \simeq G^n \cdot \mathrm{id}_{[\![U]\!]}$. Then, for every $a \in H([\![K]\!])$,

$$\mathrm{ord}_G\big(f_*(a)\big) \geq \mathrm{ord}_G\big(g_* \circ f_*(a)\big) = \mathrm{ord}_G\big(G^n \cdot a\big) \geq \mathrm{ord}_G(a) - n.$$

Taking the maximum over $T(H(\llbracket K \rrbracket))$, we get

$$0 = \mathfrak{u}_G(U) = \max_{a \in T(H(\llbracket K \rrbracket))} \operatorname{ord}_G\big(f_*(a)\big) \geq \max_{a \in T(H(\llbracket K \rrbracket))} \operatorname{ord}_G(a) - n = \mathfrak{u}_G(K) - n.$$

This shows that $\mathfrak{u}_G(K) \leq n = \lambda(K)$. ∎

The two previous lemmas, combined with the fact that clearly $\lambda(U) = 0$ (using the maps $f = g = \mathrm{id}_{\llbracket U \rrbracket}$), show that λ detects the unknot, as claimed in Proposition 1.8. A more direct proof may also be given as follows.

Proof of Proposition 1.8. Let K be a knot. It follows from the definition of λ that $\lambda(K) = 0$ if and only if $\llbracket K \rrbracket$ is ungraded chain homotopy equivalent to $\llbracket U \rrbracket$. Since, even as an ungraded module, Khovanov homology detects the unknot, the latter condition holds if and only if $K = U$. ∎

In Definition 3.2, we saw how to define λ for any $\mathbb{Z}[G]$-complex. Similarly, one can define the invariants $\mathfrak{u}_G$, $\mathfrak{u}_h$, $\mathfrak{u}_X$, $\mathfrak{u}_{(X,p)}$ on chain complexes over $\mathcal{M}_R$, where R is equal to $\mathbb{Z}[G]$, $\mathbb{F}_2[h]$, $\mathbb{Q}[X]$, $\mathbb{F}_p[X]$, respectively. Namely, if $C \in \mathrm{Kom}(\mathcal{M}_R)$ and $\eta = G, h, X$ and (X, p), then $\mathfrak{u}_\eta(C)$ is the maximal order of η-torsion in the homology $H(C) \in R$-Mod of C. Lemma 3.27 also holds for complexes over $\mathcal{M}_{\mathbb{Z}[G]}$.

We now state a few properties of $\mathfrak{u}_G$, $\mathfrak{u}_h$, $\mathfrak{u}_X$, $\mathfrak{u}_{(X,p)}$.

Lemma 3.28. *Let $R_G = \mathbb{Z}[G]$, $R_h = \mathbb{F}_2[h]$, $R_X = \mathbb{Q}[X]$, $R_{(X,p)} = \mathbb{F}_p[X]$, and let A, B be chain complexes over $\mathcal{M}_{R_\eta}$ for $\eta = G, h, X$ and (X, p). We have the following.*

(i) $\mathfrak{u}_\eta(A \oplus B) = \max(\mathfrak{u}_\eta(A), \mathfrak{u}_\eta(B))$.

(ii) *For $\eta = h, X$ and (X, p),*

$$\mathfrak{u}_\eta(A \otimes B) = \begin{cases} \max(\mathfrak{u}_\eta(A), \mathfrak{u}_\eta(B)) & \text{if } p_A > 0 \text{ and } p_B > 0, \\ \mathfrak{u}_\eta(A) & \text{if } p_A = 0 \text{ and } p_B > 0, \\ \mathfrak{u}_\eta(B) & \text{if } p_A > 0 \text{ and } p_B = 0, \\ \min(\mathfrak{u}_\eta(A), \mathfrak{u}_\eta(B)) & \text{if } p_A = p_B = 0, \end{cases}$$

where, given a complex C over $\mathcal{M}_{R_\eta}$, p_C is the number of pawn summands in the decomposition of C into pawns and knights.

Proof. The first statement is clear. For the second one, we use the fact that R_η is a PID. As noted in Section 3.3, this implies that

$$A \simeq ♙^{\oplus p_A} \oplus ♘(a_1) \oplus \cdots \oplus ♘(a_n),$$

with $a_i = X^{k_i}$ (if $\eta = X$ or (X, p)) or $a_i = h^{k_i}$ (if $\eta = h$) and $k_i \leq k_{i+1}$. Similarly,

$$B \simeq ♙^{\oplus p_B} \oplus ♘(b_1) \oplus \cdots \oplus ♘(b_m),$$

with $b_j = X^{l_j}$ or $b_j = h^{l_j}$ and $l_j \leq l_{j+1}$. One checks that

$$\mathfrak{u}_\eta(\text{♙}) = 0 \quad \text{and} \quad \mathfrak{u}_\eta(\text{♘}(a_i)) = k_i;$$

therefore, by point (i), $\mathfrak{u}_\eta(A) = \max(\mathfrak{u}_\eta(\text{♙}), \mathfrak{u}_\eta(\text{♘}(a_1)), \ldots, \mathfrak{u}_\eta(\text{♘}(a_n))) = k_n$. Similarly, $\mathfrak{u}_\eta(B) = l_m$. Now,

$$A \otimes B \simeq \bigoplus_{\substack{i \in \{0,\ldots,n\} \\ j \in \{0,\ldots,m\}}} A_i \otimes B_j,$$

with $A_0 = \text{♙}^{\oplus p_A}$, $B_0 = \text{♙}^{\oplus p_B}$ and $A_i = \text{♘}(a_i)$, $B_j = \text{♘}(b_j)$ for $i, j > 0$. It is a simple exercise to check that $\mathfrak{u}_\eta(\text{♘}(a_i) \otimes \text{♘}(b_j)) = \min(k_i, l_j)$. It follows that

$$\begin{aligned} &\mathfrak{u}_\eta(A \otimes B) \\ &\quad = \max_{i,j}(\mathfrak{u}_\eta(A_i \otimes B_j)) \\ &\quad = \max(\mathfrak{u}_\eta(A_0 \otimes B_m), \mathfrak{u}_\eta(A_n \otimes B_0), \mathfrak{u}_\eta(A_n \otimes B_m)) \\ &\quad = \max(\mathfrak{u}_\eta(\text{♙}^{\oplus p_A} \otimes \text{♘}(b_m)), \mathfrak{u}_\eta(\text{♘}(a_n) \otimes \text{♙}^{\oplus p_B}), \mathfrak{u}_\eta(\text{♘}(a_n) \otimes \text{♘}(b_m))). \end{aligned}$$

Then, statement (ii) can be deduced from the following:

$$\mathfrak{u}_\eta(\text{♙}^{\oplus p_A} \otimes \text{♘}(b_m)) = \begin{cases} \mathfrak{u}_\eta(\text{♘}(b_m)) = l_m & \text{if } p_A > 0, \\ 0 & \text{if } p_A = 0, \end{cases}$$

$$\mathfrak{u}_\eta(\text{♘}(a_n) \otimes \text{♙}^{\oplus p_B}) = \begin{cases} \mathfrak{u}_\eta(\text{♘}(a_n)) = k_n & \text{if } p_B > 0, \\ 0 & \text{if } p_B = 0, \end{cases}$$

$$\mathfrak{u}_\eta(\text{♘}(a_n) \otimes \text{♘}(b_m)) = \min(k_n, l_m). \qquad \blacksquare$$

Remark 3.29. As a consequence, of (ii) of Lemma 3.28, if $\eta = h, X$ or (X, p) and K and J are two knots,

$$\mathfrak{u}_\eta(K\#J) = \max(\mathfrak{u}_\eta(K), \mathfrak{u}_\eta(J)), \tag{3.1}$$

as the unreduced chain complexes over $\mathcal{M}_{R_\eta}$ associated to knots always contain a pawn summand (more precisely, exactly one pawn summand if $\eta = X$ or (X, p), and exactly two if $\eta = h$).

Note that neither statement (ii) of Lemma 3.28 nor equation (3.1) holds in general for $\mathfrak{u}_G$. This is due to the fact that, for $\eta \neq G$, the $\mathfrak{u}_\eta$ are defined over PIDs, while $\mathfrak{u}_G$ is not (cf. Section 3.3). Later on in this article, Remarks 4.5 and 4.9 will provide us with examples of knots K, J such that $\mathfrak{u}_G(K\#J) < \max(\mathfrak{u}_G(K), \mathfrak{u}_G(J))$, and others where $\mathfrak{u}_G(K\#J) > \max(\mathfrak{u}_G(K), \mathfrak{u}_G(J))$: if K_1, K_2 satisfy (1) of Proposition 4.4 and J satisfies (2), then

$$\mathfrak{u}_G(K_1) = \mathfrak{u}_G(K_2) = \mathfrak{u}_G(J) = 1,$$

but $\mathfrak{u}_G(K_1\#K_2) = 2 = \mathfrak{u}_G(K_1) + \mathfrak{u}_G(K_2)$ and

$$\mathfrak{u}_G((K_1\#K_2)\#J) = 1 = \mathfrak{u}_G(K_1\#K_2) - \mathfrak{u}_G(J).$$

Therefore, the best that we can hope for is that $\mathfrak{u}_G(K\#J) \leq \mathfrak{u}_G(K) + \mathfrak{u}_G(J)$.

3.5. λ of thin knots

For a homogeneous element x of a doubly graded chain complex, with quantum degree q and homological degree t, let the δ-degree of x be $q - 2t$. It was noted early on in the development of Khovanov homology that the rational Khovanov homology of alternating links is supported in a single δ-degree [26]. This led to various different notions of thinness and homological width of links; see, e.g., [47, 48]. In this article, we call a knot *thin* if their reduced integral Khovanov homology consists of free modules supported in a single δ-degree (as already defined in the introduction). Let us prove in this subsection that λ of non-trivial thin knots is 1.

Lemma 3.30. *If a chain complex $C \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$ decomposes (ignoring gradings) as a sum of one* ♙ *and finitely many* $♘(G)$ *pieces, then $\lambda(C) \leq 1$.*

Proof. Since $\lambda(♙) = 0$ and $\lambda(♘(G), 0) = 1$, this follows from Corollary 3.5. ■

Lemma 3.31. *Let K be a knot whose reduced integral Khovanov homology is torsion free. Then, $[\![K]\!]$ is homotopy equivalent to a chain complex $C \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$ of free shifted $\mathbb{Z}[G]$-modules such that the Poincaré polynomial of C is equal to the Poincaré polynomial of reduced integral Khovanov homology of K.*

Proof. Start by picking an arbitrary chain complex $C' \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$ that is homotopy equivalent to $[\![K]\!]$. Consider the chain complex $C' \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/(G)$. This is a chain complex over the integers, whose homology is isomorphic to reduced integral Khovanov homology of K. In particular, it has torsion-free homology by assumption. One may select bases for the chain groups of the complex $C' \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/(G)$, with respect to which the matrices of the differentials are in Smith normal form. Because homology is torsion free, all the entries of these matrices are 0 or 1. Gaussian elimination (see, e.g., Lemma 5.8) of all the entries equal to 1 yields a homotopy equivalence between $C' \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/(G)$ and a complex Z with trivial differentials. So, Z is isomorphic to the reduced integral Khovanov homology of K.

Now, one may lift the bases of $C' \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/(G)$ to homogeneous bases of C'. Since the matrices of the differentials of C' have homogeneous entries, it follows that if a matrix entry of a differential of $C' \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/(G)$ equals 1, then the corresponding matrix entry of the corresponding differential of C' also equals 1. Therefore, one may lift the homotopy equivalence constructed above, obtaining a homotopy equivalence between C' and a complex $C \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$ such that $C \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/(G)$ is isomorphic to Z. It follows that C and the reduced integral Khovanov homology of K have the same Poincaré polynomial, as desired. ∎

Lemma 3.32. *For all thin knots K, $[\![K]\!]$ is up to degree shifts homotopy equivalent to a sum of one ♙ piece and finitely many ♘(G) pieces.*

Proof. By Lemma 3.31, we may pick a chain complex $C \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$ that is homotopy equivalent to $[\![K]\!]$ and has the same Poincaré polynomial as reduced integral Khovanov homology of K. Since the latter is supported on a single δ-degree, so is C. Choosing arbitrary bases for the chain modules of C, it follows that every entry of the matrices of the differentials is an integer multiple of G. Similarly, as in the proof of Lemma 3.31, one may choose new bases for the modules of C such that the matrices of the differentials equal G times a matrix in Smith normal form. Consequently, ignoring gradings C decomposes into a direct sum of ♙ and ♘(aG) pieces, with a priori varying $a \in \mathbb{Z}_{>0}$. By Proposition 2.20, there is exactly one ♙ piece, and all other pieces are ♘(G) pieces. ∎

Proof. Lemmas 3.32 and 3.30 imply $\lambda(K) \leq 1$, whereas Proposition 1.8 implies $\lambda(K) \geq 1$. ∎

Remark 3.33. Note that Lemma 3.32 also provides a proof (at least for knots) for Bar-Natan's 'structural conjecture' that all alternating links are 'Khovanov basic' [5, Conjecture 1].

In [8], upper bounds for $\mathfrak{u}_X$, $\mathfrak{u}_h$, and $\mathfrak{u}_{(X,p)}$ are given in terms of the homological width of Khovanov homologies. This motivates the following question.

Question 3.34. Let K be a knot such that $[\![K]\!]$ is homotopy equivalent to a complex supported in n adjacent δ-degrees. Does then $\lambda(K) \leq n$ follow?

4. Calculations of $\mathbb{Z}[G]$-homology and the λ-invariant

4.1. Proof of Theorem 1.2: λ can be arbitrarily big

The purpose of this subsection is to show that our invariant λ can grow arbitrarily. More precisely, as claimed in Theorem 1.2, for all $n \in \mathbb{N}$, we will define a knot K such that $\lambda(K) = n$.

We saw that $\mathbb{Z}[G]$-homology is bigraded. However, our invariant λ does not depend on the quantum grading; therefore, we will omit quantum shifts in this section.

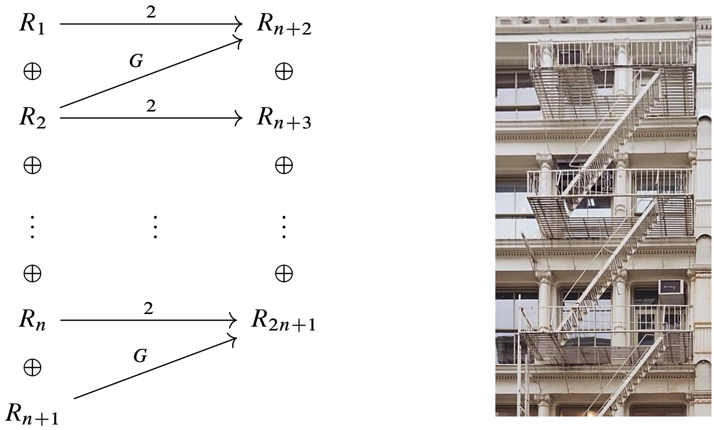


Figure 10. The staircase S_n of rank $2n+1$. Here, $R_i = \mathbb{Z}[G]$ for all i.

Definition 4.1. For every $n \in \mathbb{Z}_{>0}$, the *staircase of rank* $2n+1$, denoted by S_n, is defined as the chain complex

$$0 \to C_0 \xrightarrow{d_{S_n}} C_1 \to 0,$$

where $C_0 = (\mathbb{Z}[G])^{n+1}, C_1 = (\mathbb{Z}[G])^n$, and

$$d_{S_n} = \begin{pmatrix} 2 & G & & & 0 \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 2 & G \end{pmatrix}.$$

We can represent a staircase S_n as shown in Figure 10.

Let us prove a few lemmas about staircases.

Lemma 4.2. *Let* S_n *be a staircase of rank* $2n+1$*. Then,* $\lambda(S_n) = \mathfrak{u}_G(S_n) = n$.

Proof. We will first show that $\lambda(S_n) \leq n$ by finding ungraded chain maps $f\colon S_n \to [\![U]\!]$, $g\colon [\![U]\!] \to S_n$ such that $f \circ g \simeq G^n \cdot \mathrm{id}_{[\![U]\!]}$ and $g \circ f \simeq G^n \cdot \mathrm{id}_{S_n}$.

Since f, g need not respect the homological degree, we will simply consider S_n as a pair $(S_n = R_1 \oplus \cdots \oplus R_{2n+1}, d\colon S_n \to S_n)$, with $R_i = \mathbb{Z}[G]$ as in Figure 10 and

$$d = \begin{array}{c} \scriptstyle n+1 \\ \scriptstyle n \end{array}\left(\begin{array}{c|c} 0 & 0 \\ \hline d_{S_n} & 0 \end{array}\right), \quad \text{columns: } n+1,\ n.$$

Define

$$
\begin{aligned}
f &= \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}, \\
g &= \begin{pmatrix} G^n & -2G^{n-1} & 4G^{n-2} & \cdots & (-2)^n & 0 & \cdots & 0 \end{pmatrix}^{\mathsf{T}}.
\end{aligned}
$$

It is easy to check that f and g are ungraded chain maps (interestingly, they also respect the homological degree, so they are actually graded chain maps) and $f \circ g = G^n \cdot \mathrm{id}_{[\![U]\!]}$. We need to verify that $g \circ f \simeq G^n \cdot \mathrm{id}_{S_n}$. We have

$$
g \circ f = \left(\begin{array}{c|c}
G^n & \\
-2G^{n-1} & \\
\vdots & \\
(-2)^n & \mathbf{0} \\
0 & \\
\vdots & \\
0 &
\end{array}\right)_{2n+1} \quad (\text{right block: } 2n \text{ columns}),
$$

so let us show that

$$
G^n \cdot \mathrm{id}_{S_n} - g \circ f = \left(\begin{array}{c|cccc}
0 & 0 & \cdots & \cdots & 0 \\
2G^{n-1} & G^n & & & \mathbf{0} \\
\vdots & & \ddots & & \\
-(-2)^n & & & \ddots & \\
0 & & & & \\
\vdots & & & & \\
0 & \mathbf{0} & & & G^n
\end{array}\right)_{2n+1} \quad (\text{right block: } 2n \text{ columns})
$$

is nullhomotopic. We define $h\colon S_n \to S_n$ as

$$
h = \left(\begin{array}{c|cccc}
 & 0 & & & \\
 & G^{n-1} & & \mathbf{0} & \\
\mathbf{0} & -2G^{n-1} & \ddots & & \\
 & \vdots & \ddots & \ddots & \\
 & (-2)^{n-1} & \cdots & -2G^{n-2} & G^{n-1} \\
\hline
\mathbf{0} & & \mathbf{0} & &
\end{array}\right) \begin{matrix} n+1 \\ \\ n \end{matrix}
\quad (\text{column blocks: } n+1,\ n).
$$

It is easy to see that

$$d \circ h + h \circ d = G^n \cdot \mathrm{id}_{S_n} - g \circ f.$$

Therefore, $G^n \cdot \mathrm{id}_{S_n} - g \circ f$ is nullhomotopic and $g \circ f \simeq G^n \cdot \mathrm{id}_{S_n}$.

We now show that $\lambda(S_n) \geq n$. Since $\lambda \geq \mathfrak{u}_G$, it is enough to find an element $x \in H([\![S_n]\!])$ that has G-torsion order n. Let $x = (1 \quad 0 \quad \cdots \quad 0)^\intercal \in C_1(S_n) = \mathbb{Z}[G]^n$. Since

$$G^n \cdot x = d_{S_n}\big((0 \quad G^{n-1} \quad -2G^{n-2} \quad \cdots \quad (-2)^{n-1})^\intercal\big),$$

clearly $\mathrm{ord}_G(x) \leq n$. We will now show that the inequality $\mathrm{ord}_G(x) \geq n$ also holds, i.e., that $G^k \cdot x \neq 0$ for $k < n$. Let $k \in \mathbb{Z}_{\geq 0}$ and $a = (a_1 \quad \cdots \quad a_{n+1})^\intercal \in C_0(S_n) = \mathbb{Z}[G]^{n+1}$ such that

$$d_{S_n}(a) = G^k \cdot x.$$

Let us prove that this implies $k \geq n$. The equation

$$\begin{pmatrix} 2a_1 + Ga_2 \\ 2a_2 + Ga_3 \\ \vdots \\ 2a_n + Ga_{n+1} \end{pmatrix} = d_{S_n}(a) = G^k \cdot x = \begin{pmatrix} G^k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

yields the following:

$$\begin{array}{lllll} 2a_n + Ga_{n+1} = 0 & \Rightarrow & a_{n+1} = -2b_n, & a_n = Gb_n & \text{for } b_n \in \mathbb{Z}[G] \\ 2a_{n-1} + Ga_n = 0 & \Rightarrow & a_n = -2Gb_{n-1}, & a_{n-1} = G^2 b_{n-1} & \text{for } b_{n-1} \in \mathbb{Z}[G] \\ \quad\vdots & & & \vdots & \quad\vdots \\ 2a_2 + Ga_3 = 0 & \Rightarrow & a_3 = -2G^{n-2}b_2, & a_2 = G^{n-1}b_2 & \text{for } b_2 \in \mathbb{Z}[G] \\ 2a_1 + Ga_2 = G^k & \Rightarrow & 2a_1 + G^n b_2 = G^k & \Rightarrow k \geq n. & \end{array}$$

This proves that $\mathrm{ord}_G(x) = n$, so $\lambda(S_n) \geq \mathfrak{u}_G(S_n) \geq n$. It follows that

$$\lambda(S_n) = \mathfrak{u}_G(S_n) = n. \qquad \blacksquare$$

Lemma 4.3. *Forgetting about quantum shifts, we have*

$$\mathrm{S}_1 \otimes \mathrm{S}_n \cong \mathrm{S}_{n+1} \oplus (\text{♘}(G) \otimes \text{♘}(2))^{\oplus n}.$$

Proof. Let $C = S_1 \otimes S_n$. The complex C is isomorphic to

$$\begin{array}{ccccc} (\mathbb{Z}[G])^{2n+2} & \xrightarrow{\quad A \quad} & (\mathbb{Z}[G])^{3n+1} & \xrightarrow{\quad B \quad} & (\mathbb{Z}[G])^n. \\ \| & & \| & & \| \\ C_0 & & C_1 & & C_2 \end{array}$$

In order to describe the maps A, B, we have to choose a basis for C_0, C_1, C_2. Let us follow the notation of Figure 10 for S_1 and S_n, and denote by a_i the generators of the R_i belonging to S_1 and by b_i the generators of the R_i in S_n. Then, C_0, C_1, C_2 are, respectively, generated by

$$\begin{pmatrix} a_1 \otimes b_1 \\ \vdots \\ a_1 \otimes b_{n+1} \\ \hline a_2 \otimes b_1 \\ \vdots \\ a_2 \otimes b_{n+1} \end{pmatrix}, \quad \begin{pmatrix} a_1 \otimes b_{n+2} \\ \vdots \\ a_1 \otimes b_{2n+1} \\ \hline a_2 \otimes b_{n+2} \\ \vdots \\ a_2 \otimes b_{2n+1} \\ \hline a_3 \otimes b_1 \\ \vdots \\ a_3 \otimes b_{n+1} \end{pmatrix}, \quad \begin{pmatrix} a_3 \otimes b_{n+2} \\ \vdots \\ a_3 \otimes b_{2n+1} \end{pmatrix}.$$

We can now write out the differentials of C:

$$A = \begin{array}{c} \begin{array}{cc} \scriptstyle n+1 & \scriptstyle n+1 \end{array} \\ \left(\begin{array}{c|c} d_{S_n} & 0 \\ \hline 0 & d_{S_n} \\ \hline 2 \cdot \mathbb{1}_{n+1} & G \cdot \mathbb{1}_{n+1} \end{array}\right) \begin{array}{l} \scriptstyle n \\ \scriptstyle n \\ \scriptstyle n+1 \end{array} \end{array}, \quad B = \begin{array}{c} \begin{array}{ccc} \scriptstyle n & \scriptstyle n & \scriptstyle n+1 \end{array} \\ \left(\begin{array}{c|c|c} 2 \cdot \mathbb{1}_n & G \cdot \mathbb{1}_n & -d_{S_n} \end{array}\right)_{n} \end{array},$$

where d_{S_n} is the $n \times (n+1)$-matrix introduced in Definition 4.1.

Let now

$$C' = \mathrm{S}_{n+1} \oplus (\text{♘}(G) \otimes \text{♘}(2))^{\oplus n}.$$

This complex is given by

$$\begin{array}{ccccc} (\mathbb{Z}[G])^{n+2} & \xrightarrow{d_{S_{n+1}}} & (\mathbb{Z}[G])^{n+1} & & \\ \oplus & & \oplus & & \\ (\mathbb{Z}[G])^{n} & \xrightarrow{A''} & (\mathbb{Z}[G])^{2n} & \xrightarrow{B''} & (\mathbb{Z}[G])^{n} \\ \shortparallel & & \shortparallel & & \shortparallel \\ C'_0 & \xrightarrow{A'} & C'_1 & \xrightarrow{B'} & C'_2 \end{array}$$

where

$$A'' = \begin{pmatrix} G & & & 0 \\ 2 & & & \\ & G & & \\ & 2 & \ddots & \\ & & & G \\ 0 & & & 2 \end{pmatrix},$$

$$B'' = \begin{pmatrix} 0 & & & & -2 & G \\ & & & \cdots & & \\ & & -2 & G & & \\ -2 & G & & & & 0 \end{pmatrix}$$

and

$$A' = \begin{array}{c} \\ n+1 \\ 2n \end{array} \begin{array}{c} \begin{array}{cc} n+2 & n \end{array} \\ \left(\begin{array}{c|c} d_{S_{n+1}} & 0 \\ \hline 0 & A'' \end{array}\right) \end{array},$$

$$B' = \begin{array}{c} \begin{array}{cc} n+1 & 2n \end{array} \\ \left(\begin{array}{c|c} 0 & B'' \end{array}\right) n. \end{array}$$

Our goal is therefore to find a change of basis to obtain C' from C. We will do this in two steps: we first define a change of basis from C to $S_{n+1} \oplus \widetilde{C}$ for some complex

$$\widetilde{C} = (\widetilde{C}_0 \xrightarrow{\widetilde{A}} \widetilde{C}_1 \xrightarrow{\widetilde{B}} \widetilde{C}_2);$$

then, we do a second change of basis that yields $(♘(G) \otimes ♘(2))^{\oplus n}$ from $\widetilde{C}$.

For the first step, we have to find two invertible matrices M, N (over $\mathbb{Z}[G]$) of dimension $2n+2$ and $3n+1$, respectively, such that

$$NAM = \left(\begin{array}{c|c} d_{S_{n+1}} & 0 \\ \hline 0 & \widetilde{A} \end{array}\right) = \begin{array}{c} \begin{array}{cc} n+2 & n \end{array} \\ \left(\begin{array}{cccc|c} 2 & G & & & \\ & \ddots & \ddots & & 0 \\ & & 2 & G & \\ \hline & 0 & & & \widetilde{A} \end{array}\right) \begin{array}{c} n+1 \\ \\ 2n \end{array} \end{array} \tag{4.1}$$

and

$$BN^{-1} = \overset{\quad n+1 \qquad\quad 2n}{\left(\begin{array}{c|c} \;0\; & \widetilde{B} \end{array}\right)_{n}}. \tag{4.2}$$

We define M as follows:

$$M = \overset{\qquad n+1 \qquad\qquad\qquad n+1}{\left(\begin{array}{cccc|cccc} & \mathbb{1}_{n+1} & & & & 0 & & \\ \hline 0 & 1 & & & & & & 1 \\ & & \ddots & & & & \cdot^{\cdot^{\cdot}} & \\ & & & 1 & & & & \\ & & & 0 & 1 & & & \end{array}\right)\begin{matrix} n+1 \\ \\ n+1 \end{matrix}}.$$

We get

$$AM = \overset{\qquad\qquad n+2 \qquad\qquad\qquad\qquad\qquad n}{\left(\begin{array}{cccccc|cccc} 2 & G & & & & 0 & & & & \\ & \ddots & \ddots & & & \vdots & & 0 & & \\ & & & 2 & G & 0 & & & & \\ \hline 0 & 2 & G & & & & & & G & 2 \\ \vdots & & \ddots & \ddots & & & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 0 & & & & 2 & G & G & & & \\ & & & & & & 2 & & & \\ \hline 2 & G & & & & 0 & & & & G \\ & \ddots & \ddots & & & \vdots & & & \cdot^{\cdot^{\cdot}} & \\ & & & 2 & G & 0 & G & & & \\ \hline 0 & \cdots & 0 & & 2 & G & 0 & \cdots & & 0 \end{array}\right)\begin{matrix} n \\ \\ n \\ \\ n \end{matrix}}.$$

In order to obtain the right-hand side of equation (4.1) from AM, we have to cancel all the pairs $2\ G$ in the blocks highlighted in light red (this can be easily done by subtracting rows of the dark blue blocks), move r_{3n+1} to row $n+1$ (where r_i indicates the i-th row), and slide down $r_{n+1}, \ldots, r_{3n}$ consequently. Let us call N the matrix expressing these operations, i.e.,

$$N = Q \cdot P_n^{3n} \cdots P_1^{2n+1} \cdot P_{3n+1}^{2n} \cdot P_n^{2n-1} \cdots P_2^{n+1},$$

where P_b^a is obtained from $\mathbb{1}_{3n+1}$ by replacing r_a with $r_a - r_b$, and Q is the matrix expressing the appropriate reordering of rows $r_{n+1}, \ldots, r_{3n+1}$. It is thus clear that N satisfies equation (4.1). It is straightforward to check that N^{-1} also satisfies (4.2). Thus, we showed that

$$S_1 \otimes S_n \cong S_{n+1} \oplus \widetilde{C}.$$

The next step is to define a change of basis from $\widetilde{C}$ to $(\text{♘}(G) \otimes \text{♘}(2))^{\oplus n}$. For that purpose, we have to explicitly describe $\widetilde{A}$ and $\widetilde{B}$:

$$\widetilde{A} = \left(\begin{array}{cccc} & & G & 2 \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ G & \cdot^{\cdot^{\cdot}} & & \\ 2 & & & \\ \hline & & & G \\ & & \cdot^{\cdot^{\cdot}} & \\ G & & & \end{array}\right)\begin{array}{l} \\ n \\ \\ \\ n \\ \\ \end{array}, \quad \widetilde{B} = \overset{\qquad\quad n \qquad\qquad\qquad\qquad n}{\left(\begin{array}{ccc|cccc} G & & & -2 & -G & & \\ & \ddots & & & \ddots & \ddots & \\ & & & & & \ddots & -G \\ & & G & & & & -2 \end{array}\right)}.$$

We can easily obtain A'' from $\widetilde{A}$ by removing the G entries in the light red block (by subtracting rows of the dark blue block) and then reordering the rows appropriately. These operations are expressed by the matrix

$$L = \widetilde{Q} \cdot P_{2n}^{n-1} \cdots P_{n+2}^{1},$$

where P_b^a is obtained from $\mathbb{1}_{2n}$ by replacing r_a with $r_a - r_b$ and Q is the matrix expressing the appropriate reordering of the rows. One can check that $L\widetilde{A} = A''$ and $\widetilde{B}L^{-1} = B''$. This concludes the proof of the lemma. ∎

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Lemmas 4.2 and 4.3, together with the fact that

$$[\![K_1 \# K_2]\!] \cong [\![K_1]\!] \otimes [\![K_2]\!]$$

for any two knots K_1, K_2 (see (2.4)), are enough to construct knots with arbitrarily big λ. Indeed, let us consider the knot $K = 14n19265$. This knot was used by Seed to show that $s(K) \neq s_{\mathbb{F}_2}(K)$ [31, 46], where s is the classical Rasmussen invariant over $\mathbb{Q}$ and $s_{\mathbb{F}_2}$ is the invariant computed over $\mathbb{F}_2$. We observe using `khoca` and `homca` that the $\mathbb{Z}[G]$-complex $[\![K]\!]$ decomposes into a sum of a staircase S_1 and finitely many $\text{♘}(G)$ and $\text{♘}(G) \otimes \text{♘}(2)$. Therefore, by Corollary 3.5,

$$\lambda(K) \leq \max(\lambda(S_1), \quad \lambda(\text{♘}(G), 0), \quad \lambda(\text{♘}(G) \otimes \text{♘}(2), 0)) = 1.$$

Since $K \neq U$, it follows that $\lambda(K) = 1$. By Proposition 1.9, given $n \in \mathbb{Z}_{>0}$, we then have

$$\lambda(K^{\# n}) \leq n \cdot \lambda(K) = n.$$

On the other hand, it follows from Lemma 4.3 that $[\![K^{\# n}]\!] \cong S_n \oplus C$ for some chain complex C. We know that $\mathfrak{u}_G(S_n \oplus C) = \max(\mathfrak{u}_G(S_n), \mathfrak{u}_G(C))$ (cf. Lemma 3.28), so

$$\lambda(K^{\#n}) = \lambda(S_n \oplus C) \geq \mathfrak{u}_G(S_n \oplus C) \geq \mathfrak{u}_G(S_n) = n.$$

This proves that $\lambda(K^{\#n}) = n$ for all $n \geq 0$. ∎

The fact that $\lambda(K^{\#n}) = n$ will also follow from Proposition 4.4.

4.2. Further calculations

Proposition 4.4. *Let $K_1, \dots, K_n$ and $J_1, \dots, J_m$ be knots such that*

(1) *for all $i = 1, \dots, n$ the complex $[\![K_i]\!]$ splits as a sum of one staircase S_1 and finitely many $\text{♘}(G)$ and $\text{♘}(G) \otimes \text{♘}(2)$ pieces,*

(2) *for all $j = 1, \dots, m$ the complex $[\![J_j]\!]$ decomposes into a sum of one dual staircase $\overline{S_1}$ and finitely many $\text{♘}(G)$ and $\text{♘}(G) \otimes \text{♘}(2)$.*

Let the empty # be equal to the unknot. Then,

$$\lambda\big(\mathop{\#}_{i\leq n} K_i \# \mathop{\#}_{j\leq m} J_j\big) = \begin{cases} |n-m| & \text{if } n \neq m \text{ and } m,n \geq 0, \\ 1 & \text{if } n = m \neq 0, \\ 0 & \text{if } n = m = 0. \end{cases}$$

Remark 4.5. Using `khoca` and `homca`, one finds that there are many knots satisfying requirements (1) or (2) of Proposition 4.4. For instance, one can take any knot with up to 15 crossings such that $s_{\mathbb{F}_2} \neq s_{\mathbb{F}_3}$. One of those is the above-mentioned knot $14n19265$, and a complete list is given in [44, 45].

We also note that if a knot K satisfies condition (1) of Proposition 4.4, then its mirror image $-K$ will satisfy condition (2), and vice versa.

For the proof of Proposition 4.4, we will need the following lemmas.

Lemma 4.6. *Ignoring quantum shifts, we have*

$$\text{♘}(G) \otimes S_n \cong \text{♘}(G) \otimes \overline{S_n} \cong (\text{♘}(G) \otimes \text{♘}(2))^{\oplus n} \oplus \text{♘}(G)$$

and

$$\text{♘}(2) \otimes S_n \cong \text{♘}(2) \otimes \overline{S_n} \cong (\text{♘}(G) \otimes \text{♘}(2))^{\oplus n} \oplus \text{♘}(2).$$

Proof. We proceed very similarly to the proof of Lemma 4.3. We will only prove that

$$\text{♘}(G) \otimes S_n \cong (\text{♘}(G) \otimes \text{♘}(2))^{\oplus n} \oplus \text{♘}(G),$$

as the proofs of the remaining statements are very similar.

The complex ♘$(G) \otimes S_n$ is isomorphic to

$$(\mathbb{Z}[G])^{n+1} \xrightarrow{A} (\mathbb{Z}[G])^{2n+1} \xrightarrow{B} (\mathbb{Z}[G])^{n}$$

with

$$A = \left(\begin{array}{cccc} 2 & G & & \\ & \ddots & \ddots & \\ & & 2 & G \\ \hline G & & & \\ & \ddots & & \\ & & & G \end{array}\right)\begin{array}{l} n \\ \\ n+1 \end{array}, \quad B = \left(\begin{array}{ccc|cccc} G & & & -2 & -G & & \\ & \ddots & & & \ddots & \ddots & \\ & & G & & & -2 & -G \end{array}\right)$$

where, in B, the left block has n columns and the right block has $n+1$ columns,

and basis given by

$$\begin{pmatrix} a_1 \otimes b_1 \\ \vdots \\ a_1 \otimes b_{n+1} \end{pmatrix}, \quad \begin{pmatrix} a_1 \otimes b_{n+2} \\ \vdots \\ a_1 \otimes b_{2n+1} \\ \hline a_2 \otimes b_1 \\ \vdots \\ a_2 \otimes b_{n+1} \end{pmatrix}, \quad \begin{pmatrix} a_2 \otimes b_{n+2} \\ \vdots \\ a_2 \otimes b_{2n+1} \end{pmatrix}.$$

(Here, b_i denotes the generator of R_i in S_n and a_1, a_2 are, respectively, the generators of $\mathbb{Z}[G]$ in homological degrees 0 and 1 of ♘(G).)

The statement of the lemma holds if we can find a change of basis N such that

$$NA = \left(\begin{array}{cccc|c} G & & & & 0 \\ 2 & & & & \vdots \\ & G & & & \vdots \\ & 2 & \ddots & & \vdots \\ & & \ddots & G & \vdots \\ & & & 2 & 0 \\ \hline 0 & \cdots & \cdots & 0 & G \end{array}\right) \tag{4.3}$$

and

$$BN^{-1} = \left(\begin{array}{cccccc|c} -2 & G & & & & & 0 \\ & & -2 & G & & & \vdots \\ & & & \ddots & \ddots & & \vdots \\ & & & & -2 & G & 0 \end{array}\right). \tag{4.4}$$

In order to obtain the right-hand side of (4.3) from A, we need to perform the following operations: get rid of the G entries in the light red block of A (by subtracting rows of the dark blue block) and then reorder the rows appropriately. (Let us call Q the matrix expressing this second step.) Then, the change of basis

$$N = Q \cdot P^{n}_{2n+1} \cdots P^{1}_{n+2}$$

clearly satisfies equation (4.3). Some simple calculations show that N also satisfies (4.4). ∎

Lemma 4.7. *Let $z \in \mathbb{Z}[G]$ and $a, b \in \mathbb{Z}_{\geq 0}$ with $a \leq b$. Then,*

$$\text{♘}(z^a) \otimes \text{♘}(z^b) \cong \text{♘}(z^a) \oplus \text{♘}(z^a)$$

(quantum shifts are omitted).

Proof. The complex $\text{♘}(z^a) \otimes \text{♘}(z^b)$ is isomorphic to

$$\mathbb{Z}[G] \xrightarrow{(z^b\ z^a)^{\intercal}} \mathbb{Z}[G] \oplus \mathbb{Z}[G] \xrightarrow{(z^a\ -z^b)} \mathbb{Z}[G].$$

Consider the matrix $N = \left(\begin{smallmatrix} 1 & -z^{b-a} \\ 0 & 1 \end{smallmatrix}\right)$. We have

$$N \cdot \begin{pmatrix} z^b \\ z^a \end{pmatrix} = \begin{pmatrix} 0 \\ z^a \end{pmatrix}, \quad \begin{pmatrix} z^a & -z^b \end{pmatrix} \cdot N^{-1} = \begin{pmatrix} z^a & 0 \end{pmatrix}.$$

The matrix N is therefore the desired change of basis from $\text{♘}(z^a) \otimes \text{♘}(z^b)$ to $\text{♘}(z^a) \oplus \text{♘}(z^a)$. ∎

Lemma 4.8. *The following are isomorphisms of (ungraded) chain complexes:*

$$S_1 \otimes \overline{S_1} \cong (\text{♘}(G) \otimes \text{♘}(2))^{\oplus 2} \oplus \text{♙}, \tag{4.5}$$

$$(\text{♘}(G) \otimes \text{♘}(2)) \otimes S_1 \cong (\text{♘}(G) \otimes \text{♘}(2)) \otimes \overline{S_1} \cong (\text{♘}(G) \otimes \text{♘}(2))^{\oplus 3}, \tag{4.6}$$

$$(\text{♘}(G) \otimes \text{♘}(2)) \otimes (\text{♘}(G) \otimes \text{♘}(2)) \cong (\text{♘}(G) \otimes \text{♘}(2))^{\oplus 4}, \tag{4.7}$$

$$\text{♘}(G) \otimes (\text{♘}(G) \otimes \text{♘}(2)) \cong (\text{♘}(G) \otimes \text{♘}(2))^{\oplus 2}. \tag{4.8}$$

Proof. The chain complex $S_1 \otimes \overline{S_1}$ can be written as

$$(\mathbb{Z}[G])^2 \xrightarrow{A} (\mathbb{Z}[G])^5 \xrightarrow{B} (\mathbb{Z}[G])^2,$$

where

$$A = \begin{pmatrix} G & 0 \\ 2 & 0 \\ 0 & G \\ 0 & 2 \\ 2 & G \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & G & 0 & -G \\ 0 & 2 & 0 & G & -2 \end{pmatrix}.$$

Consider the change of basis matrices

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 \end{pmatrix}$$

and let $A' = M^{-1}AL$ and $B' = BM$. One can verify that A' and B' are the differentials of the chain complex

$$(\text{♘}(2) \otimes \text{♘}(G))^{\oplus 2} \oplus \text{♙} \cong (\text{♘}(G) \otimes \text{♘}(2))^{\oplus 2} \oplus \text{♙},$$

which proves equation (4.5).

The first isomorphism of equation (4.6) is given by Lemma 4.6. The following shows that $(\text{♘}(G) \otimes \text{♘}(2)) \otimes S_1 \cong (\text{♘}(G) \otimes \text{♘}(2))^{\oplus 3}$:

$$\begin{aligned} \text{♘}(G) \otimes \text{♘}(2) \otimes S_1 &\cong \text{♘}(G) \otimes ((\text{♘}(G) \otimes \text{♘}(2)) \oplus \text{♘}(2)) \\ &\cong (\text{♘}(G) \otimes \text{♘}(G) \otimes \text{♘}(2)) \oplus (\text{♘}(G) \otimes \text{♘}(2)) \\ &\cong ((\text{♘}(G) \oplus \text{♘}(G)) \otimes \text{♘}(2)) \oplus (\text{♘}(G) \otimes \text{♘}(2)) \\ &\cong (\text{♘}(G) \otimes \text{♘}(2)) \oplus (\text{♘}(G) \otimes \text{♘}(2)) \oplus (\text{♘}(G) \otimes \text{♘}(2)) \\ &\cong (\text{♘}(G) \otimes \text{♘}(2))^{\oplus 3}, \end{aligned}$$

where the first isomorphism is given by Lemma 4.6 and the third by Lemma 4.7. Lastly, equations (4.7) and (4.8) follow easily from Lemma 4.7. ■

We can now turn to the proof of Proposition 4.4.

Proof of Proposition 4.4. We remind the reader that for two knots K_1, K_2 we have $[\![K_1 \# K_2]\!] \cong [\![K_1]\!] \otimes [\![K_2]\!]$ (see (2.4)). Let $L = \underset{i \leq n}{\#} K_i \# \underset{j \leq m}{\#} J_j$. If $n = m = 0$, then

$$\lambda(L) = \lambda(U) = 0.$$

We now consider $\{n, m\} \neq \{0\}$. Using equations (4.5) to (4.8), Lemmas 4.6 and 4.7, we find that for all $i, j \geq 1$ the complex $[\![K_i \# J_j]\!]$ splits as a sum of the following pieces:

$$\text{♙}, \quad \text{♘}(G), \quad \text{♘}(G) \otimes \text{♘}(2).$$

The same pieces also give a decomposition of $[\![\underset{i,j \geq 1}{\#} (K_i \# J_j)]\!]$.

If $n = m \neq 0$, then $L = \underset{1 \leq i \leq m}{\#} (K_i \# J_i)$. Using Corollary 3.5 and the fact that

$$\lambda(\text{♘}(G), 0) = \lambda(\text{♘}(G) \otimes \text{♘}(2), 0) = 1,$$

one obtains

$$\lambda(L) = \lambda\Big(\mathop{\#}_{1\le i\le m}(K_i \# J_i)\Big) \le \max\big(\lambda(\pawn),\ \lambda(\knight(G), 0),\ \lambda(\knight(G) \otimes \knight(2), 0)\big) = 1.$$

We also have $\lambda(L) \ge 1$ by Proposition 1.8, since $L \ne U$. This shows that $\lambda(L) = 1$.

Let now $n > m \ge 0$. We have

$$L = \mathop{\#}_{j\le m}(K_j \# J_j) \# \mathop{\#}_{m+1\le i\le n} K_i.$$

It is easy to see, using equations (4.6) to (4.8), Lemmas 4.3, 4.6, and 4.7 that $\llbracket \mathop{\#}_{m+1\le i\le n} K_i \rrbracket$ splits as a sum of

$$\knight(G), \quad \knight(G) \otimes \knight(2), \quad S_{n-m}.$$

Now,

$$\llbracket L \rrbracket \cong \llbracket \mathop{\#}_{j\le m}(K_j \# J_j) \rrbracket \otimes \llbracket \mathop{\#}_{m+1\le i\le n} K_i \rrbracket,$$

therefore equations (4.6) to (4.8), along with Lemmas 4.6 and 4.7, show that the same pieces also give a decomposition of $\llbracket L \rrbracket$. Thus, in order to prove that $\lambda(L) \le n - m$, all we have to do is apply Corollary 3.5, which yields

$$\lambda(L) \le \max(\lambda(S_{n-m}), \lambda(\knight(G), 0), \lambda(\knight(G) \otimes \knight(2), 0)) = n - m.$$

The inequality $\lambda(L) \ge n - m$ also holds: the complex $\llbracket \mathop{\#}_{j\le m}(K_j \# J_j) \rrbracket$ has a $\pawn$ piece, and $\llbracket \mathop{\#}_{m+1\le i\le n} K_i \rrbracket$ has a S_{n-m} piece, so there is a piece $S_{n-m} \cong \pawn \otimes S_{n-m}$ in $\llbracket L \rrbracket$. Using that $\mathfrak{u}_G(S_{n-m}) = n - m$ (cf. Lemma 4.2) and Lemma 3.28, one finds $\lambda(L) \ge \mathfrak{u}_G(L) = n - m$. It follows that $\lambda(L) = n - m$.

Lastly, let $m > n \ge 0$. Then,

$$L = \mathop{\#}_{i\le n}(K_i \# J_i) \# \mathop{\#}_{n+1\le j\le m} J_j,$$

and the only pieces appearing in $\llbracket L \rrbracket$ are

$$\knight(G), \quad \knight(G) \otimes \knight(2), \quad \overline{S_{m-n}}.$$

It follows that the pieces appearing in $\llbracket -L \rrbracket = \llbracket \overline{L} \rrbracket$ are

$$\knight(G), \quad \knight(G) \otimes \knight(2), \quad S_{m-n}.$$

Hence, by Proposition 1.9 and looking at the proof of the case $n > m$ just above, one finds $\lambda(L) = \lambda(-L) = m - n$. ∎

Remark 4.9. It is easy to see from the above proof that a similar result as Proposition 4.4 also holds for $\mathfrak{u}_G$. Namely, if $K_1, \dots, K_n, J_1, \dots, J_m$ satisfy conditions (1) and (2) of Proposition 4.4, we have

$$\mathfrak{u}_G\big(\underset{i\leq n}{\#} K_i \# \underset{j\leq m}{\#} J_j\big) = \begin{cases} n-m & \text{if } n > m \geq 0, \\ 1 & \text{if } n = m \neq 0 \text{ or } m > n \geq 0, \\ 0 & \text{if } n = m = 0. \end{cases}$$

The partial difference is due to the fact that $\mathfrak{u}_G(\overline{S_k}) = 0$, while $\lambda(\overline{S_k}) = \lambda(S_k) = k$ for all $k \geq 1$.

4.3. λ of small knots

We start this subsection by computing λ for all knots with up to 10 crossings.

Proposition 1.14. *For all knots up to* 10 *crossings, we have* $\lambda = 1$*, except for the knots* 8_{19}, 10_{124}, 10_{128}, 10_{139}, 10_{152}, 10_{154}, 10_{161}, *where* $\lambda = 2$.

Proof. By Proposition 1.10, if a knot is thin, then $\lambda = 1$, so it suffices to look at knots which are not thin. Among the knots with up to 10 crossings, there are twelve knots that are thick:

$$8_{19},\ 9_{42},\ 10_{124},\ 10_{128},\ 10_{132},\ 10_{136},\ 10_{139},\ 10_{145},\ 10_{152},\ 10_{153},\ 10_{154},\ 10_{161}.$$

Using `khoca` and `homca`, one can compute that the $\mathbb{Z}[G]$-complex of the knots 9_{42}, 10_{132}, 10_{136}, 10_{145}, 10_{153} decomposes into a sum of a ♙ and several ♘(G) pieces; hence, $\lambda = 1$ by Lemma 3.30. The $\mathbb{Z}[G]$-complex of the remaining knots 8_{19}, 10_{124}, 10_{128}, 10_{139}, 10_{152}, 10_{154}, 10_{161} decomposes into a sum of a ♙, several ♘(G) pieces, and a single ♘(G^2) piece. Using Corollary 3.5 and Lemma 3.28, one obtains $\lambda = 2$ for these knots. ∎

A natural question to ask when introducing a new invariant is how it compares to other already existing invariants. For example, how does λ compare to the classical 3-genus g of a knot K? We know that λ is a lower bound for the unknotting number u, while g can be a lower or upper bound for u depending on the knot. For instance, Lee–Lee [25] showed that, for all knots with braid index ≤ 3, the inequality $u(K) \leq g(K)$ holds. However, this is no longer true for knots with braid index ≥ 4: as pointed out in their work, there are six knots with braid-index 4 and at most 9 crossings for which $u > g$ holds. How does λ fit into this scheme? For knots up to 12 crossings, we can provide the following answer.

Proposition 4.10. *For all knots up to* 12 *crossings, the* 3*-genus* g *is an upper bound for* λ.

K	$u(K)$	$g(K)$
9_{46}	2	1
$11n_{139}$	2	1
$12n_{203}$	3 or 4	3
$12n_{260}$	2 or 3	2
$12n_{404}$	2 or 3	2
$12n_{432}$	2 or 3	2
$12n_{554}$	3	2
$12n_{642}$	3 or 4	2
$12n_{764}$	3 or 4	3
$12n_{809}$	1, 2 or 3	2
$12n_{851}$	3 or 4	3

Table 1. Non-quasi-alternating prime knots with up to 12 crossings for which (possibly) $g < u$ holds.

Proof. Since λ is a lower bound for the unknotting number u, it is sufficient to consider knots with up to 12 crossings for which (possibly) $g < u$ holds. Using that quasi-alternating knots are thin and that for thin knots $\lambda = 1$ (cf. Proposition 1.10), there are 11 non-quasi-alternating knots with at most 12 crossings with (possibly) $g < u$. They were found using Livingston's wonderful KnotInfo [32] and Jablan's table of quasi-alternating knots for up to 12 crossings [16]. The knots are listed in Table 1.

A computation using `khoca` and its extension `homca` showed that the $\mathbb{Z}[G]$-complex of all knots in Table 1 decomposes into ♙ and ♘(G) summands. By Lemma 3.30, this implies that $\lambda = 1$ for all knots in Table 1. ∎

Proposition 4.10 raises the following question.

Question 4.11. Does $\lambda(K) \leq g(K)$ hold for all knots K?

5. Rational tangles and the λ-invariant

5.1. The $\mathbb{Z}[G]$-homology of rational tangles

Definition 5.1. A 4-ended (oriented or unoriented) tangle T is called *rational* if the pair (B, T) is homeomorphic to $(D^2 \times [0,1], \{(-\frac{1}{2}, 0), (\frac{1}{2}, 0)\} \times [0,1])$, drawn in Figure 11.

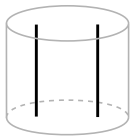


Figure 11. The rational tangle $(D^2 \times [0,1], \{(-\frac{1}{2}, 0), (\frac{1}{2}, 0)\} \times [0,1])$.

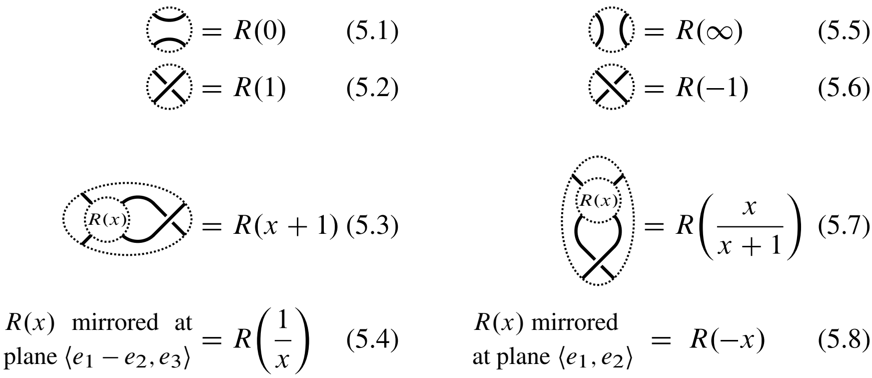


Figure 12. The recursive definition of the bijection R between $\mathbb{Q} \cup \{\infty\}$ and equivalence classes of unoriented rational tangles. In (5.4) and (5.8), e_1, e_2, e_3 denote the standard basis vectors of $\mathbb{R}^3$.

Let us briefly summarize Conway's famous one-to-one correspondence

$$R\colon \mathbb{Q} \cup \{\infty\} \to \{\text{unoriented rational tangles}\}/\text{equivalence}.$$

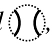
See, e.g., [9] for an introduction to this topic. Let us work with unoriented tangles in the unit ball $B_0 \subset \mathbb{R}^3 \subset S^3$ with the four end points $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}, 0)$ and base point $(-1/\sqrt{2}, -1/\sqrt{2}, 0)$. Generically, the projection to $D^2 \times \{0\}$ yields tangle diagrams; these are the tangle diagrams we consider in what follows. Then, R may be defined by the rules in Figure 12 (where we set $1/\infty = 0$ and $1/0 = \infty = \infty + 1 = -\infty$). By a slight abuse of notation, we denote by $R(x)$ both the tangle and the tangle diagram (both well defined up to equivalence).

As stated, these rules are consistent and determine the correspondence R completely, but they are somewhat redundant: for example, (5.5) to (5.7) can be derived from the other rules. For simplicity, we will now focus on rational tangles T such that $R^{-1}(T) \in \mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$ (in particular excluding $R(0)$ and $R(\infty)$). The one-to-one correspondence between such rational tangles and $\mathbb{Q}^+$ is completely determined by (5.2), (5.3), (5.4), or by (5.2), (5.3), (5.7).

Thompson [50] has computed Khovanov homology of all oriented rational tangles. More precisely, he shows that the complex of a rational tangle in Bar-Natan's category of cobordisms with dots is homotopy equivalent to a so-called zigzag complex, which may be computed recursively. As a crucial ingredient for the proof of Theorem 1.1, we require the analog of Thompson's theorem for Bar-Natan's theory *without* dots (stated below as Theorem 5.6), which is more general than the dotted theory. Fortunately, Thompson's proof carries over mutatis mutandis to that more general homology theory.

Let us mention that Kotelskiy, Watson, and Zibrowius have given an elegant way to compute the Bar-Natan complex over $\mathbb{F}_2$ of rational tangles using immersed curves [22, Example 6.2]. Potentially, their methods also work over the integers, which would give an alternative proof of Theorem 5.6.

Let us now have a look at the category $\mathrm{Cob}^{3}_{/l}{}^{\bullet}(4)$ of 4-ended crossingless tangle diagrams with base point and cobordisms between them. Recall that for two end points, we have found that $\mathrm{Cob}^{3}_{/l}{}^{\bullet}(2)$ is equivalent to $\mathcal{M}_{\mathbb{Z}[G]}$ (see Section 2). For four end points, a similar strategy of delooping and simplifying cobordisms leads to the following, which is just a reformulation of [22, Theorem 1.1].

Theorem 5.2. *Consider the $\mathbb{Z}[G]$-enriched category with the two objects*  *and* , *and graded $\mathbb{Z}[G]$-morphism modules generated by the identity cobordisms, which we both denote by I, and the saddle cobordisms*

 $\to$  *and*  $\to$ ,

which we both denote by S, and compositions of these morphisms, modulo the relations

$$S^3 = GS.$$

Then, the inclusion of the additive graded closure of this category into $\mathrm{Cob}^{3}_{/l}{}^{\bullet}(4)$ *is an equivalence of categories.* ∎

This theorem gives us a compact notation for $\mathrm{Cob}^{3}_{/l}{}^{\bullet}(4)$: objects are isomorphic to shifted sums of  and , and morphisms are equal to $\mathbb{Z}[G]$-linear combinations of I, S, and S^2. For convenience, we write $D := S^2 - G$ (following [22]). Note that

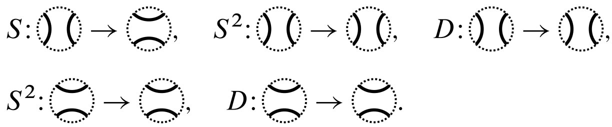
$$SD = DS = 0 \quad \text{and} \quad D^2 = -GD.$$

For the rest of the section, we will for the most part omit homological and quantum gradings without further mention.

Definition 5.3. A *zigzag complex* is a graded chain complex $(\bigoplus_{i=0}^{n} A_i, \sum_{i=1}^{n} d_i)$ over $\mathrm{Mat}(\mathrm{Cob}^{3}_{/l}{}^{\bullet}(4))$ satisfying the following.

(i) Each A_i is either  or  with a quantum and homological degree shift.

(ii) Each d_i has domain and target $A_{i-1} \to A_i$ or $A_i \to A_{i-1}$ and is one of the following five maps:



(iii) Two consecutive differentials d_i, d_{i+1} (no matter what their domain and target are) are either S and D, or S^2 and D.

(iv) There is at least one differential S.

The A_i and d_i are considered as part of the data of the zigzag complex. We say that the zigzag complex $(\bigoplus_{i=0}^{n} A_{n-i}, \sum_{i=1}^{n} d_{n+1-i})$ is obtained from $(\bigoplus_{i=0}^{n} A_i, \sum_{i=1}^{n} d_i)$ by *reindexing*. Note that reindexing does not change the isomorphism type of the chain complexes.

We would like to depict zigzag complexes using the following type of graphs.

Definition 5.4. Let us consider a directed finite graph with two *types* of vertices, $\bullet$ and $\circ$. Let us call an edge connecting a $\bullet$ and a $\circ$ vertex a *saddle edge*. Such a graph is called a *zigzag graph* if it satisfies the following conditions.

(i) The graph has the shape of a line; i.e., there are exactly two vertices of valency 1 (which we call the *ends*), and all other vertices have valency 2.

(ii) There is a partition of edges into *odd* and *even* edges such that all saddle edges are odd, and if two edges are adjacent, then one of them is odd and the other one even.

(iii) All saddle edges are directed like this: $\circ \to \bullet$.

(iv) There is at least one saddle edge.

Note that because there is at least one saddle edge, the partition of edges into odd and even edges is unique.

Definition 5.5. The *graph of a zigzag complex* is the zigzag graph with a vertex $\bullet$ or $\circ$ corresponding to each A_i that is a shift of  or , respectively, and one directed edge corresponding to each d_i.

One easily checks that the graph of a zigzag complex really is a zigzag graph. Moreover, every zigzag graph is the graph of a zigzag complex; and the graph of a zigzag complex determines the zigzag complex up to reindexing, and up to global shifts in homological and quantum degree. The correspondence between zigzag complexes and zigzag graphs is summarized in Figure 13.

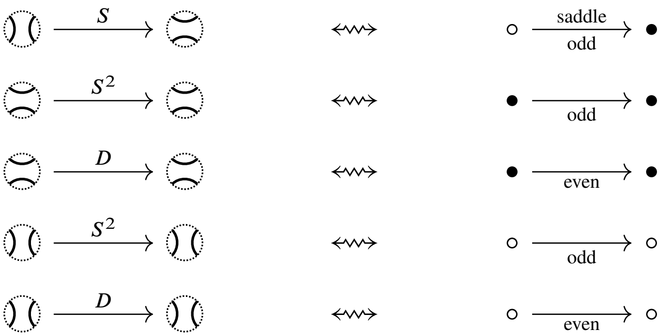


Figure 13. Summary of the correspondence between objects and differentials of zigzag complexes (left column) and vertices and edges of zigzag graphs (right columns).

Let us now recursively define a zigzag graph $zz(x)$ for all positive rational numbers $x \in \mathbb{Q}^+$ by the following rules:

(ZZ1) $zz(1) = \circ \to \bullet$.

(ZZ2) $zz(1/x)$ is obtained from $zz(x)$ by switching $\circ$ and $\bullet$ and reversing the directions of all edges.

(ZZ3) $zz(x+1)$ is obtained from $zz(x)$ by replacing each edge as shown in Table 2.[11]

Note that these cases are exhaustive, since two adjacent $\bullet$-vertices cannot both be ends (because there is at least one saddle edge), and since a saddle edge is always directed from $\circ$ to $\bullet$.

Let us check that zz is well defined. Indeed, $\circ \to \bullet$ is a zigzag graph, and one may verify that (ZZ2) and (ZZ3) map zigzag graphs to zigzag graphs. Every positive rational number can be obtained from 1 by a sequence of $x \mapsto 1/x$ and $x \mapsto x+1$. Moreover, that sequence is unique up to inserting or removing two consecutive $x \mapsto 1/x$. Since applying (ZZ2) twice has no effect, (ZZ1), (ZZ2), and (ZZ3) indeed define $zz(x)$ for every positive rational x.

We are now ready to state our generalization of Thompson's theorem.

Theorem 5.6. *Let $R(x)$ be the unoriented rational tangle corresponding to a positive rational number x. Let T be the tangle $R(x)$ equipped with some orientation o. Then, the Bar-Natan complex $[T]^\bullet$ is homotopy equivalent to a zigzag complex with graph $zz(x)$.*

[11] After the article at hand had appeared as a preprint, it was observed that the left-hand side of the fourth row in Table 2 (end $\bullet \xrightarrow[\text{even}]{} \bullet$ not an end) does not actually occur in any $zz(x)$, and so, the fourth row may be disregarded [29, Lemma A.1].

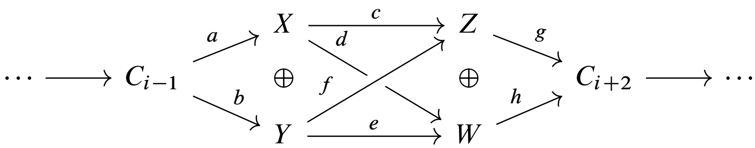
Replace	by
$\bullet \xrightarrow[\text{odd}]{} \bullet$	$\bullet \longleftarrow \circ \longrightarrow \circ \longrightarrow \bullet$
not an end $\bullet \xrightarrow[\text{even}]{} \bullet$ not an end	$\bullet \longrightarrow \bullet$
not an end $\bullet \xrightarrow[\text{even}]{} \bullet$ end	$\bullet \longrightarrow \bullet \longleftarrow \circ$
end $\bullet \xrightarrow[\text{even}]{} \bullet$ not an end	$\circ \longrightarrow \bullet \longrightarrow \bullet$
$\circ \longrightarrow \bullet$	$\circ \longrightarrow \circ \longrightarrow \bullet$
$\circ \longrightarrow \circ$	$\circ \longrightarrow \circ$

Table 2. How to obtain $zz(x+1)$ from $zz(x)$: each edge e in $zz(x)$ falls into a unique one of the six cases shown in the left column of the table. Apply the rule, i.e., replace e by the graph Γ_e in the right column in the same row. In this way, each of the two vertices v, w adjacent to e in $zz(x)$ are replaced by vertices v_e, w_e in $zz(x+1)$, namely, the leftmost and the rightmost vertex in Γ_e. If two edges e and f of $zz(x)$ are adjacent to a common vertex v, identify the vertices v_e and v_f in $zz(x+1)$. Note that this is possible since v_e and v_f always have the same type: in fact, v_e has the same type as v if v (or equivalently, v_e) is not an end.[11]

Remark 5.7. Since the Bar-Natan complex of the mirror image of a tangle is isomorphic to the dual of the Bar-Natan complex of that tangle, Theorem 5.6 yields a rather simple representative of the homotopy equivalence class of the Bar-Natan complex of any rational tangle, up to global shifts in homological and quantum degree. These shifts depend on the orientation of the tangle; since they do not matter for our work on λ, we will neglect them. Thompson computes the shifts in [50, Theorem 5.1].

The proof will use Bar-Natan's computation method of delooping and Gaussian elimination [6]. We have described delooping in Figure 5 in Section 2.3. By Gaussian elimination, we mean the following.

Lemma 5.8. *Assume that (C, d) is a chain complex in some additive category taking the following form:*



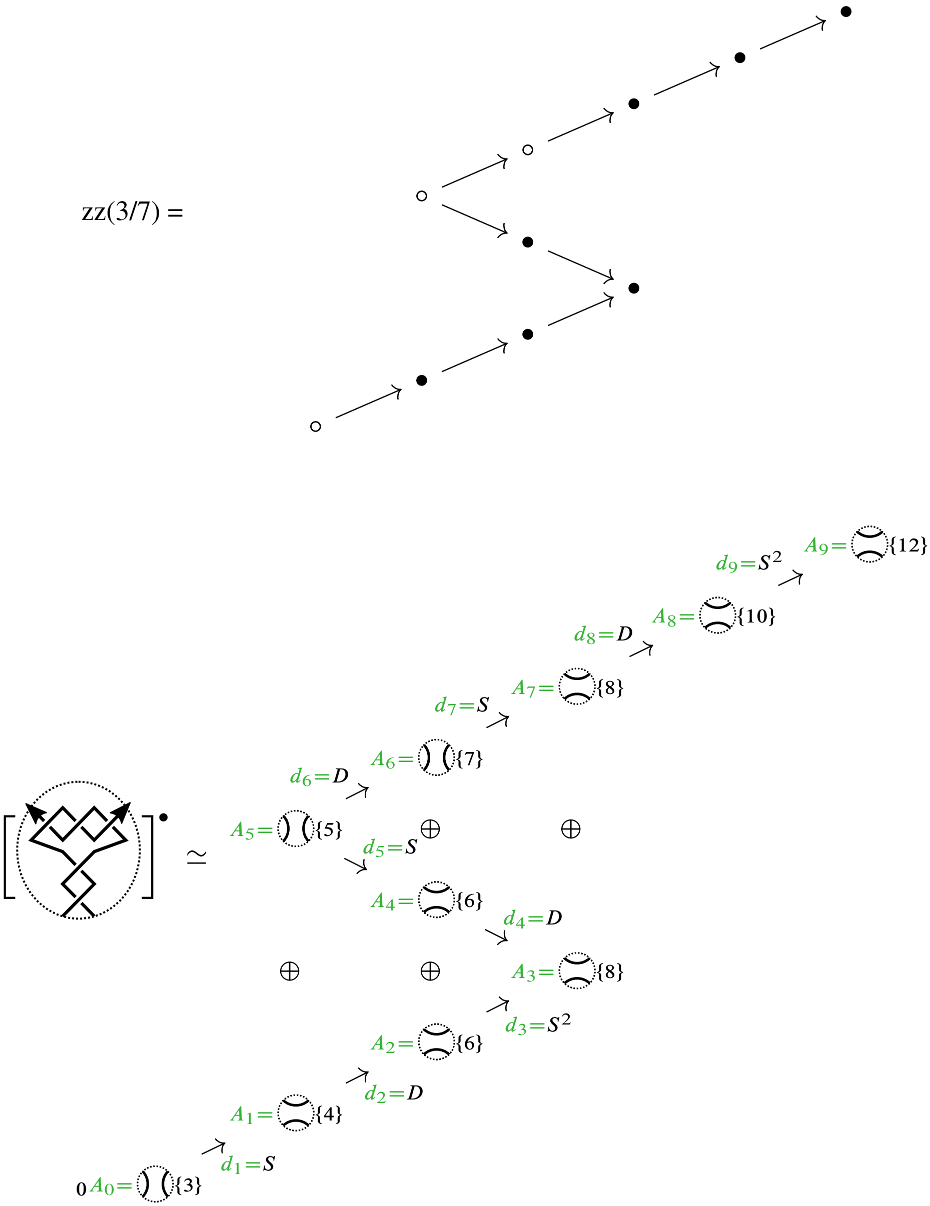


Figure 14. An illustration of the correspondence between zigzag graphs and zigzag complexes and of Theorem 5.6. On the top is shown the zigzag graph $zz(3/7)$; on the bottom is shown a zigzag complex that is homotopy equivalent to the Bar-Natan complex of the rational tangle $R(3/7)$ endowed with some orientation. The left subscript gives the homological degree.

with e an isomorphism. Then, C is homotopy equivalent to

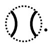
$$\cdots \to C_{i-1} \xrightarrow{a} X \xrightarrow{c-fe^{-1}d} Z \xrightarrow{g} C_{i+2} \to \cdots . \qquad \blacksquare$$

Using Gaussian elimination as stated in the above lemma, one may eliminate the domain and target of an isomorphism in a chain complex by paying the price of introducing a new differential $f \circ e^{-1} \circ d$.

Proof of Theorem 5.6. We proceed by induction over the number of transformations

$$y \mapsto 1/y \quad \text{and} \quad y \mapsto y+1$$

necessary to reach x from 1. For $x = 1$, observe that $[T]^\bullet$ is homotopy equivalent (in fact isomorphic) to a zigzag complex with graph equal to $zz(1)$, by the definition of the Bar-Natan complex of a tangle consisting of a single crossing.

So, let us now assume that the statement holds for x. To show that the statement also holds for $1/x$, observe that the tangle diagrams $R(x)$ and $R(1/x)$ are related by mirroring at the line $\mathbb{R}\binom{1}{-1}$. Equivalently, one may first rotate $R(x)$ by $\pi/2$, getting a tangle diagram D_T, and then switch all crossings in D_T, getting $R(1/x)$. For all three tangle diagrams $R(x)$, D_T, and $R(1/x)$, the base point is the lower left end point (as it is for all four-ended tangles, by the convention fixed at the beginning of this section). Let C be the zigzag complex homotopy equivalent to $[R(x)]^\bullet$, provided by the induction hypothesis. Since the Bar-Natan complex is defined geometrically, a complex C' homotopy equivalent to $[D_T]^\bullet$ may be obtained from C simply by rotating all crossingless tangle diagrams and all cobordisms in C by $\pi/2$. This switches  and . The effect on cobordisms is a little more subtle: since the base point always remains at the lower left end point, rotation by $\pi/2$ does not commute with the action of G. Thus, one finds (using the $4Tu$-relation, see Figure 3) that the rotation sends I to I, S to S, D:  $\to$  to D:  $\to$ , but sends

D:  $\to$  to $-D$:  $\to$ .

However, because of the linear shape of C', there is a simple change of basis that gets rid of the introduced minus signs, yielding a complex C'' isomorphic to C'. Finally, $[R(1/x)]^\bullet$ is homotopy equivalent to the dual of C''; naively, the dual of C'' is obtained by simply reversing the direction of all differentials. As an upshot of this discussion, the dual of C'' is a zigzag complex corresponding to the zigzag graph obtained from $zz(x)$ by switching $\circ$ and $\bullet$ (the effect of the rotation) and redirecting all arrows (the effect of switching all crossings). This shows that the statement holds for $1/x$.

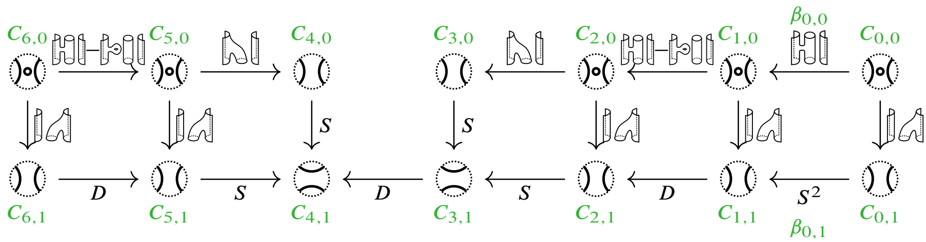
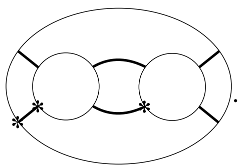


Figure 15. An example of a grid complex appearing in the proof of Theorem 5.6. The lower row is a zigzag complex corresponding to $zz(5/2)$.

Let us now show that the statement holds for $x+1$. Let T' be the tangle $R(x+1)$ equipped with the orientation induced by the orientation of T. Let $\mathcal{D}$ be the 2-input planar arc diagram



Then, T' is equivalent to $\mathcal{D}(T, R(1))$, and so, $[T']^\bullet \simeq \mathcal{D}([T]^\bullet, B)$, for B the Bar-Natan complex of a single crossing, which is, up to global shifts, equal to

$B_0 =$  $\xrightarrow{S}$  $\{1\} = B_1.$

Less formally, $[T']^\bullet$ is homotopy equivalent to the chain complex obtained by taking the tensor product of the chain complexes $[T]^\bullet$ and B, while at the same time (on the level of tangles) gluing two pairs of end points. By the induction hypothesis, $[T]^\bullet$ is homotopy equivalent to a zigzag complex $(\bigoplus_{i=0}^n A_i, \sum_{i=1}^n d_i)$ with graph $zz(x)$. Thus, the complex $[T']^\bullet$ is homotopy equivalent to the complex

$$\bigoplus_{\substack{i\in\{0,\dots,n\}\\ j\in\{0,1\}}} C_{ij}$$

with $C_{ij} = A_i \otimes B_j$ and the following differentials:

$$\alpha_i\colon C_{i,0} \to C_{i,1}$$

given by $(-1)^i \operatorname{id}_{A_i} \otimes S$, and $\beta_{i,j}$ a map $C_{i,j} \to C_{i-1,j}$ or $C_{i-1,j} \to C_{i,j}$ given by $d_i \otimes \operatorname{id}_{B_j}$. One may conveniently depict this complex in a grid with two rows and $n+1$ columns. An example is shown in Figure 15. The lower row of this grid is the subcomplex $(C_{i,1}, \beta_{i,1})$, which equals the complex $(A_i\{1\}, d_i)$. In the upper row, on

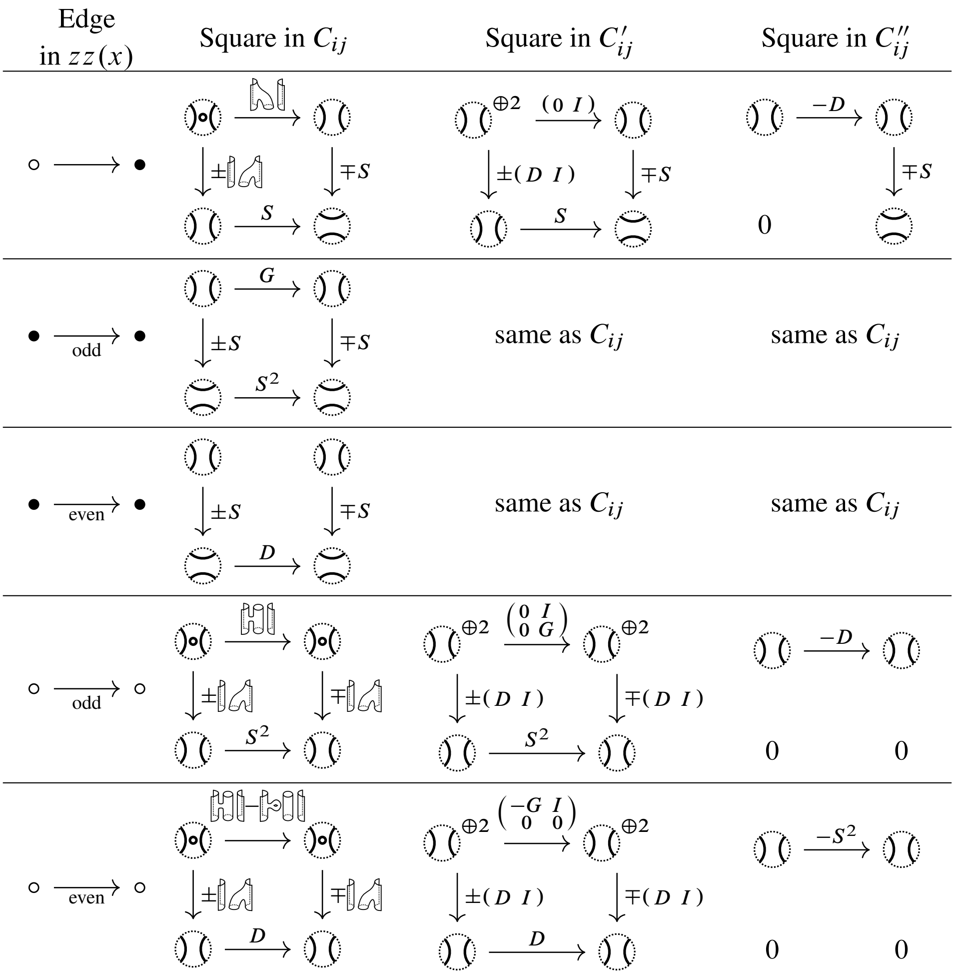


Table 3. Simplifications done in the proof of Theorem 5.6.

the other hand, $C_{i,0} =$)(if $A_i =$ ⌣⌢ and $C_{i,0} =$)∘(if $A_i =$)(; moreover, one easily checks the correspondence between maps in the lower row and the upper row shown in the first two columns of Table 3. Note that for these and the following calculations, it matters where the base point is (in the south west), and where the single crossing is attached to T to form T' (on the right). Our strategy is now to simplify C_{ij} step by step: first by delooping, then by Gaussian elimination, and finally by basis changes. Thereby, we will produce a homotopy equivalence between C_{ij} and a zigzag complex with graph $zz(x + 1)$.

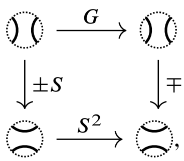
Let us first simplify C_{ij} by delooping. This yields another grid complex C'_{ij} with identical objects, except that is replaced by $C'_{i0} = \oplus $. The lower rows C'_{i0} and C_{i0} are identical, including the differentials. The third column of Table 3 shows how the upper row C'_{i1} is determined by the lower row C'_{i0}. All of the vertical differentials $\alpha'_i: C'_{i1} \to C'_{i0}$ are either $(-1)^i S$ or equal to

$$(-1)^i \begin{pmatrix} D & I \end{pmatrix}.$$

We leave it to the reader to verify these calculations.

As a second step, let us simultaneously eliminate all the I entries in α'_i by Gaussian elimination and denote the resulting grid complex by C''_{ij}. Table 3 shows how C''_{ij} is determined by C'_{ij}. Once again, we invite the reader to check these calculations. Note that if there is a differential δ in C'_{ij} whose target is one of the objects in the lower row, then there are only two possible cases. The first case is that the domain of δ is equal to the domain of one of the maps I that are being eliminated. In that case, Gaussian elimination produces a new differential, which is the reason that the squares of C''_{ij} shown in the last two rows of Table 3 are $-D$ and $-S^2$. The second case is that the domain of δ is another object in the lower row. (Since there are no edges $\bullet \to \circ$ in $zz(x)$, it cannot be a object.) But in that case, the domain itself is being eliminated, so in this case, Gaussian elimination does not produce new differentials.

The third step is to apply changes of basis. At each square W of C''_{ij} arising from an odd edge in $zz(x)$ between $\bullet$ vertices, i.e., a square



we apply the change of basis shown in Figure 16. Note that Figure 16 also shows the two squares adjacent to W. Of course, if W is at one end of the complex, then one of those squares does not exist. The point is that several of those basis changes are compatible and may be made simultaneously because the basis change maps on the adjacent objects are the identity I.

This results in a grid complex C'''_{ij}. Since C'''_{ij} is homotopy equivalent to C_{ij}, it just remains to verify that C'''_{ij} is isomorphic to a zigzag complex corresponding to $zz(x+1)$. The first two columns of Table 4 summarize how C'''_{ij} is determined by $zz(x)$. From that table, it should be evident that C'''_{ij} is a 'linear' complex; i.e., it is isomorphic to a chain complex of the form $(\bigoplus_i A_i, \sum_i d_i)$, with each A_i either or , and each d_i a map $A_{i-1} \to A_i$ or $A_i \to A_{i-1}$. It is not quite a zigzag complex, yet, since the differentials are equal to $\pm S$, $\pm S^2$, $\pm D$. However, due to the linear

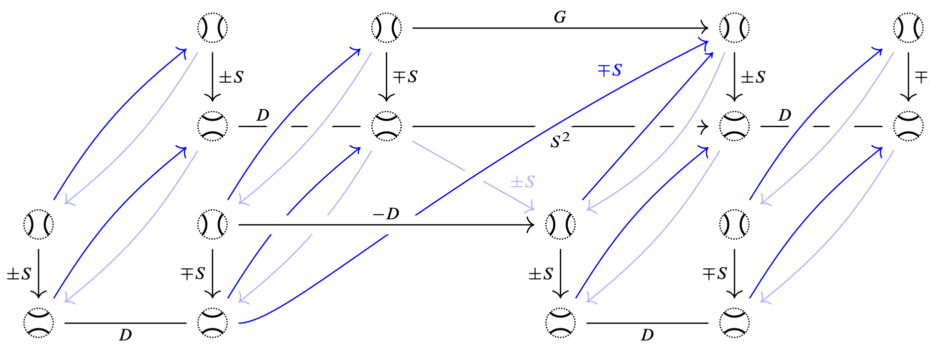


Figure 16. The change of basis that is required in the proof of Theorem 5.6. The back face is the complex C''_{ij}, and the front face is the simplified complex C'''_{ij}. Differentials are drawn in black. The light and dark blue arrows show mutually inverse isomorphisms between those two complexes. Unlabeled arrows signify identity maps I. The arrows labeled D on the lower level, on the left, have no arrowhead because they may point in either direction (either both point to the right, or both to the left). The same holds for the arrows labeled D on the lower level, on the right.

shape of the complex, one easily finds a basis change that gets rid of all minus signs. So, C'''_{ij} truly is the zigzag complex of some zigzag graph Γ.

How edges of $zz(x)$ determine small subgraphs of Γ is shown in the third column of Table 4. However, the way these subgraphs fit together is somewhat cumbersome: subgraphs coming from adjacent edges in $zz(x)$ need to be glued together either along a single $\circ$-vertex, or along a vertical downwards edge from $\circ$ to $\bullet$. As a remedy, we remove the two $\circ$-vertices from the subgraph induced by an even $\bullet \to \bullet$ edge, unless those $\circ$-vertices are at one of the ends of the graph. After this reinterpretation, we obtain the subgraphs shown in the fourth column of Table 4. (In the fourth column, we also disregard the grid structure, which was still shown in the third column.) To form Γ from the subgraphs in the fourth column, one simply glues along a single vertex. Now, the first and fourth column of Table 4 are precisely identical with the rules of Table 2. This shows that

$$\Gamma = zz(x+1),$$

concluding the proof. ∎

5.2. The λ-distance between rational tangles

Next, let us turn to Theorem 1.1. Let us break up the proof into a sequence of lemmas.

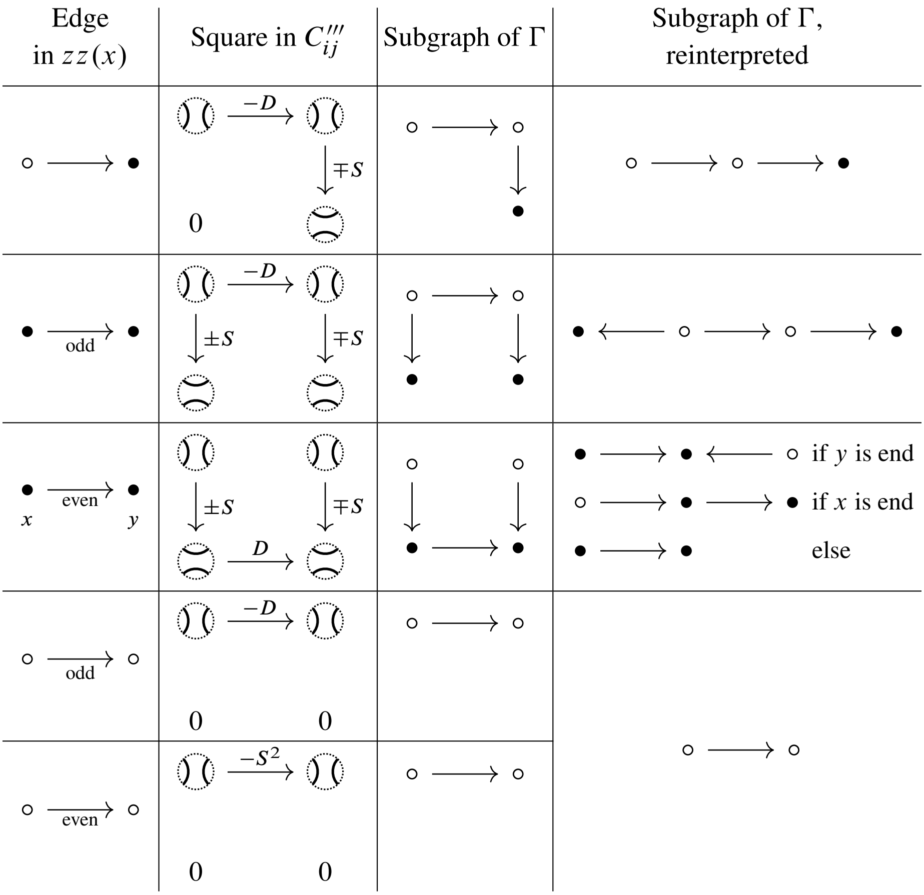


Table 4. From $zz(x)$ to $zz(x+1)$ in the proof of Theorem 5.6.

Lemma 5.9. *Let $x \in \mathbb{Q}^+$. For every non-saddle edge e of the zigzag graph $zz(x)$, there is a subgraph Γ_e of $zz(x)$ as follows for some $n \geq 1$ (in what follows, a missing arrowhead means that the edge's direction is unknown):*

$$\Gamma_e = \begin{array}{ccccccc} A_1 & \text{——} & A_2 & \text{——} & \cdots & \text{——} & A_n \\ {\scriptstyle e}\downarrow & & & & & & \\ B_1 & \text{——} & B_2 & \text{——} & \cdots & \text{——} & B_n \end{array}$$

such that, for each i with $1 \leq i < n$, the vertices A_i and B_i are of the same type ($\circ$ or $\bullet$), and the edges between A_i and A_{i+1} and between B_i and B_{i+1} are either both directed to the right, or both to the left (in the above drawing of Γ_e); and such that moreover, one of the following statements is true

(i) *A_n and B_n are of the same type, A_n has no outgoing external edge (i.e., an edge towards a vertex in $zz(x) \setminus \Gamma_e$), and B_n has no incoming external edge.*

(ii) *$n \geq 2$, and*

$$\begin{array}{ccc} A_{n-1} \text{ —— } A_n & & \circ \longrightarrow \circ \\ & \textit{looks like} & \\ B_{n-1} \text{ —— } B_n & & \circ \longrightarrow \bullet. \end{array}$$

(iii) *$n \geq 2$, and*

$$\begin{array}{ccc} A_{n-1} \text{ —— } A_n & & \bullet \longleftarrow \circ \\ & \textit{looks like} & \\ B_{n-1} \text{ —— } B_n & & \bullet \longleftarrow \bullet. \end{array}$$

Proof. We proceed by induction over the number of transformations $y \mapsto 1/y$ and $y \mapsto y + 1$ necessary to reach x from 1. For $x = 1$, there is no non-saddle edge in $zz(x)$, so the statement is trivially true.

Now, assume that the statement holds for $zz(x)$, and let us show that it holds for $zz(1/x)$ too. Since $zz(1/x)$ arises from $zz(x)$ by reversing edge directions and switching $\circ$ and $\bullet$, there is a canonical one-to-one correspondence between edges e of $zz(x)$ and edges f of $zz(1/x)$. The subgraph Γ_e of $zz(x)$ then corresponds to a subgraph Γ_f of $zz(1/x)$. Note that in order for f to be pointing downwards, one needs to turn Γ_f upside down. Then, one sees that Γ_e satisfies (i), (ii), (iii) if and only if Γ_f satisfies (i), (iii), (ii), respectively.

Finally, assume that the statement holds for $zz(x)$, and let us prove the statement for $zz(x + 1)$. Every edge of $zz(x + 1)$ comes from some edge e of $zz(x)$ by applying the rules in Table 2. So, let us consider the six rules shown in Table 2 case by case.[12] In each case, e gives rise to some number (between one and three) of edges in $zz(x + 1)$, exactly one of which is a non-saddle edge. That edge, which we denote by f, is the one for which we need to check the existence of a subgraph Γ_f as in the statement of the lemma. In each case (except when e is a saddle edge), denote by Γ_e a subgraph of $zz(x)$ as in the statement of the lemma, satisfying (i), (ii), or (iii).

[12] We invite the reader who is eager to check the following proof in detail to have a printed copy of Table 2 handy. Also, they might be well advised to prepare themself a modified version of Table 2, in which all subgraphs are rotated by 180° so that they can quickly find out the fate of edges pointing to the left.

(1) *e is an odd edge between $\bullet$-vertices.* Applying the rules in Table 2 to every edge in the subgraph

$$\Gamma_e = \begin{array}{ccccccccc} \bullet & \text{——} & A_2 & \text{——} & \cdots & \text{——} & A_n \\ {\scriptstyle e}\big\downarrow & & & & & & \\ \bullet & \text{——} & B_2 & \text{——} & \cdots & \text{——} & B_n, \end{array}$$

transforms it into a subgraph Γ' of $zz(x+1)$ that looks as follows:

$$\Gamma' = \begin{array}{ccccccccc} \circ & \longrightarrow & \bullet & \text{——} & A'_3 & \text{——} & \cdots & \text{——} & A'_k \\ {\scriptstyle f}\big\downarrow & & & & & & & & \\ \circ & \longrightarrow & \bullet & \text{——} & B'_3 & \text{——} & \cdots & \text{——} & B'_\ell. \end{array}$$

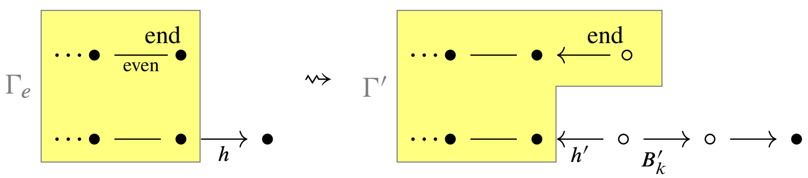
We consider the following list of exhaustive subcases.

(a) *Γ_e satisfies* (i) *with n odd.* In this case, let us show that $\Gamma_f = \Gamma'$ satisfies (i). Firstly, since the top and bottom row of Γ_e look the same and the last edge in each row is not an even edge between $\bullet$ vertices, the top and bottom rows of Γ' look the same too. More precisely, we have $k = \ell$, A'_i, and B'_i are of the same type for $3 \le i \le k$, and horizontal edges in the first and second rows of Γ' point in the same directions. Secondly, since A_n has no outgoing external edge, neither does A'_k. And since B_n has no incoming external edge and no outgoing external odd edge between $\bullet$ vertices, B'_k has no incoming external edge either. So, Γ' satisfies (i).

(b) *Γ_e satisfies* (i) *with n even, and both A_{n-1} and A_n are of type $\circ$*, same as case (a).

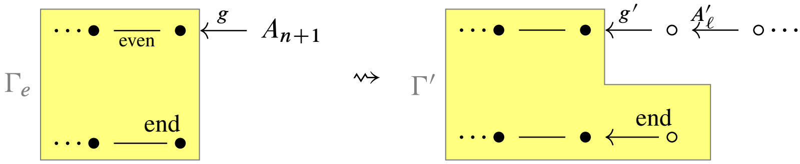
(c) *Γ_e satisfies* (i) *with n even, A_{n-1} and A_n are both of type $\bullet$, and A_n and B_n are ends*, same as case (a).

(d) *Γ_e satisfies* (i) *with n even, A_{n-1} and A_n are both of type $\bullet$, A_n is an end, and B_n is not.* Since B_n is not an end and has no external incoming edge, it must have an external outgoing edge h, which must go to a vertex B_{n+1} of type $\bullet$. Applying the rules of Table 2 to $\Gamma_e \cup \{h, B_{n+1}\}$ yields the following (note that $k = \ell + 1$):



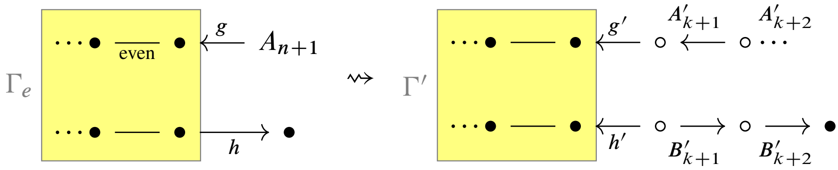
Now, $\Gamma_f = \Gamma' \cup \{h', B'_k\}$ satisfies (i).

(e) Γ_e *satisfies* (i) *with n even, A_{n-1} and A_n are both of type $\bullet$, A_n is not an end, and B_n is.* Since A_n is not an end and has no external outgoing edge, it must have an external incoming edge g from a vertex A_{n+1}. Applying the rules of Table 2 to $\Gamma_e \cup \{g, A_{n+1}\}$ yields the following (note that $\ell = k + 1$):



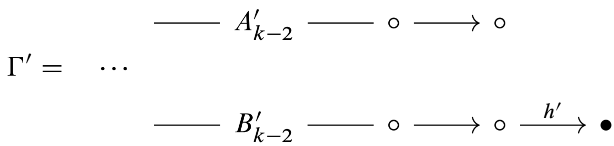
Now, $\Gamma_f = \Gamma' \cup \{g', A'_\ell\}$ satisfies (i).

(f) Γ *satisfies* (i) *with n even, A_{n-1} and A_n are both of type $\bullet$, and neither A_n nor B_n are ends.* This is basically the synthesis of the two previous cases. As before, there are an external incoming edge g from some vertex A_{n+1} to A_n and an external outgoing edge h from B_n to some vertex B_{n+1}. Applying the rules of Table 2 to $\Gamma_e \cup \{g, A_{n+1}, h, B_{n+1}\}$ yields the following (note that $k = \ell$):



Now, $\Gamma_f = \Gamma' \cup \{g', A'_{k+1}, h', B'_{k+1}\}$ satisfies (i).

(g) Γ *satisfies* (ii). In this case, we have $\ell = k + 1$ and



If A_n does not have an external outgoing edge, then neither does A'_k, and thus,

$$\Gamma_f = \Gamma' \setminus \{B'_{k+1}, h'\}$$

satisfies (i). If A_n has an external outgoing edge g towards a vertex A_{n+1}, then A'_k also has an external outgoing edge g' towards a vertex A'_{k+1}, which must be of type $\circ$. Then,

$$\Gamma_f = \Gamma' \cup \{g', A'_{k+1}\}$$

satisfies (ii).

(h) Γ_e *satisfies* (iii). In this case, we have $\ell = k + 1$ and

$$\Gamma' = \quad \cdots \begin{array}{l} \text{——} \; A'_{k-3} \; \text{——} \; \bullet \longleftarrow \circ \longleftarrow \circ \\[2em] \text{——} \; B'_{k-3} \; \text{——} \; \bullet \longleftarrow \circ \longleftarrow \circ \xrightarrow{h'} \bullet \end{array}$$

We may proceed exactly as in case (g) above.

(2) *e is an even edge between $\bullet$-vertices, none of which are an end.* Applying the rules in Table 2 to every edge in the subgraph

$$\Gamma_e = \begin{array}{ccccccc} \bullet & \text{——} & A_2 & \text{——} & \cdots & \text{——} & A_n \\ {\scriptstyle e}\big\downarrow & & & & & & \\ \bullet & \text{——} & B_2 & \text{——} & \cdots & \text{——} & B_n \end{array}$$

transforms it into a subgraph Γ' of $zz(x+1)$ that looks as follows:

$$\Gamma' = \begin{array}{ccccccc} \bullet & \text{——} & A'_2 & \text{——} & \cdots & \text{——} & A'_k \\ {\scriptstyle f}\big\downarrow & & & & & & \\ \bullet & \text{——} & B'_2 & \text{——} & \cdots & \text{——} & B'_\ell. \end{array}$$

Now, one may proceed exactly as in case (1).

(3) *e is an even edge between $\bullet$-vertices, directed towards an end.* In Γ_e, B_1 is an end. Thus, we must have $n = 1$ and Γ satisfying (i). So, in $zz(x)$, there is an edge between A_1 and a vertex A_2, directed towards A_1. The rules of Table 2 transform $A_2 \to A_1 \to B_1$ into a subgraph Γ' of $zz(x+1)$ that looks as follows:

$$\Gamma' = \begin{array}{lllll} \bullet & \longleftarrow & \circ & \longleftarrow & \circ \cdots \\ {\scriptstyle f}\big\downarrow & & & & \\ \bullet & \longleftarrow & \circ \text{ end.} & & \end{array}$$

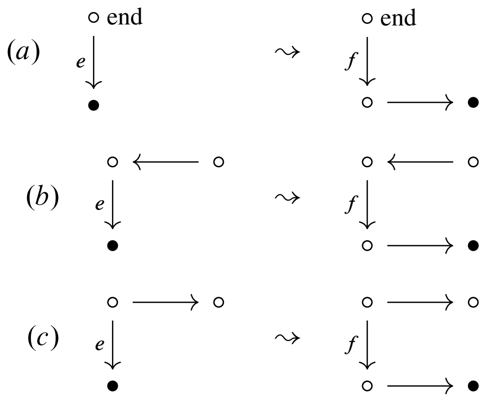
Depending on the type of A_2, the first row may have a fourth column, or not. But either way, the first two columns of Γ' form a subgraph Γ_f of $zz(x+1)$ satisfying (i) with $n = 2$.

(4) *e is an even edge between $\bullet$-vertices, directed away from an end.* In Γ, A_1 is an end. Thus, we must have $n = 1$ and Γ satisfying (i). So, in $zz(x)$, there is an edge between B_1 and a vertex B_2, directed away from B_1. Since B_1 is a $\bullet$-vertex, so is B_2. Let us inspect the transformation given by Table 2:

$$\begin{array}{lll} \bullet \text{ end} & & \\ {\scriptstyle e}\big\downarrow & & \\ \bullet & \longrightarrow & \bullet \end{array} \quad \rightsquigarrow \quad \begin{array}{lllllll} \bullet & \longleftarrow & \circ \text{ end} & & & & \\ {\scriptstyle f}\big\downarrow & & & & & & \\ \bullet & \longleftarrow & \circ & \longrightarrow & \circ & \longrightarrow & \bullet \end{array}$$

The first two columns of the right-hand side form a subgraph Γ_f of $zz(x+1)$, satisfying (i) with $n = 2$.

(5) *e is a saddle edge.* Note that we do not have a graph Γ_e in this case. Nevertheless, let us denote by A_1 the $\circ$-vertex adjacent to e. Let us inspect the transformation given by Table 2 in each of the three cases that (a) A_1 is an end, (b) A_1 has an incoming edge, and (c) A_1 has an outgoing edge:



In cases (a) and (b), the edge f and its end points form a subgraph Γ_f of $zz(x+1)$ satisfying (i), with $n = 1$. In case (c), the whole right-hand side is a subgraph Γ_f of $zz(x+1)$, satisfying (ii) with $n = 2$.

(6) *e is an edge between two $\circ$-vertices.* Treat this case similarly as (1) and (2). ■

Lemma 5.10. *Let $x \in \mathbb{Q}^+$, and let $(C, d) = (\bigoplus_{i=0}^n A_i, \sum_{i=1}^n d_i)$ be a zigzag complex corresponding to $zz(x)$. Then, for every $i \in \{1, \dots, n\}$, there exists a homotopy $h\colon C \to C$ such that $h \circ d + d \circ h = f \cdot (\mathrm{id}_{A_i} + \mathrm{id}_{A_{i-1}})$ with $f = S^2$ if d_i is odd (i.e., $d_i = S$ or $d_i = S^2$), and $f = D$ if d_i is even, i.e., $d_i = D$.*

Proof. By reindexing C if necessary, we assume without loss of generality that d_i is a map $A_{i-1} \to A_i$.

If d_i is S, then let h be given by $S\colon A_i \to A_{i-1}$. Then, for $h \circ d_j$ to be nonzero, the target of d_j and the domain of h must match; this happens only if $j = i$, or $j = i + 1 \le n$ and d_{i+1} is a map $A_{i+1} \to A_i$. In the latter case, we nevertheless have $h \circ d_{i+1} = 0$, since h is S and d_{i+1} is D. Similarly, one sees that $d_j \circ h = 0$ unless $j = i$. Overall, we find

$$h \circ d + d \circ h = \sum_{j=1}^n h \circ d_j + d_j \circ h = h \circ d_i + d_i \circ h = S^2 \cdot (\mathrm{id}_{A_i} + \mathrm{id}_{A_{i-1}})$$

as desired.

If d_i is not S, denote by e the edge in $zz(x)$ corresponding to d_i. Since e is not a saddle edge, by the previous Lemma 5.9, there is a subgraph Γ satisfying (i), (ii), or (iii). The part of C corresponding to Γ is the following (drawn in black):

$$\begin{array}{ccccccc} A_{i-1} & \xrightarrow{d_{i-1}} & A_{i-2} & \text{———} & \cdots & \xrightarrow{d_{i-k+1}} & A_{i-k} \\ {\scriptstyle d_i}\downarrow \uparrow{\scriptstyle h_0} & & \uparrow{\scriptstyle h_1} & & & & \uparrow{\scriptstyle h_{k-1}} \\ A_i & \xrightarrow[d_{i+1}]{} & A_{i+1} & \text{———} & \cdots & \xrightarrow[d_{i+k-1}]{} & A_{i+k-1}. \end{array} \tag{5.9}$$

Let the homotopy h, drawn in (5.9) in red and dashed, be defined as the sum

$$h = \sum_{j=0}^{k-1} h_j \colon A_{i+j} \to A_{i-j-1}$$

with h_j equal to $(-1)^{i-j}$ times the identity cobordism if the domain and target of h_j are both  or both  and h_j equal to $(-1)^{i-j}$ times S if one of the domain and target of h_j is , and the other is . Note that the latter case only happens if Γ satisfies (ii) or (iii) and

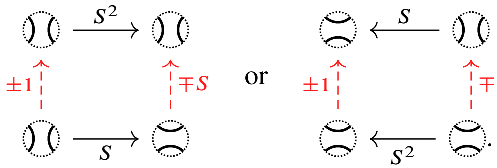
$$j = k - 1.$$

Now, $h \circ d + d \circ h$ is equal to the sum of the following terms $\alpha, \beta_j, \gamma_j, \delta$ (all other compositions of h_j and d_k vanish because target and domain do not match):

$$\begin{aligned} &\underbrace{d_i \circ h_0 + h_0 \circ d_i}_{\alpha} \\ &+ \sum_{j=1}^{k-1} \underbrace{d_{i-j} \circ h_{j-1} + h_j \circ d_{i+j}}_{\beta_j} \\ &+ \sum_{j=1}^{k-1} \underbrace{d_{i-j} \circ h_j + h_{j-1} \circ d_{i+j}}_{\gamma_j} \\ &+ \underbrace{d_{i-k} \circ h_{k-1} + h_{k-1} \circ d_{i+k}}_{\delta}. \end{aligned}$$

Now, observe that α equals $f \cdot (\mathrm{id}_{A_i} + \mathrm{id}_{A_{i-1}})$ with $f = S^2$ if d_i is S^2 and $f = D$ if d_i is D. So, it just remains to show that the terms β_j, γ_j, and δ are 0. For each j, one of β_j and γ_j is 0 because targets and domains do not match; and the other term is 0 because the squares in (5.9) anticommute. (Remember that d_{i-j} and d_{i+j} both point to the left, or both to the right.) This is also true for the last square in case that

Γ satisfies (ii) or (iii), in which case that square, respectively, looks like



Finally, in case Γ satisfies (i), δ is 0 because targets and domains mismatch. If Γ satisfies (ii) or (iii), δ is 0 either for the same reason, or because h_{k-1} is S and d_{i-j} and d_{i+j} are D. ■

The following lemma is well known (see, e.g., [7,23]), and it can easily be checked inductively.

Lemma 5.11. *For $i \in \{1, 2\}$, let p_i and q_i be coprime integers. Then, $R(p_1/q_1)$ and $R(p_2/q_2)$ have the same connectivity if and only if $p_1 \equiv p_2 \pmod 2$ and $q_1 \equiv q_2 \pmod 2$.* ■

Let us call an end of a zigzag graph *even* or *odd* depending on whether the unique edge adjacent to it is even or odd.

Lemma 5.12. *Let $x = p/q$ with p, q positive and coprime. Then, the following hold.*

(i) *$zz(x)$ has an even end iff p or q is even.*

(ii) *$zz(x)$ has an odd $\circ$-end iff p is odd.*

(iii) *$zz(x)$ has an odd $\bullet$-end iff q is odd.*

Proof. Let us show this by induction over the number of transformations $y \mapsto 1/y$ and $y \mapsto y+1$ necessary to reach x from 1. For $x = 1$, $zz(x)$ has both a $\circ$-end and a $\bullet$-end, and $p = q = 1$ are both odd, so the statement holds. Let us now assume that the statement holds for x. The zigzag graph $zz(1/x)$ is obtained from $zz(x)$ by switching $\circ$ and $\bullet$ and reversing all edges. This switches $\circ$- and $\bullet$-ends and does not change the parity of edges. This corresponds to the fact that

$$x \mapsto 1/x$$

switches the parity of p and q. Thus, the statement holds for $1/x$.

To check the statement for $x+1$, one needs to analyze the effect of (ZZ3) on parity and the type of ends ($\bullet$ or $\circ$). The possible configurations for $zz(x)$ are listed in the first two columns of Table 5.

The third and fourth columns show the respective resulting configurations for $zz(x+1)$. The third column is straightforward. To verify the fourth column, one needs to refer to Table 2. From the above table, one sees that the induction statement holds for $x+1$. This concludes the proof. ■

parities of p, q	ends of $zz(x)$	parities of $p+q, q$	ends of $zz(x+1)$
odd, odd	odd $\circ$, odd $\bullet$	even, odd	even $\circ$, odd $\bullet$
even, odd	even $\circ$, odd $\bullet$	odd, odd	odd $\circ$, odd $\bullet$
even, odd	even $\bullet$, odd $\bullet$	odd, odd	odd $\circ$, odd $\bullet$
odd, even	even $\bullet$, odd $\circ$	odd, even	even $\circ$, odd $\circ$
odd, even	even $\circ$, odd $\circ$	odd, even	even $\circ$, odd $\circ$

Table 5. Effect of $x \mapsto x+1$ on parities and ends.

The next lemma is the heart of the proof. It is the analog of [2, Lemmas 3.1 and 3.2].

Lemma 5.13. *Let $p/q \in \mathbb{Q}^+$ with both p and q odd, and let C be a zigzag complex corresponding to $zz(p/q)$. Let C' be the complex*

$C_0' =$  $\xrightarrow{S}$  $\{1\} = C_1'$,

which is (up to global shifts) the Bar-Natan complex of $R(-1)$ equipped with some orientation. Then, there are ungraded chain maps $f: C \to C'$ and $g: C' \to C$ such that

$$g \circ f \simeq G \cdot \mathrm{id}_C \quad \text{and} \quad f \circ g \simeq G \cdot \mathrm{id}_{C'}.$$

Proof. Let us write

$$(C, d) = \left(\bigoplus_{i=0}^{n} A_i, \sum_{i=1}^{n} d_i \right).$$

By reindexing this zigzag if necessary, we may assume that the end of Γ corresponding to A_0 is the $\circ$-end. We are going to define f as sum of two ungraded chain maps $\alpha: A_0 \to C_1'$ and $\gamma: A_n \to C_0'$ and g as sum of two ungraded chain maps $\beta: C_1' \to A_0$ and $\delta: C_0' \to A_n$. See Figure 17 for an example. Namely, let β and γ be $-D$, and let α and δ be 1. One may check inductively using Table 2 (see [15, Remark 12.14] for details) that the $\circ$-end of Γ has an outgoing differential, and the $\bullet$-end has an incoming differential; in other words, d_1 is a map $A_0 \to A_1$, and d_n is a map $A_{n-1} \to A_n$. Combined with the fact that d_1 and d_n are both S or S^2, this implies that f and g are ungraded chain maps. Now, one calculates that

$$\begin{aligned} f \circ g &= \alpha \circ \beta + \gamma \circ \delta \\ &= (-D: C_0' \to C_0') + (-D: C_1' \to C_1'). \end{aligned}$$

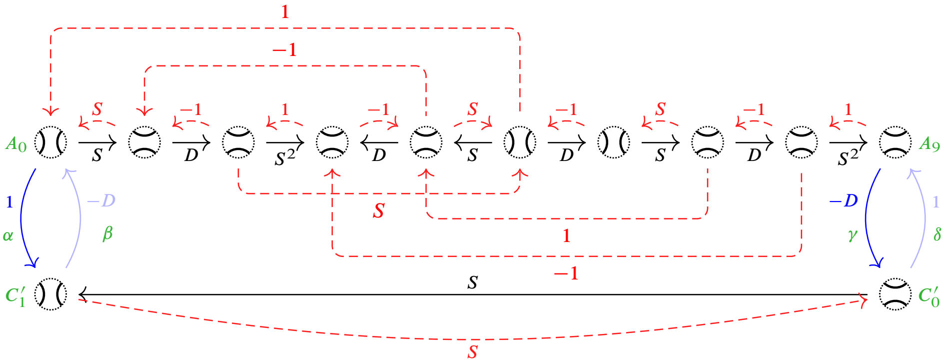


Figure 17. Illustration of the proof of Lemma 5.13. In the top row is shown a zigzag complex C with graph $zz(3/7)$ (compare Figure 14). In the bottom row is shown the complex C'. In light and dark blue are shown the ungraded chain maps $f\colon C \to C'$ (going down) and $g\colon C' \to C$ (going up). In red and dashed are indicated the required homotopies. Homological and quantum degree shifts are omitted from the diagram.

Let $h'\colon C' \to C'$ be $S\colon C_1' \to C_0'$. Then, $d' \circ h' + h' \circ d' = G \cdot \mathrm{id}_{C'} - f \circ g$, so $f \circ g$ is homotopic to G, as desired.

Similarly, one finds $g \circ f = (-D\colon A_0 \to A_0) + (-D\colon A_n \to A_n)$. By Lemma 5.10, for each i, there exists a homotopy $h_i\colon C \to C$ such that $h_i \circ d + d \circ h_i$ equals $u \cdot (\mathrm{id}_{A_i} + \mathrm{id}_{A_{i-1}})$ with $u = S^2$ if d_i is odd and $u = D$ if d_i is even. Let $h = \sum_i (-1)^{i+1} h_i$. Now, one sees that

$$h \circ d + d \circ h = G \cdot \mathrm{id}_C - g \circ f,$$

which concludes the proof. ∎

We now need to examine rational replacements (first seen in Definition 1.3) more closely.

Definition 5.14. Two unoriented links $L, L' \subset S^3$ are related by a *rational replacement* if, after an isotopy, there exists a ball $B \subset S^3$ whose boundary sphere intersects L and L' transversely such that $L \setminus B^\circ = L' \setminus B^\circ$, and the two tangles $T = L \cap B$ and $T' = L' \cap B$ are rational. If T and T' have the same connectivity, we say that the rational replacement is *proper*. If there is a homeomorphism between B and the unit ball that sends T to $R(x)$ and T' to $R(y)$ for some $x, y \in \mathbb{Q} \cup \{\infty\}$, we speak of an *$x$ by y rational replacement*.

It is a frequently used fact that a crossing change may be seen as -1 by 1 rational replacement, but also as 0 by 2 rational replacement. The following lemma generalizes this.

Lemma 5.15. *Let S, T be rational tangles in a ball B, let $x, y \in \mathbb{Q} \cup \{\infty\}$, and let φ be a homeomorphism of B to the unit ball B_0 such that $\varphi(S) = R(x)$ and $\varphi(T) = R(y)$. Then, there exist $z \in \{-1\} \cup [0, \infty)$ and a homeomorphism $\varphi' \colon B \to B_0$ such that $\varphi'(S) = R(-1)$ and $\varphi'(T) = R(z)$.*

Proof. Let ψ_1 be a homeomorphism of B_0 with $\psi_1(R(x)) = R(\infty)$. Then,

$$\psi_1(R(y)) = R(y')$$

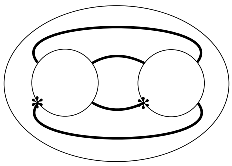
for some y'. Let ψ_2 be a homeomorphism of B_0 such that $\psi_2(R(\infty)) = R(\infty)$ and $\psi_2(R(y')) = R(y'')$ with $y'' \in (0, 1] \cup \{\infty\}$. Such a ψ_2 may be constructed by adding a certain number of twists to the right side of the ball. Finally, let ψ_3 be the homeomorphism of B_0 that sends $R(w)$ to $R(1/w - 1)$ for all $w \in \mathbb{Q} \cup \{\infty\}$. Then, $\psi_3 \circ \psi_2 \circ \psi_1 \circ \varphi(S) = R(-1)$ and $\psi_3 \circ \psi_2 \circ \psi_1 \circ \varphi(T) = R(z)$ for $z \in \{-1\} \cup [0, \infty)$, as desired. ■

Proposition 3.14. *Fix a ball and four end points on its boundary, where one of them is distinguished as base point. On the set of equivalence classes of unoriented rational tangles in that ball with fixed connectivity, the pseudometric given by λ is in fact equal to the discrete metric. That is to say, $\lambda(S, T) = 1$ for inequivalent rational tangles S and T.*

Proof. By Lemma 5.15, there exist $x \in \{-1\} \cup [0, \infty)$ and a homeomorphism that sends S to $R(-1)$ and T to $R(x)$. Since S and T are not equivalent, we have $x \neq -1$. Since the connectivities of S and T are the same, Lemma 5.11 implies that $x = p/q$ with both p and q odd (in particular, $p/q \neq 0$). By Proposition 3.10, λ is equivariant under homeomorphisms, and so, we have $\lambda(S, T) = \lambda(R(-1), R(x))$. So, it will be sufficient to show that $\lambda(R(-1), R(x)) = 1$.

By Theorem 5.6, $[R(x)]^\bullet$ is homotopy equivalent to a zigzag complex C with graph $zz(x)$. By Lemma 5.13, there are ungraded chain maps $f \colon [R(-1)]^\bullet \to C$ and $g \colon C \to [R(-1)]^\bullet$ with $g \circ f \simeq G \cdot \mathrm{id}_{[R(-1)]^\bullet}$ and $f \circ g \simeq G \cdot \mathrm{id}_C$, showing $\lambda(R(-1), R(x)) \leq 1$.

Let $\mathcal{D}$ be the following 2-input planar arc diagram:



Then, $\mathcal{D}(R(-1), R(2))$ is the unknot, and $\mathcal{D}(R(x), R(2))$ is the two-bridge knot K corresponding to $x + 2 = (p + 2q)/q$. Since $x + 2 > 1$, this is a non-trivial knot, and so, we have $\lambda(K) > 0$ because λ detects the unknot (see Proposition 1.8).[13] Overall, using Lemma 3.7, we get

$$\lambda(R(-1), R(x)) \geq \lambda(\mathcal{D}(R(-1), R(2)), \mathcal{D}(R(x), R(2))) = \lambda(K) > 0.$$

This concludes the proof. ∎

Proof of Theorem 1.1. To show $\lambda(K) \leq u_q(K)$, it is sufficient to show the following: if two knots K and J are related by a proper rational replacement, then $\lambda(K, J) \leq 1$. So, let knots K, J related by a proper rational replacement be given. By definition, there exists a 4-ended tangle T such that K is the union of T with a rational tangle S and J is the union of T with a another rational tangle S'. Since the replacement is proper, S and S' have the same connectivity. So, $\lambda(S, S') \leq 1$ by Proposition 3.14, and thus, $\lambda(K, J) \leq 1$ by Proposition 3.12. ∎

A. Proof of Proposition 2.8

This appendix is devoted to the proof of the following proposition.

Proposition 2.8. *The functor* $B\colon \mathrm{Mat}(\mathcal{E}) \to \mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))$ *given by inclusion is an equivalence of categories.*

Proof. We need to check that B is faithful, full, and dense. As explained before, density of B follows directly from delooping. To show that B is faithful and full, we are going to look at the morphism spaces

$$\begin{aligned}&\hom_{\mathcal{E}}(D_{T_0}, D_{T_0}),\\ &\hom_{\mathrm{Cob}^3(2)}(B(D_{T_0}), B(D_{T_0})),\\ &\hom_{\mathrm{Cob}^3_{/l}(2)}(B(D_{T_0}), B(D_{T_0})),\end{aligned}$$

where D_{T_0} is the diagram of the trivial 2-ended tangle T_0.

Let G, Σ_i, $i \in \mathbb{Z}_{\geq 0}$ be formal variables. We introduce a grading on $\mathbb{Z}[G]$ and $\mathbb{Z}[G, \Sigma_0, \Sigma_1, \Sigma_2, \ldots]$ by setting $\deg G = -2$ and $\deg \Sigma_i = 2 - 2i$. Then, there is an isomorphism of graded Abelian groups[14]

$$\hom_{\mathcal{E}}(D_{T_0}, D_{T_0}) \cong \mathbb{Z}[G]$$

[13] Here, we do not even need Kronheimer–Mrowka's theorem that Khovanov homology detects the unknot, but only the (much easier) theorem that Khovanov homology detects the unknot among two-bridge knots. (In fact, already the Jones polynomial can be seen to accomplish that.)

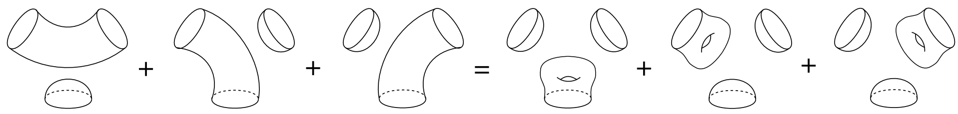


Figure 18. The $3S_2$-relation.

given by mapping a connected cobordism of genus k to G^k. Similarly, there is an isomorphism of graded Abelian groups[14]

$$\hom_{\mathrm{Cob}^3(2)}(B(D_{T_0}), B(D_{T_0})) \cong \mathbb{Z}[G, \Sigma_0, \Sigma_1, \Sigma_2, \ldots],$$

given by mapping a cobordism, which consists of the marked component with genus k and a disjoint union of n_i many closed surfaces of genus i, to the product

$$G^k \prod_{i=0}^{\infty} \Sigma_i^{n_i}.$$

In order to determine $\hom_{\mathrm{Cob}^3_{/l}(2)}(B(D_{T_0}), B(D_{T_0}))$, we need to understand how the local relations S, T, and $4Tu$ in $\mathrm{Cob}^3_{/l}(2)$ affect the ring $\mathbb{Z}[G, \Sigma_0, \Sigma_1, \Sigma_2, \ldots]$. Introducing S and T translates to $\Sigma_0 = 0$ and $\Sigma_1 = 2$. Next, we replace $4Tu$ with the equivalent $3S_2$ relation (cf. [5, Section 11.4] and Figure 18) which is easier to handle as there are at most three surfaces involved. We name the surfaces in the relation A, B, C in a clockwise manner starting top left. Suppose that $g(A) = a$, $g(B) = b$, $g(C) = c$. Let M_n be the curtain with genus $g(M_n) = n$.

Suppose first that $A = B = M_0 \neq C$. In this case, the $3S_2$ relation translates to

$$G\Sigma_c + G^c + G^c = \Sigma_{c+1} + G\Sigma_c + G\Sigma_c \Leftrightarrow \Sigma_{c+1} = 2G^c - G\Sigma_c.$$

By the S relation, we have $\Sigma_0 = 0$ and thus $\Sigma_1 = 2G^0 - G \cdot 0 = 2$, which coincides with the T relation. By induction, we therefore obtain the relation

$$\Sigma_c = \begin{cases} 2G^{c-1}, & c \text{ odd}, \\ 0, & c \text{ even}, \end{cases} \tag{*}$$

giving us a surjection $\hom_{\mathrm{Cob}^3_{/l}(2)}(B(D_{T_0}), B(D_{T_0})) \twoheadrightarrow \mathbb{Z}[G]$. We claim that there are no other relations introduced, i.e., that this surjection is an isomorphism. For this, we check all possible general cases of the $3S_2$ relation.

[14]This is in fact an isomorphism of graded rings if we declare multiplication in $\hom_{\mathcal{E}}$ (resp., $\hom_{\mathrm{Cob}^3(2)}$) as composition of cobordisms.

Case 1: A, B, C are three different closed surfaces (i.e., none of them is a curtain M_n). In this case, $3S_2$ translates to

$$\Sigma_{a+b}\Sigma_c + \Sigma_{a+c}\Sigma_b + \Sigma_{b+c}\Sigma_a = \Sigma_{a+1}\Sigma_b\Sigma_c + \Sigma_a\Sigma_{b+1}\Sigma_c + \Sigma_a\Sigma_b\Sigma_{c+1}.$$

If all $a \equiv b \equiv c \bmod 2$ or if a is odd and b, c are even, both sides of the equation vanish after applying (*). On the other hand, if a is even and b, c are odd, then (*) gives us

$$4G^{a+b+c-2} + 4G^{a+b+c-2} + 0 = 8G^{a+b+c-2} + 0 + 0,$$

showing that there is no new relation introduced. Since $3S_2$ is symmetric in A, B, C, no other parities of a, b, c need to be checked in this case.

Case 2: $A = B \neq C$ and none of them is a curtain M_n. In this case, we have

$$\Sigma_{a+1}\Sigma_c + \Sigma_{a+c} + \Sigma_{a+c} = \Sigma_a\Sigma_{c+1} + \Sigma_{a+1}\Sigma_c + \Sigma_{a+1}\Sigma_c.$$

If $a \equiv c \bmod 2$, both sides of the equation vanish after applying (*). If a is even and c is odd, we obtain

$$4G^{a+c-1} + 0 + 0 = 0 + 2G^{a+c-1} + 2G^{a+c-1},$$

showing that there is no new relation introduced. Again, by symmetry of the $3S_2$ relation, no other cases of a and c need to be checked.

Case 3: $A = B = C$ and none of them is a curtain M_n. In this case, we have

$$3\Sigma_{a+1} = 3\Sigma_{a+1},$$

showing immediately that there are no new relations introduced.

Case 4: $A = M_a$, B, C are three different surfaces. The $3S_2$ relation translates to

$$G^{a+b}\Sigma_c + G^{a+c}\Sigma_b + G^a\Sigma b + c = G^{a+1}\Sigma_b\Sigma_c + G^a\Sigma_{b+1}\Sigma_c + G^a\Sigma_b\Sigma_{c+1}.$$

If b, c are even, both sides vanish after applying (*). If b is even and c is odd, we obtain

$$2G^{a+b+c-1} + 0 + 2G^{a+b+c-1} = 0 + 4G^{a+b+c-1} + 0,$$

and if b is odd and c is even, we get

$$0 + 2G^{a+b+c-1} + 2G^{a+b+c-1} = 0 + 0 + 4G^{a+b+c-1}.$$

In both cases, no new relations are introduced.

Case 5: $A = B = M_a \neq C$. In this case, we get

$$G^{a+1}\Sigma_c + G^{a+c} + G^{a+c} = G^{a+1}\Sigma_c + G^{a+1}\Sigma_c + G^a\Sigma_{c+1}.$$

If c is odd, applying (*) yields

$$2G^{a+c} + G^{a+c} + G^{a+c} = 2G^{a+c} + 2G^{a+c} + 0,$$

and if c is even,

$$0 + G^{a+c} + G^{a+c} = 0 + 0 + 2G^{a+c}.$$

In both cases, no new relations are introduced.

Case 6: $A = M_a$, $B = C = M_c$, $M_a \neq M_c$. We have

$$G^{a+c} + G^{a+c} + G^a\Sigma_{c+1} = G^{a+1}\Sigma_c + G^a\Sigma_{c+1} + G^a\Sigma_{c+1}.$$

If c is odd, we obtain after applying (*)

$$G^{a+c} + G^{a+c} + 0 = 2G^{a+c} + 0 + 0,$$

and if c is even,

$$G^{a+c} + G^{a+c} + 2G^{a+c} = 0 + 2G^{a+c} + 2G^{a+c}.$$

In both cases, no new relations are introduced.

Case 7: $A = B = C = M_a$. As in the third case, we have

$$3G^a = 3G^a,$$

showing immediately that there are no new relations introduced.

The above shows that there are isomorphisms

$$\varphi : \hom_{\mathcal{E}}(D_{T_0}, D_{T_0}) \xrightarrow{\cong} \mathbb{Z}[G],$$
$$\psi : \hom_{\mathrm{Cob}^3_{/l}(2)}(B(D_{T_0}), B(D_{T_0})) \xrightarrow{\cong} \mathbb{Z}[G].$$

Consider the diagram

$$\begin{array}{ccc} & \mathbb{Z}[G] & \\ {}^{\varphi}\swarrow & & \searrow^{\psi} \\ \hom_{\mathcal{E}}(D_{T_0}, D_{T_0}) & \xrightarrow{B} & \hom_{\mathrm{Cob}^3_{/l}(2)}(B(D_{T_0}), B(D_{T_0})). \end{array}$$

By construction, this diagram commutes, i.e., $B \circ \varphi = \psi$. Since both φ and ψ are isomorphisms, B has to be an isomorphism as well. ∎

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