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Controlled K-theory and K-homology

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Abstract. Motivated by the idea that our access to spacetime is limited by the resolution of our measuring device, we give a new description of K-homology with a finite resolution. G. Yu introduced a C^*-algebra, known as the localization algebra, and showed that for any finite-dimensional simplicial complex X endowed with the spherical metric, the K-theory of the localization algebra is isomorphic to the K-homology of X. We give a coarse graining version of this theorem using controlled K-theory (also known as quantitative K-theory). Namely, instead of considering families of operators whose propagations converge to 0 as done in the definition of the localization algebra, we prove that for each dimension n, there exists a threshold $r_n > 0$ such that the K-homology of an n-dimensional finite simplicial complex X is isomorphic to a certain group of equivalence classes of operators whose propagation is less than r_n. This picture also enables us to represent any element in the K-homology group $K_*(X)$ by a finite matrix for a finite simplicial complex X.

1. Introduction

In this paper, we introduce a quantitative picture of the K-homology group $K_*(X)$ for a finite simplicial complex X, which enables us to present every element in the K-homology group by a finite-dimensional matrix following the idea of coarse graining in physics as discussed in [2]. In [2], based on the idea that we can only determine the underlying metric space up to a finite resolution, A. Connes and W. D. van Suijlekom proposed a noncommutative geometric framework to encode our limited resolution of physical measurements using a tolerance relation (symmetric and reflexive but not transitive relation). We approach K-homology based on a tolerance relation $\mathcal{R} = \{(x, y) : d(x, y) < r\} \subset X \times X$ for a fixed $r > 0$.

The K-homology group has already several well-known descriptions. Its first picture was given by Kasparov in [5]. He introduced the notion of the Fredholm modules and defined K-homology to be a certain group of equivalence classes of them, which is a topological invariant. The idea of the Fredholm module originated with a functional analytic abstraction of elliptic operators by Atiyah [1].

Another picture was given by G. Yu in [12]. It is known that the K-theory of Roe algebras $C^*(X)$ depends only on the large-scale structure of X, so it is not isomorphic to K-homology in general. But Yu defined a C^*-algebra $C_L^*(X)$ called the localization

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algebra generated by uniformly continuous bounded functions $f : [1, \infty) \to C^*(X)$ such that propagations of operators $\{f(t)\}_t$ converge to 0. The underlying idea is that if we restrict our attention to operators whose propagations are small, then we can recover local structures of X that we lost by forming the Roe algebra. Actually, he showed that the K-theory of the localization algebra is isomorphic to K-homology for finite-dimensional simplicial complexes. In [8], Y. Qiao and J. Roe showed that this holds for any locally compact metric space if the module is very ample. The goal of this paper is to realize K-homology using the controlled K-theory groups $K_*^{\varepsilon,r}(C^*(X))$. Instead of considering families of projections or unitaries whose propagations converge to 0, we only consider projections and unitaries whose propagations are smaller than a fixed threshold r. This is the coarse graining picture of K-homology in the same spirit as [2]. The recent progress to approximate KK-theory of C^*-algebras by controlled K-theory can be found in [11], but this paper focuses more on elementary examples (K-homology of finite simplicial complex) to provide a description using finite matrices.

We recall the notions of quasi-projection and quasi-unitary, which will be used in the rest of the paper. An operator p over $C^*(X)$ is called an (ε, r) quasi-projection if the propagation of p is at most r and p satisfies $\|p^2 - p\| < \varepsilon$ and $p = p^*$. An operator u over $C^*(X)$ is called an (ε, r) quasi-unitary if the propagation of u is at most r and u satisfies $\|u^*u - 1\| < \varepsilon$ and $\|uu^* - 1\| < \varepsilon$. The controlled K-groups $K_0^{\varepsilon,r}(C^*(X))$ and $K_1^{\varepsilon,r}(C^*(X))$ are defined as certain homotopy classes of (ε, r) quasi-projections and quasi-unitaries of matrices over $C^*(X)$; see Definition 2.7. For $0 < \varepsilon \le \varepsilon' < \frac{1}{4}$ and $0 < r \le r'$, we have a canonical forgetful map

$$\iota^{(\varepsilon,r),(\varepsilon',r')} : K_*^{\varepsilon,r}(C^*(X)) \to K_*^{\varepsilon',r'}(C^*(X)).$$

The image $\iota^{(\varepsilon,r),(\varepsilon',r')}(K_*^{\varepsilon,r}(C^*(X)))$ of this map can be regarded as a group generated by (ε, r) quasi-unitaries or quasi-projections with a relaxed equivalence relation given by the larger parameters (ε', r'). We denote this group by $K_*^{(\varepsilon,r),(\varepsilon',r')}(C^*(X))$. Then we can state the main theorem.

Theorem 1.1. *For each n, there exist a constant $\lambda_n > 1$, a function $h_n : (0, \frac{1}{4\lambda_n}) \to [1, \infty)$ and constants r_n and ε_n depending only on n such that for any n-dimensional finite simplicial complex X, we have*

$$K_*(X) \cong K_*^{(\varepsilon,r),(\lambda_n\varepsilon, h_n(\varepsilon)r)}(C^*(X)) \tag{1.1}$$

for all (ε, r) with $0 < \varepsilon < \varepsilon_n$ and $0 < r < r_n$.

The main technical tool in the proof of Theorem 1.1 is an asymptotically exact Mayer–Vietoris sequence [7] to extract information from simplicial complexes whose dimension is smaller than that of X. Note that the main idea of the proof appeared in [14], but our main theorem is the precise statement of [14, Theorem 4.7].

If X is compact, then its Roe algebra is the set of all compact operators, which is the inductive limit of matrix algebras. Therefore, each K-homology element of a finite simplicial complex can be expressed by a finite matrix via the isomorphism (1.1), as explained in Remark 6.7 with a geometric picture of discretization.

2. Roe algebras and controlled K-theory

In this section, we define some basic concepts such as support of operators, propagation, Roe algebra, filtered C^*-algebra and controlled K-theory.

For any Hilbert space H, we denote the set of all compact operators by $\mathcal{K}(H)$.

Definition 2.1. Let X be a locally compact metric space and H_X a separable Hilbert space. We say H_X is an ample X-module if there is a non-degenerate $*$-homomorphism

$$\rho : C_0(X) \to B(H_X)$$

such that $\rho(f)$ is not a compact operator for any $f \in C_0(X) \setminus \{0\}$. When it is clear from the context, we will write f in place of $\rho(f)$.

Definition 2.2. Let H_X be an X-module and H_Y be a Y-module. For $T \in B(H_X, H_Y)$, we define a subset $\mathrm{supp}(T) \subset Y \times X$ to be the complement of the set of all $(y, x) \in Y \times X$ such that there exist two functions $f \in C_0(X)$ and $g \in C_0(Y)$ such that

$$gTf = 0, \quad f(x) \neq 0, \quad g(y) \neq 0.$$

Also, we define the propagation of T by

$$\mathrm{prop}(T) := \sup\{d(x, y); (y, x) \in \mathrm{supp}(T)\} \in [0, \infty].$$

We now define a C^*-algebra $C^*(X)$, which will be used throughout this paper.

Definition 2.3. Let H_X be an ample non-degenerate X-module. We define the following algebras:

$$\mathbb{C}[H_X] := \{T \in B(H_X); Tf \in \mathcal{K}(H_X) \text{ for any } f \in C_0(X) \text{ and } \mathrm{prop}(T) < \infty\}$$
$$C^*(H_X) := \overline{\mathbb{C}[H_X]},$$

where the closure is taken with respect to the operator norm in $B(H_X)$.

When the module H_X is clear from the context, we will write $C^*(X)$ instead of $C^*(H_X)$.

Definition 2.4. A filtered C^*-algebra A is a C^*-algebra equipped with a family $(A_r)_{r>0}$ of closed linear subspaces parametrized by positive numbers r such that

(1) $A_r \subset A_{r'}$ if $r \leq r'$,

(2) $A_r^* = A_r$,

(3) $A_r \cdot A_{r'} \subset A_{r+r'}$,

(4) the subalgebra $\bigcup_{r>0} A_r$ is dense in A.

If A is unital, we also require that the identity 1 is an element of A_r for every positive number r.

Remark 2.5. A Roe algebra $C^*(X)$ is filtered by propagation of operators, namely $C^*(X)_r$ is the closure of the set of operators whose propagation is at most r.

We can define the controlled K-theory for filtered C^*-algebras, where "control" refers to control of propagations.

Definition 2.6. Let A be a unital filtered C^*-algebra and ε and r be positive numbers. We define the sets of (ε, r) quasi-projections and (ε, r) quasi-unitaries in A, respectively, by

$$P^{\varepsilon,r}(A) := \{p \in A_r; p^* = p, \|p^2 - p\| < \varepsilon\}$$
$$U^{\varepsilon,r}(A) := \{u \in A_r; \|u^*u - 1\| < \varepsilon, \|uu^* - 1\| < \varepsilon\}.$$

For any positive integer n, we set $P_n^{\varepsilon,r}(A) = P^{\varepsilon,r}(M_n(A))$ and $U_n^{\varepsilon,r}(A) = U^{\varepsilon,r}(M_n(A))$. Then there are canonical inclusions

$$P_n^{\varepsilon,r}(A) \hookrightarrow P_{n+1}^{\varepsilon,r}(A); \quad p \mapsto \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$U_n^{\varepsilon,r}(A) \hookrightarrow U_{n+1}^{\varepsilon,r}(A); \quad u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

With respect to these inclusions, their unions are denoted by

$$P_\infty^{\varepsilon,r} = \bigcup_{n=1}^{\infty} P_n^{\varepsilon,r}$$

and

$$U_\infty^{\varepsilon,r} = \bigcup_{n=1}^{\infty} U_n^{\varepsilon,r}.$$

Definition 2.7. Let A be a filtered unital C^*-algebra, r and ε be positive numbers with $\varepsilon < \frac{1}{4}$. We define the following equivalence relations on $P_\infty^{\varepsilon,r}(A) \times \mathbb{N}$ and $U_\infty^{\varepsilon,r}(A)$:

(1) For $p, q \in P_\infty^{\varepsilon,r}(A)$ and $\ell, \ell' \in \mathbb{N}$, $(p, \ell) \sim_{\varepsilon,r} (q, \ell')$ if there exists a positive integer k such that $\mathrm{diag}(p, I_{k+\ell'})$ and $\mathrm{diag}(q, I_{k+\ell})$ are homotopic in $P_\infty^{\varepsilon,r}(A)$.

(2) For $u, v \in U_\infty^{\varepsilon,r}(A)$, $u \sim_{3\varepsilon,2r} v$ if u and v are homotopic in $U_\infty^{3\varepsilon,2r}(A)$.
By these equivalence relations, the controlled K-groups are defined by

$$K_0^{\varepsilon,r}(A) := (P_\infty^{\varepsilon,r}(A) \times \mathbb{N})/\sim_{\varepsilon,r}$$
$$K_1^{\varepsilon,r}(A) := U_\infty^{\varepsilon,r}(A)/\sim_{3\varepsilon,2r}.$$

If A is non-unital, by letting $\widetilde{A} = (A_r + \mathbb{C})_r$ be the unitization of A, define

$$K_0^{\varepsilon,r}(A) := \{[p,\ell]_{\varepsilon,r} \in P_\infty^{\varepsilon,r}(\widetilde{A}) \times \mathbb{N}/\sim_{\varepsilon,r};\ \chi(\rho_A(p)) = \ell\}$$
$$K_1^{\varepsilon,r}(A) := U_\infty^{\varepsilon,r}(\widetilde{A})/\sim_{3\varepsilon,2r},$$

where $\rho_A : \widetilde{A} \twoheadrightarrow \mathbb{C}$ is the projection onto the scalar and χ is the characteristic function of the interval $[\frac{1}{2}, \infty)$. Also, we define the additive structures by

$$[p,\ell]_{\varepsilon,r} + [q,\ell']_{\varepsilon,r} = [\mathrm{diag}(p,q), \ell+\ell']_{\varepsilon,r}$$
$$[u]_{3\varepsilon,2r} + [v]_{3\varepsilon,2r} = [\mathrm{diag}(u,v)]_{3\varepsilon,2r}.$$

Remark 2.8. With these notations, $K_*^{\varepsilon,r}(A)$ is an abelian group. The reason to consider $(3\varepsilon, 2r)$-homotopy is to make $K_1^{\varepsilon,r}$ an abelian group [6, Lemma 1.15 and Remark 1.17].

Remark 2.9. Let A be a filtered unital C^*-algebra. For $0 < \delta < \frac{1}{4}$, $0 < \varepsilon < \frac{1}{4}$ and $r > 0$, assume p is an (ε, r) quasi-projection. If $p' \in A_r$ and $\|p - p'\| < \delta$, then p' is an $(\varepsilon + 5\delta, r)$ quasi-projection and the homotopy which connects p and p' linearly is a homotopy through $(\varepsilon + 5\delta, r)$ quasi-projections. This is because we have

$$\begin{aligned}\|p'^2 - p'\| &\le \|p'^2 - p'p\| + \|p'p - p^2\| + \|p^2 - p\| + \|p - p'\| \\ &< 2\delta + 2\delta + \varepsilon + \delta = 5\delta + \varepsilon,\end{aligned}$$

and for any $t \in [0,1]$ by applying the above formula to $p_t = (tp + (1-t)p')$,

$$\|p_t^2 - p_t\| < 5\|p - p_t\| + \varepsilon \le 5\delta + \varepsilon.$$

The corresponding statement is also true for quasi-unitaries.

Definition 2.10. We have the following maps from the controlled K-theory to K-theory. Assume A is a unital filtered C^*-algebra.

Let $\chi = \chi_{[\frac{1}{2},\infty)}$ be the characteristic function of $[\frac{1}{2}, \infty)$. If $p \in M_n(A)$ is self-adjoint and satisfies $\|p^2 - p\| < \frac{1}{4}$, then $\frac{1}{2} \notin \sigma(p)$. By continuous functional calculus, we have a map

$$\kappa : K_0^{\varepsilon,r}(A) \to K_0(A); \quad [p,\ell] \mapsto [\chi(p)] - \ell \cdot [1].$$

If $u \in M_n(A)$ satisfies $\|uu^* - 1\| < \frac{1}{4}$ and $\|u^*u - 1\| < \frac{1}{4}$, then u is invertible. We define

$$\kappa : K_1^{\varepsilon,r}(A) \to K_1(A); \quad [u] \mapsto \left[\frac{u}{(u^*u)^{\frac{1}{2}}}\right].$$

These two maps are called the comparison maps after [3, Definitions 4.1 and 4.4].

3. Overview of quantitative objects

In this section, we introduce a general framework to deal with controlled K-theory as a family of groups $(K_*^{\varepsilon,r}(C^*(X)))_{\varepsilon,r}$ systematically. Everything up to Definition 3.5 is contained in [7, Section 1]. In Definition 3.6, we introduce a notion of equivalence between two quantitative objects. By using this concept, for an $(n-1)$-dimensional finite simplicial complex Z and an n-dimensional finite simplicial complex $\widetilde{Z}$, we can replace $K_*^{\varepsilon,r}(C^*(\widetilde{Z}))$ by $K_*^{\varepsilon,r}(C^*(Z))$ in a long exact sequence to use induction hypothesis on the dimension.

In the subsequent definitions, we define quantitative objects and morphisms between them abstractly. Main examples of quantitative objects are controlled K-theory of Roe algebras.

Definition 3.1. A quantitative object is a family $\mathcal{O} = (O^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}$ of abelian groups, together with group homomorphisms

$$\iota^{\varepsilon,\varepsilon',r,r'} : O^{\varepsilon,r} \longrightarrow O^{\varepsilon',r'}$$

for $0<\varepsilon\le\varepsilon'<\frac{1}{4}$ and $0<r\le r'$ such that

(1) $\iota^{\varepsilon,\varepsilon,r,r} = Id_{O^{\varepsilon,r}}$,
(2) $\iota^{\varepsilon',\varepsilon'',r',r''}\circ\iota^{\varepsilon,\varepsilon',r,r'} = \iota^{\varepsilon,\varepsilon'',r,r''}$ for any $0<\varepsilon\le\varepsilon'\le\varepsilon''<\frac{1}{4}$ and $0<r\le r'\le r''$.

These maps $\iota^{\varepsilon,\varepsilon',r,r'}$ are called structure maps.

Remark 3.2. We apply the abstract terminology of quantitative object to controlled K-groups $O^{\varepsilon,r} = K_*^{\varepsilon,r}(C^*(X))$ for locally compact metric space X. In this case, the structure maps $\iota^{\varepsilon,\varepsilon',r,r'}$ are the forgetful maps which assign an equivalence class $[p,\ell]$ to the equivalence class $[p,\ell]$ represented by the same pair (p,ℓ).

Definition 3.3. A control pair is a pair (λ,h) satisfying

(1) $\lambda>1$,
(2) $h:(0,\frac{1}{4\lambda})\to(1,+\infty)$ is a map such that there exists a non-increasing map $g:(0,\frac{1}{4\lambda})\to(1,+\infty)$, with $h\le g$.

For two control pairs (λ,h) and (λ',h'), we define their composition $(\lambda,h)*(\lambda',h') = (\lambda\lambda',h*h')$ by

$$h*h' : \left(0,\frac{1}{4\lambda\lambda'}\right)\to(1,+\infty);\quad \varepsilon\mapsto h(\lambda'\varepsilon)h'(\varepsilon).$$

For a control pair (λ,h) and a quantitative object $\mathcal{O} = (O^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}$, we sometimes write its structure map as $\iota^{(\varepsilon,r),(\lambda,h)(\varepsilon,r)}$ in the sense of $\iota^{\varepsilon,\lambda\varepsilon,r,h(\varepsilon)r}$.

Definition 3.4. Let (λ,h) be a control pair, r a positive number and $\mathcal{O} = (O^{\varepsilon,s})_{0<\varepsilon<\frac{1}{4},s>0}$ and $\mathcal{O}' = (O'^{\varepsilon,s})_{0<\varepsilon<\frac{1}{4},s>0}$ be quantitative objects. A (λ,h)-controlled morphism of order r

$$\mathcal{F}:\mathcal{O}\to\mathcal{O}'$$

is a family $\mathcal{F} = (F^{\varepsilon,s})_{0<\varepsilon<\frac{1}{4\lambda},0<s<\frac{r}{h(\varepsilon)}}$ of group homomorphisms

$$F^{\varepsilon,s} : O^{\varepsilon,s} \to O'^{\lambda\varepsilon,h(\varepsilon)s}$$

which are compatible with structure maps.

The next definition is about the exactness of a sequence of morphisms.

Definition 3.5. Let (λ, h) be a control pair, and let $\mathcal{O}$, $\mathcal{O}'$ and $\mathcal{O}''$ be quantitative objects. Let

$$\mathcal{F} : \mathcal{O} \to \mathcal{O}'$$

be an $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$-controlled morphism, and let

$$\mathcal{G} : \mathcal{O}' \to \mathcal{O}''$$

be an $(\alpha_{\mathcal{G}}, k_{\mathcal{G}})$-controlled morphism. Then the composition

$$\mathcal{O} \xrightarrow{\mathcal{F}} \mathcal{O}' \xrightarrow{\mathcal{G}} \mathcal{O}''$$

is said to be (λ, h)-exact at $\mathcal{O}'$ of degree r if $\mathcal{G} \circ \mathcal{F} = 0$ and for any $0 < \varepsilon < \frac{1}{4\max\{\lambda\alpha_{\mathcal{F}},\alpha_{\mathcal{G}}\}}$, any $0 < s < \frac{1}{k_{\mathcal{F}}(h(\lambda\varepsilon))r}$ and any $y \in O'^{\varepsilon,s}$ such that $G^{\varepsilon,s}(y) = 0$, there exists an element $x \in O^{\lambda\varepsilon,h(\varepsilon)s}$ such that

$$F^{\lambda\varepsilon,h(\lambda\varepsilon)s}(x) = \iota^{\varepsilon,\alpha_{\mathcal{F}}\lambda\varepsilon,s,k_{\mathcal{F}}(h(\lambda\varepsilon))s}(y),$$

as the following diagram:

$$\begin{array}{ccccc} & & y \in O'^{\varepsilon,s} & \xrightarrow{G^{\varepsilon,s}} & O''^{\alpha_{\mathcal{G}}\varepsilon,k_{\mathcal{G}}(\varepsilon)s} \ni G^{\varepsilon,s}(y) = 0 \\ & & \Big\downarrow \iota & & \\ x \in O^{\lambda\varepsilon,h(\varepsilon)s} & \xrightarrow{F^{\lambda\varepsilon,h(\varepsilon)s}} & O'^{\alpha_{\mathcal{F}}\lambda\varepsilon,k_{\mathcal{F}}(h(\lambda\varepsilon))s} & & \end{array}$$

Next, we define the equivalence between quantitative objects.

Definition 3.6. Let (λ, h) be a control pair. A (λ, h)-controlled morphism of degree r between quantitative objects $\mathcal{O} = (O^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}$ and $\mathcal{O}' = (O'^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}$

$$\mathcal{F} : \mathcal{O} \longrightarrow \mathcal{O}'$$

is said to be a (λ, h)-controlled equivalence morphism between $\mathcal{O}$ and $\mathcal{O}'$ if there exists a (λ, h)-controlled morphism

$$\mathcal{G} : \mathcal{O}' \longrightarrow \mathcal{O}$$

such that for all $s > 0$ with $h(\lambda\varepsilon)h(\varepsilon)s < r$, we have

$$G^{\lambda\varepsilon,h(\varepsilon)s} \circ F^{\varepsilon,s} = \iota^{\varepsilon,\lambda^2\varepsilon,s,h(\lambda\varepsilon)h(\varepsilon)s}$$
$$F^{\lambda\varepsilon,h(\varepsilon)s} \circ G^{\varepsilon,s} = \iota^{\varepsilon,\lambda^2\varepsilon,s,h(\lambda\varepsilon)h(\varepsilon)s}.$$

If such a control pair (λ, h) and (λ, h)-controlled morphisms $\mathcal{F}$ and $\mathcal{G}$ exist, then we say that the two quantitative objects $\mathcal{O}$ and $\mathcal{O}'$ are asymptotically equivalent.

Remark 3.7. In some literature, each subspace A_r is not required to be closed in Definition 2.4, and we can define the controlled K-theory in the same way without assuming it (e.g., [6]). Here we remark that this difference is not important K-theoretically. Let A be a unital filtered C^*-algebra with possibly "non-closed" subspaces A_r, and we encode a different filtration on the same C^*-algebra by $A'_r = \overline{A_r}$. Then one can show that two quantitative objects $(K_*^{\varepsilon,r}(A))_{\varepsilon,r}$ and $(K_*^{\varepsilon,r}(A'))_{\varepsilon,r}$ are asymptotically equivalent. We have a natural map

$$\iota^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \to K_*^{\varepsilon,r}(A'); \quad [p,n] \mapsto [p,n].$$

We show the map

$$\sigma^{\varepsilon,r} : P_\infty^{\varepsilon,r}(A') \to K_0^{2\varepsilon,r}(A); \quad p \mapsto [q]$$

is well defined on the K-theoretic level, where $q \in M_\infty(A_r)$ is any element which satisfies $\|p - q\| < \frac{1}{15}\varepsilon$. If we have a homotopy $(p_t)_t$ through $P_\infty^{\varepsilon,r}(A')$, then there exists k such that we have $\|p_t - p_{t'}\| < \frac{1}{15}\varepsilon$ whenever $|t - t'| \leq \frac{1}{k}$. Take a partition of the interval

$$0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$$

with $t_j = \frac{j}{k}$, and for each $j = 0, 1, \ldots, k$ we can take $q_j \in A_r$ such that $\|p_{t_j} - q_j\| < \frac{1}{20}\varepsilon$. By linearly interpolating q_j's at t_j, we can define another homotopy $(q_t)_t$. Then $(q_t)_t$ satisfies $\|p_t - q_t\| < \frac{1}{5}\varepsilon$ for all t, so especially $(q_t)_t$ is a homotopy through $P_\infty^{2\varepsilon,r}(A)$ by Remark 2.9. Therefore, we have a map

$$\sigma^{\varepsilon,r} : K_*^{\varepsilon,r}(A') \to K_*^{2\varepsilon,r}(A); \quad [p,n]_{\varepsilon,r} \mapsto [q,n]_{2\varepsilon,r}.$$

Clearly compositions of $\iota^{\varepsilon,r}$ and $\sigma^{\varepsilon,r}$ are the same as forgetful maps.

Let

$$\mathcal{O}_0 \xrightarrow{\mathcal{F}} \mathcal{O}_1 \xrightarrow{\mathcal{G}} \mathcal{O}_2$$

be a sequence of (α, k)-controlled morphism $\mathcal{F}$ and (β, l)-controlled morphism $\mathcal{G}$. Assume $\mathcal{O}'_i$ is a controlled object which is asymptotically equivalent to $\mathcal{O}_i$ via (λ, h)-controlled morphisms

$$\mathcal{H}_i : \mathcal{O}_i \longrightarrow \mathcal{O}'_i$$
$$\mathcal{J}_i : \mathcal{O}'_i \longrightarrow \mathcal{O}_i$$

for $i = 0, 1, 2$. We can define a $(\lambda_1, h_1) = (\lambda, h) * (\alpha, k) * (\lambda, h)$-controlled morphism $\mathcal{F}' = (F'^{\varepsilon,r})$ and $(\lambda_2, h_2) = (\lambda, h) * (\beta, l) * (\lambda, h)$-controlled morphism $\mathcal{G}' = (G'^{\varepsilon,r})$ by

$$F'^{\varepsilon,r} : O_0'^{\varepsilon,r} \xrightarrow{J_0^{\varepsilon,r}} O_0^{\lambda\varepsilon,h(\varepsilon)r} \xrightarrow{F^{\lambda\varepsilon,h(\varepsilon)r}} O_1^{\alpha\lambda\varepsilon,k(\lambda\varepsilon)h(\varepsilon)r} \xrightarrow{H_1^{\alpha\lambda\varepsilon,k(\lambda\varepsilon)h(\varepsilon)r}} O_1'^{\lambda_1\varepsilon,h_1(\varepsilon)r}$$

$$G'^{\varepsilon,r} : O_1'^{\varepsilon,r} \xrightarrow{J_1^{\varepsilon,r}} O_1^{\lambda\varepsilon,h(\varepsilon)r} \xrightarrow{G^{\lambda\varepsilon,h(\varepsilon)r}} O_2^{\beta\lambda\varepsilon,l(\lambda\varepsilon)h(\varepsilon)r} \xrightarrow{H_2^{\beta\lambda\varepsilon,k(\lambda\varepsilon)h(\varepsilon)r}} O_2'^{\lambda_2\varepsilon,h_2(\varepsilon)r}$$

so that the following controlled diagram commutes:

$$\begin{array}{ccccc} \mathcal{O}_0 & \xrightarrow{\mathcal{F}} & \mathcal{O}_1 & \xrightarrow{\mathcal{G}} & \mathcal{O}_2 \\ \cong\uparrow & & \cong\uparrow & & \cong\uparrow \\ \mathcal{O}_0' & \xrightarrow{\mathcal{F}'} & \mathcal{O}_1' & \xrightarrow{\mathcal{G}'} & \mathcal{O}_2' \end{array}$$

We can easily see that the exactness of $(\mathcal{F}, \mathcal{G})$ passes to $(\mathcal{F}', \mathcal{G}')$.

Lemma 3.8. *If $(\mathcal{F}, \mathcal{G})$ is (δ, p)-exact, then there exists a controlled pair (δ', p') such that $(\mathcal{F}', \mathcal{G}')$ is (δ', p')-exact.*

4. Functoriality and homotopy invariance of $K_*^{\varepsilon,r}(C^*(X))$

In this section, we formulate how a coarse map between two locally compact metric spaces induces a map between their controlled K-groups, and we prove the homotopy invariance of this induced map under certain conditions. It is well known that the K-theory of the Roe algebra is a coarse functor: if we have a coarse map between two locally compact metric spaces X and Y, then we have an induced map

$$f_* : K_*(C^*(X)) \to K_*(C^*(Y)).$$

We give a controlled version of this morphism. After that, we discuss the homotopy invariance of this functor. This is analogous to [13, Lemma 4.8], which states the homotopy invariance of K-theory of a different C^*-algebra $C^*_{L,0}(X)$. We show an analogous statement for $C^*(X)$ by a similar argument.

Definition 4.1. Let (X, d_X) and (Y, d_Y) be locally compact metric spaces and $f : X \to Y$ be a Borel map between them. For each positive number $r \geq 0$, we define the expansion function of f by

$$\omega_f(r) := \sup\{d_Y(f(x_1), f(x_2)); d_X(x_1, x_2) < r\}.$$

We say that f is a coarse map if $\omega_f(r) < \infty$ for each $r \geq 0$ and $f^{-1}(K)$ is a bounded set for each bounded subset $K \subset Y$.

Definition 4.2. Let X and Y be two locally compact metric spaces, H_X and H_Y be ample X and Y-modules, respectively, $f : X \to Y$ be a coarse map and $\delta > 0$. An isometry

$$V_f : H_X \longrightarrow H_Y$$

is called a δ-cover of f if $d(y, f(x)) < \delta$ for any $(y, x) \in \operatorname{supp}(V_f) \subset Y \times X$.

Remark 4.3. For any X, Y, f and δ as above, a δ-cover V_f of f exists. This is proven in [13, Lemma 2.4].

The proof of the next theorem is provided by [4, Proposition 6.3.12].

Theorem 4.4. *Under the setting of Definition 4.2, for any $\delta > 0$, a δ-cover V_f exists and* $\mathrm{Ad}(V_f)$ *restricts to*

$$\mathrm{Ad}(V_f) : C^*(X) \longrightarrow C^*(Y),$$

and the induced map on K-theory

$$f_* := \mathrm{Ad}(V_f)_* : K_*(C^*(X)) \longrightarrow K_*(C^*(Y))$$

does not depend on δ and a δ-cover V_f of f.

Remark 4.5. Assume $f : X \to Y$ is a coarse map and V_f is a δ-cover of f, then we have

$$\mathrm{prop}(\mathrm{Ad}(V_f)(T)) < \omega_f(r) + 2\delta$$

for any $T \in B(H_X)$ whose propagation is at most r. Therefore, we can define

$$\mathrm{Ad}(V_f)_* : K_*^{\varepsilon,r}(C^*(X)) \longrightarrow K_*^{\varepsilon,\omega_f(r)+2\delta}(C^*(Y)).$$

Lemma 4.6. *Fix two positive numbers δ and r. Let $f : X \to Y$ be coarse maps with $\omega_f(r) < R$ and $V_f, V_f' : H_X \to H_Y$ be δ-cover maps of f. Then we have*

$$\iota^{(\varepsilon,R+2\delta),(\varepsilon,R+8\delta)} \circ \mathrm{Ad}(V_f)_* = \iota^{(\varepsilon,R+2\delta),(\varepsilon,R+8\delta)} \circ \mathrm{Ad}(V_f')_*,$$

as maps from $K_*^{\varepsilon,r}(C^*(X))$ to $K_*^{\varepsilon,R+8\delta}(C^*(Y))$.

Proof. Let U be the unitary

$$U = \begin{pmatrix} 1 - V_f V_f^* & V_f V_f'^* \\ V_f' V_f^* & 1 - V_f' V_f'^* \end{pmatrix},$$

which is an element in the multiplier of $M_2(C^*(Y))$, and we have

$$\mathrm{Ad}\, U \begin{pmatrix} \mathrm{Ad}(V_f)(T) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Ad}(V_f')(T) \end{pmatrix}$$

for any $T \in C^*(H_X)$. Since $\text{prop}(U) < 2\delta$, conjugation by the family of unitaries

$$\widetilde{U}_t := \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & U^* \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \in M_4(C^*(Y))$$

defines a homotopy between $\text{diag}(\text{Ad}(V_f)(p), 0, 0, 0)$ and $\text{diag}(0, \text{Ad}(V'_f)(p), 0, 0)$ through $(\varepsilon, R + 8\delta)$-projections for any (ε, r) quasi-projection p. Hence,

$$[\text{Ad}(V_f)(p)] = [\text{Ad}(V'_f)(p)] \in K_0^{\varepsilon, R+8\delta}(C^*(Y)). \qquad \blacksquare$$

Remark 4.7. For a coarse map $f : X \to Y$, with $\omega_f(r) < R$ and any $\delta > 0$, we have a well-defined homomorphism

$$\text{Ad}(V_f)_* : K_*^{\varepsilon,r}(C^*(H_X)) \longrightarrow K_*^{\varepsilon,R+8\delta}(C^*(H_Y))$$

independently of the choice of δ-cover V_f of f.

We use this remark in the following way. Let H_X and H'_X be ample X-modules. For each $\delta > 0$, there exist δ-covers of id_X

$$V_1 : H_X \to H'_X \quad \text{and} \quad V_2 : H'_X \to H_X.$$

Then since V_2V_1 and V_1V_2 are 2δ-covers of id_X, by Remark 4.7, $\text{Ad}(V_1)_*$ and $\text{Ad}(V_2)_*$ are asymptotically inverse to each other in the following sense.

Proposition 4.8. *Under the above setting,*

$$\text{Ad}(V_2)_* \circ \text{Ad}(V_1)_* : K_*^{\varepsilon,r}(C^*(H_X)) \to K_*^{\varepsilon,r+16\delta}(C^*(H'_X)) \to K_*^{\varepsilon,r+32\delta}(C^*(H_X))$$

is equal to the forgetful map $\iota^{(\varepsilon,r),(\varepsilon,r+32\delta)}$ *and the composition*

$$\text{Ad}(V_1)_* \circ \text{Ad}(V_2)_* : K_*^{\varepsilon,r}(C^*(H'_X)) \to K_*^{\varepsilon,r+16\delta}(C^*(H_X)) \to K_*^{C\varepsilon,r+32\delta}(C^*(H'_X))$$

is equal to the forgetful map $\iota^{(\varepsilon,r),(\varepsilon,r+32\delta)}$.

Next, we discuss homotopy invariance.

Definition 4.9. Two coarse maps $f, g : X \to Y$ are said to be strongly Lipschitz homotopic if there exists a continuous homotopy

$$F : [0,1] \times X \longrightarrow Y$$

such that

(1) $F(t, \cdot) : X \to Y$ is a proper map for each $t \in [0, 1]$,

(2) the family $\{F(t, \cdot) : X \to Y\}$ is uniformly Lipschitz, namely each $F(t, \cdot)$ is Lipschitz with a Lipschitz constant c that is independent of $t \in [0, 1]$,

(3) $\{F(\cdot, x) : [0,1] \to Y\}_{x \in X}$ is uniformly equicontinuous,

(4) $F(0, \cdot) = f$ and $F(1, \cdot) = g$.

The next theorem is almost the same as [13, Lemma 4.8], where the analogous statement for the C^*-algebra $C^*_{L,0}(X)$ is shown.

Theorem 4.10. *Let $f, g : X \to Y$ be strongly Lipschitz homotopic via F whose uniform Lipschitz constant is bounded by c and V_f, and let V_g be their δ-covers of f and g, respectively. Then*

$$\mathrm{Ad}(V_f)_* = \mathrm{Ad}(V_g)_* : K_*^{\varepsilon,r}(C^*(X)) \longrightarrow K_*^{21\varepsilon,5(cr+2\delta)}(C^*(Y)).$$

Proof. We will show this for $* = 1$, and a similar argument works for $* = 0$. We can take a partition of the interval

$$0 = t_0 < t_1 < \cdots < t_{l-1} < t_l = 1$$

such that $\|F(t_j, x) - F(t_{j+1}, x)\| < \delta$ for all $x \in X$ and $j = 1, 2, \dots, l-1$ by the third condition. We write $f_j := F(t_j, \cdot)$. For each $j = 1, 2, \dots, l$, we can take a δ-cover $V_j : H_X \to H_Y$ of f_j. Let u be an (ε, r) quasi-unitary over $\widetilde{C}^*(X)$. Define

$$u_i := (\mathrm{Ad}\, V_{f_i})(u) \quad (i = 0, 1, \dots, l),$$

where $\mathrm{Ad}\, V_{f_i}$ is a unital extension of the $*$-homomorphism $\mathrm{Ad}\, V_{f_i} : C^*(X) \to C^*(Y)$. For each i, define $w_i := u_i u_l^*$, and consider $(3\varepsilon, 2(cr + 2\delta))$ quasi-unitary operators acting on $\bigoplus_{i=0}^{l}(H_Y \oplus H_Y)$:

$$a := \bigoplus_{i=0}^{l}(w_i \oplus I) = (w_0 \oplus I) \oplus (w_1 \oplus I) \oplus \cdots \oplus (w_{l-1} \oplus I) \oplus (w_l \oplus I)$$

$$\begin{aligned} b &:= \bigoplus_{i=0}^{l-1}(w_{i+1} \oplus I) \oplus (w_l \oplus I) \\ &= (w_1 \oplus I) \oplus (w_2 \oplus I) \oplus \cdots \oplus (w_l \oplus I) \oplus (w_l \oplus I) \end{aligned}$$

$$c := (w_l \oplus I) \oplus \bigoplus_{i=1}^{l}(w_i \oplus I) = (w_l \oplus I) \oplus (w_1 \oplus I) \oplus (w_2 \oplus I) \oplus \cdots \oplus (w_l \oplus I).$$

For $t \in [0,1]$, we define an isometry $V_{i,i+1}(t) : (H_X \oplus H_X) \to (H_Y \oplus H_Y)$ by

$$V_{i,i+1}(t) = R(t) \begin{pmatrix} V_i & 0 \\ 0 & V_{i+1} \end{pmatrix} R(t)^*,$$

where $R(t)$ is the $\frac{\pi}{2}t$-rotation matrix $\begin{pmatrix} \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \\ -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix}$. Note that since $V_{i,i+1}(t)$ is a 2δ-cover of f_i with respect to ample modules $(H_X \oplus H_X)$ and $(H_Y \oplus H_Y)$, the operator $\mathrm{Ad}(V_{i,i+1}(t))(u \oplus I)$ is $(\varepsilon, cr + 4\delta)$ quasi-unitary and

$$\begin{aligned} \mathrm{Ad}(V_{i,i+1}(0))(u \oplus I) &= u_i \oplus I \\ \mathrm{Ad}(V_{i,i+1}(1))(u \oplus I) &= u_{i+1} \oplus I. \end{aligned}$$

Therefore,

$$\left(\bigoplus_{i=0}^{l-1} \mathrm{Ad}(V_{i,i+1}(t))(u \oplus I) \bigoplus (u_l \oplus I)\right) \circ \bigoplus_{i=0}^{l} (u_l^* \oplus I)$$

is a homotopy through $(3\varepsilon, 2(cr + 4\delta))$ quasi-unitaries between a and b. We can show that b and c are $(3\varepsilon, 2(cr + 2\delta))$ quasi-unitary-homotopic by rotating the first l-coordinates and the last coordinate. Therefore, a and c are homotopic through $(3\varepsilon, 2(cr + 4\delta))$ quasi-unitaries.

Note that since $\|u_0 - w_l^* w_0 u_l\| \le 4\varepsilon$, the two $(21\varepsilon, 5(cr + 4\delta))$ quasi-unitaries

$$(u_0 \oplus I) \oplus \bigoplus_{k=1}^{l} (I \oplus I) \quad \text{and} \quad c^* a \left((u_l \oplus I) \oplus \bigoplus_{k=1}^{l} (I \oplus I) \right)$$

are homotopic through $(21\varepsilon, 5(cr + 4\delta))$ quasi-unitaries by Remark 2.9. Similarly,

$$a^* a \left((u_l \oplus I) \oplus \bigoplus_{k=1}^{l} (I \oplus I) \right) \quad \text{and} \quad (u_l \oplus I) \oplus \bigoplus_{k=1}^{l} (I \oplus I)$$

are homotopic through $(21\varepsilon, 5(cr + 4\delta))$ quasi-unitaries. We can construct a homotopy through $(6\varepsilon, 5(cr + 4\delta))$ quasi-unitaries between

$$c^* a \left((u_l \oplus I) \oplus \bigoplus_{k=1}^{l} (I \oplus I) \right) \quad \text{and} \quad a^* a \left((u_l \oplus I) \oplus \bigoplus_{k=1}^{l} (I \oplus I) \right)$$

using the homotopy between a and c. This proves that

$$\mathrm{Ad}(V_f)_*([u]) = \mathrm{Ad}(V_g)_*([u]) \in K_1^{21\varepsilon, 5(cr+4\delta)}(C^*(H_Y)). \qquad \blacksquare$$

5. Asymptotic Mayer–Vietoris exact sequence of controlled K-theory

In this section, we decompose an n-dimensional simplicial complex X into two pieces to obtain a Mayer–Vietoris sequence of controlled K-theories. First, we recall the condition of decomposition of filtered C^*-algebra to obtain a controlled Mayer–Vietoris sequence following [7], and then we show that this decomposition of a simplicial complex satisfies the conditions. Our purpose in this section is to build a long asymptotically exact sequence to obtain information of $K_*^{\varepsilon,r}(C^*(X))$ from simplicial complexes whose dimension is less than $\dim X$. With respect to a different filtration (so-called Lipschitz filtration), the same technique is used in [9].

We encode the spherical metric on each simplex [10, Definition 7.2.1].

Definition 5.1 ([7, Definition 2.6 and Remark 2.7]). Let A be a filtered C^*-algebra and let r be a positive number. A completely coercive decomposition pair of degree r for A is a pair (Δ_1, Δ_2) of closed linear subspaces of A_r such that there exists a positive number C satisfying the following:

For any positive number s with $s \leq r$, any positive integer n and any $x \in M_n(A_s)$, there exist $y \in M_n(A_s \cap \Delta_1)$ and $z \in M_n(A_s \cap \Delta_2)$ with

$$\|y\| \leq C\|x\|, \quad \|z\| \leq C\|x\| \quad \text{and} \quad x = y + z.$$

The constant C is called the coercity of the decomposition.

Definition 5.2 ([7, Definitions 2.8 and 2.13]). Let A be a filtered C^*-algebra, r be any positive number and Δ be a closed subspace of A_r. A filtered C^*-subalgebra $B = (B \cap A_s)_s$ of A is called an r-neighborhood of Δ if it contains the subspace

$$N_{\Delta}^{(r,5r)} = \Delta + A_{5r} \cdot \Delta + \Delta \cdot A_{5r} + A_{5r} \cdot \Delta \cdot A_{5r}.$$

Definition 5.3 ([7, Definition 2.15]). Let A be a C^*-algebra. A pair (S_1, S_2) of subsets of A is said to satisfy the complete intersection approximation (CIA) property if there exists $C > 0$ such that for any positive number ε, any positive integer n and any $x \in M_n(S_1)$ and $y \in M_n(S_2)$ with $\|x - y\| \leq \varepsilon$, there exists $z \in M_n(S_1 \cap S_2)$ such that

$$\|z - x\| \leq C\varepsilon, \quad \|z - y\| \leq C\varepsilon.$$

The constant C is called the coercity of the pair (S_1, S_2).

Definition 5.4 ([7, Definition 2.16]). Let r be any positive number. An r-controlled weak Mayer–Vietoris pair for a filtered C^*-algebra A is a quadruple $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ such that

(1) (Δ_1, Δ_2) is a completely coercive decomposition pair for A of order r with coercity C,

(2) A_{Δ_i} is an r-neighborhood of Δ_i for $i = 1, 2$,

(3) the pair $(A_{\Delta_1,s}, A_{\Delta_2,s})$ has the CIA property for any positive number $s \leq r$ with coercity C,

for a positive number $C > 0$. The number C is called the coercity of the r-controlled Mayer–Vietoris pair $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$.

Now we can state the main technical tool of this section. For the definition of (λ, h)-exactness, we refer the reader to Definition 3.5.

Theorem 5.5 ([7, Theorem 3.10]). *For any positive number C, there exists a control pair (λ, h) such that for any filtered C^*-algebra A, any positive number r and any r-controlled* *weak Mayer–Vietoris pair* $(\Delta_1, \Delta_2, A_{\Delta_1}, A_{\Delta_2})$ *for* A *of order* r *with coercity* C, *we have a six-term* (λ, h)*-exact sequence of order* r *for the quantitative objects:*

$$\begin{array}{ccccc} \mathcal{K}_0(A_{\Delta_1} \cap A_{\Delta_2}) & \longrightarrow & \mathcal{K}_0(A_{\Delta_1}) \bigoplus \mathcal{K}_0(A_{\Delta_2}) & \longrightarrow & \mathcal{K}_0(A) \\ \partial \uparrow & & & & \partial \downarrow \\ \mathcal{K}_1(A) & \longrightarrow & \mathcal{K}_1(A_{\Delta_1}) \bigoplus \mathcal{K}_1(A_{\Delta_2}) & \longrightarrow & \mathcal{K}_1(A_{\Delta_1} \cap A_{\Delta_2}) \end{array}$$

Let X be an n-dimensional simplicial complex. For each n-simplex Y in X, we define

$$Y_1 := \left\{ x \in Y; d(c_Y, x) \le \frac{1 + \frac{1}{10}}{2} \right\}$$
$$Y_2 := \left\{ x \in Y; d(c_Y, x) \ge \frac{1 - \frac{1}{10}}{2} \right\},$$

where c_Y is the center of Y. We decompose X into two subsets:

$$X_1 := \bigcup \{Y_1; Y \text{ is an } n\text{-dimensional simplex in } X\}$$

and

$$X_2 := \bigcup \{Y_2; Y \text{ is an } n\text{-dimensional simplex in } X\}$$
$$\cup \bigcup \{Y; Y \text{ is a simplex in } X \text{ with } \dim Y \le (n-1)\}.$$

We denote by W_r the r-neighborhood of $W \subset X$. Then $(X_1)_{\frac{1}{10}}$, $(X_2)_{\frac{1}{10}}$ and $(X_1 \cap X_2)_{\frac{1}{10}}$ are strongly Lipschitz homotopic to a finite 0-dimensional complex Z_1 and $(n-1)$-dimensional complexes Z_2 and Z, respectively, with a uniform Lipschitz constant c_n depending only on n. Make a choice of strong Lipschitz homotopy equivalence maps:

$$\begin{aligned} f_1 &: (X_1)_{\frac{1}{10}} \longrightarrow Z_1, & g_1 &: Z_1 \longrightarrow (X_1)_{\frac{1}{10}} \\ f_2 &: (X_2)_{\frac{1}{10}} \longrightarrow Z_2, & g_2 &: Z_2 \longrightarrow (X_2)_{\frac{1}{10}} \\ f &: (X_1 \cap X_2)_{\frac{1}{10}} \longrightarrow Z, & g &: Z \longrightarrow (X_1 \cap X_2)_{\frac{1}{10}}. \end{aligned}$$

By Theorem 4.10, (f_1, g_1), (f_2, g_2) and (f, g) induce asymptotically inverse maps to each other on controlled K-theories. We denote

$$\Delta_i := C^*(X_i), \quad A_i := C^*((X_i)_{\frac{1}{10}})$$

for each $i = 1, 2$. We show the quadruple $(\Delta_1, \Delta_2, A_1, A_2)$ satisfies the condition to be a $\frac{1}{50}$-controlled Mayer–Vietoris pair for $C^*(X)$.

(1) Show the pair (Δ_1, Δ_2) is a completely coercive decomposition pair of degree $\frac{1}{50}$. Take any positive number r with $0 < r < \frac{1}{50}$ and $x \in (C^*(X))_r$. We denote $\Pi_1 = X_1 \setminus (X_1 \cap X_2)$, $\Pi_2 = X_1 \cap X_2$ and $\Pi_3 = X_2 \setminus (X_1 \cap X_2)$. In terms of this disjoint

decomposition, we write $x = (x_{i,j})_{1\le i,j\le 3}$, where $x_{i,j} = \chi_{\Pi_i} x \chi_{\Pi_j}$. Since x can be approximated by operators whose propagation is at most r, x has the form

$$x = \begin{pmatrix} x_{1,1} & x_{1,2} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} \\ 0 & x_{3,2} & x_{3,3} \end{pmatrix}.$$

We define

$$x_1 = \begin{pmatrix} x_{1,1} & x_{1,2} & 0 \\ x_{2,1} & x_{2,2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x_{2,3} \\ 0 & x_{3,2} & x_{3,3} \end{pmatrix}.$$

Then we have $x_i \in \Delta_i$, $x = x_1 + x_2$, $\|x_i\| \le 4\|x\|$ for $i = 1, 2$. If $x \in M_n((C^*(X))_r)$, we can regard x as an operator on $\bigoplus_{k=1}^n H_X$ with the diagonal module structure so that we can apply the same argument.

(2) Clearly A_i is a $\frac{1}{50}$-neighborhood of Δ_i.

(3) Show the pair $((A_1)_r, (A_2)_r)$ has the CIA property for any $0 \le r \le \frac{1}{50}$. Let $\varepsilon > 0$, $x \in (A_1)_r$ and $y \in (A_2)_r$ with $\|x - y\| < \varepsilon$. We denote $\Sigma_1 = (X_1)_{\frac{1}{10}+r} \setminus ((X_1)_{\frac{1}{10}+r} \cap (X_2)_{\frac{1}{10}+r})$, $\Sigma_2 = (X_1)_{\frac{1}{10}+r} \cap (X_2)_{\frac{1}{10}+r}$ and $\Sigma_3 = (X_2)_{\frac{1}{10}+r} \setminus ((X_1)_{\frac{1}{10}+r} \cap (X_2)_{\frac{1}{10}+r})$. In terms of this disjoint decomposition, we write $x = (x_{i,j})_{1\le i,j\le 3}$ and $y = (y_{i,j})_{1\le i,j\le 3}$, where $x_{i,j} = \chi_{\Sigma_i} x \chi_{\Sigma_j}$ and similarly for y. Then

$$x - y = \begin{pmatrix} x_{1,1} & x_{1,2} & 0 \\ x_{2,1} & x_{2,2} - y_{2,2} & -y_{2,3} \\ 0 & -y_{3,2} & -y_{3,3} \end{pmatrix}.$$

Define

$$z = \frac{x_{2,2} + y_{2,2}}{2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{x_{2,2}+y_{2,2}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in (A_1)_r \cap (A_2)_r.$$

Then

$$\|x - z\| = \left\| \begin{pmatrix} x_{1,1} & x_{1,2} & 0 \\ x_{2,1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{x_{2,2}-y_{2,2}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\| \le 4\varepsilon$$

and similarly for y. If x and y are matrices, we can reduce it to the case we have just proven as we did in (1). Therefore, the pair $((A_1)_r, (A_2)_r)$ has CIA property with coercity 4.

Therefore, by Theorem 5.5, there exists a control pair (δ, p) such that we have the following (δ, p) exact sequence of order $\frac{1}{50}$:

$$\begin{array}{ccccc} \mathcal{K}_1(A_1 \cap A_2) & \xrightarrow{k} & \mathcal{K}_1(A_1) \bigoplus \mathcal{K}_0(A_2) & \xrightarrow{l} & \mathcal{K}_1(C^*(X)) \\ \partial\uparrow & & & & \downarrow\partial \\ \mathcal{K}_0(C^*(X)) & \xleftarrow{l} & \mathcal{K}_0(A_1) \bigoplus \mathcal{K}_0(A_2) & \xleftarrow{k} & \mathcal{K}_0(A_1 \cap A_2) \end{array}$$

Note that, since the set of operators supported on $(X_1)_{\frac{1}{10}} \cap (X_2)_{\frac{1}{10}}$, whose propagations are at most r, is a dense subspace of $(A_1 \cap A_2)_r$, and $(X_1)_{\frac{1}{10}} \cap (X_2)_{\frac{1}{10}}$ is strongly Lipschitz homotopic to $(X_1 \cap X_2)_{\frac{1}{10}}$, the quantitative object $\mathcal{K}_*(A_1 \cap A_2)$ is asymptotically equivalent to $\mathcal{K}_*(C^*(Z))$ with a control pair depending only on the dimension. By Lemma 3.8 and Theorem 4.10, there exist control pairs (λ_n, h_n) and (δ'_n, p'_n) depending only on n such that there exist (λ_n, h_n)-morphisms k, l and ∂ in the following sequence, which are (δ'_n, p'_n)-exact of order s_n depending only on n:

$$\begin{array}{ccccc} \mathcal{K}_1(C^*(Z)) & \xrightarrow{k} & \mathcal{K}_1(C^*(Z_1)) \bigoplus \mathcal{K}_0(C^*(Z_2)) & \xrightarrow{l} & \mathcal{K}_1(C^*(X)) \\ \partial \uparrow & & & & \downarrow \partial \\ \mathcal{K}_0(C^*(X)) & \xleftarrow{l} & \mathcal{K}_0(C^*(Z_1)) \bigoplus \mathcal{K}_0(C^*(Z_2)) & \xleftarrow{k} & \mathcal{K}_0(C^*(Z)) \end{array} \tag{5.1}$$

6. Quantitative description of K-homology

In this section, we show the main theorem, which states that the K-homology $K_*(X)$ of a finite simplicial complex X can be realized as an image under a forgetful map of the controlled K-theory $K_*^{\varepsilon,r}(C^*(X))$. We show this using the asymptotic version of the five-lemma between Mayer–Vietoris sequences of K-homology and controlled K-theory. For this purpose, it is convenient to reformulate the Mayer–Vietoris sequence of K-homology in terms of controlled K-theory so that the diagram commutes. For the next lemma, we give the definition of the localization algebra $C_L^*(X)$, which is defined in [12].

Definition 6.1. For an ample X-module H_X, we define the algebraic localization algebra $\mathbb{C}_L[H_X]$ to be the algebra consisting of uniformly continuous and bounded functions from $[1, \infty)$ to the algebraic Roe algebra $\mathbb{C}[H_X]$ and the localization algebra $C_L^*(H_X)$ to be the completion of $\mathbb{C}_L[H_X]$ by the sup-norm. Again, when it is clear from the context, we will write $C_L^*(X)$ in place of $C_L^*(H_X)$. The localization algebra is also filtered by the supremum of propagations, that is,

$$C_L^*(X)_r = \left\{ f \in \mathbb{C}_L[H_X];\ \sup_{t \in [1,\infty)} \operatorname{prop}(f(t)) \le r \right\}$$

or its closure (see Remark 3.7).

Then Yu showed that the K-theory of the localization algebra is isomorphic to K-homology for any finite-dimensional simplicial complex.

Theorem 6.2 ([12, Theorem 3.2]). *If X is a finite-dimensional simplicial complex endowed with a spherical metric, then we have an isomorphism $K_*(X) \cong K_*(C_L^*(X))$.*

To use the same Mayer–Vietoris sequence for K-homology and controlled K-theory, we formulate K-homology in terms of controlled K-theory.

Lemma 6.3. *For any* $0 < \varepsilon < \frac{1}{8}$ *and* $r > 0$, *we have*

$$K_*(C_L^*(X)) = K_*^{\varepsilon,r}(C_L^*(X)).$$

Proof. Let H_X be an ample X-module. Since $\mathbb{C}_L[H_X]$ is dense in $C_L^*(X)$, by [3, Proposition 4.9], we have an isomorphism

$$K_*^{\varepsilon}(\mathbb{C}_L[H_X]) \cong K_*(C_L^*(X)),$$

where $K_*^{\varepsilon}(\mathbb{C}_L[H_X])$ is defined analogously to $K_*^{\varepsilon,r}(A)$ by replacing a filtered C^*-algebra A by the filtered algebra $\mathbb{C}_L[H_X]$ and r by ∞ in Definition 2.7. So it suffices to construct an isomorphism

$$K_*^{\varepsilon,r}(C_L^*(X)) \cong K_*^{\varepsilon}(\mathbb{C}_L[H_X]).$$

We construct an inverse to the forgetful map $\iota^{\varepsilon,r} : K_*^{\varepsilon,r}(C_L^*(X)) \to K_*^{\varepsilon}(\mathbb{C}_L[H_X])$. Take any ε-projection $(p_t)_{t\in[1,\infty)}$ over $\mathbb{C}_L[H_X]$. By the definition of $\mathbb{C}_L[H_X]$, there exists $N \in [1, \infty)$ such that

$$\mathrm{prop}(p_t) < r \quad \text{for all } t \geq N. \tag{6.1}$$

We define

$$\sigma^{\varepsilon,r} : P_\infty^{\varepsilon,r}(\mathbb{C}_L[H_X]) \to K_*^{2\varepsilon,r}(C_L^*(X)); \quad [(p_t)_t] \mapsto [(p_{N+t})_t]$$

independently of the choice of N satisfying (6.1). We show that this is well defined on the K-theoretic level. Assume we have a homotopy $\{(p(s)_t)_t\}_{s\in[0,1]}$ through ε-projections over $\mathbb{C}_L[H_X]$ parametrized by s. We show $\sigma^{\varepsilon,r}([p(0)_t]) = \sigma^{\varepsilon,r}([p(1)_t])$. The issue is that we cannot take N uniformly for s in general. We take k such that $\sup_{t\in[1,\infty)} \|(p(s)_t)_t - (p(s')_t)_t\| < \frac{1}{15}\varepsilon$ whenever $\|s - s'\| < \frac{1}{k}$ and a partition of the interval

$$0 = s_0 < s_1 < \cdots < s_{k-1} < s_k = 1,$$

with $s_j = \frac{j}{k}$ for $j = 0, 1, \ldots, k$. For each j, we can take N_j such that

$$\mathrm{prop}(p(s_j)_t) < r \quad \text{for all } t \geq N_j.$$

We define a new path $q(s)_t$ by linearly connecting $p(s_j)$'s. Then, by Remark 2.9, $q(s)$ is a path through 2ε-projections and

$$\mathrm{prop}(q(s)_t) < r \quad \text{for all } t \geq M := \max\{N_j\}_{j=1}^k.$$

Therefore, $\sigma^{\varepsilon,r}([p(0)_t]) = \sigma^{\varepsilon,r}([p(1)_t]) \in K_*^{2\varepsilon,r}(C_L^*(X))$. So we have a map on K-theory, which is still denoted by

$$\sigma^{\varepsilon,r} : K_*^{\varepsilon}(\mathbb{C}_L[H_X]) \to K_*^{2\varepsilon,r}(C_L^*(X)); \quad [(p_t)_t] \mapsto [(p_{N+t})_t].$$

The statement follows from the following commutative diagram:

$$\begin{array}{ccc} K_*^{\varepsilon}(\mathbb{C}_L[H_X]) & & \\ \cong\uparrow & \nwarrow^{\iota^{\varepsilon,r}} & \\ K_*^{\frac{\varepsilon}{2}}(\mathbb{C}_L[H_X]) & \xrightarrow{\sigma^{\frac{\varepsilon}{2},r}} & K_*^{\varepsilon,r}(C_L^*(X)) \end{array}$$

∎

Definition 6.4. Let X be a locally compact metric space, and let (α, k) be a control pair. We define a new quantitative object $\mathcal{K}_*^{(\alpha,k)}(C^*(X)) = (K_*^{(\alpha,k),(\varepsilon,r)}(C^*(X)))_{\varepsilon,r}$ by

$$K_*^{(\alpha,k),(\varepsilon,r)}(C^*(X)) = \iota^{\varepsilon,\alpha\varepsilon,r,k(\varepsilon)r}(K_*^{\varepsilon,r}(C^*(X))) \subset K_*^{\varepsilon\alpha,k(\varepsilon)r}(C^*(X)).$$

We call the group $K_*^{(\alpha,k),(\varepsilon,r)}(C^*(X))$ the (α, k)-relaxed (ε, r)-controlled K-theory of $C^*(X)$ because it is generated by the same generators as $K_*^{\varepsilon,r}(C^*(X))$ with a relaxed homotopy relation. Let Y be another locally compact metric space and $\mathcal{F} = (F^{\varepsilon,r})$ be a (λ, h)-controlled morphism from $(K_*^{\varepsilon,r}(C^*(X)))_{\varepsilon,r}$ to $(K_*^{\varepsilon,r}(C^*(Y)))_{\varepsilon,r}$. Then we define a control pair (λ, h') by

$$h'(\varepsilon) = \frac{h(\alpha\varepsilon)k(\varepsilon)}{k(\lambda\varepsilon)}$$

and a (λ, h')-controlled morphism $\mathcal{F}' = (F'^{\varepsilon,r})$ as the restriction of

$$F^{\alpha\varepsilon,k(\varepsilon)r} : K_*^{\alpha\varepsilon,k(\varepsilon)r}(C^*(X)) \to K_*^{\lambda\alpha\varepsilon,h(\alpha\varepsilon)k(\varepsilon)r}(C^*(Y))$$

from $K_*^{(\alpha,k),(\varepsilon,r)}(C^*(X))$ to $K_*^{(\alpha,k),(\lambda\varepsilon,h'(\varepsilon)r)}(C^*(Y))$.

Remark 6.5. The (δ, l)-exactness of $\mathcal{K}_*(\cdot)$ passes to the (δ, l')-exactness of $\mathcal{K}_*^{(\alpha,k)}(\cdot)$, where

$$l'(\varepsilon) = \frac{l(\alpha\varepsilon)k(\varepsilon)}{k(\delta\varepsilon)}.$$

Here we can prove the main theorem: the quantitative description of K-homology.

Theorem 6.6. *For each $n \in \mathbb{N}$, there exist a control pair (λ_n, h_n) and a pair of positive numbers (ε_n, r_n) depending only on n such that for any n-dimensional finite simplicial complex X endowed with a spherical metric, we have*

$$K_*(X) \cong K_*^{(\lambda_n,h_n),(\varepsilon,r)}(C^*(X))$$

for any $(\varepsilon, r) < (\varepsilon_n, r_n)$.

Proof. We show this by induction on n. First assume $n = 0$. In this case, X is a set of finitely many points $X = \{p_1, p_2, \ldots, p_N\}$. Let r_0 be the minimum of the distances between any two points:

$$r_0 := \min\{d(p_i, p_j);\ 1 \le i, j \le N,\ i \ne j\} \in (0, \infty].$$

(Note that in the spherical metric, the distance between points in different connected components is infinite. But for convenience, we allow it to be finite in this theorem, just to apply the induction hypothesis to Z at a later step.) Since any operator whose propagation is at most r_0 have to be supported on a single point, for any $0 < \varepsilon < \frac{1}{4}$ and $0 < r < r_0$, we have

$$K_*^{\varepsilon,r}(C^*(X)) = \bigoplus_{j=1}^{N} K_*^{\varepsilon,r}(C^*(\{p_j\})) = \bigoplus_{j=1}^{N} K_*(C^*(\{p_j\})) = \bigoplus_{j=1}^{N} K_*(\mathcal{K}(H)),$$

which is the same as the K-homology group of $X = \{p_1, p_2, \dots, p_n\}$. In this case, we can take $(\lambda_0, h_0) = (1, \mathrm{id})$ to have $K_*^{(\lambda_0,h_0),(\varepsilon,r)}(X) \cong K_*(X)$ for $0 < \varepsilon < \frac{1}{4}$ and $0 < r < r_0$. Next assume that the statement holds for $\dim X = 0, 1, \dots, n-1$. By the commutative diagram (5.1) and Remark 6.5, we have the following diagram of (δ_n, l_n)-exact sequence of (α_n, k_n)-morphisms for some control pairs (δ_n, l_n) and (α_n, k_n) of degree s_{n-1}:

$$\begin{array}{ccc}
\cdots \to K_*(Z) & \longrightarrow & K_*(Z_1) \bigoplus K_*(Z_2) \to \\
\Big\downarrow \mathrm{ev}_* & & \Big\downarrow \mathrm{ev}_* \\
\cdots \to \mathcal{K}_*^{(\lambda_{n-1},h_{n-1})}(Z) & \longrightarrow & \mathcal{K}_*^{(\lambda_{n-1},h_{n-1})}(Z_1) \bigoplus \mathcal{K}_*^{(\lambda_{n-1},h_{n-1})}(Z_2) \to \\
\to K_*(X) & \longrightarrow & K_{*+1}(Z) \to \cdots \\
\Big\downarrow \mathrm{ev}_* & & \Big\downarrow \mathrm{ev}_* \\
\to \mathcal{K}_*^{(\lambda_{n-1},h_{n-1})}(X) & \longrightarrow & \mathcal{K}_{*+1}^{(\lambda_{n-1},h_{n-1})}(Z) \to \cdots
\end{array}$$

Note that the Mayer–Vietoris sequence of K-homology comes from that of controlled K-theory via the isomorphism in Lemma 6.3. The vertical maps are induced by the evaluation maps at 1:

$$(\mathrm{ev}_*)_{(\varepsilon,r)}^{n-1} : K_*(Y) = K_*^{\varepsilon,r}(C_L^*(Y)) \to K_*^{(\lambda_{n-1},h_{n-1}),(\varepsilon,r)}(Y)$$

for $Y = X, Z_1, Z_2, Z$. Therefore, the above diagram is commutative because of the construction after [7, Remark 3.4]. By the induction hypothesis, there exists $(\varepsilon_{n-1}, r_{n-1})$ such that $K_*^{(\lambda_{n-1},h_{n-1}),(\varepsilon,r)}(Y)$ is stable and isomorphic to $K_*(Y)$ if $(\varepsilon, r) < (\varepsilon_{n-1}, r_{n-1})$ and $r < s_{n-1}$ for $Y = Z, Z_1, Z_2$. We take (ε, r) such that

$$(\lambda_{n-1}, h_{n-1})(\varepsilon, r) < (\varepsilon_{n-1}, r_{n-1}) \tag{6.2}$$

and

$$(\alpha_n, k_n)(\varepsilon, r) < (\varepsilon_{n-1}, r_{n-1}). \tag{6.3}$$

Then, by diagram chasing, we can show that for any $y \in K_*^{(\lambda_{n-1},h_{n-1}),(\varepsilon,r)}(X)$, there exists $x \in K_*(X)$ such that

$$(\mathrm{ev}_*)_{(\alpha_n\delta_n\varepsilon, k_n(\delta_n\varepsilon)l_n(\varepsilon)r)}^{n-1}(x) = \iota^{(\varepsilon,r),(\alpha_n,k_n)*(\delta_n,l_n)(\varepsilon,r)}(y).$$

So, if we define $(\lambda_n, h_n) := (\alpha_n, k_n) * (\delta_n, l_n) * (\lambda_{n-1}, h_{n-1})$,

$$(\mathrm{ev}_*)^n_{(\varepsilon,r)} : K_*(X) \to K_*^{(\lambda_n,h_n),(\varepsilon,r)}(X)$$

is surjective for all (ε, r) with (6.2) and (6.3). We now show that $(\mathrm{ev}_*)^n_{(\varepsilon,r)}$ is injective for small (ε, r). Let $(\varepsilon', r') := \left(\frac{\lambda_n}{\lambda_{n-1}}\varepsilon, \frac{h_n(\varepsilon)}{h_{n-1}(\frac{\lambda_n}{\lambda_{n-1}}\varepsilon)} r\right)$ so that it satisfies

$$(\lambda_{n-1}\varepsilon', h_{n-1}(\varepsilon')r') = (\lambda_n \varepsilon, h_n(\varepsilon)r). \tag{6.4}$$

Note that we have a commutative diagram:

$$\begin{array}{ccc} K_*(Y) & \xrightarrow{(\mathrm{ev}_*)_{\varepsilon,r}} & K_*^{\varepsilon,r}(C^*(Y)) \\ {\scriptstyle (\mathrm{ev}_*)_{\varepsilon',r'}}\downarrow & & \downarrow{\scriptstyle \iota^{\varepsilon,\lambda_n\varepsilon,r,h_n(\varepsilon)r}} \\ K_*^{\varepsilon',r'}(C^*(Y)) & \xrightarrow{\iota^{\varepsilon',\varepsilon'\lambda_{n-1},r',h_{n-1}(\varepsilon_1)r'}} & K_*^{\lambda_n\varepsilon,h_n(\varepsilon)r}(C^*(Y)) \end{array}$$

By the induction hypothesis, if we choose (ε', r') small enough, then for $Y = Z, Z_1, Z_2$, the composition $\iota^{\varepsilon',\varepsilon'\lambda_{n-1},r',h_{n-1}(\varepsilon)r'} \circ (\mathrm{ev}_*)_{\varepsilon',r'}$ is an isomorphism onto

$$K_*^{(\lambda_n,h_n),(\varepsilon,r)}(C^*(Y)) \subset K_*^{\lambda_n\varepsilon,h_n(\varepsilon)r}(C^*(Y))$$

by (6.4). Therefore,

$$(\mathrm{ev}_*)^n_{(\varepsilon,r)} : K_*(Y) \to K_*^{(\lambda_n,h_n),(\varepsilon,r)}(Y)$$

is also isomorphic for (ε, r) with $(\varepsilon', r') < (\varepsilon_{n-1}, r_{n-1})$ besides (6.2) and (6.3). Then by the following diagram chasing of commutative asymptotic exact sequence of degree s_n

$$\begin{array}{ccc} \cdots \to K_*(Z) & \longrightarrow & K_*(Z_1) \bigoplus K_*(Z_2) \to \\ \downarrow & & \downarrow \\ \cdots \to \mathcal{K}_*^{(\lambda_n,h_n)}(Z) & \longrightarrow & \mathcal{K}_*^{(\lambda_n,h_n)}(Z_1) \bigoplus \mathcal{K}_*^{(\lambda_n,h_n)}(Z_2) \to \\ \to K_{*+1}(X) & \longrightarrow & K_{*+1}(Z) \to \cdots \\ \downarrow & & \downarrow \\ \to \mathcal{K}_{*+1}^{(\lambda_n,h_n)}(X) & \longrightarrow & \mathcal{K}_{*+1}^{(\lambda_n,h_n)}(Z) \to \cdots \end{array}$$

we can show

$$(\mathrm{ev}_*)^n_{(\varepsilon,r)} : K_*(X) \to K_*^{(\lambda_n,h_n),(\varepsilon,r)}(X)$$

is also injective for sufficiently small (ε, r) depending only on n. ■

Remark 6.7. We continue assuming X is a finite n-dimensional simplicial complex. Let Y be a countable dense subset of X, which is an inductive limit of finite subsets (Y_i)

of Y, and let H be any infinite-dimensional separable Hilbert space. We take an increasing sequence $(H_i)_i$ of finite-dimensional subspaces H_i of H such that $H = \overline{\bigcup_i H_i}$, and we denote by $Q_i \in B(\ell^2(Y) \otimes H)$ the orthogonal projection onto $\ell^2(Y_i) \otimes H_i$. We have a unital ample representation of $C(X)$ on $\ell^2(Y) \otimes H$ given by a multiplication on $\ell^2(Y)$ and the identity for H, and this representation can be restricted to each $\ell^2(Y_i) \otimes H_i$.

In this case, the Roe algebra $C^*(\ell^2(Y) \otimes H)$ is the set of compact operators $\mathcal{K}(\ell^2(Y) \otimes H) = \varinjlim B(\ell^2(Y_i) \otimes H_i)$. Therefore, for any (ε, r)-projection p over $C^*(\ell^2(Y) \otimes H)$ and $\delta > 0$, we can find i such that $\|p - Q_i p Q_i\| \leq \delta$. By Remark 2.9, p and $Q_i p Q_i$ represent the same element in $K_0^{(\lambda_n, h_n),(\varepsilon,r)}(X)$ for sufficiently small δ.

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References

[1] M. F. Atiyah, Global theory of elliptic operators. In *Proceedings of the international conference on functional analysis and related topics, Tokyo, 1969*, pp. 21–30, University of Tokyo Press, Tokyo, 1969 Zbl 0193.43601 MR 0266247

[2] A. Connes and W. D. van Suijlekom, Tolerance relations and operator systems. *Acta Sci. Math. (Szeged)* **88** (2022), no. 1–2, 101–129 Zbl 1538.46087 MR 4500502

[3] E. Guentner, R. Willett, and G. Yu, Dynamical complexity and controlled operator K-theory. *Astérisque* **451** (2024), 1–89 Zbl 1551.19001 MR 4793582

[4] N. Higson and J. Roe, *Analytic K-homology*. Oxford Math. Monogr., Oxford University Press, Oxford, 2000 Zbl 0968.46058 MR 1817560

[5] G. G. Kasparov, Topological invariants of elliptic operators. I. K-homology (in Russian). *Izv. Akad. Nauk SSSR Ser. Mat.* **39** (1975), no. 4, 796–838 Zbl 0328.58016. English translation: *Math. USSR-Izv.* **9** (1975), no. 4, 751–792 Zbl 0337.58006 MR 0488027

[6] H. Oyono-Oyono and G. Yu, On quantitative operator K-theory. *Ann. Inst. Fourier (Grenoble)* **65** (2015), no. 2, 605–674 Zbl 1329.19009 MR 3449163

[7] H. Oyono-Oyono and G. Yu, Quantitative K-theory and the Künneth formula for operator algebras. *J. Funct. Anal.* **277** (2019), no. 7, 2003–2091 Zbl 1475.19004 MR 3989138

[8] Y. Qiao and J. Roe, On the localization algebra of Guoliang Yu. *Forum Math.* **22** (2010), no. 4, 657–665 Zbl 1204.19005 MR 2661442

[9] J. Wang, Z. Xie, and G. Yu, Decay of scalar curvature on uniformly contractible manifolds with finite asymptotic dimension. *Comm. Pure Appl. Math.* **77** (2024), no. 1, 372–440 Zbl 1536.53097 MR 4666628

[10] R. Willett and G. Yu, *Higher index theory*. Cambridge Stud. Adv. Math. 189, Cambridge University Press, Cambridge, 2020 Zbl 1471.19001 MR 4411373

[11] R. Willett and G. Yu, Controlled KK-theory, and a Milnor exact sequence. [v1] 2020, [v3] 2024, arXiv:2011.10906v3, to appear in *Doc. Math.*

[12] G. Yu, Localization algebras and the coarse Baum–Connes conjecture. *K-Theory* **11** (1997), no. 4, 307–318 Zbl 0888.46047 MR 1451759

[13] G. Yu, The Novikov conjecture for groups with finite asymptotic dimension. *Ann. of Math. (2)* **147** (1998), no. 2, 325–355 Zbl 0911.19001 MR 1626745

[14] G. Yu, A characterization of the image of the Baum–Connes map. In E. Blanchard, D. Ellwood, M. Khalkhali, M. Marcolli, H. Moscovici, and S. Popa (eds.), *Quanta of maths: non commutative geometry conference in honor of Alain Connes, Institut Henri Poincaré, Institut des Hautes Études ScientiÅ ques, Institut de Mathématiques de Jussieu, Paris, France, March 29–April 6, 2007*, pp. 649–657, Clay Math. Proc. 11, American Mathematical Society, Providence, RI, 2010 Zbl 1216.19008 MR 2732068

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