

Equivariant bundles and absorption

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Abstract. For a locally compact group G and a strongly self-absorbing G -algebra (\mathcal{D}, δ) , we obtain a new characterization of tensorial absorption of (\mathcal{D}, δ) using almost equivariant completely positive maps into the underlying algebra. The main technical tool to obtain this characterization is the existence of almost equivariant lifts for equivariant completely positive maps, proved in recent work of the authors with Thomsen. This characterization is then used to show that an equivariant $C(X)$ -algebra with $\dim_{\text{cov}}(X) < \infty$ is (\mathcal{D}, δ) -stable if and only if all of its fibers are, extending a result of Hirshberg, Rørdam, and Winter to the equivariant setting. The condition on the dimension of X is known to be necessary, and we show that it can be removed if, for example, the bundle is locally trivial.

1. Introduction

The study of $C(X)$ -algebras has attracted a great deal of attention, as their global structure is closely related to that of the fibers. Indeed, there are many results which state that, under varying sets of assumptions, a $C(X)$ -algebra satisfies a given property whenever all the fibers do; for example, see [1, 4, 10], etc. (The converse is usually also true and often easier to prove.)

We work in the equivariant setting, and for a locally compact group G we consider G - $C(X)$ -algebras, that is, $C(X)$ -algebras endowed with a fiberwise action of G . For example, whenever A is a unital G -algebra for which $\text{Prim}(A)/G$ is Hausdorff, then A can be naturally described in these terms. Equivariant bundles arise naturally in practice, even if one is only interested in actions on *simple* C^* -algebras. For example, if a compact group G acts on a unital C^* -algebra A with finite Rokhlin dimension with commuting towers (see [9]), then the continuous part of $A_\infty \cap A'$ is a G - $C(X)$ -algebra for a suitable compact space X . A description of the space X in this context is in general complicated; see, for example, the comments after [10, Definition 2.2].

This work is originally motivated by a preservation result by Hirshberg, Rørdam, and Winter [14], stating that for a given strongly self-absorbing C^* -algebra \mathcal{D} , a $C(X)$ -algebra with $\dim_{\text{cov}}(X) < \infty$ absorbs \mathcal{D} if and only if each fiber does. Recall (see [18, 19]) that a separable, unital G -algebra (\mathcal{D}, δ) is said to be *strongly self-absorbing* if the first factor embedding $\mathcal{D} \rightarrow \mathcal{D} \otimes \mathcal{D}$ is approximately G -unitarily equivalent to an equivariant

isomorphism. This notion was introduced by Szabó in order to systematize the study of absorption properties in the dynamical setting, as well as laying the foundations for the study of internal structures of dynamical systems; see, for example, [17]. When G is the trivial group, one recovers the notion of a *strongly self-absorbing C^* -algebra* introduced by Toms and Winter [22]. As it turns out, the property of being strongly self-absorbing (either for actions or for C^* -algebra) is a very rare one.¹ On the other hand, the property of *tensorially absorbing* a strongly self-absorbing C^* -algebra or action² is a much more common and robust feature, which moreover has strong connections to the structure theory of these objects. In the non-equivariant setting, these ties are by now very well understood and have been extensively exploited both in the context of simple, nuclear C^* -algebras and Elliott’s classification program (e.g., through Winter’s celebrated “UHF-stable to \mathcal{Z} -stable” isomorphism theorem [24, Theorem 7.1]), and also for nonsimple C^* -algebras (e.g., in the impressive Dixmier–Douady theory for continuous fields of strongly self-absorbing C^* -algebras of Dadarlat and Pennig [5]). The equivariant setting is by comparison less explored, although there are some very promising results particularly in the case of actions of amenable group on simple C^* -algebras; see [11, 20]. In this context, actions of nonamenable groups exhibit a somewhat wilder behavior; see [13].

The main result of this work deals with equivariant $C(X)$ -algebras whose fibers absorb a fixed strongly self-absorbing G -algebra and allows one to deduce tensorial absorption of the given strongly self-absorbing G -algebra. Dynamical systems of this form appear naturally when studying the structure of actions on (possibly simple) \mathcal{Z} -stable C^* -algebras; see [7, 10, 12, 17]. Absorption of the trivial action on the Jiang–Su algebra is particularly useful, since it opens the doors of a possible classification of actions. Indeed, showing absorption of well-behaved objects is a common feature in most of the classification results for group actions, and our results are a contribution in this direction. A concrete application is given in Theorem B.

Particular cases of the following theorem have been proved in these works using ad hoc methods, and our result gives the desired outcome in greater generality.

Theorem A (see Theorem 3.13). *Let G be a second countable, locally compact group, let X be a compact metrizable space, let (A, α) be a separable, unital G - $C(X)$ -algebra, and let (\mathcal{D}, δ) be a unitarily regular strongly self-absorbing G -algebra. If X is finite-dimensional, then (A, α) is (\mathcal{D}, δ) -stable if and only if all of its fibers are (\mathcal{D}, δ) -stable. The condition on $\dim_{\text{cov}}(X)$ is necessary and can be removed if the bundle is locally (\mathcal{D}, δ) -stable (in particular, if the bundle is locally trivial).*

¹The list of all known strongly self-absorbing C^* -algebras is very short and consists of the Cuntz algebras \mathcal{O}_2 and \mathcal{O}_∞ , the Jiang–Su algebra \mathcal{Z} , UHF-algebras of the form $M_{\mathfrak{p}^\infty}$ for a supernatural number \mathfrak{p} , and tensor products of the latter with \mathcal{O}_∞ . This list is conjecturally complete, and this is the case module, the so-called universal coefficient theorem; see [21, Corollary 6.7].

²If (\mathcal{D}, δ) is a strongly self-absorbing G - C^* -algebra, we say that another G - C^* -algebra (A, α) is (\mathcal{D}, δ) -absorbing, or (\mathcal{D}, δ) -stable, if the G - C^* -algebras (A, α) and $(A \otimes \mathcal{D}, \alpha \otimes \delta)$ are cocycle conjugate; see Definition 2.3. The choice of the tensor product here is irrelevant, since \mathcal{D} is automatically nuclear; see, for example, [22, Theorem 1.6].

Unitary regularity (see Definition 3.11) is a very mild condition which in practice serves as an equivariant analog of K_1 -injectivity. Unlike K_1 -injectivity, however, it is not known whether unitary regularity is automatic for strongly self-absorbing actions, although this may turn out to be the case. On the other hand, unitary regularity is known to hold for a wide family of strongly self-absorbing actions: this is the case, for example, for $(\mathbb{Z}, \text{id}_{\mathbb{Z}})$ -stable actions (see [19, Proposition 1.20]) and also for arbitrary strongly self-absorbing actions of amenable groups (see [11, Theorem B]). Theorem A is thus readily applicable to a large class of strongly self-absorbing actions.

The proof of Theorem A roughly follows the arguments in [14], adapted to our setting. Since we work with G -algebras, we have to keep track of equivariance conditions at all steps of the constructions. A second and technically much more challenging task is that of keeping track of continuity conditions for the action. Indeed, whenever one wishes to apply a reindexation argument in the (central) sequence algebra, or a diagonal argument, the topology of the acting group may lead to problems. This is because the most direct ways of lifting relations in the sequence algebras A_∞ or $F_\infty(A)$ (see Definition 2.4 for their definitions) will only keep track of finitely many elements of the group, which is in general not enough for most purposes.

The starting point of our proof for Theorem A is a local characterization of (\mathcal{D}, δ) -absorption, in terms of almost equivariant completely positive contractive maps into the coefficient algebra (Theorem 2.7), which is done in Section 2. The result is of a technical nature, but it is very useful in practice, since the equivalent characterizations (particularly the last two) are very different in flavor, making them very complementary. The equivalence between (1), (3), and (4) of Theorem 2.7 is a finitary reformulation of [18, Theorem 3.7], which in the case of discrete groups is probably an easy exercise. For continuous groups, some additional care is needed, as we explained above. (It is possible to work with *finite* subsets of G instead, by adding a condition controlling the continuity modulus of α on the range of ψ ; see part (4) of Theorem 2.7.) On the other hand, the equivalence between (1) and (5) is less direct, and it is the formulation in (5) that is most useful for the purpose of proving Theorem A.

We also give a concrete application of Theorem A, by obtaining a new characterization of absorption of a strongly self-absorbing, unitarily regular G - C^* -algebra in terms of order zero maps which are jointly unital. Particular cases of the following result have appeared in the literature before, particularly in connection to finiteness of the Rokhlin dimension (with commuting towers). The definition of Kirchberg's corrected central sequence algebra $F_\infty(A)$ is given in Definition 2.4.

Theorem B (see Theorem 3.14). *Let G be a second countable, locally compact group, let A be a separable C^* -algebra, let $\alpha: G \rightarrow \text{Aut}(A)$ be an action, and let $\delta: G \rightarrow \text{Aut}(\mathcal{D})$ be a unitarily regular strongly self-absorbing action. Then (A, α) is (\mathcal{D}, δ) -stable if and only if for some $d \geq 0$ there exist equivariant completely positive contractive order zero maps*

$$\rho_0, \dots, \rho_d: (\mathcal{D}, \delta) \rightarrow (F_\infty(A), F_\infty(\alpha))$$

with commuting ranges and satisfying $\rho_0(1) + \cdots + \rho_d(1) = 1$.

Throughout this work, and particularly in the proof of Theorem 2.7, the existence of almost equivariant lifts for equivariant completely positive maps, obtained in [6], is a crucial tool. For compact or even amenable groups, the said lifting results are easier to obtain and have been used in the past to prove particular cases of Theorem A. Our biggest contribution is thus in the nonamenable setting, where averaging arguments cannot be carried out, and one instead has to resort to more delicate arguments; see specifically [6, Section 2]. In the light of the wild behavior that nonamenable groups may exhibit in relation to, for example, $(\mathbb{Z}, \text{id}_{\mathbb{Z}})$ -stability [13], the fact that our results hold for arbitrary groups, even though they ultimately depend on an averaging argument, is at least somewhat surprising.

2. Absorption of strongly self-absorbing actions

In this section, we recall the definition and some basic facts about strongly self-absorbing actions and obtain a new characterization of absorption of such an action in terms of almost equivariant and almost multiplicative completely positive maps into the given algebra; see Theorem 2.7.

Given a locally compact group G , a G -algebra is a pair (A, α) consisting of a C^* -algebra A and a group homomorphism $\alpha: G \rightarrow \text{Aut}(A)$, also called an *action*, which is continuous with respect to the point-norm topology in $\text{Aut}(A)$.

We begin by reviewing some results obtained in [6] together with Thomsen, about lifts of equivariant completely positive contractive maps.

2.1. Lifts of equivariant completely positive maps

Let (A, α) and (B, β) be G -algebras, let I be a G -invariant ideal in B with associated quotient map $\pi: B \rightarrow B/I$, and let $\varphi: A \rightarrow B/I$ be an equivariant completely positive contractive map:

$$\begin{array}{ccc} & & B \\ & \nearrow \psi & \downarrow \pi \\ A & \xrightarrow{\varphi} & B/I \end{array}$$

If A is nuclear (as it will be in the situations we are interested in), then the Choi–Effros lifting theorem [3] guarantees the existence of a completely positive contractive map $\psi: A \rightarrow B$ making the above diagram commutative. Since we work in the equivariant category, we will be interested in obtaining lifts for φ which are “almost” equivariant. We will control the failure of equivariance of a lift ψ by obtaining small bounds for the quantity

$$\max_{a \in F_A} \max_{g \in K_G} \|(\psi \circ \alpha_g)(a) - (\beta_g \circ \psi)(a)\|,$$

for a finite subset $F_A \subseteq A$ and a compact subset $K_G \subseteq G$. We call the above quantity the *equivariance modulus* of ψ with respect to F_A and K_G .

In fact, we will need to work in a more general setting, and we will in particular not assume that φ is equivariant. Hence, we do not expect to find almost equivariant lifts, and instead we will be interested in finding lifts whose equivariance moduli are controlled by those of φ . This was achieved jointly with Thomsen in [6], and we recall the precise statement for use in this work.

Theorem 2.1. *Let G be a second countable, locally compact group, and let (A, α) and (B, β) be G -algebras with A separable. Let I be a β -invariant ideal in B . Denote by $\pi: B \rightarrow B/I$ the canonical quotient map and by $\bar{\beta}: G \rightarrow \text{Aut}(B/I)$ the induced action. Let $\varphi: A \rightarrow B/I$ be a nuclear completely positive contractive map.*

Given $\varepsilon > 0$, a finite subset $F_A \subseteq A$ and a compact subset $K_G \subseteq G$, there exists a completely positive contractive map $\psi: A \rightarrow B$ with $\pi \circ \psi = \varphi$, satisfying

$$\|(\psi \circ \alpha_g)(a) - (\beta_g \circ \psi)(a)\| \leq \|(\varphi \circ \alpha_g)(a) - (\bar{\beta}_g \circ \varphi)(a)\| + \varepsilon$$

for all $g \in K_G$ and $a \in F_A$.

Proof. This follows immediately from combining the Choi–Effros lifting theorem [3] with [6, Theorem 3.4]. ■

When A , B , and φ are unital, one cannot always construct lifts as above which are moreover unital; see [6, Section 4]. In our setting, a unital, asymptotically equivariant lifts exist if and only if there is a sequence of asymptotically G -equivariant completely positive unital maps $A \rightarrow B$, which is in particular the case if B admits a G -invariant state.

2.2. Strongly self-absorbing actions

For a unital C^* -algebra D , we write $\mathcal{U}(D)$ for its unitary group.

Let (A, α) and (B, β) be G -algebras, and let $\tilde{\beta}$ denote the strictly continuous extension of β to $M(B)$. Recall that two equivariant homomorphisms $\varphi, \psi: (A, \alpha) \rightarrow (B, \beta)$ are said to be *approximately G -unitarily equivalent*, written $\varphi \approx_{G,u} \psi$, if for every $\varepsilon > 0$, every compact subset $K_G \subseteq G$, and all finite subsets $F_A \subseteq A$ and $F_B \subseteq B$, there exists a unitary $u \in M(B)$ such that $\|\varphi(a) - u\psi(a)u^*\| < \varepsilon$ for all $a \in F_A$ and

$$\max_{b \in F_B} \max_{g \in K_G} \|\tilde{\beta}_g(u)b - ub\| < \varepsilon \quad \text{and} \quad \max_{b \in F_B} \max_{g \in K_G} \|b\tilde{\beta}_g(u) - bu\| < \varepsilon.$$

Equivalently, there exists a net $(u_j)_{j \in J}$ in $\mathcal{U}(M(B))$ such that $\tilde{\beta}_g(u_j) - u_j$ converges strictly to zero in $M(B)$, uniformly on compact subsets of G , and $\text{Ad}(u_j) \circ \psi$ converges pointwise to φ in norm.

Definition 2.2. Let G be a second countable, locally compact group, and let (\mathcal{D}, δ) be a G -algebra. We say that (\mathcal{D}, δ) is *strongly self-absorbing*, if \mathcal{D} is separable, unital, and infinite-dimensional, and there is an equivariant isomorphism $(\mathcal{D}, \delta) \cong (\mathcal{D} \otimes \mathcal{D}, \delta \otimes \delta)$ which is approximately G -unitarily equivalent to the (equivariant) first factor embedding $\text{id}_{\mathcal{D}} \otimes 1: (\mathcal{D}, \delta) \rightarrow (\mathcal{D} \otimes \mathcal{D}, \delta \otimes \delta)$.

If (\mathcal{D}, δ) is a strongly self-absorbing G -algebra, then clearly \mathcal{D} is strongly self-absorbing in the sense of [22]. In particular, \mathcal{D} is simple and nuclear; see [22, Proposition 1.5 and Theorem 1.6].

The property of being strongly self-absorbing is not a common one, and most of the naturally occurring examples of (G) -algebras fail to be strongly self-absorbing. On the other hand, the property of *absorbing* a given strongly self-absorbing G -algebra is a much more common and useful one. Here, absorption is always meant up to cocycle conjugacy, in the following sense.

Definition 2.3. Let G be a locally compact group, let (A, α) and (B, β) be G -algebras, and let $\omega: G \rightarrow \mathcal{U}(M(A))$ be a strictly continuous map.

- (a) We say that ω is a *1-cocycle for α* if $\omega_g \alpha_g(\omega_h) = \omega_{gh}$ for all $g, h \in G$. In this case, we let $\alpha^\omega: G \rightarrow \text{Aut}(A)$ be the action given by $\alpha_g^\omega = \text{Ad}(\omega_g) \circ \alpha_g$ for all $g \in G$.
- (b) We say that α and β are *cocycle conjugate*, denoted $\alpha \simeq_{cc} \beta$, if there exists a 1-cocycle ω for α such that α^ω is conjugate to β .

If (D, δ) is another G -algebra, we say that (A, α) *absorbs* (D, δ) , or that (A, α) is (D, δ) -*stable*, if $\alpha \otimes \delta \simeq_{cc} \alpha$.

When (\mathcal{D}, δ) is strongly self-absorbing, δ -absorption can be characterized in terms of central sequence algebras; see [18, Theorem 3.7]. In this section, we use Theorem 2.1 to obtain a new characterization of this property in terms of completely positive maps into the given algebra (instead of its central sequence); see Theorem 2.7. Our result is an equivariant version of [14, Theorem 4.1], but we warn the reader that there are several additional technicalities in the equivariant setting, mostly related to the lack of continuity of the induced action on the (central) sequence algebra.

Definition 2.4. Let A be a C^* -algebra. We write A_∞ for its *sequence algebra*, that is, $A_\infty = \ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A)$, and we write $\pi_A: \ell^\infty(\mathbb{N}, A) \rightarrow A_\infty$ for the canonical quotient map. Identifying A with the subalgebra of $\ell^\infty(\mathbb{N}, A)$ consisting of the constant sequences, and with its image in A_∞ , we write $A_\infty \cap A'$ for the relative commutant, which we call the *central sequence algebra*. We let

$$\text{Ann}(A, A_\infty) = \{x \in A_\infty : xA = Ax = \{0\}\}$$

denote the *annihilator* of A in A_∞ . Then $\text{Ann}(A, A_\infty)$ is an ideal in $A_\infty \cap A'$ (but not in general an ideal in A_∞). We let $F_\infty(A)$ denote the corresponding quotient and write $\kappa_A: A_\infty \cap A' \rightarrow F_\infty(A)$ for the canonical quotient map.

Assume that A is σ -unital. If $(a_n)_{n \in \mathbb{N}}$ is an approximate unit for A , and denoting by $a \in \ell^\infty(\mathbb{N}, A)$ the element it determines, then $\kappa_A(\pi_A(a))$ is a (and hence *the*) unit for $F_\infty(A)$.

Let G be a locally compact group, and let $\alpha: G \rightarrow \text{Aut}(A)$ be a continuous action. For each $g \in G$, the automorphism α_g induces canonical automorphisms of $\ell^\infty(\mathbb{N}, A)$, $A_\infty \cap A'$, and $F_\infty(A)$, which will be denoted by α_g^∞ , $(\alpha_\infty)_g$, and $F_\infty(\alpha)_g$, respectively. The resulting maps α^∞ , α_∞ , and $F(\alpha)$ are group homomorphisms and make the quotient maps π_A and κ_A equivariant, but they fail in general to be *continuous* actions. In order to lighten the notation, we will denote the continuous parts of $\ell^\infty(\mathbb{N}, A)$, A_∞ , and $F_\infty(A)$ by $\ell_\alpha^\infty(\mathbb{N}, A)$, $A_{\infty, \alpha}$, and $F_{\infty, \alpha}(A)$, respectively. Observe that the continuous part of $A_\infty \cap A'$ is just $A_{\infty, \alpha} \cap A'$.

Since the lifting results presented in the previous subsection apply only to continuous actions and surjective maps, the following will be needed in the sequel.

Proposition 2.5. *Let G be a locally compact group, and let (A, α) be a G -algebra. Then $\pi_A(\ell_\alpha^\infty(\mathbb{N}, A)) = A_{\infty, \alpha}$, and hence the restriction $\pi_A: \ell_\alpha^\infty(\mathbb{N}, A) \rightarrow A_{\infty, \alpha}$ of π_A to the continuous part of $\ell^\infty(\mathbb{N}, A)$ is an equivariant quotient map. If A is separable, then $\kappa_A(A_{\infty, \alpha} \cap A') = F_{\infty, \alpha}(A)$ and thus $\kappa_A: A_{\infty, \alpha} \cap A' \rightarrow F_{\infty, \alpha}(A)$ is also an equivariant quotient map.*

Proof. The result for π_A is a consequence of [2, Theorem 2], since it follows from the said theorem that the preimage under π_A of any element in $A_{\infty, \alpha}$ already belongs to $\ell_\alpha^\infty(\mathbb{N}, A)$. When A is separable, the claim about κ_A follows from [19, Lemma 4.7]. ■

2.3. Local characterizations of absorption

In Theorem 2.7, we give new characterizations for a G -algebra (A, α) to absorb a given strongly self-absorbing G -algebra in terms of almost equivariant maps into A . The following notation will be used in its statement and proof and will also be used repeatedly in the following section.

Notation 2.6. Let G be a locally compact group, let (A, α) and (D, δ) be G -algebras with D unital, and let $\psi: D \rightarrow A$ be a completely positive contractive map. Given finite subsets $F_D \subseteq D$ and $F_A \subseteq A$, a compact subset $K_G \subseteq G$, and $\varepsilon > 0$, we say that ψ is

- (U) (F_A, ε) -unital, if $\|\psi(1)a - a\| < \varepsilon$ for all $a \in F_A$;
- (C) (F_D, F_A, ε) -central, if $\|a\psi(d) - \psi(d)a\| < \varepsilon$ for all $a \in F_A$ and $d \in F_D$;
- (M) (F_D, F_A, ε) -multiplicative, if $\|a(\psi(dd') - \psi(d)\psi(d'))\| < \varepsilon$ for all $a \in F_A$ and all $d, d' \in F_D$;
- (E) (F_D, K_G, ε) -equivariant, if $\max_{g \in K_G} \|\alpha_g(\psi(d)) - \psi(\delta_g(d))\| < \varepsilon$ for all $d \in F_D$.

Whenever the tuple $(F_D, F_A, K_G, \varepsilon)$ is clear from the context, we will refer to the above conditions as (U), (C), (M), and (E).

In the proof of the following theorem, we will repeatedly apply the lifting results from Subsection 2.1 to the quotient maps π_A and κ_A . When lifting along π_A , we will need the most general lifting result from Theorem 2.1, where the given map is not assumed to be equivariant.

We make some comments about part (3) of Theorem 2.7. It does not seem possible to give a “finitistic” characterization (i.e., in terms of finite sets and $\varepsilon > 0$) of the existence of a map into the *continuous part* of A_∞ (or of $F_\infty(A)$). Indeed, for a map into A_∞ to land in the continuous part, the individual maps into A ought to be *equicontinuous*, and this cannot be described in a finitistic manner. The solution that we adopted takes advantage of the fact that the resulting map into A_∞ will have the property of being equivariant, and thus the continuity moduli of the codomain action are controlled by the continuity moduli of the domain action.

Theorem 2.7. *Let G be a second countable, locally compact group, and let (A, α) be a separable G -algebra. For a strongly self-absorbing G -algebra (\mathcal{D}, δ) , consider the following conditions:*

- (1) (A, α) is (\mathcal{D}, δ) -stable.
- (2) *There exists a completely positive contractive map $\sigma: \mathcal{D} \rightarrow \ell^\infty(\mathbb{N}, A)$ such that the range of $\pi_A \circ \sigma$ is contained in $A_{\infty, \alpha} \cap A'$ and $\pi_A \circ \sigma: \mathcal{D} \rightarrow A_{\infty, \alpha}$ is equivariant, and such that $\kappa_A \circ \pi_A \circ \sigma$ is a unital homomorphism.*
- (3) *For all finite sets $F_{\mathcal{D}} \subseteq \mathcal{D}$ and $F_A \subseteq A$, all compact subsets $K_G \subseteq G$, and $\varepsilon > 0$, there is a completely positive contractive map $\psi: \mathcal{D} \rightarrow A$ which satisfies conditions (U), (C), (M), and (E) for $(F_{\mathcal{D}}, F_A, K_G, \varepsilon)$.*
- (4) *For all finite sets $F_{\mathcal{D}} \subseteq \mathcal{D}$, $F_A \subseteq A$ and $F_G \subseteq G$, every compact neighborhood of the unit $N \subseteq G$, and every $\varepsilon > 0$, there is a completely positive contractive map $\psi: \mathcal{D} \rightarrow A$ which satisfies conditions (U), (C), (M), and (E) for $(F_{\mathcal{D}}, F_A, F_G, \varepsilon)$ as well as*

$$\max_{h \in N} \|\alpha_h(\psi(d)) - \psi(d)\| \leq \max_{h \in N} \|\delta_h(d) - d\| + \varepsilon$$

for all $d \in F_{\mathcal{D}}$.

- (5) *Given finite subsets $F_{\mathcal{D}} \subseteq \mathcal{D}$ and $F_A \subseteq A$, a compact subset $K_G \subseteq G$, and $\varepsilon > 0$, there exist homomorphisms $\mu: \mathcal{D} \rightarrow M(A)$ and $\sigma: A \rightarrow A$ with $\mu(1) = 1$, satisfying:*
 - (5a) $\sigma(a)\mu(d) = \mu(d)\sigma(a)$ for all $d \in \mathcal{D}$ and all $a \in A$;
 - (5b) $\|\sigma(a) - a\| < \varepsilon$ for all $a \in F_A$;
 - (5c) $\max_{g \in F_G} \|\mu(\delta_g(d))\alpha_g(a) - \alpha_g(\mu(d)a)\| < \varepsilon\|a\|$ for all $a \in A$ and $d \in F_{\mathcal{D}}$;
 - (5d) $\max_{g \in F_G} \|\sigma(\alpha_g(a)) - \alpha_g(\sigma(a))\| < \varepsilon$ for all $a \in F_A$.

Then (1), (2), (3), and (4) are equivalent, and (5) implies all of them. If A is unital, then they are all equivalent.

Proof. (1) implies (2): Let $(F_{\mathcal{D}}^{(n)})_{n \in \mathbb{N}}$ and $(F_A^{(n)})_{n \in \mathbb{N}}$ be increasing sequences of finite subsets of \mathcal{D} and A , respectively, with dense union, and such that $\bigcup_{n \in \mathbb{N}} F_{\mathcal{D}}^{(n)}$ is self-adjoint and closed under scalar multiplication by elements from $\mathbb{Q}[i]$. Let $(K_G^{(n)})_{n \in \mathbb{N}}$ be an increasing sequence of compact subsets of G with $G = \bigcup_{n \in \mathbb{N}} K_G^{(n)}$.

Since (A, α) is (\mathcal{D}, δ) -absorbing, we may use [18, Theorem 3.7] to fix a unital, equivariant homomorphism $\Psi: (\mathcal{D}, \delta) \rightarrow (F_{\infty, \alpha}(A), F_{\infty}(\alpha))$. Then Ψ is nuclear, because so is \mathcal{D} . By Theorem 2.1 and Proposition 2.5, for every $n \in \mathbb{N}$, there exists a completely positive contractive map $\psi_n: \mathcal{D} \rightarrow A_{\infty, \alpha} \cap A'$ satisfying $\kappa_A \circ \psi_n = \Psi$ and

$$\sup_{g \in K_G^{(n)}} \|\psi_n(\delta_g(d)) - (\alpha_{\infty})_g(\psi_n(d))\| < \frac{1}{n}$$

for all $d \in F_{\mathcal{D}}^{(n)}$. Use Theorem 2.1 to find a completely positive contractive map $\sigma_n: \mathcal{D} \rightarrow \ell^{\infty}(\mathbb{N}, A)$ with

- (i) $\pi_A \circ \sigma_n = \psi_n$ and
- (ii) $\max_{g \in K_G^{(n)}} \|\sigma_n(\delta_g(d)) - \alpha_g^{\infty}(\sigma_n(d))\| < 1/n$ for all $d \in F_{\mathcal{D}}^{(n)}$.

Fix $n \in \mathbb{N}$. Since the image of ψ_n commutes with A , we have

$$\lim_{k \rightarrow \infty} \|\sigma_n(d)_k a - a \sigma_n(d)_k\| = 0$$

for all $d \in \mathcal{D}$ and all $a \in A$. In particular, there exists $m_n \in \mathbb{N}$ such that for all $k \geq m_n$ we have $\|\sigma_n(d)_k a - a \sigma_n(d)_k\| < 1/n$ for all $d \in F_{\mathcal{D}}^{(n)}$ and all $a \in F_A^{(n)}$. Thus, upon redefining the first $m_n - 1$ entries of σ_n to be zero, we may also assume that

- (iii) $\|\sigma_n(d)_k a - a \sigma_n(d)_k\| < 1/n$ for all $k \in \mathbb{N}$, all $d \in F_{\mathcal{D}}^{(n)}$, and all $a \in F_A^{(n)}$.

Fix $n \in \mathbb{N}$. Since σ_n and σ_1 are both lifts of Ψ , there exists $k_n \in \mathbb{N}$ such that whenever $k \geq k_n$, we have

- (iv) $\|a(\sigma_n(d)_k - \sigma_1(d)_k)\| < 1/n$ for all $a \in F_A^{(n)}$ and all $d \in F_{\mathcal{D}}^{(n)}$.

Without loss of generality, we may assume that $(k_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence. Define a completely positive contractive map $\sigma: \mathcal{D} \rightarrow \ell^{\infty}(\mathbb{N}, A)$ by setting

$$\sigma(d)_k = \begin{cases} 0, & \text{if } k < k_1; \\ \sigma_1(d)_k, & \text{if } k_1 \leq k < k_2; \\ \vdots & \\ \sigma_n(d)_k, & \text{if } k_n \leq k < k_{n+1}; \\ \vdots & \end{cases}$$

for all $d \in \mathcal{D}$ and all $k \in \mathbb{N}$. Set $\psi = \pi_A \circ \sigma: \mathcal{D} \rightarrow A_{\infty}$. Then ψ is completely positive and contractive as well, and its range is contained in $A_{\infty} \cap A'$ by condition (iii). Condition (ii) implies that ψ is equivariant, and thus its range is contained in the continuous part $A_{\infty, \alpha} \cap A'$.

We claim that $\kappa_A \circ \psi = \kappa_A \circ \pi_A \circ \sigma_1$. By continuity, given $m \in \mathbb{N}$, $a \in F_A^{(m)}$, and $d \in F_{\mathcal{D}}^{(m)}$, it suffices to check that

$$\lim_{k \rightarrow \infty} \|a(\sigma_1(d)_k - \sigma(d)_k)\| = 0.$$

Let $\varepsilon > 0$, and find $n \geq m$ with $1/n < \varepsilon$. Given $k \geq k_n$, let $r \in \mathbb{N}$ satisfy $k_r \leq k < k_{r+1}$. (Note that $r \geq n$.) Using that $a \in F_A^{(r)}$ and $d \in F_{\mathcal{D}}^{(r)}$, we have

$$a\sigma_1(d)_k \stackrel{(iv)}{\approx}_{\frac{1}{r}} a\sigma_r(d)_k = a\sigma(d)_k.$$

We deduce that $\|a(\sigma_1(d)_k - \sigma(d)_k)\| < \varepsilon$ for all $k \geq k_n$. Since ε is arbitrary, this proves the claim.

We deduce that $\kappa_A \circ \psi = \kappa_A \circ \pi_A \circ \sigma_1 = \Psi$, so $\kappa_A \circ \psi$ is a unital homomorphism. We have proved (2).

(2) implies (3): Let $\varepsilon > 0$, let $F_A \subseteq A$ and $F_{\mathcal{D}} \subseteq \mathcal{D}$ be finite subsets, and let $K_G \subseteq G$ be a compact subset. Without loss of generality, we assume that F_A consists of contractions. Let $\sigma: \mathcal{D} \rightarrow \ell^\infty(\mathbb{N}, A)$ be a map as in the statement, and set $\varphi = \pi_A \circ \sigma$. Since \mathcal{D} is nuclear, the map $\varphi: \mathcal{D} \rightarrow A_{\infty, \alpha} \cap A'$ is also nuclear. By Theorem 2.1 and Proposition 2.5, there exists a completely positive contractive map $\Psi: \mathcal{D} \rightarrow \ell_{\alpha}^\infty(\mathbb{N}, A)$ which is $(F_{\mathcal{D}}, K_G, \varepsilon)$ -equivariant and satisfies $\pi_A \circ \Psi = \varphi$. For $n \in \mathbb{N}$, let $\psi_n: \mathcal{D} \rightarrow A$ denote the composition of Ψ with the evaluation map at $n \in \mathbb{N}$. By definition of the norm on $F_\infty(A)$, we have

- $\lim_{n \rightarrow \infty} a\psi_n(1) = a$ for all $a \in A$;
- $\lim_{n \rightarrow \infty} \|a\psi_n(d) - \psi_n(d)a\| = 0$ for all $a \in A$ and all $d \in \mathcal{D}$;
- $\lim_{n \rightarrow \infty} \|a(\psi_n(dd') - \psi_n(d)\psi_n(d'))\| = 0$ for all $a \in A$ and all $d, d' \in \mathcal{D}$.

Additionally, since Ψ is $(F_{\mathcal{D}}, K_G, \varepsilon)$ -equivariant, we also have

- $\limsup_{n \rightarrow \infty} \|\psi_n(\delta_g(d)) - \alpha_g(\psi_n(d))\| < \varepsilon$ for $g \in K_G$ and $d \in F_{\mathcal{D}}$.

The proof now follows by setting $\psi = \psi_n$ for n large enough.

(3) implies (4): This is immediate, since any map satisfying (U), (C), (M), and (E) for $(F_{\mathcal{D}}, F_A, F_G \cup N, \varepsilon)$ also satisfies the displayed inequality in (4).

(4) implies (1): Let $(F_{\mathcal{D}}^{(n)})_{n \in \mathbb{N}}$ and $(F_A^{(n)})_{n \in \mathbb{N}}$ be increasing sequences of finite subsets of \mathcal{D} and A , respectively, with dense union. Using that G is locally compact and second countable, let $(F_G^{(n)})_{n \in \mathbb{N}}$ be an increasing sequence of finite subsets of G whose union $G^{(0)}$ is a dense subgroup of G , and let $(N^{(n)})_{n \in \mathbb{N}}$ be a decreasing sequence of compact neighborhoods of 1_G which form a basis of neighborhoods for 1_G . For each $n \in \mathbb{N}$, let $\psi_n: \mathcal{D} \rightarrow A$ be a completely positive contractive map which satisfies conditions (U), (C), (M), and (E) from Notation 2.6 with respect to $(F_{\mathcal{D}}^{(n)}, F_A^{(n)}, F_G^{(n)}, 1/n)$, and such that

$$\max_{h \in N^{(n)}} \|\alpha_h(\psi_n(d)) - \psi_n(d)\| \leq \max_{h \in N^{(n)}} \|\delta_h(d) - d\| + \frac{1}{n} \quad (2.1)$$

for all $d \in F_{\mathcal{D}}^{(n)}$.

Let $\psi: \mathcal{D} \rightarrow \ell^\infty(\mathbb{N}, A)$ be given by $\psi(d)(n) = \psi_n(d)$ for all $d \in \mathcal{D}$ and all $n \in \mathbb{N}$. Then the image of ψ is contained in $A_\infty \cap A'$ by condition (C) in Notation 2.6. Set $\Psi = \kappa_A \circ \pi_A \circ \psi$. Then Ψ is a unital homomorphism by conditions (U) and (M), and it satisfies

$$\Psi \circ \delta_g = F_\infty(\alpha)_g \circ \Psi \quad (2.2)$$

for all $g \in G^{(0)}$, by condition (E). On the other hand, for fixed $d \in \mathcal{D}$, the function $G \rightarrow F_\infty(A)$ given by $g \mapsto F_\infty(\alpha)_g(\Psi(d))$ is continuous at 1_G (and thus at every point of G) by (2.1). It follows that the range of Ψ is contained in $\overline{F_{\infty, \alpha}(A)}$. By continuity, we deduce that the identity in (2.2) holds also for every $g \in \overline{G^{(0)}} = G$. In other words, Ψ is equivariant. It then follows from [18, Theorem 3.7] that (A, α) is (\mathcal{D}, δ) -stable, as desired.

It follows that conditions (1), (2), (3), and (4) are equivalent.

(5) implies (1): Let $(F_{\mathcal{D}}^{(n)})_{n \in \mathbb{N}}$ and $(F_A^{(n)})_{n \in \mathbb{N}}$ be increasing sequences of finite subsets of \mathcal{D} and A , respectively, whose unions are dense in the respective unit balls, and let $(K_G^{(n)})_{n \in \mathbb{N}}$ be an increasing sequence of compact subsets of G whose union equals G . For each $n \in \mathbb{N}$, let $\mu_n: \mathcal{D} \rightarrow M(A)$ and $\sigma_n: A \rightarrow A$ be homomorphisms satisfying conditions (5a) through (5d) for $F_{\mathcal{D}}^{(n)}$, $F_A^{(n)}$, $K_G^{(n)}$, and $\varepsilon_n = 1/n$.

Use the unnumbered lemma at the top of page 152 of [16] to find an approximate unit $(x_n)_{n \in \mathbb{N}}$ for A which satisfies

- (i) $\|ax_n^{1/2} - a\| < 1/n$ for all $a \in F_A^{(n)}$;
- (ii) $\|x_n^{1/2}\mu_n(d) - \mu_n(d)x_n^{1/2}\| < 1/n$ for all $d \in F_{\mathcal{D}}^{(n)}$;
- (iii) $\|\alpha_g(x_n^{1/2}) - x_n^{1/2}\| < 1/n$ for all $g \in K_G^{(n)}$.

By (i), we have

$$(iv) \quad \|ax_n^{1/2} - x_n^{1/2}a\| < \frac{2}{n}$$

for all $a \in F_A^{(n)}$. Define a map $\psi: \mathcal{D} \rightarrow \ell^\infty(\mathbb{N}, A)$ by

$$\psi(d)_n = x_n^{1/2}\mu_n(d)x_n^{1/2}$$

for all $d \in \mathcal{D}$ and all $n \in \mathbb{N}$. We claim that the range of $\pi_A \circ \psi$, which is a priori a subset of A_∞ , is contained in $A_\infty \cap A'$. Fix $n \in \mathbb{N}$, let $d \in F_{\mathcal{D}}^{(n)}$, and let $a \in F_A^{(n)}$. For $m \in \mathbb{N}$ with $m \geq n$, we have

$$\begin{aligned} \psi(d)_m a &= x_m^{1/2}\mu_m(d)x_m^{1/2}a \\ &\stackrel{(iv)}{\approx} \frac{2}{m} x_m^{1/2}\mu_m(d)ax_m^{1/2} \\ &\stackrel{(5b)}{\approx} \frac{1}{m} x_m^{1/2}\mu_m(d)\sigma_m(a)x_m^{1/2} \\ &\stackrel{(5a)}{=} x_m^{1/2}\sigma_m(a)\mu_m(d)x_m^{1/2} \\ &\stackrel{(5b)}{\approx} \frac{1}{m} x_m^{1/2}a\mu_m(d)x_m^{1/2} \\ &\stackrel{(iv)}{\approx} \frac{1}{m} ax_m^{1/2}\mu_m(d)x_m^{1/2} = a\psi(d)_m. \end{aligned}$$

Thus, $\lim_{m \rightarrow \infty} \|\psi(d)_m a - a \psi(d)_m\| = 0$. By density of the linear spans of $\bigcup_{n \in \mathbb{N}} F_{\mathcal{D}}^{(n)}$ and $\bigcup_{n \in \mathbb{N}} F_A^{(n)}$ in \mathcal{D} and A , respectively, it follows that the range of $\pi_A \circ \psi$ is contained in $A_\infty \cap A'$.

Set $\varphi = \kappa_A \circ \pi_A \circ \psi: \mathcal{D} \rightarrow F_\infty(A)$. Then φ is completely positive and contractive, because so is ψ . We claim that φ is a unital, equivariant homomorphism.

To check that φ is equivariant, fix $n \in \mathbb{N}$ and let $d \in F_{\mathcal{D}}^{(n)}$, $a \in F_A^{(n)}$, and $g \in K_G^{(n)}$ be given. For $m \geq n$, we have

$$\begin{aligned} \psi(\delta_g(d))_m a &= x_m^{1/2} \mu_m(\delta_g(d)) x_m^{1/2} a \\ &\stackrel{(iv)}{\approx} \frac{2}{m} x_m^{1/2} \mu_m(\delta_g(d)) a x_m^{1/2} \\ &\stackrel{(5c)}{\approx} \frac{1}{m} x_m^{1/2} \alpha_g(\mu_m(d) \alpha_{g^{-1}}(a)) x_m^{1/2} \\ &\stackrel{(iii)}{\approx} \frac{1}{m} \alpha_g(x_m^{1/2} \mu_m(d)) a x_m^{1/2} \\ &\stackrel{(iv)}{\approx} \frac{2}{m} \alpha_g(x_m^{1/2} \mu_m(d)) x_m^{1/2} a \\ &\stackrel{(iii)}{\approx} \frac{1}{m} \alpha_g(x_m^{1/2} \mu_m(d) x_m^{1/2}) a \\ &= \alpha_g(\psi(d)_m) a. \end{aligned}$$

We deduce that $\lim_{m \rightarrow \infty} \|(\psi(\delta_g(d))_m - \alpha_g(\psi(d)_m))a\| = 0$. By the definition of $F_\infty(A)$, this implies that $\varphi = \kappa_A \circ \pi_A \circ \psi$ is equivariant.

Recall that whenever $(a_n)_{n \in \mathbb{N}}$ is an approximate unit in A , regarded as an element $a \in \ell^\infty(\mathbb{N}, A)$, then $\kappa_A(\pi_A(a))$ is the unit of $F_\infty(A)$. Since

$$\varphi(1) = (\kappa_A \circ \pi_A \circ \psi)(1) = \kappa_A(\pi_A((a_n)_{n \in \mathbb{N}})),$$

it follows that φ is unital. Finally, to check that φ is a homomorphism, fix $n \in \mathbb{N}$ and let $d, e \in F_{\mathcal{D}}^{(n)}$, and $a \in F_A^{(n)}$ be given. For every $m \geq n$, we have

$$\begin{aligned} \psi(de)_m a &= x_m^{1/2} \mu_m(de) x_m^{1/2} a \\ &= x_m^{1/2} \mu_m(d) \mu_m(e) x_m^{1/2} a \\ &\stackrel{(i)}{\approx} \frac{2}{m} x_m^{1/2} \mu_m(d) \mu_m(e) x_m^{3/2} a \\ &\stackrel{(ii)}{\approx} \frac{2}{m} x_m^{1/2} \mu_m(d) x_m \mu_m(e) x_m^{1/2} a \\ &= \psi(d)_m \psi(e)_m a. \end{aligned}$$

Thus, $\lim_{m \rightarrow \infty} \|\psi(de)_m a - \psi(d)_m \psi(e)_m a\| = 0$. By definition of $F_\infty(A)$, this shows that $\varphi = \kappa_A \circ \pi_A \circ \psi$ is a homomorphism.

Since φ is equivariant, its image is contained in $F_{\infty, \alpha}(A)$. Thus, the equivalence between (ii) and (iii) in [18, Theorem 3.7] implies that (A, α) is (\mathcal{D}, δ) -stable, as desired.

Assume now that A is unital; we will show that (1) implies (5). Let $\varepsilon > 0$, let finite subsets $F_{\mathcal{D}} \subseteq \mathcal{D}$, $F_A \subseteq A$, and let a compact subset $K_G \subseteq G$ be given. Without loss of generality, we assume that F_A and $F_{\mathcal{D}}$ contain only contractions. Set $\widetilde{F_A} = F_A \cup \bigcup_{g \in K_G} \alpha_g(F_A)$, which is a compact subset of A .

By the equivalence between (i) and (ii) in [18, Theorem 3.7], the G -algebras (A, α) and $(A \otimes \mathcal{D}, \alpha \otimes \delta)$ are strongly cocycle conjugate. Using [18, Lemma 4.2], find an isomorphism $\theta_A: A \otimes \mathcal{D} \rightarrow A$ satisfying

$$\max_{g \in K_G} \|\alpha_g \circ \theta_A - \theta_A \circ (\alpha_g \otimes \delta_g)\| < \frac{\varepsilon}{4}. \quad (2.3)$$

Let $F'_{\mathcal{D}} \subseteq \mathcal{D}$ be a finite subset containing $F_{\mathcal{D}}$, and such that for all $a \in \widetilde{F_A}$ there exist $n \in \mathbb{N}$ and elements $a_1, \dots, a_n \in A$ and $d_1, \dots, d_n \in F'_{\mathcal{D}}$ such that

$$\left\| a - \sum_{j=1}^n \theta_A(a_j \otimes d_j) \right\| < \frac{\varepsilon}{4}. \quad (2.4)$$

Let $\psi: (\mathcal{D} \otimes \mathcal{D}, \delta \otimes \delta) \rightarrow (\mathcal{D}, \delta)$ be an equivariant isomorphism such that ψ^{-1} is G -approximately unitarily equivalent to the first factor embedding, and let $v \in \mathcal{U}(\mathcal{D})$ be a unitary satisfying $\max_{g \in K_G} \|\delta_g(v) - v\| < \varepsilon/8$ and $\|v\psi(d \otimes 1)v^* - d\| < \varepsilon/8$ for all $d \in F'_{\mathcal{D}}$. Set $\theta_{\mathcal{D}} = \text{Ad}(v) \circ \psi$. Given $g \in K_G$, we have

$$\begin{aligned} \|\delta_g \circ \theta_{\mathcal{D}} - \theta_{\mathcal{D}} \circ (\delta \otimes \delta)_g\| &= \|\delta_g \circ \text{Ad}(v) \circ \psi - \text{Ad}(v) \circ \psi \circ (\delta \otimes \delta)_g\| \\ &= \|\delta_g \circ \text{Ad}(v) - \text{Ad}(v) \circ \delta_g\| < \frac{\varepsilon}{4}. \end{aligned} \quad (2.5)$$

Moreover, for $d \in F_{\mathcal{D}}$, we have

$$\theta_{\mathcal{D}}(d \otimes 1) = v\psi(d \otimes 1)v^* \approx_{\frac{\varepsilon}{8}} d. \quad (2.6)$$

Define $\sigma: A \rightarrow A$ by setting

$$\sigma = \theta_A \circ (\text{id}_A \otimes \theta_{\mathcal{D}}) \circ (\theta_A^{-1} \otimes 1).$$

Then σ is a homomorphism. Fix $a \in \widetilde{F_A}$. Find $n \in \mathbb{N}$ and elements $a_1, \dots, a_n \in A$ and $d_1, \dots, d_n \in F'_{\mathcal{D}}$ satisfying the inequality in (2.4). Then

$$\begin{aligned} \sigma(a) &= \theta_A \circ (\text{id}_A \otimes \theta_{\mathcal{D}})(\theta_A^{-1}(a) \otimes 1) \\ &\stackrel{(2.4)}{\approx_{\frac{\varepsilon}{4}}} \sum_{j=1}^n \theta_A \circ (\text{id}_A \otimes \theta_{\mathcal{D}})(a_j \otimes d_j \otimes 1) \\ &\stackrel{(2.6)}{\approx_{\frac{\varepsilon}{4}}} \sum_{j=1}^n \theta_A(a_j \otimes d_j) \stackrel{(2.4)}{\approx_{\frac{\varepsilon}{4}}} a, \end{aligned} \quad (2.7)$$

thus establishing condition (5b) in the statement. In order to check (5d), let $g \in K_G$ and $a \in F_A$. Using that $\alpha_g(a) \in \widetilde{F_A}$, we have

$$\sigma(\alpha_g(a)) \stackrel{(2.7)}{\approx_{\frac{\varepsilon}{4}}} \alpha_g(a) \stackrel{(2.7)}{\approx_{\frac{\varepsilon}{4}}} \alpha_g(\sigma(a)),$$

as desired. Define $\mu: \mathcal{D} \rightarrow A$ by

$$\mu(d) = \theta_A(1_A \otimes \theta_{\mathcal{D}}(1 \otimes d))$$

for all $d \in \mathcal{D}$. Then μ is a unital homomorphism. We proceed to check condition (5a) in the statement. Given $a \in A$ and $d \in \mathcal{D}$, we have

$$\begin{aligned} \theta_A^{-1}(\sigma(a)\mu(d)) &= [(\text{id}_A \otimes \theta_{\mathcal{D}})(\theta_A^{-1}(a) \otimes 1)][(\text{id}_A \otimes \theta_{\mathcal{D}})(1_A \otimes 1_{\mathcal{D}} \otimes d)] \\ &= (\text{id}_A \otimes \theta_{\mathcal{D}})\left(\underbrace{(\theta_A^{-1}(a) \otimes 1)(1_A \otimes 1_{\mathcal{D}} \otimes d)}_{=(1_A \otimes 1_{\mathcal{D}} \otimes d)(\theta_A^{-1}(a) \otimes 1)}\right) = \theta_A^{-1}(\mu(d)\sigma(a)), \end{aligned}$$

as desired. Finally, to check condition (5c), it suffices to take $a = 1$. Let $g \in K_G$ and $d \in F_{\mathcal{D}}$. Then

$$\begin{aligned} \mu(\delta_g(d)) &= \theta_A(1 \otimes \theta_{\mathcal{D}}(1 \otimes \delta_g(d))) \\ &\stackrel{(2.5)}{\approx_{\frac{\varepsilon}{4}}} \theta_A(1 \otimes \delta_g(\theta_{\mathcal{D}}(1 \otimes d))) \\ &\stackrel{(2.3)}{\approx_{\frac{\varepsilon}{4}}} \alpha_g(\theta_A(1 \otimes \theta_{\mathcal{D}}(1 \otimes d))) \\ &= \alpha_g(\mu(d)). \end{aligned}$$

We conclude that $\max_{g \in K_G} \|\mu(\delta_g(d))\alpha_g(a) - \alpha_g(\mu(d)(a))\| < \varepsilon\|a\|$ for all $a \in A$ and all $d \in F_{\mathcal{D}}$, as desired. This finishes the proof. ■

When working with norm-approximations, there is usually little to no difference between considering finite subsets or norm-compact subsets of a given C*-algebra. In particular, the sets $F_{\mathcal{D}}$ and F_A in conditions (3), (4), and (5) of the previous theorem could equivalently be chosen to be compact. For the subset of G , the situation is a bit more subtle: one may replace the compact subset K_G with a finite one, at the price of having to control the continuity moduli on neighborhoods of the unit of G . We included condition (4) in the above theorem because in practice one is often only able to get approximations on a finite subset of G . For example, most of the arguments one performs in the sequence algebra will only be able to keep track of finitely many group elements at a time. In such situations, it suffices to verify (almost) equivariance on a finite subset as long as we have control over the relevant continuity moduli.

For future reference, we end this section with some comments about conditions (U) and (M) in the unital case.

Remark 2.8. If A is unital, then the witness $a \in F_A$ in part (3) of Theorem 2.7 is unnecessary in conditions (U) and (M) (see Notation 2.6), and they become

$$(U) \quad \|\psi(1) - 1\| < \varepsilon \qquad (M) \quad \|\psi(dd') - \psi(d)\psi(d')\| < \varepsilon.$$

If, in addition, A admits an α -invariant state, then condition (U) can be replaced by $\psi(1) = 1$; see the comments after Theorem 2.1.

3. Equivariant continuous bundles and absorption

In this section, we study continuous bundles of G -algebras, also known as G - $C(X)$ -algebras; see Definition 3.1. We are mostly interested in determining when such a G - $C(X)$ -algebra absorbs a given strongly self-absorbing G -algebra (\mathcal{D}, δ) . By [19, Corollary 2.8], a necessary condition is that the fibers of the bundle absorb (\mathcal{D}, δ) . When the base space X has finite covering dimension and (\mathcal{D}, δ) satisfies a mild technical condition called *unitary regularity*, we show that the converse also holds. Hence, continuous bundles of (\mathcal{D}, δ) -absorbing G -algebras are themselves (\mathcal{D}, δ) -absorbing. This is an equivariant version of [14, Theorem 4.6], and our proof depends crucially on the characterization of (\mathcal{D}, δ) -absorption obtained in Section 2.

The above-mentioned result is known to fail if X has infinite covering dimension, already in the non-equivariant setting (i.e., for the trivial actions on A and \mathcal{D}); see [14, Example 4.8]. However, even when X is infinite-dimensional, if (A, α) is *locally* (\mathcal{D}, δ) -absorbing, then (A, α) is (\mathcal{D}, δ) -absorbing as well; see the comments at the end of this section.

We begin by introducing some necessary definitions. For a C^* -algebra A , we denote by $Z(A)$ its center.

Definition 3.1. Let A be a unital C^* -algebra, and let X be a compact Hausdorff space. We say that A is a $C(X)$ -algebra, if there is a unital homomorphism $\rho: C(X) \rightarrow Z(A)$. In order to lighten the notation, we usually omit ρ and denote $\rho(f)(a)$ by $f \cdot a$ for $f \in C(X)$ and $a \in A$.

If G is a locally compact group and (A, α) is a G -algebra, then we say that it is a G - $C(X)$ -algebra if $C(X) \subseteq A^G$.

In practice, we think of G - $C(X)$ -algebras as being bundles over the base space X , where each fiber is naturally a G -algebra. This is made precise as follows.

Notation 3.2. Let G be a locally compact group, let X be a compact Hausdorff space, and let (A, α) be a G - $C(X)$ -algebra. For a closed subset $Y \subseteq X$, we set $J_Y = C_0(X \setminus Y) \cdot A$, which is a G -invariant ideal in A . We let A_Y denote the corresponding quotient, with quotient map $\pi_Y: A \rightarrow A_Y$. Then α naturally induces an action α^Y on A_Y , and π_Y is equivariant. Moreover, (A_Y, α^Y) can be regarded either as a G - $C(X)$ -algebra or as a G - $C(Y)$ -algebra in a natural way.

Remark 3.3. When Y is a singleton set $Y = \{x\}$, we abbreviate A_Y to A_x and α^Y to $\alpha^{(x)}$, and we call the G -algebra $(A_x, \alpha^{(x)})$ the *fiber* of (A, α) over x . For $a \in A$, we write a_x for $\pi_x(a)$ and observe that

$$\|a\| = \sup_{x \in X} \|a_x\|.$$

Moreover, the function $x \mapsto \|a_x\|$ is upper semicontinuous, which means that for every $\varepsilon > 0$, the set $\{x \in X : \|a_x\| < \varepsilon\}$ is open in X .

We will prove Theorem 3.13 using the new characterization of (\mathcal{D}, δ) -absorption obtained in Theorem 2.7. In particular, we will construct completely positive contractive maps $\mathcal{D} \rightarrow A$ satisfying the conditions in part (3) of the said theorem. In doing so, we will first produce, for a closed subset $Y \subseteq X$, maps $\mathcal{D} \rightarrow A$ which satisfy the said conditions *when composed* with the quotient map $\pi_Y: A \rightarrow A_Y$. We isolate this notion in the following auxiliary definition (see Notation 2.6).

Definition 3.4. Let G be a second countable, locally compact group, let (\mathcal{D}, δ) be a strongly self-absorbing G -algebra, let X be a compact Hausdorff space, and let (A, α) be a separable G - $C(X)$ -algebra. Let finite sets $F_{\mathcal{D}} \subseteq \mathcal{D}$ and $F_A \subseteq A$ and a compact subset $K_G \subseteq G$ be given. For a closed subset $Y \subseteq X$ and $\varepsilon > 0$, a completely positive contractive map $\psi: \mathcal{D} \rightarrow A$ is said to be $(F_{\mathcal{D}}, F_A, K_G, \varepsilon)$ -regular for Y if for every $y \in Y$, the composition $\pi_y \circ \psi$ satisfies conditions (U), (C), (M), and (E) with respect to $(F_{\mathcal{D}}, \pi_y(F_A), K_G, \varepsilon)$.

The above definition is an adaptation to the equivariant setting of [14, Definition 4.2], where condition (E) from Notation 2.6 is incorporated to bound the failure of equivariance for ψ for $(F_{\mathcal{D}}, K_G)$. We do not need to control the continuity modulus of α on $\psi(F_{\mathcal{D}})$, but for this it is necessary that we work with compact subsets of G , as opposed to just finite subsets. Alternatively, if we worked instead with a finite subset $F_G \subseteq G$ and required condition (E) to hold for $(F_{\mathcal{D}}, F_G)$, we would need to add a continuity-type condition over a compact neighborhood of the identity in G along the lines of the displayed inequality in part (4) of Theorem 2.7.

Remark 3.5. In the context of Definition 3.4, by Remark 3.3 a map $\psi: \mathcal{D} \rightarrow A$ is $(F_{\mathcal{D}}, F_A, K_G, \varepsilon)$ -regular for all of X if and only if it satisfies conditions (U), (C), (M), and (E) with respect to $(F_{\mathcal{D}}, F_A, K_G, \varepsilon)$. In particular, it follows from Theorem 2.7 that (A, α) is (\mathcal{D}, δ) -absorbing if and only if for all choices of $(F_{\mathcal{D}}, F_A, K_G, \varepsilon)$, there exists a $(F_{\mathcal{D}}, F_A, K_G, \varepsilon)$ -regular map for all of X .

We begin by establishing the existence of regular maps in the presence of (\mathcal{D}, δ) -absorbing fibers.

Lemma 3.6. *Let the assumptions and notation be as in Definition 3.4, and suppose that there is $x \in X$ such that $(A_x, \alpha^{(x)})$ is (\mathcal{D}, δ) -absorbing. Then there exist a*

compact neighborhood Y of x and a completely positive map $\psi: \mathcal{D} \rightarrow A$ which is $(F_{\mathcal{D}}, F_A, K_G, \varepsilon)$ -regular for Y .

Proof. Since $(A_x, \alpha^{(x)})$ is (\mathcal{D}, δ) -absorbing, it follows from Theorem 2.7 that there exists a completely positive contractive map $\varphi: \mathcal{D} \rightarrow A_x$ which satisfies conditions (U), (C), (M), and (E) with respect to the tuple $(F_{\mathcal{D}}, \pi_x(F_A), K_G, \varepsilon)$. The map φ is automatically nuclear, since so is \mathcal{D} . Let $\psi: \mathcal{D} \rightarrow A$ be a lift of φ as in the conclusion of Theorem 2.1 for the tuple $(F_{\mathcal{D}}, K_G, \varepsilon)$. It is then immediate to check that ψ is $(F_{\mathcal{D}}, F_A, K_G, \varepsilon)$ -regular for $\{x\}$. By upper semicontinuity of the norm function (see Notation 3.2), it follows that ψ is regular not just for $\{x\}$, but for a neighborhood of it. The result then follows by local compactness of X . ■

Our strategy to prove Theorem 3.13 for G - $C(X)$ -algebras is inspired by the arguments used in [14] and consists in “gluing” the regular maps obtained in Lemma 3.6 to get a map which is regular for all of X . In general, there may be topological obstructions to this gluing if the boundaries of the sets over which one wishes to glue have complicated homotopy groups. We will thus make some simplifications, which we proceed to explain.

If a compact Hausdorff space X embeds homeomorphically into another compact Hausdorff space Z , then any G - $C(X)$ -algebra (A, α) is naturally a G - $C(Z)$ -algebra, by composing the original structure map with the quotient map $C(Z) \rightarrow C(X)$. With this structure, the fiber of (A, α) over a point in X (regarded as a point in Z) is the original one, and the fiber over a point in $Z \setminus X$ is the zero G -algebra. In particular, all the Z -fibers are (\mathcal{D}, δ) -stable and are unital and separable if all the X -fibers are. Note, however, that A is not in general a continuous $C(Z)$ -algebra, even if it is a continuous $C(X)$ -algebra.

When X is compact and finite-dimensional, then by [15, Theorem V.3], there exists $m \in \mathbb{N}$ such that X embeds homeomorphically into the cube $[0, 1]^m$. Using induction, we will see that the heart of the argument lies in establishing the case $m = 1$. For G - $C([0, 1])$ -algebras, the following implies that one only needs to worry about gluing regular maps over a single point.

Lemma 3.7. *Let G be a locally compact group, let (\mathcal{D}, δ) be a strongly self-absorbing G -algebra, let (A, α) be a G - $C([0, 1])$ -algebra such that $(A_t, \alpha^{(t)})$ is (\mathcal{D}, δ) -absorbing for all $t \in [0, 1]$, and let a tuple $(F_{\mathcal{D}}, F_A, K_G, \varepsilon)$ as in Definition 3.4 be given. Then there exist $n \in \mathbb{N}$, points $0 = t_0 < t_1 < \cdots < t_n = 1$ and completely positive contractive maps $\psi_1, \dots, \psi_n: \mathcal{D} \rightarrow A$ such that ψ_j is $(F_{\mathcal{D}}, F_A, K_G, \varepsilon)$ -regular for $[t_{j-1}, t_j]$ for all $j = 1, \dots, n$.*

Proof. For every $t \in [0, 1]$, use Lemma 3.6 to find a closed interval Y_t containing t in its interior, and a map $\psi_t: \mathcal{D} \rightarrow A$ which is $(F_{\mathcal{D}}, F_A, K_G, \varepsilon)$ -regular for Y_t . Since the interiors of the Y_t cover $[0, 1]$, the result follows by compactness. ■

Gluing the regular maps obtained above will require better control over the norm estimates in (U), (C), (M), and (E) from Definition 3.4 over the endpoints; see the proof of Lemma 3.12. The following notion will be helpful in this task.

Definition 3.8. Let G be a locally compact group, let (\mathcal{D}, δ) be a strongly self-absorbing G -algebra, let (A, α) be a unital, separable G - $C([0, 1])$ -algebra, and let Y be a closed subinterval in $[0, 1]$. Let finite subsets $F_{\mathcal{D}} \subseteq F'_{\mathcal{D}} \subseteq \mathcal{D}$, $F_A \subseteq A$, a compact subset $K_G \subseteq G$, and tolerances $0 < \varepsilon' < \varepsilon$ be given. We say that a completely positive contractive map $\psi: \mathcal{D} \rightarrow A$ is $(F_{\mathcal{D}}, F_A, K_G, \varepsilon, F'_{\mathcal{D}}, \varepsilon')$ -regular for Y , if ψ is $(F_{\mathcal{D}}, F_A, K_G, \varepsilon)$ -regular for Y , and there is a compact neighborhood of the endpoints of Y for which ψ is $(F'_{\mathcal{D}}, F_A, K_G, \varepsilon')$ -regular.

In the context of the above definition, and since A is assumed to be unital, following Remark 2.8 we will use conditions (U) and (M) without the multiplicative witness $a \in F_A$.

The gluing procedure explained after Lemma 3.6 will be accomplished in the following two lemmas. In the first one, we show that, given two regular maps ψ_1 and ψ_2 that we wish to glue at $s \in [0, 1]$, we can find two other regular maps v_1 and v_2 such that, one localized at s , the maps ψ_j and v_j approximately commute for $j = 1, 2$, and v_1 is close to v_2 .

We follow the arguments in [14], but stress the fact that working in the equivariant setting makes the arguments necessarily more technical, since we must keep track of equivariance and (at least implicitly) continuity. We omit those computations similar to the ones in [14] and give details for everything related to the actions.

Lemma 3.9. *Let the notation and assumptions be as in Definition 3.8. Let $r, s, t \in [0, 1]$ with $r < s < t$, and let $\psi_1, \psi_2: \mathcal{D} \rightarrow A$ be completely positive contractive maps which are $(F_{\mathcal{D}}, F_A, K_G, \varepsilon)$ -regular for $[r, s]$ and $[s, t]$, respectively. If $(A_s, \alpha^{(s)})$ is (\mathcal{D}, δ) -stable, then there exist completely positive contractive maps*

$$\varphi_1, \varphi_2, v_1, v_2: \mathcal{D} \rightarrow A \quad \text{and} \quad \mu_1, \mu_2: \mathcal{D} \otimes \mathcal{D} \rightarrow A$$

such that the following are satisfied for $j = 1, 2$ and $d, d' \in F_{\mathcal{D}}$:

- (1) φ_1 and φ_2 are $(F_{\mathcal{D}}, F_A, K_G, \varepsilon)$ -regular for $[r, s]$ and $[s, t]$, respectively.
- (2) v_j is $(F_{\mathcal{D}}, F_A, K_G, 3\varepsilon)$ -regular for $\{s\}$.
- (3) $\|\varphi_j(d)_s v_j(d')_s - v_j(d')_s \varphi_j(d)_s\| < 2\varepsilon$.
- (4) $\|\varphi_j(d)_s v_j(d')_s - \mu_j(d \otimes d')_s\| < \varepsilon$.
- (5) $\|\alpha_g^{(s)}(\mu_j(d \otimes d')_s) - \mu_j(\delta_g(d) \otimes \delta_g(d'))_s\| < \varepsilon$ for all $g \in K_G$.
- (6) $\|v_1(d)_s - v_2(d)_s\| < 2\varepsilon$.

If ψ_1 and ψ_2 are $(F_{\mathcal{D}}, F_A, K_G, \varepsilon, F'_{\mathcal{D}}, \varepsilon')$ -regular for $[r, s]$ and $[s, t]$, respectively, then the same can be arranged for φ_1 and φ_2 . Moreover, one can arrange that v_1 and v_2 are $(F'_{\mathcal{D}}, F_A, K_G, 3\varepsilon')$ -regular for $\{s\}$, and that conditions (3) through (6) hold if one replaces $F_{\mathcal{D}}$ with $F'_{\mathcal{D}}$ and ε with ε' .

Proof. Without loss of generality, we assume that $F_{\mathcal{D}}$ and F_A consist of positive contractions, that $F_{\mathcal{D}}$ contains the unit of \mathcal{D} , that F_A contains the unit of A , and that K_G contains

the unit of G . Using that F_A and $F_{\mathcal{D}}$ are finite, and that K_G is compact, find $0 < \varepsilon_0 < \varepsilon$ such that ψ_1 and ψ_2 are $(F_{\mathcal{D}}, F_A, K_G, \varepsilon_0)$ -regular for $[r, s]$ and $[s, t]$, respectively. Explicitly, with $I_1 = [r, s]$ and $I_2 = [s, t]$ for $j = 1, 2$, we have (see Remark 2.8):

- (U) $_j$ $\|\psi_j(1)_x - 1_x\| < \varepsilon_0$ for all $x \in I_j$;
- (C) $_j$ $\|a_x \psi_j(d)_x - \psi_j(d)_x a_x\| < \varepsilon_0$ for all $a \in F_A$, $d \in F_{\mathcal{D}}$, and $x \in I_j$;
- (M) $_j$ $\|\psi_j(dd')_x - \psi_j(d)_x \psi_j(d')_x\| < \varepsilon_0$ for all $d, d' \in F_{\mathcal{D}}$, and $x \in I_j$;
- (E) $_j$ $\max_{g \in K_G} \|\alpha_g^{(x)}(\psi_j(d)_x) - \psi_j(\delta_g(d))_x\| < \varepsilon_0$ for $d \in F_{\mathcal{D}}$ and $x \in I_j$.

Equivalently, for $j = 1, 2$ and all $x \in I_j$, the map $\pi_x \circ \psi_j$ satisfies (U), (C), (M), and (E) with respect to $(F_{\mathcal{D}}, \pi_x(F_A), K_G, \varepsilon_0)$. Set $\tilde{\varepsilon} = (\varepsilon - \varepsilon_0)/6$, set $\widetilde{F_{\mathcal{D}}} = \bigcup_{g \in K_G} \delta_g(F_{\mathcal{D}})$, which is a compact subset of \mathcal{D} , and set

$$\widetilde{F_A} = F_A \cup \psi_1(\widetilde{F_{\mathcal{D}}}) \cup \psi_1(\widetilde{F_{\mathcal{D}}})\psi_1(\widetilde{F_{\mathcal{D}}}) \cup \psi_2(\widetilde{F_{\mathcal{D}}}) \cup \psi_2(\widetilde{F_{\mathcal{D}}})\psi_2(\widetilde{F_{\mathcal{D}}}),$$

which is a compact subset of A . Since $(A_s, \alpha^{(s)})$ is (\mathcal{D}, δ) -stable and unital, it satisfies the conditions in part (5) of Theorem 2.7. We thus fix homomorphisms $\mu: \mathcal{D} \rightarrow A_s$ and $\sigma: A_s \rightarrow A_s$ with $\mu(1) = 1$, satisfying:

- (5a) $\sigma(a_s)\mu(d) = \mu(d)\sigma(a_s)$ for all $d \in \mathcal{D}$ and all $a \in A$;
- (5b) $\|\sigma(a_s) - a_s\| < \tilde{\varepsilon}$ for all $a \in \widetilde{F_A}$;
- (5c) $\max_{g \in K_G} \|\alpha_g^{(s)}(\mu(d)) - \mu(\delta_g(d))\| < \tilde{\varepsilon}$ for $d \in F_{\mathcal{D}}$;
- (5d) $\max_{g \in K_G} \|\sigma(\alpha_g^{(s)}(a_s)) - \alpha_g^{(s)}(\sigma(a_s))\| < \tilde{\varepsilon}$ for all $a \in \widetilde{F_A}$.

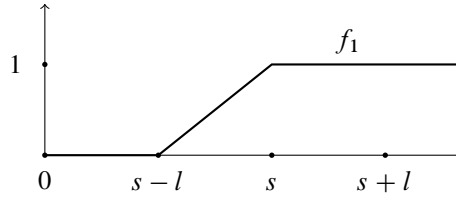
Fix $j = 1, 2$. Using (twice) that $\sigma \circ \pi_s \approx_{\tilde{\varepsilon}} \pi_s$ on $F_A \cup \psi_j(F_{\mathcal{D}})$ by condition (5b), one easily shows that $\sigma \circ \pi_s \circ \psi_j$ satisfies conditions analogous to (U) $_j$, (C) $_j$, (M) $_j$, and (E) $_j$, with $\varepsilon_0 + 2\tilde{\varepsilon}$ instead of ε_0 . We check this for (E) $_j$ and leave the other ones to the reader. Let $d \in F_{\mathcal{D}}$ and let $g \in K_G$. Using that $\delta_g(d) \in \widetilde{F_{\mathcal{D}}}$ at the fourth step, we get

$$\begin{aligned} \alpha_g^{(s)}((\sigma \circ \pi_s \circ \psi_j)(d)) &= \alpha_g^{(s)}(\sigma(\psi_j(d)_s)) \stackrel{(5b)}{\approx_{\tilde{\varepsilon}}} \alpha_g^{(s)}(\psi_j(d)_s) \stackrel{(E)_j}{\approx_{\varepsilon_0}} \psi_j(\delta_g(d))_s \\ &\stackrel{(5b)}{\approx_{\tilde{\varepsilon}}} \sigma(\psi_j(\delta_g(d))_s) = (\sigma \circ \pi_s \circ \psi_j)(\delta_g(d)), \end{aligned}$$

as desired. By upper semicontinuity of the norm, there exists $l > 0$ with $[s - l, s + l] \subseteq I_1 \cup I_2$ such that $\sigma \circ \pi_x \circ \psi_j$ satisfies similar estimates for all $x \in [s - l, s + l]$. Let $\tilde{\psi}_j: \mathcal{D} \rightarrow A$ be a completely positive contractive lift of $\sigma \circ \pi_s \circ \psi_j$. Then $\tilde{\psi}_j$ is $(F_{\mathcal{D}}, F_A, K_G, \varepsilon_0 + 2\tilde{\varepsilon})$ -regular for $[s - l, s + l]$. Moreover, for all $d \in F_{\mathcal{D}}$, we have

$$\tilde{\psi}_j(d)_s = \sigma(\psi_j(d)_s) \stackrel{(5b)}{\approx_{\tilde{\varepsilon}}} \psi_j(d)_s, \quad (3.1)$$

so by reducing l we may also assume that $\|\tilde{\psi}_j(d)_x - \psi_j(d)_x\| < \tilde{\varepsilon}$ for all $d \in F_{\mathcal{D}}$ and all $x \in [s - l, s + l]$. Let $f_1: \mathbb{R} \rightarrow [0, 1]$ be given by the following graph:



and set $f_2(s+x) = f_1(s-x)$ for all $x \in \mathbb{R}$. We regard f_1 and f_2 as elements in $C([0, 1])$ by restricting them to $[0, 1]$. For $j = 1, 2$, define a completely positive contractive map $\varphi_j: \mathcal{D} \rightarrow A$ by

$$\varphi_j(d) = (1 - f_j) \cdot \psi_j(d) + f_j \cdot \tilde{\psi}_j(d) \quad (3.2)$$

for all $d \in \mathcal{D}$.

Claim 1. The map φ_j is $(F_{\mathcal{D}}, F_A, K_G, \varepsilon_0 + 2\tilde{\varepsilon})$ -regular for I_j . We prove this for $j = 1$, since the proof for $j = 2$ is analogous. Note that φ_1 agrees with ψ_1 on $[0, s-l]$, and with $\tilde{\psi}_1$ on $[s, s+l]$. Thus, it suffices to prove regularity of φ_1 for $[s-l, s]$. Conditions (U), (C), and (M) are explicitly verified in [14, Lemma 4.4], so we only prove (E). Given $d \in F_{\mathcal{D}}$, $g \in K_G$, and $x \in [s-l, s]$, we have

$$\begin{aligned} \alpha_g^{(x)}(\varphi_1(d)_x) &= (1 - f_1(x)) \underbrace{\alpha_g^{(x)}(\psi_1(d)_x)}_{\approx_{\varepsilon_0} \psi_1(\delta_g(d))_x} + f_1(x) \underbrace{\alpha_g^{(x)}(\tilde{\psi}_1(d)_x)}_{\approx_{\varepsilon_0+2\tilde{\varepsilon}} \tilde{\psi}_1(\delta_g(d))_x} \\ &\approx_{\varepsilon_0+2\tilde{\varepsilon}} ((1 - f_1(x))\psi_1(\delta_g(d))_x + f_1(x)\tilde{\psi}_1(\delta_g(d))_x) \\ &= \varphi_1(\delta_g(d))_x, \end{aligned}$$

where the above bracketed approximations use condition (E)₁ and the fact that $\tilde{\psi}_1$ is $(F_{\mathcal{D}}, F_A, K_G, \varepsilon_0 + 2\tilde{\varepsilon})$ -regular for $[s-l, s+l]$, respectively. We deduce that φ_1 satisfies condition (E)_j, as desired. This proves the claim and establishes condition (1) in the statement.

We proceed to the construction of the map μ_j . Since

$$\pi_s \circ \varphi_j = f_j(s)\pi_s \circ \tilde{\psi}_j = \sigma \circ \pi_s \circ \psi_j,$$

it follows that the range of $\pi_s \circ \varphi_j$ is contained in the range of σ , and thus by (5a) we have

$$\varphi_j(d)_s \mu(d') = \mu(d') \varphi_j(d)_s \quad (3.3)$$

for all $d, d' \in \mathcal{D}$. Thus, there exists a well-defined completely positive contractive map $\bar{\mu}_j: \mathcal{D} \otimes \mathcal{D} \rightarrow A_s$ which is determined by $\bar{\mu}_j(d \otimes d') = \varphi_j(d)_s \mu(d')$ for all $d, d' \in \mathcal{D}$. Let $\bar{\nu}_j: \mathcal{D} \rightarrow A_s$ be given by $\bar{\nu}_j(d) = \bar{\mu}_j(1 \otimes d)$ for all $d \in \mathcal{D}$. Use the Choi–Effros lifting theorem to find completely positive contractive maps

$$\mu_j: \mathcal{D} \otimes \mathcal{D} \rightarrow A \quad \text{and} \quad \nu_j: \mathcal{D} \rightarrow A$$

satisfying $\pi_s \circ \mu_j = \bar{\mu}_j$ and $\pi_s \circ \nu_j = \bar{\nu}_j$.

Claim 2. The map $\pi_s \circ v_j$ satisfies conditions (U), (C), (M), and (E) with respect to $(F_{\mathcal{D}}, \pi_s(F_A), K_G, 3\varepsilon)$. As in Claim 1, conditions (U), (C), and (M) are checked in the proof of Lemma 4.4 in [14], so we only prove (E). By Claim 1, condition (U) for φ_j gives $\varphi_j(1)_s \approx_{\varepsilon_0+2\varepsilon} 1$. Using this at the last step, we get

$$v_j(d)_s = \bar{v}_j(d) = \bar{\mu}_j(1 \otimes d) = \varphi_j(1)_s \mu(d) \approx_{\varepsilon} \mu(d) \quad (3.4)$$

for all $d \in \mathcal{D}$. For $g \in K_G$ and $d \in F_{\mathcal{D}}$, we have

$$\begin{aligned} \alpha_g^{(s)}(v_j(d)_s) &\stackrel{(3.4)}{\approx_{\varepsilon}} \alpha_g^{(s)}(\mu(d)) \\ &\stackrel{(5c)}{\approx_{\varepsilon}} \mu(\delta_g(d)) \\ &\stackrel{(3.4)}{\approx_{\varepsilon}} v_j(\delta_g(d))_s. \end{aligned}$$

This proves the claim and establishes condition (2) in the statement.

We check condition (3). Let $d, d' \in F_{\mathcal{D}}$, and recall that $F_{\mathcal{D}}$ was assumed to consist of contractions. Then

$$\begin{aligned} \varphi_j(d)_s v_j(d')_s &\stackrel{(3.4)}{\approx_{\varepsilon}} \varphi_j(d)_s \mu(d') \\ &\stackrel{(3.3)}{=} \mu(d') \varphi_j(d)_s \\ &\stackrel{(3.4)}{\approx_{\varepsilon}} v_j(d')_s \varphi_j(d)_s, \end{aligned}$$

as desired. To check condition (4), let $d, d' \in F_{\mathcal{D}}$. Then

$$\mu_j(d \otimes d')_s = \varphi_j(d)_s \mu(d') \stackrel{(3.4)}{\approx_{\varepsilon}} \varphi_j(d)_s v_j(d')_s,$$

as desired. We turn to condition (5), so fix $d, d' \in F_{\mathcal{D}}$, and $g \in K_G$. Then

$$\begin{aligned} \alpha_g^{(s)}(\mu_j(d \otimes d')_s) &= \alpha_g^{(s)}(\mu(d') \varphi_j(d)_s) \\ &\stackrel{(5c)}{\approx_{\varepsilon}} \mu(\delta_g(d')) \alpha_g^{(s)}(\varphi_j(d)_s) \\ &\stackrel{\text{Claim 1}}{\approx_{\varepsilon_0+2\varepsilon}} \mu(\delta_g(d')) \varphi_j(\delta_g(d))_s \\ &= \mu_j(\delta_g(d) \otimes \delta_g(d'))_s, \end{aligned}$$

as desired. To prove condition (6), let $d \in F_{\mathcal{D}}$. Then

$$v_1(d)_s \stackrel{(3.4)}{\approx_{\varepsilon}} \mu(d) \stackrel{(3.4)}{\approx_{\varepsilon}} v_2(d)_s,$$

as desired. This finishes the proof of the first part of the lemma.

The last assertion in the lemma is proved identically as above, noticing that all the estimates we made at the point $\{s\}$ only depend on the regularity of ψ_1 and ψ_2 at $\{s\}$. We omit the details. ■

We will need the following notation, which is borrowed from [18].

Notation 3.10. Let G be a second countable, locally compact group, and let (D, δ) be a unital G -algebra. Following [18], for a compact set $K \subseteq G$ and $\varepsilon > 0$, we set

$$D_{\varepsilon, K}^{\delta} = \{d \in D : \|\delta_g(d) - d\| < \varepsilon \text{ for all } g \in K\}$$

and $\mathcal{U}(D_{\varepsilon, K}^{\delta}) = \mathcal{U}(D) \cap D_{\varepsilon, K}^{\delta}$. Finally, we let $\mathcal{U}_0(D_{\varepsilon, K}^{\delta})$ denote the connected component of 1_D in $\mathcal{U}(D_{\varepsilon, K}^{\delta})$.

We now recall a technical condition for unital G -algebras called *unitary regularity*; see [19, Definition 2.17]. For the trivial G -action on D , the property reduces to the fact that the commutator subgroup of $\mathcal{U}(D)$ is contained in $\mathcal{U}_0(D)$.

Definition 3.11. Let G be a locally compact group, and let (D, δ) be a unital G -algebra. We say that δ is *unitarily regular*, if for every compact set $K \subseteq G$ and $\varepsilon > 0$, there exists $\gamma > 0$ such that the commutator uvu^*v^* belongs to $\mathcal{U}_0(D_{\varepsilon, K}^{\delta})$ for every $u, v \in \mathcal{U}(D_{\gamma, K}^{\delta})$.

If G is compact, then the above definition reduces to the assertion that the commutator subgroup of $\mathcal{U}(D^{\delta})$ is contained in $\mathcal{U}_0(D^{\delta})$. Recall that a C^* -algebra D is said to be K_1 -injective if the canonical map $\mathcal{U}(D)/\mathcal{U}_0(D) \rightarrow K_1(D)$ is injective. By [19, Proposition 2.19], a unital G -algebra (D, δ) is unitarily regular whenever the fixed point algebra of the sequence algebra D_{∞} is K_1 -injective. Using this, it is shown in the said proposition that any action absorbing $\text{id}_{\mathbb{Z}}$ tensorially is unitarily regular.

In the equivariant setting, unitary regularity serves as a replacement of K_1 -injectivity for strongly self-absorbing G -actions. Even though K_1 -injectivity for strongly self-absorbing C^* -algebras is automatic by [23, Theorem 3.1], the analogous result for strongly self-absorbing G -algebras remains open (although it has been confirmed when G is amenable; see [11]).

For a subset F of a C^* -algebra and $n \in \mathbb{N}$, we let F^n denote the set of all n -fold products of elements in F .

Lemma 3.12. Let G be a second countable, locally compact group, let (\mathcal{D}, δ) be a strongly self-absorbing, unitarily regular G -algebra, let $F_{\mathcal{D}} \subseteq \mathcal{D}$ be a finite subset, let $K_G \subseteq G$ be a compact subset, and let $\varepsilon > 0$ be given. Then there exist a finite set $F'_{\mathcal{D}} \supseteq F_{\mathcal{D}}$ and $0 < \varepsilon' < \varepsilon$ with the following property.

Let (A, α) be a separable, unital G - $C([0, 1])$ -algebra with (\mathcal{D}, δ) -stable fibers, let $r < s < t$ in $[0, 1]$, let $F_A \subseteq A$ be a finite subset, and let $\psi_1, \psi_2: \mathcal{D} \rightarrow A$ be completely positive contractive maps that are $(F_{\mathcal{D}}, F_A, K_G, \varepsilon, F'_{\mathcal{D}}, \varepsilon')$ -regular on $[r, s]$ and $[s, t]$, respectively. Then there is a completely positive contractive map $\Psi: \mathcal{D} \rightarrow A$ which is $(F_{\mathcal{D}}, F_A, K_G, \varepsilon, F'_{\mathcal{D}}, \varepsilon')$ -regular for $[r, t]$.

Proof. Without loss of generality, we assume that $F_{\mathcal{D}}$ contains 1 and that K_G contains 1_G . Set $\widetilde{F_{\mathcal{D}}} = \bigcup_{g \in K_G} \delta_g(F_{\mathcal{D}} \cdot F_{\mathcal{D}})$, which is a compact subset of \mathcal{D} . Using that

(\mathcal{D}, δ) is unitarily regular and self-absorbing, we apply [19, Lemma 3.10] to find a continuous unitary path $u: [0, 1] \rightarrow \mathcal{U}(\mathcal{D} \otimes \mathcal{D})$ with $u_0 = 1_{\mathcal{D} \otimes \mathcal{D}}$ such that it satisfies $\max_{g \in K_G} \|(\delta \otimes \delta)_g(u_x) - u_x\| < \varepsilon/9$ for all $x \in [0, 1]$, and such that

$$\|u_1(d \otimes 1)u_1^* - 1 \otimes d\| < \frac{\varepsilon}{9} \quad (3.5)$$

for all $d \in \widetilde{F_{\mathcal{D}}}$. Using that $(u_x)_{x \in [0, 1]}$ is norm-compact, find $m \in \mathbb{N}$ and $v_{k, \ell}, w_{k, \ell} \in \mathcal{D}$ for $k, \ell = 1, \dots, m$ such that, with $y_k = \sum_{\ell=1}^m v_{k, \ell} \otimes w_{k, \ell}$, for any $x \in [0, 1]$ there is $k \in \{1, \dots, m\}$ with $\|u_x - y_k\| < \varepsilon/9$. Without loss of generality, we may assume that $\|y_k\| \leq 1$. Note that

$$\max_{g \in K_G} \|(\delta \otimes \delta)_g(y_k) - y_k\| < \frac{2\varepsilon}{9} \quad (3.6)$$

for all $k = 1, \dots, m$. Set $\varepsilon' = \varepsilon/144m^4$. With $V = \{v_{k, \ell}, w_{k, \ell} : k, \ell = 1, \dots, m\}$, set

$$F'_{\mathcal{D}} = \bigcup_{g \in K_G} \delta_g((F_{\mathcal{D}} \cup V \cup V^*)^6).$$

Then $F_{\mathcal{D}} \subseteq \widetilde{F_{\mathcal{D}}} \subseteq F'_{\mathcal{D}}$.

Let (A, α) be a separable, unital G - $C([0, 1])$ -algebra with (\mathcal{D}, δ) -stable fibers, let $r < s < t$ in $[0, 1]$, let $F_A \subseteq A$ be a finite subset, and let $\psi_1, \psi_2: \mathcal{D} \rightarrow A$ be completely positive contractive maps that are $(F_{\mathcal{D}}, F_A, K_G, \varepsilon, F'_{\mathcal{D}}, \varepsilon')$ -regular on $[r, s]$ and $[s, t]$, respectively. Without loss of generality, we assume that F_A contains the unit of A . For $j = 1, 2$, let $\varphi_j, \nu_j: \mathcal{D} \rightarrow A$ and $\mu_j: \mathcal{D} \otimes \mathcal{D} \rightarrow A$ be completely positive contractive maps as in the conclusion of Lemma 3.9. Using upper semicontinuity of the norm, and since φ_1 and φ_2 are $(F'_{\mathcal{D}}, F_A, K_G, \varepsilon')$ -regular for $\{s\}$, find $l > 0$ such that, with $I = [s - 3l, s + 3l]$, we have $I \subseteq [s, t]$, the maps φ_1, φ_2 are $(F'_{\mathcal{D}}, F_A, K_G, \varepsilon')$ -regular for I and conditions (3) through (6) in Lemma 3.9 hold not just for s , but for all $x \in I$. In particular, condition (5) in Lemma 3.9 gives

$$\max_{g \in K_G} \|\alpha_g^{(x)}(\mu_j(d \otimes d')_x) - \mu_j((\delta \otimes \delta)_g(d \otimes d'))_x\| < \varepsilon' \quad (3.7)$$

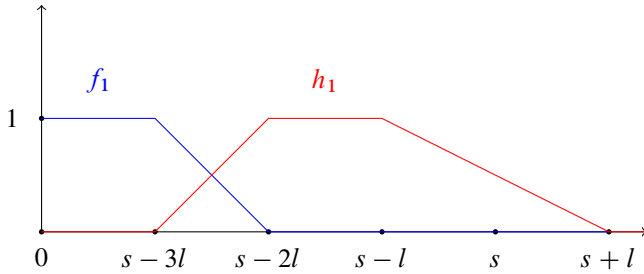
for all $x \in I$ and all $d, d' \in F'_{\mathcal{D}}$. For $j = 1, 2$, define a completely positive contractive map $\sigma_j: C([0, 1]) \otimes \mathcal{D} \otimes \mathcal{D} \rightarrow A$ by

$$\sigma_j(f \otimes d \otimes d') = f \cdot \mu_j(d \otimes d')$$

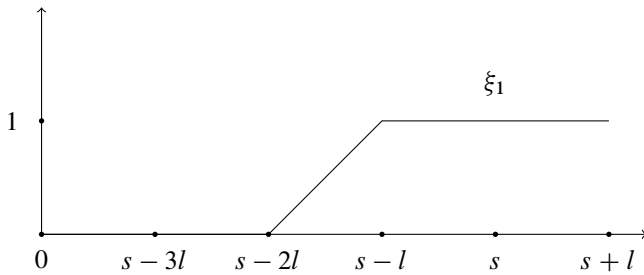
for all $f \in C([0, 1])$ and $d, d' \in \mathcal{D}$. It is then immediate to check that

$$\sigma_j(\eta)_x = \mu_j(\eta(x))_x \quad (3.8)$$

for all $\eta \in C([0, 1]) \otimes \mathcal{D} \otimes \mathcal{D}$ and all $x \in [0, 1]$. Let $f_1, h_1: \mathbb{R} \rightarrow [0, 1]$ be given by the following graphs:



Set $f_2(s+x) = f_1(s-x)$ and $h_2(s+x) = h_1(s-x)$ for all $x \in \mathbb{R}$. Note that $f_1 + h_1 + h_2 + f_2 = 1$ and that $f_1(f_2 + h_2) = 0 = f_2(f_1 + h_1)$. Let $\xi_1: \mathbb{R} \rightarrow [0, 1]$ be given by



and set $\xi_2(s+x) = \xi_1(s-x)$ for all $x \in \mathbb{R}$. We regard $f_1, f_2, h_1, h_2, \xi_1$, and ξ_2 as elements in $C([0, 1])$ by restricting them to $[0, 1]$. For $j = 1, 2$, set

$$v_j = u \circ \xi_j: [0, 1] \rightarrow \mathcal{U}(\mathcal{D} \otimes \mathcal{D}),$$

and let $\theta_j: \mathcal{D} \rightarrow A$ be the completely positive contractive map defined by

$$\theta_j(d) = (\sigma_j \circ \text{Ad}(v_j))(1_{C([0,1])} \otimes d \otimes 1)$$

for all $d \in \mathcal{D}$. Finally, define $\Psi: \mathcal{D} \rightarrow A$ by

$$\Psi(d) = f_1\varphi_1(d) + h_1\theta_1(d) + h_2\theta_2(d) + f_2\varphi_2(d)$$

for all $d \in \mathcal{D}$. We claim that Ψ is $(F_{\mathcal{D}}, F_A, K_G, \varepsilon, F'_{\mathcal{D}}, \varepsilon')$ -regular for $[r, t]$.

Observe that $\Psi(d)_{[0, s-3l]} = \varphi_1(d)_{[0, s-3l]}$ and $\Psi(d)_{[s+3l, 1]} = \varphi_2(d)_{[s+3l, 1]}$. Since φ_1 and φ_2 are $(F_{\mathcal{D}}, F_A, K_G, \varepsilon, F'_{\mathcal{D}}, \varepsilon')$ -regular for $[r, s]$ and $[s, t]$, respectively, it suffices to prove that Ψ is $(F_{\mathcal{D}}, F_A, K_G, \varepsilon)$ -regular for $[s-3l, s+3l]$.

The proof of Lemma 4.5 in [14] shows that Ψ satisfies conditions (U), (C), and (M) with respect to $(F_{\mathcal{D}}, F_A, \varepsilon)$ for $[s-3l, s+3l]$. Thus, it suffices to check the following condition for all $d \in F_{\mathcal{D}}$ and for all $x \in [s-3l, s+3l]$:

$$(E)_x \quad \max_{g \in K_G} \|\alpha_g^{(x)}(\Psi(d)_x) - \Psi(\delta_g(d))_x\| < \varepsilon.$$

We divide the proof into the following cases:

Case I. $x \in [s - 3l, s - 2l]$. Then $v_1(x) = 1$, and thus $\theta_1(d)_x = \varphi_1(d)_x$ for all $d \in \mathcal{D}$. It follows that $\Psi(d)_x = f_1(x)\varphi_1(d)_x + h_1(x)\varphi_1(d)_x = \varphi_1(d)_x$, as $f_1 + h_1 = 1$ on $[s - 3l, s - 2l]$. Since φ_1 satisfies condition (E)_x above, the result follows in this case.

Case II. $x \in [s - 2l, s - l]$. Then $f_1(x) = f_2(x) = h_2(x) = 0$ and $h_1(x) = 1$. Hence, $\Psi(d)_x = \theta_1(d)_x$ for all $d \in \mathcal{D}$. Find $k \in \{1, \dots, m\}$ with $\|v_1(x) - y_k\| < \varepsilon/9$, and note that $y_k(d \otimes 1)y_k^*$ belongs to $F'_\mathcal{D} \otimes F'_\mathcal{D}$ for all $d \in F_\mathcal{D}$ (recall that $F_\mathcal{D}$ contains $1_\mathcal{D}$). For $d \in \mathcal{D}$, it follows that

$$\begin{aligned} \Psi(d)_x &= \sigma_1(v_1(1 \otimes d \otimes 1)v_1^*)_x \\ &\stackrel{(3.8)}{=} \mu_1(v_1(x)(d \otimes 1)v_1(x)^*)_x \\ &\approx_{\frac{2\varepsilon}{9}} \mu_1(y_k(d \otimes 1)y_k^*)_x. \end{aligned} \quad (3.9)$$

For $g \in K_G$ and $d \in F_\mathcal{D}$, we get

$$\begin{aligned} \alpha_g^{(x)}(\Psi(d)_x) &\stackrel{(3.9)}{\approx_{\frac{2\varepsilon}{9}}} \alpha_g^{(x)}(\mu_1(y_k(d \otimes 1)y_k^*)_x) \\ &\stackrel{(3.7)}{\approx_{\varepsilon'}} \mu_1((\delta \otimes \delta)_g(y_k(d \otimes 1)y_k^*)_x) \\ &\stackrel{(3.6)}{\approx_{\frac{\varepsilon}{3}}} \mu_1(y_k(\delta_g(d) \otimes 1)y_k^*)_x \stackrel{(3.9)}{\approx_{\frac{2\varepsilon}{9}}} \Psi(\delta_g(d))_x. \end{aligned}$$

It follows that $\|\alpha_g^{(x)}(\Psi(d)_x) - \Psi(\delta_g(d))_x\| < \varepsilon$, thus establishing (E)_x, as desired.

Case III. $x \in [s - l, s + l]$. Note that $f_1(x) = f_2(x) = 0$ and $v_1(x) = v_2(x) = u_1$. Using this at the first and third steps, respectively, for $d \in \widetilde{F_\mathcal{D}}$, it follows that

$$\begin{aligned} \Psi(d)_x &= \sum_{j=1}^2 h_j(x)\sigma_j(v_j(1 \otimes d \otimes 1)v_j^*)_x \\ &\stackrel{(3.8)}{=} \sum_{j=1}^2 h_j(x)\mu_j(v_j(x)(d \otimes 1)v_j(x)^*)_x \\ &= \sum_{j=1}^2 h_j(x)\mu_j(u_1(d \otimes 1)u_1^*)_x \\ &\stackrel{(3.5)}{\approx_{\frac{2\varepsilon}{9}}} \sum_{j=1}^2 h_j(x)\mu_j(1 \otimes d)_x. \end{aligned} \quad (3.10)$$

For $g \in K_G$ and $d \in F_\mathcal{D}$, and using at the last step that $\delta_g(d)$ is in $\widetilde{F_\mathcal{D}}$, we get

$$\begin{aligned} \alpha_g^{(x)}(\Psi(d)_x) &\stackrel{(3.10)}{\approx_{\frac{2\varepsilon}{9}}} \sum_{j=1}^2 h_j(x)\alpha_g^{(x)}(\mu_j(1 \otimes d)_x) \\ &\stackrel{(3.7)}{\approx_{\varepsilon'}} \sum_{j=1}^2 h_j(x)\mu_j(1 \otimes \delta_g(d))_x \stackrel{(3.10)}{\approx_{\frac{\varepsilon}{9}}} \Psi(\delta_g(d))_x. \end{aligned}$$

This proves (E)_x, as desired.

Case IV. $x \in [s + l, s + 2l]$. This is proved identically to Case II, by exchanging the subscripts 1 and 2 everywhere.

Case V. $x \in [s + 2l, s + 3l]$. This is proved identically to Case I, since $\Psi(d)_x = \varphi_2(d)_x$ for all $d \in \mathcal{D}$. ■

We are now ready to prove the main result of this section, which is an equivariant version of [14, Theorem 4.6]. As mentioned in the introduction, particular cases of it were proved in [12, Theorem 4.28] and [17, Corollary 3.7].

Theorem 3.13. *Let G be a second countable, locally compact group, let (\mathcal{D}, δ) be a unitarily regular strongly self-absorbing G -algebra, let X be a compact metrizable space, and let (A, α) be a separable, unital G - $C(X)$ -algebra. Assume that the covering dimension of X is finite. Then (A, α) is (\mathcal{D}, δ) -stable if and only if $(A_x, \alpha^{(x)})$ is (\mathcal{D}, δ) -stable for all $x \in X$.*

Proof. The “only if” implication follows from [19, Corollary 2.8] (regardless of the covering dimension of X).

We prove the “if” implication. Using that X is compact and finite-dimensional, together with comments preceding Lemma 3.7, we may assume that there is $m \in \mathbb{N}$ with $X = [0, 1]^m$. We prove the result by induction on m .

Since the case $m = 0$ is trivial, let us assume that the result holds for $m - 1$ and prove it for m . Let $[0, 1]^m \rightarrow [0, 1]$ be the canonical projection onto the first coordinate, and use this map to regard (A, α) as a G - $C([0, 1])$ -algebra in such a way that the fiber over $x \in [0, 1]$ is $A_{\{x\} \times [0, 1]^{m-1}}$. By the inductive assumption, these fibers are (\mathcal{D}, δ) -stable. In other words, in order to establish the inductive step, it suffices to prove that a G - $C([0, 1])$ -algebra is (\mathcal{D}, δ) -stable whenever its fibers are. Thus, we assume from now on that $X = [0, 1]$.

Fix finite subsets $F_{\mathcal{D}} \subseteq \mathcal{D}$ and $F_A \subseteq A$, a compact subset $K_G \subseteq G$, and tolerance $\varepsilon > 0$. We will produce a completely positive contractive map $\psi: \mathcal{D} \rightarrow A$ which is $(F_{\mathcal{D}}, F_A, K_G, \varepsilon)$ -regular for all of $[0, 1]$. Once we do this, the result will follow from Theorem 2.7 (see Remark 3.5).

Let $F'_{\mathcal{D}} \supseteq F_{\mathcal{D}}$ and $\varepsilon' < \varepsilon$ be as in the conclusion of Lemma 3.12 for $(F_{\mathcal{D}}, K_G, \varepsilon)$. Apply Lemma 3.7 to the tuple $(F'_{\mathcal{D}}, F_A, K_G, \varepsilon')$ to find $n \in \mathbb{N}$, points $0 = t_0 < t_1 < \dots < t_n = 1$, and completely positive contractive maps $\psi_j: \mathcal{D} \rightarrow A$ for $j = 1, \dots, n$ such that ψ_j is $(F'_{\mathcal{D}}, F_A, K_G, \varepsilon')$ -regular for $[t_{j-1}, t_j]$.

Apply Lemma 3.12 to ψ_1 and ψ_2 to find a completely positive contractive map $\Psi_2: \mathcal{D} \rightarrow A$ which is $(F_{\mathcal{D}}, F_A, K_G, \varepsilon, F'_{\mathcal{D}}, \varepsilon')$ -regular for $[0, t_2]$. Applying Lemma 3.12 to Ψ_2 and ψ_3 , find a completely positive contractive map $\Psi_3: \mathcal{D} \rightarrow A$ which is $(F_{\mathcal{D}}, F_A, K_G, \varepsilon, F'_{\mathcal{D}}, \varepsilon')$ -regular for $[t_0, t_3]$. Repeating this procedure, we arrive at a completely positive map $\Psi_n: \mathcal{D} \rightarrow A$ which is $(F_{\mathcal{D}}, F_A, K_G, \varepsilon, F'_{\mathcal{D}}, \varepsilon')$ -regular for $[0, 1]$. Thus, Ψ_n satisfies conditions (U), (C), (M), and (E) from Notation 2.6 with respect to $(F_{\mathcal{D}}, F_A, K_G, \varepsilon)$, as desired. This concludes the proof. ■

The argument used in [14, Proposition 4.11] can be adapted to our setting to show that for general compact metrizable (not necessarily finite-dimensional) X , a G - $C(X)$ -algebra (A, α) absorbs a unitarily regular strongly self-absorbing G -algebra (\mathcal{D}, δ) if and only if it is *locally* (\mathcal{D}, δ) -absorbing (meaning that for every $x \in X$ there is a compact neighborhood Y_x of it such that the quotient G -algebra (A_{Y_x}, α^{Y_x}) is (\mathcal{D}, δ) -stable). The non-trivial implication follows from the fact that, under these assumptions, one can show along the lines of [14, Proposition 4.9] that (A, α) is an iterated equivariant pullback of (\mathcal{D}, δ) -stable G -algebras, and the said pullbacks are (\mathcal{D}, δ) -stable by [19, Corollary 2.8 and Theorem 5.9] (for which unitary regularity is necessary). This applies, in particular, whenever (A, α) is equivariantly locally trivial, in the natural sense.

We close this work with a concrete application of Theorem 3.13. The result we present below has been proved in some particular cases (see, e.g., [12, Theorem 4.35]), and we give here a general version of it. Note that the case $d = 0$ is precisely the equivalence between (ii) and (iii) in [18, Theorem 3.7].

Theorem 3.14. *Let G be a second countable, locally compact group, let A be a separable C^* -algebra, let $\alpha: G \rightarrow \text{Aut}(A)$ be an action, and let $\delta: G \rightarrow \text{Aut}(\mathcal{D})$ be a unitarily regular strongly self-absorbing action. Then (A, α) is (\mathcal{D}, δ) -stable if and only if for some $d \geq 0$ there exist equivariant completely positive contractive order zero maps*

$$\rho_0, \dots, \rho_d: (\mathcal{D}, \delta) \rightarrow (F_\infty(A), F_\infty(\alpha))$$

with commuting ranges and satisfying $\rho_0(1) + \dots + \rho_d(1) = 1$.

Proof. As explained above, if (A, α) is (\mathcal{D}, δ) -stable, then we may take $d = 0$ in the statement; see [18, Theorem 3.7].

We prove the converse, so let $d \geq 0$ and ρ_0, \dots, ρ_d be as in the theorem. Denote by C the minimal unitization of the cone over \mathcal{D} , and let G act on it via the natural extension of δ , using the trivial action in the cone direction. Set $E = \bigotimes_{j=0}^d C$, endowed with the diagonal G -action. Denote by $t \in C_0((0, 1])$ the canonical inclusion of $(0, 1]$ into \mathbb{C} and regard t as an element in C . For $j = 0, \dots, d$, let $t_j \in E$ denote the simple tensor that has t in its j th position and 1 otherwise, and note that t_j is G -invariant. Let I denote the ideal in E generated by $1 - (t_0 + \dots + t_d)$, and note that I is G -invariant. Denote by $B = E/I$ the associated quotient and by $\beta: G \rightarrow \text{Aut}(B)$ the associated induced action.

Using [9, Corollary 2.10] (see also [25]), we let $\varphi_0, \dots, \varphi_d: C \rightarrow F_\infty(A)$ the equivariant homomorphisms that satisfy $\varphi_j(a \otimes t) = \rho_j(a)$ for all $a \in \mathcal{D}$ and all $j = 0, \dots, d$. Since the ranges of $\varphi_0, \dots, \varphi_d$ commute, the universal property of the tensor product gives us a unital, equivariant homomorphism $\varphi: E \rightarrow F_\infty(A)$. Finally, since

$$\varphi(t_0 + \dots + t_d) = \rho_0(1) + \dots + \rho_d(1) = 1,$$

it follows that $I \subseteq \ker(\varphi)$ and hence there is an induced unital, equivariant homomorphism $\rho: (B, \beta) \rightarrow (F_\infty(A), F_\infty(\alpha))$.

Denoting by X the spectrum of the center of $B = E/I$, we deduce that X is a closed subspace of $[0, 1]^{d+1}$, so in particular it is metrizable and finite-dimensional. Note that B is a $C(X)$ -algebra. Since all fibers of B are equivariantly isomorphic to (\mathcal{D}, δ) , it follows from Theorem 3.13 that (B, β) absorbs (\mathcal{D}, δ) tensorially. In particular, there is a unital, equivariant homomorphism $(\mathcal{D}, \delta) \rightarrow (B_\infty, \beta_\infty)$, so we obtain a unital, equivariant homomorphism

$$(\mathcal{D}, \delta) \rightarrow (F_\infty(F_\infty(A)), F_\infty(F_\infty(\alpha))).$$

Using that \mathcal{D} is separable and G is second countable, an easy reindexation argument as in [8, Lemma 4.18], or an application of countable saturation for the equivariant ultrapower $(F_\infty(A), F_\infty(\alpha))$, gives a unital, equivariant homomorphism $(\mathcal{D}, \delta) \rightarrow (F_\infty(A), F_\infty(\alpha))$. Now the result follows from the equivalence between (ii) and (iii) in [18, Theorem 3.7]. ■

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