

Absolute values of BMOA functions

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Abstract. The paper contains a complete characterization of the moduli of BMOA functions. These are described explicitly by a certain Muckenhoupt-type condition involving Poisson integrals. As a consequence, it is shown that an outer function with BMO modulus need not belong to BMOA. Some related results are obtained for the Bloch space.

1. Introduction.

Let \mathbb{D} denote the disk $\{z \in \mathbb{C} : |z| < 1\}$, \mathbb{T} its boundary, and m the normalized arclength measure on \mathbb{T} . Further, let μ_z be the harmonic measure associated with a point $z \in \mathbb{D}$, so that

$$d\mu_z(\zeta) \stackrel{\text{def}}{=} \frac{1 - |z|^2}{|\zeta - z|^2} dm(\zeta), \quad \zeta \in \mathbb{T}.$$

The space BMO consists, by definition, of all functions $f \in L^1(\mathbb{T}, m)$ satisfying

$$\|f\|_* \stackrel{\text{def}}{=} \sup_{z \in \mathbb{D}} \int |f(\zeta) - f(z)| d\mu_z(\zeta) < \infty,$$

where $f(z)$ stands for $\int f d\mu_z$. Alternative characterizations of BMO, as well as a systematic treatment of the subject, can be found in [G, Chapter VI] or [K, Chapter X]. Meanwhile, let us only recall that the

Garsia norm

$$\|f\|_G \stackrel{\text{def}}{=} \sup_{z \in \mathbb{D}} \left(\int |f|^2 d\mu_z - |f(z)|^2 \right)^{1/2},$$

defined originally for $f \in L^2(\mathbb{T}, m)$, is in fact an equivalent norm on BMO.

We shall also be concerned with the analytic subspace

$$\text{BMOA} \stackrel{\text{def}}{=} \text{BMO} \cap H^1$$

(as usual, we denote by H^p , $0 < p \leq \infty$, the classical Hardy spaces of the disk). It is well known that

$$H^\infty \subset \text{BMOA} \subset \bigcap_{0 < p < \infty} H^p.$$

Now one of the basic facts about H^p spaces (see *e.g.* [G, Chapter II]) is this: In order that a function $\varphi \geq 0$, living almost everywhere on \mathbb{T} , coincide with the modulus of some nonzero H^p function, it is necessary and sufficient that $\varphi \in L^p(\mathbb{T}, m)$ and

$$(1.1) \quad \int \log \varphi dm > -\infty.$$

On the other hand, the very natural (and perhaps no less important) problem of characterizing the moduli of functions in BMOA seems to have been unsolved (or unposed?) until now, and the present paper is intended to fill that gap.

Thus, we look at a measurable function $\varphi \geq 0$ on \mathbb{T} and ask whether

$$(1.2) \quad \varphi = |f|, \quad \text{for some } f \in \text{BMOA}, f \neq 0.$$

The two immediate necessary conditions are (1.1) and

$$(1.3) \quad \varphi \in \text{BMO}.$$

(To see that (1.2) implies (1.3), use the following simple fact: If for any $z \in \mathbb{D}$ there is a number $c(z)$ such that

$$(1.4) \quad \sup_{z \in \mathbb{D}} \int |\varphi(\zeta) - c(z)| d\mu_z(\zeta) < \infty,$$

then $\varphi \in \text{BMO}$. Now, given that (1.2) holds, (1.4) is obviously fulfilled with $c(z) = |f(z)|$.) However, we shall see that (1.1) and (1.3) together are not yet sufficient for (1.2) to hold.

Assuming that (1.1) holds true, we consider the *outer function* \mathcal{O}_φ given by

$$\mathcal{O}_\varphi(z) \stackrel{\text{def}}{=} \exp \left(\int \frac{\zeta + z}{\zeta - z} \log \varphi(\zeta) dm(\zeta) \right), \quad z \in \mathbb{D},$$

and note that (1.2) is equivalent to saying that

$$(1.5) \quad \mathcal{O}_\varphi \in \text{BMOA}.$$

Indeed, since $|\mathcal{O}_\varphi| = \varphi$ almost everywhere on \mathbb{T} , the implication (1.5) implies (1.2) is obvious. The converse is also true, because the outer factor of a BMOA function must itself belong to BMOA (in fact, if $f = FI$ with $F \in H^2$ and I an inner function, then it is easy to see that $\|f\|_G \geq \|F\|_G$). The problem has thus been reduced to ascertaining when (1.5) holds.

In this paper we point out a new crucial condition (reminiscent, to some extent, of the Muckenhoupt (A_p) condition, cf. [G, Chapter VI]) which characterizes, together with (1.1) and (1.3), the non-negative functions φ with $\mathcal{O}_\varphi \in \text{BMOA}$; this is contained in Section 2 below. Further, in Section 3, we exhibit an example of a BMO function $\varphi \geq 0$ with $\log \varphi \in L^1(\mathbb{T}, m)$ for which our Muckenhoupt-type condition fails. In other words, we show that the obvious necessary conditions (1.1) and (1.3) alone do not ensure the inclusion $\mathcal{O}_\varphi \in \text{BMOA}$. Finally, in Section 4 we find out when an outer function with BMO modulus lies in the Bloch space \mathcal{B} .

2. Outer functions in BMOA.

Given a function $\varphi \in L^1(\mathbb{T}, m)$, $\varphi \geq 0$, we recall the notation

$$\varphi(z) \stackrel{\text{def}}{=} \int \varphi d\mu_z, \quad z \in \mathbb{D},$$

and introduce, for a fixed $M > 0$, the level set

$$\Omega(\varphi, M) \stackrel{\text{def}}{=} \{z \in \mathbb{D} : \varphi(z) \geq M\}.$$

In order to avoid confusion, let us point out the notational distinction between

$$\varphi(z)^p \stackrel{\text{def}}{=} (\varphi(z))^p = \left(\int \varphi d\mu_z \right)^p$$

and

$$\varphi^p(z) \stackrel{\text{def}}{=} (\varphi^p)(z) = \int \varphi^p d\mu_z$$

(here $p > 0$ and $z \in \mathbb{D}$). Finally, we need the function

$$\log^- t \stackrel{\text{def}}{=} \begin{cases} \log \frac{1}{t}, & 0 < t < 1, \\ 0, & t \geq 1. \end{cases}$$

Our main result is

Theorem 1. *Suppose that $\varphi \in \text{BMO}$, $\varphi \geq 0$, and*

$$\int \log \varphi dm > -\infty.$$

The following are equivalent.

- i) $\mathcal{O}_\varphi \in \text{BMOA}$.
- ii) *For some $M > 0$, one has*

$$\sup \left\{ \varphi(z)^2 \int \log^- \varphi d\mu_z : z \in \Omega(\varphi, M) \right\} < \infty.$$

REMARK. The latter is vaguely reminiscent of the well-known Muckenhoupt (A_p) condition [G, Chapter VI] which can be written in the form

$$\sup \left\{ \varphi(z)^\tau \int \varphi^{-\tau} d\mu_z : z \in \mathbb{D} \right\} < \infty,$$

where $\tau = 1/(p-1)$ and $1 < p < \infty$.

The proof of Theorem 1 makes use of the following elementary fact.

Lemma 1. *The function*

$$R(u) \stackrel{\text{def}}{=} \log \frac{1}{u} + u - 1, \quad u > 0,$$

is nonnegative and satisfies

$$R(u) \leq 2(u - 1)^2, \quad \text{for } u \geq \frac{1}{2}.$$

Indeed, since $R(u)$ is the remainder term in the first order Taylor formula for $\log 1/u$, when expanded about the point $u = 1$, one has

$$R(u) = \frac{1}{2\xi^2}(u - 1)^2,$$

where $\xi = \xi(u)$ is a suitable point between u and 1.

We also cite, as Lemma 2, the “harmonic measure version” of the classical John-Nirenberg theorem (see Section 2 and Exercise 18 in [G, Chapter VI]).

Lemma 2. *There are absolute constants $C > 0$ and $c > 0$ such that*

$$\mu_z\{\zeta \in \mathbb{T} : |f(\zeta) - f(z)| > \lambda\} \leq C \exp\left(-\frac{c\lambda}{\|f\|_*}\right),$$

whenever $z \in \mathbb{D}$, $f \in \text{BMO}$ and $\lambda > 0$ (here again $f(z) \stackrel{\text{def}}{=} \int f d\mu_z$).

PROOF OF THEOREM 1. Since $\varphi \in \text{BMO}$, we know that

$$(2.1) \quad \|\varphi\|_G^2 = \sup_{z \in \mathbb{D}} (\varphi^2(z) - \varphi(z)^2) < \infty.$$

Similarly, condition i) of Theorem 1 is equivalent to

$$\|\mathcal{O}_\varphi\|_G^2 = \sup_{z \in \mathbb{D}} (\varphi^2(z) - |\mathcal{O}_\varphi(z)|^2) < \infty$$

and hence, in view of (2.1), to

$$(2.2) \quad \sup_{z \in \mathbb{D}} (\varphi(z)^2 - |\mathcal{O}_\varphi(z)|^2) < \infty.$$

In order to ascertain when (2.2) holds, we note that

$$|\mathcal{O}_\varphi(z)| = \exp\left(\int \log \varphi d\mu_z\right) = \varphi(z) e^{-J(z)},$$

where

$$J(z) \stackrel{\text{def}}{=} \log \varphi(z) - \int \log \varphi d\mu_z ,$$

and rewrite (2.2) in the form

$$(2.3) \quad \sup_{z \in \mathbb{D}} \varphi(z)^2 (1 - e^{-2J(z)}) < \infty .$$

We remark that $J(z) \geq 0$ by Jensen's inequality. Further, we claim that (2.3) is equivalent to the following condition

$$(2.4) \quad \sup \{ \varphi(z)^2 J(z) : z \in \Omega(\varphi, M) \} < \infty , \quad \text{for some } M > 0 .$$

Indeed, to deduce (2.3) from (2.4), one uses the inequality $1 - e^{-x} \leq x$ and the obvious fact that

$$\sup \{ \varphi(z)^2 (1 - e^{-2J(z)}) : z \in \mathbb{D} \setminus \Omega(\varphi, M) \} \leq M^2 .$$

Conversely, to show that (2.3) implies (2.4), let K be the value of the supremum in (2.3) and put $M \stackrel{\text{def}}{=} \sqrt{2K}$. It then follows from (2.3) that

$$\sup \{ J(z) : z \in \Omega(\varphi, M) \} < \infty ,$$

and so $1 - e^{-2J(z)}$ is comparable to $J(z)$ as long as $z \in \Omega(\varphi, M)$.

We have thus reduced condition i) to (2.4), and we now proceed by looking at (2.4) more closely. To this end, we fix a point $z \in \Omega(\varphi, 2)$ and introduce the sets

$$E_1 = E_1(z) \stackrel{\text{def}}{=} \left\{ \zeta \in \mathbb{T} : \varphi(\zeta) \geq \frac{1}{2} \varphi(z) \right\}$$

and

$$E_2 = E_2(z) \stackrel{\text{def}}{=} \mathbb{T} \setminus E_1 .$$

Using the function $R(u)$ from Lemma 1, we write

$$\begin{aligned} J(z) &= \int \log \frac{\varphi(z)}{\varphi(\zeta)} d\mu_z(\zeta) \\ &= \int \left(\log \frac{\varphi(z)}{\varphi(\zeta)} + \frac{\varphi(\zeta) - \varphi(z)}{\varphi(z)} \right) d\mu_z(\zeta) \\ (2.5) \quad &= \int R\left(\frac{\varphi(\zeta)}{\varphi(z)}\right) d\mu_z(\zeta) \\ &= I_1(z) + I_2(z) , \end{aligned}$$

where

$$I_j(z) \stackrel{\text{def}}{=} \int_{E_j} R\left(\frac{\varphi(\zeta)}{\varphi(z)}\right) d\mu_z(\zeta), \quad j = 1, 2.$$

Now if $\zeta \in E_1$ then $\varphi(\zeta)/\varphi(z) \geq 1/2$, and Lemma 1 tells us that

$$R\left(\frac{\varphi(\zeta)}{\varphi(z)}\right) \leq 2 \left(\frac{\varphi(\zeta) - \varphi(z)}{\varphi(z)}\right)^2.$$

Integrating, we get

$$I_1(z) \leq \frac{2}{\varphi(z)^2} \int (\varphi(\zeta) - \varphi(z))^2 d\mu_z(\zeta) \leq \frac{2}{\varphi(z)^2} \|\varphi\|_G^2,$$

so that

$$(2.6) \quad I_1(z) = O\left(\frac{1}{\varphi(z)^2}\right).$$

In order to estimate $I_2(z)$, we observe that

$$\begin{aligned} \mu_z(E_2) &= \mu_z\left\{\zeta : \varphi(\zeta) < \frac{1}{2}\varphi(z)\right\} \\ &= \mu_z\left\{\zeta : \varphi(z) - \varphi(\zeta) > \frac{1}{2}\varphi(z)\right\} \\ (2.7) \quad &\leq \mu_z\left\{\zeta : |\varphi(z) - \varphi(\zeta)| > \frac{1}{2}\varphi(z)\right\} \\ &\leq C \exp\left(-\frac{c\varphi(z)}{2\|\varphi\|_*}\right), \end{aligned}$$

as follows from Lemma 2. Besides, for $\zeta \in E_2$ one obviously has

$$(2.8) \quad \left|\frac{\varphi(\zeta) - \varphi(z)}{\varphi(z)}\right| = 1 - \frac{\varphi(\zeta)}{\varphi(z)} \leq 1$$

and

$$(2.9) \quad \log \frac{\varphi(z)}{\varphi(\zeta)} \geq \log 2 > 0.$$

Further, we set

$$\begin{aligned} E_2^+ &\stackrel{\text{def}}{=} \{\zeta \in E_2 : \varphi(\zeta) \geq 1\}, \\ E_2^- &\stackrel{\text{def}}{=} \{\zeta \in E_2 : \varphi(\zeta) < 1\}, \end{aligned}$$

and

$$S(z) \stackrel{\text{def}}{=} \int_{E_2} \frac{\varphi(\zeta) - \varphi(z)}{\varphi(z)} d\mu_z(\zeta) + \int_{E_2^+} \log \frac{\varphi(z)}{\varphi(\zeta)} d\mu_z(\zeta) \\ + \mu_z(E_2^-) \log \varphi(z).$$

We have then

$$(2.10) \quad I_2(z) = S(z) + \int_{E_2^-} \log \frac{1}{\varphi(\zeta)} d\mu_z(\zeta).$$

Using (2.8) and (2.9), we see that

$$\left| \int_{E_2} \frac{\varphi(\zeta) - \varphi(z)}{\varphi(z)} d\mu_z(\zeta) \right| \leq \mu_z(E_2)$$

and

$$0 \leq \int_{E_2^+} \log \frac{\varphi(z)}{\varphi(\zeta)} d\mu_z(\zeta) \leq \mu_z(E_2^+) \log \varphi(z).$$

Consequently,

$$(2.11) \quad |S(z)| \leq \mu_z(E_2) + (\mu_z(E_2^+) + \mu_z(E_2^-)) \log \varphi(z) \\ = \mu_z(E_2) (1 + \log \varphi(z)) \\ \leq C \exp\left(-\frac{c\varphi(z)}{2\|\varphi\|_*}\right) (1 + \log \varphi(z)),$$

where the last inequality relies on (2.7). The function

$$t \mapsto t^2 \exp(-at) (1 + \log t), \quad t \geq 2,$$

being bounded for any fixed $a > 0$, we conclude from (2.11) that

$$S(z) = O\left(\frac{1}{\varphi(z)^2}\right).$$

Together with (2.10), this means that

$$(2.12) \quad I_2(z) = O\left(\frac{1}{\varphi(z)^2}\right) + \int_{E_2^-} \log \frac{1}{\varphi(\zeta)} d\mu_z(\zeta).$$

A juxtaposition of (2.5), (2.6) and (2.12) now yields

$$(2.13) \quad J(z) = O\left(\frac{1}{\varphi(z)^2}\right) + \int_{E_2^-} \log \frac{1}{\varphi(\zeta)} d\mu_z(\zeta).$$

Finally, recalling the assumption $z \in \Omega(\varphi, 2)$, we note that

$$E_2^- = \{\zeta \in \mathbb{T} : \varphi(\zeta) < 1\}$$

(indeed, if $\zeta \in \mathbb{T}$ and $\varphi(\zeta) < 1$, then $\varphi(\zeta) < \varphi(z)/2$, so that $\zeta \in E_2$). Thus, (2.13) can be rewritten as

$$J(z) = O\left(\frac{1}{\varphi(z)^2}\right) + \int \log^- \varphi d\mu_z,$$

and this relation has been actually verified for $z \in \Omega(\varphi, 2)$.

It now follows that condition (2.4) (in which one can safely replace the words “for some $M > 0$ ” by “for some $M > 2$ ”) holds if and only if

$$\sup \left\{ \varphi(z)^2 \int \log^- \varphi d\mu_z : z \in \Omega(\varphi, M) \right\} < \infty,$$

for some $M > 0$; we have thus arrived at ii). On the other hand, we have seen that (2.4) is a restatement of i). The desired equivalence relation is therefore established.

We proceed by pointing out a few corollaries of Theorem 1.

Corollary 1. *Let $\varphi \in \text{BMO}$, $\varphi \geq 0$, and $\int \log \varphi dm > -\infty$. If $\mathcal{O}_\varphi \in \text{BMOA}$ and $0 < p < 1$, then $\mathcal{O}_{\varphi^p} (= \mathcal{O}_\varphi^p) \in \text{BMOA}$.*

PROOF. Since $\varphi \in \text{BMO}$, we have also $\varphi^p \in \text{BMO}$ (this is easily deduced from the inequality $|a^p - b^p| \leq |a - b|^p$, valid for $a, b \geq 0$ and $0 < p < 1$). By Theorem 1, the inclusion $\mathcal{O}_\varphi \in \text{BMOA}$ yields

$$(2.14) \quad \sup \left\{ \varphi(z)^2 \int \log^- \varphi d\mu_z : z \in \Omega(\varphi, M) \right\} < \infty,$$

for some $M > 0$, and hence also for some $M \geq 1$. Hölder’s inequality gives

$$\varphi^p(z) \leq \varphi(z)^p, \quad z \in \mathbb{D},$$

whence

$$\varphi^p(z) \leq \varphi(z), \quad z \in \Omega(\varphi, 1),$$

and

$$\Omega(\varphi^p, M^p) \subset \Omega(\varphi, M).$$

Therefore, (2.14) with $M \geq 1$ implies the condition

$$\sup \left\{ (\varphi^p(z))^2 \int \log^- \varphi^p d\mu_z : z \in \Omega(\varphi^p, M^p) \right\} < \infty,$$

which in turn means, by Theorem 1, that $\mathcal{O}_{\varphi^p} \in \text{BMOA}$.

Corollary 2. *Let $\varphi \in \text{BMO}$, $\varphi \geq 0$, and $\int \log \varphi dm > -\infty$. Assume, in addition, that φ possesses (after a possible correction on a set of zero measure) the following property: For some $\varepsilon > 0$, the set $\{\zeta \in \mathbb{T} : \varphi(\zeta) \leq \varepsilon\}$ is closed and consists of continuity points for φ . Then $\mathcal{O}_\varphi \in \text{BMOA}$.*

PROOF. We may put $\varepsilon = 1$ (otherwise, consider the function $\varphi_1 \stackrel{\text{def}}{=} \varphi/\varepsilon$). Thus, we are assuming that the set

$$K \stackrel{\text{def}}{=} \{\zeta \in \mathbb{T} : \varphi(\zeta) \leq 1\}$$

is closed, while φ is continuous at every point of K . We now claim that

$$(2.15) \quad K \cap \text{clos } \Omega(\varphi, 2) = \emptyset.$$

Indeed, if $\zeta_0 \in K \cap \text{clos } \Omega(\varphi, 2)$, then one could find a sequence $\{z_n\} \subset \mathbb{D}$ such that $\varphi(z_n) \geq 2$ and $z_n \rightarrow \zeta_0$. On the other hand, since φ is continuous at ζ_0 , we would have $\lim_{n \rightarrow \infty} \varphi(z_n) = \varphi(\zeta_0) \leq 1$, a contradiction.

From (2.15) it follows that

$$\delta \stackrel{\text{def}}{=} \text{dist}(K, \Omega(\varphi, 2)) > 0.$$

Hence, for $z \in \Omega(\varphi, 2)$, one has

$$(2.16) \quad \begin{aligned} \int \log^- \varphi d\mu_z &= \int_K \frac{1 - |z|^2}{|\zeta - z|^2} \log \frac{1}{\varphi(\zeta)} dm(\zeta) \\ &\leq \frac{1 - |z|^2}{\delta^2} \|\log \varphi\|_{L^1(\mathbb{T}, m)}. \end{aligned}$$

An easy estimate for the Poisson integral of a BMO function gives

$$(2.17) \quad \varphi(z) = O\left(\log \frac{2}{1-|z|}\right), \quad z \in \mathbb{D}.$$

Combining (2.16) and (2.17) yields

$$(2.18) \quad \begin{aligned} &\varphi(z)^2 \int \log^- \varphi \, d\mu_z \\ &\leq \text{const} \left(\log \frac{2}{1-|z|}\right)^2 \frac{1-|z|^2}{\delta^2} \|\log \varphi\|_{L^1(\mathbb{T}, m)}, \end{aligned}$$

for all $z \in \Omega(\varphi, 2)$. Since

$$(1-|z|^2) \left(\log \frac{2}{1-|z|}\right)^2 = O(1), \quad z \in \mathbb{D},$$

the right-hand side of (2.18) is bounded by a constant independent of z . Thus,

$$\sup \left\{ \varphi(z)^2 \int \log^- \varphi \, d\mu_z : z \in \Omega(\varphi, 2) \right\} < \infty,$$

and the desired conclusion follows by Theorem 1.

Corollary 3. *If $\varphi \in \text{BMO}$ and $\text{ess inf}_{\zeta \in \mathbb{T}} \varphi(\zeta) > 0$, then $\mathcal{O}_\varphi \in \text{BMOA}$.*

PROOF. For a suitable $\varepsilon > 0$ one has $\{\zeta \in \mathbb{T} : \varphi(\zeta) \leq \varepsilon\} = \emptyset$, so it only remains to apply Corollary 2.

3. An outer function with BMO modulus that does not belong to BMOA.

Although Theorem 1 provides a complete characterization of the moduli of BMOA functions, one may still ask whether the obvious necessary conditions (1.1) and (1.3) are also sufficient for \mathcal{O}_φ to be in BMOA (equivalently, whether condition ii) of Theorem 1 follows automatically from (1.1) and (1.3)). An affirmative answer might perhaps seem plausible in light of corollaries 2 and 3 above. However, we are now going to construct an example that settles the question in the negative. In other words, we prove

Theorem 2. *There is a nonnegative function $\varphi \in \text{BMO}$ with*

$$\int \log \varphi \, dm > -\infty$$

such that $\mathcal{O}_\varphi \notin \text{BMOA}$.

Actually, we find it more convenient to deal with the space $\text{BMO}(\mathbb{R})$ of the real line, defined as the set of functions $f \in L^1(\mathbb{R}, dt/(1+t^2))$ with

$$\|f\|_* \stackrel{\text{def}}{=} \sup_{z \in \mathbb{C}_+} \int_{\mathbb{R}} |f(t) - f(z)| \, d\mu_z(t) < \infty.$$

Here \mathbb{C}_+ denotes the upper half-plane $\{\text{Im } z > 0\}$, the harmonic measure μ_z is now given by

$$d\mu_z(t) = \frac{1}{\pi} \frac{\text{Im } z}{|t - z|^2} dt, \quad z \in \mathbb{C}_+, \quad t \in \mathbb{R},$$

and $f(z)$ stands for $\int_{\mathbb{R}} f \, d\mu_z$. The subspace $\text{BMOA}(\mathbb{C}_+)$ consists, by definition, of those $f \in \text{BMO}(\mathbb{R})$ for which $f(z)$ is holomorphic on \mathbb{C}_+ . Using the conformal invariance of BMO (see [G, Chapter VI]), one can restate Theorem 2 as follows.

Theorem 2'. *There is a nonnegative function $\varphi \in \text{BMO}(\mathbb{R})$ with*

$$\int_{\mathbb{R}} \frac{\log \varphi(t)}{1+t^2} dt > -\infty$$

such that the outer function

$$\mathcal{O}_\varphi(z) \stackrel{\text{def}}{=} \exp \left(\frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{z-t} + \frac{t}{t^2+1} \right) \log \varphi(t) dt \right), \quad z \in \mathbb{C}_+,$$

fails to belong to $\text{BMOA}(\mathbb{C}_+)$.

The proof will rely on the following auxiliary result.

Lemma 3. *Let E and I be two (finite and nondegenerate) subintervals of \mathbb{R} having the same center and satisfying*

$$\frac{|E|}{|I|} \stackrel{\text{def}}{=} \sigma < 1$$

(here $|\cdot|$ denotes length). Then there exists a function $\psi \in \text{BMO}(\mathbb{R})$ such that

$$(3.1) \quad 0 \leq \psi \leq 1, \quad \text{almost everywhere on } \mathbb{R},$$

$$(3.2) \quad \psi|_E = 1, \quad \psi|_{\mathbb{R} \setminus I} = 0,$$

and

$$(3.3) \quad \|\psi\|_* \leq C \left(\log \frac{1}{\sigma} \right)^{-1},$$

where $C > 0$ is some absolute constant.

PROOF OF LEMMA 3. By means of a linear mapping, the general case is reduced to the special one where $E = [-\sigma, \sigma]$ and $I = [-1, 1]$. This done, we define the function ψ by (3.2) and by

$$\psi(t) = \frac{\log |t|}{\log \sigma}, \quad \sigma < |t| \leq 1.$$

Now (3.1) is obvious, while (3.3) follows from the well-known facts that $\log |t| \in \text{BMO}(\mathbb{R})$ and that $\text{BMO}(\mathbb{R})$ is preserved by truncations (see Section 1 and Exercise 1 in [G, Chapter VI]).

REMARK. A more general (and much more difficult) version of Lemma 3, where E is an arbitrary measurable set contained in the middle third of I , is due to Garnett and Jones [GJ]; see also Exercise 19 in [G, Chapter VI]. We have, nonetheless, found it worthwhile to include a short proof of the version required.

PROOF OF THEOREM 2'. For $k = 1, 2, \dots$, set $\sigma_k \stackrel{\text{def}}{=} \exp(-k^2)$ and let the numbers

$$0 = a_1 < b_1 < a_2 < b_2 < \dots$$

be such that

$$b_k - a_k = \sigma_k \quad \text{and} \quad a_{k+1} - b_k = k^{-5/4} \sigma_k.$$

Consider the intervals $I_k \stackrel{\text{def}}{=} [a_k, b_k]$ and $J_k \stackrel{\text{def}}{=} [b_k, a_{k+1}]$. Further, let

$$(3.4) \quad x_k \stackrel{\text{def}}{=} \frac{a_k + b_k}{2}, \quad y_k \stackrel{\text{def}}{=} \sigma_k^2$$

and

$$E_k \stackrel{\text{def}}{=} \left[x_k - \frac{1}{2} y_k, x_k + \frac{1}{2} y_k \right].$$

Since $|E_k| = \sigma_k^2 = \sigma_k |I_k|$, Lemma 3 provides, for every $k \in \mathbb{N}$, a function $\psi_k \in \text{BMO}(\mathbb{R})$ such that

$$\begin{aligned} 0 \leq \psi_k \leq 1, & \quad \text{on } \mathbb{R}, \\ \psi_k|_{E_k} = 1, & \quad \psi_k|_{\mathbb{R} \setminus I_k} = 0 \end{aligned}$$

and

$$\|\psi_k\|_* \leq C \left(\log \frac{1}{\sigma_k} \right)^{-1}.$$

Finally, we set

$$\alpha_k \stackrel{\text{def}}{=} k^{3/4}, \quad \beta_k \stackrel{\text{def}}{=} \exp \left(- \frac{1}{\sigma_k} \right)$$

and define the sought-after function φ by

$$\varphi \stackrel{\text{def}}{=} \chi_{\mathbb{R} \setminus \cup_k J_k} + \sum_k (\alpha_k \psi_k + \beta_k \chi_{J_k})$$

(here, as usual, χ_A stands for the characteristic function of the set A). In order to show that φ enjoys the required properties, we have to verify several claims.

Claim 1. $\varphi \in \text{BMO}(\mathbb{R})$.

This follows at once from the inclusions

$$\varphi - \sum_k \alpha_k \psi_k \in L^\infty(\mathbb{R})$$

and

$$\sum_k \alpha_k \psi_k \in \text{BMO}(\mathbb{R}),$$

where the latter holds true because

$$\sum_k \alpha_k \|\psi_k\|_* \leq C \sum_k \alpha_k \left(\log \frac{1}{\sigma_k} \right)^{-1} = C \sum_k k^{-5/4} < \infty.$$

Claim 2. $\log \varphi \in L^1(\mathbb{R}, dt/(1 + t^2))$.

Indeed, since $\varphi(t) < 1$ if and only if $t \in \bigcup_k J_k$, we have

$$\int \log^- \varphi dt = \sum_k \int_{J_k} \log \frac{1}{\varphi} dt = \sum_k |J_k| \log \frac{1}{\beta_k} = \sum_k k^{-5/4} < \infty .$$

Thus $\log^- \varphi \in L^1(\mathbb{R}, dt)$. Observing, in addition, that $\log \varphi = 0$ outside the finite interval

$$S \stackrel{\text{def}}{=} \bigcup_k I_k \cup \bigcup_k J_k$$

and noting that Claim 1 implies $\varphi \in L^1(S, dt)$, whence also

$$\log^+ \varphi (= |\log \varphi| - \log^- \varphi) \in L^1(S, dt) ,$$

we eventually conclude that

$$\log \varphi \in L^1(\mathbb{R}, dt) .$$

A stronger version of Claim 2 is thus established.

Claim 3. *For every $M > 0$, one has*

$$(3.5) \quad \sup \left\{ \varphi(z)^2 \int \log^- \varphi d\mu_z : z \in \mathbb{C}_+, \varphi(z) \geq M \right\} = \infty .$$

To verify (3.5), we set $z_k \stackrel{\text{def}}{=} x_k + i y_k$ (here x_k and y_k are defined by (3.4)) and show that both

$$(3.6) \quad \lim_{k \rightarrow \infty} \varphi(z_k) = \infty$$

and

$$(3.7) \quad \lim_{k \rightarrow \infty} \varphi(z_k)^2 \int \log^- \varphi d\mu_{z_k} = \infty .$$

To this end, we first note that $\mu_{z_k}(E_k) = \text{const}$, and so

$$(3.8) \quad \varphi(z_k) = \int \varphi d\mu_{z_k} \geq \int_{E_k} \varphi d\mu_{z_k} = (\alpha_k + 1) \mu_{z_k}(E_k) \geq \text{const } \alpha_k ,$$

which proves (3.6). Further, we write

$$(3.9) \quad \int \log^- \varphi d\mu_{z_k} \geq \int_{J_k} \log^- \varphi d\mu_{z_k} = \mu_{z_k}(J_k) \log \frac{1}{\beta_k}.$$

Together with the simple fact that

$$\mu_{z_k}(J_k) \geq \text{const } |J_k|,$$

the inequality (3.9) gives

$$(3.10) \quad \int \log^- \varphi d\mu_{z_k} \geq \text{const } |J_k| \log \frac{1}{\beta_k}.$$

Finally, combining (3.8) and (3.10), we obtain

$$\varphi(z_k)^2 \int \log^- \varphi d\mu_{z_k} \geq \text{const } \alpha_k^2 |J_k| \log \frac{1}{\beta_k} = \text{const } k^{1/4}.$$

This proves (3.7), and hence also Claim 3. In view of Theorem 1 (which admits an obvious restatement for $\text{BMO}(\mathbb{R})$), Claim 3 is equivalent to saying that

$$\mathcal{O}_\varphi \notin \text{BMOA}(\mathbb{C}_+),$$

so the proof is complete.

4. Outer functions with BMO moduli lying in the Bloch space.

Recall that the Bloch space \mathcal{B} is defined to be the set of analytic functions f on \mathbb{D} with

$$\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} \sup_{z \in \mathbb{D}} (1 - |z|) |f'(z)| < \infty$$

(see [ACP] for a detailed discussion of this class). We now supplement Theorem 1 from Section 2 with the following result.

Theorem 3. *Let*

$$(4.1) \quad \varphi \in \text{BMO}, \varphi \geq 0, \text{ and } \int \log \varphi dm > -\infty.$$

Suppose that, for some $M > 0$,

$$(4.2) \quad \sup \left\{ \varphi(z) \int \log^- \varphi \, d\mu_z : z \in \Omega(\varphi, M) \right\} < \infty .$$

Then $\mathcal{O}_\varphi \in \mathcal{B}$.

The proof hinges on

Lemma 4. *If φ satisfies (4.1), then*

$$(4.3) \quad (1 - |z|) |\mathcal{O}'_\varphi(z)| \leq \text{const} + 2 \varphi(z) \int \log^- \varphi \, d\mu_z ,$$

whenever $z \in \Omega(\varphi, 2)$; the constant on the right depends only on φ .

PROOF OF LEMMA 4. Differentiating the equality

$$\mathcal{O}_\varphi(z) = \exp \left(\int \frac{\zeta + z}{\zeta - z} \log \varphi(\zeta) \, dm(\zeta) \right), \quad z \in \mathbb{D},$$

gives

$$(4.4) \quad \begin{aligned} \mathcal{O}'_\varphi(z) &= \mathcal{O}_\varphi(z) \int \frac{2\zeta}{(\zeta - z)^2} \log \varphi(\zeta) \, dm(\zeta) \\ &= \mathcal{O}_\varphi(z) \int \frac{2\zeta}{(\zeta - z)^2} \log \frac{\varphi(\zeta)}{\varphi(z)} \, dm(\zeta), \end{aligned}$$

where we have also used the fact that

$$\int \frac{2\zeta}{(\zeta - z)^2} \, dm(\zeta) = 0 .$$

From (4.4) one gets

$$(4.5) \quad (1 - |z|) |\mathcal{O}'_\varphi(z)| \leq 2 |\mathcal{O}_\varphi(z)| \int \left| \log \frac{\varphi(\zeta)}{\varphi(z)} \right| d\mu_z(\zeta),$$

and we proceed by looking at the integral on the right. Following the strategy employed in the proof of Theorem 1, we set

$$E_1 = E_1(z) \stackrel{\text{def}}{=} \left\{ \zeta \in \mathbb{T} : \varphi(\zeta) \geq \frac{1}{2} \varphi(z) \right\}$$

and

$$E_2 = E_2(z) \stackrel{\text{def}}{=} \mathbb{T} \setminus E_1 .$$

Using the elementary inequality

$$|\log u| \leq 2|u - 1|, \quad u \geq \frac{1}{2},$$

we obtain

$$\begin{aligned} \int_{E_1} \left| \log \frac{\varphi(\zeta)}{\varphi(z)} \right| d\mu_z(\zeta) &\leq 2 \int_{E_1} \left| \frac{\varphi(\zeta)}{\varphi(z)} - 1 \right| d\mu_z(\zeta) \\ (4.6) \qquad \qquad \qquad &\leq \frac{2}{\varphi(z)} \int |\varphi(\zeta) - \varphi(z)| d\mu_z(\zeta) \\ &\leq \frac{2}{\varphi(z)} \|\varphi\|_* . \end{aligned}$$

Repeating again some steps from the proof of Theorem 1, we introduce the sets

$$\begin{aligned} E_2^+ &\stackrel{\text{def}}{=} \{\zeta \in E_2 : \varphi(\zeta) \geq 1\}, \\ E_2^- &\stackrel{\text{def}}{=} \{\zeta \in E_2 : \varphi(\zeta) < 1\}, \end{aligned}$$

and note that, since $z \in \Omega(\varphi, 2)$ (which is assumed from now on), we actually have

$$(4.7) \qquad E_2^- = \{\zeta \in \mathbb{T} : \varphi(\zeta) < 1\} .$$

This done, we write

$$\begin{aligned} \int_{E_2} \left| \log \frac{\varphi(\zeta)}{\varphi(z)} \right| d\mu_z(\zeta) &= \int_{E_2} \log \frac{\varphi(z)}{\varphi(\zeta)} d\mu_z(\zeta) \\ (4.8) \qquad \qquad \qquad &= \mu_z(E_2) \log \varphi(z) + \int_{E_2^+} \log \frac{1}{\varphi(\zeta)} d\mu_z(\zeta) \\ &\quad + \int_{E_2^-} \log \frac{1}{\varphi(\zeta)} d\mu_z(\zeta) . \end{aligned}$$

The estimate (2.7) from Section 2 says

$$(4.9) \qquad \mu_z(E_2) \leq C \exp \left(- \frac{c \varphi(z)}{2 \|\varphi\|_*} \right) ,$$

where $C > 0$ and $c > 0$ are certain absolute constants. Besides, we obviously have

$$(4.10) \quad \int_{E_2^+} \log \frac{1}{\varphi(\zeta)} d\mu_z(\zeta) \leq 0$$

and

$$(4.11) \quad \int_{E_2^-} \log \frac{1}{\varphi(\zeta)} d\mu_z(\zeta) = \int \log^- \varphi(\zeta) d\mu_z(\zeta)$$

(the latter relies on (4.7)). Using (4.9), (4.10) and (4.11) to estimate the right-hand side of (4.8), we get

$$(4.12) \quad \int_{E_2} \left| \log \frac{\varphi(\zeta)}{\varphi(z)} \right| d\mu_z(\zeta) \leq C \exp \left(-\frac{c \varphi(z)}{2 \|\varphi\|_*} \right) \log \varphi(z) + \int \log^- \varphi d\mu_z .$$

Since

$$\sup_{t \geq 2} t e^{-at} \log t < \infty ,$$

for any $a > 0$, (4.12) implies

$$(4.13) \quad \int_{E_2} \left| \log \frac{\varphi(\zeta)}{\varphi(z)} \right| d\mu_z(\zeta) \leq \frac{\text{const}}{\varphi(z)} + \int \log^- \varphi d\mu_z .$$

Combining (4.6) and (4.13) yields

$$(4.14) \quad \int_{\mathbb{T}} \left| \log \frac{\varphi(\zeta)}{\varphi(z)} \right| d\mu_z(\zeta) \leq \frac{\text{const}}{\varphi(z)} + \int \log^- \varphi d\mu_z .$$

Finally, substituting (4.14) into the right-hand side of (4.5) and noting that $|\mathcal{O}_\varphi(z)| \leq \varphi(z)$ (say, by Jensen's inequality), one eventually arrives at (4.3).

PROOF OF THEOREM 3. Let $M \geq 2$ be a number for which (4.2) holds. Further, set $\psi \stackrel{\text{def}}{=} \sqrt{\varphi}$. Then $\psi \in \text{BMO}$, $\log \psi \in L^1(\mathbb{T}, m)$, and

$$\psi(z)^2 \leq \varphi(z) , \quad z \in \mathbb{D} .$$

In particular,

$$\Omega(\psi, \sqrt{M}) \subset \Omega(\varphi, M)$$

(similar observations were made in the proof of Corollary 1 in Section 2). Condition (4.2) therefore yields

$$\sup \left\{ \psi(z)^2 \int \log^- \psi \, d\mu_z : z \in \Omega(\psi, \sqrt{M}) \right\} < \infty.$$

By Theorem 1, it follows that $\mathcal{O}_\psi \in \text{BMOA}$. Since $\text{BMOA} \subset \mathcal{B}$, we also know that $\mathcal{O}_\psi \in \mathcal{B}$. In order to derive the required estimate

$$(4.15) \quad |\mathcal{O}'_\varphi(z)| \leq \text{const} (1 - |z|)^{-1},$$

we distinguish two cases.

Case 1. $z \in \mathbb{D} \setminus \Omega(\varphi, M)$.

We have then

$$|\mathcal{O}_\psi(z)| \leq \psi(z) \leq \varphi(z)^{1/2} \leq \sqrt{M},$$

and so

$$|\mathcal{O}'_\varphi(z)| = |(\mathcal{O}_\psi^2)'(z)| = 2 |\mathcal{O}_\psi(z)| |\mathcal{O}'_\psi(z)| \leq 2 \sqrt{M} \|\mathcal{O}_\psi\|_{\mathcal{B}} (1 - |z|)^{-1}.$$

Case 2. $z \in \Omega(\varphi, M)$.

Since $\Omega(\varphi, M) \subset \Omega(\varphi, 2)$, a juxtaposition of (4.3) and (4.2) immediately yields

$$(1 - |z|) |\mathcal{O}'_\varphi(z)| \leq \text{const} < \infty.$$

Thus, (4.15) is established for all $z \in \mathbb{D}$, and the proof is complete.

Before proceeding with our final result, we point out two elementary facts.

Lemma 5. *Let φ satisfy (4.1). For any $M > 0$, the following are equivalent.*

$$(a) \quad \sup \left\{ \varphi(z)^2 \int \log^- \varphi \, d\mu_z : z \in \Omega(\varphi, M) \right\} < \infty.$$

$$(b) \quad \sup \left\{ \varphi^2(z) \int \log^- \varphi \, d\mu_z : z \in \Omega(\varphi, M) \right\} < \infty.$$

PROOF. Since $\varphi(z)^2 \leq \varphi^2(z)$, the implication (b) implies (a) is obvious. Conversely, let C be the value of the supremum in (a). For $z \in \Omega(\varphi, M)$, condition (a) implies

$$\int \log^- \varphi \, d\mu_z \leq \frac{C}{M^2}$$

and hence

$$(\varphi^2(z) - \varphi(z)^2) \int \log^- \varphi \, d\mu_z \leq C M^{-2} \|\varphi\|_G^2,$$

which leads to (b).

Lemma 6. *Let $\psi \in \text{BMO}$, $\psi \geq 0$. Suppose the numbers $M > 0$ and $M_1 > 0$ are related by*

$$(4.16) \quad M_1 = M^2 + \|\psi\|_G^2.$$

Then $\Omega(\psi^2, M_1) \subset \Omega(\psi, M)$.

PROOF. If $\psi^2(z) \geq M_1$, then

$$\psi(z)^2 = \psi^2(z) - (\psi^2(z) - \psi(z)^2) \geq M_1 - \|\psi\|_G^2 = M^2,$$

so that $\psi(z) \geq M$.

Now we are in a position to prove

Theorem 4. *If $f \in \text{BMOA}$ is an outer function with $|f|^2 \in \text{BMO}$, then $f^2 \in \mathcal{B}$.*

PROOF. Set $\psi \stackrel{\text{def}}{=} |f|$ and $\varphi \stackrel{\text{def}}{=} \psi^2$, so that $f = \mathcal{O}_\psi$ and $f^2 = \mathcal{O}_\varphi$. Since $\mathcal{O}_\psi \in \text{BMOA}$, Theorem 1 yields

$$(4.17) \quad \sup \left\{ \psi(z)^2 \int \log^- \psi \, d\mu_z : z \in \Omega(\psi, M) \right\} < \infty$$

with some $M > 0$. By Lemma 5, we can replace $\psi(z)^2$ by $\psi^2(z)$ ($= \varphi(z)$); by Lemma 6, the arising condition will remain valid if we replace $\Omega(\psi, M)$ by the smaller set $\Omega(\varphi, M_1)$, where M_1 is defined by (4.16). Consequently, (4.17) implies

$$\sup \left\{ \varphi(z) \int \log^- \varphi \, d\mu_z : z \in \Omega(\varphi, M_1) \right\} < \infty.$$

Since $\varphi \in \text{BMO}$, the desired conclusion that $\mathcal{O}_\varphi \in \mathcal{B}$ now follows by Theorem 3.

REMARKS. 1) Of course, there are outer functions $f \in \text{BMOA}$ with $f^2 \notin \mathcal{B}$. For example, this happens for $f(z) = \log(1 - z)$, where \log is the branch determined by $\log 1 = 2\pi i$.

2) Let $\varphi \geq 0$ on \mathbb{T} . Recalling Muckenhoupt's (A_p) condition (see Section 2 above), we have the implications

$$\varphi \in \text{BMO} \cap (A_{3/2}) \text{ implies } \mathcal{O}_\varphi \in \text{BMOA}$$

and

$$\varphi \in \text{BMO} \cap (A_2) \text{ implies } \mathcal{O}_\varphi \in \mathcal{B}.$$

To see why, use Theorems 1 and 3 together with the inequality $\tau \log^- \varphi \leq \varphi^{-\tau}$ ($\tau > 0$). It would be interesting to determine the full range of p 's for which $\varphi \in \text{BMO} \cap (A_p)$ implies $\mathcal{O}_\varphi \in \text{BMOA}$ or $\mathcal{O}_\varphi \in \mathcal{B}$.

3) There used to be a question whether there existed a function lying in all H^p classes with $0 < p < \infty$ and in \mathcal{B} , but not in BMOA . Various constructions (based on different ideas) of such functions were given in [CCS], [HT] and [D2]. Our current results show how to construct an *outer* function with these properties. Namely, it suffices to find a function φ satisfying (4.1) and (4.2), with some $M > 0$, but such that

$$\sup \left\{ \varphi(z)^2 \int \log^- \varphi d\mu_z : z \in \Omega(\varphi, M) \right\} = \infty,$$

for all $M > 0$. (An explicit example can be furnished in the spirit of Section 3 above.) This done, one has $\mathcal{O}_\varphi \in \bigcap_{0 < p < \infty} H^p$ (because $\varphi \in \bigcap_{0 < p < \infty} L^p$) and $\mathcal{O}_\varphi \in \mathcal{B} \setminus \text{BMOA}$, as readily seen from Theorems 1 and 3.

4) While this paper deals with outer functions only, in [D1] and [D2] we have studied the interaction between the outer and inner factors of BMOA functions. Besides, we have characterized in [D3], [D4], [D5] the moduli of analytic functions in some other popular classes, such as Lipschitz and Besov spaces. In this connection, see also [Sh, Chapter II]. Finally, we mention the recent paper [D6], which is close in spirit to the current one.

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