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Deformation and K-theoretic index formulae on boundary groupoids

Yu Qiao and Bing Kwan So

Abstract. Motivated by investigating K-theoretic index formulae for boundary groupoids of the form

$$\mathcal{H} = M_0 \times M_0 \sqcup G \times M_1 \times M_1 \rightrightarrows M = M_0 \sqcup M_1,$$

where G is an exponential Lie group, we introduce the notion of a *deformation from the pair groupoid*, which makes sense for general Lie groupoids. Once there exists a deformation from the pair groupoid for a (general) Lie groupoid $\mathcal{G} \rightrightarrows M$, we are able to construct explicitly a deformation index map relating the analytic index on $\mathcal{G}$ and the index on the pair groupoid $M \times M$, which in turn enables us to establish index formulae for (fully) elliptic (pseudo)differential operators on $\mathcal{H}$ by applying the numerical index formula of M. J. Pflaum, H. Posthuma, and X. Tang. In particular, we find that the index is given by the Atiyah–Singer integral but does not involve any η-term in the higher codimensional cases. These results recover and generalize our previous results for renormalizable boundary groupoids via renormalized traces.

1. Introduction

The Atiyah–Singer index theorem, one of the greatest mathematics achievements in the twentieth century, states that the analytic index of an elliptic differential operator is equal to its topological counterpart.

There have been many results generalizing the Atiyah–Singer index theorem to other pseudodifferential calculi, constructed for different purposes [2, 7, 20, 23, 30, 36]. These pseudodifferential calculi can be realized as groupoid pseudodifferential calculi on certain groupoid $\mathcal{G} \rightrightarrows M$ with M compact. Then the subalgebras of operators of order zero and $-\infty$ can be completed to C^*-algebras $\mathfrak{U}(\mathcal{G})$ and $C^*(\mathcal{G})$, respectively, yielding the short exact sequence

$$0 \to C^*(\mathcal{G}) \to \mathfrak{U}(\mathcal{G}) \to \mathfrak{U}/C^*(\mathcal{G}) \to 0. \tag{1.1}$$

After passing to the six-term exact sequence, one defines the analytic index of any elliptic operator to be its image in $K_0(C^*(\mathcal{G}))$ under the boundary map. For continuous family groupoids, the analytic index is similarly defined by Lauter, Monthubert, and Nistor in [25].

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In the 1990s, Connes gave a “Lie groupoid proof” of the Atiyah–Singer index theorem in his famous book [17, Section 2]. As the starting point, given a closed manifold M, Connes constructed the so-called *tangent groupoid*, which is a kind of *deformation* of the pair groupoid $M \times M$. This construction induces a map (which we shall denote by $\mathrm{ind}_{\mathcal{T}(M\times M)}$ below) from the K_0-group of the tangent bundle to

$$K_0(C^*(M \times M)) \cong K_0(\mathcal{K}),$$

where $\mathcal{K}$ denotes the C^*-algebra of compact operators. Applying a mapping cone argument as described in, for example, [32, Theorem 1.2] and [12, Section 4.1], one proves that the index map corresponding to (1.1) is equal to $\mathrm{ind}_{\mathcal{T}(M\times M)}$ composed with the principal symbol. Next, Connes constructed a topological index map, by embedding M into $\mathbb{R}^N$ for some large enough N, and by considering Thom isomorphism and Morita equivalence (between groupoids). Then he showed that these two index maps coincide.

Connes’ proof of the Atiyah–Singer index theorem triggered a large number of subsequent works in index theory through Lie groupoids. For instance, based on the analysis and index problems on manifolds with corners, many groupoids were constructed [9, 11, 20, 27, 31, 35]. These constructions and index theorems depend heavily on the existence of boundary defining functions and embedding the manifold under question into a cube instead of $\mathbb{R}^N$. Meanwhile, Androulidakis and Skandalis associate the holonomy groupoid to a singular foliation and investigated corresponding properties [3], which opens the door to explore singular foliations via Lie groupoids. Alternatively, one can study the simplified index problem on Lie groupoids by pairing with cohomology [10, 16, 37–39].

To study the Fredholm index of fully elliptic operators on manifolds with boundary, Carrillo Rouse, Lescure, and Monthubert replace the tangent bundle in the adiabatic groupoid construction by some “non-commutative tangent bundle” [10]. In that case, it is (the C^*-algebra of) a subgroupoid of the adiabatic groupoid of the b-stretched product groupoid associated to a manifold M with boundary M_1; explicitly,

$$(M_1 \times M_1 \times \mathbb{R}) \times (0,1) \sqcup TM \times \{0\} \rightrightarrows M \times [0,1]$$

(it is a continuous family groupoid). Similar arguments are utilized by Debord, Lescure, and Nistor for the case of conical pseudomanifolds [19].

In this paper, we consider index formulae for pseudodifferential operators on Lie groupoids $\mathcal{H} \rightrightarrows M$ with two orbits, which are isomorphic as abstract groupoids (i.e., sets with groupoid operations) to

$$M_0 \times M_0 \sqcup G \times M_1 \times M_1 \rightrightarrows M_0 \sqcup M_1 = M, \tag{1.2}$$

where $M_0 = M \setminus M_1$ is an open dense subset, and where $G \times M_1 \times M_1 \rightrightarrows M_1$ is the product of the pair groupoid $M_1 \times M_1$ and the Lie group G as a groupoid over a single point. We shall further assume that the isotropy group G is an exponential Lie group, of dimension equal to the codimension of the manifold M_1 in M.

Definition 1.1. In this paper, we shall call these Lie groupoids *boundary groupoids with two leaves and exponential isotropy*. Throughout the paper, we shall use $\mathcal{H} \rightrightarrows M$ to denote such a groupoid.

Note that a groupoid of the form (1.2) may carry different Lie groupoid structures, and we shall regard them as different objects (see Proposition 3.3 and Example 3.5 below). These Lie groupoids are the simplest example of boundary groupoids [42], and are often holonomy groupoids integrating some singular foliations [1,2,4,5,18]. In this case, it is not clear whether an embedding analogous to the manifold with corners exists. Fortunately, the K-theory of these Lie groupoid C^*-algebras is computed in [12]. Namely, for *all* boundary groupoids with two leaves and exponential isotropy $\mathcal{H} \rightrightarrows M$, we have

$$\begin{aligned} &K_0(C^*(\mathcal{H})) \cong \mathbb{Z} \text{ and } K_1(C^*(\mathcal{H})) \cong \mathbb{Z}, && \text{if } M_1 \text{ is of odd codimension} \geq 3; \\ &K_0(C^*(\mathcal{H})) \cong \mathbb{Z} \oplus \mathbb{Z} \text{ and } K_1(C^*(\mathcal{H})) \cong \{0\}, && \text{if } M_1 \text{ is of even codimension,} \end{aligned}$$

regardless of Lie structures (see (2.1) below). Hence in order to derive an index formula for elliptic operators (or just their principal symbols), it suffices to produce one integer in the odd codimension case and two integers in the even codimension case, which would completely describe its index in $K_0(C^*(\mathcal{H}))$. Moreover, the pushforward induced by the *extension map*

$$K_0(C^*(M_0 \times M_0)) \xrightarrow{\varepsilon_{\mathcal{H},M_0}} K_0(C^*(\mathcal{H}))$$

is an isomorphism in the odd case and is injective in the even case. This implies that the Fredholm index of a fully elliptic operator on $\mathcal{H}$ is also determined by its $K_0(C^*(\mathcal{H}))$ index.

In the special case when a renormalized trace can be defined, one can derive an index formula [40] using renormalization techniques similar to that of [33]. It was found that in the odd case with codimension ≥ 3, the η-term of the renormalized index formula vanishes; hence both the Fredholm index and the $K_0(C^*(\mathcal{H}))$ index are just given by the Atiyah–Singer integral. Moreover, one could expect a deeper description of the relationship between the isomorphism $K_\bullet(C^*(M \times M)) \cong K_\bullet(C^*(\mathcal{H}))$ and the vanishing of the η-term, which we shall prove in this paper.

In this paper, we take a different approach. Denote by $\mathcal{G} \rightrightarrows M$ a *general* Lie groupoid with unit space M. We shall introduce the notion of a *deformation from the pair groupoid* (see Definition 3.1), where we deform the pair groupoid $M \times M$ to our desired groupoid $\mathcal{G} \rightrightarrows M$. Here, we would like to point out two major differences between our definition and that for the tangent groupoid and the adiabatic groupoid, even for the construction of the deformation to the normal cone [21–23]:

(1) For the tangent groupoid, the fiber at $t = 0$ is the tangent bundle of M; whereas, for a deformation from the pair groupoid, the fiber at $t = 0$ is the groupoid $\mathcal{G}$, usually *not* a vector bundle.

(2) Given a closed manifold M, there is always the associated tangent groupoid; but a deformation from the pair groupoid for $\mathcal{G} \rightrightarrows M$ may not exist. (Thus, we exploit a little obstruction theory for the existence of a deformation from the pair groupoid.)

Hence, unlike the tangent groupoid and adiabatic groupoid, our idea seems to go backwards. That is, given a Lie groupoid $\mathcal{G} \rightrightarrows M$, we ask if $\mathcal{G} \rightrightarrows M$ can be obtained by deforming the pair groupoid $M \times M$; in other words, we are looking for a bigger Lie groupoid $\mathcal{D}$ which realizes such a deformation.

For a Lie groupoid $\mathcal{G} \rightrightarrows M$, once a deformation $\mathcal{D}$ from the pair groupoid $M \times M$ exists, we shall construct a deformation index map

$$\mathrm{ind}_{\mathcal{D}} : K_0(C^*(\mathcal{G})) \to K_0(C^*(M \times M \rightrightarrows M)) \cong \mathbb{Z},$$

which is the key ingredient to establish the following theorem.

Theorem 1.2 (Theorem 3.8). *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and A the associated Lie algebroid of $\mathcal{G}$. Suppose that a deformation $\mathcal{D}$ from the pair groupoid $M \times M$ exists. Then one has the commutative diagram*

$$\begin{array}{ccc} K_0(C^*(A)) & \xrightarrow{\mathrm{ind}_{\mathcal{T}(\mathcal{G})}} & K_0(C^*(\mathcal{G})) \\ \downarrow \cong & & \downarrow \mathrm{ind}_{\mathcal{D}} \\ K_0(C^*(TM)) & \xrightarrow{\mathrm{ind}_{\mathcal{T}(M\times M)}} & K_0(C^*(M \times M)), \end{array}$$

where the top map is just the analytic index map constructed via the adiabatic groupoid and the bottom map is the Atiyah–Singer index.

The above theorem can be used to simplify index problems on Lie groupoids to those on the pair groupoid of the unit space. In particular, for a boundary groupoid with two orbits and exponential isotropy, we have the following theorem, which identifies index maps on such groupoids.

Theorem 1.3 (Theorem 3.9). *Suppose that $\mathcal{H} \rightrightarrows M$ is any boundary groupoid with two orbits and exponential isotropy, and that a deformation from the pair groupoid exists for $\mathcal{H}$. Then the composition*

$$K_0(C^*(M_0 \times M_0)) \cong \mathbb{Z} \xrightarrow{\varepsilon_{\mathcal{H},M_0}} K_0(C^*(\mathcal{H})) \xrightarrow{\mathrm{ind}_{\mathcal{D}}} K_0(C^*(M \times M)) \cong \mathbb{Z}$$

is an isomorphism.

As applications, we apply the pairing considered in [38] to obtain our index formulae on boundary groupoids with two leaves and exponential isotropy.

Theorem 1.4 (Theorems 4.3 and 4.6). *Suppose that $\mathcal{H} \rightrightarrows M$ is any boundary groupoid with two orbits and exponential isotropy, and that a deformation from the pair groupoid exists for $\mathcal{H}$. Let Ψ be an elliptic pseudodifferential operator on $\mathcal{H}$. One has the following index formulae:*

(1) *If* $\dim G \geq 3$ *is odd, then*

$$\operatorname{ind}(\Psi) = \operatorname{ind}_{\mathcal{T}(M\times M)}(\partial[\sigma(\Psi)]) = \int_{T^*M} \langle \hat{A}(T^*M) \wedge \operatorname{ch}(\sigma[\Psi]), \Omega_{\pi^!TM}\rangle;$$

(2) *If* $\dim G$ *is even, then*

$$\operatorname{ind}(\Psi) = \int_{T^*M} \langle \hat{A}(T^*M) \wedge \operatorname{ch}(\sigma[\Psi]), \Omega_{\pi^!TM}\rangle \oplus \int_{T^*M_1\times\mathfrak{g}} \langle \hat{A}(T^*M_1 \oplus \mathfrak{g}) \wedge \operatorname{ch}(\sigma[\Psi]|_{M_1}), \Omega'_{\pi^!TM_1}\rangle,$$

where $\mathfrak{g}$ *is the Lie algebra of* G.

Again, it is interesting to observe that the above index formulae do not depend on the Lie structure of the boundary groupoid in question.

1.1. Structure of the paper

Section 2 is devoted to reviewing basic definitions and facts related to Lie groupoids, such as Lie algebroids, boundary groupoids, submanifold groupoids, and the tangent groupoid. In Section 3, given a Lie groupoid $\mathcal{G} \rightrightarrows M$, we define the notion of *a deformation from the pair groupoid* to $\mathcal{G}$, show that such a deformation exists for submanifold groupoids, and briefly discuss obstructions to the existence of such deformation groupoids. Then, under the assumption that a deformation $\mathcal{D}$ from the pair groupoid to $\mathcal{G}$ exists, we construct explicitly an index map

$$\operatorname{ind}_{\mathcal{D}} : K_0(C^*(\mathcal{G})) \to K_0(C^*(M \times M)) \cong \mathbb{Z},$$

which is similar to that of the tangent (or adiabatic) groupoid, and show that it is compatible with the analytic index map, and hence implies Theorem 1.2. Finally, in Section 4, after reviewing the Atiyah–Singer formula appearing in [38], we combine these results with Theorem 1.2 to establish index formulae for (fully) elliptic (pseudo)differential operators on boundary groupoids with two orbits and exponential isotropy $\mathcal{H} \rightrightarrows M$. These index formulae are essentially the Atiyah–Singer formula; hence we give a K-theoretic proof of the results in [40] without using a renormalized trace.

2. Preliminaries

2.1. Lie groupoids, Lie algebroids, pseudodifferential operators on Lie groupoids, and groupoid C^*-algebras

We first review some basic knowledge of Lie groupoids, Lie algebroids, pseudodifferential calculus on Lie groupoids, and groupoid C^*-algebras [2,25,26,28,29,32,36,41,42].

Definition 2.1. A Lie groupoid $\mathcal{G} \rightrightarrows M$ consists of the following data:

(i) manifolds M, called the space of units, and $\mathcal{G}$;

(ii) a unit map (inclusion) $\mathbf{u} : M \to \mathcal{G}$;

(iii) submersions $\mathbf{s}, \mathbf{t} : \mathcal{G} \to M$, called the source and target maps respectively, satisfying

$$\mathbf{s} \circ \mathbf{u} = \mathrm{id}_M = \mathbf{t} \circ \mathbf{u};$$

(iv) a multiplication map $\mathbf{m} : \mathcal{G}^{(2)} := \{(g,h) \in \mathcal{G} \times \mathcal{G} : \mathbf{s}(g) = \mathbf{t}(h)\} \to \mathcal{G}$, $(g,h) \mapsto gh$, which is associative and satisfies

$$\mathbf{s}(gh) = \mathbf{s}(h), \quad \mathbf{t}(gh) = \mathbf{t}(g), \quad g(\mathbf{u} \circ \mathbf{s}(g)) = g = (\mathbf{u} \circ \mathbf{t}(g))g;$$

(v) an inverse diffeomorphism $\mathbf{i} : \mathcal{G} \to \mathcal{G}$, $g \mapsto g = g^{-1}$, such that $\mathbf{s}(g^{-1}) = \mathbf{t}(g)$, $\mathbf{t}(g^{-1}) = \mathbf{s}(g)$, and

$$gg^{-1} = \mathbf{u}(\mathbf{t}(g)), \quad g^{-1}g = \mathbf{u}(\mathbf{s}(g)).$$

All maps above are assumed to be smooth.

Definition 2.2. A *homomorphism* between Lie groupoids $\mathcal{G} \rightrightarrows M_1$ and $\mathcal{H} \rightrightarrows M_2$ is by definition a functor $\phi : \mathcal{G} \to \mathcal{H}$ which is smooth both on the unit space M_1 and on $\mathcal{G}$. Two Lie groupoids $\mathcal{G}$ and $\mathcal{H}$ are said to be *isomorphic* if there are homomorphisms $\phi : \mathcal{G} \to \mathcal{H}$ and $\psi : \mathcal{H} \to \mathcal{G}$ such that $\psi \circ \phi$ and $\phi \circ \psi$ are identity homomorphisms on $\mathcal{G}$ and $\mathcal{H}$ respectively.

Lie groupoids are closely related to Lie algebroids. Here we recall the definition.

Definition 2.3. A *Lie algebroid A over a manifold M is a vector bundle A over M, together with a Lie algebra structure on the space $\Gamma^\infty(A)$ of the smooth sections of A and a bundle map $\nu : A \to TM$, called the* anchor *map, such that*

$$[X, fY] = f[X,Y] + (\nu(X)f)Y,$$

for all smooth sections X and Y of A and any smooth function f on M.

Given a Lie groupoid $\mathcal{G}$ with units M, we can associate a Lie algebroid $A(\mathcal{G})$ to $\mathcal{G}$ as follows. (For more details, see [28, 29].) The $\mathbf{s}$-vertical sub-bundle of $T\mathcal{G}$ for $\mathbf{s} : \mathcal{G} \to M$ is denoted by $T^{\mathbf{s}}(\mathcal{G})$ and called simply the $\mathbf{s}$-vertical bundle for $\mathcal{G}$. It is an involutive distribution on $\mathcal{G}$ whose leaves are the components of the $\mathbf{s}$-fibers of $\mathcal{G}$. (Here involutive distribution means that $T^{\mathbf{s}}(\mathcal{G})$ is closed under the Lie bracket, i.e., if $X, Y \in \mathfrak{X}(\mathcal{G})$ are sections of $T^{\mathbf{s}}(\mathcal{G})$, then the vector field $[X, Y]$ is also a section of $T^{\mathbf{s}}(\mathcal{G})$.) Hence we obtain

$$T^{\mathbf{s}}\mathcal{G} = \operatorname{Ker} \mathbf{s}_* = \bigcup_{x \in M} T\mathcal{G}_x \subset T\mathcal{G}.$$

The *Lie algebroid* of $\mathcal{G}$, denoted by $A(\mathcal{G})$ (or simply A sometimes), is defined to be $T^{\mathbf{s}}(\mathcal{G})\big|_M$, the restriction of the $\mathbf{s}$-vertical tangent bundle to the set of units M. In this case, we say that $\mathcal{G}$ *integrates* $A(\mathcal{G})$.

Remark 2.4. Given a Lie algebroid A, there may not exist a Lie groupoid integrating A. When A is almost regular (i.e., ν has constant rank on an open dense subset of M), Debord's quasigraphoid construction can be used to integrate A [18]. If in particular A is of the form as in (iv) of Definition 2.12 with $\mathfrak{g}_k$ solvable, then the resulting groupoid (which is a quasigraphoid) is necessarily of the form

$$\mathcal{G} = (M_0 \times M_0) \sqcup \big(\sqcup_i M_i \times M_i \times G \big),$$

as in (ii) of Definition 2.12. However, Debord's quasigraphoid is not always Hausdorff. Fortunately, the result of [12] follows from the composition series (see Lemma 2.11 below) and Thom isomorphism, which only requires G to be *locally Hausdorff* and M to be Hausdorff.

If, furthermore, all leaves of A are simply connected, then Nistor's gluing construction [34] integrates A to a Hausdorff Lie groupoid $\mathcal{G}$ with simply connected **s**-fibers. Moreover, uniqueness implies $\mathcal{G}$ is again necessarily of the form as in (ii) of Definition 2.12.

Since we shall only consider almost-regular Lie algebroids, we can always integrate them into Lie groupoids, where the K-theoretic calculation can be applied.

Example 2.5. The Lie algebroid of the pair groupoid $M \times M$ is the tangent bundle TM with the usual Lie bracket on vector fields, and the anchor map is the identity.

Let $E \to M$ be a vector bundle. Recall [36] that an m-th order pseudodifferential operator on $\mathcal{G}$ is a right invariant, smooth family $P = \{P_x\}_{x \in M}$, where each P_x is an m-th order classical pseudodifferential operator on sections of $\mathbf{t}^* E \to \mathbf{s}^{-1}(x)$. We denote by $\Psi^m(\mathcal{G}, E)$ (resp. $D^m(\mathcal{G}, E)$) the algebra of uniformly supported, order m classical pseudodifferential operators (resp. differential operators).

Recall [25] that one defines the strong norm for $P \in \Psi^0(\mathcal{G}, E)$ by

$$\|P\| := \sup_{\rho} \|\rho(P)\|,$$

where ρ ranges over all bounded $*$-representations of $\Psi^0(\mathcal{G}, E)$ satisfying

$$\|\rho(P)\| \leq \sup_{x \in M} \Big\{ \int_{\mathbf{s}^{-1}(x)} |\kappa_P(g)| \, \mu_x, \int_{\mathbf{s}^{-1}(x)} |\kappa_P(g^{-1})| \, \mu_x \Big\}$$

whenever $P \in \Psi^{-\dim M - 1}(\mathcal{G}, E)$ with (continuous) kernel κ_P.

Definition 2.6. The C^*-algebras $\mathfrak{U}(\mathcal{G})$ and $C^*(\mathcal{G})$ are defined to be the completion of $\Psi^0(\mathcal{G}, E)$ and $\Psi^{-\infty}(\mathcal{G}, E)$ respectively with respect to the strong norm $\| \cdot \|$.

One also defines the reduced C^*-algebras $\mathfrak{U}_r(\mathcal{G})$ and $C^*_r(\mathcal{G})$ by completing $\Psi^0(\mathcal{G}, E)$ and $\Psi^{-\infty}(\mathcal{G}, E)$ respectively with respect to the reduced norm

$$\|P\|_r := \sup_{x \in M} \big\{ \|P_x\|_{L^2(\mathbf{s}^{-1}(x))} \big\}.$$

Recall that if the strong and reduced norms coincide, then $\mathcal{G}$ is called *(metrically) amenable*, which is the case for the groupoids we shall consider.

2.2. Invariant submanifolds and composition series

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid.

Definition 2.7. Let S be any subset of M. We denote by

$$\mathcal{G}_S := \mathbf{s}^{-1}(S) \cap \mathbf{t}^{-1}(S)$$

the *reduction* of $\mathcal{G}$ to S. The reduction $\mathcal{G}_S$ is a subgroupoid of $\mathcal{G}$. In particular, if $S = \{x\}$, then $\mathcal{G}_x := \mathcal{G}_S$ is called the *isotropy group* at x.

If $S \subseteq M$ is an embedded submanifold such that $\mathbf{s}^{-1}(S) = \mathbf{t}^{-1}(S)$, we say that $S \subset M$ is an *invariant submanifold.*

Definition 2.8. Given a closed invariant submanifold S of $\mathcal{G}$, for any groupoid pseudodifferential operator $P = \{P_x\}_{x \in M} \in \Psi^m(\mathcal{G}, E)$, we define the restriction

$$P|_{\mathcal{G}_S}(P) := \{P_x\}_{x \in \mathcal{G}_S} \in \Psi^m(\mathcal{G}_S, E).$$

The restriction extends to a map from $\mathfrak{U}(\mathcal{G})$ to $\mathfrak{U}(\mathcal{G}_S)$ and also from $C^*(\mathcal{G})$ to $C^*(\mathcal{G}_S)$. We denote both such restriction maps, and also the induced K-group homomorphisms, by $r_{\mathcal{G},S}$.

Notation 2.9. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and let U be an open subset of M. Then $\mathcal{G}_U := \mathbf{s}^{-1}(U) \cap \mathbf{t}^{-1}(U) \rightrightarrows U$ is an open subgroupoid of $\mathcal{G}$. Any element in $C^*(\mathcal{G}_U)$ extends to $C^*(\mathcal{G})$ by 0. We denote such an extension map by $\varepsilon_{\mathcal{G},U}$. It is a homomorphism of C^*-algebras and hence induces a map from $K_\bullet(C^*(\mathcal{G}_U))$ to $K_\bullet(C^*(\mathcal{G}))$, which we shall still denote by $\varepsilon_{\mathcal{G},U}$.*

Now suppose we are given a groupoid $\mathcal{G} \rightrightarrows M$ (with M not necessarily compact), and invariant submanifolds $M_0, M_1, \ldots, M_r$, such that their closures $\bar{M}_i$ are also invariant submanifolds that furthermore satisfy

$$M = \bar{M}_0 \supset \bar{M}_1 \supset \cdots \supset \bar{M}_r$$

(such a setting is natural for boundary groupoids in Definition 2.12 below). For simplicity, we shall denote $\bar{\mathcal{G}}_i := \mathcal{G}_{\bar{M}_i}$. Denote by SA' the sphere sub-bundle of the dual of the Lie algebroid $A(\mathcal{G})$ of $\mathcal{G}$.

Definition 2.10. Let $\sigma : \Psi^m(\mathcal{G}) \to C^\infty(SA')$ denote the principal symbol map. For each $i = 1, \ldots, r$, define the *joint symbol maps*

$$\mathbf{j}_i : \Psi^m(\mathcal{G}) \to C^\infty(SA') \oplus \Psi^m(\bar{\mathcal{G}}_i), \quad \mathbf{j}_i(P) := (\sigma(P), P|_{\bar{\mathcal{G}}_i}).$$

The map $\mathbf{j}_i$ extends to a homomorphism from $\mathfrak{U}(\mathcal{G})$ to $C_0(SA^*) \oplus \mathfrak{U}(\bar{\mathcal{G}}_i)$.

We say that $P \in \Psi^m(\mathcal{G})$ is *elliptic* if $\sigma(P)$ is invertible, and it is called *fully elliptic* if $\mathbf{j}_1(P)$ is invertible (which implies P is elliptic).

Denote by $\mathcal{J}_0 := \overline{\Psi^{-1}(\mathcal{G})} \subset \mathfrak{U}(\mathcal{G})$, and denote by $\mathcal{J}_i \subset \mathfrak{U}(\mathcal{G})$, $i = 1, \dots, r$, the null space of $\mathbf{j}_{r-i+1}$.

By construction, it is clear that

$$\mathcal{J}_0 \supset \mathcal{J}_1 \supset \cdots \supset \mathcal{J}_r.$$

Also, any uniformly supported kernel in $\Psi^{-\infty}(\mathcal{G}_{\bar{M}_i \setminus \bar{M}_j}, E|_{\bar{M}_i \setminus \bar{M}_j})$ can be extended to a kernel in $\Psi^{-\infty}(\mathcal{G}_{\bar{M}_i}, E|_{\bar{M}_i})$ by zero. This induces a $*$-algebra homomorphism from $C^*(\mathcal{G}_{\bar{M}_i \setminus \bar{M}_j})$ to $C^*(\mathcal{G}_{\bar{M}_i})$. We shall use the following key fact.

Lemma 2.11. [25, Lemma 2 and Theorem 3] *One has the short exact sequences*

$$0 \to \mathcal{J}_{i+1} \to \mathcal{J}_i \to C^*(\mathcal{G}_{\bar{M}_i \setminus \bar{M}_{i+1}}) \to 0,$$
$$0 \to C^*(\mathcal{G}_{\bar{M}_i \setminus \bar{M}_j}) \to C^*(\bar{\mathcal{G}}_i) \to C^*(\bar{\mathcal{G}}_j) \to 0, \quad \forall\, j > i.$$

2.3. Boundary groupoids and submanifold groupoids

In this paper we are interested in some more specific classes of groupoids. To begin with, let us recall the definition of boundary groupoids in [42, 43].

Definition 2.12. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with M compact. We say that $\mathcal{G}$ is a boundary groupoid if:

- (i) The singular foliation defined by the anchor map $\nu : A \to TM$ has finite number of leaves $M_0, M_1, \dots, M_r \subset M$ (which are invariant submanifolds), such that $\dim M = \dim M_0 > \dim M_1 > \cdots > \dim M_r$;
- (ii) For all $k = 0, 1, \dots, r$, $\bar{M}_k := M_k \cup \cdots \cup M_r$ is a closed submanifold of M;
- (iii) For $k = 0$, $\mathcal{G}_0 := \mathcal{G}_{M_0}$ is the pair groupoid, and for $k = 1, 2, \dots, r$, we have $\mathcal{G}_k := \mathcal{G}_{M_k} \cong G_k \times M_k \times M_k$ for some Lie group G_k;
- (iv) For each $k = 0, 1, \dots, r$, there exists a unique sub-bundle $\bar{A}_k \subset A|_{\bar{M}_k}$ such that $\bar{A}_k|_{M_k} = \mathrm{Ker}(\nu|_{M_k}) (= \mathfrak{g}_k \times M_k)$.

Boundary groupoids are closely related to Fredholm groupoids and blowup groupoids. Roughly speaking, Fredholm groupoids are those on which Fredholmness of a pseudodifferential operator is completely characterized by its ellipticity and invertibility at the boundary. For the definition and basic properties of Fredholm groupoids, one may consult [13–15]. The basic result relevant to our discussion is that boundary groupoids are often amenable and Fredholm groupoids.

Lemma 2.13. [26, see Lemma 7] *For any boundary groupoid of the form $\mathcal{G} = (M_0 \times M_0) \cup (\mathbb{R}^q \times M_1 \times M_1)$,*

$$C^*(\mathcal{G}) \cong C^*_r(\mathcal{G}).$$

In other words, $\mathcal{G}$ is metrically amenable.

Moreover, as the additive group $\mathbb{R}^q$ is amenable, the product groupoid $\mathbb{R}^q \times M_1 \times M_1$ is topologically amenable. By [15, Theorem 4.3], we have the following proposition.

Proposition 2.14. *The boundary groupoid $\mathcal{G} = (M_0 \times M_0) \cup (\mathbb{R}^q \times M_1 \times M_1)$ is a Fredholm groupoid.*

Given a boundary groupoid, one naturally considers the sequence of invariant submanifolds

$$M \supset \bar{M}_1 \supset \cdots \supset \bar{M}_r,$$

where $\bar{M}_i$ is given to be as in (ii) of Definition 2.12. We have the short exact sequence

$$0 \to C^*(\mathcal{G}_{\bar{M}_i \setminus \bar{M}_j}) \to C^*(\bar{\mathcal{G}}_i) \to C^*(\bar{\mathcal{G}}_j) \to 0, \quad \forall\, j > i,$$

which induces the following K-theory six-term exact sequences.

$$\begin{array}{ccccc} K_1(C^*(\mathcal{G}_{M_i})) & \longrightarrow & K_1(C^*(\bar{\mathcal{G}}_i)) & \longrightarrow & K_1(C^*(\bar{\mathcal{G}}_{i+1})) \\ \uparrow & & & & \downarrow \\ K_0(C^*(\bar{\mathcal{G}}_{i+1})) & \longleftarrow & K_0(C^*(\bar{\mathcal{G}}_i)) & \longleftarrow & K_0(C^*(\mathcal{G}_{M_i})) \end{array} \tag{2.1}$$

If $r = 1$, there is only one exact sequence in (2.1). Moreover, if G_1 is solvable, connected, and simply connected (i.e., exponential), then Connes' Thom isomorphism readily gives the K-groups of (the C^*-algebra of) $\bar{\mathcal{G}}_1 = G \times M_1 \times M_1$. Using these facts, Carrillo Rouse and the second author computed the K-theory of boundary groupoids with two leaves and exponential isotropy, as follows.

Lemma 2.15. *Suppose that a boundary groupoid $\mathcal{G}$ is of the form $\mathcal{G} = (M_0 \times M_0) \cup (G \times M_1 \times M_1)$ with G an exponential Lie group (i.e., $\mathcal{G}$ is a boundary groupoid with two leaves and exponential isotropy). One has*

$$\begin{array}{llll} K_0(C^*(\mathcal{G})) \cong \mathbb{Z} & \textit{and} & K_1(C^*(\mathcal{G})) \cong \mathbb{Z}, & \textit{if } \dim G \geq 3 \textit{ odd}; \\ K_0(C^*(\mathcal{G})) \cong \mathbb{Z} \oplus \mathbb{Z} & \textit{and} & K_1(C^*(\mathcal{G})) \cong \{0\}, & \textit{if } \dim G \textit{ even}. \end{array}$$

Next, we recall *submanifold groupoids*, which form a subclass of boundary groupoids in [12].

Example 2.16 (Submanifold groupoids). Suppose that M_1 is a closed embedded submanifold of M of codimension $q \geq 2$. Let $f \in C^\infty(M)$ be any smooth function which vanishes on M_1 and is strictly positive over $M_0 = M \setminus M_1$ (if M is connected then $M_0 = M \setminus M_1$ is still connected and f must either be strictly positive or strictly negative on M_0). Then there exists a Lie algebroid structure over

$$A_f := TM \to M$$

with anchor map

$$(z, w) \mapsto (z, f(z) \cdot w), \quad \forall (z, w) \in TM,$$

and Lie bracket on sections

$$[X, Y]_{A_f} := f[X, Y] + (X \cdot f)Y - (Y \cdot f)X.$$

This Lie algebroid is almost injective (since the anchor is injective, in fact an isomorphism, over the open dense subset $M_0 = M \setminus M_1$), and hence it integrates to a Lie groupoid that is a quasigraphoid (see [18, Theorems 2 and 3]).

By [18] or [34], A_f integrates to a Lie groupoid of the form

$$\mathcal{H} = M_0 \times M_0 \sqcup \mathbb{R}^q \times M_1 \times M_1 \rightrightarrows M = M_0 \sqcup M_1,$$

a boundary groupoid with two leaves and exponential isotropy. This groupoid is called a submanifold groupoid.

Example 2.17 (Renormalizable boundary groupoids). For even more specific examples of submanifold groupoids, in [40], we considered

$$f = r^N$$

for some fixed even integers N, where $r := d(M_1, \cdot)$ is the Riemannian distance function from M_1 with respect to some metric. The resulting groupoid is called a *renormalizable boundary groupoid*. In [40], it was shown that the η-term for renormalizable groupoids vanishes in the index formula via the method of renormalized trace.

2.4. The tangent groupoid and the adiabatic groupoid

Let us recall that for a closed manifold M, Connes [17] constructed the tangent groupoid, which is a Lie groupoid of the form

$$\mathcal{T}(M \times M) := M \times M \times (0, 1] \sqcup TM \times \{0\} \rightrightarrows M \times [0, 1],$$

where the differentiable structure of $\mathcal{T}(M \times M)$ is defined by fixing some Riemannian metric on M and then using the following maps from some open subsets of $TM \times [0, 1]$ to $\mathcal{T}(M \times M)$ as charts:

$$\mathbf{x}(p, X, \epsilon) := (p, \exp_p(-\epsilon X), \epsilon), \text{ for } \epsilon > 0,$$
$$\mathbf{x}(p, X, 0) := (p, X, 0).$$

The tangent groupoid construction generalizes to any Lie groupoid $\mathcal{G}$. One considers the adiabatic groupoid

$$\mathcal{T}(\mathcal{G}) := \mathcal{G} \times (0, 1] \sqcup A \times \{0\} \rightrightarrows M \times [0, 1],$$

where A is the Lie algebroid of $\mathcal{G}$. Debord and Skandalis [23] further generalized the construction by replacing the set of units $M \subset \mathcal{G}$ with arbitrary subgroupoid $\mathcal{H} \subset \mathcal{G}$, and considered gluing $\mathcal{N}^{\mathcal{G}}_{\mathcal{H}}$, the normal bundle of $\mathcal{H}$ in $\mathcal{G}$, to $\mathcal{G} \times \mathbb{R} \setminus \{0\}$. The resulting object is naturally a Lie groupoid, which they call the deformation to the normal cone.

For simplicity, we return to the case of the adiabatic groupoid $\mathcal{T}(\mathcal{G})$. One naturally constructs the index map

$$\mathrm{ind}_{\mathcal{T}(\mathcal{G})} := r_{\mathcal{T}(\mathcal{G}), M \times \{1\}} \circ r^{-1}_{\mathcal{T}(\mathcal{G}), M \times \{0\}}, \tag{2.2}$$

where $r_{\mathcal{T}(\mathcal{G}),M\times\{1\}} : C^*(\mathcal{T}(\mathcal{G})) \to C^*(\mathcal{G})$ and $r_{\mathcal{T}(\mathcal{G}),M\times\{0\}} : C^*(\mathcal{T}(\mathcal{G})) \to C^*(A)$ are respectively the restriction maps to the subgroupoids of $\mathcal{T}(\mathcal{G})$ over $M \times \{1\}$ and $M \times \{0\}$.

Let Ψ be any classical elliptic pseudodifferential operator on $\mathcal{G} \rightrightarrows M$. Its principal symbol $\sigma(\Psi)$ is an invertible element in $C(SA')$, where A' is the dual bundle of A and SA' denotes the sphere bundle of A'. We identify $C(A')$, the algebra of continuous functions on A' vanishing at infinity, with $C^*(A)$, the (fiberwise) convolution C^*-algebra, through Fourier transform to obtain

$$K_0(C(A')) \cong K_0(C^*(A)),$$

and let

$$\partial : K_1(SA') \to K_0(C(A')) \cong K_0(C^*(A))$$

be the connecting map induced by the short exact sequence

$$0 \to C(A') \to C(\bar{A}') \to (SA') \to 0$$

(where $\bar{A}'$ denotes the bundle of solid balls in A'). Then by [12], the analytic index of Ψ is given by

$$\mathrm{ind}_{\mathcal{T}(\mathcal{G})}(\partial[\sigma(\Psi)]) \in C^*(\mathcal{G}),$$

which, if Ψ is of order 0, coincides with the image of $[\Psi] \in K_1(\mathfrak{U}(\mathcal{G}))$ under the connecting map induced by the short exact sequence

$$0 \to C^*(\mathcal{G}) \to \mathfrak{U}(\mathcal{G}) \to \mathfrak{U}/C^*(\mathcal{G}) \to 0.$$

3. The deformation groupoid and the deformation index map

Motivated by the tangent groupoid construction in the previous section, we introduce the following definition, which plays a central role in the paper.

Definition 3.1. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, not necessarily a boundary groupoid. A *deformation from the pair groupoid* $M \times M$ *to* $\mathcal{G}$ is a Lie groupoid $\mathcal{D} \rightrightarrows M \times [0, 1]$ that is isomorphic as an abstract groupoid to

$$M \times M \times (0, 1] \sqcup \mathcal{G} \rightrightarrows M \times (0, 1] \sqcup M$$

(where $M \times M \times (0, 1] \rightrightarrows M \times (0, 1]$ is the product of the pair groupoid and the set $(0, 1]$).

Example 3.2. Let M be a closed manifold. Regard $TM \to M$ as a Lie groupoid; then Connes' tangent groupoid, $\mathcal{T}(M \times M)$, gives a deformation from the pair groupoid to TM.

3.1. Construction of deformations from the pair groupoid

The following fact is another motivation to introduce the above definition.

Proposition 3.3. *There exists a deformation from the pair groupoid for submanifold groupoids defined in Example 2.16.*

Proof. We use the same notation as in Example 2.16. To construct the groupoid $\mathcal{D} \rightrightarrows M \times [0,1]$, we begin with defining the Lie algebroid of $\mathcal{D}$. We set $\tilde{A}_f \to M \times [0,1]$ to be the vector bundle pullback of A_f ($\cong TM$). Sections of $\tilde{A}_f$ are just vector fields on $M \times [0,1]$ which are tangential to the M direction. Hence one can define the anchor map

$$\nu((z,s),w) = ((z,s),(f(z)+s)\cdot w), \quad \forall (z,w) \in TM,\ s \in [0,1],$$

and Lie bracket on sections by

$$[X,Y]_{\tilde{A}_f} := (f+s)[X,Y] + (X\cdot f)Y - (Y\cdot f)X.$$

When restricted to $M \times \{0\}$, the Lie algebroid $\tilde{A}_f|_{M\times\{0\}}$ is isomorphic to A_f. On $M \times \{s\}$ for each $s > 0$, since $f+s$ is positive and bounded away from 0, ν defines an isomorphism

$$\tilde{A}_f|_{M\times\{s\}} \cong TM.$$

Using [18] and [34, Theorem 1], one observes that $\tilde{A}_f$ integrates to a Lie groupoid of the form

$$\mathcal{D} = \mathcal{H} \sqcup (M \times M \times (0,1]) \rightrightarrows M \times [0,1],$$

which is what we need. ■

3.2. Obstructions to the existence of deformations from the pair groupoid

In this subsection we point out some notable necessary conditions for $\mathcal{G}$ in order for a deformation from the pair groupoid to exist.

Proposition 3.4. *Suppose that $\mathcal{G} \rightrightarrows M$ is a Lie groupoid such that a deformation from the pair groupoid exists. Denote by A the Lie algebroid of $\mathcal{G}$. Then we have*

$$A \cong TM,$$

as vector bundles.

Proof. Let $\mathcal{D} \rightrightarrows M \times [0,1]$ be a deformation from the pair groupoid. Denote its Lie algebroid by $\tilde{A} \to M \times [0,1]$. Then one has

$$\tilde{A}|_{M\times\{0\}} \cong A, \quad \tilde{A}|_{M\times\{1\}} \cong TM.$$

Hence an isomorphism can be constructed by, say, fixing a connection $\tilde{\nabla}$ on $\tilde{A}$ and then parallel transporting along the family of curves $(p,t), t \in [0,1]$ for each $p \in M$. ■

The existence of a deformation from the pair groupoid also imposes necessary conditions on the structural vector fields of $\mathcal{G}$ (i.e. the image of the anchor map). For simplicity, we consider the case when M_1 is a single point $\{p\}$. Suppose a deformation $\mathcal{D}$ from the pair groupoid for $\mathcal{G}$ exists. Denote the Lie algebroids of $\mathcal{G}$ and $\mathcal{D}$ by A and $\tilde{A}$ respectively, and their anchor maps by ν and respectively $\tilde{\nu}$. Then it is obvious that

$$\tilde{A}\big|_{M\times(0,1]} \cong TM \times (0,1],$$
$$\tilde{A}\big|_{M\times\{0\}} \cong A.$$

Suppose that s is a nowhere vanishing local section of A around p, such that $\nu(s)(p) = 0$. Then the section s extends to some nowhere vanishing local section $\tilde{s}$ of $\tilde{A}$ over an open neighborhood of $\{p\} \times [0,1]$. Because $\tilde{\nu}$ is injective except at $(p,0)$, it follows that $\tilde{\nu}(\tilde{s})$ is a family of local vector fields on M parameterized by $[0,1]$, which is nonvanishing on $M \times \{t\}$, $1 \geq t > 0$, and has an isolated zero at p on $M \times \{0\}$. This implies that the Hopf–Poincaré index of $\nu(s) = \tilde{\nu}(\tilde{s})\big|_{M\times\{0\}}$ equals zero. Hence, we are able to construct the following counterexample which shows that a boundary groupoid may not possess a deformation from the pair groupoid.

Example 3.5. Let $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the 2-sphere. As a set, we identify

$$\mathbb{S}^2 = \{(0,0,-1)\} \sqcup \mathbb{R}^2$$

by stereographic projection. The vector fields

$$X := \tfrac{\partial}{\partial u}, \quad Y := \tfrac{\partial}{\partial v}$$

on $\mathbb{R}^2$ both extend smoothly to $\mathbb{S}^2$ by 0. Indeed, using stereographic coordinates on $\mathbb{S}^2 \setminus \{(0,0,1)\}$, one has

$$X = -((u')^2 - (v')^2)\tfrac{\partial}{\partial u'} - 2u'v'\tfrac{\partial}{\partial v'}$$

and a similar expression for Y. These vector fields commute and together integrate to an action of $\mathbb{R}^2$ on $\mathbb{S}^2$, and one obtains the action groupoid $\mathcal{G} = \mathbb{S}^2 \rtimes \mathbb{R}^2$. Thus the structural vector fields of the tangent Lie algebroid of $\mathcal{G}$ are spanned by $\{X, Y\}$, which contradicts the above necessary condition that the Hopf–Poincaré index is equal to zero.

Lastly, note that the action groupoid $\mathbb{R}^2 \rtimes \mathbb{R}^2$ (by right translation) is isomorphic to the pair groupoid $\mathbb{R}^2 \times \mathbb{R}^2$; therefore $\mathcal{G}$ is a boundary groupoid.

Remark 3.6. The discussion above exhibits that there are certain obstructions to the existence of such deformations for boundary groupoids. At present, we do not know of other necessary or sufficient conditions that guarantee the existence of such deformations in general, which will be left as future work.

3.3. The deformation index map

In this subsection, we always assume that a deformation from the pair groupoid

$$\mathcal{D} := \mathcal{G} \sqcup (M \times M \times (0,1]) \rightrightarrows M \times [0,1]$$

exists for the given Lie groupoid $\mathcal{G} \rightrightarrows M$.

We shall use $\mathcal{D}$ to construct an index, similar to [17, 44]. Observe that $\mathcal{D}$ has closed saturated subgroupoids $\mathcal{D}|_{M\times\{0\}} \cong \mathcal{G}$, $\mathcal{D}|_{M\times\{1\}} \cong M \times M$. Because the sequence

$$K_\bullet(C^*(\mathcal{D}|_{M\times(0,1]})) \cong K_\bullet(C^*(M \times M \times (0,1])) \cong 0 \xrightarrow{\varepsilon_{\mathcal{D},M\times(0,1]}} K_\bullet(C^*(\mathcal{D})) \xrightarrow{r_{\mathcal{D},M\times\{0\}}} K_\bullet(C^*(\mathcal{G})) \xrightarrow{\partial} K_{\bullet+1}(C^*(\mathcal{D}|_{M\times(0,1]})) \cong 0$$

is exact, the map $r_{\mathcal{D},M\times\{0\}}$ is invertible, where the extension map $\varepsilon_{\mathcal{D},M\times(0,1]}$ is defined in Notation 2.9, and the restriction map $r_{\mathcal{D},M\times\{0\}}$ is defined in Definition 2.8.

Hence it is natural to introduce the following index map.

Definition 3.7. The deformation index map is defined to be

$$\mathrm{ind}_{\mathcal{D}} := r_{\mathcal{D},M\times\{1\}} \circ r^{-1}_{\mathcal{D},M\times\{0\}} : K_\bullet(C^*(\mathcal{G})) \to K_\bullet(C^*(M \times M)).$$

Using this deformation index map $\mathrm{ind}_{\mathcal{D}}$, we establish the following 'index comparison' theorem, which is one of the main results in the paper.

Theorem 3.8. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Suppose there exists a deformation groupoid $\mathcal{D}$ from the pair groupoid $M \times M$. Then one has the commutative diagram*

$$\begin{array}{ccc} K_0(C^*(A)) & \xrightarrow{\mathrm{ind}_{\mathcal{T}(\mathcal{G})}} & K_0(C^*(\mathcal{G})) \\ \downarrow \cong & & \downarrow \mathrm{ind}_{\mathcal{D}} \\ K_0(C^*(TM)) & \xrightarrow{\mathrm{ind}_{\mathcal{T}(M\times M)}} & K_0(C^*(M \times M)), \end{array}$$

where the top map is just the analytic index map constructed via the adiabatic groupoid and the bottom map is the Atiyah–Singer index.

Proof. Let $\mathcal{T}(\mathcal{D}) \rightrightarrows (M \times [0,1]) \times [0,1]$ be the adiabatic groupoid of $\mathcal{D}$. Recall that as a set, $\mathcal{T}(\mathcal{D}) = \mathcal{D} \times (0,1] \sqcup \tilde{A}$. Hence $\mathcal{T}(\mathcal{D})$ restricted to $(M \times \{0\}) \times [0,1]$ and $(M \times \{1\}) \times [0,1]$ are respectively

$$\mathcal{T}(\mathcal{G}) \rightrightarrows M \times [0,1] = \mathcal{G} \times (0,1] \sqcup A,$$
$$\text{and } \mathcal{T}(M \times M) \rightrightarrows M \times [0,1] = M \times M \times (0,1] \sqcup TM,$$

the adiabatic groupoid of $\mathcal{G}$ (respectively the adiabatic groupoid of $M \times M$). One can further restrict $\mathcal{T}(\mathcal{G})$ and $\mathcal{T}(M \times M)$ to $M \times \{0\}$ or $M \times \{1\}$, resulting in the commutative diagram

$$\begin{array}{ccccc}
C^*(A) & \xleftarrow{r_{\mathcal{T}(\mathcal{G}),M\times\{0\}}} & C^*(\mathcal{T}(\mathcal{G})) & \xrightarrow{r_{\mathcal{T}(\mathcal{G}),M\times\{1\}}} & C^*(\mathcal{G}) \\
\uparrow r_{\tilde{A},M\times\{0\}} & & \uparrow r_{\mathcal{T}(\mathcal{D}),(M\times\{0\})\times[0,1]} & & \uparrow r_{\mathcal{D},M\times\{0\}} \\
C^*(\tilde{A}) & \xleftarrow{r_{\mathcal{T}(\mathcal{D}),(M\times[0,1])\times\{0\}}} & C^*(\mathcal{T}(\mathcal{D})) & \xrightarrow{r_{\mathcal{T}(\mathcal{D}),(M\times[0,1])\times\{1\}}} & C^*(\mathcal{D}) \\
\downarrow r_{\tilde{A},M\times\{1\}} & & \downarrow r_{\mathcal{T}(\mathcal{D}),(M\times\{1\})\times[0,1]} & & \downarrow r_{\mathcal{D},M\times\{1\}} \\
C^*(TM) & \xleftarrow[r_{\mathcal{T}(M\times M),M\times\{0\}}]{} & C^*(\mathcal{T}(M\times M)) & \xrightarrow[r_{\mathcal{T}(M\times M),M\times\{1\}}]{} & C^*(M\times M).
\end{array}$$

Then we pass to the corresponding K-group maps. Observe that $r_{\mathcal{T}(\mathcal{D}),(M\times\{0\})\times[0,1]}$ fits in the six-term exact sequence

$$K_\bullet(C^*(\mathcal{T}(\mathcal{D})_{(M\times(0,1])\times[0,1]})) \xrightarrow{\varepsilon_{\mathcal{T}(\mathcal{D}),(M\times(0,1])\times[0,1]}} K_\bullet(C^*(\mathcal{T}(\mathcal{D}))) \xrightarrow{r_{\mathcal{T}(\mathcal{D}),(M\times\{0\})\times[0,1]}} K_\bullet(C^*(\mathcal{T}(\mathcal{G}))) \to \cdots,$$

and moreover, we have

$$\mathcal{T}(\mathcal{D})_{(M\times(0,1])\times[0,1]} \cong \mathcal{T}(\mathcal{D}_{M\times(0,1]}) \cong \mathcal{T}(M\times M\times(0,1]) \cong \mathcal{T}(M\times M)\times(0,1],$$

whose convolution C^*-algebra is contractible. Therefore the map $r_{\mathcal{T}(\mathcal{D}),(M\times\{0\})\times[0,1]}$ is invertible. Similarly, we have that the maps $r_{\mathcal{T}(\mathcal{G}),M\times\{0\}}$, $r_{\tilde{A},M\times\{0\}}$, $r_{\mathcal{T}(\mathcal{D}),(M\times[0,1])\times\{0\}}$, $r_{\mathcal{T}(M\times M),M\times\{0\}}$, and $r_{\mathcal{D},M\times\{0\}}$ are all isomorphisms between corresponding K-groups.

Recall (2.2) that

$$\begin{aligned}
\mathrm{ind}_{\mathcal{T}(\mathcal{G})} &:= r_{\mathcal{T}(\mathcal{G}),M\times\{1\}} \circ r^{-1}_{\mathcal{T}(\mathcal{G}),M\times\{0\}}, \\
\mathrm{ind}_{\mathcal{T}(M\times M)} &:= r_{\mathcal{T}(M\times M),M\times\{1\}} \circ r^{-1}_{\mathcal{T}(M\times M),M\times\{0\}}
\end{aligned}$$

are just Connes' analytic index maps. Lastly, observe from the proof of Proposition 3.4 that $\tilde{A} \cong A\times[0,1]$. Hence for any $u \in C^*(A)$ that is furthermore a Schwartz function, let $\tilde{u} \in C^*(\tilde{A})$ be the pullback of u (i.e., $\tilde{u}$ is just u on each $A\times\{s\}$). Then

$$u \mapsto \tilde{u}\big|_{M\times\{1\}}$$

induces the K-theory map

$$r_{\tilde{A},M\times\{1\}} \circ r^{-1}_{\tilde{A},M\times\{0\}}$$

(which is equivalent to identifying $A \cong TM$). Hence one ends up with the commutative diagram

$$\begin{array}{ccc}
K_0(C^*(A)) & \xrightarrow{\mathrm{ind}_{\mathcal{T}(\mathcal{G})}} & K_0(C^*(\mathcal{G})) \\
\downarrow \cong & & \downarrow \mathrm{ind}_{\mathcal{D}} \\
K_0(C^*(TM)) & \xrightarrow{\mathrm{ind}_{\mathcal{T}(M\times M)}} & K_0(C^*(M\times M)),
\end{array}$$

which completes the proof. ∎

We are in position to prove the following theorem for boundary groupoids with two orbits and exponential isotropy, which implies Theorem 1.3. Recall that the restriction map r and the extension map ε are defined in Definition 2.8 and Notation 2.9, respectively.

Theorem 3.9. *Let $\mathcal{H} = M_0 \times M_0 \sqcup G \times M_1 \times M_1 \rightrightarrows M = M_0 \sqcup M_1$ be a boundary groupoid with two orbits and exponential isotropy. Suppose further that there exists a deformation $\mathcal{D}$ from the pair groupoid $M \times M$ for $\mathcal{H}$. Then one has*

$$\mathrm{ind}_{\mathcal{D}} \circ \varepsilon_{\mathcal{H},M_0} = \varepsilon_{M\times M,M_0}.$$

Proof. Observe that

$$M_0 \times M_0 \times [0,1] = \mathcal{D}_{M_0\times[0,1]}$$

is an open subgroupoid. For any $u \in C^*(M_0 \times M_0)$, let $\tilde{u} \in C^*(M_0 \times M_0 \times [0,1])$ be the pullback of u by the projection to the $M_0 \times M_0$ factor (i.e., $\tilde{u}$ is just u on each $M_0 \times M_0 \times \{s\}$). Extend $\tilde{u}$ to $\mathcal{D}$ by 0 and one gets $\varepsilon_{\mathcal{D},M_0\times[0,1]}(\tilde{u})$. When restricted to $s = 0$ and $s = 1$, it is clear that $\varepsilon_{\mathcal{D},M_0\times[0,1]}(\tilde{u})$ equals

$$\varepsilon_{\mathcal{H},M_0}(u) \quad \text{and} \quad \varepsilon_{M\times M,M_0}(u),$$

respectively. Passing to K-theory, one obtains

$$r_{\mathcal{D},M\times\{0\}} \circ \varepsilon_{\mathcal{D},M_0\times[0,1]}([\tilde{u}]) = \varepsilon_{\mathcal{H},M_0}([u]),$$

and

$$r_{\mathcal{D},M\times\{1\}} \circ \varepsilon_{\mathcal{D},M_0\times[0,1]}([\tilde{u}]) = \varepsilon_{M\times M,M_0}([u]),$$

for any class $[u] \in K_\bullet(M_0 \times M_0)$. Hence, we have

$$\begin{aligned}\mathrm{ind}_{\mathcal{D}} \circ \varepsilon_{\mathcal{H},M_0}([u]) &= r_{\mathcal{D},M\times\{1\}} \circ r^{-1}_{\mathcal{D},M\times\{0\}} \circ r_{\mathcal{D},M\times\{0\}} \circ \varepsilon_{\mathcal{D},M_0\times[0,1]}([\tilde{u}]) \\ &= \varepsilon_{M\times M,M_0}([u]),\end{aligned}$$

which completes the proof. ∎

4. Fredholm and K-theoretic index of (fully) elliptic operators on boundary groupoids

In this section, we combine the map in Theorem 1.2 and the Atiyah–Singer index formula to compute the index. Before doing that, we briefly recall the Atiyah–Singer index formula; in particular, we shall use the version appearing in [38, Theorem 5.1].

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and A the associated Lie algebroid of $\mathcal{G}$. Define the line bundle

$$L := \wedge^{\mathrm{top}} T^*M \otimes \wedge^{\mathrm{top}} A.$$

We suppose further that $\mathcal{G}$ is unimodular, i.e., there exists an invariant nowhere vanishing section Ω of L. Then the characteristic map χ_Ω defines a map from the groupoid cohomology to cyclic cohomology by [38, Equation (5)]

$$\chi_\Omega(\varphi_1 \otimes \cdots \otimes \varphi_k)(a_0 \otimes \cdots \otimes a_k) := \int_M \Big(\int_{g_0 \cdots g_k = 1_x} a_0(g_0)\varphi_1(g_1)a_1(g_1) \cdots \varphi_k(g_k)a_k(g_k) \Big) \Omega(x)$$

(on the level of cochains), for any groupoid cocycle $\varphi_1 \otimes \cdots \otimes \varphi_k$. Then one combines χ_Ω with the canonical pairing between cyclic homology and cohomology, the Connes–Chern character map, which maps from $K_0(\Psi_c^{-\infty}(\mathcal{G}))$ to the cyclic homology of $\Psi_c^{-\infty}(\mathcal{G})$, and the van Est map, to obtain a pairing

$$\langle [\psi], \alpha \rangle_\Omega \in \mathbb{C}$$

for any $[\psi] \in K_0(\Psi_c^{-\infty}(\mathcal{G})), \alpha \in H^\bullet(A, \mathbb{C})$.

Then the main result of [38, Theorem 5.1] states that such pairing can be computed by the following formula.

Lemma 4.1. *For any elliptic pseudodifferential operator Ψ and $\alpha \in H^{2k}(A, \mathbb{C})$, we have*

$$\langle \operatorname{ind}(\Psi), \alpha \rangle_\Omega = (2\pi\sqrt{-1})^{-k} \int_{A'} \big\langle \pi^*\alpha \wedge \hat{A}(A') \wedge \operatorname{ch}(\sigma[\Psi]), \Omega_{\pi^! A} \big\rangle,$$

where $\hat{A}(A') \in H^\bullet(TM)$ is the A-hat genus, $\operatorname{ch} : K_0(C(A')) \to H^{even}(TM)$ *is the Chern character (which can be defined by, say, the Chern–Weil construction), and $\pi^! A$ is the pullback Lie algebroid of A along the projection $\pi : A' \to M$.*

Let us consider the particular case when $\mathcal{G} = M \times M$ is the pair groupoid, hence $A = TM$, and L is trivial. One obtains an invariant nowhere vanishing section Ω of L by considering

$$\Omega := C dx^1 \wedge \cdots \wedge dx^{\dim M} \otimes \tfrac{\partial}{\partial x^1} \wedge \cdots \wedge \tfrac{\partial}{\partial x^{\dim M}} \tag{4.1}$$

locally, for any constant $C \neq 0$. Also, the constant function 1 obviously defines a class $[1] \in H^0(A, \mathbb{C})$. Applying Lemma 4.1, one gets

$$\langle \operatorname{ind}(\Psi), 1 \rangle_\Omega = \int_{T^*M} \langle \hat{A}(T^*M) \wedge \operatorname{ch}(\sigma[\Psi]), \Omega_{\pi^! TM} \rangle. \tag{4.2}$$

Since $\operatorname{ind}(\Psi)$ is an integer, by choosing an appropriate normalization C for Ω, we have

$$\operatorname{ind}(\Psi) = \langle \operatorname{ind}(\Psi), 1 \rangle_\Omega,$$

and the identity (4.2) simplifies to

$$\operatorname{ind}(\Psi) = \int_{T^*M} \langle \hat{A}(T^*M) \wedge \operatorname{ch}(\sigma[\Psi]), \Omega_{\pi^! TM} \rangle. \tag{4.3}$$

Remark 4.2. In order to apply Lemma 4.1, one needs a nontrivial $\mathcal{G}$-invariant Lie algebroid class. By [24] the construction of such a class is nontrivial, because the Lie algebroid associated to a boundary groupoid is in general not unimodular. However, in the following subsections we shall use a deformation from the pair groupoid to simplify the problem to that of a regular groupoid, which is clearly unimodular, so that Lemma 4.1 can be applied.

In the sequel, we focus on boundary groupoids with two orbits and exponential isotropy, i.e., of the form

$$\mathcal{H} := M_0 \times M_0 \sqcup G \times M_1 \times M_1 \rightrightarrows M = M_0 \sqcup M_1, \tag{4.4}$$

where G is an exponential Lie group, and assume that a deformation from the pair groupoid exists for $\mathcal{H}$.

4.1. The odd codimension case

Recall that if the isotropy subgroup G is exponential and of odd dimension ≥ 3, then

$$K_0(M_0 \times M_0) \cong \mathbb{Z} \xrightarrow{\varepsilon_{\mathcal{H},M_0}} C^*(\mathcal{H}) \cong \mathbb{Z}$$

is an isomorphism. Identifying $K_0(M_0 \times M_0) \cong K_0(M \times M)$ using $\varepsilon_{M\times M,M_0}$, one sees from Theorem 1.3 that $\mathrm{ind}_{\mathcal{D}}$ is the inverse of $\varepsilon_{\mathcal{H},M_0}$.

Given any elliptic pseudodifferential operator Ψ, in order to compute the integer

$$\mathrm{ind}(\Psi) = \mathrm{ind}_{\mathcal{T}(\mathcal{H})}(\partial[\sigma(\Psi)]),$$

we apply Theorem 1.2 to obtain

$$\mathrm{ind}(\Psi) = \mathrm{ind}_{\mathcal{T}(\mathcal{H})}(\partial[\sigma(\Psi)]) = \mathrm{ind}_{\mathcal{T}(M\times M)}(\partial[\sigma(\Psi)]),$$

where we regard $\partial[\sigma] \in K_0(C^*(TM))$ on the rightmost expression. Observe that

$$\mathrm{ind}_{\mathcal{T}(M\times M)}(\partial[\sigma])$$

is just the analytic index of the pair groupoid. Hence we apply Lemma 4.1 (with the normalization of (4.3)) to conclude the following theorem.

Theorem 4.3. *Suppose that $\mathcal{H} \rightrightarrows M$ is a boundary groupoid with two orbits and exponential isotropy with* $\dim G \geqslant 3$ *odd, and a deformation from the pair groupoid exists for $\mathcal{H}$. Let Ψ be an elliptic pseudodifferential operator on $\mathcal{H}$. Then one has the index formula*

$$\mathrm{ind}(\Psi) = \mathrm{ind}_{\mathcal{T}(M\times M)}(\partial[\sigma(\Psi)]) = \int_{T^*M} \langle \hat{A}(T^*M) \wedge \mathrm{ch}(\sigma[\Psi]), \Omega_{\pi^! TM} \rangle. \tag{4.5}$$

Remark 4.4. By Proposition 3.3, we see that a deformation from the pair groupoid always exists for submanifold groupoids. Because renormalizable boundary groupoids form a special class of submanifold groupoids, Theorem 4.3 applies and implies that the index is given only by the Atiyah–Singer term. Comparing with the results of [7, 8] (for higher codimension) and also [40], one sees that the η-term vanishes for elliptic pseudodifferential operators on $\mathcal{G}$. Therefore we give a completely new proof that generalizes these previous results.

Example 4.5. Recall [26, Section 6] that for any Lie groupoid the vertical de Rham operator d is an (invariant) groupoid differential operator. Its vector representation computes its Lie algebroid cohomology. Fixing a Riemannian metric on A, one can then define the formal adjoint d^* of d, which leads one to construct the Euler operator $d + d^*$, and also the signature operator. More generally, one can construct generalized Dirac operators.

For illustration here, we only consider the case when the Lie algebroid A is spin, and $\mathcal{H}$ is of the form (4.4) with M_1 of odd codimension, where a deformation from the pair groupoid exists. Let S_A be the spinor bundle of A. Following [26, Section 6], one constructs the (groupoid) Dirac operator D_A associated with the Levi-Civita connection. On the other hand, Proposition 3.4 implies that M is also a spin manifold, and the spinor bundle S_{TM} is isomorphic to S_A. Hence it is standard to construct the Dirac operator D associated with the Levi-Civita connection. One sees from its explicit construction that the principal symbol of D_A is equal to that of D (however, D_A and D are very different as groupoid differential operators). Hence by Theorem 3.8, the index of D_A is the same as the index of D. Theorem 4.3 then gives an explicit formula for $\mathrm{ind}(D_A)$. Lastly, recall [6, Section 3] that one can generalize the construction of the Dirac operator by tensoring S_A with a vector bundle W, and in this case the Chern form $\mathrm{ch}(\sigma[D_A])$ can be written explicitly using the twisting curvature.

4.2. The even codimension case

On the other hand, if G is of even dimension, one has the short exact sequence

$$0 \to K_0(M_0 \times M_0) \cong \mathbb{Z} \xrightarrow{\varepsilon_{\mathcal{H},M_0}} K_0(C^*(\mathcal{H})) \xrightarrow{r_{\mathcal{H},M_1}} K_0(C^*(\mathcal{H}_{M_1})) \cong \mathbb{Z} \to 0,$$

and Theorem 1.3 canonically identifies

$$K_0(C^*(\mathcal{H})) \cong \mathbb{Z} \oplus \mathbb{Z}$$

via $\mathrm{ind}_{\mathfrak{D}} \oplus r_{\mathcal{H},M_1}$. For the $\mathrm{ind}_{\mathfrak{D}}$ component, the arguments for the odd case apply without any change, and the result is the same Atiyah–Singer integral (4.5). To compute

$$r_{\mathcal{H},M_1} \circ \mathrm{ind}_{\mathcal{T}(\mathcal{H})}(\partial[\sigma]),$$

we observe that restriction of $\mathcal{T}(\mathcal{H})$ to $M_1 \times [0,1]$ is just $\mathcal{T}(\mathcal{H}_1)$, which is the adiabatic groupoid of $\mathcal{H}_1 = G \times M_1 \times M_1$; hence we obtain the commutative diagram

$$\begin{array}{ccccc}
C^*(A) & \xleftarrow{r_{\mathcal{T}(\mathcal{H}),M\times\{0\}}} & C^*(\mathcal{T}(\mathcal{H})) & \xrightarrow{r_{\mathcal{T}(\mathcal{H}),M\times\{1\}}} & C^*(\mathcal{H}) \\
\downarrow{\scriptstyle r_{A,M_1}} & & \downarrow{\scriptstyle r_{\mathcal{T}(\mathcal{H}),M_1\times[0,1]}} & & \downarrow{\scriptstyle r_{\mathcal{H},M_1}} \\
C^*(A|_{M_1}) & \xleftarrow[r_{\mathcal{T}(\mathcal{H}_1),M_1\times\{0\}}]{} & C^*(\mathcal{T}(\mathcal{H}_1)) & \xrightarrow[r_{\mathcal{T}(\mathcal{H}_1),M\times\{1\}}]{} & C^*(\mathcal{H}_1),
\end{array}$$

which implies

$$r_{\mathcal{H},M_1} \circ \mathrm{ind}_{\mathcal{T}(\mathcal{H})}(\partial[\sigma]) = \mathrm{ind}_{\mathcal{T}(\mathcal{H}_1)}(\partial[\sigma]|_{M_1}).$$

The right-hand side of the above can, in turn, be computed by Lemma 4.1 with $\mathcal{H}_1 = M_1 \times M_1 \times G$ as the groupoid:

$$\mathrm{ind}_{\mathcal{T}(\mathcal{H}_1)}(\partial[\sigma]|_{M_1}) = \int_{T^*M_1 \times \mathfrak{g}} \big\langle \hat{A}(T^*M_1 \oplus \mathfrak{g}) \wedge \mathrm{ch}(\sigma[\Psi]|_{M_1}), \Omega'_{\pi^! TM_1} \big\rangle,$$

where $\Omega' \in \Gamma^\infty(\wedge^{\mathrm{top}}\mathfrak{g} \otimes \wedge^{\mathrm{top}} T^*M_1 \otimes \wedge^{\mathrm{top}} TM_1)$ is a suitably normalized, invariant nowhere vanishing section defined in the same manner as in Equation (4.1), and $\mathfrak{g}$ is the Lie algebra of the isotropy group G.

To conclude, we have arrived at an index formula for elliptic pseudodifferential operators on $\mathcal{H}$.

Theorem 4.6. *Suppose that $\mathcal{H} \rightrightarrows M$ is a boundary groupoid with two orbits and exponential isotropy, i.e., of the form (4.4) with $\dim G$ even, and a deformation from the pair groupoid exists for $\mathcal{H}$. Let Ψ be an elliptic pseudodifferential operator on $\mathcal{H}$. One has the index formula*

$$\begin{aligned}
\mathrm{ind}(\Psi) &= \int_{T^*M} \big\langle \hat{A}(T^*M) \wedge \mathrm{ch}(\sigma[\Psi]), \Omega_{\pi^! TM} \big\rangle \\
&\qquad \oplus \int_{T^*M_1 \times \mathfrak{g}} \big\langle \hat{A}(T^*M_1 \oplus \mathfrak{g}) \wedge \mathrm{ch}(\sigma[\Psi]|_{M_1}), \Omega'_{\pi^! TM_1} \big\rangle \\
&\in \mathbb{Z} \oplus \mathbb{Z} \\
&\cong K_0(C^*(\mathcal{H})).
\end{aligned}$$

4.3. The Fredholm index for fully elliptic operators

Lastly, recall the definition of fully elliptic operators in Definition 2.10. We suppose in this subsection that Ψ is a fully elliptic operator on $\mathcal{H} \rightrightarrows M$, where $\mathcal{H}$ is a boundary groupoid with two orbits and exponential isotropy.

Corollary 4.7. *The Fredholm index of Ψ is*

$$\mathrm{ind}_F(\Psi) = \int_{T^*M} \big\langle \hat{A}(T^*M) \wedge \mathrm{ch}(\sigma[\Psi]), \Omega_{\pi^! TM} \big\rangle.$$

Proof. Recall that Ψ is invertible modulo $C^*(M_0 \times M_0) \cong \mathcal{K}$ and its Fredholm index lies in $K_0(C^*(M_0 \times M_0))$. By Theorems 1.3, 1.2, and [12], we have

$$\begin{aligned}\operatorname{ind}_F(\Psi) &= \operatorname{ind}_{\mathcal{D}} \circ \varepsilon_{\mathcal{H},M_0}(\operatorname{ind}_F(\Psi)) \\ &= \operatorname{ind}_{\mathcal{D}}(\operatorname{ind}_{\mathcal{T}(\mathcal{H})}(\sigma(\Psi))) \\ &= \int_{T^*M} \langle \hat{A}(T^*M) \wedge \operatorname{ch}(\sigma[\Psi]), \Omega_{\pi^! TM} \rangle. \end{aligned}$$ ∎

Remark 4.8. One can replace the pair groupoid in Definition 3.1 by other groupoids. Most of the arguments in Section 4 still work in a more general setting. It would be interesting to see which class of groupoids can arise from this kind of (nontrivial) deformation, and what index formulae one can obtain.

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Yu Qiao
School of Mathematics and Statistics, Shaanxi Normal University, 620 West Chang'an Ave, Chang'an District, 710119 Xi'an, Shaanxi, P. R. China; yqiao@snnu.edu.cn

Bing Kwan So
School of Mathematics, Jilin University, 2699 Qianjin Ave, 130012 Changchun, Jilin, P. R. China; bkso@graduate.hku.hk