J. Noncommut. Geom. 19 (2025), 1515–1543
DOI 10.4171/JNCG/586

Quadratic algebras associated with exterior 3-forms

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Abstract. This paper is devoted to the study of the quadratic algebras with relations generated by superpotentials which are exterior 3-forms. Such an algebra is regular if and only if it is Koszul and is then a 3-Calabi–Yau domain. After some general results, we investigate the case of the algebras generated in low dimensions n with $n \leq 7$. We show that whenever the ground field is algebraically closed, all these algebras associated with 3-regular exterior 3-forms are regular and are thus 3-Calabi–Yau domains. This result does not generalize to dimensions n with $n \geq 8$: we describe a counterexample in dimension $n = 8$.

1. Introduction

Throughout $\mathbb{K}$ is a field and all vector spaces, algebras, tensor products, etc. are over $\mathbb{K}$. By an algebra without other specifications, we mean a unital associative $\mathbb{K}$-algebra. We use the Einstein convention of summation over repeated up-down indices in the formulas.

Let $\mathcal{A} = \bigoplus_{n\geq 0} \mathcal{A}_n$ be an $\mathbb{N}$-graded connected algebra; $\mathcal{A}_0 = \mathbb{K}\mathbb{1}$ is the trivial module and is identified with $\mathbb{K}$. The projective dimension of the trivial module is the global dimension of $\mathcal{A}$ and is also its Hochschild dimension (in homology as well as in cohomology) [2, 6]. The algebra $\mathcal{A}$ is said to be *regular* if its global dimension is finite, say, $\mathrm{gldim}(\mathcal{A}) = D$ $(\in \mathbb{N})$, and if

$$\mathrm{Ext}^k(\mathbb{K}, \mathcal{A}) = \begin{cases} \mathbb{K} & \text{if } k = D, \\ 0 & \text{if } k \neq D, \end{cases}$$

i.e., $\mathrm{Ext}^k(\mathbb{K}, \mathcal{A}) = \delta^k_D \mathbb{K}$.

In the following, we will be concerned about the connected algebras freely finitely generated in degree 1 with a finite number of relations r_i of degrees ≥ 2, thus by algebras of the form

$$\mathcal{A} = \mathbb{K}\langle x^1, \dots, x^n\rangle / [\{r_i\}],$$

where $\mathbb{K}\langle x^1, \dots, x^n\rangle$ is the free connected algebra generated by the x^k $(k \in \{1, \dots, n\})$ and where $[S]$ denotes the ideal of $\mathbb{K}\langle x^1, \dots, x^n\rangle$ generated by $S \subset \mathbb{K}\langle x^1, \dots, x^n\rangle$. Notice that $\mathbb{K}\langle x^1, \dots, x^n\rangle$ is canonically isomorphic to the tensor algebra $T(\mathbb{K}^n)$ of $\mathbb{K}^n$. When

Mathematics Subject Classification 2020: 16S37.
Keywords: algebra, noncommutative geometry.

all the relations r_i are of the same degree N, $\mathcal{A}$ is said to be an *N-homogeneous algebra*. One has the following result [3].

Proposition 1. *Let $\mathcal{A}$ be a regular algebra of global dimension D.*

(i) *If $D = 2$, then $\mathcal{A}$ is quadratic and Koszul.*

(ii) *If $D = 3$, then $\mathcal{A}$ is N-homogeneous with $N \geq 2$ and Koszul.*

The notion of Kozulity introduced in [19] for quadratic algebras has been extended in [1] for N-homogeneous algebras with $N \geq 2$.

For the case $D = 2$, one has the following complete description [23].

Proposition 2. *Let $\mathcal{A}$ be a regular algebra of global dimension 2; then,*

$$\mathcal{A} = \mathbb{K}\langle x^1, \dots, x^n \rangle / [b_{ij} x^i \otimes x^j], \tag{1.1}$$

where $b_{ij} = b(e_i, e_j)$ are the components of a nondegenerate bilinear form b on $\mathbb{K}^n$. Conversely, if b is a nondegenerate bilinear form on $\mathbb{K}^n$, Formula (1.1) defines a regular algebra of global dimension 2.

In order to state the similar result of the first part of the above proposition for the case of the global dimension $D = 3$, let us remind some definitions of [10] concerning multilinear forms on $\mathbb{K}^n$.

Let $Q \in \mathrm{GL}(n, \mathbb{K})$, and let m be an integer with $n \geq m \geq 2$. Then, an m-linear form w on $\mathbb{K}^n$ is said to be *Q-cyclic* if one has

$$w(X_1, \dots, X_m) = w(QX_m, X_1, \dots, X_{m-1})$$

for any $X_1, \dots, X_m \in \mathbb{K}^n$. An m-linear form w on $\mathbb{K}^n$ is said to be *preregular* if it satisfies the following conditions:

(i) $w(X, X_1, \dots, X_{m-1}) = 0$ for any $X_1, \dots, X_{m-1} \in \mathbb{K}^n$ implies $X = 0$,

(ii) there is an element $Q_w \in \mathrm{GL}(n, \mathbb{K})$ such that w is Q_w-cyclic.

In view of (i), Q_w is then unique and this twisted cyclicity implies that whenever $w(X_1, \dots, X_k, X, X_{k+1}, \dots, X_{m-1}) = 0$ for any $X_1, \dots, X_{m-1} \in \mathbb{K}^n$ then $X = 0$; this latter condition will be referred to as 1-*site nondegeneracy*.

It is worth noticing here that a bilinear form on $\mathbb{K}^n$ is preregular if and only if it is nondegenerate. Finally, we need another definition of [10]; namely, an m-linear form w on $\mathbb{K}^n$ will be said to be 3-*regular* if it is preregular and satisfies the following.

(iii) If $L_1, L_2 \in \mathrm{End}(\mathbb{K}^n)$ are such that

$$w(L_1X_1, X_2, X_3, \dots, X_m) = w(X_1, L_2X_2, X_3, \dots, X_m)$$

for any $X_1, \dots, X_m \in \mathbb{K}^n$, then $L_1 = L_2 = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{K}$.

We can now formulate the global dimension 3 version of the first part of Proposition 2 [9–11].

Proposition 3. *Let $\mathcal{A}$ be a regular algebra of global dimension 3; then,*

$$\mathcal{A} = \mathbb{K}\langle x^1, \dots, x^n\rangle / [\{w_{i\, i_1 \cdots i_N} x^{i_1} \otimes \cdots \otimes x^{i_N}\}],$$

where $w_{i_0 i_1 \cdots i_N} = w(e_{i_0}, \dots, e_{i_N})$ are the components of a 3-regular $(N+1)$-linear form w on $\mathbb{K}^n$ with $N \geq 2$.

We write for the relations in Proposition 3

$$\partial_i w = w_{i i_1 \cdots i_N} x^{i_1} \otimes \cdots \otimes x^{i_N}$$

for $i \in \{1, \dots, n\}$. This is a sort of derivative of w identified with

$$w = w_{i_0 \cdots i_N} x^{i_0} \otimes \cdots \otimes x^{i_N}$$

which is the generalization of the volume form and is referred to as the *superpotential* [4, 5, 22]. For the interpretation in terms of noncommutative volume, see [10, Proposition 10].

Propositions 2 and 3 generalize to higher global dimensions, but one has then to assume the N-Koszul property and one has to take higher-order derivations of the superpotential (a preregular multilinear form) to write the relations; see in [9, Theorem 4.3] and in [10, Theorem 11] and [5] for the generalization to the quiver case.

2. Exterior 3-forms

Let $(e_i)_{i\in\{1,\dots,n\}}$ be the canonical basis of $\mathbb{K}^n$, and let us equip $\mathbb{K}^n$ with the unique scalar product for which the canonical basis is orthonormal, that is, for which one has

$$(e_i, e_j) = \delta_{ij}$$

for $i, j \in \{1, \dots, n\}$. In the following, we assume that $n \geq 3$ since we are interested in exterior 3-forms.

Let α be an exterior 3-form on $\mathbb{K}^n$. To $\alpha \in \bigwedge^3 \mathbb{K}^n$ one associates n endomorphisms A_k of $\mathbb{K}^n$ by setting

$$A_k e_i = \sum_j \alpha_{jki} e_j = (A_k)^j_i e_j \tag{2.1}$$

for $k, i \in \{1, \dots, n\}$, where $\alpha_{jki} = \alpha(e_j, e_k, e_i)$ are the components of α. Since they are antisymmetric, the A_k are n elements of the Lie algebra $\mathfrak{so}(n, \mathbb{K})$ of $SO(n, \mathbb{K})$. The 3-form α is cyclic and therefore α is preregular if and only if it is *nondegenerate*, that is,

$$i_X(\alpha) = 0 \Rightarrow X = 0 \tag{2.2}$$

for $X \in \mathbb{K}^n$, where the 2-form $i_X(\alpha)$ is defined by

$$i_X(\alpha)(Y, Z) = \alpha(X, Y, Z)$$

for $Y, Z \in \mathbb{K}^n$. Condition (2.2) reads

$$A_k X = 0 \quad \forall k \Rightarrow X = 0$$

in terms of the A_k. Finally, α is 3-regular if and only if

$$MA_k = A_k N \quad \forall k \Rightarrow M = N = \lambda \mathbb{1}$$

for $M, N \in \mathrm{End}(\mathbb{K}^n)$ with $\lambda \in \mathbb{K}$. Thus, if α is 3-regular, the system (A_k) is irreducible (in view of the Schur lemma) which implies in particular that the $A_k \in \mathfrak{so}(n, \mathbb{K})$ generate the algebra $\mathrm{End}(\mathbb{K}^n)$ of the endomorphisms of $\mathbb{K}^n$ which contains as subspace the whole Lie algebra $\mathfrak{so}(n, \mathbb{K})$ of the antisymmetric endomorphisms of $\mathbb{K}^n$. However, one should be aware of the fact that this does not mean that $\mathfrak{so}(n, \mathbb{K})$ is generated *as Lie algebra* by the A_k. For instance, the Lie algebra $\mathfrak{g}_2(\mathbb{K})$ has an irreducible representation in $\mathbb{K}^7$ although it is a proper Lie sub-algebra of $\mathfrak{so}(7, \mathbb{K})$. In our general setting, there is no rule as shown by the 2 families of examples given below.

In the next section, we will define exterior 3-forms $\alpha^{(p)}$ in dimensions $2p + 1$ which are 3-regular and such that the corresponding $A_k^{(p)} \in \mathfrak{so}(2p + 1, \mathbb{K})$ generate the whole Lie algebra $\mathfrak{so}(2p + 1, \mathbb{K})$ of antisymmetric $(2p + 1) \times (2p + 1)$ matrices with coefficient in $\mathbb{K}$.

On the other hand, let α_{ijk} be the structure constants of the compact real form of a simple complex Lie algebra $\mathfrak{g}$ of dimension n in its standard orthonormal basis. Then, the α_{ijk} define a real 3-regular 3-form $\alpha \in \bigwedge^3 \mathbb{R}^n$ and the associated A_k span the adjoint representation of $\mathfrak{g}$, that is, the corresponding Lie-sub-algebra $\mathfrak{a}$ of $\mathfrak{so}(n, \mathbb{R})$. The Jacobi identity reads then

$$(A_k)^j_{i_1} \alpha_{j i_2 i_3} + (A_k)^j_{i_2} \alpha_{i_1 j i_3} + (A_k)^j_{i_3} \alpha_{i_1 i_2 j} = 0$$

for any $k \in \{1, \dots, n\}$, which means that the Lie sub-algebra $\mathfrak{a}$ of $\mathfrak{so}(n, \mathbb{R})$ generated by the A_k preserves the exterior 3-form α. This is clearly not the case for the whole Lie algebra $\mathfrak{so}(n, \mathbb{R})$ except for the dimension $n = 3$, i.e., for $\mathfrak{g} = \mathfrak{a}_1$ which corresponds to the exterior 3-form $\alpha^{(1)}$. The 3-regularity of α follows from the irreducibility of the adjoint representation of $\mathfrak{g}$.

Remark. The dimension 4 is an exception since any 3-form $\alpha \in \bigwedge^3 \mathbb{K}^4$ is of the form $\alpha = i_X(\mathrm{vol}_4)$ for some $X \in \mathbb{K}^4$, where the volume form $\mathrm{vol}_4 \in \bigwedge^4 \mathbb{K}^4$ is defined by $\mathrm{vol}_4(e_1, e_2, e_3, e_4) = 1$ which implies that $\mathrm{vol}_4(e_i, e_j, e_k, e_\ell) = \varepsilon_{ijk\ell}$. Therefore, an element $\alpha = i_X(\mathrm{vol}_4) \in \bigwedge^3 \mathbb{K}^4$ is degenerate since then $i_X(\alpha) = 0$ for $X \in \mathbb{K}^4$ with $X \neq 0$ if $\alpha \neq 0$ and thus no nontrivial $\alpha \in \bigwedge^3 \mathbb{K}^4$ can be 3-regular. This has a counterpart on the side of the Lie algebra $\mathfrak{so}(4, \mathbb{K})$ which is not simple since $\mathfrak{so}(4, \mathbb{K}) = \mathfrak{so}(3, \mathbb{K}) \oplus \mathfrak{so}(3, \mathbb{K})$ while $\mathfrak{so}(3, \mathbb{K})$ and $\mathfrak{so}(n, \mathbb{K})$ for $n \geq 5$ are simple Lie algebras.

The first interesting nontrivial case can only occur in dimension 5, and it turns out that it consists in the $\mathrm{GL}(5, \mathbb{K})$ orbit of the exterior 3-form $\alpha^{(2)}$.

3. Exterior 3-forms as superpotentials

In this paper, we are interested in the regularity of quadratic algebras with relations generated by exterior 3-forms, that is, by algebras of the form

$$\mathcal{A} = \mathbb{K}\langle x^1, \dots, x^n \rangle / [\{\alpha_{ijk} x^j \otimes x^k\}], \tag{3.1}$$

where the α_{ijk} are the components of an exterior 3-form α, i.e., are completely antisymmetric. Since by Proposition 3 we know that the 3-regularity of α is a necessary condition for the regularity of $\mathcal{A}$, we are led to introduce a list of 3-regular 3-forms α in $\mathbb{K}^n$ and to study the regularity of the corresponding quadratic algebras given by (3.1). We will use freely the following lemma.

Lemma 4. *Let α be an exterior 3-regular 3-form on $\mathbb{K}^n$. Then, the quadratic algebra*

$$\mathcal{A} = \mathbb{K}\langle x^1, \dots, x^n \rangle / [\{\partial_i \alpha\}]$$

is a domain and the following statements are equivalent:

(i) $\mathcal{A}$ *is Koszul,*

(ii) $\mathcal{A}$ *is regular,*

(iii) $\mathcal{A}$ *is 3-Calabi–Yau.*

Sketch of proof. The algebra $\mathcal{A}$ is a domain because it is the universal enveloping algebra of a Lie algebra since the relations involve only commutators [15]. The equivalence (i)⇔(ii) follows directly from [10, Proposition 16]. Concerning the last equivalence, since α is cyclic, $\mathcal{A}$ is Calabi–Yau [12] whenever it is regular [4]. ■

Let $\alpha \in \bigwedge^3 \mathbb{K}^n$, and let $\mathcal{A}$ be the quadratic algebra defined by (3.1). One defines an antisymmetric $n \times n$-matrix $A(x)$ with entries in $\mathcal{A}$ by setting

$$A(x) = A_k x^k, \tag{3.2}$$

where the A_k are the matrices given by (2.1), that is, by $(A_k)^i_j = \alpha_{ikj}$. The relations $\partial_i \alpha$ of $\mathcal{A}$ can be expressed as

$$\begin{pmatrix} \partial_1 \alpha \\ \vdots \\ \partial_n \alpha \end{pmatrix} = A(x) \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} \tag{3.3}$$

or simply by $\partial x = A(x)x$, where ∂x and x denote the corresponding columns. Let us assume now that α is 3-regular; then, the augmented Koszul complex of $\mathcal{A}$ reads as follows from [10]:

$$0 \to \mathcal{A} \xrightarrow{x^t} \mathcal{A}^n \xrightarrow{A(x)} \mathcal{A}^n \xrightarrow{x} \mathcal{A} \xrightarrow{\varepsilon} \mathbb{K} \to 0,$$

where x^t means multiplication in $\mathcal{A}$ by the transposed x^t of the column x, $A(x)$ means multiplication in $\mathcal{A}$ of the line $\mathcal{A}^n$ of elements of $\mathcal{A}$ with the $n \times n$-matrix $A(x)$ of elements of $\mathcal{A}$, x means multiplication in $\mathcal{A}$ of the line $\mathcal{A}^n$ of elements of $\mathcal{A}$ by the column x while ε is the projection onto degree 0.

The exactness of the sequence

$$0 \to \mathcal{A} \xrightarrow{x^t} \mathcal{A}^n$$

follows from the fact that $\mathcal{A}$ is a domain while the exactness of the sequence

$$\mathcal{A}^n \xrightarrow{A(x)} \mathcal{A}^n \xrightarrow{x} \mathcal{A} \xrightarrow{\varepsilon} \mathbb{K} \to 0$$

just expresses the definition of $\mathcal{A}$ by generators and relations. Therefore, $\mathcal{A}$ is Koszul if and only if the sequence

$$\mathcal{A} \xrightarrow{x^t} \mathcal{A}^n \xrightarrow{A(x)} \mathcal{A}^n$$

is exact.

It is clear that the quadratic algebras $\mathcal{A}$ of the form (3.1) do only depend up to isomorphism on the 3-forms α up to the $\mathrm{GL}(n,\mathbb{K})$ action. Thus, we are only interested in the $\mathrm{GL}(n,\mathbb{K})$ orbits of 3-forms in $\mathbb{K}^n$. Notice that the 3-regularity of a 3-form is an invariant notion as well as the regularity of the corresponding algebra. In Section 4, we will analyze the orbits of nondegenerate exterior 3-forms for $\mathbb{K} = \mathbb{C}$ by choosing a convenient element in each orbit for dimension n with $3 \leq n \leq 7$, using the results of [8]. We will then select the 3-regular orbits and corresponding 3-regular representative exterior 3-forms. The regularity of the corresponding quadratic algebras will be investigated in Section 5.

We now analyze a particular family of exterior 3-forms $\alpha^{(p)} \in \bigwedge^3 \mathbb{K}^{2p+1}$ and show that the corresponding quadratic algebras $\mathcal{A}^{(p)}$ are regular of global dimension 3. Let us define $\alpha^{(p)}$ by

$$\alpha^{(p)} = \sum_{m=1}^{p} \theta^m \wedge \theta^{m+p} \wedge \theta^{2p+1} = \alpha_{ijk}\theta^i \otimes \theta^j \otimes \theta^k, \tag{3.4}$$

where (θ^i) is the dual basis of the basis (e_j) of $\mathbb{K}^{2p+1}$. It is easily verified that $\alpha^{(p)}$ is 3-regular. The corresponding superpotential is $\alpha^{(p)} = \alpha_{ijk}x^i \otimes x^j \otimes x^k$, and the relations $\partial_i\alpha^{(p)}$ of the corresponding quadratic algebra $\mathcal{A}^{(p)}$ read

$$\begin{aligned} \partial_m\alpha^{(p)} &= x^{m+p} \otimes x^{2p+1} - x^{2p+1} \otimes x^{m+p}, \\ \partial_{m+p}\alpha^{(p)} &= x^{2p+1} \otimes x^m - x^m \otimes x^{2p+1} \end{aligned} \tag{3.5}$$

for $1 \leq m \leq p$ and

$$\partial_{2p+1}\alpha^{(p)} = \sum_{r=1}^{p} x^r \otimes x^{r+p} - x^{r+p} \otimes x^r \tag{3.6}$$

and imply that x^{2p+1} is in the center of $\mathcal{A}^{(p)}$ in view of (3.5) and that the x^ℓ for $\ell \in \{1,\dots,2p\}$ span the quadratic sub-algebra

$$\mathcal{B}^{(p)} = \mathbb{K}\langle x^1,\dots,x^{2p}\rangle/[\partial_{2p+1}\alpha^{(p)}] \tag{3.7}$$

of $\mathcal{A}^{(p)}$. Since $\partial_{2p+1}\alpha^{(p)}$ given by (3.6) is obviously the superpotential corresponding to a nondegenerate bilinear form on $\mathbb{K}^{2p}$, $\mathcal{B}^{(p)}$ is a regular quadratic algebra of global dimension 2. Thus, one has

$$\mathcal{A}^{(p)} = \mathcal{B}^{(p)} \otimes \mathbb{K}[x^{2p+1}],$$

which implies that $\mathcal{A}^{(p)}$ is regular of global dimension 3.

Let us now show that the $(2p+1)\times(2p+1)$ antisymmetric matrices $A_k^{(p)} \in \mathfrak{so}(2p+1,\mathbb{K})$ associated (via (2.1)) to $\alpha^{(p)} \in \bigwedge^3 \mathbb{K}^{2p+1}$ generate as Lie algebra the whole Lie algebra $\mathfrak{so}(2p+1,\mathbb{K})$. One defines

$$A(u) = \begin{pmatrix} 0_p & -u^{2p+1}\mathbb{1}_p & \begin{matrix} u^{2p} \\ \vdots \\ u^{p+1} \end{matrix} \\ u^{2p+1}\mathbb{1}_p & 0_p & \begin{matrix} -u^{1} \\ \vdots \\ -u^{p} \end{matrix} \\ -u^{2p}\cdots - u^{p+1} & u^1\cdots u^p & 0 \end{pmatrix}$$

by setting

$$A(u) = A_k^{(p)} u^k$$

for $u \in \mathbb{K}^{2p+1}$. Then, by computing the commutator $[A(u), A(v)]$ for $u, v \in \mathbb{K}^{2p+1}$, one verifies that the components of the antisymmetric matrix $[A(u), A(v)]$ are just a permutation of the components of the exterior product $u \wedge v$. It is clear that the linear span of the $u \wedge v$ for $u, v \in \mathbb{K}^{2p+1}$ is the whole Lie algebra $\mathfrak{so}(2p+1,\mathbb{K})$ of antisymmetric $(2p+1)\times(2p+1)$-matrices, which is therefore also the case for the $[A(u), A(v)]$ for $u, v \in \mathbb{K}^{2p+1}$.

We summarize the above results concerning the $\alpha^{(p)}$ by the following proposition.

Proposition 5. *Let $\alpha^{(p)} \in \bigwedge^3 \mathbb{K}^{2p+1}$ be defined by (3.4), and let $\mathcal{A}^{(p)}$ be the associated quadratic algebra as in (3.1) and $A_k^{(p)}$ the corresponding endomorphisms of $\mathbb{K}^{2p+1}$ as in (2.1). Then, one has the following.*

(1) *The quadratic algebra $\mathcal{A}^{(p)}$ is regular of global dimension* 3.

(2) *The $A_k^{(p)}$, $k \in \{1,\dots,2p+1\}$ generate the Lie-algebra $\mathfrak{so}(2p+1,\mathbb{K})$.*

4. Exterior 3-forms in $\mathbb{K}^n$ with $3 \le n \le 7$

In this section, $\mathbb{K}$ is algebraically closed. We will use the results of [8] for $3 \le n \le 7$ and the choices of representative elements in $\bigwedge^3 \mathbb{K}^n$ for the $\mathrm{GL}(n, \mathbb{K})$-orbits of nondegenerate (i.e., preregular) exterior 3-forms; see also in [13]. However, the analysis of the regularity of these elements is independent of the assumption that $\mathbb{K}$ is algebraically closed.

In dimension $n = 3$, the only nontrivial orbit is the one of $\alpha^{(1)} = \theta^1 \wedge \theta^2 \wedge \theta^3$. The corresponding superpotential is

$$\alpha^{(1)} = \varepsilon_{ijk} x^i \otimes x^j \otimes x^k,$$

and the quadratic algebra $\mathcal{A}^{(1)}$ is the commutative algebra

$$\mathcal{A}^{(1)} = \mathbb{K}[x^1, x^2, x^3]$$

of polynomials in the x^i, $i \in \{1, 2, 3\}$. By setting $A^{(1)}(x^1, x^2, x^3) = A^{(1)}_k x^k$, one has

$$A^{(1)}(x^1, x^2, x^3) = \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{pmatrix} \tag{4.1}$$

for the corresponding matrix.

In dimension $n = 4$, all exterior 3-forms are degenerate.

In dimension $n = 5$, the only orbit of nondegenerate exterior 3-forms is the one of $\alpha^{(2)} = (\theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^4) \wedge \theta^5 = \alpha^{(2)}_{ijk} \theta^i \wedge \theta^j \wedge \theta^k$ which is 3-regular. Moreover, the associated quadratic algebra $\mathcal{A}^{(2)}$ is regular of global dimension 3 (see the last section).

In dimension $n = 6$, there are 2 orbits of nondegenerate exterior 3-forms: namely, the orbit of $\gamma = \theta^1 \wedge \theta^2 \wedge \theta^3 + \theta^4 \wedge \theta^5 \wedge \theta^6$ and the orbit of

$$\omega = \theta^1 \wedge \theta^2 \wedge \theta^6 + \theta^3 \wedge \theta^1 \wedge \theta^5 + \theta^2 \wedge \theta^3 \wedge \theta^4.$$

None of these 3-forms is 3-regular, so the associated quadratic algebras $\mathcal{A}_\gamma$ and $\mathcal{A}_\omega$ cannot be regular. Indeed, γ is disconnected, that is, the sum of two exterior 3-forms in two different spaces $\mathbb{K}^3$ with bases (e_1, e_2, e_3) and (e_4, e_5, e_6), respectively, so the commutant of the corresponding

$$A_\gamma(x) = \begin{pmatrix} A^{(1)}(x^1, x^2, x^3) & 0_3 \\ 0_3 & A^{(1)}(x^4, x^5, X^6) \end{pmatrix}$$

contains the matrices

$$\begin{pmatrix} \lambda \mathbb{1}_3 & 0_3 \\ 0_3 & \mu \mathbb{1}_3 \end{pmatrix}$$

for any $\lambda, \mu \in \mathbb{K}$ which implies that γ is not 3-regular, while for ω, one has

$$A_\omega(x) = \begin{pmatrix} A^{(1)}(x^4, x^5, x^6) & -A^{(1)}(x^1, x^2, x^3)^t \\ A^{(1)}(x^1, x^2, x^3) & 0_3 \end{pmatrix},$$

where $A^{(1)}$ is defined by (4.1), so one has for any $\mu \in \mathbb{K}$

$$\begin{pmatrix} 0_3 & \mu \mathbb{1}_3 \\ 0_3 & 0_3 \end{pmatrix} A_\omega = A_\omega \begin{pmatrix} 0_3 & 0_3 \\ \mu \mathbb{1}_3 & 0_3 \end{pmatrix},$$

which implies that ω is not 3-regular.

In dimension $n = 7$, the situation is much more elaborated. There are five orbits of non-degenerate exterior 3-forms, namely, the orbits of the following exterior 3-forms:

$$\begin{aligned}
\rho &= \theta^1 \wedge \theta^2 \wedge \theta^3 + \theta^2 \wedge \theta^4 \wedge \theta^6 + \theta^3 \wedge \theta^5 \wedge \theta^7, \\
\beta &= (\theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^4) \wedge \theta^5 + \theta^1 \wedge \theta^7 \wedge \theta^2 + \theta^3 \wedge \theta^6 \wedge \theta^4, \\
\alpha^{(3)} &= (\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^5 + \theta^3 \wedge \theta^6) \wedge \theta^7, \\
\alpha^{(3)\prime} &= \alpha^{(3)} + \theta^1 \wedge \theta^2 \wedge \theta^3, \\
\alpha^{(3)\prime\prime} &= \alpha^{(3)} + \theta^1 \wedge \theta^2 \wedge \theta^3 + \theta^4 \wedge \theta^5 \wedge \theta^6,
\end{aligned}$$

where we have used the notation of Section 3 for $\alpha^{(3)}$. It is worth noticing here that one can also express β as

$$\beta = \alpha^{(2)} + \theta^1 \wedge \theta^7 \wedge \theta^2 + \theta^3 \wedge \theta^6 \wedge \theta^4$$

with the notation of Section 3. The correspondence with the notations of [8] is $\alpha^{(1)} = f_1, \alpha^{(2)} = f_2, \gamma = f_3, \omega = f_4, \rho = f_5, \beta = f_7, \alpha^{(3)} = f_8, \alpha^{(3)\prime} = f_6$, and $\alpha^{(3)\prime\prime} = f_9$.

The exterior 3-forms ρ and β are 3-regular while $\alpha^{(3)}$, $\alpha^{(3)\prime}$, and $\alpha^{(3)\prime\prime}$ span an affine plane

$$\alpha^{(t_0,t_1,t_2)} = t_0 \alpha^{(3)} + t_1 \alpha^{(3)\prime} + t_2 \alpha^{(3)\prime\prime}, \quad t_0 + t_1 + t_2 = 1$$

of 3-regular exterior 3-forms to which are associated a corresponding family $\mathcal{A}^{(t_0,t_1,t_2)}$ of quadratic 3-Calabi–Yau algebras of the type investigated in [14]. Namely, the $\mathcal{A}^{(t_0,t_1,t_2)}$ are the cross-products of the 2-Calabi–Yau algebra $\mathcal{B}^{(3)}$ defined by (3.7) with the derivations $t_1\delta_1 + t_2\delta_2$, (i.e., are the Ore extensions of $\mathcal{B}^{(3)}$ associated with data $(I, t_1\delta_1 + t_2\delta_2)$). Notice that $\alpha^{(3)\prime\prime}$ is in the orbit of the generic 3-forms, so the associated quadratic algebra is isomorphic to the algebra with G_2-symmetry defined in [21].

For $\alpha = \rho$, the matrix $A(x)$ given by (3.2) reads

$$A_\rho(x) = \begin{pmatrix}
0 & -x^3 & x^2 & 0 & 0 & 0 & 0 \\
x^3 & 0 & -x^1 & -x^6 & 0 & x^4 & 0 \\
-x^2 & x^1 & 0 & 0 & -x^7 & 0 & x^5 \\
0 & x^6 & 0 & 0 & 0 & -x^2 & 0 \\
0 & 0 & x^7 & 0 & 0 & 0 & -x^3 \\
0 & -x^4 & 0 & x^2 & 0 & 0 & 0 \\
0 & 0 & -x^5 & 0 & x^3 & 0 & 0
\end{pmatrix} \tag{4.2}$$

and the relations of the associated quadratic algebra $\mathcal{A}_\rho$ read

$$\begin{aligned}
\partial_1\rho &= x^2\otimes x^3 - x^3\otimes x^2,\\
\partial_2\rho &= x^3\otimes x^1 - x^1\otimes x^3 + x^4\otimes x^6 - x^6\otimes x^4,\\
\partial_3\rho &= x^1\otimes x^2 - x^2\otimes x^1 + x^5\otimes x^7 - x^7\otimes x^5,\\
\partial_4\rho &= x^6\otimes x^2 - x^2\otimes x^6,\\
\partial_5\rho &= x^7\otimes x^3 - x^3\otimes x^7,\\
\partial_6\rho &= x^2\otimes x^4 - x^4\otimes x^2,\\
\partial_7\rho &= x^3\otimes x^5 - x^5\otimes x^3.
\end{aligned}$$

One can show by direct computation (or by using computer) that, for matrices $M, N \in M_7(\mathbb{K})$, the relation

$$MA_\rho(x) = A_\rho(x)N$$

implies that $M = N = \lambda\mathbb{1}_7$ for some $\lambda \in \mathbb{K}$, so the exterior 3-form ρ is 3-regular.

For $\alpha = \beta$, one has for the corresponding matrix $A(x) = A_\beta(x)$

$$A_\beta(x) = \begin{pmatrix} 0 & x^7 & -x^5 & 0 & x^3 & 0 & -x^2\\ -x^7 & 0 & 0 & -x^5 & x^4 & 0 & x^1\\ x^5 & 0 & 0 & x^6 & -x^1 & -x^4 & 0\\ 0 & x^5 & -x^6 & 0 & -x^2 & x^3 & 0\\ -x^3 & -x^4 & x^1 & x^2 & 0 & 0 & 0\\ 0 & 0 & x^4 & -x^3 & 0 & 0 & 0\\ x^2 & -x^1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.3}$$

and the relations of the associated quadratic algebra $\mathcal{A}_\beta$ read

$$\begin{aligned}
\partial_1\beta &= x^3\otimes x^5 - x^5\otimes x^3 + x^7\otimes x^2 - x^2\otimes x^7,\\
\partial_2\beta &= x^4\otimes x^5 - x^5\otimes x^4 + x^1\otimes x^7 - x^7\otimes x^1,\\
\partial_3\beta &= x^5\otimes x^1 - x^1\otimes x^5 + x^6\otimes x^4 - x^4\otimes x^6,\\
\partial_4\beta &= x^5\otimes x^2 - x^2\otimes x^5 + x^3\otimes x^6 - x^6\otimes x^3,\\
\partial_5\beta &= x^1\otimes x^3 - x^3\otimes x^1 + x^2\otimes x^4 - x^4\otimes x^2,\\
\partial_6\beta &= x^4\otimes x^3 - x^3\otimes x^4,\\
\partial_7\beta &= x^2\otimes x^1 - x^1\otimes x^2.
\end{aligned}$$

Thus, it follows that $\mathcal{A}_\beta$ has 2 non-intersecting sub-algebras, namely, the quadratic sub-algebra $\mathcal{B}$ generated by x^1, x^2, x^3, x^4 with relations $\partial_5\beta$, $\partial_6\beta$, $\partial_7\beta$ and the sub-algebra $\mathcal{C} = \mathbb{K}\langle x^5, x^6, x^7\rangle$ freely generated by x^5, x^6, x^7 which is isomorphic to the tensor algebra $T(\mathbb{K}^3)$ and is a Koszul algebra of global dimension one. Concerning the sub-algebra $\mathcal{B}$, it is a Koszul algebra of global dimension 2. Indeed, by introducing the

3×4-matrix $B(x)$ with coefficients in $\mathcal{B}$ defined by

$$B(x) = \begin{pmatrix} -x^3 & -x^4 & x^1 & x^2 \\ 0 & 0 & x^4 & -x^3 \\ x^2 & -x^1 & 0 & 0 \end{pmatrix},$$

the augmented Koszul complex of $\mathcal{B}$ reads

$$0 \to \mathcal{B}^3 \xrightarrow{B(x)} \mathcal{B}^4 \xrightarrow{x} \mathcal{B} \xrightarrow{\varepsilon} \mathbb{K} \to 0$$

and, by using the fact that $\mathcal{B}$ is a domain since it is a universal enveloping algebra of a Lie algebra, it is easy to show that this is an exact sequence, that is, that the mapping $\mathcal{B}^3 \to \mathcal{B}^4$ is injective. Thus, $\mathcal{A}_\beta$ is generated by the two non-intersecting sub-algebra $\mathcal{B}$ and $\mathcal{C}$ which are Koszul of global dimensions $\mathrm{gldim}(\mathcal{B}) = 2$ and $\mathrm{gldim}(\mathcal{C}) = 1$. The relations $\partial_1\beta, \partial_2\beta, \partial_3\beta, \partial_4\beta$ give commutation relations between the generators of $\mathcal{B}$ and generators of $\mathcal{C} = \mathbb{K}\langle x^5, x^6, x^7\rangle$.

Finally, one can show by using computer that for matrices $M, N \in M_7(\mathbb{K})$ the relation

$$MA_\beta(x) = A_\beta(x)N$$

implies $M = N = \lambda \mathbb{1}_7$ for some $\lambda \in \mathbb{K}$, which means that β is 3-regular.

We now discuss the cases of $\alpha^{(3)\prime}$ and $\alpha^{(3)\prime\prime}$.

For $\alpha^{(3)}$, this is the discussion of Section 3 for $p = 3$. One has $\mathcal{A}^{(3)} = \mathcal{B}^{(3)} \otimes \mathbb{K}[x^7]$ which is the cross product of $\mathcal{B}^{(3)}$ with the trivial derivation $\delta_0 = 0$.

For $\alpha^{(3)\prime}$, one has

$$A^{(3)\prime}(x) = \begin{pmatrix} 0 & -x^3 & x^2 & -x^7 & 0 & 0 & x^4 \\ x^3 & 0 & -x^1 & 0 & -x^7 & 0 & x^5 \\ -x^2 & x^1 & 0 & 0 & 0 & -x^7 & x^6 \\ x^7 & 0 & 0 & 0 & 0 & 0 & -x^1 \\ 0 & x^7 & 0 & 0 & 0 & 0 & -x^2 \\ 0 & 0 & x^7 & 0 & 0 & 0 & -x^3 \\ -x^4 & -x^5 & -x^6 & x^1 & x^2 & x^3 & 0 \end{pmatrix}$$

for the corresponding matrix $A(x) = A^{(3)\prime}(x)$, and the relations of the associated quadratic algebra $\mathcal{A}^{(3)\prime}$ read

$$\begin{aligned}
\partial_1\alpha^{(3)\prime} &= x^4 \otimes x^7 - x^7 \otimes x^4 + x^2 \otimes x^3 - x^3 \otimes x^2, \\
\partial_2\alpha^{(3)\prime} &= x^5 \otimes x^7 - x^7 \otimes x^5 + x^3 \otimes x^1 - x^1 \otimes x^3, \\
\partial_3\alpha^{(3)\prime} &= x^6 \otimes x^7 - x^7 \otimes x^6 + x^1 \otimes x^2 - x^2 \otimes x^1, \\
\partial_4\alpha^{(3)\prime} &= \partial_4\alpha^{(3)} = x^7 \otimes x^1 - x^1 \otimes x^7, \\
\partial_5\alpha^{(3)\prime} &= \partial_5\alpha^{(3)} = x^7 \otimes x^2 - x^2 \otimes x^7, \\
\partial_6\alpha^{(3)\prime} &= \partial_6\alpha^{(3)} = x^7 \otimes x^3 - x^3 \otimes x^7, \\
\partial_7\alpha^{(3)\prime} &= x^1 \otimes x^4 - x^4 \otimes x^1 + x^2 \otimes x^5 - x^5 \otimes x^2 + x^3 \otimes x^6 - x^6 \otimes x^3 \\
&= \partial_7\alpha^{(3)}.
\end{aligned}$$

Now, let $\mathcal{B}^{(3)}$ be the quadratic algebra $\mathcal{B}^{(p)}$ of Section 3 for $p = 3$, that is,

$$\mathcal{B}^{(3)} = \mathbb{K}\langle x^1, \dots, x^6\rangle/[\partial_7\alpha^{(3)}]$$

which is a regular algebra of global dimension 2; then,

$\delta_1 x^1 = x^2 \otimes x^3 - x^3 \otimes x^2$, $\delta_1 x^2 = x^3 \otimes x^1 - x^1 \otimes x^3$, $\delta_1 x^3 = x^1 \otimes x^2 - x^2 \otimes x^1$,

$\delta_1 x^4 = 0$, $\delta_1 x^5 = 0$, and $\delta_1 x^6 = 0$ defines a derivation of degree 1 of $\mathbb{K}\langle x^1, \dots, x^6\rangle$ which satisfies

$$\delta_1(\partial_7\alpha^{(3)}) = 0$$

in view of the associativity of the tensor product. It follows that δ_1 defines a derivation of degree 1, again denoted by δ_1, of $\mathcal{B}^{(3)}$. Then, the relations of $\mathcal{A}^{(3)\prime}$ imply that $\mathcal{A}^{(3)\prime}$ is the cross product

$$\mathcal{A}^{(3)\prime} = \mathcal{B}^{(3)} \rtimes_{\delta_1} \mathbb{K}[x^7]$$

of $\mathcal{B}^{(3)}$ with the derivation δ_1, the corresponding new generator being x^7, ($\delta_1 = \mathrm{ad}(x^7)$).

One proceeds similarly for $\alpha^{(3)\prime\prime}$; one has

$$A^{(3)\prime\prime}(x) = \begin{pmatrix} 0 & -x^3 & x^2 & -x^7 & 0 & 0 & x^4 \\ x^3 & 0 & -x^1 & 0 & -x^7 & 0 & x^5 \\ -x^2 & x^1 & 0 & 0 & 0 & -x^7 & x^6 \\ x^7 & 0 & 0 & 0 & -x^6 & x^5 & -x^1 \\ 0 & x^7 & 0 & x^6 & 0 & -x^4 & -x^2 \\ 0 & 0 & x^7 & -x^5 & x^4 & 0 & -x^3 \\ -x^4 & -x^5 & -x^6 & x^1 & x^2 & x^3 & 0 \end{pmatrix}$$

while the relations of $\mathcal{A}^{(3)\prime\prime}$ read

$$\begin{aligned}
\partial_1\alpha^{(3)\prime\prime} &= \partial_1\alpha^{(3)\prime} = x^4 \otimes x^7 - x^7 \otimes x^4 + x^2 \otimes x^3 - x^3 \otimes x^2,\\
\partial_2\alpha^{(3)\prime\prime} &= \partial_2\alpha^{(3)\prime} = x^5 \otimes x^7 - x^7 \otimes x^5 + x^3 \otimes x^1 - x^1 \otimes x^3,\\
\partial_3\alpha^{(3)\prime\prime} &= \partial_3\alpha^{(3)\prime} = x^6 \otimes x^7 - x^7 \otimes x^6 + x^1 \otimes x^2 - x^2 \otimes x^1,\\
\partial_4\alpha^{(3)\prime\prime} &= x^7 \otimes x^1 - x^1 \otimes x^7 + x^5 \otimes x^6 - x^6 \otimes x^5,\\
\partial_5\alpha^{(3)\prime\prime} &= x^7 \otimes x^2 - x^2 \otimes x^7 + x^6 \otimes x^4 - x^4 \otimes x^6,\\
\partial_6\alpha^{(3)\prime\prime} &= x^7 \otimes x^3 - x^3 \otimes x^7 + x^4 \otimes x^5 - x^5 \otimes x^4,\\
\partial_7\alpha^{(3)\prime\prime} &= \partial_7\alpha^{(3)\prime} = \partial_7\alpha^{(3)}.
\end{aligned}$$

One defines a derivation δ_2 of degree 1 of $\mathbb{K}\langle x^1, \dots, x^6\rangle$ by setting

$$\begin{aligned}
\delta_2 x^1 &= x^2 \otimes x^3 - x^3 \otimes x^2,\\
\delta_2 x^2 &= x^3 \otimes x^1 - x^1 \otimes x^3,\\
\delta_2 x^3 &= x^1 \otimes x^2 - x^2 \otimes x^1,
\end{aligned}$$

$$\delta_2 x^4 = x^5 \otimes x^6 - x^6 \otimes x^5,$$
$$\delta_2 x^5 = x^6 \otimes x^4 - x^4 \otimes x^6,$$
$$\delta_2 x^6 = x^4 \otimes x^5 - x^5 \otimes x^2$$

which satisfies again

$$\delta_2(\partial_7 \alpha^{(3)}) = 0$$

in view of the associativity of the tensor product. Therefore, δ_2 passes to the quotient and defines a derivation of degree 1, again denoted by δ_2, of the regular algebra $\mathcal{B}^{(3)}$. Thus, $\mathcal{A}^{(3)\prime\prime}$ is the cross product

$$\mathcal{A}^{(3)\prime\prime} = \mathcal{B}^{(3)} \rtimes_{\delta_2} \mathbb{K}[x^7]$$

of $\mathcal{B}^{(3)}$ with the derivation δ_2, the corresponding “new" generator of degree 1 being x^7.

Remark. Since the relations of all the quadratic algebras of this paper have relations defined in terms of commutators, it follows that they are the universal enveloping algebras of graded Lie algebras generated in degree 1 as already pointed out. The cross-products with derivations considered above correspond to the universal enveloping algebra versions of semi-direct products of the graded Lie algebras with derivations of degree 1. Thus, $\mathcal{B}^{(3)}$ is the universal enveloping algebra of the corresponding graded Lie algebra $\mathfrak{b}^{(3)}$ and the $\delta_i, i \in \{0,1,2\}$ are in fact derivations of degree 1 of $\mathfrak{b}^{(3)}$ while $\mathcal{A}^{(3)}$, $\mathcal{A}^{(3)\prime}$, $\mathcal{A}^{(3)\prime\prime}$ are the universal enveloping algebras of the semi-direct products of the Lie algebra $\mathfrak{b}^{(3)}$ with the derivations $\delta_0, \delta_1, \delta_2$.

5. Regularity

The cross product with a derivation of degree 1 is a particular case of the graded Ore extension which preserves the Koszul property, and even more, it preserves the $\mathcal{K}_2$ property [7] in a very strong sense [18]. It follows that the quadratic algebras $\mathcal{A}^{(3)}$, $\mathcal{A}^{(3)\prime}$, $\mathcal{A}^{(3)\prime\prime}$ associated with the exterior 3-forms $\alpha^{(3)}$, $\alpha^{(3)\prime}$, $\alpha^{(3)\prime\prime} \in \bigwedge^3 \mathbb{K}^7$ are Koszul and therefore are regular in view of Lemma 4. This is also true for the quadratic algebras $\mathcal{A}^{(p)}$ associated with the exterior 3-forms $\alpha^{(p)} \in \bigwedge^3 \mathbb{K}^{2p+1}$ $(p \geq 1)$ as explained in the second part of Section 3. All these quadratic algebras are particular cases of the ones of [14].

It remains to discuss the case of the quadratic algebras $\mathcal{A}_\rho$ and $\mathcal{A}_\beta$ associated with the 3-forms ρ and $\beta \in \bigwedge^3 \mathbb{K}^7$.

The matrix $A_\rho(x)$ is given by (4.2), so the relations of the algebra $\mathcal{A}_\rho$ read

$$[x^2, x^3] = 0, \tag{5.1}$$
$$[x^3, x^1] + [x^4, x^6] = 0, \tag{5.2}$$
$$[x^1, x^2] + [x^5, x^7] = 0, \tag{5.3}$$
$$[x^6, x^2] = 0, \tag{5.4}$$

$$[x^7, x^3] = 0, \tag{5.5}$$
$$[x^2, x^4] = 0, \tag{5.6}$$
$$[x^3, x^5] = 0 \tag{5.7}$$

for the generators x^k ($k \in \{1, \dots, 7\}$) of $\mathcal{A}_\rho$.

Proposition 6. *The sequence*

$$\mathcal{A}_\rho \xrightarrow{x^t} \mathcal{A}_\rho^7 \xrightarrow{A_\rho(x)} \mathcal{A}_\rho^7$$

is exact.

Proof. It is sufficient to prove that

$$(a^1, a^2, a^3, a^4, a^5, a^6, a^7) A_\rho(x) = 0 \tag{5.8}$$

for $(a^k) \in \mathcal{A}_\rho^7$ implies that

$$a^k = ax^k \quad \forall k \in \{1, \dots, 7\} \tag{5.9}$$

for some $a \in \mathcal{A}_\rho$.

Equation (5.8) reads by using (4.2)

$$a^2x^3 - a^3x^2 = 0, \tag{5.10}$$
$$-a^1x^3 + a^3x^1 + a^4x^6 - a^6x^4 = 0, \tag{5.11}$$
$$a^1x^2 - a^2x^1 + a^5x^7 - a^7x^5 = 0, \tag{5.12}$$
$$-a^2x^6 + a^6x^2 = 0, \tag{5.13}$$
$$-a^3x^7 + a^7x^3 = 0, \tag{5.14}$$
$$a^2x^4 - a^4x^2 = 0, \tag{5.15}$$
$$a^3x^5 - a^5x^3 = 0. \tag{5.16}$$

For the a^k of degree 0, that is, $a^k \subset \mathbb{K}$ ($\forall k \in \{1, \dots, 7\}$), it is clear that (5.8)⇒(5.9) with $a = 0$ since the x^k are linearly independent.

In order to prove the above proposition, we first prove the following lemmas.

Lemma 7. *Assume that the $a^k \in \mathcal{A}_\rho$ with $2 \leq k \leq 7$ satisfy the relations (5.13), (5.14), (5.15), and (5.16). Then, $a^2 = ax^2$ and $a^3 = ax^3$ for some $a \in \mathcal{A}_\rho$ imply that $a^k = ax^k$ for any k with $2 \leq k \leq 7$.*

Proof. Assume $a^2 = ax^2$ and $a^3 = ax^3$; then,

(5.13) reads $(a^6 - ax^6)x^2 = 0$ in view of (5.4),

(5.14) reads $(a^7 - ax^7)x^3 = 0$ in view of (5.5),

(5.15) reads $(a^4 - ax^4)x^2 = 0$ in view of (5.6),

(5.16) reads $(a^5 - ax^5)x^3 = 0$ in view of (5.7).

Since $\mathcal{A}_\rho$ is a domain, this implies that $a^k = ax^k$ for $2 \leq k \leq 7$. ∎

Lemma 8. *Assume now that the $a^k \subset \mathcal{A}_\rho$ with $2 \le k \le 7$ satisfy* (5.10)*,* (5.13)*,* (5.14)*,* (5.15)*, and* (5.16)*. Then, there is some $a \in \mathcal{A}_\rho$ such that $a^k = ax^k$ for any $2 \le k \le 7$.*

Proof. We proceed by induction on the degree d of the a^k. This is clearly true for $d = 0$, with $a = 0$ in view of the linear independence of the x^k. Let us assume that this is true for $d \le n$, and let the degree of the a^k be $n + 1$. The relation (5.10) reads

$$(0, -a^3, a^2, 0, 0, 0, 0)x = 0,$$

which implies that there are $b^k \in \mathcal{A}_p$ for $k \in \{1, \dots, 7\}$ such that

$$(b^1, \dots, b^7)A_\rho(x) = (0, -a^3, a^2, 0, 0, 0, 0) \tag{5.17}$$

since the exactness of the sequence

$$\mathcal{A}_\rho^7 \xrightarrow{A_\rho(x)} \mathcal{A}_\rho^7 \xrightarrow{x} \mathcal{A}$$

is equivalent to the presentation of $\mathcal{A}_\rho$.

The relation (5.17) reads

$$b^2x^3 - b^3x^2 = 0, \tag{5.18}$$
$$-b^1x^3 + b^3x^1 + b^4x^6 - b^6x^4 = -a^3, \tag{5.19}$$
$$b^1x^2 - b^2x^1 + b^5x^7 - b^7x^5 = a^2, \tag{5.20}$$
$$-b^2x^6 + b^6x^2 = 0, \tag{5.21}$$
$$-b^3x^7 + b^7x^3 = 0, \tag{5.22}$$
$$b^2x^4 - b^4x^2 = 0, \tag{5.23}$$
$$b^3x^5 - b^5x^3 = 0. \tag{5.24}$$

Now, the b^k are of degree n, and therefore for $k \ge 2$, one has $b^k = bx^k$ for some $b \in \mathcal{A}_\rho$ in view of Lemma 7 and of the induction assumption. Therefore, using (5.3) and (5.2), relations (5.20) and (5.19) read $a^2 = (b^1 - bx^1)x^2$ and $a^3 = (b^1 - bx^1)x^3$ which in view of Lemma 7 implies that

$$a^k = ax^k \quad \text{for } 2 \le k \le 7$$

with $a = b^1 - bx^1$. ∎

End of proof of Proposition 6. Let $(a^1, \dots, a^7) \in \text{Ker}(A_\rho(x))$; we have proven that $a^k = ax^k$ for $k \ge 2$, but then relations (5.12) and (5.11) read

$$(a^1 - ax^1)x^2 = 0 \quad \text{and} \quad (a^1 - ax^1)x^3 = 0.$$

Each one of the last relations implies that $a^1 = ax^1$ since $\mathcal{A}_\rho$ is a domain. Thus, one has $a^k = ax^k$, $\forall k \in \{1, \dots, 7\}$. ∎

This implies that $\mathcal{A}_\rho$ is Koszul and therefore is regular and is in fact a 3-Calabi–Yau domain.

The matrix $A_\beta(x)$ is given by (4.3), and the relations of the algebra $\mathcal{A}_\beta$ read

$$[x^3,x^5]+[x^7,x^2]=0, \tag{5.25}$$
$$[x^4,x^5]+[x^1,x^7]=0, \tag{5.26}$$
$$[x^5,x^1]+[x^6,x^4]=0, \tag{5.27}$$
$$[x^5,x^2]+[x^3,x^6]=0, \tag{5.28}$$
$$[x^1,x^3]+[x^2,x^4]=0, \tag{5.29}$$
$$[x^4,x^3]=0, \tag{5.30}$$
$$[x^2,x^1]=0, \tag{5.31}$$

where it is understood that the relations are valid in the algebra $\mathcal{A}_\beta$ for the generator $x^\mu, \mu\in\{1,\dots,7\}$. The relations (5.29), (5.30), and (5.31) between x^1, x^2, x^3, x^4 define the quadratic sub-algebra $\mathcal{B}$ of $\mathcal{A}_\beta$ which is Koszul of global dimension 2, and the following lemma implies in particular the injectivity of the $\mathcal{B}$-module homomorphism

$$\mathcal{B}^3 \xrightarrow{B(x)} \mathcal{B}^4,$$

where $B(x)$ is the 3×4 matrix defined by 4 in the last section.

Lemma 9. *One has the following identity:*

$$\begin{pmatrix}-x^3 & -x^4 & x^1 & x^2\\ 0 & 0 & x^4 & -x^3\\ x^2 & -x^1 & 0 & 0\end{pmatrix}\begin{pmatrix}-x^1 & 0 & 2x^4\\ -x^2 & 0 & -2x^3\\ x^3 & 2x^2 & 0\\ x^4 & -2x^1 & 0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}u,$$

where $u\in\mathcal{B}\subset\mathcal{A}_\beta$ is given by

$$u=2(x^1x^3+x^2x^4)=2(x^3x^1+x^4x^2)=x^1x^3+x^2x^4+x^3x^1+x^4x^2;$$

the different equalities follow from (5.29).

Proof. This is a direct consequence of the relations (5.29), (5.30), and (5.31) which characterize $\mathcal{B}\subset\mathcal{A}_\beta$. ∎

Let $(a^1,a^2,a^3,a^4,a^5,a^6,a^7)\in\mathcal{A}_\beta^7$ be in the kernel of the $\mathcal{A}_\beta$-module homomorphism

$$\mathcal{A}_\beta^7 \xrightarrow{A_\beta(x)} \mathcal{A}_\beta^7$$

that means that one has

$$(a^1,a^2,a^3,a^4,a^5,a^6,a^7)A_\beta(x)=0 \tag{5.32}$$

for the $a^\mu \in \mathcal{A}_\beta$, $\mu \in \{1,\dots,7\}$. Let us cut the above relation (5.32) in two pieces as

$$(a^1,a^2,a^3,a^4)\begin{pmatrix} 0 & x^7 & -x^5 & 0\\ -x^7 & 0 & 0 & -x^5\\ x^5 & 0 & 0 & x^6\\ 0 & x^5 & -x^6 & 0\end{pmatrix} + (a^5,a^6,a^7)\begin{pmatrix} -x^3 & -x^4 & x^1 & x^2\\ 0 & 0 & x^4 & -x^3\\ x^2 & -x^2 & 0 & 0\end{pmatrix} = 0 \tag{5.33}$$

and

$$(a^1,a^2,a^3,a^4)\begin{pmatrix} x^3 & 0 & -x^2\\ x^4 & 0 & x^1\\ -x^1 & -x^4 & 0\\ -x^2 & x^3 & 0\end{pmatrix} = 0 \tag{5.34}$$

which separates (a^1,a^2,a^3,a^4) and (a^5,a^6,a^7). We now use Lemma 9 to express a^5, a^6, a^7 in terms of a^1, a^2, a^3, a^4. By applying the 4×3-matrix

$$\begin{pmatrix} -x^1 & 0 & 2x^4\\ -x^2 & 0 & -2x^3\\ x^3 & 2x^2 & 0\\ x^4 & -2x^1 & 0\end{pmatrix}$$

on both terms of (5.33), one obtains

$$(a^1,a^2,a^3,a^4)C = (a^5,a^6,a^7)u, \tag{5.35}$$

where the 4×3-matrix C which coefficients in $\mathcal{A}_\beta$ is given by

$$C = \begin{pmatrix} -(x^7x^2+x^5x^3) & -2x^5x^2 & -2x^7x^3\\ x^7x^1 - x^5x^4 & 2x^5x^1 & -2x^7x^4\\ -x^5x^1 + x^6x^4 & -2x^6x^1 & 2x^5x^4\\ -(x^5x^2+x^6x^3) & -2x^6x^2 & -2x^5x^3\end{pmatrix}. \tag{5.36}$$

It is obvious that a solution of (5.34) is $(a^1a^2a^3a^4) = a(x^1,x^2,x^3,x^4)$ with $a \in \mathcal{A}_\beta$. It follows then from (5.35) and (5.36) that one has the implication

$$(a^1,a^2,a^3,a^4) = a(x^1,x^2,x^3,x^4) \Rightarrow (a^5,a^6,a^7) = a(x^5,x^6,x^7) \tag{5.37}$$

as easily verified by using the relations (5.25), (5.26), (5.27), and (5.28).

Lemma 10. *The algebra $\mathcal{A}_\beta$ is bigraded by giving bidegree $(1,0)$ to x^1, x^2, x^3, x^4 and bidegree $(0,1)$ to x^5, x^6, x^7.*

Proof. This is clear since the relations are homogeneous in the bidegree either of bidegree $(2,0)$ or of bidegree $(1,1)$. ∎

In the following, we will refer to the degree in (x^5, x^6, x^7) as the *second degree* and our proof of the regularity of the quadratic algebra $\mathcal{A}_\beta$ will be based on the induction with respect to this second-degree $p \in \mathbb{N}$. More precisely, we will prove by induction on the second-degree $p \in \mathbb{N}$ the following statement.

Proposition 11. *Let a^1, a^2, a^3, a^4 be 4 elements of $\mathcal{A}_\beta$. Then, (a^1, a^2, a^3, a^4) satisfies Relation (5.34) if and only if one has*

$$(a^1, a^2, a^3, a^4) = a(x^1, x^2, x^3, x^4)$$

for some $a \in \mathcal{A}_\beta$.

The following lemma is the step 0 of the induction.

Lemma 12. *Assume that $a^1, a^2, a^3, a^4 \in \mathcal{A}_\beta$ are of second degree 0. Then, (a^1, a^2, a^3, a^4) satisfies Relation (5.34) if and only if one has*

$$(a^1, a^2, a^3, a^4) = a(x^1, x^2, x^3, x^4)$$

for some $a \in \mathcal{A}_\beta$ of second degree 0.

Proof. To assume that a^k is of second degree 0 is the same as to assume $a^k \in \mathcal{B} \subset \mathcal{A}_\beta$. Since $\mathcal{B}$ is Koszul of global dimension 2, one has the exact sequence

$$0 \to \mathcal{B}^3 \xrightarrow{B(x)} \mathcal{B}^4 \xrightarrow{x} \mathcal{B} \to \mathbb{K} \to 0,$$

from which the result will follow. Indeed, Relation (5.34) reads

$$\begin{aligned}(-a^3, -a^4, a^1, a^2)x &= 0,\\ (0, 0, a^4, -a^3)x &= 0,\\ (a^2, -a^1, 0, 0)x &= 0,\end{aligned}$$

which in $\mathcal{B}$ is equivalent to

$$\begin{aligned}(-a^3, -a^4, a^1, a^2) &= (u^1, u^2, u^3)B(x),\\ (0, 0, a^4, -a^3) &= (v^1, v^2, v^3)B(x),\\ (a^2, -a^1, 0, 0) &= (w^1, w^2, w^3)B(x),\end{aligned}$$

in view of the above exact sequence. From the second relation and the fact that $\mathcal{B}$ is a domain, it follows that $(a^3, a^4) = r(x^3, x^4)$ and from the third relation that $(a^1, a^2) = s(x^1, x^2)$ with $r, s \in \mathcal{B}$. Finally, by using the first relation, it follows that $r = s(= a)$. ■

Let us come back to the general case. Equation (5.34) is equivalent to the equations

$$-a^3x^1 - a^4x^2 + a^1x^3 + a^2x^4 = 0, \tag{5.38}$$

$$a^4x^3 - a^3x^4 = 0, \tag{5.39}$$

$$a^2x^1 - a^1x^2 = 0. \tag{5.40}$$

Equation (5.38) is equivalent to

$$(-a^3, -a^4, a^1, a^2, 0, 0, 0) = (b^1, \dots, b^7) A_\beta(x)$$

for some elements $b^1, \dots, b^7$ of $\mathcal{A}_\beta$. It follows that one has

$$\begin{aligned} a^1 &= b^5 x^1 + b^6 x^4 - b^1 x^5 - b^4 x^6, \\ a^2 &= b^5 x^2 - b^6 x^3 - b^2 x^5 + b^3 x^6, \\ a^3 &= b^5 x^3 - b^7 x^2 - b^3 x^5 + b^2 x^7, \\ a^4 &= b^5 x^4 + b^7 x^1 - b^4 x^5 - b^1 x^7 \end{aligned} \tag{5.41}$$

and that

$$(b^1, b^2, b^3, b^4) \begin{pmatrix} x^3 & 0 & -x^2 \\ x^4 & 0 & x^1 \\ -x^1 & -x^4 & 0 \\ -x^2 & x^3 & 0 \end{pmatrix} = 0. \tag{5.42}$$

We now assume (induction hypothesis) that for any $a^i \in \mathcal{A}_\beta$ $i \in \{1,2,3,4\}$ of second-degree $q \leq p$ the relation (5.34) implies

$$(a^1, a^2, a^3, a^4) = a(x^1, x^2, x^3, x^4)$$

for some $a \in \mathcal{A}_\beta$, and let us assume that the a^i in (5.41) are of second-degree $q = p + 1$. Then, it follows that the b^i are of second-degree $q = p$ for $i \in \{1,2,3,4\}$ and therefore that $b^i = bx^i$ for some $b \in \mathcal{A}_\beta$ for $i \in \{1,2,3,4\}$ in view of (5.42) and of the induction hypothesis. Then, (5.41) implies that

$$\begin{aligned} a^1 &= (b^5 - bx^5) x^1 + (b^6 - bx^6) x^4, \\ a^2 &= (b^5 - bx^5) x^2 - (b^6 - bx^6) x^3, \\ a^3 &= (b^5 - bx^5) x^3 - (b^7 - bx^7) x^2, \\ a^4 &= (b^5 - bx^5) x^4 + (b^7 - bx^7) x^1, \end{aligned}$$

where we have used the relations (5.27), (5.28), (5.25), and (5.26). That is,

$$\begin{aligned} a^1 &= ax^1 + rx^4, \\ a^2 &= ax^2 - rx^3, \\ a^3 &= ax^3 - sx^2, \end{aligned}$$

and

$$a^4 = ax^4 + sx^1$$

with $a, r, s \in \mathcal{A}_\beta$. By insertion of these expressions again in (5.39) and (5.40), one obtains

$$\begin{aligned} sx^1x^3 + sx^2x^4 &= s(x^1x^3 + x^2x^4) = 0, \\ -rx^3x^1 - rx^4x^2 &= -r(x^1x^3 + x^2x^4) = 0, \end{aligned}$$

which implies that $r = s = 0$ since $\mathcal{A}_\beta$ is a domain. So, $a^i = ax^i$ for $i \in \{1,2,3,4\}$ which achieves the proof of $(a^1, a^2, a^3, a^4) = a(x^1, x^2, x^3, x^4)$ by induction on the second degree p. ■

From the implication (5.37), it follows finally that one has

$$(a^1, \dots, a^7) = a(x^1, \dots, a^7)$$

for some $a \in \mathcal{A}_\beta$ whenever $(a^1, \dots, a^7)$ is in the kernel of $A_\beta(x)$. This implies that $\mathcal{A}_\beta$ is Koszul and therefore is regular of global dimension 3 or equivalently here is 3-Calabi–Yau.

In summary, we have proved the following result.

Theorem 13. *The quadratic algebras associated with the* 3*-regular exterior* 3*-forms* ρ, β, $\alpha^{(3)}$, $\alpha^{(3)'}$, $\alpha^{(3)''}$ *in dimension* 7 *and with the exterior* 3*-forms* $\alpha^{(p)}$ *in dimensions* $2p+1$ *for* $p \in \mathbb{N}$ *with* $p \geq 1$ *are regular and are therefore* 3*-Calabi–Yau domains.*

Using the fact that the 3-regular exterior 3-forms that we have considered contain representative elements in all orbits of 3-regular exterior 3-forms in $\mathbb{K}^n$ for $n \leq 7$ whenever $\mathbb{K}$ is algebraically closed, one has the following theorem.

Theorem 14. *Assume that* $\mathbb{K}$ *is algebraically closed, and let* α *be an exterior* 3*-regular* 3*-form on* $\mathbb{K}^n$ *with* $n \leq 7$*. Then, the quadratic algebra*

$$\mathcal{A} = \mathbb{K}\langle x^1, \dots, x^n \rangle / [\{\partial_i \alpha\}]$$

is regular which implies that it is a 3*-Calabi–Yau domain.*

It is worth noticing here that, for $n \leq 7$, we have shown that such exterior 3-regular 3-forms only exist in dimensions $n = 3, 5$, and 7.

In the next paper of this series, we will investigate a sequence of quadratic algebras of the above type associated with the sequence of the simple complex Lie algebras.

6. Exterior 3-forms in $\mathbb{K}^n$ with $n \geq 8$

A natural question is as follows: does Theorem 14 remain true for exterior 3-forms of rank $n > 7$? Or, in other words, is the quadratic algebra associated with a 3-regular exterior 3-form in $\mathbb{K}^n$ with $n \geq 8$ a Koszul algebra?

It turns out that the answer is negative. Indeed, in [20], it is pointed out that the case XII of rank 8 of the book of G. B. Gurevich [13], namely, the exterior 3-form

$$\omega = \theta^2 \wedge \theta^1 \wedge \theta^7 + \theta^2 \wedge \theta^5 \wedge \theta^3 + \theta^2 \wedge \theta^6 \wedge \theta^4 + \theta^8 \wedge \theta^1 \wedge \theta^4 + \theta^7 \wedge \theta^3 \wedge \theta^4$$

which is 3-regular leads to an associated quadratic algebra which is not Koszul and therefore not regular. Let us explain this fact.

First, the 3-regularity of ω follows from a tedious but straightforward calculation. The relations of the quadratic algebra $\mathcal{A}_\omega = \mathcal{A}$ associated with ω read

$$\begin{cases} \partial_1\omega : [x^2, x^7] + [x^8, x^4] = 0, \\ \partial_2\omega : [x^7, x^1] + [x^3, x^5] + [x^4, x^6] = 0, \\ \partial_3\omega : [x^7, x^4] + [x^5, x^2] = 0, \\ \partial_4\omega : [x^3, x^7] + [x^1, x^8] + [x^6, x^2] = 0, \\ \partial_5\omega : [x^2, x^3] = 0, \\ \partial_6\omega : [x^2, x^4] = 0, \\ \partial_7\omega : [x^1, x^2] + [x^4, x^3] = 0, \\ \partial_8\omega : [x^4, x^1] = 0. \end{cases}$$

These relations show that $\mathcal{A}$ is generated by two sub-algebras: a tensor algebra

$$\mathbb{K}\langle x^5, x^6, x^7, x^8\rangle = \mathcal{T}$$

generated by x^5, x^6, x^7, x^8 and a quadratic sub-algebra $\mathcal{B}_\omega = \mathcal{B}$ generated by x^1, x^2, x^3, x^4 with relations $\partial_5\omega, \partial_6\omega, \partial_7\omega, \partial_8\omega$ while the first four relations $\partial_1\omega, \partial_2\omega, \partial_3\omega, \partial_4\omega$ of $\mathcal{A}$ give commutation relations between elements of $\mathcal{T}$ and elements of $\mathcal{B}$. The relations of $\mathcal{A}$ read $\partial\omega = A(x)x$ as explained in Section 3 (formulas (3.2) and (3.3)) where x is the column $\begin{pmatrix} cx^1 \\ \vdots \\ x^8 \end{pmatrix}$ and where the antisymmetric 8×8-matrix $A(x) = A_\omega(x)$ is given by

$$A(x) = \begin{bmatrix} 0 & x^7 & 0 & -x^8 & 0 & 0 & -x^2 & x^4 \\ -x^7 & 0 & x^5 & x^6 & -x^3 & -x^4 & x^1 & 0 \\ 0 & -x^5 & 0 & -x^7 & x^2 & 0 & x^4 & 0 \\ x^8 & -x^6 & x^7 & 0 & 0 & x^2 & -x^3 & -x^1 \\ 0 & x^3 & -x^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & x^4 & 0 & -x^2 & 0 & 0 & 0 & 0 \\ x^2 & -x^1 & -x^4 & x^3 & 0 & 0 & 0 & 0 \\ -x^4 & 0 & 0 & x^1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which reads in 4×4-matrices blocks

$$A(x) = \begin{pmatrix} A_0(x_{II}) & -B(x_I)^t \\ & \\ B(x_I) & 0 \end{pmatrix},$$

where $x_I = (x^1, x^2, x^3, x^4)^t$ and $x_{II} = (x^5, x^6, x^7, x^8)^t$.

The 3-regularity of ω implies [10] that the augmented Koszul complex of $\mathcal{A}$ reads

$$0 \to \mathcal{A}\otimes\omega \xrightarrow{d} \mathcal{A}\otimes R \xrightarrow{d} \mathcal{A}\otimes E \xrightarrow{d} \mathcal{A} \to \mathbb{K} \to 0,$$

where $E = \mathcal{A}_1$ is the space generated by the $x^1, \dots, x^8$ and where $R \subset E \otimes E$ is the space of relations of $\mathcal{A}$ generated by the $\partial_1\omega, \dots, \partial_8\omega$ while $\omega \in \bigwedge^3 E \subset E \otimes E \otimes E$ is identified as an element of $E^{\otimes^3}$. With other notations, the above Koszul complex of $\mathcal{A}$ identifies to the sequence (see in Section 3)

$$0 \to \mathcal{A} \xrightarrow{x^t} \mathcal{A}^8 \xrightarrow{A(x)} \mathcal{A}^8 \xrightarrow{x} \mathcal{A} \to \mathbb{K} \to 0,$$

where the exactness of

$$\mathcal{A}^8 \xrightarrow{A(x)} \mathcal{A}^8 \xrightarrow{x} \mathcal{A} \to \mathbb{K} \to 0$$

is automatic since it is equivalent to the presentation of $\mathcal{A}$ by generators and relations while the exactness of

$$0 \to \mathcal{A} \xrightarrow{x^t} \mathcal{A}^8$$

follows from the fact that $\mathcal{A}$ is a domain. Thus, it remains to analyze the small complex

$$\mathcal{A} \xrightarrow{x^t} \mathcal{A}^8 \xrightarrow{A(x)} \mathcal{A}^8, \tag{6.1}$$

that is, to describe the kernel $\mathrm{Ker}(A(x))$ of $A(x)$. By construction, one has

$$a_1(x^1, \dots, x^8) \in \mathrm{Ker}(A(x)), \quad \forall a_1 \in \mathcal{A} \tag{6.2}$$

since (6.1) is a complex; however, there is another injective mapping of $\mathcal{A}$ into $\mathrm{Ker}(A(x))$ which is quadratic in the x^i defined by

$$a_2(j^1(x), \dots, j^8(x)) \in \mathrm{Ker}(A(x)), \quad \forall a_2 \in \mathcal{A}, \tag{6.3}$$

where

$$\begin{aligned}
&j^1 = j^2 = j^3 = j^4 = 0,\\
&j^5 = u^1(x_I) = -(x^4)^2,\\
&j^6 = u^2(x_I) = x^2x^1 + x^4x^3 + [x^2, x^1] \ (\simeq (x^2x^1 + x^4x^3 + [x^4, x^3])),\\
&j^7 = u^3(x_I) = x^2x^4 (\simeq x^4x^2),\\
&j^8 = u^4(x_I) = (x^2)^2.
\end{aligned}$$

(The equivalences are modulo the relations $\partial_i\omega$ for $i \in \{5, 6, 7, 8\}$.) It is easy to verify that (6.2) and (6.3) belong to $\mathrm{Ker}(A(x))$, and one can show that they generate $\mathrm{Ker}(A(x))$. This implies in particular that the sequence (6.1) is not exact and therefore that $\mathcal{A}$ is not a Koszul algebra, and so cannot be regular. Notice that nevertheless one has the exact sequence

$$0 \to \mathcal{A}^2 \xrightarrow{(x^t, j)} \mathcal{A}^8 \xrightarrow{A(x)} \mathcal{A}^8 \xrightarrow{x} \mathcal{A} \to \mathbb{K} \to 0$$

which is a minimal projective resolution of the $\mathcal{A}$-module $\mathbb{K}$. This implies [6] that the global dimension of $\mathcal{A}$ is 3, $\mathrm{gldim}(\mathcal{A}) = 3$.

The origin of the above facts is that the global dimension of the sub-algebra $\mathcal{B}$ is 3, $\mathrm{gldim}(\mathcal{B}) = 3$, and that one has the minimal free resolution of $\mathbb{K}$ as $\mathcal{B}$-module

$$0 \to \mathcal{B} \xrightarrow{u} \mathcal{B}^4 \xrightarrow{B(x_I)} \mathcal{B}^4 \xrightarrow{x_I} \mathcal{B} \to \mathbb{K} \to 0$$

since the mapping $u : \mathcal{B} \to \mathcal{B}^4$ given by

$$b \mapsto b(u^1(x_I), u^2(x_I), u^3(x_I), u^4(x_I))$$

is injective and that $\mathrm{Im}(u) = \mathrm{Ker}(B(x_I))$ as easily verified.

By comparison of the above resolutions with the usual form of the minimal projective resolutions (i.e., here minimal free resolutions since we are in the graded case [6])

$$\cdots \to \mathcal{A} \otimes \mathrm{Tor}_n^{\mathcal{A}}(\mathbb{K}, \mathbb{K}) \xrightarrow{d_n} \cdots \xrightarrow{d_1} \mathcal{A} \otimes \mathrm{Tor}_0^{\mathcal{A}}(\mathbb{K}, \mathbb{K}) \to \mathbb{K} \to 0,$$

one gets that

$$\mathrm{Tor}_3^{\mathcal{A}}(\mathbb{K}, \mathbb{K}) = \mathrm{Tor}_{3,3}^{\mathcal{A}}(\mathbb{K}, \mathbb{K}) \oplus \mathrm{Tor}_{3,4}^{\mathcal{A}}(\mathbb{K}, \mathbb{K}),$$
$$\mathrm{Tor}_3^{\mathcal{B}}(\mathbb{K}, \mathbb{K}) = \mathrm{Tor}_{3,4}^{\mathcal{B}}(\mathbb{K}, \mathbb{K})$$

with

$$\dim(\mathrm{Tor}_{3,3}^{\mathcal{A}}(\mathbb{K}, \mathbb{K})) = \dim(\mathrm{Tor}_{3,4}^{\mathcal{A}}(\mathbb{K}, \mathbb{K})) = \dim(\mathrm{Tor}_{3,4}^{\mathcal{B}}(\mathbb{K}, \mathbb{K})) = 1$$

and

$$\mathrm{Tor}_n^{\mathcal{A}}(\mathbb{K}, \mathbb{K}) = \mathrm{Tor}_n^{\mathcal{B}}(\mathbb{K}, \mathbb{K}) = 0$$

for $n \geq 4$. It follows that the Poincaré double series

$$P_{\mathcal{A}}(r, s) = \sum_{k,\ell} \dim(\mathrm{Tor}_{k,\ell}^{\mathcal{A}}(\mathbb{K}, \mathbb{K})) r^k s^\ell$$

of $\mathcal{A}$ and $\mathcal{B}$ read

$$P_{\mathcal{A}}(r, s) = 1 + 8rs + 8r^2s^2 + r^3s^3 + r^3s^4,$$
$$P_{\mathcal{B}}(r, s) = 1 + 4rs + 4r^2s^2 + r^3s^4,$$

from which one obtains the Hilbert series

$$H_{\mathcal{A}}(t) = \sum_n \dim(\mathcal{A}_n) t^n$$

of $\mathcal{A}$ and $\mathcal{B}$

$$H_{\mathcal{A}}(t) = 1/1 - 8t + 8t^2 - t^3 - t^4$$

which is the result of [20], and

$$H_{\mathcal{B}}(t) = 1/1 - 4t + 4t^2 - t^4$$

by using the general formula

$$H_{\mathcal{A}}(t) P_{\mathcal{A}}(-1, t) = 1$$

and the above results.

7. Conclusion and further prospects

It is worth noticing here that for the 3-linear forms which are 3-regular there is already an example in dimension 3 for which the associated quadratic algebra is not regular [11] which shows that the conjecture of [10] is wrong. However, this counterexample is very asymmetric from the point of view of the group of permutations between the coordinates. This is why it was interesting to investigate the case of the 3-regular exterior 3-forms for which the complete antisymmetry is a maximal symmetry for the group of permutations between the coordinates. There we have shown that the first counterexample occurs only in dimension 8.

Nevertheless, it is interesting to consider natural families of 3-regular exterior 3-forms as the family of 3-regular $\alpha^{(p)} \in \bigwedge^3 \mathbb{K}^{2p+1}$ investigated in Section 3 which leads to associated regular quadratic algebras (see Proposition 5 in Section 3). A very interesting such family is the family of canonical 3-forms of simple Lie algebras. Let us explain this point.

Let $\mathfrak{g}$ be a finite-dimensional Lie-algebra and let ω be the 3-linear form on $\mathfrak{g}$ defined by

$$\omega(X, Y, Z) = \mathrm{Tr}([\mathrm{ad}(X), \mathrm{ad}(Y)]\, \mathrm{ad}(Z))$$

for $X, Y, Z \in \mathfrak{g}$, where Tr(.) denotes the trace of the linear endomorphisms of $\mathfrak{g}$. By definition, $\omega(X, Y, Z)$ is antisymmetric in X and Y; furthermore, the symmetry of the trace implies

$$\omega(X, Y, Z) = \omega(Z, X, Y)$$

for any $X, Y, Z \in \mathfrak{g}$ as well as the ad-invariance of ω. Thus, ω is an invariant exterior 3-form on $\mathfrak{g}$ ($\omega \in \bigwedge_I^3 \mathfrak{g}^*$) which will be referred to as *the canonical 3-form of* $\mathfrak{g}$. (The above terminology is the same as the one of [16], while in [17], for instance, ω is referred to as the *Cartan 3-form of* $\mathfrak{g}$.) This invariant exterior 3-form is closely related to the structure constants of $\mathfrak{g}$. Indeed, let (E_k) be an arbitrary basis of $\mathfrak{g}$; the relations of $\mathfrak{g}$ read in this basis

$$[E_k, E_\ell] = C_{k\ell}{}^r E_r,$$

where the $C_{k\ell}{}^r \in \mathbb{K}$ are the corresponding structure constants of $\mathfrak{g}$. Then, the components $\omega_{k\ell m} = \omega(E_k, E_\ell, E_m)$ of ω read

$$\omega_{k\ell m} = C_{k\ell}{}^r K_{rm},$$

where $K_{rm} = K(E_r, E_m)$ are the components of *the Killing form* of $\mathfrak{g}$ defined by

$$K(X, Y) = \mathrm{Tr}(\mathrm{ad}(X)\, \mathrm{ad}(Y))$$

for any $X, Y \in \mathfrak{g}$.

In the sequel, we assume that $\mathbb{K} = \mathbb{C}$ so that the Lie algebra $\mathfrak{g}$ is a complex Lie algebra.

Lemma 15. *The canonical* 3*-form* ω *of* $\mathfrak{g}$ *is nondegenerate if and only if* $\mathfrak{g}$ *is a semisimple Lie algebra.*

Proof. This is a consequence of the following connection between ω and the Killing form of $\mathfrak{g}$:

$$\omega(X, Y, Z) = K(X, [Y, Z]) \tag{7.1}$$

for any $X, Y, Z \in \mathfrak{g}$. Indeed, if $i_X\omega = 0$, then $K(X, [YZ]) = 0$ for any $Y, Z \in \mathfrak{g}$, so if $i_X\omega = 0$ implies $X = 0$, then $K(X, [Y, Z]) = 0$ for any $Y, Z \in \mathfrak{g}$ implies $X = 0$. Therefore, if ω is nondegenerate, then a fortiori the Killing form is nondegenerated so that $\mathfrak{g}$ is semi-simple. Conversely, if $\mathfrak{g}$ is semi-simple $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, the relation (7.1) implies that the nondegeneracy of ω is equivalent to the one of K. ∎

The content of Lemma 15 above is the same as the one of [17, Lemma 2.1].

As already pointed out, ω nondegenerate is equivalent to ω preregular for an exterior 3-form.

Theorem 16. *The canonical* 3*-form* ω *of* $\mathfrak{g}$ *is* 3*-regular if and only if* $\mathfrak{g}$ *is a simple Lie algebra.*

Proof. The Lie algebra $\mathfrak{g}$ is simple if and only if its adjoint representation is irreducible. This irreducibility is equivalent in the present context to the fact that the commutant $\mathrm{ad}(\mathfrak{g})'$ of $\mathrm{ad}(\mathfrak{g})$ consists of multiples of the identity mapping of $\mathfrak{g}$ onto itself, that is, to the condition

$$\begin{aligned} &[L, \mathrm{ad}\, X] = 0 \quad \forall X \in \mathfrak{g} \\ &\text{implies} \\ &L = \lambda I \quad \text{with } \lambda \in \mathbb{C} \end{aligned} \tag{7.2}$$

for any linear endomorphism L of $\mathfrak{g}$, where I denotes the identity mapping of $\mathfrak{g}$.

One may assume that $\mathfrak{g}$ is semisimple since both ω 3-regular or $\mathfrak{g}$ simple implies the semisimplicity of $\mathfrak{g}$. The Killing form K of $\mathfrak{g}$ is then nondegenerate. Now,

$$\omega(LX, Y, Z) - \omega(X, MY, Z) = 0 \quad \forall X, Y, Z \in \mathfrak{g}$$

is equivalent to

$$K(Z, [LX, Y] - [X, MY]) = 0 \quad \forall X, Y, Z \in \mathfrak{g}$$

in view of relation (7.1) and finally is equivalent to

$$[LX, Y] - [X, MY] = 0 \quad \forall X, Y \in \mathfrak{g}$$

since K is nondegenerate. Thus, the assumption that ω is 3-regular is equivalent to the condition

$$\begin{aligned} &[LX, Y] - [X, MY] = 0 \quad \forall X, Y \in \mathfrak{g} \\ &\text{implies} \\ &L = M = \lambda I \text{ with } \lambda \in \mathbb{C} \end{aligned} \tag{7.3}$$

for the linear endomorphisms L and M of $\mathfrak{g}$, where I denotes the identity mapping of $\mathfrak{g}$.

Let us first prove that the 3-regularity of ω implies the simplicity of $\mathfrak{g}$, that is, that one has the implication (7.3)⇒(7.2). By setting $M = L$ in condition (7.3), one has that

$$\mathrm{ad}(X)LY + \mathrm{ad}(Y)LX = 0$$

implies $L = \lambda I$. Now, if $L \in \mathrm{ad}(\mathfrak{g})'$, one has

$$\mathrm{ad}(X)LY + \mathrm{ad}(Y)L(X) = L([X,Y] + [Y,X]) = 0$$

and therefore $L = \lambda I$. This means that $\mathfrak{g}$ is simple whenever its canonical 3-form ω is 3-regular.

Let us show that conversely the simplicity of $\mathfrak{g}$ implies the 3-regularity of its canonical 3-form ω.

So, let $\mathfrak{g}$ be a simple n-dimensional complex Lie algebra, and let the E_k ($k \in \{1,\dots,n\}$) be a basis of $\mathfrak{g}$ which is orthonormal for the Killing form, i.e., such that

$$K(E_k, E_\ell) = \mathrm{Tr}(\mathrm{ad}(E_k)\,\mathrm{ad}(E_\ell)) = \delta_{k\ell}$$

for $k, \ell \in \{1,\dots,n\}$. The relations of $\mathfrak{g}$ read then

$$[E_k, E_\ell] = \sum_m \omega_{kl\ell m} E_m,$$

where the $\omega_{k\ell m}$ are the components of the canonical 3-form ω in the basis (E_k). The matrix components of $\mathrm{ad}(E_k)$ are given by

$$\mathrm{ad}(E_k)^m_\ell = \omega_{k\ell m}$$

$\forall k, \ell, m \in \{1,\dots,n\}$ and the orthonormality of the E_k reads in terms of the $\omega_{k\ell m}$

$$\sum_{ij} \omega_{ijk}\omega_{ij\ell} = \delta_{k\ell} \tag{7.4}$$

$\forall k, \ell \in \{1,\dots,n\}$ while the Jacobi identity reads

$$\sum_m (\omega_{ijm}\omega_{k\ell m} + \omega_{kim}\omega_{j\ell m} + \omega_{jkm}\omega_{i\ell m}) = 0 \tag{7.5}$$

$\forall i, j, k, \ell \in \{1,\dots,n\}$.

Let L and M be two linear endomorphisms of $\mathfrak{g}$ satisfying

$$\omega(LX, Y, Z) = \omega(X, MY, Z) \quad \forall X, Y, Z \in \mathfrak{g}$$

which reads in components

$$L^r_i \omega_{rjk} = M^s_j \omega_{isk}$$

$\forall i, j, k$, and let us show that this implies that $L = M = \lambda I$. The antisymmetry of ω_{ijk} implies

$$L^r_i \omega_{rji} = 0, \tag{7.6}$$

while (7.4) implies

$$L_i^r = \sum_{jk} M_j^s \omega_{isk} \omega_{rjk} \Rightarrow \mathrm{Tr}(L) = \mathrm{Tr}(M)$$

which leads by using (7.5) and (7.6) to

$$L_i^r = \sum_{jk} M_j^s \omega_{srk} \omega_{ijk} = L_r^t \omega_{tjk} \omega_{ijk} = L_r^i$$

for $r, i \in \{1, \dots, n\}$. Similarly, one gets $M_j^s = M_s^j$ for $s, j \in \{1, \dots, n\}$. This implies that

$$\mathrm{ad}(E_k) L = M \,\mathrm{ad}(E_k)$$

and by transposition

$$\mathrm{ad}(E_k) M = L \,\mathrm{ad}(E_k) \quad \forall k,$$

so one has $[\mathrm{ad}(E_k), L + M] = 0$ which by irreducibility of the adjoint representation (simplicity of $\mathfrak{g}$) implies that $L + M = 2\lambda I$ for some $\lambda \in \mathbb{C}$. Then,

$$\mathrm{ad}(E_k)(L - \lambda I) = -(L - \lambda I)\,\mathrm{ad}(E_k),$$

which implies that $[\mathrm{ad}(E_k), \mathrm{ad}(E_\ell)](L - \lambda I) = 0$, so finally (by simplicity)

$$L = M = \lambda I$$

which means that the canonical 3-form ω is 3-regular. ∎

Thus, the family of canonical 3-forms of simple Lie algebras is a family of 3-regular exterior 3-forms. It is natural to investigate the regularity of the associated quadratic algebras. For the lowest-dimensional simple Lie algebra $\mathfrak{a}_1 = \mathfrak{sl}(2)$ which is of dimension 3, it is straightforward to show that the associated quadratic algebra is the polynomial algebra $\mathbb{C}[x^1, x^2, x^3]$ which is regular of global dimension 3. For the next simple Lie algebra $\mathfrak{a}_2 = \mathfrak{sl}(3)$ which is of dimension 8, one can show with a computer that the associated quadratic algebra is Koszul of global dimension 3, but an analytic proof is not straightforward. To go further on, one must use the root space decomposition since it is clear that these quadratic algebras are in fact associated with the irreducible root systems. This work is in progress.

Acknowledgments. MDV thanks Roland Berger for useful discussion.

Funding. This work was supported by the project MTM2017-86987-P from MICIU (Spain) and the group FQM 211 from Junta de Andalucía.

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Received 25 November 2023; revised 13 December 2023.

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