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Compact convergence, deformation of the L^2-$\overline{\partial}$-complex and canonical K-homology classes

Francesco Bei

Abstract. Let (X, γ) be a compact, irreducible Hermitian complex space of complex dimension m and with $\dim(\mathrm{sing}(X)) = 0$. Let $(F, \tau) \to X$ be a Hermitian holomorphic vector bundle over X, and let us denote by $\overline{\eth}_{F,m,\mathrm{abs}}$ the rolled-up operator of the maximal L^2-$\overline{\partial}$-complex of F-valued $(m, \bullet)$-forms. Let $\pi : M \to X$ be a resolution of singularities, g a metric on M, $E := \pi^* F$ and $\rho := \pi^* \tau$. In this paper, under quite general assumptions on τ, we prove the following equality of analytic K-homology classes $[\overline{\eth}_{F,m,\mathrm{abs}}] = \pi_*[\overline{\eth}_{E,m}]$, with $\overline{\eth}_{E,m}$ the rolled-up operator of the L^2-$\overline{\partial}$-complex of E-valued $(m, \bullet)$-forms on M. Our proof is based on functional analytic techniques developed in Kuwae and Shioya (2003) and provides an explicit homotopy between the even unbounded Fredholm modules induced by $\overline{\eth}_{F,m,\mathrm{abs}}$ and $\overline{\eth}_{E,m}$.

Contents

1. Introduction

Let (X, γ) be a compact and irreducible Hermitian complex space of complex dimension m. The existence and geometric interpretation of analytic K-homology classes induced by the Hodge–Dolbeault operator have been investigated in various papers, see, e.g., [6, 7, 9, 12, 20]. Concerning the existence the main obstacle is due to the lack of a satisfactory picture for the L^2 theory of the Hodge–Dolbeault operator $\overline{\partial} + \overline{\partial}^t$. To the best of our knowledge, there are only few cases in which $\overline{\partial} + \overline{\partial}^t$ is known to have self-adjoint extensions with an entirely discrete spectrum. The first is when $m = 2$ (with no assumptions on $\mathrm{sing}(X)$) and one considers $\overline{\eth}_{\mathrm{rel}}$, that is, the rolled-up operator of the

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minimal L^2-$\overline{\partial}$-complex of $(0, \bullet)$-forms, see [3, Theorem 3.1]. The second case arises when $\dim(\mathrm{sing}(X)) = 0$ (with no assumptions on m) and one considers $\overline{\eth}_{\mathrm{abs/rel}}$, that is, the rolled-up operator of either the minimal or the maximal L^2-$\overline{\partial}$-complex of $(0, \bullet)$-forms, see [22, Theorem 1.2] and [2, Theorem 5.2]. In all the previous cases it is known that the operator $\overline{\eth}_{\mathrm{abs/rel}}$ gives rise to an unbounded even Fredholm module and thus to a class $[\overline{\eth}_{\mathrm{abs/rel}}] \in KK_0(C(X), \mathbb{C})$, see [7, Propositions 3.6, 3.7 and 3.8]. Moreover, for the class $[\overline{\eth}_{\mathrm{rel}}]$, the following interesting geometric interpretation is given in [7, Theorem 4.1] and [20, Proposition 5.1]: given an arbitrarily fixed resolution of singularities $\pi : M \to X$ and a Hermitian metric g on M it holds

$$\pi_*[\overline{\eth}_M] = [\overline{\eth}_{\mathrm{rel}}] \tag{1}$$

in $KK_0(C(X), \mathbb{C})$ with $\overline{\eth}_M$ the rolled-up operator of the L^2-$\overline{\partial}$-complex of $(0, \bullet)$-forms on (M, g). In this paper, we have a twofold aim: we prove an equality similar to (1), but for a different operator; and in the proof, we develop a completely different approach compared with the one used in [7, 20]. More precisely, let (X, γ) be a compact and irreducible Hermitian complex space with $\dim(\mathrm{sing}(X)) = 0$ endowed with a Hermitian holomorphic vector bundle $(F, \tau) \to X$. Let

$$\overline{\eth}_{E,m,\mathrm{abs}} : L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \gamma, \tau) \to L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \gamma, \tau)$$

be the rolled-up operator of the maximal L^2-$\overline{\partial}$-complex of F-valued $(m, \bullet)$-forms. Let $\pi : M \to X$ be an arbitrarily fixed resolution of X, let g be a Hermitian metric on M, and let $E = \pi^* F$ and $\rho := \pi^*\tau$. Our main result, Theorem 4.1, shows that under quite general assumptions on τ, we have the following equality in $KK_0(C(X), \mathbb{C})$:

$$\pi_*[\overline{\eth}_{E,m}] = [\overline{\eth}_{F,m,\mathrm{abs}}] \tag{2}$$

with $\overline{\eth}_{E,m} : L^2\Omega^{m,\bullet}(M, E, g, \rho) \to L^2\Omega^{m,\bullet}(M, E, g, \rho)$ the rolled-up operator of the L^2-$\overline{\partial}$-complex of E-valued $(m, \bullet)$-forms on M and $[\overline{\eth}_{E,m}]$ the corresponding class in $KK_0(C(M), \mathbb{C})$. Certainly, the reader familiar with the topic will notice immediately that (2) can be proved quickly by adopting the same strategy used in [7, 20] and that goes back to [12], which is based on the short exact sequence $0 \to KK_0(C(\mathrm{sing}(X)), \mathbb{C}) \to KK_0(C(X), \mathbb{C}) \to KK_0(C_0(\mathrm{reg}(X)), \mathbb{C}) \to 0$ and, crucially, employs the fact that $\dim(\mathrm{sing}(X)) = 0$. Nevertheless, there are at least two reasons that we believe make our paper interesting. First, our result holds in a more general framework than [7, 20] since we allow the twist with a Hermitian holomorphic vector bundle. Second, our proof is entirely different. Indeed, we prove the equality (2) in a direct way by constructing an explicit homotopy between the unbounded even Fredholm modules induced by $\overline{\eth}_{E,m}$ and $\overline{\eth}_{F,m,\mathrm{rel}}$. Our construction relies on functional analytic techniques, developed in [18], to tackle spectral convergence problems. Let us now give more details by describing how the paper is organised. Section 2 contains background material, such as the basic properties of Hermitian metrics and the corresponding L^2-metrics, the functional analytic framework that will be used in the rest of the paper, and the main definitions and properties of analytic K-homology. Section 3 contains the main technical results of this paper. In the first part, we consider a compact complex manifold M of complex dimension m endowed with a holomorphic Hermitian vector bundle $(E,\rho)\to M$. We equip M with g_t, $t\in[0,1]$, a family of Hermitian metrics on M that degenerates to a positive semidefinite Hermitian pseudometric $h:=g_0$ as $t\to 0$ (we refer to Section 3 for a precise formulation). We then consider the operators

$$\overline{\partial}^{g_1,g_s}_{E,m,q,\max}: L^2\Omega^{m,q}(M,E,g_1,\rho)\to L^2\Omega^{m,q+1}(M,E,g_s,\rho)$$

and

$$\overline{\partial}^{g_s,h}_{E,m,q,\max}: L^2\Omega^{m,q}(M,E,g_s,\rho)\to L^2\Omega^{m,q+1}(A,E|_A,h|_A,\rho|_A)$$

with A the open and dense subset of M, where h is positive definite. Under quite general assumptions on the family g_t we show that both $\overline{\partial}^{g_1,g_s}_{E,m,q,\max}$ and $\overline{\partial}^{g_s,h}_{E,m,q,\max}$ have a well-defined and compact Green operator

$$G_{\overline{\partial}^{g_1,g_s}_{E,m,q,\max}}: L^2\Omega^{m,q+1}(M,E,g_s,\rho)\to L^2\Omega^{m,q}(M,E,g_1,\rho)$$

and

$$G_{\overline{\partial}^{g_s,h}_{E,m,q,\max}}: L^2\Omega^{m,q+1}(A,E|_A,h|_A,\rho|_A)\to L^2\Omega^{m,q}(M,E,g_s,\rho)$$

and that when $s\to 0$

$$G_{\overline{\partial}^{g_1,g_s}_{E,m,q,\max}}\to G_{\overline{\partial}^{g_1,h}_{E,m,q,\max}}\quad\text{and}\quad G_{\overline{\partial}^{g_s,h}_{E,m,q,\max}}\to G_{\overline{\partial}^{h,h}_{E,m,q,\max}}$$

both in the compact sense, see Definition 2.3. The above convergence results are then used to prove that the resolvent of $\overline{\eth}_{E,m}$ can be continuously deformed with respect to the operator norm to the resolvent of a self-adjoint operator unitarily equivalent to $\overline{\eth}_{F,m,\text{abs}}$. We point out that the results of this section hold without any assumption on $\operatorname{sing}(X)$. The constraint $\dim(\operatorname{sing}(X))=0$ arises only in Section 4 as a sufficient condition to have well-defined even unbounded Fredholm modules, see Remark 4.1. Finally, in the last section, we use the above convergence and deformation results to give a direct proof of (2). We conclude this introduction with a last comment that enlightens another possibly interesting feature of our approach. The strategy adopted in this paper could potentially lead to a better version of (2). More precisely, if one can prove that $G_{\overline{\partial}^{g_s,g_s}_{E,m,q}}$ is compactly convergent to $G_{\overline{\partial}^{h,h}_{E,m,q,\max}}$ as $s\to 0$, then (2) would hold true without assumptions on $\operatorname{sing}(X)$, see Remark 4.1. Unfortunately, we do not have such a strong convergence result at our disposal.

2. Background material

2.1. Hermitian metrics and L^2-metrics

Let (M,J) be a complex manifold of complex dimension m and let g and h be Hermitian metrics on M. Let $F\in C^\infty(M,\operatorname{End}(TM))$ be the endomorphism of the tangent bundle such that $h(\cdot,\cdot) = g(F\cdot,\cdot)$ and let $F_{\mathbb{C}} \in C^{\infty}(M, \mathrm{End}(TM \otimes \mathbb{C}))$ be the $\mathbb{C}$-linear endomorphism induced by F on the complexified tangent bundle. Since F commutes with J, it follows that both $T^{1,0}M$ and $T^{0,1}M$ are preserved by $F_{\mathbb{C}}$. We denote the corresponding restrictions with $F^{1,0}_{\mathbb{C}} := F_{\mathbb{C}}|_{T^{1,0}M}$ and $F^{0,1}_{\mathbb{C}} := F_{\mathbb{C}}|_{T^{0,1}M}$. Let now g^* and h^* be Hermitian metrics induced by g and h on T^*M, respectively. We have $h^*(\cdot,\cdot) = g^*((F^{-1})^t\cdot,\cdot)$ with $(F^{-1})^t$ the transpose of F^{-1}, that is, the endomorphism of T^*M induced by F^{-1}. Let $G \in C^{\infty}(M, \mathrm{End}(T^*M))$ be defined as $G := (F^{-1})^t$ and let $G_{\mathbb{C}}$, $G^{1,0}_{\mathbb{C}}$ and $G^{0,1}_{\mathbb{C}}$ be the $\mathbb{C}$-linear endomorphisms induced by G and acting on $T^*M \otimes \mathbb{C}$, $T^{1,0,*}M$ and $T^{0,1,*}M$, respectively. Let us now denote by $g_{\mathbb{C}}$ and $h_{\mathbb{C}}$ Hermitian metrics on $TM \otimes \mathbb{C}$ induced by g and h, respectively. Let $h^*_{\mathbb{C}}$, $h^*_{a,b}$, $g^*_{\mathbb{C}}$ and $g^*_{a,b}$ be Hermitian metrics on $T^*M \otimes \mathbb{C}$ and $\Lambda^{a,b}(M)$ induced by $h_{\mathbb{C}}$ and $g_{\mathbb{C}}$, respectively. Clearly, $h^*_{a,b} = h^*_{a,0} \otimes h^*_{0,b}$, $g^*_{a,b} = g^*_{a,0} \otimes g^*_{0,b}$ and $h^*_{a,0}(\cdot,\cdot) = g^*_{a,0}(G^{a,0}_{\mathbb{C}}\cdot,\cdot)$, $h^*_{0,b}(\cdot,\cdot) = g^*_{0,b}(G^{0,b}_{\mathbb{C}}\cdot,\cdot)$, $h^*_{a,b}(\cdot,\cdot) = g^*_{a,b}(G^{a,0}_{\mathbb{C}} \otimes G^{0,b}_{\mathbb{C}}\cdot,\cdot)$, where $G^{0,b}_{\mathbb{C}} \in C^{\infty}(M, \mathrm{End}(\Lambda^{0,b}(M)))$ and $G^{a,0}_{\mathbb{C}} \in C^{\infty}(M, \mathrm{End}(\Lambda^{a,0}(M)))$ are the endomorphisms induced in the natural way by $G^{0,1}_{\mathbb{C}}$ and $G^{1,0}_{\mathbb{C}}$, respectively. Now, we consider a holomorphic vector bundle $E \to M$ endowed with a Hermitian metric ρ. Let us denote by $g^*_{a,b,\rho}$ and $h^*_{a,b,\rho}$ Hermitian metrics induced by $g^*_{a,b}$, $h^*_{a,b}$ and ρ on $\Lambda^{a,b}(M) \otimes E$, respectively. Let $S^{a,b} \in C^{\infty}(M, \mathrm{End}(\Lambda^{a,b}(M) \otimes E))$ given by $G^{a,0}_{\mathbb{C}} \otimes G^{0,b}_{\mathbb{C}} \otimes \mathrm{Id}$. Then, we have

$$h^*_{a,b,\rho}(\cdot,\cdot) = g^*_{a,b,\rho}(S^{a,b}\cdot,\cdot).$$

Let $\Omega^{p,q}_c(M, E)$ be the space of smooth E-valued (p, q)-forms with compact support, let $L^2\Omega^{p,q}(M, E, g, \rho)$ be the Hilbert space of E-valued L^2-(p, q)-forms over M with respect to g and ρ, and with self-explanatory notation let us also consider $L^2\Omega^{p,q}(M, E, h, \rho)$. Given any $\phi, \psi \in \Omega^{a,b}_c(M, E)$, we can describe the L^2-product induced by h in terms of the L^2-product induced by g as follows:

$$\begin{aligned}\langle \psi, \phi \rangle_{L^2\Omega^{a,b}(M,E,h,\rho)} &= \int_M h^*_{a,b,\rho}(\psi, \phi)\,\mathrm{dvol}_h \\ &= \int_M g^*_{a,b,\rho}(S^{a,b}\psi, \phi)\sqrt{\det(F)}\,\mathrm{dvol}_g. \end{aligned} \tag{3}$$

Let us now point out some consequences of (3). If $\psi \in \Omega^{0,b}_c(M, E)$, we have

$$\begin{aligned}\langle \psi, \psi \rangle_{L^2\Omega^{0,b}(M,E,h,\rho)} &= \int_M h^*_{0,b,\rho}(\psi, \psi)\,\mathrm{dvol}_h \\ &= \int_M g^*_{0,b,\rho}(S^{0,b}\psi, \psi)\sqrt{\det(F)}\,\mathrm{dvol}_g \\ &= \int_M g^*_{0,b,\rho}\big((G^{0,b}_{\mathbb{C}} \otimes \mathrm{Id})\psi, \psi\big)\sqrt{\det(F)}\,\mathrm{dvol}_g \\ &\leq \int_M |G^{0,b}_{\mathbb{C}}|_{g^*_{0,b}} g^*_{0,b,\rho}(\psi, \psi)\sqrt{\det(F)}\,\mathrm{dvol}_g\ ,\end{aligned}$$

where $|G^{0,b}_{\mathbb{C}}|_{g^*_{0,b}} : M \to \mathbb{R}$ is the function that assigns to each $p \in M$ the pointwise operator norm of $G^{0,b}_{\mathbb{C},p} : \Lambda^{0,b}_p(M) \to \Lambda^{0,b}_p(M)$ with respect to $g^*_{0,b}$, that is,

$$|G^{0,b}_{\mathbb{C}}|_{g^*_{0,b}}(p) = \sup_{0\neq v\in\Lambda^{0,b}_p(M)} \sqrt{\frac{g^*_{0,b}(G^{0,b}_{\mathbb{C}} v, G^{0,b}_{\mathbb{C}} v)}{g^*_{0,b}(v, v)}}.$$

In particular, if $|G^{0,b}_{\mathbb{C}}|_{g^*_{0,b}} \sqrt{\det(F)} \in L^\infty(M)$, we obtain

$$\begin{aligned}\langle \psi, \psi\rangle_{L^2\Omega^{0,b}(M,E,h,\rho)} &\leq \int_M |G^{0,b}_{\mathbb{C}}|_{g^*_{0,b}} g^*_{0,b,\rho}(\psi, \psi) \sqrt{\det(F)}\,\mathrm{dvol}_g \\ &\leq \big\| |G^{0,b}_{\mathbb{C}}|_{g^*_{0,b}} \sqrt{\det(F)}\big\|_{L^\infty(M)} \int_M g^*_{0,b,\rho}(\psi, \psi)\,\mathrm{dvol}_g \\ &= \big\| |G^{0,b}_{\mathbb{C}}|_{g^*_{0,b}} \sqrt{\det(F)}\big\|_{L^\infty(M)} \langle \psi, \psi\rangle_{L^2\Omega^{0,b}(M,E,g,\rho)}.\end{aligned}$$

When $\psi \in \Omega^{m,b}_c(M)$, we have

$$\begin{aligned}\langle \psi, \psi\rangle_{L^2\Omega^{m,b}(M,E,h,\rho)} &= \int_M h^*_{m,b,\rho}(\psi, \psi)\,\mathrm{dvol}_h \\ &= \int_M g^*_{m,b,\rho}(S^{m,b}\psi, \psi) \sqrt{\det(F)}\,\mathrm{dvol}_g \\ &= \int_M g^*_{m,b,\rho}\big((\det(G^{1,0}_{\mathbb{C}}) \otimes G^{0,b}_{\mathbb{C}} \otimes \mathrm{Id}\big)\psi, \psi\big) \sqrt{\det(F)}\,\mathrm{dvol}_g \\ &= \int_M g^*_{m,b,\rho}\big((\mathrm{Id} \otimes G^{0,b}_{\mathbb{C}} \otimes \mathrm{Id})\psi, \psi\big)\,\mathrm{dvol}_g \\ &\leq \int_M |G^{0,b}_{\mathbb{C}}|_{g^*_{0,b}} g^*_{m,b,\rho}(\psi, \psi)\,\mathrm{dvol}_g.\end{aligned} \tag{4}$$

Thus, whenever $|G^{0,b}_{\mathbb{C}}|_{g^*_{0,b}} \in L^\infty(M)$, we have

$$\langle \psi, \psi\rangle_{L^2\Omega^{m,b}(M,E,h,\rho)} \leq \big\| |G^{0,b}_{\mathbb{C}}|_{g^*_{0,b}}\big\|_{L^\infty(M)} \langle \psi, \psi\rangle_{L^2\Omega^{m,b}(M,E,g,\rho)},$$

whereas if there exists a positive constant c, such that $g^*_{0,b}(G^{0,b}_{\mathbb{C}}\cdot,\cdot) \geq c g^*_{0,b}(\cdot,\cdot)$, then

$$g^*_{m,b,\rho}\big((\mathrm{Id} \otimes G^{0,b}_{\mathbb{C}} \otimes \mathrm{Id})\psi, \psi\big) \geq c g^*_{m,b,\rho}(\psi, \psi)$$

and therefore,

$$\langle \psi, \psi\rangle_{L^2\Omega^{m,b}(M,E,h,\rho)} \geq c\langle \psi, \psi\rangle_{L^2\Omega^{m,b}(M,E,g,\rho)}. \tag{5}$$

Let us denote by $\overline{\partial}_{E,p,q} : \Omega^{p,q}(M, E) \to \Omega^{p,q+1}(M, E)$ the Dolbeault operator acting on E-valued (p, q)-forms. When we look at $\overline{\partial}_{E,p,q} : L^2\Omega^{p,q}(M, E, g, \rho) \to L^2\Omega^{p,q+1}(M, E, h, \rho)$ as an unbounded and densely defined operator with domain $\Omega^{p,q}_c(M, E)$, we denote by

$$\overline{\partial}^{g,h}_{p,q,\max/\min} : L^2\Omega^{p,q}(M, E, g, \rho) \to L^2\Omega^{p,q+1}(M, E, h, \rho),$$

respectively its maximal and minimal extension.The former is the closed extension defined in the distributional sense: $\omega \in \mathcal{D}(\overline{\partial}^{g,h}_{E,p,q,\max})$ if $\omega \in L^2\Omega^{p,q}(M, E, g, \rho)$ and $\overline{\partial}_{E,p,q}\omega$, applied in the distributional sense, lies in $L^2\Omega^{p,q+1}(M, E, h, \rho)$. The latter is defined as the graph closure of $\Omega^{p,q}_c(M, E)$ in $L^2\Omega^{p,q}(M, E, g, \rho)$, with respect to the graph norm of $\overline{\partial}_{p,q} : \Omega^{p,q}_c(M, E) \to L^2\Omega^{p,q+1}(M, E, h, \rho)$. With $\overline{\partial}^{g,h,t}_{E,p,q} : \Omega^{p,q+1}_c(M) \to \Omega^{p,q}_c(M)$ we denote the formal adjoint of $\overline{\partial}_{E,p,q}$ with respect to Hermitian metrics g on $\Omega^{p,q}_c(M, E)$ and h on $\Omega^{p,q+1}_c(M, E)$; with $\overline{\partial}^{g,h,t}_{E,p,q,\max/\min} : L^2\Omega^{p,q+1}(M, E, h, \rho) \to L^2\Omega^{p,q}(M, E, g, \rho)$ we denote the corresponding maximal and minimal extensions. Note that

$$\overline{\partial}^{g,h,t}_{E,p,q,\max} = (\overline{\partial}^{g,h}_{E,p,q,\min})^* \quad \text{and} \quad \overline{\partial}^{g,h,t}_{E,p,q,\min} = (\overline{\partial}^{g,h,t}_{E,p,q,\max})^*. \tag{6}$$

We conclude this section with the following.

Proposition 2.1. *Let (M, J) be a complex manifold of complex dimension m and let $(E, \rho) \to M$ be a Hermitian holomorphic vector bundle over M. Let g_1, g_2, h_1 and h_2 be four Hermitian metrics on M such that $h_1 \leq c_1 g_1$ and $h_2 \leq c_2 g_2$, with c_1 and c_2 positive constants. Finally, let us consider the operators*

$$\overline{\partial}^{g_1,g_2}_{E,m,q,\max} : L^2\Omega^{m,q}(M, E, g_1, \rho) \to L^2\Omega^{m,q+1}(M, E, g_2, \rho),$$
$$\overline{\partial}^{h_1,h_2}_{E,m,q,\max} : L^2\Omega^{m,q}(M, E, h_1, \rho) \to L^2\Omega^{m,q+1}(M, E, h_2, \rho).$$

If $\omega \in \mathcal{D}(\overline{\partial}^{h_1,h_2}_{E,m,q,\max})$, then $\omega \in \mathcal{D}(\overline{\partial}^{g_1,g_2}_{E,m,q,\max})$ and $\overline{\partial}^{h_1,h_2}_{E,m,q,\max}\omega = \overline{\partial}^{g_1,g_2}_{E,m,q,\max}\omega$. Moreover, the induced inclusion

$$\mathcal{D}(\overline{\partial}^{h_1,h_2}_{E,m,q,\max}) \hookrightarrow \mathcal{D}(\overline{\partial}^{g_1,g_2}_{E,m,q,\max})$$

is continuous with respect to the corresponding graph norms.

Proof. Since we assumed $h_1 \leq c_1 g_1$ and $h_2 \leq c_2 g_2$, it follows from (4) that (5) holds true for both g_1, h_1 and g_2, h_2. Thus, the identity $\mathrm{Id} : \Omega^{m,q}_c(M, E) \to \Omega^{m,q}_c(M, E)$ induces continuous inclusions

$$\begin{aligned} L^2\Omega^{m,q}(M, E, h_1, \rho) &\hookrightarrow L^2\Omega^{m,q}(M, E, g_1, \rho) \\ L^2\Omega^{m,q}(M, E, h_2, \rho) &\hookrightarrow L^2\Omega^{m,q}(M, E, g_2, \rho) \end{aligned} \tag{7}$$

for each $q = 0, \ldots, m$. Let now $\omega \in \mathcal{D}(\overline{\partial}^{h_1,h_2}_{E,m,q,\max})$ and let $\eta = \overline{\partial}^{h_1,h_2}_{E,m,q,\max}\omega$. This is equivalent to saying that

$$(-1)^{m+q+1} \int_M \omega \wedge \overline{\partial}_{E^*,0,m-q-1}\phi = \int \eta \wedge \phi$$

for any $\phi \in \Omega^{0,m-q-1}_c(M, E^*)$. Thanks to (7), the above equality implies that $\omega \in \mathcal{D}(\overline{\partial}^{g_1,g_2}_{E,m,q,\max})$ and that $\eta = \overline{\partial}^{g_1,g_2}_{E,m,q,\max}\omega$. Finally, again by (7), we can conclude that the induced inclusion $\mathcal{D}(\overline{\partial}^{h_1,h_2}_{E,m,q,\max}) \hookrightarrow \mathcal{D}(\overline{\partial}^{g_1,g_2}_{E,m,q,\max})$ is continuous with respect to the corresponding graph norms. ∎

2.2. Functional analytic prerequisites

We briefly recall some functional analytic tools that will be used later. All the material is taken from [18]. We refer to it for an in-depth treatment. Let $\{H_n\}_{n\in\mathbb{N}}$ be a sequence of infinite dimensional separable complex Hilbert spaces. Let H be another infinite dimensional separable complex Hilbert space. Let us denote by $\langle\cdot,\cdot\rangle_{H_n}$, $\|\cdot\|_{H_n}$, $\langle\cdot,\cdot\rangle_H$ and $\|\cdot\|_H$ the corresponding inner products and norms. Let $\mathcal{C} \subseteq H$ be a dense subset. Assume that for every $n \in \mathbb{N}$, there exists a linear map $\Phi_n : \mathcal{C} \to H_n$. We will say that H_n converges to H as $n \to \infty$ if and only if

$$\lim_{n\to\infty} \|\Phi_n u\|_{H_n} = \|u\|_H \tag{8}$$

for any $u \in \mathcal{C}$.

Assumption. In the following definitions and propositions, we will always assume that the sequence $\{H_n\}_{n\in\mathbb{N}}$ *converges* to H.

Definition 2.1. Let $u \in H$ and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \in H_n$ for each $n \in \mathbb{N}$. We say that u_n strongly converges to u as $n \to \infty$ if there exists a net $\{v_\beta\}_{\beta\in\mathcal{B}} \subset \mathcal{C}$ tending to u in H such that

$$\lim_{\beta} \limsup_{n\to\infty} \|\Phi_n v_\beta - u_n\|_{H_n} = 0. \tag{9}$$

Definition 2.2. Let $u \in H$ and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \in H_n$ for each $n \in \mathbb{N}$. We say that u_n weakly converges to u as $n \to \infty$ if

$$\lim_{n\to\infty} \langle u_n, w_n\rangle_{H_n} = \langle u, w\rangle_H$$

for any $w \in H$ and any sequence $\{w_n\}_{n\in\mathbb{N}}$, $w_n \in H_n$, strongly convergent to w.

Proposition 2.2. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \in H_n$ for each $n \in \mathbb{N}$. Assume that there exists a positive real number, c, such that $\|u_n\|_{H_n} \le c$ for every $n \in \mathbb{N}$. There then exists a subsequence $\{u_m\}_{m\in\mathbb{N}} \subset \{u_n\}_{n\in\mathbb{N}}$, $u_m \in H_m$, weakly convergent to some element $u \in H$.*

Proof. See [18, Lemma 2.2]. ∎

Proposition 2.3. *Let $\{u_n\}_{n\in\mathbb{N}}$, $u_n \in H_n$, be a sequence weakly convergent to some element $u \in H$. There then exists a positive real number, ℓ, such that*

$$\sup_{n\in\mathbb{N}} \|u_n\|_{H_n} \le \ell \qquad \textit{and} \qquad \|u\|_H \le \liminf_{n\to\infty} \|u_n\|_{H_n}.$$

Proof. See [18, Lemma 2.3]. ∎

We now have the following remark. Consider the case of a constant sequence of infinite dimensional separable complex Hilbert spaces $\{H_n\}_{n\in\mathbb{N}}$, that is, for each $n \in \mathbb{N}$, $H_n = H$, $\mathcal{C} = H$ and $\Phi_n : \mathcal{C} \to H_n$ is nothing but the identity $\mathrm{Id} : H \to H$. Then

Definitions 2.1 and 2.2 coincide with ordinary notions of convergence in H and weak convergence in H. Indeed, let $\{v_n\} \subset H$ be a sequence converging to some $v \in H$. Then by taking the constant net $\{v_\beta\}_{\beta\in B} \subset H$, $v_\beta := v$ as a net in H converging to v, we have

$$\lim_{\beta} \limsup_{n\to\infty} \|\Phi_n v_\beta - v_n\|_{H_n} = \limsup_{n\to\infty} \|v - v_n\|_H = 0.$$

Therefore, $v_n \to v$ strongly in the sense of Definition 2.1. Conversely, let us assume that for some net $\{v_\beta\}_{\beta\in B} \subset H$ tending to v in H we have

$$\lim_{\beta} \limsup_{n\to\infty} \|\Phi_n v_\beta - v_n\|_{H_n} = 0.$$

Given any $\beta \in B$ we have $\|v - v_n\|_H \leq \|v - v_\beta\|_H + \|v_\beta - v_n\|_H$. Therefore, for every $\beta \in B$

$$\limsup_{n\to\infty} \|v - v_n\|_H \leq \|v - v_\beta\|_H + \limsup_{n\to\infty} \|v_\beta - v_n\|_H$$

and finally,

$$\limsup_{n\to\infty} \|v - v_n\|_H \leq \lim_{\beta} \|v - v_\beta\|_H + \lim_{\beta} \limsup_{n\to\infty} \|v_\beta - v_n\|_H = 0.$$

Therefore, $v_n \to v$ in H and thus we showed that Definition 2.1 coincides with the ordinary notion of convergence in H. This, in turn, implies immediately that Definition 2.2 coincides with the standard definition of weak convergence in H. We now have the following.

Definition 2.3. A sequence of bounded operators $B_n : H_n \to H_n$ compactly converges to a bounded operator $B : H \to H$ if $B_n(u_n) \to B(u)$ strongly as $n \to \infty$ for any sequence $\{u_n\}_{n\in\mathbb{N}}$, $u_n \in H_n$, weakly convergent to $u \in H$.

Given a Hilbert space H and a bounded operator $T : H \to H$, we denote by $\|T\|_{\mathrm{op}}$ the operator norm of T. We recall the following fact.

Proposition 2.4. *Let H be a separable Hilbert space and let B and $\{B_n\}_{n\in\mathbb{N}}$ be bounded operators acting on H. Assume that for each weakly convergent sequence $\{v_n\}_{n\in\mathbb{N}} \subset H$, $v_n \rightharpoonup v$ as $n \to \infty$ to some $v \in H$, we have $\|B_n v_n - Bv\|_H \to 0$ as $n \to \infty$. Then $\|B_n - B\|_{\mathrm{op}} \to 0$ as $n \to \infty$ and B is compact.*

Proof. See [18, Lemma 2.8]. ∎

Finally, we conclude this section by recalling some well-known facts about Green operators. Let H_1 and H_2 be separable Hilbert spaces whose Hilbert products are denoted by $\langle\ ,\ \rangle_{H_1}$ and $\langle\ ,\ \rangle_{H_2}$. Let $T : H_1 \to H_2$ be an unbounded, densely defined and closed operator with domain $\mathcal{D}(T)$. Assume that $\mathrm{im}(T)$ is closed. Let $T^* : H_2 \to H_1$ be the adjoint of T. Then $\mathrm{im}(T^*)$ is closed as well, and we have the following orthogonal decompositions: $H_1 = \ker(T) \oplus \mathrm{im}(T^*)$ and $H_2 = \ker(T^*) \oplus \mathrm{im}(T)$. The Green operator of T,

$$G_T : H_2 \to H_1$$

is then the operator defined by the following assignments: if $u \in \ker(T^*)$, then $G_T(u) = 0$, if $u \in \operatorname{im}(T)$, then $G_T(u) = v$, where v is the unique element in $\mathcal{D}(T) \cap \operatorname{im}(T^*)$ such that $T(v) = u$. We have that $G_T : H_2 \to H_1$ is a bounded operator. Moreover, if $H_1 = H_2$ and T is self-adjoint, then G_T is also self-adjoint. If $H_1 = H_2$ and T is self-adjoint and non-negative, that is, $\langle Tu, u \rangle_{H_1} \geq 0$ for each $u \in \mathcal{D}(T)$, then G_T is self-adjoint and non-negative as well. Finally, we recall the following property, which is straightforward to check.

Proposition 2.5. *Let $T : H_1 \to H_2$ be as above. The following two properties are then equivalent:*

(1) *$G_T : H_2 \to H_1$ is a compact operator;*

(2) *the inclusion $\mathcal{D}(T) \cap \operatorname{im}(T^*) \hookrightarrow H_1$, where $\mathcal{D}(T) \cap \operatorname{im}(T^*)$ is endowed with the graph norm of T, is a compact operator.*

2.3. Analytic K-homology classes

We now recall the definition of $KK_0(C(X), \mathbb{C})$. We invite the interested reader to consult [13] for a thorough exposition. Let Z be a second countable compact space and let $C(Z)$ be the corresponding C^*-algebra of continuous complex-valued functions. An even Fredholm module over $C(Z)$ is a triplet (H, ρ, F) satisfying the following properties:

(1) H is a separable Hilbert space.

(2) ρ is a representation $\rho : C(Z) \to \mathcal{B}(H)$ of $C(Z)$ as bounded operators on H.

(3) F is an operator on H such that for all $f \in C(Z)$:

$$(F^2 - \mathrm{Id}) \circ \rho(f), \ (F - F^*) \circ \rho(f) \text{ and } [F, \rho(f)] \text{ lie in } \mathcal{K}(H),$$

where $\mathcal{K}(H) \subset \mathcal{B}(H)$ is the space of compact operators.

(4) The Hilbert space H is equipped with a $\mathbb{Z}_2$-grading $H = H^+ \oplus H^-$ in such a way that for each $f \in C(Z)$, the operator $\rho(f)$ is even-graded, while the operator F is odd-graded.

Let (H_1, ρ_1, F_1) and (H_2, ρ_2, F_2) be two even Fredholm modules over $C(Z)$. A *unitary equivalence* between them is a grading-zero unitary isomorphism $u : H_1 \to H_2$, which intertwines the representations ρ_1 and ρ_2 and the operators F_1 and F_2.

Given two even Fredholm modules (H, ρ, F_0) and (H, ρ, F_1) over $C(Z)$, an *operator homotopy* between them is a family of Fredholm modules (H, ρ, F_t) parameterised by $t \in [0, 1]$ in such a way that the representation ρ, the Hilbert space H and its grading structures remain constant but the operator F_t varies with t and the function $[0, 1] \to \mathcal{B}(H)$, $t \mapsto F_t$ is operator norm continuous. In this case, we will say that (H, ρ, F_0) and (H, ρ, F_1) are operator homotopic.

The notion of direct sum for even Fredholm modules is defined naturally: one takes the direct sum of the Hilbert spaces, representations and operators F. The zero even Fredholm module has zero Hilbert space, zero representation and zero operator.

Now, we can recall Kasparov's definition of K-homology. The K-homology group $KK_0(C(Z), \mathbb{C})$ is the abelian group with one generator $[x]$ for each unitary equivalence class of even Fredholm modules over $C(Z)$ and with the following relations:

- if x_0 and x_1 are operator homotopic even Fredholm modules, then $[x_0] = [x_1]$ in $KK_0(C(Z), \mathbb{C})$,
- if x_0 and x_1 are any two even Fredholm modules, then $[x_0 + x_1] = [x_0] + [x_1]$ in $KK_0(C(Z), \mathbb{C})$.

Now, we go on by recalling the notion of *even unbounded Fredholm module* over the C^*-algebra $C(Z)$. This is a triplet (H, υ, D) such that:

(1) H is a separable Hilbert space endowed with a unitary $*$-representation $\upsilon : C(Z) \to \mathcal{B}(H)$; D is an unbounded, densely defined and self-adjoint linear operator on H;

(2) there is a dense $*$-subalgebra $\mathcal{A} \subset C(Z)$, such that, for all $a \in \mathcal{A}$, the domain of D is invariant by $\upsilon(a)$ and $[D, \upsilon(a)]$ extends to a bounded operator on H;

(3) $\upsilon(a)(1 + D^2)^{-1}$ is a compact operator on H for any $a \in \mathcal{A}$;

(4) H is equipped with a grading $\tau = \tau^*$, $\tau^2 = \mathrm{Id}$, such that $\tau \circ \upsilon = \upsilon \circ \tau$ and $\tau \circ D = -\tau \circ D$. In other words, τ commutes with υ and anti-commutes with D.

We now recall the following important result, see [1, Proposition 2.2].

Proposition 2.6. *Let (H, υ, D) be an even unbounded Fredholm module over $C(Z)$. Then*

$$\big(H, \upsilon, D \circ (\mathrm{Id} + D^2)^{-\frac{1}{2}}\big)$$

is an even Fredholm module over $C(Z)$.

In what follows, given an even unbounded Fredholm module as above, by the notation $[D]$ we will mean the class induced by the even Fredholm module $(H, \upsilon, D \circ (\mathrm{Id} + D^2)^{-\frac{1}{2}})$ in $KK_0(C(Z), \mathbb{C})$.

Proposition 2.7. *Let (H, υ, D_t) with $t \in [0, 1]$ be a family of even unbounded Fredholm modules over $C(Z)$ with respect to a fixed dense $*$-subalgebra $\mathcal{A} \subset C(Z)$. Assume that:*

(1) *for each $a \in \mathcal{A}$ the map $[0, 1] \to B(H)$, $t \mapsto [D_t, \upsilon(a)]$ is continuous with respect to the strong operator topology;*

(2) *the map $[0, 1] \to B(H)$, $t \mapsto (i + D_t)^{-1}$ is continuous with respect to the operator norm.*

Then the following equality:

$$[D_0] = [D_1]$$

holds in $KK_0(C(Z), \mathbb{C})$.

Proof. This follows by [1, Remark 2.5 (iv)] and [17, §6]. See also [14]. ∎

We conclude this section with the following.

Lemma 2.1. *Let H be a separable Hilbert space and let D_t, $t \in [0,1]$, be a family of unbounded, densely defined and self-adjoint operators with closed range such that*

(1) $\|G_{D_t} - G_{D_0}\|_{\text{op}} \to 0$ *as* $t \to 0$*;*

(2) $\|\pi_{K,t} - \pi_{K,0}\|_{\text{op}} \to 0$ *as* $t \to 0$*, with* $\pi_{K,t} : H \to \ker(D_t)$ *denoting the orthogonal projection on* $\ker(D_t)$ *for each* $t \in [0,1]$.

Then

$$\lim_{t\to 0} \left\|(D_t + i)^{-1} - (D_0 + i)^{-1}\right\|_{\text{op}} = 0.$$

Proof. First, we need to recall the following formulas: let $D : H \to H$ be an arbitrarily fixed unbounded, densely defined and self-adjoint operator with closed range. Let us denote the resolvent of D, $(D+i)^{-1} : H \to H$, with R_D. Then

$$R_D|_{\text{im}(D)} = G_D \circ (G_D + i)^{-1}|_{\text{im}(D)}.$$

Let us show the above claim. First, we point out that $\text{im}(D)$ is preserved by the action of G_D and R_D. Let now $\beta \in \text{im}(D)$ be arbitrarily fixed and let $\alpha := G_D\beta$. Then $D\alpha + i\alpha = \beta + i\alpha$ and consequently

$$G_D(\beta) = \alpha = R_D(D\alpha + i\alpha) = R_D(\beta + i\alpha) = R_D(\beta + iG_D\beta) = R_D((\text{Id} + iG_D)\beta),$$

that is, $G_D|_{\text{im}(D)} = R_D \circ (\text{Id} + iG_D)|_{\text{im}(D)}$. Since $G_D : H \to H$ is self-adjoint, we know that $\text{Id} + iG_D : H \to H$ is invertible. Note that $(\text{Id} + iG_D)(\text{im}(D)) = \text{im}(D)$. Therefore, $(\text{Id} + iG_D)|_{\text{im}(D)} : \text{im}(D) \to \text{im}(D)$ is invertible and thus $((\text{Id} + iG_D)|_{\text{im}(D)})^{-1} = (\text{Id} + iG_D)^{-1}|_{\text{im}(D)}$. In this way, we can conclude that

$$R_D|_{\text{im}(D)} = G_D \circ (\text{Id} + iG_D)^{-1}|_{\text{im}(D)}.$$

Let $\pi_{\text{im},t} : H \to \text{im}(D_t)$ be the orthogonal projection on $\text{im}(D_t)$ for each $t \in [0,1]$. Since $R_{D_t}|_{\ker(D_t)} = -i\,\text{Id}$ we have

$$\begin{aligned}
&\left\|(D_t + i)^{-1} - (D_0 + i)^{-1}\right\|_{\text{op}} \\
&\quad= \left\|(D_t + i)^{-1} \circ (\pi_{K,t} + \pi_{\text{im},t}) - (D_0 + i)^{-1} \circ (\pi_{K,0} + \pi_{\text{im},0})\right\|_{\text{op}} \\
&\quad\le \left\|(D_t + i)^{-1} \circ \pi_{\text{im},t} - (D_0 + i)^{-1} \circ \pi_{\text{im},0}\right\|_{\text{op}} \\
&\qquad+ \left\|(D_t + i)^{-1} \circ \pi_{K,t} - (D_0 + i)^{-1} \circ \pi_{K,0}\right\|_{\text{op}} \\
&\quad\le \left\|G_{D_t} \circ (\text{Id} + iG_{D_t})^{-1} \circ \pi_{\text{im},t} - G_{D_0} \circ (\text{Id} + iG_{D_0})^{-1} \circ \pi_{\text{im},0}\right\|_{\text{op}} + \left\|\pi_{K,t} - \pi_{K,0}\right\|_{\text{op}}.
\end{aligned}$$

By assumption, we know that $\|\pi_{K,t} - \pi_{K,0}\|_{\text{op}} \to 0$ as $t \to 0$. This clearly implies that $\|\pi_{\text{im},t} - \pi_{\text{im},0}\|_{\text{op}} \to 0$ as $t \to 0$, since $\pi_{\text{im},t} = \text{Id} - \pi_{K,t}$. We also know that

$\|G_{D_t} - G_{D_0}\|_{\mathrm{op}} \to 0$, as $t \to 0$ and this tells us that $\|(G_{D_t} - i)^{-1} - (G_{D_0} - i)^{-1}\|_{\mathrm{op}} \to 0$ as $t \to 0$, see [23, Theorem VIII.18]. Obviously, this in turn implies that

$$\|(\mathrm{Id} + iG_{D_t})^{-1} - (\mathrm{Id} + iG_{D_0})^{-1}\|_{\mathrm{op}} \to 0$$

as $t \to 0$. We can thus conclude that both

$$\lim_{t\to 0} \left\| G_{D_t} \circ (\mathrm{Id} + iG_{D_t})^{-1} \circ \pi_{\mathrm{im},t} - G_{D_0} \circ (\mathrm{Id} + iG_{D_0})^{-1} \circ \pi_{\mathrm{im},0} \right\|_{\mathrm{op}} = 0$$

$$\lim_{t\to 0} \left\| \pi_{K,t} - \pi_{K,0} \right\|_{\mathrm{op}} = 0$$

and therefore,

$$\lim_{t\to 0} \left\| (D_t + i)^{-1} - (D_0 + i)^{-1} \right\|_{\mathrm{op}} = 0$$

as desired. ■

3. Deformation of the L^2-$\overline{\partial}$-complex

This section is divided into two subsections and contains the main technical results of this paper.

3.1. Compact convergence of the Green operators

Let (M, J) be a compact complex manifold of complex dimension m endowed with a Hermitian pseudometric h. We recall that a Hermitian pseudometric on M is a positive semidefinite Hermitian product on M, strictly positive over an open and dense subset $A \subset M$. Let $(E, \rho) \to M$ be a Hermitian holomorphic vector bundle over M. We make the following assumptions:

- $(A, g|_A)$ is *parabolic* with respect to some Riemannian metric g on M.

Note that, since M is compact and parabolicity is a stable property through quasi-isometries, we can conclude that if $(A, g|_A)$ is parabolic with respect to some Riemannian metric g on M, then $(A, g|_A)$ is parabolic with respect to any Riemannian metric g on M. Moreover, since $(A, g|A)$ is parabolic, we know that $M \setminus A$ has zero Lebesgue measure, see [25, Theorem 3.4 and Proposition 3.1].

- The L^2-$\overline{\partial}$ cohomology group

$$H^{m,q+1}_{2,\overline{\partial}_{\max}}(A, E|_A, h|_A, \rho|_A) := \ker(\overline{\partial}^{h,h}_{E,m,q+1,\max})/\mathrm{im}(\overline{\partial}^{h,h}_{E,m,q,\max})$$

  is finite dimensional.

Note that, since $H^{m,q+1}_{2,\overline{\partial}_{\max}}(A, E|_A, h|_A, \rho|_A)$ is finite dimensional, the image of the operator

$$\overline{\partial}^{h,h}_{E,m,q,\max} : L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A) \to L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho|_A)$$

is closed. Let g be an arbitrarily fixed Hermitian metric on M. As a first step, we recall the following result, see [2, Proposition 3.2].

Proposition 3.1. *In the above setting, the following three operators coincide:*

$$\overline{\partial}^{g,g}_{E,p,q,\max/\min} : L^2\Omega^{p,q}(A, E|_A, h|_A, \rho|_A) \to L^2\Omega^{p,q+1}(A, E|_A, h|_A, \rho|_A),$$

$$\overline{\partial}^{g,g}_{E,p,q} : L^2\Omega^{p,q}(M, E, g, \rho) \to L^2\Omega^{p,q+1}(M, E, g, \rho), \tag{10}$$

where (10) *is the unique closed extension of* $\overline{\partial}_{E,p,q} : \Omega^{p,q}(M, E) \to \Omega^{p,q+1}(M, E)$ *viewed as an unbounded and densely defined operator acting between* $L^2\Omega^{p,q}(M, E, g, \rho)$ *and* $L^2\Omega^{p,q+1}(M, E, g, \rho)$.

Proposition 3.2. *In the above setting, the operator*

$$\overline{\partial}^{g,h}_{E,m,q,\max} : L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A) \to L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho|_A) \tag{11}$$

has closed range and the corresponding Green operator

$$G_{\overline{\partial}^{g,h}_{E,m,q,\max}} : L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho|_A) \to L^2\Omega^{m,q}(A, E|_A, g|_A, \rho|_A) \tag{12}$$

is compact.

Proof. First we note that $\mathrm{im}(\overline{\partial}^{g,h}_{E,m,q,\max})$ is a closed subspace of $L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho|_A)$. Indeed, Proposition 2.1 tells us that $\mathrm{im}(\overline{\partial}^{h,h}_{E,m,q,\max}) \subset \mathrm{im}(\overline{\partial}^{g,h}_{E,m,q,\max})$, and by the fact that $H^{m,q+1}_{2,\overline{\partial}_{\max}}(M, h)$ is finite dimensional we deduce that the quotient $\ker(\overline{\partial}^{h,h}_{E,m,q,\max})/\mathrm{im}(\overline{\partial}^{g,h}_{E,m,q,\max})$ is finite dimensional too, which in turn implies that $\mathrm{im}(\overline{\partial}^{g,h}_{E,m,q,\max})$ is closed. Hence, $G_{\overline{\partial}^{g,h}_{E,m,q,\max}}: L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho) \to L^2\Omega^{m,q}(A, E|_A, g|_A, \rho|_A)$ exists. Let us consider $\overline{\partial}^{g,h,t}_{E,m,q,\min} : L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho|_A) \to L^2\Omega^{m,q}(A, E|_A, g|_A, \rho|_A)$. Since $\mathrm{im}(\overline{\partial}^{g,h}_{E,m,q,\max}) \subset L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho|_A)$ is closed by (6), we know that $\mathrm{im}(\overline{\partial}^{g,h,t}_{E,m,q,\min}) \subset L^2\Omega^{m,q}(A, E|_A, g|_A, \rho|_A)$ is closed too. Let us then define $B := \mathcal{D}(\overline{\partial}^{g,h}_{E,m,q,\max}) \cap \mathrm{im}(\overline{\partial}^{g,h,t}_{E,m,q,\min})$. If we endow $\mathcal{D}(\overline{\partial}^{g,h}_{E,m,q,\max})$ with the corresponding graph product, then B becomes a closed subspace of $\mathcal{D}(\overline{\partial}^{g,h}_{E,m,q,\max})$, and we have the following orthogonal decomposition:

$$\mathcal{D}(\overline{\partial}^{g,h}_{E,m,q,\max}) = \ker(\overline{\partial}^{g,h}_{E,m,q,\max}) \oplus B.$$

According to Proposition 2.5, the compactness of (12) amounts to showing that $B \hookrightarrow L^2\Omega^{m,q}(A, E|_A, g|_A, \rho|_A)$ is a compact inclusion, with B endowed with the corresponding graph norm as above. To this aim, let us now consider the operator defined in (10). Classical elliptic theory on compact manifolds tells us that $\mathrm{im}(\overline{\partial}^{g,g}_{E,m,q}) \subset L^2\Omega^{m,q+1}(M, E, g, \rho)$ is closed and the corresponding Green operator

$$G_{\overline{\partial}^{g,g}_{E,m,q}} : L^2\Omega^{m,q+1}(M, E, g, \rho) \to L^2\Omega^{m,q}(M, E, g, \rho) \tag{13}$$

is compact. As recalled above, the compactness of (13) is equivalent to saying that the natural inclusion

$$\mathcal{D}(\overline{\partial}^{g,g}_{E,m,q}) \cap \mathrm{im}\big((\overline{\partial}^{g,g}_{E,m,q})^*\big) \hookrightarrow L^2\Omega^{m,q}(M,E,g,\rho) \tag{14}$$

is a compact operator with $\mathcal{D}(\overline{\partial}^{g,g}_{E,m,q}) \cap \mathrm{im}\big((\overline{\partial}^{g,g}_{E,m,q})^*\big)$ endowed with the corresponding graph product. Let $A := \mathcal{D}(\overline{\partial}^{g,g}_{E,m,q}) \cap \mathrm{im}\big((\overline{\partial}^{g,g}_{E,m,q})^*\big)$ and consider the corresponding orthogonal decomposition of $\mathcal{D}(\overline{\partial}^{g,g}_{E,m,q})$ with respect to the graph product

$$\mathcal{D}(\overline{\partial}^{g,g}_{E,m,q}) = \ker(\overline{\partial}^{g,g}_{E,m,q}) \oplus A.$$

Note that, thanks to Propositions 2.1 and 3.1, we know that $\mathcal{D}(\overline{\partial}^{g,h}_{E,m,q,\max}) \subset \mathcal{D}(\overline{\partial}^{g,g}_{E,m,q})$ and $\overline{\partial}^{g,g}_{E,m,q}\omega = \overline{\partial}^{g,h}_{E,m,q,\max}\omega$ for any $\omega \in \mathcal{D}(\overline{\partial}^{g,h}_{E,m,q,\max})$. We want to show now that $B \subset A$. Let us consider any $\omega \in B$ and let $\eta_1 \in \ker(\overline{\partial}^{g,g}_{E,m,q})$, $\eta_2 \in A$ be such that $\omega = \eta_1 + \eta_2$. It is clear that $\ker(\overline{\partial}^{g,h}_{E,m,q,\max}) = \ker(\overline{\partial}^{g,g}_{E,m,q})$. Therefore, we get immediately that $\eta_2 \in \mathcal{D}(\overline{\partial}^{g,h}_{E,m,q,\max})$. Moreover, for any $\varphi \in \ker(\overline{\partial}^{g,h}_{E,m,q,\max}) = \ker(\overline{\partial}^{g,g}_{E,m,q})$ we have

$$\begin{aligned}\langle \varphi, \eta_2\rangle_{L^2\Omega^{m,q}(M,E,g,\rho)} &+ \big\langle \overline{\partial}^{g,h}_{E,m,q,\max}\varphi, \overline{\partial}^{g,h}_{E,m,q,\max}\eta_2\big\rangle_{L^2\Omega^{m,q+1}(A,E|_A,h,\rho)}\\ &= \langle \varphi, \eta_2\rangle_{L^2\Omega^{m,q}(M,E,g,\rho)} = 0.\end{aligned}$$

Hence, $\eta_2 \in B$ and thus $\eta_1 = 0$ and $\eta_2 = \omega$ since $\eta_2 - \omega = \eta_1 \in \ker(\overline{\partial}^{g,h}_{E,m,q,\max}) \cap B = \{0\}$. Finally, let $\{\omega_k\}_{k\in\mathbb{N}} \subset B$ be a bounded sequence with respect to the graph norm of (11). Thanks to the inclusion $B \subset A$ and the continuous inclusion $L^2\Omega^{m,q+1}(A, E|_A, h, \rho) \hookrightarrow L^2\Omega^{m,q+1}(M, E, g, \rho)$, we know that $\{\omega_k\}_{k\in\mathbb{N}} \subset A$ and that it is bounded with respect to the graph norm of $\overline{\partial}^{g,g}_{E,m,q}$. Since (14) is a compact inclusion, there exists a subsequence $\{\psi_k\}_{k\in\mathbb{N}} \subset \{\omega_k\}_{k\in\mathbb{N}}$ and an element $\psi \in L^2\Omega^{m,q}(M,E,g,\rho) = L^2\Omega^{m,q}(A,E|_A,g|_A,\rho|_A)$ such that $\psi_k \to \psi$ in $L^2\Omega^{m,q}(A, E|_A, g|_A, \rho|_A)$ as $k \to \infty$. Summarising, given a sequence $\{\omega_k\}_{k\in\mathbb{N}} \subset B$, which is bounded with respect to the graph norm of (11), we have proved the existence of a subsequence $\{\psi_k\}_{k\in\mathbb{N}} \subset \{\omega_k\}_{k\in\mathbb{N}}$ and an element $\psi \in L^2\Omega^{m,q}(A, E|_A, g|_A, \rho|_A)$ such that $\psi_k \to \psi$ in $L^2\Omega^{m,q}(A, E|_A, g|_A, \rho|_A)$ as $k \to \infty$. We can thus conclude that the Green operator

$$G_{\overline{\partial}^{g,h}_{E,m,q,\max}} : L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho|_A) \to L^2\Omega^{m,q}(A, E|_A, g|_A, \rho|_A)$$

is compact as desired. ∎

Let us now consider $M \times [0,1]$ and let $p : M \times [0,1] \to M$ be the canonical projection. Let $g_s \in C^\infty(M \times [0,1], p^*T^*M \otimes p^*T^*M)$ be a smooth section of $p^*T^*M \otimes p^*T^*M \to M \times [0,1]$ such that:

(1) $g_s(JX, JY) = g_s(X,Y)$ for any $X, Y \in \mathfrak{X}(M)$ and $s \in [0,1]$;

(2) g_s is a Hermitian metric on M for any $s \in (0,1]$;

(3) $g_0 = h$;

(4) there exists a positive constant α such that $g_0 \leq \alpha g_s$ for each $s \in [0,1]$.

Roughly, g_s is a smooth family of J-invariant Riemannian metrics that degenerates to h at $s = 0$. Note that $(A, g_s|_A)$ is parabolic for any $s \in (0,1]$. Examples of such families of metrics are easy to build. For instance, if $f(s)$ is a smooth function on $[0,1]$ such that $f(0) = 0$, $f(1) = 1$ and $0 < f(s) \leq 1$ for each $s \in (0,1)$, then $g_s := (1 - f(s))h + f(s)g$ satisfies the above requirements, see [5, Proposition 4.2]. Let $F_s \in C^\infty(M \times [0,1], p^*\mathrm{End}(TM))$ be a section of $p^*\mathrm{End}(TM) \to M \times [0,1]$ such that $g_1(F_s\cdot,\cdot) = g_s(\cdot,\cdot)$ for each $s \in [0,1]$. Clearly, $F_1 = \mathrm{Id}$ and F_s is self-adjoint and positive definite on M with respect to g_1 for each fixed $s \in (0,1]$. Following the notations of Section 2.1, we have $G_s := (F_s^{-1})^t$ and the induced operators

$$G^{a,0}_{s,\mathbb{C}} \in C^\infty(A \times [0,1], p^*\mathrm{End}(\Lambda^{a,0}(A))) \text{ and } G^{0,b}_{s,\mathbb{C}} \in C^\infty(A \times [0,1], p^*\mathrm{End}(\Lambda^{0,b}(A))).$$

The first main goal of this subsection is to show that the family of Green operators $\{G_{\overline{\partial}^{g_1,g_s}_{E,m,q}}\}$ converges in the compact sense to $G_{\overline{\partial}^{g_1,h}_{E,m,q,\max}}$ as $s \to 0$. To prove this, we need some preliminary results.

Proposition 3.3. *There exists a suitable constant $\nu \geq 1$ such that the identity map $\mathrm{Id} : \Omega^{m,q}_c(A, E|_A) \to \Omega^{m,q}_c(A, E|_A)$ gives rise to a continuous inclusion*

$$i : L^2\Omega^{m,q}(A, E|_A, g_s|_A, \rho|_A) \hookrightarrow L^2\Omega^{m,q}(A, E|_A, g_1|_A, \rho|_A),$$

which satisfies the following inequality:

$$\|\omega\|^2_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} \leq \nu\|\omega\|^2_{L^2\Omega^{m,1}(A,E|_A,g_s|_A,\rho|_A)} \tag{15}$$

for any $s \in [0,1]$, $q = 0,\dots,m$ and $\omega \in L^2\Omega^{m,q}(A, E|_A, g_s|_A, \rho|_A)$.

Proof. This follows arguing as in [5, Proposition 3.1 and Lemma 4.1]. ∎

We also have the following family of uniform continuous inclusions.

Proposition 3.4. *There exists a suitable constant $a > 0$ such that the identity map $\mathrm{Id} : \Omega^{m,q}_c(A, E|_A) \to \Omega^{m,q}_c(A, E|_A)$ gives rise to a continuous inclusion*

$$i : L^2\Omega^{m,q}(A, E|_A, g_0|_A, \rho|_A) \hookrightarrow L^2\Omega^{m,q}(A, E|_A, g_s|_A, \rho|_A),$$

which satisfies the following inequality:

$$\|\omega\|^2_{L^2\Omega^{m,q}(A,E|_A,g_s|_A,\rho|_A)} \leq a\|\omega\|^2_{L^2\Omega^{m,q}(A,E|_A,g_0|_A,\rho|_A)} \tag{16}$$

for any $s \in [0,1]$, $q = 0,\dots,m$ and $\omega \in L^2\Omega^{m,q}(A, E|_A, g_0|_A, \rho|_A)$.

Proof. This follows arguing as in [5, Proposition 3.2]. ∎

We now recall the following convergence result.

Proposition 3.5. *Let $\{s_n\}_{n\in\mathbb{N}} \subset [0,1]$ be any sequence such that $s_n \to 0$ as $n \to \infty$. Consider the Hilbert space $L^2\Omega^{m,q}(A, E|_A, g_0|_A, \rho|_A)$ and the sequence of Hilbert spaces $\{L^2\Omega^{m,q}(A, E|_A, g_{s_n}|_A, \rho|_A)\}_{n\in\mathbb{N}}$. Let $\mathcal{C} := L^2\Omega^{m,q}(A, E|_A, g_0|_A, \rho|_A)$ and for any $n \in \mathbb{N}$, let $\Phi_n^{m,q} : \mathcal{C} \to L^2\Omega^{m,q}(A, E|_A, g_{s_n}|_A, \rho|_A)$ be the identity map $\mathrm{Id} : L^2\Omega^{m,q}(A, E|_A, g_0|_A, \rho|_A) \to L^2\Omega^{m,q}(A, E|_A, g_{s_n}|_A, \rho|_A)$, which is well defined thanks to Proposition 3.4. Then*

$$\{L^2\Omega^{m,q}(A, E|_A, g_{s_n}|_A, \rho|_A)\}_{n\in\mathbb{N}} \quad \textit{converges to} \quad L^2\Omega^{m,q}(A, E|_A, g_0|_A, \rho|_A)$$

in the sense of (8).

Proof. This follows by arguing as in [5, Proposition 3.3]. ■

We have the following immediate consequence.

Corollary 3.1. *Let $\omega \in L^2\Omega^{m,q}(A, E|_A, g_0|_A, \rho|_A)$ be arbitrarily fixed. Then the constant sequence $\{\omega_n\}_{n\in\mathbb{N}}$, $\omega_n := \omega$, viewed as a sequence where $\omega_n \in L^2\Omega^{m,q}(A, E|_A, g_{s_n}|_A, \rho|_A)$ for any $n \in \mathbb{N}$, converges strongly in the sense of Definition 2.1 to ω as $n \to \infty$.*

In the remaining part of this section, we investigate the compact convergence of the operators $G_{\overline{\partial}^{g_1,g_s}_{E,m,q}}$ and $G_{\overline{\partial}^{g_s,h}_{E,m,q}}$, as $s \to 0$. To this aim, we need to prove various preliminary properties.

Lemma 3.1. *Let $\phi \in \Omega_c^{m,q+1}(A, E|_A)$ and let $\{s_n\}_{n\in\mathbb{N}} \subset (0,1]$ be a sequence tending to 0 as $n \to \infty$. Then*

$$\overline{\partial}^{g_1,g_{s_n},t}_{E,m,q}\phi \rightharpoonup \overline{\partial}^{g_1,h,t}_{E,m,q}\phi$$

as $n \to \infty$, that is, $\{\overline{\partial}^{g_1,g_{s_n},t}_{E,m,q}\phi\}$ converges weakly to $\overline{\partial}^{g_1,h,t}_{E,m,q}\phi$ in $L^2\Omega^{m,q}(A, E|_A, g_1|_A, \rho|_A)$ as $n \to \infty$.

Proof. Let $f : A \times [0,1] \to \mathbb{R}$ be the function that assigns to any $(p,s) \in A \times [0,1]$ the square of the pointwise norm of $\overline{\partial}^{g_1,g_s,t}_{E,m,q}\phi$ in p with respect to g_1 and ρ. By the facts $g_s \in C^\infty(A \times [0,1], p^*T^*M \otimes p^*T^*M)$ and $\phi \in \Omega_c^{m,1}(A)$, we know that f is continuous on $A \times [0,1]$ and $\mathrm{supp}(f) \subset \mathrm{supp}(\phi) \times [0,1]$. In particular, $\mathrm{supp}(f)$ is a compact subset of $A \times [0,1]$. Therefore, there exists a positive constant $b \in \mathbb{R}$ such that $f(p,s) \leq b$ for any $p \in A$ and $s \in [0,1]$. This latter inequality tells us that

$$\|\overline{\partial}^{g_1,g_{s_n},t}_{E,m,q}\phi\|^2_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} = \int_A f(p,s)\,\mathrm{dvol}_{g_1} \leq b\,\mathrm{vol}_{g_1}(A)$$

for any $s \in [0,1]$. Now, as we know that $\{\|\overline{\partial}^{g_1,g_{s_n},t}_{E,m,q}\phi\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\}_{n\in\mathbb{N}}$ is a bounded sequence, to conclude the proof it is enough to fix a dense subset Z of

$L^2\Omega^{m,q}(A, E|_A, g_1|_A, \rho|_A)$ and to show that

$$\lim_{n\to\infty}\langle\omega, \overline{\partial}^{g_1,g_{s_n},t}_{E,m,q}\phi\rangle_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} = \langle\omega, \overline{\partial}^{g_1,h,t}_{E,m,q}\phi\rangle_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}$$

for any $\omega \in Z$. Let us fix $Z := \Omega^{m,q}_c(A, E|_A)$ and let $\omega \in \Omega^{m,q}_c(A, E|_A)$. Thanks to Proposition 3.5 and Corollary 3.1 we have

$$\begin{aligned}\lim_{n\to\infty}\langle\omega, \overline{\partial}^{g_1,g_{s_n},t}_{E,m,q}\phi\rangle_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} &= \lim_{n\to\infty}\langle\overline{\partial}_{E,m,q}\omega, \phi\rangle_{L^2\Omega^{m,q+1}(A,E|_A,g_{s_n}|_A,\rho|_A)}\\ &= \langle\overline{\partial}_{E,m,q}\omega, \phi\rangle_{L^2\Omega^{m,q+1}(A,E|_A,h|_A,\rho|_A)}\\ &= \langle\omega, \overline{\partial}^{g_1,h,t}_{E,m,q}\phi\rangle_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\end{aligned}$$

as desired. ■

The next lemma provides an extension of Proposition 3.1.

Lemma 3.2. *For each $s \in (0, 1]$, the following three operators coincide:*

$$\begin{aligned}\overline{\partial}^{g_1,g_s}_{E,p,q,\max/\min} &: L^2\Omega^{p,q}(A, E|_A, g|_A, \rho|_A) \to L^2\Omega^{p,q+1}(A, E|_A, g_s|_A, \rho|_A);\\ \overline{\partial}^{g_1,g_s}_{E,p,q} &: L^2\Omega^{p,q}(M, E, g_1, \rho) \to L^2\Omega^{p,q+1}(M, E, g_s, \rho), \qquad (17)\end{aligned}$$

with (17) the unique closed extension of $\overline{\partial}_{E,p,q} : \Omega^{p,q}(M, E) \to \Omega^{p,q+1}(M, E)$ viewed as an unbounded and densely defined operator acting between $L^2\Omega^{p,q}(M, E, g_1, \rho)$ and $L^2\Omega^{p,q+1}(M, E, g_s, \rho)$.

Proof. This is an immediate consequence of Proposition 3.1 and the fact that g_1 and g_s are quasi-isometric for each $s \in (0, 1]$. ■

As in the proof of Proposition 3.2, we define $B^{g_1,h}_{m,q} \subset L^2\Omega^{m,q}(A, E|, g_1|, \rho|_A)$ as

$$B^{g_1,h}_{m,q} := \left(\ker(\overline{\partial}^{g_1,h}_{E,m,q,\max})\right)^{\perp} \cap \mathcal{D}(\overline{\partial}^{g_1,h}_{E,m,q,\max}).$$

When $s \in (0, 1]$, we consider the operator (17) and in analogy with the above construction, we define $B^{g_1,g_s}_{m,q} \subset L^2\Omega^{m,q}(A, E|_A, g_1|_A, \rho|_A)$ as

$$B^{g_1,g_s}_{m,q} := \left(\ker(\overline{\partial}^{g_1,g_s}_{E,m,q})\right)^{\perp} \cap \mathcal{D}(\overline{\partial}^{g_1,g_s}_{E,m,q,}).$$

Lemma 3.3. *For each $s \in (0, 1]$, we have*

$$\ker(\overline{\partial}^{g_1,g_1}_{E,m,q}) = \ker(\overline{\partial}^{g_1,g_s}_{E,m,q}) \qquad \textit{and} \qquad B^{g_1,g_1}_{m,q} = B^{g_1,g_s}_{m,q} \quad \textit{for each } s \in (0, 1].$$

If $s = 0$, we have

$$\ker(\overline{\partial}^{g_1,g_1}_{E,m,q}) = \ker(\overline{\partial}^{g_1,h}_{E,m,q,\max}) \qquad \textit{and} \qquad B^{g_1,h}_{m,q} \subset B^{g_1,g_1}_{m,q}.$$

Proof. When $s \in (0, 1]$, the above equalities follow immediately by Lemma 3.2 and the fact that g_1 and g_s are quasi-isometric for each $s \in (0, 1]$. When $s = 0$, the above statements follow by Propositions 2.1 and 3.4. ∎

Corollary 3.2. *For each $s \in (0, 1]$, the Green operator*

$$G_{\overline{\partial}^{g_1,g_s}_{E,m,q}} : L^2\Omega^{m,q+1}(M, E, g_s, \rho) \to L^2\Omega^{m,q}(M, E, g_1, \rho)$$

exists and is compact.

Proof. This follows immediately by Lemmas 3.2 and 3.3. ∎

Lemma 3.4. *For each $s \in (0, 1]$, let $\lambda^{g_1,g_s}_{m,q}$ be defined as*

$$\lambda^{g_1,g_s}_{m,q} := \inf_{0 \neq \eta \in B^{g_1,g_s}_{m,q}} \frac{\|\overline{\partial}^{g_1,g_s}_{E,m,q}\eta\|_{L^2\Omega^{m,q+1}(A,E|_A,g_s|_A,\rho|_A)}}{\|\eta\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}}.$$

Similarly, let

$$\lambda^{g_1,h}_{m,q} := \inf_{0 \neq \eta \in B^{g_1,h}_{m,q}} \frac{\|\overline{\partial}^{g_1,h}_{E,m,q,\max}\eta\|_{L^2\Omega^{m,q+1}(A,E|_A,h|_A,\rho|_A)}}{\|\eta\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}}.$$

Let $\nu > 0$ and $a > 0$ be the constants appearing in (15) and (16), respectively. Then, we have

$$0 < \lambda^{g_1,g_1}_{m,q} \leq \sqrt{\nu}\lambda^{g_1,g_s}_{m,q} \leq \sqrt{a\nu}\lambda^{g_1,h}_{m,q}$$

for each $s \in (0, 1]$ and $q = 0, \dots, m$.

Proof. The inequality $0 < \lambda^{g_1,g_1}_{m,q}$ follows by the fact that the operator

$$\overline{\partial}^{g_1,g_1}_{E,m,q} : L^2\Omega^{m,q}(A, E|_A, g_1|_A, \rho|_A) \to L^2\Omega^{m,q+1}(A, E|_A, g_1|_A, \rho|_A)$$

has closed range. Let us show now that $\lambda^{g_1,g_1}_{m,q} \leq \sqrt{\nu}\lambda^{g_1,g_s}_{m,q}$ for each $s \in (0, 1]$. Thanks to Lemma 3.3, we know that $B^{g_1,g_s}_{m,q} = B^{g_1,g_1}_{m,q}$ and $\overline{\partial}^{g_1,g_1}_{E,m,q}\eta = \overline{\partial}^{g_1,g_s}_{E,m,q}\eta$ for each $s \in (0, 1]$ and $\eta \in B^{g_1,g_s}_{m,q}$. In this way, by Proposition 3.3, we obtain

$$\begin{aligned}
\sqrt{\nu}\lambda^{g_1,g_s}_{m,q} &:= \inf_{0 \neq \eta \in B^{g_1,g_s}_{m,q}} \frac{\sqrt{\nu}\|\overline{\partial}^{g_1,g_s}_{E,m,q}\eta\|_{L^2\Omega^{m,q+1}(A,E|_A,g_s|_A,\rho|_A)}}{\|\eta\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}} \\
&= \inf_{0 \neq \eta \in B^{g_1,g_1}_{m,q}} \frac{\sqrt{\nu}\|\overline{\partial}^{g_1,g_1}_{E,m,q}\eta\|_{L^2\Omega^{m,q+1}(A,E|_A,g_s|_A,\rho|_A)}}{\|\eta\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}} \\
&\geq \inf_{0 \neq \eta \in B^{g_1,g_1}_{m,q}} \frac{\|\overline{\partial}^{g_1,g_1}_{E,m,q}\eta\|_{L^2\Omega^{m,q+1}(A,E|_A,g_1|_A,\rho|_A)}}{\|\eta\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}} \\
&= \lambda^{g_1,g_1}_{m,q}.
\end{aligned}$$

We now tackle the remaining inequality. By Lemma 3.3 we know that $B^{g_1,h}_{m,q} \subset B^{g_1,g_s}_{m,q}$ and $\overline{\partial}^{g_1,g_s}_{E,m,q}\eta = \overline{\partial}^{g_1,h}_{E,m,q,\max}\eta$ for each $s \in (0,1]$ and $\eta \in B^{g_1,h}_{m,q}$. In this way, thanks to Proposition 3.4, we obtain

$$\begin{aligned}
\sqrt{a}\lambda^{g_1,h}_{m,q} &:= \inf_{0\neq\eta\in B^{g_1,h}_{m,q}} \frac{\sqrt{a}\left\|\overline{\partial}^{g_1,h}_{E,m,q,\max}\eta\right\|_{L^2\Omega^{m,q+1}(A,E|_A,h|_A,\rho|_A)}}{\|\eta\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}} \\
&\geq \inf_{0\neq\eta\in B^{g_1,h}_{m,q}} \frac{\left\|\overline{\partial}^{g_1,g_s}_{E,m,q}\eta\right\|_{L^2\Omega^{m,q+1}(A,E|_A,g_s|_A,\rho|_A)}}{\|\eta\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}} \\
&\geq \inf_{0\neq\eta\in B^{g_1,g_s}_{m,q}} \frac{\left\|\overline{\partial}^{g_1,g_s}_{E,m,q,}\eta\right\|_{L^2\Omega^{m,q+1}(A,E|_A,g_s|_A,\rho|_A)}}{\|\eta\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}} \\
&= \lambda^{g_1,g_s}_{m,q}
\end{aligned}$$

for each $s \in (0,1]$. ∎

We now have the first main result of this section.

Theorem 3.1. *Let $\{s_n\}_{n\in\mathbb{N}} \subset (0,1]$ be any sequence such that $s_n \to 0$ as $n \to \infty$ and let*

$$G_{\overline{\partial}^{g_1,g_{s_n}}_{E,m,q}} : L^2\Omega^{m,q+1}(M,E,g_{s_n},\rho) \to L^2\Omega^{m,q}(M,E,g_1,\rho)$$

be the Green operator of $\overline{\partial}^{g_1,g_{s_n}}_{E,m,q} : L^2\Omega^{m,q}(M,E,g_{s_1},\rho) \to L^2\Omega^{m,q+1}(M,E,g_{s_n},\rho)$. Then

$$G_{\overline{\partial}^{g_1,g_{s_n}}_{E,m,q}} \to G_{\overline{\partial}^{g_1,h}_{E,m,q,\max}} \quad \textit{compactly as } n \to \infty.$$

Proof. Let $\{\eta_{s_n}\}_{n\in\mathbb{N}}$ with $\eta_{s_n} \in L^2\Omega^{m,q+1}(A,E|_A,g_{s_n}|_A,\rho|_A)$ be a weakly convergent sequence to some $\eta \in L^2\Omega^{m,q+1}(A,E|_A,h|_A,\rho|_A)$, as $n \to \infty$. Let $\eta_{s_n,1}$ be the orthogonal projection of η_{s_n} on $\operatorname{im}(\overline{\partial}^{g_1,g_{s_n}}_{E,m,q})$ and let $\eta_{s_n,2}$ be the orthogonal projection of η_{s_n} on $\left(\operatorname{im}(\overline{\partial}^{g_1,g_{s_n}}_{E,m,q})\right)^{\perp}$. Analogously, let η_1 and η_2 be the orthogonal projection of η on $\operatorname{im}(\overline{\partial}^{g_1,h}_{E,m,q,\max})$ and $\left(\operatorname{im}(\overline{\partial}^{g_1,h}_{E,m,q,\max})\right)^{\perp}$, respectively. We have $G_{\overline{\partial}^{g_1,g_{s_n}}_{E,m,q}}\eta_{s_n,2} = 0 = G_{\overline{\partial}^{g_1,h}_{E,m,q,\max}}\eta_2$ for each $n \in \mathbb{N}$. Thus, to prove this proposition, we have to show that

$$G_{\overline{\partial}^{g_1,g_{s_n}}_{E,m,q}}\eta_{s_n,1} \to G_{\overline{\partial}^{g_1,h}_{E,m,q,\max}}\eta_1 \quad \text{in} \quad L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A) \text{ as } n \to \infty.$$

As a first step, we observe that there exists a constant $c > 0$ such that

$$\|\eta_{s_n,1}\|_{L^2\Omega^{m,q+1}(A,E|_A,g_{s_n}|_A,\rho|_A)} \leq \|\eta_{s_n}\|_{L^2\Omega^{m,q+1}(A,E|_A,g_{s_n}|_A,\rho|_A)} \leq c$$

for each $n \in \mathbb{N}$, see Proposition 2.3. Now consider the sequence $\left\{G_{\overline{\partial}^{g_1,g_{s_n}}_{E,m,q}}\eta_{s_n}\right\}_{n\in\mathbb{N}}$. By construction, we have $G_{\overline{\partial}^{g_1,g_{s_n}}_{E,m,q}}\eta_{s_n} \in B^{g_1,g_{s_n}}_{m,q}$ and so by Lemma 3.3, we obtain $G_{\overline{\partial}^{g_1,g_{s_n}}_{E,m,q}}\eta_{s_n} \in B^{g_1,g_1}_{m,q}$ and

$$\overline{\partial}^{g_1,g_1}_{E,m,q}(G_{\overline{\partial}^{g_1,g_{s_n}}_{E,m,q}}\eta_{s_n}) = \overline{\partial}^{g_1,g_{s_n}}_{E,m,q}(G_{\overline{\partial}^{g_1,g_{s_n}}_{E,m,q}}\eta_{s_n}) = \eta_{s_n,1}.$$

In particular, by applying Proposition 3.3, we obtain

$$\begin{aligned}&\left\|\overline{\partial}^{g_1,g_1}_{E,m,q}(G_{\overline{\partial}^{g_1,g_{s_n}}_{E,m,q}}\eta_{s_n})\right\|_{L^2\Omega^{m,q+1}(A,E|_A,g_1|_A,\rho|_A)}\\&=\|\eta_{s_n,1}\|_{L^2\Omega^{m,q+1}(A,E|_A,g_1|_A,\rho|_A)}\\&\leq\sqrt{\nu}\|\eta_{s_n,1}\|_{L^2\Omega^{m,q+1}(A,E|_A,g_{s_n}|_A,\rho|_A)}\leq\sqrt{\nu}c.\end{aligned}$$

Moreover, by Lemma 3.4, we have

$$\begin{aligned}\left\|G_{\overline{\partial}^{g_1,g_{s_n}}_{E,m,q}}\eta_{s_n}\right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}&=\left\|G_{\overline{\partial}^{g_1,g_{s_n}}_{E,m,q}}\eta_{s_n,1}\right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\\&\leq\frac{1}{\lambda^{g_1,g_s}_{m,q}}\|\eta_{s_n,1}\|_{L^2\Omega^{m,q+1}(A,E|_A,g_{s_n}|_A,\rho|_A)}\leq\frac{\sqrt{\nu}}{\lambda^{g_1,g_1}_{m,q}}c\end{aligned}$$

for each $n\in\mathbb{N}$. We have just shown that $\left\{G_{\overline{\partial}^{g_1,g_{s_n}}_{E,m,q}}\eta_{s_n}\right\}\subset B^{g_1,g_1}_{m,q}$ is a bounded sequence with respect to the graph norm of $\overline{\partial}^{g_1,g_1}_{E,m,q}$. Since $G_{\overline{\partial}^{g_1,g_1}_{E,m,q}}$ is compact, there exists a subsequence $\{r_n\}_{n\in\mathbb{N}}\subset\{s_n\}_{n\in\mathbb{N}}$ and elements $\psi\in L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)$ and $\chi\in L^2\Omega^{m,q+1}(A,E|_A,h|_A,\rho|_A)$, such that

$$\begin{aligned}G_{\overline{\partial}^{g_1,g_1}_{E,m,q}}\eta_{r_n,1}&\to\psi \text{ in } L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho)\text{ as } n\to\infty\\\eta_{r_n,1}&\to\chi\text{ weakly as } n\to\infty.\end{aligned}$$

Now, to complete the proof, we have to show that

$$\psi\in B^{g_1,h}_{m,q}\quad\text{and}\quad\overline{\partial}^{g_1,h}_{E,m,q,\max}\psi=\chi=\eta_1.$$

Let $\phi\in\Omega^{m,q+1}_c(A,E|_A)$ be arbitrarily fixed. By Lemma 3.1, we have

$$\begin{aligned}&\langle\psi,\overline{\partial}^{g_1,h,t}_{E,m,q}\phi\rangle_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\\&=\lim_{n\to\infty}\langle G_{\overline{\partial}^{g_1,g_{r_n}}_{E,m,q}}\eta_{r_n,1},\overline{\partial}^{g_1,g_{r_n},t}_{E,m,q}\phi\rangle_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\\&=\lim_{n\to\infty}\langle\overline{\partial}^{g_1,g_{r_n}}_{E,m,q}(G_{\overline{\partial}^{g_1,g_{r_n}}_{E,m,q}}\eta_{r_n,1}),\phi\rangle_{L^2\Omega^{m,q+1}(A,E|_A,g_{r_n}|_A,\rho|_A)}\\&=\lim_{n\to\infty}\langle\eta_{r_n,1},\phi\rangle_{L^2\Omega^{m,q+1}(A,E|_A,g_{r_n}|_A,\rho|_A)}\\&=\langle\chi,\phi\rangle_{L^2\Omega^{m,q+1}(A,E|_A,h,\rho|_A)}.\end{aligned}$$

This shows that $\psi\in\mathcal{D}(\overline{\partial}^{g_1,h}_{E,m,q,\max})$ and that $\overline{\partial}^{g_1,h}_{E,m,q,\max}\psi=\chi$. Now, considering any $\alpha\in\ker(\overline{\partial}^{g_1,h}_{E,m,q,\max})$ and keeping in mind that $\ker(\overline{\partial}^{g_1,h}_{E,m,q,\max})=\ker(\overline{\partial}^{g_1,g_s}_{E,m,q})$ for each $s\in(0,1]$, we have

$$\langle\alpha,\psi\rangle_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}=\lim_{n\to\infty}\langle\alpha,G_{\overline{\partial}^{g_1,g_{r_n}}_{E,m,q}}\eta_{r_n,1}\rangle_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}=0.$$

Hence, we can conclude that $\psi \in B^{g_1,h}_{m,q}$ and that $\overline{\partial}^{g_1,h}_{E,m,q,\max}\psi = \chi$. We are left to show that $\eta_1 = \chi$. Let $\xi \in \operatorname{im}(\overline{\partial}^{g_1,h}_{E,m,q,\max})$ be arbitrarily fixed. Keeping in mind that $\operatorname{im}(\overline{\partial}^{g_1,h}_{E,m,q,\max}) \subset \operatorname{im}(\overline{\partial}^{g_1,g_{s_n}}_{E,m,q})$ for each $n \in \mathbb{N}$, we have

$$\begin{aligned}\langle \xi, \chi\rangle_{L^2\Omega^{m,q+1}(A,E|_A,h|_A,\rho|_A)} &= \lim_{n\to\infty}\langle \xi, \eta_{r_n,1}\rangle_{L^2\Omega^{m,q+1}(A,E|_A,g_{r_n}|_A,\rho|_A)}\\ &= \lim_{n\to\infty}\langle \xi, \eta_{r_n}\rangle_{L^2\Omega^{m,q+1}(A,E|_A,g_{r_n}|_A,\rho|_A)}\\ &= \langle \xi, \eta\rangle_{L^2\Omega^{m,q+1}(A,E|_A,h|_A,\rho|_A)}\\ &= \langle \xi, \eta_1\rangle_{L^2\Omega^{m,q+1}(A,E|_A,h|_A,\rho|_A)}.\end{aligned}$$

In conclusion, for each $\xi \in \operatorname{im}(\overline{\partial}^{g_1,h}_{E,m,q,\max})$, we have

$$\langle \xi, \chi\rangle_{L^2\Omega^{m,q+1}(A,E|_A,h|_A,\rho|_A)} = \langle \xi, \eta_1\rangle_{L^2\Omega^{m,q+1}(A,E|_A,h|_A,\rho|_A)}$$

and so we can conclude that $\eta_1 = \chi$. Therefore, we have shown that $\psi \in B^{g_1,h}_{m,q}$ and that $\overline{\partial}^{g_1,h}_{E,m,q,\max}\psi = \chi = \eta_1$, that is, $\psi = G_{\overline{\partial}^{g_1,h}_{E,m,q,\max}}\eta$. Summarising, given a weakly convergent sequence $\eta_{s_n} \to \eta$ as $n \to \infty$ with $\eta_{s_n} \in L^2\Omega^{m,q+1}(A, E|_A, g_{s_n}|_A, \rho|_A)$ and $\eta \in L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho|_A)$, we have proved the existence of a subsequence $\{r_n\}_{n\in\mathbb{N}}$ such that

$$G_{\overline{\partial}^{g_1,g_{r_n}}_{E,m,q}}\eta_{r_n} \to G_{\overline{\partial}^{g_1,h}_{E,m,q,\max}}\eta$$

in $L^2\Omega^{m,q}(A, g_1|_A, E|_A, \rho|_A)$ as $n \to \infty$. Now if we fix an arbitrary subsequence $\{\ell_n\}_{n\in\mathbb{N}} \subset \{s_n\}_{n\in\mathbb{N}}$ and we repeat the above argument with $\{\eta_{\ell_n}\}_{n\in\mathbb{N}}$, we obtain a further subsequence $\{t_n\}_{n\in\mathbb{N}} \subset \{\ell_n\}_{n\in\mathbb{N}}$ such that $G_{\overline{\partial}^{g_1,g_{t_n}}_{E,m,q}}\eta_{t_n} \to G_{\overline{\partial}^{g_1,h}_{E,m,q,\max}}\eta$ in $L^2\Omega^{m,q}(A, g_1|_A, E|_A, \rho|_A)$ as $n \to \infty$. Clearly, this allows us to conclude that

$$G_{\overline{\partial}^{g_1,g_{s_n}}_{E,m,q}}\eta_{s_n} \to G_{\overline{\partial}^{g_1,h}_{E,m,q,\max}}\eta \quad \text{in} \quad L^2\Omega^{m,q}(A, g_1|_A, E|_A, \rho|_A) \quad \text{as} \quad n \to \infty$$

and therefore,

$$G_{\overline{\partial}^{g_1,g_{s_n}}_{E,m,q}} \to G_{\overline{\partial}^{g_1,h}_{E,m,q,\max}} \quad \text{compactly as} \quad n \to \infty.$$ ■

The next goal is to establish the compact convergence of the sequence $\{G_{\overline{\partial}^{g_{s_n},h}_{E,m,q,\max}}\}$ to $G_{\overline{\partial}^{h,h}_{E,m,q,\max}}$. To do this, we need other auxiliary results.

Lemma 3.5. *Let $\phi \in \Omega^{m,q+1}_c(A, E|_A)$ and let $\{s_n\}_{n\in\mathbb{N}} \subset (0, 1]$ be a sequence tending to 0 as $n \to \infty$. Then*

$$\overline{\partial}^{g_{s_n},h,t}_{E,m,q}\phi \to \overline{\partial}^{h,h,t}_{E,m,q}\phi$$

strongly as $n \to \infty$.

Proof. As a first step, we want to show that over A and for each $s \in [0, 1]$ the operator $\overline{\partial}^{g_s,h,t}_{E,m,q}$ can be written as the composition of $\overline{\partial}^{g_1,h,t}_{E,m,q}$ with an endomorphism of $\Lambda^{m,q}(A)\otimes E$ that depends smoothly on s. We then use this decomposition to tackle the above limit. As usual, let $p : A\times[0,1]\to A$ be the left projection and let $S_s^{m,q}\in C^{\infty}(A\times[0,1],\mathrm{End}(p^*\Lambda^{m,q}(A)\otimes p^*E))$ be defined as $S_s^{m,q}:=\det(G_{\mathbb{C},s}^{1,0})\otimes G_{\mathbb{C},s}^{0,q}\otimes \mathrm{Id}$. Note that $S_s^{m,q}$ is the family of endomorphisms of $p^*\Lambda^{m,q}(A)\otimes p^*E$ such that $g^*_{s,m,q,\rho}(\bullet,\bullet)=g^*_{1,m,q,\rho}(S_s^{m,q}\bullet,\bullet)$. The previous equality tells us that

$$g^*_{s,m,q,\rho}((S_s^{m,q})^{-1}\bullet,\bullet)=g^*_{1,m,q,\rho}(\bullet,\bullet),$$

which in turn implies that $(S_s^{m,q})^{-1}$ is fiberwise self-adjoint with respect to $g^*_{s,m,q,\rho}$ for each fixed $s\in[0,1]$. Besides $S_s^{m,q}$, let us also introduce $T_s^{m,q}:=\mathrm{Id}\otimes G_{\mathbb{C},s}^{0,q}\otimes\mathrm{Id}$. Clearly, also $T_s^{m,q}\in C^{\infty}(A\times[0,1],\mathrm{End}(p^*\Lambda^{m,q}(A)\otimes p^*E))$. Given any $\varphi\in\Omega_c^{m,q}(A,E|_A)$ and $\phi\in\Omega_c^{m,q+1}(A,E|_A)$, we have

$$\begin{aligned}
&\langle\overline{\partial}_{E,m,q}\varphi,\phi\rangle_{L^2\Omega^{m,q+1}(A,E|_A,h|_A,\rho|_A)}\\
&=\langle\varphi,\overline{\partial}^{g_1,h,t}_{E,m,q}\phi\rangle_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\\
&=\int_A g^*_{1,m,q,\rho}(\varphi,\overline{\partial}^{g_1,h,t}_{E,m,q}\phi)\,\mathrm{dvol}_{g_1}\\
&=\int_A g^*_{s,m,q,\rho}\big((S_s^{m,q})^{-1}\varphi,\overline{\partial}^{g_1,h,t}_{E,m,q}\phi\big)\mathrm{det}^{-\frac{1}{2}}(F_s)\,\mathrm{dvol}_{g_s}\\
&=\int_A g^*_{s,m,q,\rho}\big(\varphi,(S_s^{m,q})^{-1}(\overline{\partial}^{g_1,h,t}_{E,m,q}\phi)\big)\mathrm{det}^{-\frac{1}{2}}(F_s)\,\mathrm{dvol}_{g_s}\\
&=\int_A g^*_{s,m,q,\rho}\big(\varphi,\big(\det(G_{\mathbb{C},s}^{1,0})\otimes G_{\mathbb{C},s}^{0,q}\otimes\mathrm{Id}\big)^{-1}(\overline{\partial}^{g_1,h,t}_{E,m,q}\phi)\big)\mathrm{det}^{-\frac{1}{2}}(F_s)\,\mathrm{dvol}_{g_s}\\
&=\int_A g^*_{s,m,q,\rho}\big(\varphi,(\mathrm{Id}\otimes G_{\mathbb{C},s}^{0,q}\otimes\mathrm{Id})^{-1}(\overline{\partial}^{g_1,h,t}_{E,m,q}\phi)\big)\,\mathrm{dvol}_{g_s}\\
&=\int_A g^*_{s,m,q,\rho}\big(\varphi,(T_s^{m,q})^{-1}(\overline{\partial}^{g_1,h,t}_{E,m,q}\phi)\big)\,\mathrm{dvol}_{g_s}\\
&=\langle\varphi,(T_s^{m,q})^{-1}(\overline{\partial}^{g_1,h,t}_{E,m,q}\phi)\rangle_{L^2\Omega^{m,q}(A,E|_A,h|_A,g_s|_A)}.
\end{aligned}$$

Summarising, we have shown that for each $\varphi\in\Omega_c^{m,q}(A,E|_A)$ and $\phi\in\Omega_c^{m,q+1}(A,E|_A)$, we have

$$\langle\overline{\partial}_{E,m,q}\varphi,\phi\rangle_{L^2\Omega^{m,q+1}(A,E|_A,h|_A,\rho|_A)}=\langle\varphi,(T_s^{m,q})^{-1}(\overline{\partial}^{g_1,h,t}_{E,m,q}\phi)\rangle_{L^2\Omega^{m,q}(A,E|_A,g_s|_A,\rho|_A)}$$

and thus we can conclude that, for each $s\in[0,1]$, it holds

$$\overline{\partial}^{g_s,h,t}_{E,m,q}=(T_s^{m,q})^{-1}\circ\overline{\partial}^{g_1,h,t}_{E,m,q}. \tag{18}$$

In particular, for $s=0$, we have

$$\overline{\partial}^{h,h,t}_{E,m,q}=(T_0^{m,q})^{-1}\circ\overline{\partial}^{g_1,h,t}_{E,m,q}.$$

We are now in position to show that given any arbitrarily fixed $\phi \in \Omega_c^{m+1,q}(A, E|_A)$ and a sequence $\{s_n\}_{n\in\mathbb{N}} \subset (0,1]$ with $s_n \to 0$ as $n \to \infty$, we have $\overline{\partial}_{E,m,q}^{g_{s_n},h,t}\phi \to \overline{\partial}_{E,m,q}^{h,h,t}\phi$ strongly as $n \to \infty$. To put it differently, thanks to (18), Proposition 3.5 and Corollary 3.1, we have to show that

$$\lim_{n\to\infty}\left\|(T_{s_n}^{m,q})^{-1}(\overline{\partial}_{E,m,q}^{g_1,h,t}\phi) - \Phi_{s_n}^{m,q}((T_0^{m,q})^{-1}(\overline{\partial}_{E,m,q}^{g_1,h,t}\phi))\right\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)} = 0.$$

Let us denote $\psi := \overline{\partial}_{E,m,q}^{g_1,h,t}\phi$. Since

$$\Phi_{s_n}^{m,q} : L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A) \to L^2\Omega^{m,q}(A, E|_A, g_{s_n}|_A, \rho|_A)$$

is nothing but the continuous inclusion induced by the identity $\mathrm{Id} : \Omega_c^{m,q}(A, E|_A) \to \Omega_c^{m,q}(A, E|_A)$, the above limit amounts to proving that

$$\lim_{n\to\infty}\left\|((T_{s_n}^{m,q})^{-1} - (T_0^{m,q})^{-1})(\psi)\right\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)} = 0.$$

Let us define the function $f : A \times [0,1] \to \mathbb{R}$ as

$$f(p,s) := \left|((T_s^{m,q})^{-1} - (T_0^{m,q})^{-1})(\psi)\right|^2_{g_{s,m,q,\rho}^*}(p).$$

In other words, f is the function that assigns to each $p \in A$ and $s \in [0,1]$ the square of the pointwise norm of the section $(T_s^{m,q})^{-1}(\psi) - (T_0^{m,q})^{-1}(\psi)$ in p with respect to g_s and ρ. Note that $f \in C_c^\infty(A \times [0,1], \mathbb{R})$ and for each fixed $p \in A$ we have

$$\lim_{s\to 0} f(p,s) = 0. \tag{19}$$

In this way, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\left\|(T_{s_n}^{m,q})^{-1}(\overline{\partial}_{E,m,q}^{g_1,h,t}\phi) - (T_0^{m,q})^{-1}(\overline{\partial}_{E,m,q}^{g_1,h,t}\phi)\right\|^2_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}\\
&\quad= \lim_{n\to\infty}\left\|((T_{s_n}^{m,q})^{-1} - (T_0^{m,q})^{-1})(\psi)\right\|^2_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}\\
&\quad= \lim_{n\to\infty}\int_A \left|((T_{s_n}^{m,q})^{-1} - (T_0^{m,q})^{-1})(\psi)\right|^2_{g_{s_n,m,q,\rho}^*} \mathrm{dvol}_{g_{s_n}}\\
&\quad= \lim_{n\to\infty}\int_A f(p,s_n)\sqrt{\det(F_{s_n})}\,\mathrm{dvol}_{g_1}.
\end{aligned}$$

Observe now that $\det(F_s) \in C^\infty(A \times [0,1], \mathbb{R})$ and therefore, we have $f(p,s)\sqrt{\det(F_s)} \in C_c(A \times [0,1], \mathbb{R})$. Thus, by the fact that $\mathrm{vol}_{g_1}(A) < \infty$, we can apply the Lebesgue dominated convergence theorem and (19) to conclude that

$$\lim_{n\to\infty}\int_A f(p,s_n)\sqrt{\det(F_{s_n})}\,\mathrm{dvol}_{g_1} = \int_A \lim_{n\to\infty} f(p,s_n)\sqrt{\det(F_{s_n})}\,\mathrm{dvol}_{g_1} = 0.$$

Summarising, we proved that $\overline{\partial}_{E,m,q}^{g_{s_n},h,t}\phi \to \overline{\partial}_{E,m,q}^{h,h,t}\phi$ strongly as $n \to \infty$, as required. ■

Lemma 3.6. *Let* $\{s_n\}_{n\in\mathbb{N}} \subset (0,1]$ *be a sequence with* $s_n \to 0$ *as* $n \to \infty$. *Let* $\{\psi_{s_n}\}_{n\in\mathbb{N}}$, *with* $\psi_{s_n} \in \ker(\overline{\partial}^{g_{s_n},h}_{E,m,q,\max})$, *be a weakly convergent sequence to some*

$$\psi \in L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A)$$

as $n \to \infty$. *Then* $\psi \in \ker(\overline{\partial}^{h,h}_{E,m,q,\max})$.

Proof. To prove that $\psi \in \ker(\overline{\partial}^{h,h}_{E,m,q,\max})$, we have to show that

$$\langle \psi, \overline{\partial}^{h,h,t}_{E,m,q}\phi\rangle_{L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)} = 0$$

for each $\phi \in \Omega^{m,q+1}_c(A, E|_A)$. Thanks to Lemma 3.5, we have

$$\begin{aligned}\langle \psi, \overline{\partial}^{h,h,t}_{E,m,q}\phi\rangle_{L^2\Omega^{m,q}(A,E|_A,h,\rho)} &= \lim_{n\to\infty}\langle \psi_{s_n}, \overline{\partial}^{g_{s_n},h,t}_{E,m,q}\phi\rangle_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}\\ &= \lim_{n\to\infty}\langle \overline{\partial}^{g_{s_n},h}_{E,m,q,\max}\psi_{s_n}, \phi\rangle_{L^2\Omega^{m,q+1}(A,E|_A,g_{s_n}|_A,\rho|_A)} = 0\end{aligned}$$

as $\psi_{s_n} \in \ker(\overline{\partial}^{g_{s_m},h}_{E,m,q,\max})$. ∎

To state the next result, we need some auxiliary notations. Let $s \in [0,1]$ and let us consider the orthogonal decomposition

$$L^2\Omega^{m,q}(A, E|_A, g_s|_A, \rho|_A) = \ker(\overline{\partial}^{g_s,h}_{E,m,q,\max}) \oplus \left(\ker(\overline{\partial}^{g_s,h}_{E,m,q,\max})\right)^{\perp}. \tag{20}$$

We denote by π_s and τ_s the orthogonal projection on $\ker(\overline{\partial}^{g_s,h}_{E,m,q,\max})$ and $\left(\ker(\overline{\partial}^{g_s,h}_{E,m,q,\max})\right)^{\perp}$, respectively.

Lemma 3.7. *Let* $\eta \in \left(\ker(\overline{\partial}^{h,h}_{E,m,q,\max})\right)^{\perp}$. *Then given any sequence* $\{s_n\}_{n\in\mathbb{N}} \subset (0,1]$ *with* $s_n \to 0$ *as* $n \to \infty$, *we have*

$$\tau_{s_n}\left(\Phi^{m,q}_{s_n}(\eta)\right) \to \eta$$

strongly as $n \to \infty$.

Proof. Let $\psi_{s_n} := \pi_{s_n}(\Phi_{s_n}(\eta))$. Then $\psi_{s_n} \in \ker(\overline{\partial}^{g_{s_n},h}_{E,m,q,\max})$ and in order to prove the above lemma, we have to show that $\psi_{s_n} \to 0$ strongly as $n \to \infty$, that is,

$$\lim_{n\to\infty}\|\psi_{s_n}\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)} = 0.$$

Thanks to (16), we know that $\{\psi_{s_n}\}$ is a bounded sequence as

$$\begin{aligned}\|\psi_{s_n}\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)} &\le \|\Phi_{s_n}(\eta)\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}\\ &\le a\|\eta\|_{L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)}\end{aligned}$$

for each $n \in \mathbb{N}$ and thus there exists a subsequence $\{\psi_{r_n}\} \subset \{\psi_{s_n}\}$ such that $\psi_{r_n} \to \psi$ weakly as $n \to \infty$ to some $\psi \in L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A)$. We claim now that $\psi = 0$ and that $\psi_{r_n} \to 0$ strongly as $n \to \infty$. Indeed, we have

$$\begin{aligned}\lim_{n\to\infty} \|\psi_{r_n}\|^2_{L^2\Omega^{m,q}(A,E|_A,g_{r_n}|_A,\rho|_A)} &= \lim_{n\to\infty} \langle \psi_{r_n}, \pi_{r_n}(\Phi_{r_n}(\eta))\rangle_{L^2\Omega^{m,q}(A,E|_A,g_{r_n}|_A,\rho|_A)} \\ &= \lim_{n\to\infty} \langle \psi_{r_n}, \Phi_{r_n}(\eta)\rangle_{L^2\Omega^{m,q}(A,E|_A,g_{r_n}|_A,\rho|_A)} \\ &= \langle \psi, \eta\rangle_{L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)} = 0.\end{aligned}$$

Note that the second-to-last equality above follows by the fact that $\psi_{r_n} \to \psi$ weakly and $\Phi_{r_n}(\eta) \to \eta$ strongly while the last equality follows by the fact that $\eta \in \big(\ker(\overline{\partial}^{h,h}_{E,m,q,\max})\big)^{\perp}$ and $\psi \in \ker(\overline{\partial}^{h,h}_{E,m,q,\max})$, see Lemma 3.6. Now, if we fix an arbitrary subsequence $\{\psi_{s'_n}\} \subset \{\psi_{s_n}\}$ and we repeat the above argument with respect to $\{\psi_{s'_n}\}$, we find a subsequence $\{\psi_{r'_n}\} \subset \{\psi_{s'_n}\}$ such that

$$\lim_{n\to\infty} \|\psi_{r'_n}\|_{L^2\Omega^{m,q}(A,E|_A,g_{r'_n}|_A,\rho|_A)} = 0.$$

Summarising, every subsequence of $\{\psi_{s_n}\}$ has a further subsequence strongly convergent to 0. We can thus conclude that $\psi_{s_n} \to 0$ strongly as $n \to \infty$. ■

We now have all the ingredients to prove the next result.

Theorem 3.2. *Let* $\{s_n\}_{n\in\mathbb{N}} \subset (0, 1]$ *be any sequence such that* $s_n \to 0$ *as* $n \to \infty$ *and let*

$$G_{\overline{\partial}^{g_{s_n},h}_{E,m,q,\max}} : L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho|_A) \to L^2\Omega^{m,q}(A, E|_A, g_{s_n}|_A, \rho|_A)$$

be the Green operator of

$$\overline{\partial}^{g_{s_n},h}_{E,m,q,\max} : L^2\Omega^{m,q}(A, E|_A, g_{s_n}|_A, \rho|_A) \to L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho|_A).$$

If

$$G_{\overline{\partial}^{h,h}_{E,m,q,\max}} : L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho|_A) \to L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A)$$

is compact and $\operatorname{im}(\overline{\partial}^{h,h}_{E,m,q,\max}) = \operatorname{im}(\overline{\partial}^{g_1,h}_{E,m,q,\max})$, *then*

$$G_{\overline{\partial}^{g_{s_n},h}_{E,m,q,\max}} \to G_{\overline{\partial}^{h,h}_{E,m,q,\max}} \quad \textit{compactly as } n \to \infty.$$

Proof. First, we observe that by the assumption $\operatorname{im}(\overline{\partial}^{h,h}_{E,m,q,\max}) = \operatorname{im}(\overline{\partial}^{g_1,h}_{E,m,q,\max})$, we obtain immediately $\operatorname{im}(\overline{\partial}^{h,h}_{E,m,q,\max}) = \operatorname{im}(\overline{\partial}^{g_s,h}_{E,m,q,\max})$ for each $s \in (0, 1]$, as g_s and $g_{s'}$ are quasi-isometric for every $s, s' \in (0, 1]$. Now, let $\{\alpha_n\}_{n\in\mathbb{N}} \subset L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho|_A)$ be a weakly convergent sequence to some $\alpha \in L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho|_A)$, that is, $\alpha_n \rightharpoonup \alpha$ in $L^2\Omega^{m,q+1}(A, E|_A, h, \rho|_A)$ as $n \to \infty$. Let us define $\beta_n := G_{\overline{\partial}^{h,h}_{E,m,q,\max}} \alpha_n$ and $\beta := G_{\overline{\partial}^{h,h}_{E,m,q,\max}} \alpha$. Since we assumed that

$$G_{\overline{\partial}^{h,h}_{E,m,q,\max}} : L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho) \to L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A)$$

is compact, we know that $\beta_n \to \beta$ in $L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A)$ as $n \to \infty$. Let us now define $\beta_{s_n} := \Phi_{s_n}(\beta_n) \in L^2\Omega^{m,q}(A, E|_A, g_{s_n}|_A, \rho|_A)$. Thanks to Proposition 2.1, we know that $\beta_{s_n} \in \mathcal{D}(\overline{\partial}^{g_{s_n},h}_{E,m,q,\max})$ and

$$\overline{\partial}^{g_{s_n},h}_{E,m,q,\max}\beta_{s_n} = \overline{\partial}^{h,h}_{E,m,q,\max}\beta_n = \alpha_{n,1}$$

with $\alpha_{n,1}$ the orthogonal projection of α_n on $\mathrm{im}(\overline{\partial}^{h,h}_{E,m,q,\max})$. Note that the above equalities and the fact that $\mathrm{im}(\overline{\partial}^{h,h}_{E,m,q,\max}) = \mathrm{im}(\overline{\partial}^{g_{s_n},h}_{E,m,q,\max})$ for each $n \in \mathbb{N}$ imply

$$\tau_{s_n}(\beta_{s_n}) = G_{\overline{\partial}^{g_{s_n},h}_{E,m,q,\max}}\alpha_n$$

for each $n \in \mathbb{N}$, with τ_{s_n} defined in (20). We are in a position to prove that $G_{\overline{\partial}^{g_{s_n},h}_{E,m,q,\max}}\alpha_n \to \beta$ strongly as $n \to \infty$. Thanks to Proposition 3.5 and Corollary 3.1, this means that we have to show that

$$\lim_{n\to\infty}\left\|\Phi_{s_n}(\beta) - G_{\overline{\partial}^{g_{s_n},h}_{E,m,q,\max}}\alpha_n\right\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)} = 0.$$

We have

$$\begin{aligned}
&\left\|\Phi_{s_n}(\beta) - G_{\overline{\partial}^{g_{s_n},h}_{E,m,q,\max}}\alpha_n\right\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}\\
&\quad= \left\|\Phi_{s_n}(\beta) - \tau_{s_n}(\beta_{s_n})\right\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}\\
&\quad= \left\|\Phi_{s_n}(\beta) - \tau_{s_n}(\Phi_{s_n}(\beta)) + \tau_{s_n}(\Phi_{s_n}(\beta)) - \tau_{s_n}(\beta_{s_n})\right\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}\\
&\quad\leq \left\|\Phi_{s_n}(\beta) - \tau_{s_n}(\Phi_{s_n}(\beta))\right\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}\\
&\qquad+ \left\|\tau_{s_n}(\Phi_{s_n}(\beta)) - \tau_{s_n}(\beta_{s_n})\right\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}\\
&\quad\leq \left\|\Phi_{s_n}(\beta) - \tau_{s_n}(\Phi_{s_n}(\beta))\right\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}\\
&\qquad+ \left\|\Phi_{s_n}(\beta) - \Phi_{s_n}(\beta_n)\right\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}\\
\text{(by (16))}&\quad\leq \left\|\Phi_{s_n}(\beta) - \tau_{s_n}(\Phi_{s_n}(\beta))\right\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}\\
&\qquad+ a\left\|\beta - \beta_n\right\|_{L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)}.
\end{aligned}$$

We have already seen above that

$$\lim_{n\to\infty}\left\|\beta - \beta_n\right\|_{L^2\Omega^{m,q}(A,E|_A,h,\rho)} = 0.$$

Moreover, by applying Lemma 3.7, we know that

$$\lim_{n\to\infty}\left\|\Phi_{s_n}(\beta) - \tau_{s_n}(\Phi_{s_n}(\beta))\right\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho)} = 0.$$

Summarising, we proved that

$$\lim_{n\to\infty}\left\|\Phi_{s_n}(\beta) - G_{\overline{\partial}^{g_{s_n},h}_{E,m,q,\max}}\alpha_n\right\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho)} = 0$$

and so, we can conclude that $G_{\overline{\partial}^{g_{s_n},h}_{E,m,q,\max}} \to G_{\overline{\partial}^{h,h}_{E,m,q,\max}}$ compactly as $n \to \infty$. ∎

As in Theorem 3.2, we continue to assume that $\operatorname{im}(\overline{\partial}^{h,h}_{E,m,q,\max}) = \operatorname{im}(\overline{\partial}^{g_1,h}_{E,m,q,\max})$. For each $s \in (0,1]$ let us consider the following orthogonal decomposition:

$$\begin{aligned} &L^2\Omega^{m,q}(A, E|_A, g_s|_A, \rho|_A) \\ &\quad = \left(\ker(\overline{\partial}^{g_s,h}_{E,m,q,\max}) \cap \ker(\overline{\partial}^{g_1,g_s,t}_{E,m,q-1})\right) \oplus \operatorname{im}(\overline{\partial}^{g_1,g_s}_{E,m,q-1}) \oplus \operatorname{im}(\overline{\partial}^{g_s,h,t}_{E,m,q,\min}) \end{aligned} \tag{21}$$

and let

$$\pi^{m,q}_{K,s} : L^2\Omega^{m,q}(A, E|_A, g_s|_A, \rho|_A) \to \ker(\overline{\partial}^{g_s,h}_{E,m,q,\max}) \cap \ker(\overline{\partial}^{g_1,g_s,t}_{E,m,q-1})$$

and

$$\pi^{m,q}_{I,s} : L^2\Omega^{m,q}(A, E, g_s, \rho) \to \operatorname{im}(\overline{\partial}^{g_1,g_s}_{E,m,q-1})$$

be the orthogonal projections on $\ker(\overline{\partial}^{g_s,h}_{E,m,q,\max}) \cap \ker(\overline{\partial}^{g_1,g_s,t}_{E,m,q-1})$ and $\operatorname{im}(\overline{\partial}^{g_1,g_s}_{E,m,q-1})$ induced by (21). For $s = 0$, we consider the orthogonal decomposition

$$\begin{aligned} &L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A) \\ &\quad = \left(\ker(\overline{\partial}^{h,h}_{E,m,q,\max}) \cap \ker(\overline{\partial}^{g_1,h,t}_{E,m,q-1,\min})\right) \oplus \operatorname{im}(\overline{\partial}^{g_1,h}_{E,m,q-1,\max}) \oplus \operatorname{im}(\overline{\partial}^{h,h,t}_{E,m,q,\min}) \end{aligned} \tag{22}$$

and the corresponding orthogonal projections

$$\pi^{m,q}_{K,0} : L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A) \to \ker(\overline{\partial}^{h,h}_{E,m,q,\max}) \cap \ker(\overline{\partial}^{g_1,h,t}_{E,m,q-1})$$

and

$$\pi_{I,0} : L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A) \to \operatorname{im}(\overline{\partial}^{g_1,h}_{E,m,q-1}).$$

The last goal of this subsection is to show that $\pi^{m,q}_{k,s_n} \to \pi^{m,q}_{k,0}$ compactly, as $n \to +\infty$.

Lemma 3.8. *Let $\{s_n\}_{n\in\mathbb{N}} \subset (0,1]$ be a sequence with $s_n \to 0$ as $n \to \infty$. Let $\psi \in \ker(\overline{\partial}^{h,h}_{E,m,q,\max}) \cap \ker(\overline{\partial}^{g_1,h,t}_{E,m,q-1,\min})$ be arbitrarily fixed. Then*

$$\pi^{m,q}_{K,s_n}(\Phi^{m,q}_{s_n}(\psi)) \to \psi$$

strongly as $n \to \infty$.

Proof. Thanks to Corollary 3.1, we know that $\Phi^{m,q}_{s_n}(\psi) \to \psi$ strongly as $n \to \infty$. Thus, we need to show that both $\pi^{m,q}_{I,s_n}(\Phi^{m,q}_{s_n}(\psi))$ and $\Phi^{m,q}_{s_n}(\psi) - \pi^{m,q}_{K,s_n}(\Phi^{m,q}_{s_n}(\psi)) - \pi^{m,q}_{I,s_n}(\Phi^{m,q}_{s_n}(\psi))$ converge strongly to 0 as $n \to \infty$. By Proposition 2.1, we know that $\Phi^{m,q}_{s_n}(\psi) \in \ker(\overline{\partial}^{g_{s_n},h}_{E,m,q,\max})$ for each n. Hence,

$$\Phi^{m,q}_{s_n}(\psi) - \pi^{m,q}_{K,s_n}(\Phi^{m,q}_{s_n}(\psi)) - \pi^{m,q}_{I,s_n}(\Phi^{m,q}_{s_n}(\psi)) = 0$$

for each n and consequently

$$\Phi^{m,q}_{s_n}(\psi) - \pi^{m,q}_{K,s_n}(\Phi^{m,q}_{s_n}(\psi)) - \pi^{m,q}_{I,s_n}(\Phi^{m,q}_{s_n}(\psi)) \to 0$$

strongly as $n \to \infty$. Concerning $\pi^{m,q}_{I,s_n}(\Phi^{m,q}_{s_n}(\psi))$, we know that

$$\begin{aligned}\left\|\pi^{m,q}_{I,s_n}(\Phi^{m,q}_{s_n}(\psi))\right\|_{L^2\Omega^{M,q}(A,E|_A,g_{s_n}|_A,\rho|_A)} &\leq \left\|\Phi^{m,q}_{s_n}(\psi)\right\|_{L^2\Omega^{M,q}(A,E|_A,g_{s_n}|_A,\rho|_A)} \\ &\leq a\left\|\psi\right\|_{L^2\Omega^{M,q}(A,E_A,h|_A,\rho|_A)},\end{aligned}$$

see (16). Thus, $\{\pi^{m,q}_{I,s_n}(\Phi^{m,q}_{s_n}(\psi))\}_{n\in\mathbb{N}}$ is a bounded sequence and therefore by Proposition 2.2 there exists a subsequence $\{\pi^{m,q}_{I,s'_n}(\Phi^{m,q}_{s'_n}(\psi))\}_{n\in\mathbb{N}} \subset \{\pi^{m,q}_{I,s_n}(\Phi^{m,q}_{s_n}(\psi))\}_{n\in\mathbb{N}}$ and an element $\psi_0 \in L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A)$ such that

$$\pi^{m,q}_{I,s'_n}(\Phi^{m,q}_{s'_n}(\psi)) \to \psi_0$$

weakly as $n \to \infty$. Since $\pi^{m,q}_{I,s'_n}(\Phi^{m,q}_{s'_n}(\psi)) \in \operatorname{im}(\overline{\partial}^{g_1,g_{s'_n}}_{E,m,q-1})$ and $\pi^{m,q}_{I,s'_n}(\Phi^{m,q}_{s'_n}(\psi)) \to \psi_0$ weakly as $n \to \infty$, we can argue, as in the proof of Theorem 3.1, to conclude that $\psi_0 \in \operatorname{im}(\overline{\partial}^{g_1,h}_{E,m,q,\max})$. We want to show that $\psi_0 = 0$ and that $\pi^{m,q}_{I,s'_n}(\Phi^{m,q}_{s'_n}(\psi)) \to 0$ strongly as $n \to \infty$. Let $\beta \in \operatorname{im}(\overline{\partial}^{g_1,h}_{E,m,q,\max})$ be arbitrarily fixed. We have

$$\begin{aligned}&\langle\psi_0,\beta\rangle_{L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)} \\ &\quad= \lim_{n\to\infty}\langle\pi^{m,q}_{I,s'_n}(\Phi^{m,q}_{s'_n}(\psi)),\Phi^{m,q}_{s'_n}(\beta)\rangle_{L^2\Omega^{m,q}(A,E|_A,g_{s'_n}|_A,\rho|_A)} \\ &\quad= \lim_{n\to\infty}\langle\pi^{m,q}_{I,s'_n}(\Phi^{m,q}_{s'_n}(\psi)) + \pi^{m,q}_{K,s'_n}(\Phi^{m,q}_{s'_n}(\psi)),\Phi_{s'_n}(\beta)\rangle_{L^2\Omega^{m,q}(A,E|_A,g_{s'_n}|_A,\rho|_A)} \\ &\quad= \lim_{s\to\infty}\langle\Phi^{m,q}_{s'_n}(\psi),\Phi^{m,q}_{s'_n}(\beta)\rangle_{L^2\Omega^{m,q}(A,E|_A,g_{s'_n}|_A,\rho|_A)} \\ &\quad= \langle\psi,\beta\rangle_{L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)} = 0.\end{aligned}$$

Note that the first equality above follows by the fact that $\pi^{m,q}_{I,s'_n}(\Phi^{m,q}_{s'_n}(\psi)) \to \psi_0$ weakly and $\Phi^{m,q}_{s'_n}(\beta) \to \beta$ strongly. The second equality is a consequence of the fact that $\Phi^{m,q}_{s}(\beta) \in \operatorname{im}(\overline{\partial}^{g_1,g_s}_{E,m,q})$ as $\operatorname{im}(\overline{\partial}^{g_1,h}_{E,m,q,\max}) \subset \operatorname{im}(\overline{\partial}^{g_1,g_s}_{E,m,q})$ for each $0 < s \leq 1$. Finally, the third equality follows by the fact that $\Phi^{m,q}_{s'_n}(\psi) - \pi^{m,q}_{K,s'_n}(\Phi^{m,q}_{s'_n}(\psi)) - \pi^{m,q}_{I,s'_n}(\Phi^{m,q}_{s'_n}(\psi)) = 0$ for each n. We can thus conclude that $\psi_0 = 0$ as $\langle\psi_0,\beta\rangle_{L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)} = 0$ for any arbitrarily fixed $\beta \in \operatorname{im}(\overline{\partial}^{g_1,h}_{E,m,q,\max})$. We are left to show that $\pi^{m,q}_{I,s'_n}(\Phi_{s'_n}(\psi)) \to 0$ strongly as $n \to \infty$. To this aim we have

$$\begin{aligned}&\lim_{n\to\infty}\langle\pi^{m,q}_{I,s'_n}(\Phi^{m,q}_{s'_n}(\psi)),\pi^{m,q}_{I,s'_n}(\Phi^{m,q}_{s'_n}(\psi))\rangle_{L^2\Omega^{m,q}(A,E|_A,g_{s'_n}|_A,\rho|_A)} \\ &\quad= \lim_{n\to\infty}\langle\pi^{m,q}_{I,s'_n}(\Phi^{m,q}_{s'_n}(\psi)),\pi^{m,q}_{K,s'_n}(\Phi^{m,q}_{s'_n}(\psi)) + \pi^{m,q}_{I,s'_n}(\Phi^{m,q}_{s'_n}(\psi))\rangle_{L^2\Omega^{m,q}(A,E|_A,g_{s'_n}|_A,\rho|_A)} \\ &\quad= \lim_{n\to\infty}\langle\pi^{m,q}_{I,s'_n}(\Phi^{m,q}_{s'_n}(\psi)),\Phi^{m,q}_{s'_n}(\psi)\rangle_{L^2\Omega^{m,q}(A,E|_A,g_{s'_n}|_A,\rho|_A)} \\ &\quad= \langle\psi_0,\psi\rangle_{L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)} = 0.\end{aligned}$$

In conclusion, $\pi^{m,q}_{I,s'_n}(\Phi^{m,q}_{s'_n}(\psi)) \to 0$ strongly as $n \to \infty$. Now, if we fix an arbitrary subsequence $\{r_n\}_{n\in\mathbb{N}} \subset \{s_n\}_{n\in\mathbb{N}}$ and we repeat the above argument, we conclude that there exists a subsequence $\{r'_n\}_{n\in\mathbb{R}} \subset \{r_n\}_{n\in\mathbb{N}}$ such that $\pi_{I,r'_n}(\Phi^{m,q}_{r'_n}(\psi)) \to 0$ strongly as $n \to \infty$. We can thus conclude that $\pi^{m,q}_{I,s_n}(\Phi^{m,q}_{s_n}(\psi)) \to 0$ strongly as $n \to \infty$ and hence that $\pi^{m,q}_{K,s_n}(\Phi^{m,q}_{s_n}(\psi)) \to \psi$ strongly as $n \to \infty$. ∎

Lemma 3.9. *If* $\dim(H^{m,q}_{\overline{\partial}_E}(M,E)) = \dim(H^{m,q}_{2,\overline{\partial}_{\max}}(A, E|_A, h|_A, \rho|_A))$*, then for each* $s \in (0,1]$ *the linear map*

$$\pi^{m,q}_{K,s} \circ \Phi^{m,q}_s : \ker(\overline{\partial}^{g_1,h,t}_{E,m,q-1,\min}) \cap \ker(\overline{\partial}^{h,h}_{E,m,q,\max}) \to \ker(\overline{\partial}^{g_1,g_s,t}_{E,m,q-1}) \cap \ker(\overline{\partial}^{g_s,h}_{E,m,q,\max})$$

is an isomorphism.

Proof. By assumption, we know that g_1 and g_s are quasi-isometric for each $s \in (0,1]$ and that $\mathrm{im}(\overline{\partial}^{g_1,h}_{E,m,q-1,\max}) = \mathrm{im}(\overline{\partial}^{h,h}_{E,m,q-1,\max})$ in $L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)$. This tells us that for each $s \in (0,1]$ we have $\mathrm{im}(\overline{\partial}^{g_1,h}_{E,m,q-1,\max}) = \mathrm{im}(\overline{\partial}^{g_s,h}_{E,m,q-1,\max})$ in $L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)$ and thus

$$\Phi^{m,q}_s : L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A) \to L^2\Omega^{m,q}(A,E|_A,g_s|_A,\rho|_A)$$

induces an injective linear map between the L^2-$\overline{\partial}$-cohomology groups

$$\Phi^{m,q}_s : \ker(\overline{\partial}^{h,h}_{E,m,q,\max})/\mathrm{im}(\overline{\partial}^{g_1,h}_{E,m,q-1,\max}) \to \ker(\overline{\partial}^{g_s,h}_{E,m,q,\max})/\mathrm{im}(\overline{\partial}^{g_1,g_s}_{E,m,q-1}). \tag{23}$$

Note that we have

$$\begin{aligned}
\dim\big(H^{m,q}_{2,\overline{\partial}_{\max}}(A,E|_A,h|_A,\rho|_A)\big) &= \dim\big(\ker(\overline{\partial}^{h,h}_{E,m,q,\max})/\mathrm{im}(\overline{\partial}^{g_1,h}_{E,m,q-1,\max})\big)\\
&\le \dim\big(\ker(\overline{\partial}^{g_s,h}_{E,m,q,\max})/\mathrm{im}(\overline{\partial}^{g_1,g_s}_{E,m,q-1})\big)\\
&\le \dim\big(\ker(\overline{\partial}^{g_1,g_1}_{E,m,q})/\mathrm{im}(\overline{\partial}^{g_1,g_1}_{E,m,q-1})\big)\\
&= \dim\big(H^{m,q}_{\overline{\partial}_E}(M,E)\big)\\
&= \dim\big(H^{m,q}_{\overline{\partial}_{\max}}(A,E|_A,h|_A,\rho|_A)\big).
\end{aligned}$$

We can thus conclude that (23) is an isomorphism for each $0 < s \le 1$. Thanks to (21) and (22) we have isomorphisms

$$\ker(\overline{\partial}^{g_1,h,t}_{E,m,q-1,\min}) \cap \ker(\overline{\partial}^{h,h}_{E,m,q,\max}) \cong \ker(\overline{\partial}^{h,h}_{E,m,q,\max})/\mathrm{im}(\overline{\partial}^{g_1,h}_{E,m,q-1,\max}) \tag{24}$$

and

$$\ker(\overline{\partial}^{g_1,g_s,t}_{E,m,q-1}) \cap \ker(\overline{\partial}^{g_s,h}_{E,m,q,\max}) \cong \ker(\overline{\partial}^{g_s,h}_{E,m,q,\max})/\mathrm{im}(\overline{\partial}^{g_1,g_s}_{E,m,q-1}). \tag{25}$$

It is easy to check that if $\alpha \in \ker(\overline{\partial}^{g_1,h,t}_{E,m,q-1,\min}) \cap \ker(\overline{\partial}^{h,h}_{E,m,q,\max})$ is the unique representative in $\ker(\overline{\partial}^{g_1,h,t}_{E,m,q-1,\min}) \cap \ker(\overline{\partial}^{h,h}_{E,m,q,\max})$ of $[\alpha] \in \ker(\overline{\partial}^{h,h}_{E,m,q,\max})/\mathrm{im}(\overline{\partial}^{g_1,h}_{E,m,q-1,\max})$, then $\pi^{m,q}_{K,s}(\Phi^{m,q}_s(\alpha))$ is the unique representative in $\ker(\overline{\partial}^{g_1,g_s,t}_{E,m,q-1}) \cap \ker(\overline{\partial}^{g_s,h}_{E,m,q,\max})$ of $[\Phi^{m,q}_s(\alpha)] \in \ker(\overline{\partial}^{g_s,h}_{E,m,q,\max})/\mathrm{im}(\overline{\partial}^{g_1,g_s}_{E,m,q-1})$. The conclusion now follows immediately by (23), (24) and (25). ∎

Theorem 3.3. *In the setting of Theorem 3.2, assume in addition that* $\dim(H^{m,q}_{\overline{\partial}}(M,E)) = \dim(H^{m,q}_{2,\overline{\partial}_{\max}}(A,E,h,\rho))$. *Let* $\{s_n\}_{n\in\mathbb{N}} \subset (0,1]$ *be a sequence with* $s_n \to 0$ *as* $n \to \infty$. *Then*

$$\pi^{m,q}_{K,s} \to \pi^{m,q}_{K,0}$$

compactly as $n \to \infty$.

Proof. Let $\{\psi_1,\dots,\psi_\ell\}$ be an orthonormal basis of $\ker(\overline{\partial}^{g_1,h,t}_{E,m,q-1,\min}) \cap \ker(\overline{\partial}^{h,h}_{E,m,q,\max})$ and for each $j \in \{1,\dots,\ell\}$ let $\psi_{j,s_n} := \pi^{m,q}_{K,s_n}(\Phi^{m,q}_{s_n}(\psi_j))$. Then, by Lemmas 3.8 and 3.9 we know that $\{\psi_{1,s_n},\dots,\psi_{\ell,s_n}\}$ is a basis for $\ker(\overline{\partial}^{g_1,g_{s_n},t}_{E,m,q-1}) \cap \ker(\overline{\partial}^{g_{s_n},h}_{E,m,q,\max})$ and $\psi_{j,s_n} \to \psi_j$ strongly as $n \to \infty$. Let $\{\chi_{1,s_n},\dots,\chi_{\ell,s_n}\}$ be the basis of $\ker(\overline{\partial}^{g_1,g_{s_n},t}_{E,m,q-1}) \cap \ker(\overline{\partial}^{g_{s_n},h}_{E,m,q,\max})$ made by pairwise orthogonal elements obtained by applying the Gram–Schmidt procedure to the basis $\{\psi_{1,s_n},\dots,\psi_{\ell,s_n}\}$. Explicitly, we have

$$\begin{aligned}
\chi_{1,s_n} &= \psi_{1,s_n}\\
\chi_{2,s_n} &= \psi_{2,s_n} - \mathrm{pr}_{\chi_{1,s_n}}(\psi_{2,s_n})\\
&\dots\\
\chi_{j,s_n} &= \psi_{j,s_n} - \sum_{k=1}^{j-1} \mathrm{pr}_{\chi_{k,s_n}}(\psi_{j,s_n})\\
&\dots\\
\chi_{\ell,s_n} &= \psi_{\ell,s_n} - \sum_{k=1}^{\ell-1} \mathrm{pr}_{\chi_{k,s_n}}(\psi_{\ell,s_n}),
\end{aligned}$$

where

$$\mathrm{pr}_{\chi_{k,s_n}}(\psi_{j,s_n}) := \frac{\langle \psi_{j,s_n}, \chi_{k,s_n}\rangle_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}}{\langle \chi_{k,s_n}, \chi_{k,s_n}\rangle_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}}\chi_{k,s_n} \tag{26}$$

for each $j \in \{1,\dots,\ell\}$ and $k \in \{1,\dots,j-1\}$. Looking at (26) and arguing by induction it is easy to check that

$$\lim_{n\to\infty} \mathrm{pr}_{\chi_{k,s_n}}(\psi_{j,s_n}) = 0$$

strongly and consequently

$$\chi_{j,s_n} \to \psi_j$$

strongly as $n \to \infty$ for each $j = 1,\dots,\ell$. In particular,

$$\|\chi_{j,s_n}\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)} \to 1 = \|\psi_j\|_{L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)}$$

as $n \to \infty$ for each $j = 1,\dots,\ell$. Therefore, by defining

$$\varphi_{j,s_n} := \chi_{j,s_n}/\|\chi_{j,s_n}\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)},$$

we obtain an orthonormal basis of $\ker(\overline{\partial}^{g_1,g_{s_n},t}_{E,m,q-1}) \cap \ker(\overline{\partial}^{g_{s_n},h}_{E,m,q,\max})$ made by $\{\varphi_{1,s_n},\ldots,\varphi_{\ell,s_n}\}$ such that $\varphi_{j,s_n} \to \psi_j$ strongly as $n \to \infty$ for each $j = 1,\ldots,\ell$. Let now $\{\beta_{s_n}\}_{n\in\mathbb{N}}$, $\beta_{s_n} \in L^2\Omega^{m,q}(A, E|_A, g_{s_n}|_A, \rho|_A)$, be a weakly convergent sequence to some $\beta \in L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A)$ as $n \to \infty$. We want to show that $\pi^{m,q}_{K,s_n}\beta_{s_n} \to \pi^{m,q}_{K,0}\beta$ strongly as $n \to \infty$. We have

$$\pi^{m,q}_{K,s_n}\beta_{s_n} = \sum_{j=1}^{\ell}\langle\varphi_{j,s_n},\beta_{s_n}\rangle_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}\varphi_{j,s_n}.$$

Thanks to the first part of the proof, we know that $\varphi_{j,s_n} \to \psi_j$ strongly as $n \to \infty$. Since β_{s_n} converges weakly to β as $n \to \infty$, we get that

$$\langle\varphi_{j,s_n},\beta_{s_n}\rangle_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)} \to \langle\psi_j,\beta\rangle_{L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)}$$

as $n \to \infty$. Therefore, we can conclude that

$$\sum_{j=1}^{\ell}\langle\varphi_{j,s_n},\beta_{s_n}\rangle_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}\varphi_{j,s_n} \to \sum_{j=1}^{\ell}\langle\psi_j,\beta\rangle_{L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)}\psi_j$$

strongly as $j \to \infty$, that is,

$$\pi^{m,q}_{K,s_n}\beta_{s_n} \to \pi^{m,q}_{K,0}\beta$$

strongly as $n \to \infty$. ∎

3.2. From compact convergence to convergence in norm operator

As in Lemma 3.5, let $p : A \times [0,1] \to A$ be the left projection and let

$$S^{m,q}_s \in C^\infty(A \times [0,1], \mathrm{End}(p^*\Lambda^{m,q}(A) \otimes p^*E))$$

be defined as $S^{m,q}_s := \det(G^{1,0}_{\mathbb{C},s}) \otimes G^{0,q}_{\mathbb{C},s} \otimes \mathrm{Id}$. We recall that $S^{m,q}_s$ is the family of endomorphisms of $p^*\Lambda^{m,q}(A) \otimes p^*E$ such that $g^*_{m,q,\rho,s}(\bullet,\bullet) = g^*_{m,q,\rho,1}(S^{m,q}_s\bullet,\bullet)$. It is not difficult to see that there exists $\Gamma^{m,q}_s \in C(A \times [0,1], \mathrm{End}(p^*\Lambda^{m,q}(A) \otimes p^*E))$, that is, a continuous section of $\mathrm{End}(p^*\Lambda^{m,q}(A) \otimes p^*E) \to A \times [0,1]$, such that

(1) $(\Gamma^{m,q}_s)^2 = S^{m,q}_s$;

(2) $g^*_{1,m,q,\rho}(\Gamma^{m,q}_s\bullet,\bullet) = g^*_{1,m,q,\rho}(\bullet,\Gamma^{m,q}_s\bullet)$, that is, $\Gamma^{m,q}_s$ is fiberwise self-adjoint w.r.t. $g^*_{1,m,q,\rho}$;

(3) $g^*_{1,m,q,\rho}(\Gamma^{m,q}_s\bullet,\bullet) > 0$ whenever $\bullet \neq 0$, that is, $\Gamma^{m,q}_s$ is positive definite w.r.t. $g^*_{1,m,q,\rho}$;

see, e.g., [21, Problem 2-E, p. 24]. Note that $g^*_{1,m,q,\rho}(\Gamma^{m,q}_s\bullet,\Gamma^{m,q}_s\bullet) = g^*_{s,m,q,\rho}(\bullet,\bullet)$. In other words, $\Gamma^{m,q}_s$ is a fiberwise isometry between $p^*\Lambda^{m,q}(A) \otimes p^*E$ endowed with $g^*_{s,m,q,\rho}$ and $p^*\Lambda^{m,q}(A)\otimes p^*E$ endowed with $g^*_{m,q,\rho,1}$. Now let us define $\Psi^{m,q}_s \in C(A\times[0,1],\mathrm{End}(p^*\Lambda^{m,q}(A)\otimes p^*E))$ as

$$\Psi^{m,q}_s := (\det(F^{1,0}_{\mathbb{C},s}))^{\frac{1}{2}}\otimes \Gamma^{m,q}_s. \tag{27}$$

Let us check that

$$\Psi^{m,q}_s : L^2\Omega^{m,q}(A,E|_A,g_s|_A,\rho|_A)\to L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)$$

is an isometry for each $s\in[0,1]$. Let $\eta,\omega\in\Omega^{m,q}_c(A,E|_A)$. We have

$$\begin{aligned}\left\langle\Psi^{m,q}_s\eta,\Psi^{m,q}_s\omega\right\rangle_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} &= \int_A g^*_{1,m,q,\rho}(\Psi^{m,q}_s\eta,\Psi^{m,q}_s\omega)\,\mathrm{dvol}_{g_1}\\ &= \int_A g^*_{1,m,q,\rho}(\Gamma^{m,q}_s\eta,\Gamma^{m,q}_s\omega)\det(F^{1,0}_{\mathbb{C},s})\,\mathrm{dvol}_{g_1}\\ &= \int_A g^*_{1,m,q,\rho}(S^{m,q}_s\eta,\omega)(\det(F_s))^{\frac{1}{2}}\,\mathrm{dvol}_{g_1}\\ &= \int_A g^*_{s,m,q,\rho}(\eta,\omega)\,\mathrm{dvol}_{g_s}\\ &= \langle\eta,\omega\rangle_{L^2\Omega^{m,q}(A,E|_A,g_s|_A,\rho|_A)}.\end{aligned}$$

We now prove various properties concerning $\Psi^{m,q}_s$.

Lemma 3.10. *Given any $\eta\in L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)$, it holds*

$$\lim_{s\to 0}\left\|\Psi^{m,q}_s(\Phi^{m,q}_s(\eta))-\Psi^{m,q}_0(\eta)\right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}=0.$$

Proof. First, we deal with the case $\eta\in\Omega^{m,q}_c(A,E|_A)$. In this case

$$\begin{aligned}&\left\|\Psi^{m,q}_s(\Phi^{m,q}_s(\eta))-\Psi^{m,q}_0(\eta)\right\|^2_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\\ &\quad=\int_A g^*_{1,m,q,\rho}(\Psi^{m,q}_s\eta-\Psi^{m,q}_0\eta,\Psi^{m,q}_s\eta-\Psi^{m,q}_0\eta)\,\mathrm{dvol}_{g_1}.\end{aligned}$$

Since $\eta\in\Omega^{m,q}_c(A,E|_A)$, $\Psi^{m,q}_s\in C(A\times[0,1],p^*\mathrm{End}(p^*\Lambda^{m,q}(A)\otimes p^*E))$ and $\mathrm{vol}_{g_1}(A)<\infty$, we can apply the Lebesgue dominated convergence theorem:

$$\begin{aligned}&\lim_{s\to 0}\left\|\Psi^{m,q}_s(\Phi^{m,q}_s(\eta))-\Psi^{m,q}_0(\eta)\right\|^2_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\\ &=\lim_{s\to 0}\int_A g^*_{1,m,q,\rho}(\Psi^{m,q}_s\eta-\Psi^{m,q}_0\eta,\Psi^{m,q}_s\eta-\Psi^{m,q}_0\eta)\,\mathrm{dvol}_{g_1}\\ &=\int_A \lim_{s\to 0} g^*_{1,m,q,\rho}(\Psi^{m,q}_s\eta-\Psi^{m,q}_0\eta,\Psi^{m,q}_s\eta-\Psi^{m,q}_0\eta)\,\mathrm{dvol}_{g_1}\\ &=0.\end{aligned}$$

Now, we consider the general case $\eta \in L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A)$. Let $\varepsilon > 0$ be arbitrarily fixed and let $\varphi \in \Omega_c^{m,q}(A, E|_A)$ be such that $\|\eta - \varphi\|_{L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)} < \varepsilon$. Let a be the positive constant appearing in (16). Since $\Psi_s^{m,q} : L^2\Omega^{m,q}(A, E|_A, \rho|_A, g_s|_A) \to L^2\Omega^{m,q}(A, E|_A, \rho|_A, g_1|_A)$ is an isometry for each $s \in [0, 1]$, we have

$$\begin{aligned}
&\left\| \Psi_s^{m,q}(\Phi_s^{m,q}(\eta)) - \Psi_0^{m,q}(\eta) \right\|^2_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} \\
&\quad = \left\| \Psi_s^{m,q}(\Phi_s^{m,q}(\eta)) - \Psi_s^{m,q}(\Phi_s^{m,q}(\varphi)) + \Psi_s^{m,q}(\Phi_s^{m,q}(\varphi)) \right. \\
&\qquad \left. - \Psi_0^{m,q}(\eta) \right\|^2_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} \\
&\quad \le \left\| \Psi_s^{m,q}(\Phi_s^{m,q}(\eta)) - \Psi_s^{m,q}(\Phi_s^{m,q}(\varphi)) \right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} \\
&\qquad + \left\| \Psi_s^{m,q}(\Phi_s^{m,q}(\varphi)) - \Psi_0^{m,q}(\eta) \right\|^2_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} \\
&\quad = \left\| \eta - \varphi \right\|_{L^2\Omega^{m,q}(A,E|_A,g_s|_A,\rho|_A)} \\
&\qquad + \left\| \Psi_s^{m,q}(\Phi_s^{m,q}(\varphi)) - \Psi_0^{m,q}(\varphi) + \Psi_0^{m,q}(\varphi) - \Psi_0^{m,q}(\eta) \right\|^2_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} \\
&\quad \le a \left\| \eta - \varphi \right\|_{L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)} + \left\| \Psi_s^{m,q}(\Phi_s^{m,q}(\varphi)) - \Psi_0^{m,q}(\varphi) \right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} \\
&\qquad + \left\| \Psi_0^{m,q}(\varphi) - \Psi_0^{m,q}(\eta) \right\|^2_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} \\
&\quad \le a\varepsilon + \left\| \Psi_s^{m,q}(\Phi_s^{m,q}(\varphi)) - \Psi_0^{m,q}(\varphi) \right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} + \varepsilon.
\end{aligned}$$

Since $\phi \in \Omega_c^{m,q}(A, E|_A)$, we can conclude as above that

$$\lim_{s \to 0} \left\| \Psi_s^{m,q}(\Phi_s^{m,q}(\varphi)) - \Psi_0^{m,q}(\varphi) \right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} = 0$$

which in turn gives us

$$\limsup_{s \to 0} \left\| \Psi_s^{m,q}(\Phi_s^{m,q}(\eta)) - \Psi_0^{m,q}(\eta) \right\|^2_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} \le (a+1)\varepsilon.$$

Since ε is arbitrarily fixed, we can conclude that

$$\lim_{s \to 0} \left\| \Psi_s^{m,q}(\Phi_s^{m,q}(\eta)) - \Psi_0^{m,q}(\eta) \right\|^2_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} = 0$$

as desired. ∎

Lemma 3.11. *Let $\{s_n\}_{n\in\mathbb{N}} \subset (0, 1]$ be a sequence such that $s_n \to 0$ as $n \to \infty$. Let $\eta \in L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A)$ and let $\{\eta_{s_n}\}_{n\in\mathbb{N}}$ be a sequence such that $\eta_{s_n} \in L^2\Omega^{m,q}(A, E|_A, g_{s_n}|_A, \rho|_A)$. Then $\eta_{s_n} \to \eta$ strongly as $n \to \infty$ if and only if*

$$\lim_{n \to \infty} \left\| \Psi_{s_n}^{m,q}\eta_{s_n} - \Psi_0^{m,q}\eta \right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} = 0. \tag{28}$$

Proof. Let us assume that $\eta_{s_n} \to \eta$ strongly as $n \to \infty$. We have

$$\begin{aligned}
&\left\|\Psi^{m,q}_{s_n}\eta_{s_n} - \Psi^{m,q}_0\eta\right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\\
&\quad= \left\|\Psi^{m,q}_{s_n}\eta_{s_n} - \Psi^{m,q}_{s_n}(\Phi^{m,q}_{s_n}(\eta)) + \Psi^{m,q}_{s_n}(\Phi^{m,q}_{s_n}(\eta)) - \Psi^{m,q}_0\eta\right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\\
&\quad\leq \left\|\Psi^{m,q}_{s_n}\eta_{s_n} - \Psi^{m,q}_{s_n}(\Phi^{m,q}_{s_n}(\eta))\right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\\
&\qquad+ \left\|\Psi^{m,q}_{s_n}(\Phi^{m,q}_{s_n}(\eta)) - \Psi^{m,q}_0\eta\right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\\
&\quad= \left\|\eta_{s_n} - \Phi^{m,q}_{s_n}(\eta)\right\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho)}\\
&\qquad+ \left\|\Psi^{m,q}_{s_n}(\Phi^{m,q}_{s_n}(\eta)) - \Psi^{m,q}_0\eta\right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}.
\end{aligned}$$

As $\eta_{s_n} \to \eta$ strongly as $n \to \infty$, we know that

$$\lim_{n\to\infty}\left\|\eta_{s_n} - \Phi^{m,q}_{s_n}(\eta)\right\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)} = 0.$$

Furthermore, Lemma 3.10 tells us that

$$\lim_{n\to\infty}\left\|\Psi^{m,q}_{s_n}(\Phi^{m,q}_{s_n}(\eta)) - \Psi^{m,q}_0\eta\right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} = 0.$$

We can thus conclude that

$$\lim_{n\to\infty}\left\|\Psi^{m,q}_{s_n}\eta_{s_n} - \Psi^{m,q}_0\eta\right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} = 0.$$

Conversely, let us assume (28). We want to show that $\eta_{s_n} \to \eta$ strongly as $n \to \infty$, that is,

$$\lim_{n\to\infty}\left\|\eta_{s_n} - \Phi^{m,q}_{s_n}\eta\right\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)} = 0.$$

We have

$$\begin{aligned}
&\left\|\eta_{s_n} - \Phi^{m,q}_{s_n}\eta\right\|_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}\\
&\quad= \left\|\Psi^{m,q}_{s_n}\eta_{s_n} - \Psi^{m,q}_0\eta + \Psi^{m,q}_0\eta - \Psi^{m,q}_{s_n}(\Phi^{m,q}_{s_n}\eta)\right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\\
&\quad\leq \left\|\Psi^{m,q}_{s_n}\eta_{s_n} - \Psi^{m,q}_0\eta\right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\\
&\qquad+ \left\|\Psi^{m,q}_0\eta - \Psi^{m,q}_{s_n}(\Phi^{m,q}_{s_n}\eta)\right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}.
\end{aligned}$$

We assumed that

$$\lim_{n\to\infty}\left\|\Psi^{m,q}_{s_n}\eta_{s_n} - \Psi^{m,q}_0\eta\right\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} = 0$$

and by Lemma 3.10 we know that

$$\lim_{n\to\infty}\left|\|\Psi^{m,q}_0\eta - \Psi^{m,q}_{s_n}(\Phi^{m,q}_{s_n}\eta)\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\right| = 0.$$

We can thus conclude that $\eta_{s_n} \to \eta$ strongly as $n \to \infty$. ∎

Lemma 3.12. *Let $\{s_n\}_{n\in\mathbb{N}} \subset (0,1]$ be a sequence such that $s_n \to 0$ as $n \to \infty$. Let $\eta \in L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A)$ and let $\{\eta_{s_n}\}_{n\in\mathbb{N}}$ be a sequence such that $\eta_{s_n} \in L^2\Omega^{m,q}(A, E|_A, g_{s_n}|_A, \rho|_A)$. Then $\eta_{s_n} \to \eta$ weakly as $n \to \infty$ if and only if*

$$\Psi^{m,q}_{s_n}\eta_{s_n} \rightharpoonup \Psi^{m,q}_0\eta$$

in $L^2\Omega^{m,q}(A, E|_A, g_1|_A, \rho|_A)$ as $n \to \infty$.

Proof. Assume that $\eta_{s_n} \to \eta$ weakly as $n \to \infty$. Let $\omega \in L^2\Omega^{m,q}(A, g_1|_A, E|_A, \rho|_A)$. Thanks to Lemma 3.11, we know that $(\Psi^{m,q}_{s_n})^{-1}\omega \to (\Psi^{m,q}_0)^{-1}\omega$ strongly as $n \to \infty$. Thus, we obtain

$$\begin{aligned}\lim_{n\to\infty}\langle\Psi^{m,q}_{s_n}\eta_{s_n}, \omega\rangle_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} &= \lim_{n\to\infty}\langle\eta_{s_n}, (\Psi^{m,q}_{s_n})^{-1}\omega\rangle_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)}\\ &= \langle\eta, (\Psi^{m,q}_0)^{-1}\omega\rangle_{L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)}\\ &= \langle\Psi^{m,q}_0\eta, \omega\rangle_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\end{aligned}$$

as required. Conversely, let us assume that $\Psi^{m,q}_{s_n}\eta_{s_n} \rightharpoonup \Psi^{m,q}_0\eta$ in $L^2\Omega^{m,q}(A, E|_A, g_1|_A, \rho|_A)$ as $n \to \infty$. Let $\{\phi_{s_n}\}_{n\in\mathbb{N}}$ be a sequence such that $\phi_{s_n} \in L^2\Omega^{m,q}(A, E|_A, g_{s_n}|_A, \rho|_A)$ and $\phi_{s_n} \to \phi$ strongly to some $\phi \in L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A)$ as $n \to \infty$. Lemma 3.11 tells us that $\|\Psi^{m,q}_{s_n}\phi_{s_n} - \Psi^{m,q}_0\phi\|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} \to 0$ as $n \to \infty$. Hence, we obtain

$$\begin{aligned}\lim_{n\to\infty}\langle\eta_{s_n}, \phi_{s_n}\rangle_{L^2\Omega^{m,q}(A,E|_A,g_{s_n}|_A,\rho|_A)} &= \lim_{n\to\infty}\langle\Psi^{m,q}_{s_n}\eta_{s_n}, \Psi^{m,q}_{s_n}\phi_{s_n}\rangle_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\\ &= \langle\Psi^{m,q}_0\eta, \Psi^{m,q}_0\phi\rangle_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)}\\ &= \langle\eta, \phi\rangle_{L^2\Omega^{m,q}(A,E|_A,h|_A,\rho|_A)}\end{aligned}$$

as desired. ■

Lemma 3.13. *Let $\{s_n\}_{n\in\mathbb{N}} \subset (0,1]$ be a sequence such that $s_n \to 0$ as $n \to \infty$. Then*

$$\lim_{n\to\infty}\left\|\Psi^{m,q}_{s_n} \circ G_{\overline{\partial}^{g_{s_n},h}_{E,m,q,\max}} \circ (\Psi^{m,q+1}_0)^{-1} - \Psi^{m,q}_0 \circ G_{\overline{\partial}^{h,h}_{E,m,q,\max}} \circ (\Psi^{m,q+1}_0)^{-1}\right\|_{\mathrm{op}} = 0$$

and

$$\lim_{n\to\infty}\left\|G_{\overline{\partial}^{g_1,g_{s_n}}_{E,m,q}} \circ (\Psi^{m,q+1}_{s_n})^{-1} - G_{\overline{\partial}^{g_1,h}_{E,m,q,\max}} \circ (\Psi^{m,q+1}_0)^{-1}\right\|_{\mathrm{op}} = 0.$$

Proof. The first limit above follows immediately by Proposition 2.4, Theorem 3.2 and Lemmas 3.11 and 3.12. The second one follows immediately by Proposition 2.4, Theorem 3.1 and Lemmas 3.11 and 3.12. ■

Lemma 3.14. *Let $\{s_n\}_{n\in\mathbb{N}} \subset (0,1]$ be a sequence such that $s_n \to 0$ as $n \to \infty$. Then*

$$\lim_{n\to\infty}\left\|\Psi^{m,q+1}_0 \circ G_{\overline{\partial}^{g_{s_n},h,t}_{E,m,q,\min}} \circ (\Psi^{m,q}_{s_n})^{-1} - \Psi^{m,q+1}_0 \circ G_{\overline{\partial}^{h,h,t}_{E,m,q,\min}} \circ (\Psi^{m,q}_0)^{-1}\right\|_{\mathrm{op}} = 0$$

and

$$\lim_{n\to\infty}\left\|(\Psi^{m,q+1}_{s_n}) \circ G_{\overline{\partial}^{g_1,g_{s_n},t}_{E,m,q}} - (\Psi^{m,q+1}_0) \circ G_{\overline{\partial}^{g_1,h,t}_{E,m,q,\min}}\right\|_{\mathrm{op}} = 0.$$

Proof. Note that

$$\left(\Psi^{m,q}_{s_n}\circ G_{\overline{\partial}^{g_{s_n},h}_{E,m,q,\max}}\circ(\Psi^{m,q+1}_0)^{-1}\right)^*=\Psi^{m,q+1}_0\circ G_{\overline{\partial}^{g_{s_n},h,t}_{E,m,q,\min}}\circ(\Psi^{m,q}_{s_n})^{-1}$$

and

$$\left(\Psi^{m,q}_0\circ G_{\overline{\partial}^{h,h}_{E,m,q,\max}}\circ(\Psi^{m,q+1}_0)^{-1}\right)^*=\Psi^{m,q+1}_0\circ G_{\overline{\partial}^{h,h,t}_{E,m,q,\min}}\circ(\Psi^{m,q}_0)^{-1}.$$

Analogously,

$$\left(G_{\overline{\partial}^{g_1,g_{s_n}}_{E,m,q}}\circ(\Psi^{m,q+1}_{s_n})^{-1}\right)^*=(\Psi^{m,q+1}_{s_n})\circ G_{\overline{\partial}^{g_1,g_{s_n},t}_{E,m,q}}$$

and

$$\left(G_{\overline{\partial}^{g_1,h}_{E,m,q,\max}}\circ(\Psi^{m,q+1}_0)^{-1}\right)^*=(\Psi^{m,q+1}_0)\circ G_{\overline{\partial}^{g_1,h,t}_{E,m,q,\min}}.$$

The conclusion now follows by Lemma 3.13. ■

Let $\pi^{m,q}_{K,s}$ and $\pi^{m,q}_{K,0}$ be the projections defined in (21) and (22), respectively.

Lemma 3.15. *Let $\{s_n\}_{n\in\mathbb{N}}\subset(0,1]$ be a sequence such that $s_n\to 0$ as $n\to\infty$. Then*

$$\lim_{n\to\infty}\left\|\Psi^{m,q}_{s_n}\circ\pi^{m,q}_{K,s_n}\circ(\Psi^{m,q}_{s_n})^{-1}-\Psi^{m,q}_0\circ\pi^{m,q}_{K,0}\circ(\Psi^{m,q}_0)^{-1}\right\|_{\mathrm{op}}=0.$$

Proof. This follows by Proposition 2.4, Theorem 3.3 and Lemmas 3.11 and 3.12. ■

Now for each $s\in[0,1]$ let us consider the following complex:

$$\begin{aligned}&L^2\Omega^{m,0}(A,E|_A,g_1|_A,\rho|_A)\xrightarrow{\overline{\partial}^{g_1,g_1}_{E,m,0}}\dots\xrightarrow{\overline{\partial}^{g_1,g_1}_{E,m,q-2}}L^2\Omega^{m,q-1}(A,E|_A,g_1|_A,\rho|_A)\\&\xrightarrow{\overline{\partial}^{g_1,g_s}_{E,m,q-1,\max}}L^2\Omega^{m,q}(A,E|_A,g_s|_A,\rho|_A)\xrightarrow{\overline{\partial}^{g_s,h}_{E,m,q,\max}}L^2\Omega^{m,q+1}(A,E|_A,h|_A,\rho|_A)\\&\xrightarrow{\overline{\partial}^{h,h}_{E,m,q+1,\max}}\dots\xrightarrow{\overline{\partial}^{h,h}_{E,m,m-1,\max}}L^2\Omega^{m,m}(A,E|_A,h|_A,\rho|_A).\end{aligned}\tag{29}$$

In other words, from 0 up to $q-1$ we have the L^2-$\overline{\partial}_E$-complex with respect to g_1 and (E,ρ), from $q+1$ up m we have the maximal L^2-$\overline{\partial}_E$-complex with respect to h and (E,ρ) and the connecting piece is given by

$$\xrightarrow{\overline{\partial}^{g_1,g_s}_{E,m,q-1,\max}}L^2\Omega^{m,q}(A,E|_A,g_s|_A,\rho|_A)\xrightarrow{\overline{\partial}^{g_s,h}_{E,m,q,\max}}L^2\Omega^{m,q+1}(A,E|_A,h|_A,\rho|_A).$$

Let us now introduce the following complex:

$$\begin{aligned}&L^2\Omega^{m,0}(A,E|_A,g_1|_A,\rho|_A)\xrightarrow{\overline{\partial}^{g_1,g_1}_{E,m,0}}\dots\xrightarrow{\overline{\partial}^{g_1,g_1}_{E,m,q-2}}L^2\Omega^{m,q-1}(A,E|_A,g_1|_A,\rho|_A)\\&\xrightarrow{D^{g_1,g_s}_{m,q-1}}L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)\xrightarrow{D^{g_s,h}_{m,q}}L^2\Omega^{m,q+1}(A,E|_A,g_1|_A,\rho|_A)\\&\xrightarrow{D^{h,h}_{m,q+1}}\dots\xrightarrow{D^{h,h}_{m,m-1}}L^2\Omega^{m,m}(A,E|_A,g_1|_A,\rho|_A)\end{aligned}\tag{30}$$

with

$$D^{g_1,g_s}_{m,q-1} := \Psi^{m,q}_s \circ \overline{\partial}^{g_1,g_s}_{E,m,q-1,\max},$$
$$D^{g_s,h}_{m,q} := \Psi^{m,q+1}_0 \circ \overline{\partial}^{g_s,h}_{E,m,q,\max} \circ (\Psi^{m,q}_s)^{-1},$$
$$D^{h,h}_{m,r} := \Psi^{m,r+1}_0 \circ \overline{\partial}^{h,h}_{E,m,r,\max} \circ (\Psi^{m,r}_0)^{-1}$$

for each $r = q+1, \dots, m$. Let

$$P^{m,q}_s : L^2\Omega^{m,\bullet}(A, E|_A, g_1|_A, \rho|_A) \to L^2\Omega^{m,\bullet}(A, E, g_1, \rho) \tag{31}$$

be the rolled-up operator of the complex (30), see [8, p. 91–92]. This means nothing but

$$P^{m,q}_s|_{L^2\Omega^{m,r}(A,E|_A,g_1|_A,\rho|_A)} := \overline{\partial}^{g_1,g_1}_{E,m,r} + \overline{\partial}^{g_1,g_1,t}_{E,m,r-1},\ r = 0, \dots, q-2$$
$$P^{m,q}_s|_{L^2\Omega^{m,q-1}(A,E|_A,g_1|_A,\rho|_A)} := D^{g_1,g_s}_{m,q-1} + \overline{\partial}^{g_1,g_1,t}_{E,m,q-2}$$
$$P^{m,q}_s|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} := D^{g_s,h}_{m,q} + (D^{g_1,g_s}_{m,q-1})^*$$
$$P^{m,q}_s|_{L^2\Omega^{m,q+1}(A,E|_A,g_1|_A,\rho|_A)} := D^{h,h}_{m,q+1} + (D^{g_s,h}_{m,q})^*$$
$$P^{m,q}_s|_{L^2\Omega^{m,r}(A,E|_A,g_1|_A,\rho|_A)} := D^{h,h}_{m,r} + (D^{h,h}_{m,r-1})^*,\ r = q+2, \dots, m.$$

Note that $P^{m,q}_0 = P^{m,q-1}_1$ for each $q \in \{1, \dots, m\}$. Moreover, we have $(D^{g_1,g_s}_{m,q-1})^* = \overline{\partial}^{g_1,g_s,t}_{E,m,q-1,\min} \circ (\Psi^{m,q}_s)^{-1}$, $(D^{g_s,h}_{m,q})^* = \Psi^{m,q}_s \circ \overline{\partial}^{g_s,h,t}_{E,m,q,\min} \circ (\Psi^{m,q+1}_0)^{-1}$ and $(D^{h,h}_{m,r-1})^* = \Psi^{m,r-1}_0 \circ \overline{\partial}^{h,h,t}_{E,m,r-1,\min} \circ (\Psi^{m,r}_0)^{-1}$ with $r = q+2, \dots, m$. Furthermore, we point out that when $q = m$ and $s = 1$, the complex (30) is nothing but the L^2-$\overline{\partial}_E$-complex on M w.r.t. g_1 and ρ whereas when $q = 0 = s$, the complex (30) is unitarily equivalent to the maximal L^2-$\overline{\partial}_E$-complex on A with respect to h and (E, ρ). Consequently, in the case $q = m$ and $s = 1$, the operator (31) is the Dirac–Dolbeault operator on M w.r.t. g_1 and (E, ρ), whereas in the case $q = 0 = s$, the operator (31) is unitarily equivalent to the rolled-up operator of the maximal L^2-$\overline{\partial}_E$-complex over A with respect to h and (E, ρ). We have now the following property.

Lemma 3.16. *In the setting of Theorem 3.3, the operator $P^{m,q}_s$ is self-adjoint and has entirely discrete spectrum for each $q \in \{0, \dots, m\}$ and $s \in [0, 1]$.*

Proof. The fact that $P^{m,q}_s$ is self-adjoint follows immediately by its definition because it is the rolled-up operator of a Hilbert complex, see [8, p. 92]. The discreteness of its spectrum is an easy consequence of the fact that the complex (29) has finite cohomology, and each operator has a compact Green operator. ■

We are now in a position to prove the main result of this subsection.

Theorem 3.4. *In the above setting we have*

$$\lim_{s\to 0} \|(P^{m,q}_s + i)^{-1} - (P^{m,q}_0 + i)^{-1}\|_{\mathrm{op}} = 0$$

for each $q = 0, \dots, m$.

Proof. According to Lemma 2.1, it suffices to show that

$$\lim_{s\to 0}\|G_{P_s^{m,q}} - G_{P_0^{m,q}}\|_{\mathrm{op}} = 0,$$

$$\lim_{s\to 0}\|\pi_{K,P_s^{m,q}} - \pi_{K,P_0^{m,q}}\|_{\mathrm{op}} = 0,$$

where $\pi_{K,P_s^{m,q}} : L^2\Omega^{m,\bullet}(A, E|_A, g_1|_A, \rho|_A) \to L^2\Omega^{m,\bullet}(A, E|_A, g_1|_A, \rho|_A)$ stands for the orthogonal projection on $\ker(P_s^{m,q})$. Since $P_s^{m,q}$ is the rolled-up operator of the complex (30), it is easy to check that

$$\begin{aligned}\ker(P_s^{m,q}) &= \bigoplus_{r=0}^{m}\ker(P_s^{m,q}|_{L^2\Omega^{m,r}(A,E|_A,g_1|_A,\rho|_A)})\\ G_{P_s^{m,q}} &= \bigoplus_{r=0}^{m} G_{P_s^{m,q}}|_{L^2\Omega^{m,r}(A,E|_A,g_1|_A,\rho|_A)}.\end{aligned}\tag{32}$$

Concerning $\ker(P_s^{m,q})$, we obtain the following decomposition:

$$\begin{aligned}\ker(P_s^{m,q}) = {}&\Big(\bigoplus_{r=0}^{q-2}\ker(\overline{\partial}^{g_1,g_1}_{E,m,r})\cap\ker(\overline{\partial}^{g_1,g_1,t}_{E,m,r})\Big)\oplus\big(\ker(D^{g_1,g_s}_{m,q-1})\cap\ker(\overline{\partial}^{g_1,g_1,t}_{E,m,q-2})\big)\\ &\oplus\big(\ker(D^{g_s,h}_{m,q})\cap\ker((D^{g_1,g_s}_{m,q-1})^*)\big)\oplus\big(\ker(D^{h,h}_{m,q+1})\cap\ker((D^{g_s,h}_{m,q})^*)\big)\\ &\oplus\Big(\bigoplus_{r=q+2}^{m}\ker(D^{h,h}_r)\cap\ker((D^{h,h}_r)^*)\Big).\end{aligned}$$

Note that $\ker(D^{g_1,g_s}_{m,q-1})$ is independent on $s\in[0,1]$. Moreover, also $\ker((D^{g_s,h}_{m,q})^*)$ does not depend on $s\in[0,1]$. This latter assertion follows because by assumption $\mathrm{im}(\overline{\partial}^{g_1,h}_{E,m,q,\max}) = \mathrm{im}(\overline{\partial}^{h,h}_{E,m,q,\max})$ which in turn implies that $\mathrm{im}(\overline{\partial}^{g_s,h}_{E,m,q,\max}) = \mathrm{im}(\overline{\partial}^{h,h}_{E,m,q,\max})$ for each $s\in[0,1]$. Thus, $\mathrm{im}(D^{g_s,h}_{m,q}) = \mathrm{im}(D^{h,h}_{m,q})$ for each $s\in[0,1]$ and eventually we can conclude that $\ker((D^{g_s,h}_{m,q})^*)$ does not depend on $s\in[0,1]$ since $\ker((D^{g_s,h}_{m,q})^*) = (\mathrm{im}(D^{g_s,h}_{E,m,q}))^{\perp}$ in $L^2\Omega^{m,q+1}(A,E,h,\rho)$. Thus, in the above decomposition the only term depending on s is $\ker(D^{g_s,h}_{m,q})\cap\ker((D^{g_1,g_s}_{m,q-1})^*)$ and so the limit

$$\lim_{s\to 0}\|\pi_{K,P_s^{m,q}} - \pi_{K,P_0^{m,q}}\|_{\mathrm{op}} = 0\tag{33}$$

boils down to proving that

$$\lim_{n\to\infty}\left\|\Psi^{m,q}_{s_n}\circ\pi^{m,q}_{K,s_n}\circ(\Psi^{m,q}_{s_n})^{-1} - \Psi^{m,q}_0\circ\pi^{m,q}_{K,0}\circ(\Psi^{m,q}_0)^{-1}\right\|_{\mathrm{op}} = 0,\tag{34}$$

where $\pi^{m,q}_{K,s}$ is the orthogonal projection defined in (21). By Lemma 3.15, we already know that (34) holds true and thus (33) holds true as well. Let us go back to the second half of (32). Looking at (30), we note that the only terms in the decomposition of $G_{P_s^{m,q}}$ that depend on s are

$$G_{P_s^{m,q}}|_{L^2\Omega^{m,r}(A,E|_A,g_1|_A,\rho|_A)}, \quad r = q-1, q, q+1.$$

We have

$$\begin{aligned}
&G_{P_s^{m,q}}|_{L^2\Omega^{m,q-1}(A,E|_A,g_1|_A,\rho|_A)} = G_{\bar\partial^{g_1,g_1}_{E,m,q-2}} + G_{(D^{g_1,g_s}_{m,q-1})^*} :\\
&\quad L^2\Omega^{m,q-1}(A, E|_A, g_1|_A, \rho|_A) \to\\
&\quad L^2\Omega^{m,q-2}(A, E|_A, g_1|_A, \rho|_A) \oplus L^2\Omega^{m,q}(A, E|_A, g_1|_A, \rho|_A);\\
&G_{P_s^{m,q}}|_{L^2\Omega^{m,q}(A,E|_A,g_1|_A,\rho|_A)} = G_{D^{g_1,g_s}_{m,q-1}} + G_{(D^{g_s,h}_{m,q})^*} :\\
&\quad L^2\Omega^{m,q}(A, E|_A, g_1|_A, \rho|_A) \to\\
&\quad L^2\Omega^{m,q-1}(A, E|_A, g_1|_A, \rho|_A) \oplus L^2\Omega^{m,q+1}(A, E|_A, g_1|_A, \rho|_A);\\
&G_{P_s^{m,q}}|_{L^2\Omega^{m,q+1}(A,E|_A,g_1|_A,\rho|_A)} = G_{D^{g_s,h}_{m,q}} + G_{(D^{h,h}_{m,q+1})^*} :\\
&\quad L^2\Omega^{m,q+1}(A, E|_A, g_1|_A, \rho|_A) \to\\
&\quad L^2\Omega^{m,q}(A, E|_A, g_1|_A, \rho|_A) \oplus L^2\Omega^{m,q+2}(A, E|_A, g_1|_A, \rho|_A).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\| G_{P_s^{m,q}} - G_{P_0^{m,q}} \right\|_{\text{op}}\\
&= \left\| \bigoplus_{r=0}^{m} G_{P_s^{m,q}}|_{L^2\Omega^{m,r}(A,E|_A,g_1|_A,\rho|_A)} - \bigoplus_{r=0}^{m} G_{P_0^{m,q}}|_{L^2\Omega^{m,r}(A,E|_A,g_1|_A,\rho|_A)} \right\|_{\text{op}}\\
&\le \sum_{r=q-1}^{q+1} \left\| G_{P_s^{m,q}}|_{L^2\Omega^{m,r}(A,E|_A,g_1|_A,\rho|_A)} - G_{P_0^{m,q}}|_{L^2\Omega^{m,r}(A,E|_A,g_1|_A,\rho|_A)} \right\|_{\text{op}}\\
&\le \left\| G_{(D^{g_1,g_s}_{m,q-1})^*} - G_{(D^{g_1,h}_{m,q-1})^*} \right\|_{\text{op}} + \left\| G_{D^{g_1,g_s}_{m,q-1}} - G_{D^{g_1,h}_{m,q-1}} \right\|_{\text{op}}\\
&\quad + \left\| G_{(D^{g_s,h}_{m,q})^*} - G_{(D^{h,h}_{m,q})^*} \right\|_{\text{op}} + \left\| G_{D^{g_s,h}_{m,q}} - G_{D^{h,h}_{m,q}} \right\|_{\text{op}}.
\end{aligned}$$

Clearly, for each $s \in [0, 1]$, we have

$$\begin{aligned}
G_{D^{g_1,g_s}_{m,q-1}} &= G_{\bar\partial^{g_1,g_s}_{E,m,q-1,\max}} \circ (\Psi^{m,q}_s)^{-1},\\
G_{(D^{g_1,g_s}_{m,q-1})^*} &= \Psi^{m,q}_s \circ G_{\bar\partial^{g_1,g_s,t}_{E,m,q-1,\min}},\\
G_{D^{g_s,h}_{m,q}} &= \Psi^{m,q}_s \circ G_{\bar\partial^{g_s,h}_{E,m,q,\max}} \circ (\Psi^{m,q+1}_0)^{-1},\\
G_{(D^{g_s,h}_{m,q})^*} &= \Psi^{m,q+1}_0 \circ G_{(\bar\partial^{g_s,h,t}_{E,m,q,\min})^*} \circ (\Psi^{m,q}_s)^{-1}.
\end{aligned}$$

Thanks to Lemmas 3.13 and 3.14, we know that

$$\lim_{s\to 0} \left\| G_{D^{g_1,g_s}_{m,q-1}} - G_{D^{g_1,h}_{m,q-1}} \right\|_{\text{op}} = 0, \qquad \lim_{s\to 0} \left\| G_{(D^{g_1,g_s}_{m,q-1})^*} - G_{(D^{g_1,h}_{m,q-1})^*} \right\|_{\text{op}} = 0,$$

$$\lim_{s\to 0} \left\| G_{D^{g_s,h}_{m,q}} - G_{D^{h,h}_{m,q}} \right\|_{\text{op}} = 0, \qquad \lim_{s\to 0} \left\| G_{(D^{g_s,h}_{m,q})^*} - G_{(D^{h,h}_{m,q})^*} \right\|_{\text{op}} = 0.$$

We can thus conclude that

$$\lim_{s\to 0}\|G_{P_s^{m,q}} - G_{P_0^{m,q}}\|_{\mathrm{op}} = 0$$

as required. ■

4. Resolutions and canonical K-homology classes

Let (M, g) be a compact complex manifold of complex dimension m endowed with a Hermitian metric g. Let $(E,\rho) \to M$ be a Hermitian holomorphic vector bundle over M. For each $p \in \{0,\dots,m\}$ let us consider the Hilbert space $L^2\Omega^{p,\bullet}(M, E, g,\rho)$ endowed with the grading induced by the splitting in L^2 E-valued $(p,\bullet)$-forms with even/odd antiholomorphic degree. Furthermore, we consider the corresponding Dirac–Dolbeault operator $\overline{\eth}_{E,p} := \overline{\partial}_{E,p} + \overline{\partial}^t_{E,p}$

$$\overline{\eth}_{E,p} : L^2\Omega^{p,\bullet}(M, E, g,\rho) \to L^2\Omega^{p,\bullet}(M, E, g,\rho)$$

and the C^*-algebra $C(M) := C(M,\mathbb{C})$ acting on $L^2\Omega^{p,\bullet}(M, E, g,\rho)$ by pointwise multiplication:

$$C(M) \ni f \mapsto m_f \in \mathcal{B}(L^2\Omega^{p,\bullet}(M, E, g,\rho)) \text{ given by } m_f\psi := f\psi \tag{35}$$

for every $\psi \in L^2\Omega^{p,\bullet}(M, E, g,\rho)$. Finally, let us consider as a dense subalgebra $\mathcal{A} := C^\infty(M)$. It is well known that the triplet $(L^2\Omega^{p,\bullet}(M, E, g,\rho), m, \overline{\eth}_{E,p})$ is an even unbounded Fredholm module, see, for example, [13, §10], and thus the triplet $(L^2\Omega^{p,\bullet}(M, E, g,\rho), m, \overline{\eth}_{E,p} \circ (\mathrm{Id} + (\overline{\eth}_{E,p})^2)^{-\frac{1}{2}})$ gives a class in $KK_0(C(M),\mathbb{C})$, see Proposition 2.6. We denote this class with

$$[\overline{\eth}_{E,p}] \in KK_0(C(M),\mathbb{C})$$

and when $p = m$ we call it the canonical analytic K-homology class of M and E. In particular, when $p = m$ and E is the trivial holomorphic line bundle $M \times \mathbb{C} \to M$, we call the above class the canonical analytic K-homology class of M. This terminology is based on the fact that $\Lambda^{m,0}(M)$ is called the canonical bundle of M. We remark that since M is compact, the class $[\overline{\eth}_{E,p}]$ depends neither on g nor on ρ since all Hermitian metrics on M, as well as all Hermitian metrics on E, are quasi-isometric, see, e.g., [14]. Now, we briefly recall the notion of Hermitian complex space. Complex spaces are a classical topic in complex geometry, and we refer, for instance, to [10] for definitions and properties. We recall that a paracompact and reduced complex space X is said to be *Hermitian* if the regular part of X carries a Hermitian metric γ such that, for every point $p \in X$ there exists an open neighbourhood $U \ni p$ in X, a proper holomorphic embedding of U into a polydisc $\phi : U \to \mathbb{D}^N \subset \mathbb{C}^N$, and a Hermitian metric g on $\mathbb{D}^N$ such that $(\phi|_{\mathrm{reg}(U)})^*g = \gamma$, see, e.g., [24]. In this case, we will write (X,γ) and with a little abuse of language, we will say that γ is a *Hermitian metric on* X. Clearly, any analytic subvariety of a complex Hermitian manifold endowed with the metric induced by the ambient space metric is an example of Hermitian complex space. Note that when X is compact, all Hermitian metrics on X belong to the same quasi-isometry class. This follows easily by the lifting lemma see [11, Remark 1.30.1, p. 37]. Let now $F \to X$ be a holomorphic vector bundle of complex rank s. We assume that $F|_{\mathrm{reg}(X)}$ is equipped with a Hermitian metric τ such that for each point $p \in X$, the following property holds true: there exists an open neighbourhood U, a positive constant c and a holomorphic trivialisation $\psi : E|_U \to U \times \mathbb{C}^s$ such that, denoting by σ the Hermitian metric on $\mathrm{reg}(U) \times \mathbb{C}^s$ induced by τ through ψ, we have

$$c^{-1}\upsilon_{\mathrm{std}} \leq \sigma \leq c\upsilon_{\mathrm{std}}, \tag{36}$$

where υ_{std} is the Hermitian metric on $\mathrm{reg}(U) \times \mathbb{C}^s$ that assigns to each point of $\mathrm{reg}(U)$ the standard Euclidean Kähler metric of $\mathbb{C}^s$. In order to state the next results, we also need to recall the existence of a resolution of singularities see [15, 16]. Let X be a compact and irreducible complex space. There then exists a compact complex manifold M, a divisor with only normal crossings $D \subset M$, and a surjective holomorphic map $\pi : M \to X$ such that $\pi^{-1}(\mathrm{sing}(X)) = D$ and

$$\pi|_{M\setminus D} : M \setminus D \to X \setminus \mathrm{sing}(X)$$

is a biholomorphism. Consider now the maximal L^2-$\overline{\partial}_F$-complex

$$0 \to L^2\Omega^{m,0}(\mathrm{reg}(X), F, \gamma, \tau) \xrightarrow{\overline{\partial}^{\gamma,\gamma}_{E,m,0,\max}} \dots \xrightarrow{\overline{\partial}^{\gamma,\gamma}_{E,m,m-1,\max}} L^2\Omega^{m,m}(\mathrm{reg}(X), F, \gamma, \tau) \to 0$$

and let

$$\overline{\eth}_{F,m,\mathrm{abs}} : L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \gamma, \tau) \to L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \gamma, \tau)$$

be the corresponding rolled-up operator. Note that we can write

$$\overline{\eth}_{F,m,\mathrm{abs}} = \overline{\partial}^{\gamma,\gamma}_{F,m,\max} + \overline{\eth}^{\gamma,\gamma,t}_{F,m,\min} \tag{37}$$

with

$$\overline{\partial}^{\gamma,\gamma}_{F,m,\max} : L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \gamma, \tau) \to L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \gamma, \tau)$$

defined by

$$\overline{\partial}^{\gamma,\gamma}_{F,m,\max}|_{L^2\Omega^{m,r}(\mathrm{reg}(X),F,\gamma,\tau)} := \overline{\partial}^{\gamma,\gamma}_{E,m,r,\max}$$

for each $r = 0, \dots, m$ and with $\overline{\partial}^{\gamma,\gamma,t}_{F,m,\min}$ defined in the obvious analogous way. We have now the following.

Proposition 4.1. *Let* (X, γ) *be a compact and irreducible Hermitian complex space of complex dimension m such that* $\dim(\mathrm{sing}(X)) = 0$. *Let* $(F, \tau) \to X$ *be a Hermitian holomorphic vector bundle over* X *which satisfies* (36). *Then the triplet*

$$\left(L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \gamma, \tau), m, \overline{\eth}_{F,m,\mathrm{abs}}\right)$$

defines an even unbounded Fredholm module for $C(X)$ *and thus a class*

$$[\overline{\eth}_{F,m,\text{abs}}] \in KK_0(C(X), \mathbb{C}).$$

Moreover, this class does not depend on the particular Hermitian metric γ *that we fix on* X.

Proof. The proof follows by arguing as in [7, Proposition 3.6]. In particular, we use $S_c(X)$ defined as

$$S_c(X) := \big\{f \in C(X) \cap C^{\infty}(\text{reg}(X)) \text{ such that for each } p \in \text{sing}(X) \text{ there exists an open neighbourhood } U \text{ of } p \text{ with } f|_U = c \in \mathbb{C}\big\}$$

as a dense $*$-subalgebra of $C(X)$. Moreover, we recall that m denotes the pointwise multiplication see (35). The only point that needs an explanation is the discreteness of the spectrum of $\overline{\eth}_{F,m,\text{abs}}$. This is settled in the next lemma. ∎

Lemma 4.1. *Let* (X, γ) *be a compact and irreducible Hermitian complex space of complex dimension* m *with* $\dim(\text{sing}(X)) = 0$. *Then*

$$\overline{\eth}_{F,m,\text{abs}} : L^2\Omega^{m,\bullet}(\text{reg}(X), F, \gamma, \tau) \to L^2\Omega^{m,\bullet}(\text{reg}(X), F, \gamma, \tau)$$

has entirely discrete spectrum, with $(F, \tau) \to X$ *any Hermitian holomorphic vector bundle over* X *which satisfies* (36).

Proof. Let $\text{sing}(X) = \{p_1, \ldots, p_\ell\}$. First, we prove this lemma under some additional requirements: the holomorphic vector bundle F is endowed with a Hermitian metric τ' such that, for each $p_k \in \text{sing}(X)$, $k = 1, \ldots, \ell$ there exists an open neighbourhood U_k and a trivialisation $\chi_k : F|_{U_k} \to U_k \times \mathbb{C}^n$, with $n := \text{rnk}(F)$, such that $\chi^*(\upsilon_{\text{std}}) = \tau'$, with υ_{std} defined in (36). Clearly, we can always endow F with such a metric. Thanks to [2, Theorem 5.2] and [22, Theorem 1.2] we know that

$$\overline{\eth}_{m,\text{abs}} : L^2\Omega^{m,\bullet}(\text{reg}(X), \gamma) \to L^2\Omega^{m,\bullet}(\text{reg}(X), \gamma)$$

has entirely discrete spectrum. From this, we get immediately that the twisted Dirac–Dolbeault operator with respect to the trivial holomorphic vector bundle $\text{reg}(X) \times \mathbb{C}^n$ endowed with the standard Euclidean Kähler metric υ_{std}

$$\overline{\eth}_{\mathbb{C}^n,m,\text{abs}} : L^2\Omega^{m,\bullet}(\text{reg}(X), \text{reg}(X) \times \mathbb{C}^n, \gamma, \upsilon_{\text{std}}) \to L^2\Omega^{m,\bullet}(\text{reg}(X), \text{reg}(X) \times \mathbb{C}^n, \gamma, \upsilon_{\text{std}}) \tag{38}$$

has entirely discrete spectrum as well. Let now U_0 be an open subset of $\text{reg}(X)$ such that $\{U_0, U_1, \ldots, U_\ell\}$ is an open covering of X. We also assume that $U_0 \cap \text{sing}(X) = \emptyset$ and that $U_i \cap U_j = \emptyset$ for each $1 \le i < j \le \ell$. Let $\{\phi_0, \ldots, \phi_\ell\}$ be a partition of unity subordinated to $\{U_0, U_1, \ldots, U_\ell\}$ such that $\phi_i \in C^{\infty}(\text{reg}(X)) \cap C(X)$ for each

$i = 0, \ldots, \ell$. Note that for every $1 \leq i \leq \ell$ there exists an open neighbourhood V_i of p_i with $V_i \subset U_i$ and $\phi_i|_{V_i} = 1$. In particular, we have $d_0\phi_i \in \Omega^1_c(\mathrm{reg}(U_i))$. Consider now a sequence $\{\eta_j\}_{j\in\mathbb{N}} \subset \mathcal{D}(\overline{\eth}_{F,m,\mathrm{abs}})$, which is bounded with respect to the corresponding graph norm. Clearly, the sequence $\{\phi_0\eta_j\}_{j\in\mathbb{N}}$ still lies in $\mathcal{D}(\overline{\eth}_{F,m,\mathrm{abs}})$ and is bounded with respect to the corresponding graph norm. Moreover, the support of $\phi_0\eta_j$ is contained in U_0 for each $j \in \mathbb{N}$. Since U_0 is relatively compact in $\mathrm{reg}(X)$, we can use elliptic estimates see, e.g., [19, Lemma 1.1.17], and a Rellich-type compactness theorem to deduce the existence of a subsequence $\{\eta_{0,j}\}_{j\in\mathbb{N}} \subset \{\eta_j\}_{j\in\mathbb{N}}$ such that $\{\phi_0\eta_{0,j}\}_{j\in\mathbb{N}}$ converges in $L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \gamma, \tau')$. Consider now the sequence $\{\phi_1\psi_{0,j}\}_{j\in\mathbb{N}} \subset L^2\Omega^{m,\bullet}(\mathrm{reg}(X), \mathrm{reg}(X)\times\mathbb{C}^n, \gamma, \upsilon_{\mathrm{std}})$ with $\psi_{0,j} := (\chi_1^{-1})^*(\eta_{0,j}|_{\mathrm{reg}(U_1)})$. It is clear that the sequence $\{\phi_1\psi_{0,j}\}_{j\in\mathbb{N}}$ lies both in the domain of

$$\overline{\partial}^{\gamma,\gamma}_{\mathbb{C}^n,m,\max} : L^2\Omega^{m,\bullet}(\mathrm{reg}(U_1), \mathrm{reg}(X)\times\mathbb{C}^n, \gamma|_{\mathrm{reg}(U_1)}, \upsilon_{\mathrm{std}}|_{\mathrm{reg}(U_1)}) \to L^2\Omega^{m,\bullet}(\mathrm{reg}(U_1), \mathrm{reg}(X)\times\mathbb{C}^n, \gamma|_{\mathrm{reg}(U_1)}, \upsilon_{\mathrm{std}}|_{\mathrm{reg}(U_1)})$$

and

$$\overline{\partial}^{\gamma,\gamma,t}_{\mathbb{C}^n,m,\min} : L^2\Omega^{m,\bullet}(\mathrm{reg}(U_1), \mathrm{reg}(X)\times\mathbb{C}^n, \gamma|_{\mathrm{reg}(U_1)}, \upsilon_{\mathrm{std}}|_{\mathrm{reg}(U_1)}) \to L^2\Omega^{m,\bullet}(\mathrm{reg}(U_1), \mathrm{reg}(X)\times\mathbb{C}^n, \gamma|_{\mathrm{reg}(U_1)}, \upsilon_{\mathrm{std}}|_{\mathrm{reg}(U_1)})$$

and it is bounded in the corresponding graph norm. From the definition of minimal domain we get immediately that $\{\phi_1\psi_{0,j}\}_{j\in\mathbb{N}}$ lies in the domain of

$$\overline{\partial}^{\gamma,\gamma,t}_{\mathbb{C}^n,m,\min} : L^2\Omega^{m,\bullet}(\mathrm{reg}(X), \mathrm{reg}(X)\times\mathbb{C}^n, \gamma, \upsilon_{\mathrm{std}}) \to L^2\Omega^{m,\bullet}(\mathrm{reg}(X), \mathrm{reg}(X)\times\mathbb{C}^n, \gamma, \upsilon_{\mathrm{std}}).$$

Moreover, since ϕ_1 has compact support contained in U_1, it is not difficult to see that $\{\phi_1\psi_{0,j}\}_{j\in\mathbb{N}}$ lies also in the domain of

$$\overline{\partial}^{\gamma,\gamma}_{\mathbb{C}^n,m,\max} : L^2\Omega^{m,\bullet}(\mathrm{reg}(X), \mathrm{reg}(X)\times\mathbb{C}^n, \gamma, \upsilon_{\mathrm{std}}) \to L^2\Omega^{m,\bullet}(\mathrm{reg}(X), \mathrm{reg}(X)\times\mathbb{C}^n, \gamma, \upsilon_{\mathrm{std}}).$$

Summarising, we shown that $\{\phi_1\psi_{0,j}\}_{j\in\mathbb{N}}$ lies in the domain of (38), and it is bounded in the corresponding graph norm. Therefore, there exists a subsequence $\{\psi_{1,j}\}_{j\in\mathbb{N}} \subset \{\psi_{0,j}\}_{j\in\mathbb{N}}$ such that $\{\phi_1\psi_{1,j}\}_{j\in\mathbb{N}}$ converges in $L^2\Omega^{m,\bullet}(\mathrm{reg}(X), \mathrm{reg}(X)\times\mathbb{C}^n, \gamma, \upsilon_{\mathrm{std}})$. Eventually, we can conclude that there exists a subsequence $\{\eta_{1,j}\}_{j\in\mathbb{N}} \subset \{\eta_{0,j}\}_{j\in\mathbb{N}}$, which satisfies $(\chi_1^{-1})^*(\eta_{1,j}|_{\mathrm{reg}(U_1)}) = \psi_{1,j}$, such that the sequence $\{\phi_1\eta_{1,j}\}_{j\in\mathbb{N}}$ converges in $L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \gamma, \tau')$. Repeating this procedure up to n, we construct a subsequence $\{\eta_{n,j}\}_{j\in\mathbb{N}} \subset \{\eta_j\}_{j\in\mathbb{N}}$ such that $\{\phi_i\eta_{n,j}\}_{j\in\mathbb{N}}$ converges in $L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \gamma, \tau')$ for each $i = 0, \ldots, n$. We can thus conclude that the sequence $\{\eta_{n,j}\}_{j\in\mathbb{N}}$ converges in $L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \gamma, \tau')$ and this completes the first part of the proof. Note that, as a by-product of this first part of the proof we get that $\mathrm{im}(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max})$ is a closed subspace of $L^2\Omega^{m,q}(\mathrm{reg}(X), F, \gamma, \tau')$ for each $q = 0, \ldots, m$,

$$\big(\ker(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max}) \cap \ker(\overline{\partial}^{\gamma,\gamma,t}_{F,m,q-1,\min})\big) \cong \ker(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max})/\mathrm{im}(\overline{\partial}^{\gamma,\gamma}_{F,m,q-1,\max})$$

is a finite-dimensional vector space and

$$G^{\tau'}_{\overline{\partial}_{F,m,q,\max}} : L^2\Omega^{m,q+1}(\text{reg}(X), F, \gamma, \tau') \to L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau')$$

is a compact operator where we have denoted with $G^{\tau'}_{\overline{\partial}_{F,m,q,\max}}$ the Green operator of

$$\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max} : L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau') \to L^2\Omega^{m,q+1}(\text{reg}(X), F, \gamma, \tau').$$

Let now τ be an arbitrarily fixed Hermitian metric on $F \to X$ which satisfies (36). Since τ and τ' are quasi-isometric, we know that for each $q = 0, \ldots, m$,

$$\overline{\partial}^{\gamma,\gamma}_{F,m,q,\text{abs}} : L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau) \to L^2\Omega^{m,q+1}(\text{reg}(X), F, \gamma, \tau)$$

has closed range and

$$\big(\ker(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max}) \cap \ker(\overline{\partial}^{\gamma,\gamma,t}_{F,m,q-1,\min})\big) \cong \ker(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max})/\text{im}(\overline{\partial}^{\gamma,\gamma}_{F,m,q-1,\max})$$

is a finite-dimensional vector space. Let us now consider the following L^2-decomposition:

$$\begin{aligned} &L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau) \\ &\quad = \big(\ker(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max}) \cap \ker(\overline{\partial}^{\gamma,\gamma,t}_{F,m,q-1,\min})\big) \oplus \text{im}(\overline{\partial}^{\gamma,\gamma}_{F,m,q-1,\max}) \oplus \text{im}(\overline{\partial}^{\gamma,\gamma,t}_{F,m,q,\min}). \end{aligned}$$

We already know that $\ker(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max}) \cap \ker(\overline{\partial}^{\gamma,\gamma,t}_{F,m,q-1,\min})$ is finite dimensional. Since

$$\begin{aligned} \mathcal{D}(\overline{\eth}_{F,m,\text{abs}}) &= \bigoplus_{q=0}^{m} \big(\mathcal{D}(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max}) \cap \mathcal{D}(\overline{\partial}^{\gamma,\gamma,t}_{F,m,q-1,\min})\big) \\ \ker(\overline{\eth}_{F,m,\text{abs}}) &= \bigoplus_{q=0}^{m} \big(\ker(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max}) \cap \ker(\overline{\partial}^{\gamma,\gamma,t}_{F,m,q-1,\min})\big) \\ \text{im}(\overline{\eth}_{F,m,\text{abs}}) &= \bigoplus_{q=0}^{m} \big(\text{im}(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max}) \oplus \text{im}(\overline{\partial}^{\gamma,\gamma,t}_{F,m,q-1,\min})\big), \end{aligned}$$

we know that $\ker(\overline{\eth}_{F,m,\text{abs}})$ has finite dimension and $\text{im}(\overline{\eth}_{F,m,\text{abs}})$ is closed. Thus, in order to conclude that $\overline{\eth}_{F,m,\text{abs}} : L^2\Omega^{m,\bullet}(\text{reg}(X), F, \gamma, \tau) \to L^2\Omega^{m,\bullet}(\text{reg}(X), F, \gamma, \tau)$ has entirely discrete spectrum, it is enough to prove that the corresponding Green operator is compact see Proposition 2.5. Thanks to (37), this boils down to show that the Green operators of $\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max}$ and $\overline{\partial}^{\gamma,\gamma,t}_{F,m,q,\min}$ with respect to τ:

$$G^{\tau}_{\overline{\partial}_{F,m,q,\max}} : L^2\Omega^{m,q+1}(\text{reg}(X), F, \gamma, \tau) \to L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau) \tag{39}$$

and

$$G^{\tau}_{\overline{\partial}^t_{F,m,q-1,\min}} : L^2\Omega^{m,q-1}(\text{reg}(X), F, \gamma, \tau) \to L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau) \tag{40}$$

are both compact for each $q = 0, \ldots, m$. Note that the compactness of (40) follows from the compactness of (39). Indeed, $\overline{\partial}^{\gamma,\gamma,t}_{F,m,q-1,\min} = (\overline{\partial}^{\gamma,\gamma}_{F,m,q-1,\max})^*$ and consequently $G^{\tau}_{\overline{\partial}^t_{F,m,q-1,\min}} = \big(G^{\tau}_{\overline{\partial}_{F,m,q-1,\max}}\big)^*$. Thus, we are left to prove the compactness of (39). To this aim, we point out that since τ and τ' are quasi-isometric, we have an equality of topological vector spaces $L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau') = L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau)$. In particular, the identity map

$$\text{Id} : L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau') \to L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau)$$

is bijective, continuous with continuous inverse. Moreover, it is clear that the identity induces a continuous isomorphism, that we denote by K_q, with continuous inverse

$$L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau') \supset \text{im}(\overline{\partial}^{\gamma,\gamma}_{F,m,q-1,\max}) \stackrel{K_q}{\longrightarrow} \text{im}(\overline{\partial}^{\gamma,\gamma}_{F,m,q-1,\max}) \subset L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau),$$

with K_q defined as the restriction of Id on $\text{im}(\overline{\partial}^{\gamma,\gamma}_{F,m,q-1,\text{abs}}) \subset L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau')$. Furthermore, let us introduce the map J_q:

$$L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau') \supset \text{im}(\overline{\partial}^{\gamma,\gamma,t}_{F,m,q,\min}) \stackrel{J_q}{\longrightarrow} \text{im}(\overline{\partial}^{\gamma,\gamma,t}_{F,m,q,\min}) \subset L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau)$$

defined as $J_q := \pi^{\tau}_q \circ \text{Id}$ with π^{τ}_q the orthogonal projection $\pi^{\tau}_q : L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau) \to \text{im}(\overline{\partial}^{\gamma,\gamma,t}_{F,m,q,\min})$. Using again the fact that τ and τ' are quasi-isometric, it is not difficult to check that J_q is bounded, bijective with bounded inverse and that

$$G^{\tau}_{\overline{\partial}_{F,m,q,\max}}|_{\text{im}(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max})} : \text{im}(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max}) \to L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau)$$

equals

$$J_q \circ G^{\tau'}_{\overline{\partial}_{F,m,q,\max}} \circ K^{-1}_{q+1} : \text{im}(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max}) \to L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau).$$

Since both J_q and K^{-1}_q are continuous and

$$G_{\overline{\partial}^{\tau'}_{F,m,q,\max}} : L^2\Omega^{m,q+1}(\text{reg}(X), F, \gamma, \tau') \to L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau')$$

is compact, we obtain that

$$G^{\tau}_{\overline{\partial}_{F,m,q,\max}}|_{\text{im}(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max})} : \text{im}(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max}) \to L^2\Omega^{m,q}(\text{reg}(X), F, \gamma, \tau)$$

is compact. Finally, this implies immediately that also (39) is compact. ∎

We have now the main result of this paper.

Theorem 4.1. *Let (X,γ) be a compact and irreducible Hermitian complex space of complex dimension m such that* $\dim(\operatorname{sing}(X)) = 0$. *Let $(F,\tau) \to X$ be a Hermitian holomorphic vector bundle over X which satisfies* (36). *Let $\pi : M \to X$ be a resolution of X and let $E \to M$ be the holomorphic vector bundle defined as $E := \pi^* F$. We then have the following equality in $KK_0(C(X),\mathbb{C})$:*

$$\pi_*[\overline{\eth}_{E,m}] = [\overline{\eth}_{F,m,\text{abs}}] \in KK_0(C(X),\mathbb{C}).$$

In order to prove the above theorem, we need some preliminary results.

Lemma 4.2. *Let (X,γ) be a compact and irreducible Hermitian complex space of complex dimension m with* $\dim(\operatorname{sing}(X)) = 0$. *Let $(F,\tau) \to X$ be a Hermitian holomorphic vector bundle over X which satisfies* (36). *Let $\pi : M \to X$ be a resolution of X and let g be an arbitrarily fixed Hermitian metric on M. Let $E \to M$ be the holomorphic vector bundle defined as $E := \pi^* F$ and let $h = \pi^*\gamma$, $\rho := \pi^*\tau$ and $A := \pi^{-1}$*$(\operatorname{reg}(X))$. *Consider the operators*

$$\overline{\partial}^{g,h}_{E,m,q,\max} : L^2\Omega^{m,q}(A, E|_A, g|_A, \rho|_A) \to L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho|_A)$$

and

$$\overline{\partial}^{h,h}_{E,m,q,\max} : L^2\Omega^{m,q}(A, E|_A, h|_A, \rho|_A) \to L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho|_A).$$

Then for each $q = 0,\dots,m$, *the following equalities hold in* $L^2\Omega^{m,q+1}(A, E|_A, h|_A, \rho|_A)$:

$$\operatorname{im}(\overline{\partial}^{h,h}_{E,m,q,\max}) = \overline{\operatorname{im}(\overline{\partial}^{h,h}_{E,m,q,\max})} = \overline{\operatorname{im}(\overline{\partial}^{g,h}_{E,m,q,\max})} = \operatorname{im}(\overline{\partial}^{g,h}_{E,m,q,\max}).$$

Moreover,

$$\dim\big(H^{m,q}_{\overline{\partial}}(M,E)\big) = \dim\big(H^{m,q}_{2,\overline{\partial}_{\max}}(A, E|_A, h|_A, \rho|_A)\big).$$

Proof. First, we note that when $F = \operatorname{reg}(X) \times \mathbb{C}^n$ and $\tau = \upsilon_{\text{std}}$, the above chain of equalities is an immediate consequence of [24, Theorem 1.5]. We now tackle the general case. To this aim, we introduce the following presheaves $C^{m,q}_{F,\gamma}$ on X given by the assignments

$$C^{m,q}_{F,\gamma}(U) := \big\{\mathcal{D}(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max}) \text{ on } \operatorname{reg}(U)\big\};$$

in other words, to every open subset U of X we assign the maximal domain of $\overline{\partial}_{F,m,q}$ over $\operatorname{reg}(U)$ with respect to $F|_U$, $\gamma|_{\operatorname{reg}(U)}$ and $\tau|_{\operatorname{reg}(U)}$. We denote by $\mathcal{C}^{m,q}_{F,\gamma}$ the corresponding sheafification. Finally, let $(\mathcal{C}^{m,\bullet}_{F,\gamma}, \overline{\partial}^{\gamma,\gamma}_{F,m,\bullet})$ be the complex of sheaves where the action of $\overline{\partial}^{\gamma,\gamma}_{F,m,\bullet}$ is understood in the distributional sense. Let σ be the Hermitian metric on $\operatorname{reg}(X)$ defined as $\sigma := ((\pi|_A)^{-1})^* g$. Let us consider the corresponding complex of sheaves $(\mathcal{C}^{m,\bullet}_{F,\sigma}, \overline{\partial}^{\sigma,\sigma}_{F,m,\bullet})$. Thanks to Proposition 2.1, it is easy to check that the continuous inclusion $I : L^2\Omega^{m,q}(\operatorname{reg}(X), F, \gamma, \tau) \hookrightarrow L^2\Omega^{m,q}(\operatorname{reg}(X), F, \sigma, \tau)$ gives rise to a morphism of sheaves

$$\mathcal{I} : (\mathcal{C}^{m,\bullet}_{F,\gamma}, \overline{\partial}^{\gamma,\gamma}_{F,m,\bullet}) \to (\mathcal{C}^{m,\bullet}_{F,\sigma}, \overline{\partial}^{\sigma,\sigma}_{F,m,\bullet}). \tag{41}$$

Let $\mathcal{K}_M(E)$ be the sheaf of holomorphic sections of $K_M \otimes E \to M$. Since we assumed (36), we can argue as in the proof of [24, Theorem 1.5] to show that both $(\mathcal{C}^{m,\bullet}_{F,\gamma}, \overline{\partial}^{\gamma,\gamma}_{F,m,\bullet})$ and $(\mathcal{C}^{m,\bullet}_{F,\sigma}, \overline{\partial}^{\sigma,\sigma}_{F,m,\bullet})$ are fine resolutions of $\pi_*(\mathcal{K}_M(E))$. This in turn implies that the morphism (41) induces an isomorphism, still denoted with $\mathcal{I}$, between the cohomology groups:

$$\mathcal{I} : H^q(X, \mathcal{C}^{m,\bullet}_{F,\gamma}(X)) \to H^q(X, \mathcal{C}^{m,\bullet}_{F,\sigma}(X)), q = 0, \ldots, m, \tag{42}$$

where, by $H^q(X, \mathcal{C}^{m,\bullet}_{F,\gamma}(X))$ and $H^q(X, \mathcal{C}^{m,\bullet}_{F,\sigma}(X))$, we mean the cohomology groups of the complexes of global sections of $\mathcal{C}^{m,\bullet}_{F,\gamma}$ and $\mathcal{C}^{m,\bullet}_{F,\sigma}$, that is, the cohomology of the complexes:

$$0 \to \mathcal{C}^{m,0}_{F,\gamma}(X) \xrightarrow{\overline{\partial}^{\gamma,\gamma}_{F,m,0}} \ldots \xrightarrow{\overline{\partial}^{\gamma,\gamma}_{F,m,m-1}} \mathcal{C}^{m,m}_{F,\gamma}(X) \to 0$$

$$0 \to \mathcal{C}^{m,0}_{F,\sigma}(X) \xrightarrow{\overline{\partial}^{\sigma,\sigma}_{F,m,0}} \ldots \xrightarrow{\overline{\partial}^{\sigma,\sigma}_{F,m,m-1}} \mathcal{C}^{m,m}_{F,\sigma}(X) \to 0.$$

It is clear that on X we have the equalities $\mathcal{C}^{m,q}_{F,\gamma}(X) = \{\mathcal{D}(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max}) \text{ on } \operatorname{reg}(X)\}$ and analogously $\mathcal{C}^{m,q}_{F,\sigma}(X) = \{\mathcal{D}(\overline{\partial}^{\sigma,\sigma}_{F,m,q,\max}) \text{ on } \operatorname{reg}(X)\}$, which in turn imply the equalities

$$H^q(X, \mathcal{C}^{m,\bullet}_{F,\gamma}(X)) = H^{m,q}_{\overline{\partial}^{\gamma,\gamma}_{F,\max}}(\operatorname{reg}(X), F, \sigma, \tau)$$
$$H^q(X, \mathcal{C}^{m,\bullet}_{F,\sigma}(X)) = H^{m,q}_{\overline{\partial}^{\sigma,\sigma}_{F,\max}}(\operatorname{reg}(X), F, \sigma, \tau).$$

Therefore, by the fact that (42) is an isomorphism, we obtain that the continuous inclusion $I : L^2\Omega^{m,q}(\operatorname{reg}(X), F, \gamma, \tau) \hookrightarrow L^2\Omega^{m,q}(\operatorname{reg}(X), F, \sigma, \tau)$ induces an isomorphism between the L^2-$\overline{\partial}$ cohomology groups

$$H^{m,q}_{2,\overline{\partial}_{\max}}(\operatorname{reg}(X), F, \gamma, \tau) \cong H^{m,q}_{2,\overline{\partial}_{\max}}(\operatorname{reg}(X), F, \sigma, \tau). \tag{43}$$

By using (43), we get immediately that $\operatorname{im}(\overline{\partial}^{\gamma,\gamma}_{F,m,q,\max}) = \operatorname{im}(\overline{\partial}^{\sigma,\gamma}_{F,m,q,\max})$ and therefore, $\operatorname{im}(\overline{\partial}^{h,h}_{E,m,q,\max}) = \operatorname{im}(\overline{\partial}^{g,h}_{E,m,q,\max})$, as required. Moreover, since $H^{m,q}_{2,\overline{\partial}_{\max}}(\operatorname{reg}(X), F, \gamma, \tau)$ is finite dimensional, we have that $\operatorname{im}(\overline{\partial}^{h,h}_{E,m,q,\max})$ is closed. Hence, $\operatorname{im}(\overline{\partial}^{g,h}_{E,m,q,\max}) = \overline{\operatorname{im}(\overline{\partial}^{g,h}_{E,m,q,\max})}$ as required. Finally, as remarked above, we know that both the complexes $(\mathcal{C}^{m,\bullet}_{F,\gamma}, \overline{\partial}^{\gamma,\gamma}_{F,m,\bullet})$ and $(\mathcal{C}^{m,\bullet}_{F,\sigma}, \overline{\partial}^{\sigma,\sigma}_{F,m,\bullet})$ are fine resolutions of the sheaf $\pi_*(\mathcal{K}_M(E))$. Hence,

$$\begin{aligned}
&\dim\big(H^{m,q}_{2,\overline{\partial}_{\max}}(A, E|_A, h|_A, \rho|_A)\big)\\
&\quad = \dim\big(H^{m,q}_{2,\overline{\partial}_{\max}}(\operatorname{reg}(X), F, \gamma, \tau)\big) = \dim\big(H^q(X, \mathcal{C}^{m,\bullet}_{F,\gamma}(X))\big)\\
&\quad = \dim\big(H^q(X, \mathcal{C}^{m,\bullet}_{F,\sigma}(X))\big) = \dim\big(H^{m,q}_{2,\overline{\partial}_{\max}}(\operatorname{reg}(X), F, \sigma, \tau)\big)\\
&\quad = \dim\big(H^{m,q}_{2,\overline{\partial}_{\max}}(A, E|_A, g|_A, \rho|_A)\big) = \dim\big(H^{m,q}_{\overline{\partial}}(M, E)\big),
\end{aligned}$$

where the last equality follows by Proposition 3.1. The proof is thus complete. ∎

In order to continue, we need to introduce various tools. Let $\pi : M \to X$ be a resolution of X with $A := \pi^{-1}(\mathrm{reg}(X))$. As in the previous section, we consider $M \times [0,1]$ and the canonical projection $p : M \times [0,1] \to M$. Let $g_s \in C^\infty(M \times [0,1], p^*T^*M \otimes p^*T^*M)$ be a smooth section of $p^*T^*M \otimes p^*T^*M \to M \times [0,1]$ such that:

(1) $g_s(JX, JY) = g_s(X,Y)$ for any $X, Y \in \mathfrak{X}(M)$ and $s \in [0,1]$;

(2) g_s is a Hermitian metric on M for any $s \in (0,1]$;

(3) $g_1 = g$ and $g_0 = h$ with $h := \pi^*\gamma$;

(4) there exists a positive constant $\mathfrak{a}$ such that $g_0 \leq \mathfrak{a} g_s$ for each $s \in [0,1]$.

Let us also denote by $p : \mathrm{reg}(X) \times [0,1] \to \mathrm{reg}(X)$ the left projection and let

$$\sigma_s \in C^\infty(\mathrm{reg}(X) \times [0,1], p^*T^*\,\mathrm{reg}(X) \otimes p^*T^*\,\mathrm{reg}(X))$$

be the smooth section of $p^*T^*\,\mathrm{reg}(X) \otimes p^*T^*\,\mathrm{reg}(X) \to \mathrm{reg}(X)$ induced by g_s and π. Note that σ_s is the Hermitian metric over $\mathrm{reg}(X)$ given by $\sigma_s := ((\pi|_A)^{-1})^* g_s$ for each $s \in [0,1]$. In particular, $\sigma_1 = ((\pi|_A)^{-1})^* g$ whereas $\sigma_0 = \gamma$. Let us consider the following complex:

$$\begin{aligned} &L^2\Omega^{m,0}(\mathrm{reg}(X), F, \sigma_1, \tau) \xrightarrow{\overline{\partial}^{\sigma_1,\sigma_1}_{F,m,0}} \dots \xrightarrow{\overline{\partial}^{\sigma_1,\sigma_1}_{F,m,q-2}} L^2\Omega^{m,q-1}(\mathrm{reg}(X), F, \sigma_1, \tau) \\ &\xrightarrow{\overline{\partial}^{\sigma_1,\sigma_s}_{F,m,q-1,\max}} L^2\Omega^{m,q}(\mathrm{reg}(X), F, \sigma_s, \tau) \xrightarrow{\overline{\partial}^{\sigma_s,\gamma}_{F,m,q,\max}} L^2\Omega^{m,q+1}(\mathrm{reg}(X), F, \gamma, \tau) \\ &\xrightarrow{\overline{\partial}^{\gamma,\gamma}_{F,m,q+1,\max}} \dots \xrightarrow{\overline{\partial}^{\gamma,\gamma}_{F,m,m-1,\max}} L^2\Omega^{m,m}(\mathrm{reg}(X), F, \gamma, \tau). \end{aligned} \tag{44}$$

Let $\chi_s^{m,q} \in C(\mathrm{reg}(X) \times [0,1], \mathrm{End}(p^*\Lambda^{m,q}(\mathrm{reg}(X)) \otimes p^*F))$ be the family of endomorphisms defined as $\chi_s^{m,q} := (\pi^*)^{-1} \circ \Psi_s^{m,q} \circ \pi^*$, see (27) for the definition of $\Psi_s^{m,q}$. Clearly, $\chi_s^{m,q} : L^2\Omega^{m,q}(\mathrm{reg}(X), F, \sigma_s, \tau) \to L^2\Omega^{m,q}(\mathrm{reg}(X), F, \sigma_1, \tau)$ is an isometry for each $s \in [0,1]$. Let also define the following family of endomorphisms:

$$\chi_s^{m,\bullet} \in C(\mathrm{reg}(X) \times [0,1], \mathrm{End}(p^*\Lambda^{m,\bullet}(\mathrm{reg}(X)) \otimes p^*F)), \quad \chi_s^{m,\bullet} := \bigoplus_{r=0}^{m} \chi_s^{m,r}.$$

It is clear that $\chi_s^{m,\bullet} : L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_s, \tau) \to L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau)$ is an isometry. Following (30), we introduce the following complex:

$$\begin{aligned} &L^2\Omega^{m,0}(\mathrm{reg}(X), F, \sigma_1, \tau) \xrightarrow{\overline{\partial}^{\sigma_1,\sigma_1}_{F,m,0}} \dots \xrightarrow{\overline{\partial}^{\sigma_1,\sigma_1}_{F,m,q-2}} L^2\Omega^{m,q-1}(\mathrm{reg}(X), F, \sigma_1, \tau) \\ &\xrightarrow{D^{\sigma_1,\sigma_s}_{m,q-1}} L^2\Omega^{m,q}(\mathrm{reg}(X), F, \sigma_1, \tau) \xrightarrow{D^{\sigma_s,\gamma}_{m,q}} L^2\Omega^{m,q+1}(\mathrm{reg}(X), F, \sigma_1, \tau) \xrightarrow{D^{\gamma,\gamma}_{m,q+1}} \dots \\ &\xrightarrow{D^{\gamma,\gamma}_{m,m-1}} L^2\Omega^{m,m}(\mathrm{reg}(X), F, \sigma_1, \tau) \end{aligned} \tag{45}$$

with $D^{\sigma_1,\sigma_s}_{m,q-1} := \chi_s^{m,q} \circ \overline{\partial}^{\sigma_1,\sigma_s}_{F,m,q-1,\max}$, $D^{\sigma_s,\gamma}_{m,q} := \chi_0^{m,q+1} \circ \overline{\partial}^{\sigma_s,\gamma}_{E,m,q,\max} \circ (\chi_s^{m,q})^{-1}$ and $D^{\gamma,\gamma}_{m,r} := \chi_0^{m,r+1} \circ \overline{\partial}^{\gamma,\gamma}_{F,m,r,\max} \circ (\chi_0^{m,r})^{-1}$ for each $r = q+1, \dots, m$. Finally, let

$$Q_s^{m,q} : L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau) \to L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau) \tag{46}$$

be the rolled-up operator of the complex (45). Note that $Q_1^{m,q-1} = Q_0^{m,q}$ for each $q \in \{1, \dots, m\}$ see (31). The next lemma is the key tool to prove Theorem 4.1.

Lemma 4.3. *In the setting of Theorem 4.1, the following properties hold true:*

(1) *The triplet*

$$(L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau), m, Q_s^{m,q})$$

is an even unbounded Fredholm module over $C(X)$. We denote by $[Q_s^{m,q}]$ the corresponding class in $KK_0(C(X), \mathbb{C})$.

(2) *For each $q \in 0, \dots, m$ and $s \in [0, 1]$, we have the equality in $KK_0(C(X), \mathbb{C})$:*

$$[Q_s^{m,q}] = [Q_1^{m,q}].$$

Proof. Let $f \in C(X)$. Since in particular $f \in L^\infty(X)$, we obtain immediately that $m_f : L^2\Omega^{m,r}(\mathrm{reg}(X), F, \sigma_s, \tau) \to L^2\Omega^{m,r}(\mathrm{reg}(X), F, \sigma_s, \tau)$ is bounded for each $r \in \{0, \dots, m\}$ and $s \in [0, 1]$. Let us now fix $S_c(X)$ as a dense $*$-subalgebra of $C(X)$. Clearly, we have $\overline{\partial} f \in \Omega_c^{0,1}(\mathrm{reg}(X))$ and therefore the map $\overline{\partial} f \wedge$ given by

$$L^2\Omega^{m,r}(\mathrm{reg}(X), F, \sigma_{s_1}, \tau) \ni \eta \mapsto \overline{\partial} f \wedge \eta \in L^2\Omega^{m,r+1}(\mathrm{reg}(X), F, \sigma_{s_2}, \tau) \tag{47}$$

is continuous for any choice of $r \in \{0, \dots, m\}$ and $s_1, s_2 \in [0, 1]$. Consequently, the adjoint map $(\overline{\partial} f \wedge)^*$ given by

$$L^2\Omega^{m,r+1}(\mathrm{reg}(X), F, \sigma_{s_2}, \tau) \ni \varphi \mapsto (\overline{\partial} f \wedge)^* \varphi \in L^2\Omega^{m,r}(\mathrm{reg}(X), F, \sigma_{s_1}, \tau)$$

is continuous as well. Note that we can write the above map $(\overline{\partial} f \wedge)^*$ as $(U_{s_1}^{m,r})^{-1} \circ i_{(\nabla_1 f)^{0,1}} \circ U_{s_2}^{m,r+1}$ with

$$U_s^{m,r} := (\pi^*)^{-1} \circ S_s^{m,r} \circ \pi^*, S_s^{m,r} \in C^\infty(A \times [0, 1], \mathrm{End}(p^*\Lambda^{m,q}(A) \otimes p^*E))$$

defined in the proof of Lemma 3.5, $\nabla_1 f$ the gradient of f w.r.t. σ_1, $(\nabla_1 f)^{0,1}$ the $(0, 1)$ component of $\nabla_1 f$ and $i_{(\nabla_1 f)^{0,1}}$ the interior multiplication w.r.t. $(\nabla_1 f)^{0,1}$. Since $f \in L^\infty(X)$ and $df \in \Omega_c^1(\mathrm{reg}(X))$, we can argue, as in [4, Proposition 2.3], to conclude that m_f preserves the domain of

$$\overline{\partial}_{F,m,r,\max}^{\sigma_1,\sigma_2} : L^2\Omega^{m,r}(\mathrm{reg}(X), F, \sigma_{s_1}, \tau) \to L^2\Omega^{m,r+1}(\mathrm{reg}(X), F, \sigma_{s_2}, \tau).$$

Moreover, it is also easy to see that m_f preserves the domain of

$$\overline{\partial}_{F,m,r,\min}^{\sigma_1,\sigma_2,t} : L^2\Omega^{m,r+1}(\mathrm{reg}(X), F, \sigma_{s_1}, \tau) \to L^2\Omega^{m,r}(\mathrm{reg}(X), F, \sigma_{s_2}, \tau). \tag{48}$$

Indeed, if η lies in the domain of (48) and $\{\eta_k\}_{k \in \mathbb{N}} \in \Omega_c^{r+1}(\mathrm{reg}(X), F)$ is a sequence converging to η in the graph norm of $\overline{\partial}_{F,m,r}^{\sigma_1,\sigma_2,t}$, then $f\eta_k \to f\eta$ in $L^2\Omega^{m,r+1}(\mathrm{reg}(X), F, \sigma_2, \tau)$ as $k \to \infty$ and $\overline{\partial}_{F,m,r}^{\sigma_1,\sigma_2,t}(f\eta_k) = f\overline{\partial}_{F,m,r}^{\sigma_1,\sigma_2,t}\eta_k - (\overline{\partial} f \wedge)^*\eta_k \to f\overline{\partial}_{F,m,r,\min}^{\sigma_1,\sigma_2,t}\eta - (\overline{\partial} f \wedge)^*\eta$

in $L^2\Omega^{m,r}(\mathrm{reg}(X), F, \sigma_2, \tau)$ as $k \to \infty$. Hence, we can conclude that also $f\eta$ lies in the domain of (48). Consider now the complex (44)

$$L^2\Omega^{m,0}(\mathrm{reg}(X), F, \sigma_1, \tau) \xrightarrow{\overline{\partial}^{\sigma_1,\sigma_1}_{F,m,0}} \dots \xrightarrow{\overline{\partial}^{\sigma_1,\sigma_1}_{F,m,q-2}} L^2\Omega^{m,q-1}(\mathrm{reg}(X), F, \sigma_1, \tau)$$
$$\xrightarrow{\overline{\partial}^{\sigma_1,\sigma_s}_{F,m,q-1,\max}} L^2\Omega^{m,q}(\mathrm{reg}(X), F, \sigma_s, \tau) \xrightarrow{\overline{\partial}^{\sigma_s,\gamma}_{F,m,q,\max}} L^2\Omega^{m,q+1}(\mathrm{reg}(X), F, \gamma, \tau)$$
$$\xrightarrow{\overline{\partial}^{\gamma,\gamma}_{F,m,q+1,\max}} \dots \xrightarrow{\overline{\partial}^{\gamma,\gamma}_{F,m,m-1,\max}} L^2\Omega^{m,m}(\mathrm{reg}(X), F, \gamma, \tau)$$

and let $L_s^{m,q}$ be the corresponding rolled-up operator. By the above discussion it is now clear that m_f preserves the domain of $L_s^{m,q}$ and that $[L_s^{m,q}, m_f] = \overline{\partial} f \wedge -(\overline{\partial} f \wedge)^*$ is continuous for each $f \in S_c(X)$. Since $\chi_s^{m,r}$ is a vector bundle isometric endomorphism, we have $\chi_s^{m,r} \circ m_f = m_f \circ \chi_s^{m,r}$ and $(\chi_s^{m,r})^{-1} \circ m_f = m_f \circ (\chi_s^{m,r})^{-1}$ for each $s \in [0, 1]$, $r = 0, \dots, m$ and $f \in C(X)$. Therefore, using the above arguments, we can also conclude that for each $f \in S_c(X)$ the operator m_f preserves the domain of $Q_s^{m,q}$ and

$$[Q_s^{m,q}, m_f] : L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau) \to L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau)$$

is continuous. Furthermore, the operator $Q_s^{m,q}$ is unitarily equivalent to the operator defined in (31) through the isometry $\pi^* : L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau) \to L^2\Omega^{m,\bullet}(A, E, g_1, \rho)$. By Lemma 3.16, we can thus conclude that $Q_s^{m,q} : L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau) \to L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau)$ has entirely discrete spectrum and this is equivalent to the compactness of the resolvent. Finally, it is clear that the grading of $L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau)$, which is induced by the splitting in L^2 F-valued $(m, \bullet)$-forms with even/odd anti-holomorphic degree, commutes with m and anti-commutes with $Q_s^{m,q}$. We can therefore conclude that the triplet

$$(L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau), m, Q_s^{m,q})$$

is an even unbounded Fredholm module over $C(X)$. This concludes the proof of the first part. Now, we tackle the second part of the proof, and to do that, we use Proposition 2.7. Note that for each $s \in (0, 1]$ the metrics σ_s and σ_1 are quasi-isometric. Hence, the continuity of the map $(0, 1] \to B(L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau))$ given by $s \mapsto (Q_s^{m,q} + i)^{-1}$ with respect to the operator norm follows by arguing as in [14]. As remarked above, we have $Q_s^{m,q} = (\pi^*)^{-1} \circ P_s^{m,q} \circ \pi^*$ with $P_s^{m,q}$ defined in (31) and $\pi^* : L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau) \to L^2\Omega^{m,\bullet}(A, E, g_1, \rho)$ the isometry induced by the resolution map $\pi : M \to X$. Hence, we have

$$\begin{aligned} &\left\|(Q_s^{m,q} + i)^{-1} - (Q_0^{m,q} + i)^{-1}\right\|_{\mathrm{op}} \\ &\quad = \left\|\pi^* \circ (P_s^{m,q} + i)^{-1} \circ (\pi^*)^{-1} - \pi^* \circ (P_0^{m,q} + i)^{-1} \circ (\pi^*)^{-1}\right\|_{\mathrm{op}} \\ &\quad = \left\|(P_s^{m,q} + i)^{-1} - (P_0^{m,q} + i)^{-1}\right\|_{\mathrm{op}}. \end{aligned}$$

Thanks to Lemmas 4.1 and 4.2, we are in a position to apply Theorem 3.4 and hence, we obtain

$$\lim_{s \to 0} \left\|(P_s^{m,q} + i)^{-1} - (P_0^{m,q} + i)^{-1}\right\|_{\mathrm{op}} = 0.$$

We can thus conclude that the map $[0,1] \to B(L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau))$ given by $s \mapsto (Q_s^{m,q} + i)^{-1}$ is continuous with respect to the operator norm. This settles the second requirement of Proposition 2.7. We are left to show that for each $f \in S_c(X)$ the map $[0,1] \to B(L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau))$ given by $s \mapsto [Q_s^{m,q}, m_f]$ is continuous with respect to the strong operator topology. To this aim, it is enough to show that for any arbitrarily fixed $r = 0, \dots, m$ and $\eta \in L^2\Omega^{m,r}(\mathrm{reg}(X), F, \sigma_1, \tau)$ we have

$$\lim_{s\to 0}[Q_s^{m,q}, m_f]\eta = [Q_0^{m,q}, m_f]\eta \quad \text{in } L^2\Omega^{m,r}(\mathrm{reg}(X), F, \sigma_1, \tau). \tag{49}$$

If $0 \leq r \leq q-2$, then

$$[Q_s^{m,q}, m_f]\eta = [\overline{\partial}_{F,m,r}^{\sigma_1,\sigma_1} + \overline{\partial}_{F,m,r-1}^{\sigma_1,\sigma_1,t}, m_f]\eta = [Q_0^{m,q}, m_f]\eta$$

for each $s \in [0,1]$. Thus, (49) is obviously satisfied. If $r = q-1$, then

$$\begin{aligned}
[Q_s^{m,q}, m_f]\eta &= [D_{F,m,q-1}^{\sigma_1,\sigma_s} + \overline{\partial}_{F,m,q-2}^{\sigma_1,\sigma_1,t}, m_f]\eta \\
&= [\chi_s^{m,q} \circ \overline{\partial}_{F,m,q-1,\max}^{\sigma_1,\sigma_s} + \overline{\partial}_{F,m,q-2}^{\sigma_1,\sigma_1,t}, m_f]\eta \\
&= [\chi_s^{m,q} \circ \overline{\partial}_{F,m,q-1,\max}^{\sigma_1,\sigma_s}, m_f]\eta + [\overline{\partial}_{F,m,q-2}^{\sigma_1,\sigma_1,t}, m_f]\eta \\
&= \chi_s^{m,q} \circ [\overline{\partial}_{F,m,q-1,\max}^{\sigma_1,\sigma_s}, m_f]\eta + [\overline{\partial}_{F,m,q-2}^{\sigma_1,\sigma_1,t}, m_f]\eta \\
&= \chi_s^{m,q}(\overline{\partial} f \wedge \eta) + [\overline{\partial}_{F,m,q-2}^{\sigma_1,\sigma_1,t}, m_f]\eta.
\end{aligned}$$

Note that the term $[\overline{\partial}_{F,m,q-2}^{\sigma_1,\sigma_1,t}, m_f]\eta$ is independent on s while the equality

$$\lim_{s\to 0} \chi_s^{m,q}(\overline{\partial} f \wedge \eta) = \chi_0^{m,q}(\overline{\partial} f \wedge \eta) \quad \text{in } L^2\Omega^{m,q}(\mathrm{reg}(X), F, \sigma_1, \tau)$$

follows easily by the Lebesgue dominated convergence theorem and the fact that $\overline{\partial} f \in \Omega_c^{0,1}(\mathrm{reg}(X))$ and $\chi_s^{m,q} \in C(\mathrm{reg}(X) \times [0,1], \mathrm{End}(p^*\Lambda^{m,q}(\mathrm{reg}(X)) \otimes p^*F))$. Since

$$[Q_0^{m,q-1}, m_f]\eta = \chi_0^{m,q}(\overline{\partial} f \wedge \eta) + [\overline{\partial}_{F,m,q-2}^{\sigma_1,\sigma_1,t}, m_f]\eta,$$

we can conclude that (49) also holds true in the case $r = q-1$. If $r = q$, then

$$\begin{aligned}
[Q_s^{m,q}, m_f]\eta &= [D_{F,m,q}^{\sigma_s,\gamma} + (D_{F,m,q-1}^{\sigma_1,\sigma_s})^*, m_f]\eta \\
&= [\chi_0^{m,q+1} \circ \overline{\partial}_{F,m,q,\max}^{\sigma_s,\gamma} \circ (\chi_s^{m,q})^{-1} + \overline{\partial}_{F,m,q-1,\min}^{\sigma_1,\sigma_s,t} \circ (\chi_s^{m,q})^{-1}, m_f]\eta \\
&= [\chi_0^{m,q+1} \circ \overline{\partial}_{F,m,q,\max}^{\sigma_s,\gamma} \circ (\chi_s^{m,q})^{-1}, m_f]\eta + [\overline{\partial}_{F,m,q-1,\min}^{\sigma_1,\sigma_s,t} \circ (\chi_s^{m,q})^{-1}, m_f]\eta \\
&= \chi_0^{m,q+1} \circ [\overline{\partial}_{F,m,q,\max}^{\sigma_s,\gamma}, m_f] \circ (\chi_s^{m,q})^{-1}\eta + [\overline{\partial}_{F,m,q-1,\min}^{\sigma_1,\sigma_s,t}, m_f] \circ (\chi_s^{m,q})^{-1}\eta \\
&= \chi_0^{m,q+1}(\overline{\partial} f \wedge (\chi_s^{m,q})^{-1}\eta) + i_{(\nabla_1 f)^{0,1}}(U_s^{m,q}((\chi_s^{m,q})^{-1}\eta)).
\end{aligned}$$

Again, by the fact that $\overline{\partial} f \in \Omega_c^{0,1}(\mathrm{reg}(X))$, $U_s^{m,q}$ and

$$(\chi_s^{m,q})^{-1} \in C(\mathrm{reg}(X) \times [0,1], \mathrm{End}(p^*\Lambda^{m,q}(\mathrm{reg}(X)) \otimes p^*F))$$

and the Lebesgue dominated convergence theorem, we have

$$\lim_{s\to 0}\chi_0^{m,q+1}(\overline{\partial} f\wedge(\chi_s^{m,q})^{-1}\eta)$$
$$=\chi_0^{m,q+1}(\overline{\partial} f\wedge(\chi_0^{m,q})^{-1}\eta)\quad \text{in } L^2\Omega^{m,q+1}(\text{reg}(X),F,\sigma_1,\tau)$$

and

$$\lim_{s\to 0} i_{(\nabla_1 f)^{0,1}}(U_s^{m,q}((\chi_s^{m,q})^{-1}\eta))$$
$$=i_{(\nabla_1 f)^{0,1}}(U_0^{m,q}((\chi_0^{m,q})^{-1}\eta))\quad \text{in } L^2\Omega^{m,q-1}(\text{reg}(X),F,\sigma_1,\tau).$$

Since

$$[Q_0^{m,q},m_f]\eta=\chi_0^{m,q+1}(\overline{\partial} f\wedge(\chi_0^{m,q})^{-1}\eta)+i_{(\nabla_1 f)^{0,1}}(U_0^{m,q}((\chi_0^{m,q})^{-1}\eta)),$$

we can conclude that (49) holds true also in the case $r=q$. If $r=q+1$ we have

$$\begin{aligned}
&[Q_s^{m,q},m_f]\eta\\
&=[D^{\gamma,\gamma}_{F,m,q+1}+(D^{\sigma_s,\gamma}_{F,m,q})^*,m_f]\eta\\
&=[\chi_0^{m,q+2}\circ\overline{\partial}^{\gamma,\gamma}_{F,m,q+1,\max}\circ(\chi_0^{m,q+1})^{-1}+\chi_s^{m,q}\circ\overline{\partial}^{\sigma_s,\gamma,t}_{F,m,q,\min}\circ(\chi_0^{m,q+1})^{-1},m_f]\eta\\
&=[\chi_0^{m,q+2}\circ\overline{\partial}^{\gamma,\gamma}_{F,m,q+1,\max}\circ(\chi_0^{m,q+1})^{-1},m_f]\eta+[\chi_s^{m,q}\circ\overline{\partial}^{\sigma_s,\gamma,t}_{F,m,q,\min}\circ(\chi_0^{m,q+1})^{-1},m_f]\eta\\
&=\chi_0^{m,q+2}\circ[\overline{\partial}^{\gamma,\gamma}_{F,m,q+1,\max},m_f]\circ(\chi_0^{m,q+1})^{-1}\eta+\chi_s^{m,q}\circ[\overline{\partial}^{\sigma_s,\gamma,t}_{F,m,q,\min},m_f]\circ(\chi_0^{m,q+1})^{-1}\eta\\
&=\chi_0^{m,q+2}(\overline{\partial} f\wedge(\chi_0^{m,q+1})^{-1}\eta)+\chi_s^{m,q}((U_s^{m,q})^{-1}(i_{(\nabla_1 f)^{0,1}}(U_0^{m,q+1}((\chi_0^{m,q+1})^{-1}\eta)))).
\end{aligned}$$

Note that the first term does not depend on s whereas for the second term we have

$$\lim_{s\to 0}\chi_s^{m,q}((U_s^{m,q})^{-1}(i_{(\nabla_1 f)^{0,1}}(U_0^{m,q+1}((\chi_0^{m,q+1})^{-1}\eta))))$$
$$=\chi_0^{m,q}((U_0^{m,q})^{-1}(i_{(\nabla_1 f)^{0,1}}(U_0^{m,q+1}((\chi_0^{m,q+1})^{-1}\eta))))$$

in $L^2\Omega^{m,q}(\text{reg}(X),F,\sigma_1,\tau)$ for the same reasons explained in the previous cases. Since for $r=q+1$ we have

$$\begin{aligned}
&[Q_0^{m,q},m_f]\eta\\
&\quad=\chi_0^{m,q+2}(\overline{\partial} f\wedge(\chi_0^{m,q+1})^{-1}\eta)+\chi_0^{m,q}((U_0^{m,q})^{-1}(i_{(\nabla_1 f)^{0,1}}(U_0^{m,q+1}((\chi_0^{m,q+1})^{-1}\eta)))),
\end{aligned}$$

we can conclude that (49) holds true also in the case $r=q+1$. Finally, if $r\geq q+2$ we have

$$[Q_s^{m,q},m_f]\eta=[D^{\gamma,\gamma}_{F,m,r}+(D^{\gamma,\gamma}_{F,m,r-1})^*,m_f]\eta=[Q_0^{m,q},m_f]\eta$$

for each $s\in[0,1]$. Thus, (49) is obviously satisfied for $r\geq q+2$, which completes the proof of this lemma. ∎

Remark 4.1. The condition $\dim(\mathrm{sing}(X)) = 0$ allows us to use $S_c(X)$ as a dense $*$-subalgebra of $C(X)$ and thus the map (47) is bounded even when $s_1 \neq 0$ and $s_2 = 0$. If $\dim(\mathrm{sing}(X)) > 0$, it is not clear to us how to define a dense $*$-subalgebra of $C(X)$ such that the map (47) is bounded when $s_1 \neq 0$ and $s_2 = 0$. This problem might be overcome if one can replace Theorems 3.1 and 3.2 with a stronger convergence result, namely

$$G_{\overline{\partial}_{E,m,q}^{g_s,g_s}} \to G_{\overline{\partial}_{E,m,q,\max}^{h,h}} \tag{50}$$

compactly as $s \to 0$. Indeed, in this case one can simply consider as a dense $*$-subalgebra of $C(X)$ the space of smooth functions on X see [7, Definition 1]. Unfortunately, at the moment, we do not know how to prove (50).

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. We start by pointing out that $\pi^* : L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau) \to L^2\Omega^{m,\bullet}(A, E, g_1, \rho)$ is an isometry that makes the even bounded Fredholm modules

$$(L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau), m, Q_1^{m,m} \circ (\mathrm{Id} + (Q_1^{m,m})^2)^{-\frac{1}{2}})$$

and

$$(L^2\Omega^{m,\bullet}(M, E, g, \rho), m \circ \pi^*, \overline{\eth}_{E,m} \circ (\mathrm{Id} + (\overline{\eth}_{E,m})^2)^{-\frac{1}{2}})$$

unitarily equivalent. Therefore, $\pi_*[\overline{\eth}_{E,m}] = [Q_1^{m,m}]$. Thanks to Lemma 4.3, we know that $[Q_1^{m,m}] = [Q_0^{m,m}]$ and by construction, see (46), we have $[Q_0^{m,m}] = [Q_1^{m,m-1}]$. Again by Lemma 4.3, we have $[Q_1^{m,m-1}] = [Q_0^{m,m-1}]$ and therefore, $[Q_1^{m,m}] = [Q_0^{m,m-1}]$. Applying this procedure iteratively, we can conclude that $[Q_1^{m,m}] = [Q_0^{m,r}]$ for each $r = 0, \dots, m$. In particular, we have $[Q_1^{m,m}] = [Q_0^{m,0}]$. Finally, note that $\chi_0^{m,\bullet} : L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \gamma, \tau) \to L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau)$ is an isometry that turns the even bounded Fredholm modules

$$(L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \sigma_1, \tau), m, Q_0^{m,0} \circ (\mathrm{Id} + (Q_0^{m,0})^2)^{-\frac{1}{2}})$$

and

$$(L^2\Omega^{m,\bullet}(\mathrm{reg}(X), F, \gamma, \tau), m, \overline{\eth}_{F,m,\mathrm{abs}} \circ (\mathrm{Id} + (\overline{\eth}_{F,m,\mathrm{abs}})^2)^{-\frac{1}{2}})$$

unitarily equivalent. Therefore, $[Q_0^{m,0}] = [\overline{\eth}_{F,m,\mathrm{abs}}]$ and so, we can thus conclude that

$$\pi_*[\overline{\eth}_{E,m}] = [\overline{\eth}_{F,m,\mathrm{abs}}] \quad \text{in } KK_0(C(X), \mathbb{C})$$

as desired. ■

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Francesco Bei
Dipartimento di Matematica, Sapienza Università di Roma, Rome, Italy; bei@mat.uniroma1.it, francescobei27@gmail.com