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Exel–Pardo algebras with a twist

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Abstract. Takeshi Katsura associated a C^*-algebra $C^*_{A,B}$ to a pair of square matrices $A \geq 0$ and B of the same size with integral coefficients and gave sufficient conditions on (A, B) to be simple purely infinite (SPI). We call such a pair a KSPI pair. It follows from a result of Katsura that any separable C^*-algebra $\mathfrak{A}$ which is a cone of a map $\xi : C(S^1)^n \to C(S^1)^n$ in Kasparov's triangulated category KK is KK-isomorphic to $C^*_{A,B}$ for some KSPI pair (A, B). In this article, we introduce, for the data of a commutative ring ℓ, non-necessarily square matrices A, B and a matrix C of the same size with coefficients in the group $\mathcal{U}(\ell)$ of invertible elements, an ℓ-algebra $\mathcal{O}^C_{A,B}$, the *twisted Katsura algebra* of the triple (A, B, C). When A and B are square and C is trivial, we recover the Katsura ℓ-algebra first considered by Enrique Pardo and Ruy Exel. We show that if ℓ is a field of characteristic 0 and (A, B) is KSPI, then $\mathcal{O}^C_{A,B}$ is SPI, and that any ℓ-algebra which is a cone of a map $\xi : \ell[t, t^{-1}]^n \to \ell[t, t^{-1}]^n$ in the triangulated bivariant algebraic K-theory category kk is kk-isomorphic to $\mathcal{O}^C_{A,B}$ for some (A, B, C) as above so that (A, B) is KSPI. Katsura ℓ-algebras are examples of the Exel–Pardo algebras $L(G, E, \phi)$ associated to a group G acting on a directed graph E and a 1-cocycle $\phi : G \times E^1 \to G$. Similarly, twisted Katsura algebras are examples of the twisted Exel–Pardo ℓ-algebras $L(G, E, \phi_c)$ we introduce in the current article; they are associated to data (G, E, ϕ) as above twisted by a 1-cocycle $c : G \times E^1 \to \mathcal{U}(\ell)$. The algebra $L(G, E, \phi_c)$ can be variously described by generators and relations, as a quotient of a twisted semigroup algebra, as a twisted Steinberg algebra, as a corner skew Laurent polynomial algebra, and as a universal localization of a tensor algebra. We use each of these guises of $L(G, E, \phi_c)$ to study its K-theoretic, regularity, and (purely infinite) simplicity properties. For example, we show that if ℓ is a field of characteristic 0, G and E are countable, and E is regular, then $L(G, E, \phi_c)$ is simple whenever the Exel–Pardo C^*-algebra $C^*(G, E, \phi)$ is, and is SPI if in addition the Leavitt path algebra $L(E)$ is SPI.

1. Introduction

An *Exel–Pardo tuple* (G, E, ϕ) consists of a (directed) graph

$$E : E^1 \underset{s}{\overset{r}{\rightrightarrows}} E^0$$

together with an action of G by graph automorphisms and a 1-cocycle $\phi : G \times E^1 \to G$ satisfying $\phi(g, e)(v) = g(v)$ for all $g \in G$, $e \in E^1$, and $v \in E^0$. To an Exel–Pardo tuple

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(G, E, ϕ) and a commutative ring ℓ one associates a C^*-algebra $C^*(G, E, \phi)$ and an ℓ-algebra $L(G, E, \phi)$. For example, the Katsura C^*-algebra $C^*_{A,B}$ [23, Definition 2.2] associated to a pair of square, not necessarily finite row-finite integral matrices A and B with $A_{i,j} \geq 0$ for all i, j and such that

$$A_{i,j} = 0 \Rightarrow B_{i,j} = 0 \tag{1.1}$$

is an Exel–Pardo C^*-algebra, and its purely algebraic counterpart, the ℓ-algebra $\mathcal{O}_{A,B} = \mathcal{O}_{A,B}(\ell)$ is an Exel–Pardo ℓ-algebra. Here, $E = E_A$ is the graph with reduced incidence matrix A and both the action of $G = \mathbb{Z}$ and the cocycle ϕ are determined by B. Katsura showed that, for such (A, B), $C^*_{A,B}$ is separable, nuclear, and in the UCT class and that its (topological) K-theory is completely determined by the kernel and cokernel of the matrices $I - A^t$ and $I - B^t$. He further proved that, under certain conditions on (A, B), which we call KSPI, $C^*_{A,B}$ is simple purely infinite. Moreover, his results show that every Kirchberg algebra $\mathfrak{A}$ is isomorphic, as a C^*-algebra, to $C^*_{A,B}$ for some KSPI pair (A, B).

In the current paper, we consider Exel–Pardo tuples as above further *twisted* by a 1-cocycle $c : G \times E^1 \to \mathcal{U}(\ell)$ with values in the invertible elements of the ground ring ℓ, to which we associate an algebra $L(G, E, \phi_c)$, the *twisted Exel–Pardo algebra*. As a particular case, we obtain twisted Katsura algebras $\mathcal{O}^C_{A,B}$, where A, B are not necessarily square, integral matrices of the same size, where, as above, we assume that $A_{i,j} \geq 0$ for all i, j and that (1.1) is satisfied, and C is a matrix of the same size as A and B, but with coefficients in $\mathcal{U}(\ell)$, and such that

$$A_{i,j} = 0 \Rightarrow C_{i,j} = 1. \tag{1.2}$$

We study several properties of twisted Exel–Pardo algebras in general and of twisted Katsura algebras in particular. We define $L(G, E, \phi_c)$ by generators and relations (Section 3.4) and show that it can variously be regarded as a twisted groupoid algebra (Proposition 4.2.2), a corner skew Laurent polynomial ring (Section 8), and a universal localization (Lemma 10.7). For example, using the twisted groupoid picture and building upon the results of [4, 18, 26, 28, 29], we obtain the following simplicity criterion. Recall that a vertex $v \in E^0$ is *regular* if it emits a nonzero finite number of edges, a *sink* if it emits no edges, and an *infinite emitter* if it emits infinitely many edges. We write $\mathrm{reg}(E)$, $\mathrm{sink}(E)$, and $\inf(E) \subset E^0$ for the subsets of regular vertices, sinks, and infinite emitters. The elements of $\mathrm{sing}(E) = \mathrm{sink}(E) \cup \inf(E) = E^0 \setminus \mathrm{reg}(E)$ are the *singular* vertices of E. We say that E is *regular* if it contains no singular vertices, or equivalently if $E^0 = \mathrm{reg}(E)$.

Theorem 1.3. *Let ℓ be a field of characteristic zero and (G, E, ϕ_c) a twisted Exel–Pardo tuple with G countable and E countable and regular. If the Exel–Pardo C^*-algebra $C^*(G, E, \phi)$ is simple, then $L(G, E, \phi_c)$ is simple. If furthermore $L(E)$ is simple purely infinite, then so is $L(G, E, \phi_c)$.*

The above result specializes to twisted Katsura algebras as follows.

Corollary 1.4. *Let (A, B, C) be as above and ℓ a field of characteristic zero. Assume that (A, B) is a KSPI pair. Then, $\mathcal{O}_{A,B}^C$ is simple purely infinite.*

The description of $L(G, E, \phi_c)$ by generators and relations provides an algebra extension akin to the Cohn extension of a Leavitt path algebra or the Toeplitz extension of a graph algebra. It has the form

$$0 \to \mathcal{K}(G, E, \phi_c) \to C(G, E, \phi_c) \to L(G, E, \phi_c) \to 0. \tag{1.5}$$

We use this extension to study $L(G, E, \phi_c)$ in terms of the bivariant algebraic K-theory category kk. We show that $\mathcal{K}(G, E, \phi_c)$ and $C(G, E, \phi_c)$ are canonically kk-isomorphic to the crossed products $\ell^{\mathrm{reg}(E)} \rtimes G$ and $\ell^{E^0} \rtimes G$ (Proposition 6.2.5 and Theorem 6.3.1). Because the canonical functor $j : \mathrm{Alg}_\ell^* \to kk$ sends algebra extensions to distinguished triangles, we get that if E^0 is finite, the Cohn extension above gives rise to a distinguished triangle in kk

$$j(\ell^{\mathrm{reg}(E)} \rtimes G) \xrightarrow{f} j(\ell^{E^0} \rtimes G) \to j(L(G, E, \phi_c)). \tag{1.6}$$

We explicitly compute f in the case of twisted Katsura algebras. In this case, $G = \mathbb{Z}$ acts trivially on E^0, so for $L_1 = \ell[t, t^{-1}]$, $- \rtimes \mathbb{Z} = - \otimes L_1$ above. Since the kernel of the evaluation map $\sigma = \mathrm{Ker}(ev_1 : L_1 \to \ell)$ represents the suspension in kk, we have $j(L_1) = j(\ell) \oplus j(\ell)[-1]$ in kk. We show in Theorem 7.3 that for the reduced incidence matrix $A \in \mathbb{N}_0^{\mathrm{reg}(E)\times E^0}$ and $B \in \mathbb{Z}^{\mathrm{reg}(E)\times E^0}$ and $C \in \mathcal{U}(\ell)^{\mathrm{reg}(E)\times E^0}$ satisfying (1.1) and (1.2) the map of (1.6) has the following matricial form:

$$f : j(\ell)^{\mathrm{reg}(E)} \oplus (j(\ell)[-1])^{\mathrm{reg}(E)} \xrightarrow{\left[\begin{smallmatrix} I-A^t & C^* \\ 0 & I-B^t \end{smallmatrix}\right]} j(\ell)^{E^0} \oplus j(\ell)[-1]^{E^0}. \tag{1.7}$$

Here, $C^*_{v,w} = C^{-1}_{w,v}$. Recall from [14] that $\hom_{kk}(j(\ell)[-n], j(\ell)) = KH_n(\ell)$ is Weibel's homotopy algebraic K-theory. The coefficients of $I - A^t$, $I - B^t$, and C^* are regarded as elements of $kk(\ell, \ell) = KH_0(\ell)$ and $kk(\sigma, \ell) = KH_1(\ell)$ via the canonical maps $\mathbb{Z} \to KH_0(\ell)$ and $\mathcal{U}(\ell) \to KH_1(\ell)$, which are isomorphisms when ℓ is a field or a PID, for example. In the latter cases, we also have $kk(\ell, \sigma) = KH_{-1}(\ell) = K_{-1}(\ell) = 0$; thus, any element of $kk(L_1^m, L_1^n)$ is represented by a matrix with a zero block in the lower left corner. In fact, we prove the following theorem.

Theorem 1.8. *Let ℓ be a field or a PID, $n \geq 1$, and let $R \in \mathrm{Alg}_\ell$ such that there is a distinguished triangle*

$$j(\ell)^n \oplus j(\ell)[-1]^n \xrightarrow{g} j(\ell)^n \oplus j(\ell)[-1]^n \to j(R). \tag{1.9}$$

Then, there exist matrices $A \in M_{2n}(\mathbb{N}_0)$, $B \in M_{2n}(\mathbb{Z})$, and $C \in M_{2n}(\mathcal{U}(\ell))$ satisfying (1.1) and (1.2) with (A, B) KSPI and an isomorphism

$$j\left(\mathcal{O}_{A,B}^C\right) \cong j(R).$$

Putting Theorems 1.3 and 1.8 together, we get that if ℓ is a field of characteristic zero, then any ℓ-algebra R admitting a presentation (1.9) is kk-isomorphic to a simple purely infinite twisted Katsura algebra. An analogous result in the Kasparov KK-category of C^*-algebras follows from work of Katsura [23, Proposition 3.2], showing that any C^*-algebra with a KK-presentation of the form (1.9) is KK-isomorphic to a KSPI untwisted Katsura C^*-algebra. This uses the fact that, for Kasparov's KK and Banach algebraic K-theory, we have $KK(\mathbb{C}[-1], \mathbb{C}) = K_1^{\mathrm{top}}(\mathbb{C}) = 0$. Note however that in the purely algebraic context, it is useful to introduce the twist C, since as $kk(\ell[-1], \ell) = \mathcal{U}(\ell)$ is nonzero, the matricial form of the map g in (1.9) need not have a trivial block in the upper right corner.

Observe that applying $KH_*(-)$ to the triangle (1.6) gives a long exact sequence

$$KH_{n+1}(L(G,E,\phi_c)) \to KH_n\big(\ell^{(\mathrm{reg}(E))} \rtimes G\big) \xrightarrow{f} KH_n\big(\ell^{(E^0)} \rtimes G\big) \to KH_n(L(G,E,\phi_c)). \tag{1.10}$$

One may ask whether a similar sequence holds for Quillen's K-theory. This will be the case if every ring R appearing in (1.10) is K-regular, for in this case the comparison map $K_*(R) \to KH_*(R)$ is an isomorphism. In Section 8, we observe that when E is finite without sources, the $\mathbb{Z}$-graded algebra $L(G,E,\phi_c)$ can be regarded as a Laurent polynomial ring twisted by certain corner isomorphism ψ of the homogeneous component of degree zero. We use this to give a sufficient condition for the K-regularity of $L(G,E,\phi_c)$, where E is any row-finite graph and G acts trivially on E^0; see Theorem 8.15. As a particular case, we obtain that if $\ell[G]$ is regular supercoherent and the untwisted Exel–Pardo tuple (G,E,ϕ) is pseudo-free in the sense of [18, Definition 5.4], then $L(G,E,\phi_c)$ is K-regular. For the specific case of twisted Katsura algebras, we have the following proposition.

Proposition 1.11. *Let ℓ be a field and (A,B,C) a twisted Katsura triple. If either of the following holds, then $\mathcal{O}_{A,B}^C$ is K-regular.*

(i) $B_{v,w} = 0 \Leftrightarrow A_{v,w} = 0$.

(ii) *If $v \in \mathrm{reg}(E)$ is such that $B_{v,w} = 0$ for some $w \in r(s^{-1}\{v\})$, then $B_{v,w'} = 0$ for all $w' \in r(s^{-1}\{v\})$.*

In Section 10, we show that if E is finite, then $L(G,E,\phi_c)$ can be described as a universal localization of certain tensor algebra (Lemma 10.7), much in the same way as the Leavitt path algebra $L(E)$ is a universal localization of the usual path algebra $P(E) = T_{\ell^{E^0}}(\ell^{E^1})$. We use this to show, in Proposition 10.16, that under rather mild conditions, $L(G,E,\phi_c)$ is a regular ring, in the sense that every module over it has finite projective dimension. In particular, we get the following proposition.

Proposition 1.12. *Let ℓ be field and (A,B,C) finite square matrices satisfying (1.1) and (1.2). Then, $\mathcal{O}_{A,B}^C$ is a regular ring.*

The rest of this article is organized as follows. Section 2 starts by introducing some notation and recalling basic facts on Exel–Pardo tuples (Sections 2.1 and 2.2). Let $\mathcal{P}(E)$

be the set of all finite paths in a graph E. Lemma 2.2.4 recalls that if (G, E, ϕ) is an EP-tuple, then the G-action on E extends to an action on $\mathcal{P}(E)$ and that ϕ extends to a 1-cocycle $G \times \mathcal{P}(E) \to G$. Section 2.3 introduces twisted EP-tuples. Lemma 2.3.1 shows that there is an essentially unique way to extend a 1-cocycle $c : G \times E^1 \to G$ to a 1-cocycle $G \times \mathcal{P}(E) \to \mathcal{U}(\ell)$ that is compatible with ϕ and with the G-action. In Section 2.4, we recall the Exel–Pardo pointed inverse semigroup $\mathcal{S}(G, E, \phi)$ and show that c induces a semigroup 2-cocycle $\omega : \mathcal{S}(G, E, \phi)^2 \to \mathcal{U}(\ell)_\bullet = \mathcal{U}(\ell) \cup \{0\}$. Section 3 is devoted to introducing all the algebras of the extension (1.5). In Section 3.1, we define $C(G, E, \phi_c)$ by generators and relations and prove in Proposition 3.1.5 that it is isomorphic to the twisted semigroup algebra $\ell[\mathcal{S}(G, E, \phi), \omega]$. This automatically gives an ℓ-linear basis $\mathcal{B}$ for $C(G, E, \phi_c)$ (Corollary 3.1.10). In Section 3.2, we define $\mathcal{K}(G, E, \phi_c)$ as the two-sided ideal of $C(G, E, \phi_c)$ generated by certain elements and prove in Proposition 3.2.5 that it is isomorphic to the crossed-product of G with certain ultramatricial algebra; this again gives an ℓ-linear basis $\mathcal{B}'$ of $\mathcal{K}(G, E, \phi_c)$ (see (3.2.7)). The next subsection introduces a subset $\mathcal{B}" \subset \mathcal{B}$ such that, under certain hypothesis, $\mathcal{B}' \cup \mathcal{B}"$ is an ℓ-linear basis for $C(G, E, \phi_c)$ (Proposition 3.3.7). The hypothesis is satisfied both when the G-action on E and the 1-cocycle ϕ are trivial, and when the action satisfies a weak version of the notion of pseudo-freeness introduced in [18, Definition 5.4] which we call partial pseudo-freeness. Section 3.4 defines $L(G, E, \phi_c)$ by the extension (1.5). Then, we show that the canonical map $L(E) \to L(G, E, \phi_c)$ from the Leavitt path algebra is injective (Proposition 3.4.3) and use Proposition 3.3.7 to give a linear basis for $L(G, E, \phi_c)$ in Corollary 3.4.2 under the hypothesis of that proposition. Section 4 describes $L(G, E, \phi_c)$ as a twisted groupoid algebra in the sense of [4, 5] and uses this description to give simplicity criteria. Section 4.2 deals with the general setup of an inverse semigroup $\mathcal{S}$ acting on a space, explains how to go from a semigroup 2-cocycle $\nu : \mathcal{S} \times \mathcal{S} \to \mathcal{U}(\ell)_\bullet$ to a groupoid 2-cocycle $\tilde{\nu} : \mathcal{G} \times \mathcal{G} \to \mathcal{U}(\ell)$ on the groupoid $\mathcal{G}$ of germs, and expresses the twisted Steinberg algebra $\mathcal{A}(\mathcal{G}, \tilde{\nu})$ as a quotient of the twisted semigroup algebra $\ell[\mathcal{S}, \nu]$ (Lemma 4.1.7). Section 4.2 applies the above to the cocycle $\omega : \mathcal{S}(G, E, \phi)^2 \to \mathcal{U}(\ell)_\bullet$ and the tight groupoid $\mathcal{G}(G, E, \phi_c)$ and shows that $L(G, E, \phi_c) \cong \mathcal{A}(\mathcal{G}(S(G, E, \phi_c), \tilde{\omega}))$ (Proposition 4.2.2). Theorem 1.3 is proved as Theorem 4.3.10. Section 5 introduces twisted Katsura algebras. Corollary 1.4 is proved as Theorem 5.5. A version of the latter theorem valid over fields of arbitrary characteristic, but with additional hypothesis on the matrices A and B is proved in Proposition 5.6. Section 6 is concerned with bivariant algebraic K-theory. Section 6.1 recalls some basic facts about kk. The next two subsections are concerned with the algebras $\mathcal{K}(G, E, \phi_c)$ and $C(G, E, \phi_c)$ in the extension (1.5). Proposition 6.2.5 shows that $\mathcal{K}(G, E, \phi_c)$ is kk-isomorphic to $\ell^{(\mathrm{reg}(E))} \rtimes G$, and Theorem 6.3.1 shows that $C(G, E, \phi_c)$ is kk-isomorphic to $\ell^{(E^0)} \rtimes G$. Section 7 is about twisted Katsura algebras in kk; the map f of (1.6) is computed in this section; see Theorem 7.3 and Corollary 7.11. A short exact sequence computing homotopy K-theory $KH_*(\mathcal{O}^C_{A,B})$ is obtained in Corollary 7.9. Theorem 1.8 is a particular case of Theorem 7.13; see Corollary 7.16. Section 8 is concerned with K-regularity of $L(G, E, \phi_c)$. It starts by observing that if E is finite without sources, $L(G, E, \phi_c)$ can be regarded as a corner skew Laurent polynomial ring in the sense of [3]. This is then used in Theorem 8.15 to give a general criterion for the K-regularity of the algebra of a twisted EP-tuple (G, E, ϕ_c), where E is row-finite and G acts trivially on E^0. It applies in particular if (G, E, ϕ_c) is pseudo-free and $\ell[G]$ is regular supercoherent (Corollary 8.17). The next section investigates K-regularity of twisted Katsura algebras; Proposition 1.12 is proved as Proposition 9.2. Finally, in Section 10, still under the assumption that G acts trivially on E^0, we describe $L(G, E, \phi_c)$ as a universal localization of a certain tensor algebra (Lemma 10.7) and use this to show in Proposition 10.16 that, under rather mild conditions, $L(G, E, \phi_c)$ is a regular ring. In particular, $\mathcal{O}^C_{A,B}$ is regular (Corollary 10.17).

2. Preliminaries

2.1. Algebras

We fix a commutative unital ring ℓ. By an *algebra* we mean a symmetric ℓ-bimodule A together with an associative product $A \otimes_\ell A \to A$. For a set X, an algebra R, and a function $f : X \to R$, we write $\operatorname{supp}(f) = f^{-1}(R \setminus 0)$. We write $|Y|$ for the cardinal of a set Y. For a ring R and a set X, put

$$\begin{aligned}
\Gamma^w_X R &= \{A : X \times X \to R : |\operatorname{supp}(A(x, -))| < \infty > |\operatorname{supp}(A(-, x))|,\ \forall x \in X\},\\
\Gamma^a_X R &= \{A \in \Gamma^w_X R : \exists N \in \mathbb{N}\ |\operatorname{supp}(A(x, -))|, |\operatorname{supp}(A(-, x))| \le N\ \forall x \in X\},\\
\Gamma_X R &= \{A \in \Gamma^a_X R : |\operatorname{Im}(A)| < \infty\},\\
M_X R &= \{A \in \Gamma_X R : |\operatorname{supp}(A)| < \infty\}.
\end{aligned}$$

We regard an element of any of the sets above as an $X \times X$ matrix with coefficients in R. A matrix A is in $\Gamma^W_X R$ if every row and column of A has finite support, in $\Gamma^a_X R$ if in addition the number of elements of those supports is bounded, and in $\Gamma_X R$ if furthermore the set of entries of A is finite. A matrix A is in $M_X R$ if it has finitely many nonzero rows and finitely many nonzero columns. Observe that matricial sum and product make all four sets above into ℓ-algebras. When $R = \ell$, we drop it from the notation; thus,

$$M_X = M_X \ell, \quad \Gamma_X = \Gamma_X \ell, \quad \Gamma^a_X = \Gamma^a_X \ell, \quad \text{and} \quad \Gamma^W_X = \Gamma^W_X \ell. \tag{2.1}$$

Let $\mathcal{I}(X)$ be the inverse semigroup of all partially defined injections $X \supset \operatorname{Dom}(\sigma) \xrightarrow{\sigma} X$. If $\sigma \in \mathcal{I}(X)$ and $\operatorname{Graph}(\sigma) \subset X \times X$ is its graph, then its characteristic function lives in Γ_X:

$$U_f := \chi_{\operatorname{Graph}(\sigma)} \in \Gamma_X.$$

Let R^X be the set of all functions $X \to R$. For $a \in R^X$, let

$$\operatorname{diag}(a)_{x,y} = \delta_{x,y} a_x.$$

Observe that, for all $a \in R^X$ and $\sigma \in \mathcal{I}(X)$,

$$\Gamma^a_X R \ni \operatorname{diag}(a) U_\sigma.$$

If a takes finitely many distinct values, then $\mathrm{diag}(a)U_\sigma \in \Gamma_X R$; in fact, by [10, Lemma 2.1], the latter elements generate $\Gamma_X R$ as an ℓ-module, at least when X is countable.

2.2. Exel–Pardo tuples

Let G be a group and X a set, together with a G-action

$$G \times X \to X, \quad (g, x) \mapsto g(x).$$

Let H be a group; a 1-*cocycle* with values in H for the G-set X is a function $\psi : G \times X \to H$ such that, for every $g, h \in G$ and $x \in X$, we have

$$\psi(gh, x) = \psi(g, h(x))\psi(h, x).$$

A (directed) *graph* E consists of sets E^0 and E^1 of vertices and edges and *range* and *source* maps $r, s : E^1 \to E^0$. If $e \in E^1$, then $s(e)$ and $r(e)$ are, respectively, the *source* and the *range* of e. Let $n \geq 1$; a sequence of edges $\alpha = e_1 \cdots e_n$, such that $r(e_i) = s(e_{i+1})$ for all $1 \leq i \leq n-1$ is called a *path of length* $|\alpha| = n$, with source $s(\alpha) = s(e_1)$ and range $r(\alpha) = r(e_n)$. Vertices are regarded as paths of length 0. We write $\mathcal{P}(E)$ for the set of all paths of finite length, and, abusing notation, also for the graph with vertices E^0, edges $\mathcal{P}(E)$ and range and source maps as just defined. The set $\mathcal{P}(E)$ is partially ordered by

$$\alpha \geq \beta \Leftrightarrow (\exists \gamma)\, \beta = \alpha\gamma. \tag{2.2.2}$$

If E and F are graphs, then by a homomorphism $h : E \to F$ we understand a pair of functions $h^i : E^i \to F^i$, $i = 0, 1$ such that $r \circ h^1 = h^0 \circ r$ and $s \circ h^1 = h^0 \circ s$. A G-*graph* is a graph E equipped with an action of G by graph automorphisms. An *Exel–Pardo* 1-*cocycle* for E with values in G is a 1-cocycle $\phi : G \times E^1 \to G$ such that

$$\phi(g, e)(v) = g(v) \tag{2.2.3}$$

for all $g \in G$, $e \in E^1$, and $v \in E^0$. An *Exel–Pardo tuple* is a tuple (G, E, ϕ) consisting of a group G, a G-graph E, and a 1-cocycle as above.

Lemma 2.2.4 ([18, Proposition 2.4]). *Let (G, E, ϕ) be an Exel–Pardo tuple. Then, the G-action on E and the cocycle ϕ extend, respectively, to a G-action and a* 1*-cocycle on the path graph $\mathcal{P}(E)$ satisfying all four conditions below.*

(i) $\phi(g, v) = g$ *for all* $v \in E^0$.

(ii) $|g(\alpha)| = |\alpha|$ *for all* $\alpha \in \mathcal{P}(E)$. *The next two conditions hold for all concatenable* α, $\beta \in \mathcal{P}(E)$.

(iii) $g(\alpha\beta) = g(\alpha)\phi(g, \alpha)(\beta)$.

(iv) $\phi(g, \alpha\beta) = \phi(\phi(g, \alpha), \beta)$.

Moreover, such an extension is unique.

2.3. Twisted Exel–Pardo tuples

Write $\mathcal{U}(\ell)$ for the group of invertible elements of our ground ring ℓ. A *twisted Exel–Pardo tuple* is an Exel–Pardo tuple (G, E, ϕ) together with a 1-cocycle $c : G \times E^1 \to \mathcal{U}(\ell)$. Remark that

$$\phi_c : G \times E^1 \to \mathcal{U}(\ell)G \subset \mathcal{U}(\ell[G]), \quad \phi_c(g, e) = c(g, e)\phi(g, e)$$

is a 1-cocycle with values in the multiplicative group of the group algebra $\ell[G]$. We write (G, E, ϕ_c) for the twisted EP-tuple (G, E, ϕ, c).

Lemma 2.3.1. *Let (G, E, ϕ_c) be a twisted Exel–Pardo tuple. Then, c extends uniquely to a 1-cocycle $G \times \mathcal{P}(E) \to \mathcal{U}(\ell)$ satisfying*

$$c(g, v) = 1, \quad c(g, \alpha\beta) = c(g, \alpha)c(\phi(g, \alpha), \beta) \tag{2.3.2}$$

for all concatenable $\alpha, \beta \in \mathcal{P}(E)$.

Proof. The prescriptions (2.3.2) together with Lemma 2.2.4 dictate that we must have

$$c(g, v) = 1, \quad c(g, e_1 \cdots e_n) = c(g, e_1) \prod_{i=1}^{n-1} c(\phi(g, e_1 \cdots e_i), e_{i+1}) \tag{2.3.3}$$

for every vertex $v \in E^0$ and every path $e_1 \cdots e_n \in \mathcal{P}(E)$. We have to check that the formulas (2.3.3) define a 1-cocycle satisfying (2.3.2). It is clear that (2.3.2) is satisfied by (2.3.3) whenever α or β are vertices. Let $n, m \geq 1$, and let $e_1 \cdots e_{n+m} \in \mathcal{P}(E)$. Then, using (2.3.3) for the first identity and part (iv) of Lemma 2.2.4 for the second identity, we have

$$\begin{aligned}
&c(\phi(g, e_1, \ldots, e_n), e_{n+1} \cdots e_{n+m}) \\
&\quad = c(\phi(g, e_1, \ldots, e_n), e_{n+1}) \\
&\quad\quad \cdot \prod_{j=1}^{m-1} c(\phi(\phi(g, e_1, \ldots, e_n), e_{n+1} \cdots e_{n+j}), e_{n+j+1}) \\
&\quad = \prod_{j=0}^{m-1} c(\phi(g, e_1, \ldots, e_{n+j}), e_{n+j+1}).
\end{aligned}$$

Hence, using (2.3.3) and the identity we have just proved, we obtain

$$\begin{aligned}
&c(g, e_1 \cdots e_n)c(\phi(g, e_1, \ldots, e_n), e_{n+1} \cdots e_{n+m}) \\
&\quad = c(g, e_1) \prod_{i=1}^{n+m-1} c(\phi(g, e_1 \cdots e_i), e_{i+1}) \\
&\quad = c(g, e_1 \cdots e_{n+m}).
\end{aligned}$$

We have thus established that (2.3.3) satisfies (2.3.2); it remains to show that it is a 1-cocycle. Let $g, h \in G$, $n \geq 1$, and $e_1 \cdots e_n \in \mathcal{P}(E)$. Then, by (iii) of Lemma 2.2.4,

$$h(e_1 \cdots e_n) = h(e_1)\phi(h, e_1)(e_2) \cdots \phi(h, e_1 \cdots e_{n-1})(e_n).$$

Hence, using the above identity and the facts that ϕ is a cocycle on $\mathcal{P}(E)$ and that c is a cocycle on E^1, we obtain

$$\begin{aligned}
c(gh, e_1, \dots, e_n) &= c(gh, e_1) \prod_{i=1}^{n-1} c(\phi(gh, e_1 \cdots e_i), e_{i+1}) \\
&= c(g, h(e_1))c(h, e_1) \prod_{i=1}^{n-1} c(\phi(g, h(e_1 \cdots e_i))\phi(h, e_1 \cdots e_i), e_{i+1}) \\
&= \left(c(h, e_1) \prod_{i=1}^{n-1} c(\phi(h, e_1 \cdots e_i), e_{i+1}) \right) \\
&\quad \cdot \left(c(g, h(e_1)) \prod_{i=1}^{n-1} c(\phi(g, h(e_1)\phi(h, e_1)(e_2) \right. \\
&\qquad \left. \cdots \phi(h, e_1 \cdots e_{i-1})(e_i)), \phi(h, e_1 \cdots e_i)(e_{i+1})) \right) \\
&= c(h, e_1 \cdots e_n)c(g, h(e_1)\phi(h, e_1)(e_2) \cdots \phi(h, e_1, \dots, e_{n-1}(e_n))) \\
&= c(h, e_1 \cdots e_n)c(g, h(e_1 \cdots e_n)). \qquad \blacksquare
\end{aligned}$$

Remark 2.3.4. Let $\phi : G \times \mathcal{P}(E) \to G$ and $c : G \times \mathcal{P}(E) \to \mathcal{U}(\ell)$ be the extensions of ϕ and c given by Lemmas 2.2.4 and 2.3.1. The prescription

$$\phi_c(g, \alpha) = \phi(g, \alpha)c(g, \alpha)$$

determines a unique map

$$\phi_c : \ell[G] \times \mathcal{P}(E) \to \ell[G],$$

which is ℓ-linear on the first variable. It follows from (2.3.2) and part (iv) of Lemma 2.2.4 that

$$\phi_c(g, \alpha\beta) = \phi_c(\phi_c(g, \alpha), \beta).$$

2.4. Twists and semigroups

Let $\mathcal{S}$ be an inverse semigroup with inverse operation $s \mapsto s^*$, pointed by a zero element $\emptyset$. A (normalized) 2-*cocycle* with values in the pointed inverse semigroup

$$\mathcal{U}(\ell)_\bullet = \mathcal{U}(\ell) \cup \{0\}$$

is a map $\omega : \mathcal{S} \times \mathcal{S} \to \mathcal{U}(\ell)_\bullet$ such that, for all $s, t, u \in \mathcal{S}$ and $x \in \mathcal{S} \setminus \{\emptyset\}$,

$$\omega(st, u)\omega(s, t) = \omega(s, tu)\omega(t, u), \tag{2.4.1}$$

$$\omega(\emptyset, s) = \omega(s, \emptyset) = 0, \quad \omega(x, x^*) = \omega(xx^*, x) = \omega(x, x^*x) = 1. \tag{2.4.2}$$

The *twisted semigroup algebra* $\ell[\mathcal{S},\omega]$ is the ℓ-module $\ell[\mathcal{S}]=(\bigoplus_{s\in S}\ell s)/\ell\emptyset$ equipped with the ℓ-linear multiplication $\cdot_\omega$ induced by

$$s\cdot_\omega t = st\omega(s,t). \tag{2.4.3}$$

A straightforward calculation shows that $\ell[\mathcal{S},\omega]$ is an associative algebra and that, for every $s\in\mathcal{S}$,

$$s\cdot_\omega s^*\cdot_\omega s = s.$$

Let (G,E,ϕ_c) be a twisted Exel–Pardo tuple. Recall from [18, Definition 4.1] that there is a pointed inverse semigroup

$$\mathcal{S}(G,E,\phi)=\{(\alpha,g,\beta)\colon \alpha,\beta\in\mathcal{P}(E),\ r(\alpha)=g(r(\beta))\}\cup\{0\}.$$

Multiplication in $\mathcal{S}(G,E,\phi)$ is defined by

$$(\alpha,g,\beta)(\gamma,h,\theta)=\begin{cases}(\alpha g(\gamma_1),\phi(g,\gamma_1)h,\theta) & \text{if } \gamma=\beta\gamma_1,\\ (\alpha,g\phi(h,h^{-1}(\beta_1)),\theta h^{-1}(\beta_1)) & \text{if } \beta=\gamma\beta_1,\\ 0 & \text{otherwise.}\end{cases} \tag{2.4.4}$$

The inverse of an element of $\mathcal{S}(G,E,\phi)$ is defined by $(\alpha,g,\beta)^*=(\beta,g^{-1},\alpha)$. Define a map

$$\begin{aligned}&\omega:\mathcal{S}(G,E,\phi)\times\mathcal{S}(G,E,\phi)\to\mathcal{U}(\ell)_\bullet\\ &\omega(0,s)=\omega(s,0)=0\quad\forall s\in\mathcal{S}(G,E,\phi),\\ &\omega((\alpha,g,\beta),(\gamma,h,\theta))=\begin{cases}c(h,h^{-1}(\beta_1)) & \text{if } \beta=\gamma\beta_1\\ c(g,\gamma_1) & \text{if } \gamma=\beta\gamma_1\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{2.4.5}$$

Lemma 2.4.6. *The map ω of (2.4.5) is a 2-cocycle.*

Proof. It is clear that ω satisfies the identities (2.4.2). We have to check that the identity (2.4.1) holds for all $s,t,u\in\mathcal{S}(G,E,\phi)$. If any of s,t,u is 0, this is clear. Observe that for nonzero s,t, $\omega(s,t)=0$ exactly when $st=0$. Hence, (2.4.1) is also clear whenever $st=0$ or $tu=0$. To check the remaining cases, put $s=(\alpha,g,\beta)$, $t=(\gamma,h,\theta)$, $u=(\xi,k,\eta)$.

(1) $\beta=\gamma\beta_1$, $\xi=\theta\xi_1$. Then,

$$st=(\alpha,g\phi(h,h^{-1}(\beta_1)),\theta h^{-1}(\beta_1)),\quad tu=(\gamma h(\xi_1),\phi(h,\xi_1)k,\eta).$$

We divide this case into 3 subcases:

(a) ξ_1 and $h^{-1}(\beta_1)$ are incomparable with respect to the path order (2.2.2). Then, $h(\xi_1)$ and β_1 are incomparable also; hence, $\omega(s,tu)=\omega(st,u)=0$, so (2.4.1) holds in this case.

(b) $\xi_1 = h^{-1}(\beta_1)\xi_2$. Then, $h(\xi_1) = \beta_1\phi(h, h^{-1}(\beta_1))(\xi_2)$, and we have

$$\begin{aligned}
&\omega(st, u)\omega(s, t)\\
&\quad = c(g\phi(h, h^{-1}(\beta_1)), \xi_2)c(h, h^{-1}(\beta_1))\\
&\quad = c(g, \phi(h, h^{-1}(\beta_1))(\xi_2))c(\phi(h, h^{-1}(\beta_1)), \xi_2)c(h, h^{-1}(\beta_1))\\
&\quad = \omega(s, tu)c(h, h^{-1}(\beta_1)\xi_2)\\
&\quad = \omega(s, tu)\omega(t, u).
\end{aligned}$$

(c) $h^{-1}(\beta_1) = \xi_1\beta_2$. Then, $\beta_1 = h(\xi_1)\phi(h, \xi_1)(\beta_2)$, and we have

$$\begin{aligned}
\omega(s, tu)\omega(t, u) &= c(\phi(h, \xi_1)k, k^{-1}(\beta_2))c(h, \xi_1)\\
&= c(\phi(h, \xi_1), \beta_2)c(k, k^{-1}(\beta_2))c(h, \xi_1)\\
&= c(k, k^{-1}(\beta_2))c(h, \xi_1\beta_2)\\
&= \omega(st, u)\omega(s, t).
\end{aligned}$$

(2) $\gamma = \beta\gamma_1$ and $\xi = \theta\xi_1$. Then,

$$st = (\alpha g(\gamma_1), \phi(g, \gamma_1)h, \theta), \quad tu = (\gamma h(\xi_1), \phi(h, \xi_1)k, \eta),$$

and we have

$$\begin{aligned}
\omega(st, u)\omega(s, t) &= c(\phi(g, \gamma_1)h, \xi_1)c(g, \gamma_1)\\
&= c(\phi(g, \gamma_1), h(\xi_1))c(h, \xi_1)c(g, \gamma_1)\\
&= c(g, \gamma_1 h(\xi_1))c(h, \xi_1)\\
&= \omega(s, tu)w(t, u).
\end{aligned}$$

(3) $\beta = \gamma\beta_1$, $\theta = \xi\theta_1$. Then,

$$\begin{aligned}
&\omega(st, u)\omega(s, t)\\
&\quad = c(k, k^{-1}(\theta_1 h^{-1}(\beta_1)))c(h, h^{-1}(\beta_1))\\
&\quad = c(k, k^{-1}(\theta_1)(\phi(k^{-1}, \theta_1)h^{-1})(\beta_1))c(h, h^{-1}(\beta_1))\\
&\quad = c(k, k^{-1}(\theta_1))c(\phi(k, k^{-1}(\theta_1)), (\phi(k^{-1}, \theta_1)h^{-1})(\beta_1))c(h, h^{-1}(\beta_1))\\
&\quad = \omega(t, u)c(\phi(k, k^{-1}(\theta_1)), (\phi(k, k^{-1}(\theta_1))^{-1}h^{-1})(\beta_1))c(h, h^{-1}(\beta_1))\\
&\quad = \omega(t, u)c(h\phi(k, k^{-1}(\theta_1)), (\phi(k, k^{-1}(\theta_1))^{-1}h^{-1})(\beta_1))\\
&\quad = \omega(t, u)\omega(s, tu).
\end{aligned}$$

(4) $\gamma = \beta\gamma_1$, $\theta = \xi\theta_1$. Then,

$$\begin{aligned}
\omega(st, u)\omega(s, t) &= c(k, k^{-1}(\theta_1))c(g, \gamma_1)\\
&= \omega(t, u)\omega(s, tu).
\end{aligned}$$

∎

3. Twisted Exel–Pardo algebras via the Cohn extension

3.1. Cohn algebras

The *Cohn algebra* of the twisted Exel–Pardo tuple (G, E, ϕ_c) is the quotient $C(G, E, \phi_c)$ of the free algebra on the set

$$\{v, vg, e, eg, e^*, ge^* : v \in E^0,\ g \in G,\ e \in E^1\} \tag{3.1.1}$$

modulo the following relations:

$$\begin{aligned} v &= v1, \quad e = e1, \quad 1e^* = e^*, \quad eg = s(e)e(r(e)g), \\ ge^* &= (g(r(e))g)e^*s(e), && (3.1.2) \\ e^*f &= \delta_{e,f}r(e), \quad (vg)wh = \delta_{v,g(w)}vgh, && (3.1.3) \\ (vg)e &= \delta_{v,g(s(e))}g(e)\phi_c(g,e), \quad e^*vg = \delta_{v,s(e)}\phi_c(g, g^{-1}(e))(g^{-1}(e))^*. && (3.1.4) \end{aligned}$$

If E^0 happens to be finite, then $C(G, E, \phi_c)$ is unital with unit $1 = \sum_{v \in E^0} v$, and $g \mapsto \sum_{v \in E^0} vg$ is a group homomorphism $G \to \mathcal{U}(C(G, E, \phi_c))$. It will follow from Proposition 3.1.5 below that the latter is a monomorphism, and we will identify G with its image in $\mathcal{U}(C(G, E, \phi_c))$. Thus, if E^0 is finite, then writing $\cdot$ for the product in $C(G, E, \phi_c)$, we have $vg = v \cdot g$ and $eg = e \cdot g$. However, if E^0 is infinite, we do not identify G with any subset of $C(G, E, \phi_c)$.

Proposition 3.1.5. *Let ω be as in (2.4.5). There is an isomorphism of algebras*

$$C(G, E, \phi_c) \xrightarrow{\sim} \ell[\mathcal{S}(G, E, \phi), \omega]$$

mapping $vg \mapsto (v, g, g^{-1}(v))$, $eg \mapsto (e, g, g^{-1}(r(e)))$, $ge^* \mapsto (g(r(e)), g, e)$.

Proof. One checks that the elements

$$\begin{aligned} &(v, 1, v), \quad (v, g, g^{-1}(v)), \quad (e, 1, r(e)), \quad (e, g, g^{-1}(r(e))), \\ &(r(e), 1, e), \quad \text{and} \quad (g(r(e)), g, e) \end{aligned}$$

of $\ell[\mathcal{S}(G, E, \phi), \omega]$ satisfy the relations (3.1.2), (3.1.3), and (3.1.4); thus, there is a homomorphism as in the proposition. Next, we define its inverse. Because $e = s(e)er(e)$ in $C(G, E, \phi_c)$, we have a multiplication preserving map $\mathcal{P}(E) \to C(G, E, \phi_c)$ sending a path $\alpha = e_1 \cdots e_n$ to the product of its edges in $C(G, E, \phi_c)$. If $\alpha, \beta \in \mathcal{P}(E)$ and $g \in G$, we write

$$\alpha g \beta^* := \alpha(r(\alpha)g)\beta^* \in C(G, E, \phi_c).$$

If α or β is a vertex, we abbreviate the above as αg or $g\beta^*$. The set-theoretic map

$$a : \mathcal{S}(G, E, \phi) \to C(G, E, \phi_c), \quad a(0) = 0, \quad a(\alpha, g, \beta) = \alpha g \beta^* \tag{3.1.6}$$

induces a homomorphism of ℓ-modules $\ell[\mathcal{S}(G, E, \phi), \omega] \to C(G, E, \phi_c)$. We will show that this is in fact a homomorphism of algebras. The usual proof that the Cohn algebra

$C(E) = C(1, E, 1)$ is the semigroup algebra of $\mathcal{S}(E) = \mathcal{S}(1, E)$ shows that the relations (3.1.2) and (3.1.3) imply that if $\beta, \gamma \in \mathcal{P}(E)$, then

$$\beta^*\gamma = \begin{cases} \beta_1^* & \text{if } \beta = \gamma\beta_1, \\ \gamma_1 & \text{if } \gamma = \beta\gamma_1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.7}$$

Next, we show, by induction on $|\gamma|$, that

$$(vg)\gamma = \delta_{v,g(s(\gamma))}g(\gamma)\phi_c(g, \gamma). \tag{3.1.8}$$

If $|\gamma| \le 1$, this is clear from the second equation of (3.1.3) and the first in (3.1.4). For the inductive step, let $n \ge 1$, $|\gamma| = n + 1$, $\gamma = e_{n+1} \cdots e_1$, $w = s(\gamma)$, $u = r(\gamma)$. Then, using the first equation of (3.1.4) in the first line, the inductive step in the second, and (2.3.2) and part (iii) of Lemma 2.2.4 and Remark 2.3.4 in the last, we obtain

$$\begin{aligned}(vg)\gamma &= \delta_{v,g(w)}g(e_{n+1}) \cdot (g(r(e_{n+1}))\phi_c(g, e_{n+1})) \cdot (e_n \cdots e_1) \\ &= \delta_{v,g(w)}c(g, e_{n+1})g(e_{n+1})\phi(g, e_{n+1})(e_n \cdots e_1)\phi_c(\phi(g, e_{n+1}), e_n \cdots e_1) \\ &= \delta_{v,g(w)}g(\gamma)\phi_c(g, \gamma).\end{aligned}$$

A similar argument shows that

$$\gamma^*(vg) = \delta_{v,s(\gamma)}\phi_c(g, g^{-1}(\gamma))g^{-1}(\gamma)^*. \tag{3.1.9}$$

Next, we use (3.1.7), (3.1.8), and (3.1.9) to show that (3.1.6) induces an algebra homomorphism $\ell[\mathcal{S}(G, E, \phi), \omega] \to C(G, E, \phi_c)$. If, for example, $(\alpha, g, \beta), (\gamma, h, \theta) \in \mathcal{S}(G, E, \phi)$ and $\gamma = \beta\gamma_1$, then using (3.1.7), (3.1.8), and (2.4.4) we obtain

$$\begin{aligned}a(\alpha, g, \beta)a(\gamma, h, \theta) &= (\alpha g\beta^*)(\gamma h\theta) \\ &= \alpha g \cdot \gamma_1 h\theta \\ &= c(g, \gamma_1)\alpha g(\gamma_1)\phi(g, \gamma_1)h\theta^* \\ &= \omega((\alpha, g, \beta), (\gamma, h, \theta))a((\alpha, g, \beta)(\gamma, h, \theta)).\end{aligned}$$

The case $\beta = \gamma\beta_1$ is similar, using (3.1.9) in place of (3.1.8). Thus, a is an algebra homomorphism. It is clear that the composite of a with the map of the lemma is the identity on $C(G, E, \phi_c)$. To see that the reverse composition is the identity also, one checks that if $\alpha = e_1 \cdots e_n$ and $\beta = f_1 \cdots f_m$ and $g \in G$ are such that $g(r(\beta)) = r(\alpha)$, then the following identity holds in $\mathcal{S}(G, E, \phi)$:

$$(\alpha, g, \beta) = (e_1, 1, r(e_1)) \cdots (e_n, 1, r(e_n))(r(e_n), g, r(f_m))(r(f_m), 1, f_m) \cdots (r(f_1), 1, f_1).$$

This completes the proof. ∎

Corollary 3.1.10. *$C(G, E, \phi_c)$ is a free ℓ-module with basis*

$$\mathcal{B} = \{\alpha g\beta^* : \alpha, \beta \in \mathcal{P}(E),\ g \in G,\ g(r(\beta)) = r(\alpha)\}.$$

Remark 3.1.11. If E^0 is finite, then we may add up (3.1.8) over all $v \in E^0$ to obtain

$$g\gamma = g(\gamma)\phi_c(g, \gamma).$$

Similarly, from (3.1.9), we get

$$\gamma^* g = \phi_c(g, g^{-1}(\gamma))g^{-1}(\gamma)^*.$$

3.2. The algebra $\mathcal{K}(G, E, \phi_c)$

For each regular vertex v of E and each $g \in G$, we consider the following element of $C(G, E, \phi_c)$:

$$q_v g := vg - \sum_{s(e)=v} e\phi_c(g, g^{-1}(e))g^{-1}(e)^*. \tag{3.2.1}$$

As usual, we set $q_v = q_v 1$; observe that

$$q_v g = q_v \cdot (vg).$$

Recall that a vertex $v \in E^0$ is *regular* if $0 < |s^{-1}\{v\}| < \infty$; write $\mathrm{reg}(E) \subset E^0$ for the subset of regular vertices. Define a two-sided ideal

$$C(G, E, \phi_c) \rhd \mathcal{K}(G, E, \phi_c) = \langle q_v : v \in \mathrm{reg}(E)\rangle. \tag{3.2.2}$$

For each $v \in E^0$, let

$$\mathcal{P}_v = \{\alpha \in \mathcal{P}(E) : r(\alpha) = v\}, \quad \mathcal{P}^v = \{\alpha \in \mathcal{P}(E) : s(\alpha) = v\}. \tag{3.2.3}$$

With notations as in (2.1), the action of G on $\mathcal{P}(E)$ induces one on $\bigoplus_{v\in E^0} M_{\mathcal{P}_v}$, which maps $\bigoplus_{v\in\mathrm{reg}(E)} M_{\mathcal{P}_v}$ to itself. In particular, we can form the crossed- (or smash) product algebra

$$\Big(\bigoplus_{v\in\mathrm{reg}(E)} M_{\mathcal{P}_v}\Big) \rtimes G.$$

Multiplication in the crossed-product is defined by

$$(\varepsilon_{\alpha,\beta} \rtimes g)(\varepsilon_{\gamma,\theta} \rtimes h) = \delta_{\beta,g(\gamma)}\varepsilon_{\alpha,g(\theta)} \rtimes gh. \tag{3.2.4}$$

Proposition 3.2.5. *There is an algebra isomorphism*

$$\upsilon : \Big(\bigoplus_{v\in\mathrm{reg}(E)} M_{\mathcal{P}_v}\Big) \rtimes G \to \mathcal{K}(G, E, \phi_c), \quad \varepsilon_{\alpha,\beta} \rtimes g \mapsto \alpha(q_{r(\alpha)}g)(g^{-1}(\beta))^*.$$

In particular, $\mathcal{K}(G, E, \phi_c)$ is independent of ϕ_c up to canonical algebra isomorphism.

Proof. One checks that, for $v, w \in \mathrm{reg}(E)$ and $g, h \in G$, we have

$$(q_v g)(q_w h) = \delta_{v,g(w)}q_v(gh). \tag{3.2.6}$$

Thus, if $\alpha, \beta \in \mathcal{P}_v$ and $\gamma, \theta \in \mathcal{P}_w$, then

$$\upsilon(\varepsilon_{\alpha,\beta} \rtimes g)\upsilon(\varepsilon_{\gamma,\theta} \rtimes h) = (\alpha(q_v g)g^{-1}(\beta)^*)(\gamma(q_w h)h^{-1}(\theta)^*)$$
$$= \delta_{g(\gamma),\beta}\alpha q_v(gh)h^{-1}(\theta)^* = \upsilon(\varepsilon_{\alpha,g(\theta)} \rtimes gh).$$

Hence, υ is a homomorphism of algebras. To prove that υ is bijective, it suffices to show that

$$\mathcal{B}' = \{\alpha q_v g\beta^* :\ v \in \operatorname{reg}(E),\ g \in G,\ \alpha \in \mathcal{P}_v,\ \beta \in \mathcal{P}_{g^{-1}(v)}\} \tag{3.2.7}$$

is an ℓ-module basis of $\mathcal{K}(G, E, \phi_c)$. By Corollary 3.1.10, $\mathcal{K}(G, E, \phi_c)$ is generated by the products $x(q_v g)y$ with $x, y \in \mathcal{B}$. One checks that if $|\alpha| \geq 1$, then $\alpha^*(q_v g) = (q_v g)\alpha = 0$. It follows that $\mathcal{B}'$ generates $\mathcal{K}(G, E, \phi_c)$ as an ℓ-module. Linear independence of $\mathcal{B}'$ is derived from that of $\mathcal{B}$, just as in the case $G = 1$ ([1, Proposition 1.5.11]; see also [11, Proposición 4.3.3]). ■

Corollary 3.2.8. *The $*$-algebra $\mathcal{K}(G, E, \phi_c)$ carries a canonical G-grading, where, for each $g \in G$, the homogeneous component of degree g is*

$$\mathcal{K}(G, E, \phi_c)_g = \operatorname{span}\{\alpha(q_v g)\beta^* : v \in \operatorname{reg}(E),\ g \in G,\ \alpha \in \mathcal{P}_v,\ \beta \in \mathcal{P}_{g^{-1}(v)}\}.$$

3.3. Extending $\mathcal{B}'$ to a basis of $C(G, E, \phi_c)$

Let (G, E, ϕ_c) be a twisted EP-tuple, and let $e \in E^1$. Consider the map

$$\nabla_e : G \to E^1 \times G, \quad g \mapsto (g^{-1}(e), \phi(g, g^{-1}(e))). \tag{3.3.1}$$

In Lemma 3.3.3 below, we show equivalent conditions to the injectivity of the map ∇_e of (3.3.1). First, recall some definitions and notations. Let R be a ring, M a right R-module, and $x \in M$. Write

$$\operatorname{Ann}_R(x) = \{a \in R : xa = 0\}$$

for the *annihilator* of x. Let (G, E, ϕ_c) be a twisted EP-tuple. The ℓ-module $\ell[\mathcal{S}(G, E, \phi)]$ has two different right $\ell[G]$-module structures, induced, respectively, by multiplication in $\mathcal{S}(G, E, \phi)$ and by that in $\ell[\mathcal{S}(G, E, \phi), \omega]$:

$$(\alpha, g, \beta) \cdot h = (\alpha, g, \beta)(s(\beta), h, h^{-1}(s(\beta)))$$

and

$$(\alpha, g, \beta) \cdot_\omega h = (\alpha, g, \beta) \cdot_\omega (s(\beta), h, h^{-1}(s(\beta))).$$

Observe that if $e \in E^1$, then an element $\sum_{g \in G} \lambda_g g \in \ell[G]$ is in the annihilator of $(e, 1, e)$ with respect to the first structure if and only if $\sum_{g \in G} \lambda_g c(g, g^{-1}(e))^{-1} g$ is in the annihilator with respect to the second structure. Hence, in view of Proposition 3.1.5, the condition

$$\operatorname{Ann}_{\ell[G]}(ee^*) = 0 \tag{3.3.2}$$

does not depend on whether we regard ee^* as an element of $C(G,E,\phi)$ or of $C(G,E,\phi_c)$. Recall from [18, Section 5] that a finite path $\alpha\in\mathcal{P}(E)$ is *strongly fixed* by an element $g\in G$ if $g(\alpha)=\alpha$ and $\phi(g,\alpha)=1$.

Lemma 3.3.3. *Let (G,E,ϕ_c) be a twisted EP-tuple and $e\in E^1$. Then, the following are equivalent.*

(i) $\operatorname{Ann}_{\ell[G]}(ee^*)=0$.

(ii) $\{g\in G: g$ *fixes* e *strongly*$\}=\{1\}$.

(iii) *The map* ∇_e *of* (3.3.1) *is injective.*

Proof. The equivalence between (ii) and (iii) is [18, Proposition 5.6]. We show that (i)$\Rightarrow$(ii) and that (iii)$\Rightarrow$(i). For the purpose of this proof, we will regard ee^* as an element of $C(G,E,\phi)$. If $g\in G\setminus\{1\}$ fixes e strongly, then $0\ne g-1\in\operatorname{Ann}_{\ell[G]}(ee^*)$. Next, assume that there is a nonzero element $x=\sum_{g\in G}\lambda_g g\in\operatorname{Ann}_{\ell[G]}(ee^*)$. Then,

$$0=\sum_{g\in G}\lambda_g ee^*\cdot g=\sum_{\{h\in G,f\in G\cdot e\}}\Big(\sum_{\{g|g^{-1}(e)=f,\phi(g,f)=h\}}\lambda_g\Big)ehf^*.$$

By Corollary 3.1.10, we must have

$$\sum_{\{g|g-1(e)=f,\ \phi(g,f)=h\}}\lambda_g=0\quad(\forall f\in G\cdot e,h\in G).$$

Since $x\ne 0$, this implies that there are $g\ne g'\in G$ such that $g^{-1}(e)=(g')^{-1}(e)$ and $\phi(g,g^{-1}(e))=\phi(g',(g')^{-1}(e))$, which precisely means that ∇_e is not injective. ■

Consider the following subsets of $\operatorname{reg}(E)$:

$$\operatorname{reg}(E)_0=\{v\colon \operatorname{Im}(\nabla_e)=\{(e,1)\}\ \forall e\in s^{-1}\{v\}\},\tag{3.3.4}$$
$$\operatorname{reg}(E)_1=\{v\colon (\exists e\in s^{-1}\{v\})\ \nabla_e \text{ is injective}\}.\tag{3.3.5}$$

Let

$$\operatorname{reg}(E)\to E^1,\quad v\mapsto e_v\tag{3.3.6}$$

be a section of s such that ∇_{e_v} is injective for all $v\in\operatorname{reg}(E)_1$. Set

$$\mathcal{B}\supset\mathcal{A}=\{\alpha e_v\phi(g,g^{-1}(e_v))(\beta g^{-1}(e_v))^*,\ v\in\operatorname{reg}(E),\ g\in G\}$$
$$\cup\{\alpha vg\beta^*\colon v\in\operatorname{reg}(E)_0,\ g\in G\setminus\{1\}\}.$$

Proposition 3.3.7. *Let $\mathcal{B}'$ be as in* (3.2.7)*. Assume that* $\operatorname{reg}(E)=\operatorname{reg}(E)_0\sqcup\operatorname{reg}(E)_1$. *Set* $\mathcal{B}"=\mathcal{B}\setminus\mathcal{A}$. *Then,* $\mathcal{B}'\cup\mathcal{B}"$ *is an ℓ-module basis of* $C(G,E,\phi_c)$.

Proof. Put $\mathcal{B}'''=\mathcal{B}'\cup\mathcal{B}"$, and let M be the ℓ-submodule generated by $\mathcal{B}'''$. Let $v\in E^0$ and $g\in G$ such that $r(\alpha)=v=g(r(\beta))$. If $v\in\operatorname{reg}(E)_0$, then

$$\alpha vg\beta^*=\alpha q_v g\beta^*-\alpha q_v\beta^*+\alpha\beta^*\in M.$$

For any $v \in \operatorname{reg}(E)$,

$$\alpha e_v \phi(g, g^{-1}(e_v))(\beta g^{-1}(e_v))^* \\ = c(g^{-1}, e_v)\Big(\alpha v g \beta^* - \alpha q_v g \beta^* - \sum_{s(e)=v, e \neq e_v} \alpha e \phi_c(g, g^{-1}(e))(\beta g^{-1}(e))^*\Big) \in M.$$

Thus, $M = C(G, E, \phi_c)$. Next, we show that $\mathcal{B}'''$ is linearly independent. Let N be the linear span of $\mathcal{B}$". By Proposition 3.2.5 and Corollary 3.1.10, $\mathcal{B}'$ is a basis of $\mathcal{K} = \mathcal{K}(G, E, \phi_c)$ and $\mathcal{B}$" is a basis of N. Hence, it suffices to show that $\mathcal{K} \cap N = 0$. Suppose otherwise that $0 \neq x \in \mathcal{K} \cap N$. Let $\alpha q_v g \beta^*$ be an element of $\mathcal{B}'$ with $d = |\alpha|$ maximum among those appearing with a nonzero coefficient in the unique expression of x as an ℓ-linear combination of $\mathcal{B}'$. Because x is also a linear combination of $\mathcal{B}$", and $\alpha e_v e_v^* g \beta^* = \alpha e_v \phi(g, g^{-1}(e_v)) g^{-1}(e_v)^* \beta^* \notin \mathcal{B}$", there must be another element of $\mathcal{B}'$ appearing in the expression of x that cancels the latter term. Because d is maximum, that other element must be of the form $\alpha q_v h \beta^*$ for some $h \neq g \in G$ such that $h^{-1}(e_v) = g^{-1}(e_v)$ and $\phi(h, h^{-1}(e_v)) = \phi(g, g^{-1}(e_v))$. By our hypothesis on E, this implies that $v \in \operatorname{reg}(E)_0$. It follows that $\alpha g \beta^*$ and $\alpha h \beta^*$ appear in the unique expression of x as a linear combination of $\mathcal{B}$, and because we are assuming that $x \in N$, both basis elements must be in $\mathcal{B}$", a contradiction. This completes the proof. ■

We say that a twisted EP tuple (G, E, ϕ_c) is *pseudo-free* if the associated untwisted EP-tuple (G, E, ϕ) is pseudo-free in the sense of [18, Definition 5.4], which means precisely that the conditions of Lemma 3.3.3 hold for every $e \in E^1$. We call (G, E, ϕ_c) *partially pseudo-free* if $\operatorname{reg}(E) = \operatorname{reg}(E)_1$. In this case, we have

$$\mathcal{B}" = \mathcal{B} \setminus \{\alpha e_v \phi(g, g^{-1}(e_v))(\beta g^{-1}(e_v))^*,\ v \in \operatorname{reg}(E),\ g \in G\}. \tag{3.3.8}$$

3.4. The twisted Exel–Pardo algebra $L(G, E, \phi_c)$

Let (G, E, ϕ_c) be a twisted EP-tuple. Set

$$L(G, E, \phi_c) = C(G, E, \phi_c)/\mathcal{K}(G, E, \phi_c).$$

Thus, in view of (3.2.1) and (3.2.2) and the definition of $C(G, E, \phi_c)$, we obtain that $L(G, E, \phi_c)$ is the free algebra on the set (3.1.1) subject to the relations (3.1.2), (3.1.3), (3.1.4) and the additional relation

$$v = \sum_{s(e)=v} ee^* \ (v \in \operatorname{reg}(E)). \tag{3.4.1}$$

The relation (3.4.1) is the second *Cuntz–Krieger relation*; as is common practice in the field, we will refer to it as CK2. The first Cuntz–Krieger relation CK1 is the first of the two relations in (3.1.3).

The following is a corollary of Proposition 3.3.7.

Corollary 3.4.2. *Let (G, E, ϕ_c) be a twisted EP tuple such that*

$$\operatorname{reg}(E) = \operatorname{reg}(E)_0 \sqcup \operatorname{reg}(E)_1,$$

and let $\mathcal{B}" \subset C(G, E, \phi_c)$ be as in Proposition 3.3.7. Then, the image of $\mathcal{B}"$ is a basis of $L(G, E, \phi_c)$. In particular, the algebra extension

$$\mathcal{K}(G, E, \phi_c) \hookrightarrow C(G, E, \phi_c) \twoheadrightarrow L(G, E, \phi_c)$$

is ℓ-linearly split.

Proposition 3.4.3. *Let (G, E, ϕ_c) be any twisted EP tuple. Then, the canonical homomorphism $L(E) \to L(G, E, \phi_c)$ is injective.*

Proof. Let (3.3.6) be any section of s. By [1, Corollary 1.5.12], the following subset of $C(E)$

$$\mathcal{B}_0 = \{\alpha\beta^* : r(\alpha) = r(\beta)\} \setminus \{\alpha e_v(\beta e_v)^* : r(\alpha) = s(\alpha) = v \in \operatorname{reg}(E)\}$$

maps to a basis of $L(E)$. Hence, it suffices to show that $\mathcal{B}_0 \cup \mathcal{B}' \subset C(G, E, \phi_c)$ is linearly independent. As in the proof of Proposition 3.3.7, this amounts to showing that the ℓ-submodule $C(G, E, \phi_c) \supset N = \mathcal{K}(G, E, \phi_c) \cap \operatorname{span} \mathcal{B}_0$ is zero. If $0 \neq x \in N$, then it is a linear combination of elements of $\mathcal{B}_0 \subset \mathcal{B}$, so any element $\alpha q_v g \beta^*$ appearing with a nonzero coefficient in the unique expression of x as a linear combination of $\mathcal{B}'$ must have $g = 1$. So, $x \in \mathcal{K}(1, E, 1) \cap \operatorname{span}(\mathcal{B}_0)$ which is zero by [1, Corollary 1.5.12]. ■

4. Twisted Exel–Pardo algebras as twisted Steinberg algebras

4.1. From semigroup twists to groupoid twists

Let $(\mathcal{S}, \emptyset)$ be a pointed inverse semigroup, and let $\omega : \mathcal{S} \times \mathcal{S} \to \mathcal{U}(\ell)_\bullet$ be a 2-cocycle as in Section 2.4. Equip the smash product set

$$\begin{aligned}\mathcal{U}(\ell)_\bullet \wedge \mathcal{S} &= \mathcal{U}(\ell)_\bullet \times \mathcal{S} / \mathcal{U}(\ell)_\bullet \times \{\emptyset\} \cup \{0\} \times \mathcal{S} \\ &= \mathcal{U}(\ell) \times \mathcal{S} / \mathcal{U}(\ell) \times \{\emptyset\}.\end{aligned}$$

with the product

$$(u \wedge s)(v \wedge t) = uv\omega(s, t) \wedge st.$$

The result is a pointed inverse semigroup $\widetilde{\mathcal{S}}$, with inverse

$$(u \wedge s)^* = u^{-1} \wedge s^*.$$

We write 0 for the class of the zero element in $\widetilde{\mathcal{S}}$. The coordinate projection $\mathcal{U}(\ell) \times \mathcal{S} \to \mathcal{S}$ induces a surjective semigroup homomorphism $\pi : \widetilde{S} \to S$. Observe that $\pi^{-1}(\{\emptyset\}) = \{0\}$ and that if $p \in \mathcal{S}$ is a nonzero idempotent, we have a group isomorphism

$$\pi^{-1}(\{p\}) = \mathcal{U}(\ell) \wedge p \cong \mathcal{U}(\ell).$$

Next, let X be a locally compact, Hausdorff space, and let

$$\mathcal{I}_{\text{cont}}(X) = \{f : U \to X \mid U \subset X \text{ open, } f \text{ injective and continuous}\}$$

be the inverse semigroup of partially defined injective maps with open domains. Recall from [17, Definition 4.3] that an *action* of $\mathcal{S}$ on X is a semigroup homomorphism $\theta : \mathcal{S} \to \mathcal{I}_{\text{cont}}(X)$, $s \mapsto \theta_s$, such that

$$X = \bigcup_{s \in \mathcal{S}} \text{Dom}(\theta_s).$$

To alleviate notation, for $s \in S$ we write

$$\text{Dom}(s) = \text{Dom}(\theta_s), \quad \text{and for } x \in \text{Dom}(s),\ s(x) = \theta_s(x).$$

A *germ* for the action of $\mathcal{S}$ on X is the class $[s, x]$ of a pair $(s, x) \in \mathcal{S} \times X$ with $x \in \text{Dom}(s)$, where $[s, x] = [t, y]$ if and only if $x = y$ and there exists an idempotent $p \in \mathcal{S}$ such that $p(x) = x$ and $sp = tp$. The set of all germs forms an étale groupoid $\mathcal{G} = \mathcal{G}(\mathcal{S}, X)$ [17, Proposition 4.17], topologized by the basis of open sets

$$[s, U] = \{[s, x] : x \in U\}, \quad s \in \mathcal{S}, \quad U \subset X \quad \text{open such that } \text{Dom}(s) \supset U.$$

A germ $[s, x]$ has domain x and range $s(x)$ and the multiplication map $\mathcal{G}^{(2)} \to \mathcal{G}$ is given by

$$[s, t(x)][t, x] = [st, x].$$

Assume that an action of $\mathcal{S}$ on X is given; then, $\widetilde{\mathcal{S}}$ also acts on X via $\theta \circ \pi$, and so, we can consider the groupoids of germs

$$\mathcal{G} = \mathcal{G}(\mathcal{S}, X), \quad \widetilde{\mathcal{G}} = \mathcal{G}(\widetilde{\mathcal{S}}, X).$$

Equip $\mathcal{U}(\ell)$ with the discrete topology and regard it as a groupoid over the one-point space. The map

$$\widetilde{\mathcal{G}} \to \mathcal{U}(\ell) \times \mathcal{G}, \quad [u \wedge s, x] \mapsto (u, [s, x]) \tag{4.1.1}$$

is a homeomorphism mapping a basic open set $[u \wedge s, U]$ to the basic open $\{u\} \times [s, U]$, and the sequence

$$\mathcal{U}(\ell) \times \mathcal{G}^{(0)} \to \widetilde{\mathcal{G}} \to \mathcal{G} \tag{4.1.2}$$

is a *discrete twist* over the (possibly non-Hausdorff) étale groupoid $\mathcal{G}$, in the sense of [5, Section 2.3]. Under the homeomorphism (4.1.1), multiplication of $\widetilde{\mathcal{G}}$ corresponds to

$$(u, [s, t(x)])(v, [t, x]) = (uv\omega(s, t), [st, x]).$$

The map

$$\widetilde{\omega} : \mathcal{G}^{(2)} \to \mathcal{U}(\ell), \quad \widetilde{\omega}([s, t(x)], [t, x]) = \omega(s, t) \tag{4.1.3}$$

is a continuous, normalized 2-cocycle in the sense of [4, p. 4].

Recall that a *slice* (or *local bisection*) is an open subset $U \subset \mathcal{G}$ such that the domain and codomain functions restrict to injections on U. For example, the basic open subsets $[s,U]$ are slices [17, Proposition 4.18]. We observe moreover that the domain map induces a homeomorphism $[s,U] \overset{\sim}{\to} U$ [17, Proposition 4.15]. The groupoid $\mathcal{G}$ is *ample* if its topology has a basis of compact slices. Such is the case, for example, if X has a basis of clopen subsets, in which case we say that X is *Boolean*. If $\mathcal{G}$ is ample, its *Steinberg* algebra $\mathcal{A}(\mathcal{G})$ [27, Proposition 4.3] is the ℓ-module spanned by all characteristic functions of compact slices, equipped with the convolution product

$$f \star g([s,x]) = \sum_{t_1 t_2 = s} f[t_1, t_2(x)]g[t_2, x]. \tag{4.1.4}$$

Following [27, Definition 5.2], we say that the action of $\mathcal{S}$ on X is *ample* if X is Boolean and $\mathrm{Dom}(s)$ is a compact clopen subset for all $s \in \mathcal{S}$. In this case, the characteristic function $\chi_{[s,\mathrm{Dom}(s)]} \in \mathcal{A}(\mathcal{G})$ for every $s \in \mathcal{S}$, and the ℓ-module map

$$\ell[\mathcal{S}] \to \mathcal{A}(\mathcal{G}), \quad s \mapsto \chi_{[s,\mathrm{Dom}(s)]} \tag{4.1.5}$$

is a homomorphism of algebras. One may also equip the ℓ-module $\mathcal{A}(\mathcal{G})$ with the *twisted convolution product*

$$\begin{aligned} f \star_\omega g([s,x]) &= \sum_{t_1 t_2 = s} \tilde{\omega}([t_1, t_2(x)], [t_2, x]) f[t_1, t_2(x)]g[t_2, x] \\ &= \sum_{t_1 t_2 = s} \omega(t_1, t_2) f[t_1, t_2(x)]g[t_2, x]. \end{aligned} \tag{4.1.6}$$

The result is an algebra $\mathcal{A}(\mathcal{G}, \tilde{\omega})$. By [27, Proposition 4.3], this is the same as the twisted Steinberg algebra defined in [4, Proposition 3.2] under the assumption that $\mathcal{G}$ is Hausdorff. Moreover, by [4, Corollary 4.25], the latter also agrees with the Steinberg algebra of the discrete twist (4.1.2). Comparison of the formulas (4.1.4), (4.1.6), and (2.4.3) tells us that whenever the ℓ-linear map (4.1.5) is an algebra homomorphism $\ell[\mathcal{S}] \to \mathcal{A}(\mathcal{G})$, it is also an algebra homomorphism $\ell[\mathcal{S}, \omega] \to \mathcal{A}(\mathcal{G}, \tilde{\omega})$.

Lemma 4.1.7. *Let $\mathcal{S}$ be an inverse semigroup, $\omega : \mathcal{S} \times \mathcal{S} \to \mathcal{U}(\ell)$ a 2-cocycle, $\mathcal{S} \to \mathcal{I}_{\mathrm{cont}}(X)$ an ample action, and $\mathcal{G}$ the groupoid of germs. Let $\tilde{\omega} : \mathcal{G}^{(2)} \to \mathcal{U}(\ell)$ be as (4.1.3). Assume that the algebra homomorphism (4.1.5) is surjective with kernel $\mathcal{K}$. Then, $\mathcal{K} \lhd \ell[\mathcal{S}, \omega]$ is an ideal, and $\ell[\mathcal{S}, \omega]/\mathcal{K} \cong \mathcal{A}(\mathcal{G}, \tilde{\omega})$.*

Proof. This follows from the discussion immediately above the lemma. ∎

4.2. The twisted Exel–Pardo algebra as a twisted groupoid algebra

Let E be a graph. Recall from Section 1 that $\mathrm{sing}(E) \subset E^0$ is the set of singular (i.e., non-regular) vertices. Set

$$\mathcal{X}(E) = \{\alpha : \text{infinite path in } E\} \cup \{\alpha \in \mathcal{P}(E) : r(\alpha) \in \mathrm{sing}(E)\}.$$

For $\alpha \in \mathcal{P}(E)$, let

$$\mathfrak{X}(E) \supset \mathcal{Z}(\alpha) = \{x \in \mathfrak{X}(E) : x = \alpha y,\ y \in \mathfrak{X}(E)\} = \alpha\mathfrak{X}(E).$$

For each finite set $F \subset \alpha\mathcal{P}(E)$, let

$$\mathcal{Z}(\alpha \setminus F) = \mathcal{Z}(\alpha) \cap \Big(\bigcup_{\alpha\beta \in F} \mathcal{Z}(\alpha\beta)\Big)^c.$$

The sets $\mathcal{Z}(\alpha \setminus F)$ are a basis of compact open sets for a locally compact, Hausdorff topology on $\mathfrak{X}(E)$ [31, Theorem 2.1]. We regard the latter as a topological space equipped with this topology; it is a Boolean space. Now, assume that an Exel–Pardo tuple (G, E, ϕ) is given. Then, there is an action of $\mathcal{S}(G, E, \phi)$ on $\mathfrak{X}(E)$ such that $\mathrm{Dom}(\alpha, g, \beta) = \mathcal{Z}(\beta)$ and

$$(\alpha, g, \beta)(\beta x) = \alpha g(x) \in \mathcal{Z}(\alpha)$$

for all $x \in r(\beta)\mathfrak{X}(E)$. Let $\mathcal{G}(G, E, \phi)$ be the groupoid of germs associated to this action; it is an ample groupoid. If moreover $c : G \times E^1 \to \mathcal{U}(\ell)$ is a 1-cocycle and

$$\omega : \mathcal{S}(G, E, \phi) \times \mathcal{S}(G, E, \phi) \to \mathcal{U}(\ell)_\bullet$$

is the semigroup 2-cocycle of (2.4.5), then by (4.1.3) we also have a twist

$$\tilde{\omega} : \mathcal{G}(G, E, \phi) \times \mathcal{G}(G, E, \phi) \to \mathcal{U}(\ell). \tag{4.2.1}$$

Proposition 4.2.2. *Let (G, E, ϕ_c) be a twisted EP tuple, and let (4.2.1) be the associated groupoid cocycle. Let $L(G, E, \phi_c)$ and $\mathcal{A}(\mathcal{G}(G, E, \phi), \tilde{\omega})$ be the twisted Exel–Pardo and Steinberg algebras. Then, there is an algebra isomorphism*

$$L(G, E, \phi_c) \xrightarrow{\sim} \mathcal{A}(\mathcal{G}(G, E, \phi), \tilde{\omega}), \quad \alpha g \beta^* \mapsto \chi_{[(\alpha, g, \beta), \mathcal{Z}(\beta)]}.$$

Proof. By [25, Theorem 6.4] and the discussion preceding it, for $\mathcal{S} = \mathcal{S}(G, E, \phi)$ the homomorphism (4.1.5) (called ψ in [25]) descends to an isomorphism

$$L(G, E, \phi) = \ell[\mathcal{S}(G, E, \phi)]/\mathcal{K}(G, E, \phi) \to \mathcal{A}(G, E, \phi).$$

Hence, by Lemma 4.1.7 and Proposition 3.1.5, the same map also induces an isomorphism

$$L(G, E, \phi_c) \to \mathcal{A}(\mathcal{G}(G, E, \phi), \tilde{\omega}).$$

This is precisely the isomorphism of the proposition. ∎

4.3. Simplicity of twisted Exel–Pardo algebras

In the following proposition and elsewhere, by the *support* of a function $f : X \to \ell$, we understand the set

$$\mathrm{supp}(f) = \{x \in X : f(x) \neq 0\}.$$

Proposition 4.3.1. *Let (G, E, ϕ) be an EP-tuple with G countable and E countable and regular. Assume that the Exel–Pardo C^*-algebra $C^*(G, E, \phi)$ is simple. Then,*

(i) *$\mathcal{G}(G, E, \phi)$ is effective and minimal.*

(ii) *If ℓ is a subring of $\mathbb{C}$, then the support of every nonzero element of*

$$\mathcal{A}(\mathcal{G}(G, E, \phi)) \cong L(G, E, \phi)$$

has nonempty interior.

Proof. Because G is countable and E is countable and regular, we have $C^*(G, E, \phi) = C^*(\mathcal{G}(G, E, \phi))$ by [19, Theorems 2.5 and 3.2]. Hence, because $C^*(G, E, \phi)$ is assumed to be simple, $\mathcal{G}(G, E, \phi)$ is minimal and effective by [26, Theorem 4.10]. This proves (i). Again, by [26, Theorem 4.10], we obtain that the interior of the support of every function f in Connes' algebra $\mathcal{C}(\mathcal{G}(G, E, \phi))$ spanned by the locally continuous and compactly supported functions (defined, e.g., in [19, Section 4.1]) is nonempty. This implies (ii); since as we are assuming $\ell \subset \mathbb{C}$, we have

$$\mathcal{A}(\mathcal{G}(G, E, \phi)) = \mathcal{A}_\ell(\mathcal{G}(G, E, \phi)) \subset \mathcal{A}_{\mathbb{C}}(\mathcal{G}(G, E, \phi)) \subset \mathcal{C}(\mathcal{G}(G, E, \phi)). \qquad \blacksquare$$

Corollary 4.3.2. *If $C^*(G, E, \phi)$ is simple and ℓ is a field of characteristic zero, then $L(G, E, \phi)$ is simple.*

Proof. By Proposition 4.3.1 and [29, Theorem A], the corollary holds for $\ell = \mathbb{Q}$. This implies that it holds for any field of characteristic zero, by [29, Theorem 5.9]. ■

Let $\mathcal{G}$ be an ample étale groupoid with domain function $d : \mathcal{G}^{(1)} \to \mathcal{G}^{(0)}$. Recall from [4, p. 4] that a continuous 2-cocycle $\nu : \mathcal{G}^{(2)} \to \mathcal{U}(\ell)$ is *normalized* if it satisfies

$$\nu(\xi, d(\xi)) = \nu(d(\xi^{-1}), \xi) = 1 \quad \forall \xi \in \mathcal{G}. \tag{4.3.3}$$

Multiplication in the twisted Steinberg algebra $\mathcal{A}(\mathcal{G}, \nu)$ is defined by the twisted convolution product

$$(f \star_\nu g)(\xi) = \sum_{\{(\xi_1,\xi_2)\in\mathcal{G}^{(2)} | \xi=\xi_1\xi_2\}} \nu(\xi_1, \xi_2) f(\xi_1) g(\xi_2). \tag{4.3.4}$$

Lemma 4.3.5. *Let $\mathcal{G}$ be an ample groupoid such that $\mathcal{G}^{(0)}$ is Hausdorff and $\nu : \mathcal{G}^{(2)} \to \mathcal{U}(\ell)$ a normalized cocycle. Let $L \subset \mathcal{G}^{(0)}$ be a compact open subset, and let $f \in \mathcal{A}(\mathcal{G})$. Then,*

$$f \star_\nu \chi_L = f \star \chi_L, \quad \chi_L \star_\nu f = \chi_L \star f.$$

Proof. Immediate from (4.3.3) and (4.3.4). ■

The following proposition is a twisted version of [26, Theorem 3.11 and Corollary 3.12].

Proposition 4.3.6. *Let $\mathcal{G}$ be a second countable ample groupoid such that $\mathcal{G}^{(0)}$ is Hausdorff, ℓ a field, and $\nu : \mathcal{G}^{(2)} \to \mathcal{U}(\ell)$ a normalized 2-cocycle. Assume the following.*

(i) *$\mathcal{G}$ is effective.*

(ii) *For all $0 \neq f \in \mathcal{A}(\mathcal{G}, \nu)$, $\operatorname{supp}(f)^{\circ} \neq \emptyset$.*

Then, for every nonzero ideal $0 \neq I \lhd \mathcal{A}(\mathcal{G}, \nu)$, there exists a nonempty compact open subset $L \subset \mathcal{G}^{(0)}$ such that $\chi_L \in I$.

Proof. Let $\mathcal{A} = \mathcal{A}(\mathcal{G}, \nu)$, and let $0 \neq f \in I \lhd \mathcal{A}$ be an ideal. By the argument in the first paragraph of the proof of [26, Theorem 3.11], there exists a compact open slice $B \subset \mathcal{G}$ such that f is a nonzero constant on B. Let $g = f \star_\nu \chi_{B^{-1}}$, $b \in B$, and $u = bb^{-1}$. Then, $g(u) = \nu(b, b^{-1}) f(b) \neq 0$. By Lemma 4.3.5, the rest of the argument of the proof of [26, Theorem 3.11] now applies verbatim. ∎

The following proposition is a twisted version of one of the directions of the equivalence established in [26, Theorem 3.14].

Proposition 4.3.7. *Let $\mathcal{G}$, ν, and ℓ be as in Proposition 4.3.6. Further, assume that $\mathcal{G}$ is minimal. Then, $\mathcal{A}(\mathcal{G}, \nu)$ is simple.*

Proof. Let $0 \neq I \lhd \mathcal{A}(\mathcal{G}, \nu)$ be an ideal. By Proposition 4.3.6, there is a nonempty compact open subset $L \subset \mathcal{G}^{(0)}$ such that $\chi_L \in I$. As in the proof of [26, Theorem 3.14], we may choose, for each $x \in \mathcal{G}^{(0)}$, an element $\mathcal{G} \ni \gamma_x$ with $d(\gamma_x) \in L$ and $d(\gamma_x^{-1}) = x$ and a compact open slice $\gamma_x \in B_x$. Upon replacing B_x by $B_x L$, we may assume that

$$B_x = B_x L. \tag{4.3.8}$$

Because ν is locally constant and B_x is a compact open slice, there are $n \geq 1$ and clopen subsets $B_x(1), \dots, B_x(n) \subset B_x$ such that $(B_x \times B_x^{-1}) \cap \mathcal{G}^{(2)} = \bigsqcup_{i=1}^{n} (B_x(i) \times B_x^{-1}(i)) \cap \mathcal{G}^{(2)}$ and such that ν is constant on each $(B_x(i) \times B_x^{-1}(i)) \cap \mathcal{G}^{(2)}$. Let i be such that $\gamma_x \in B_x(i)$. Upon replacing B_x with $B_x(i)$ we may assume that ν is constant on $(B_x \times B_x^{-1}) \cap \mathcal{G}^{(2)}$. Let $L_x = B_x L B_x^{-1}$; then, $x \in L_x$, and by (4.3.8) and Lemma 4.3.5,

$$\chi_{B_x} \star_\nu \chi_L \star_\nu \chi_{B_x^{-1}} = \chi_{B_x} \star_\nu \chi_{B_x^{-1}} = \nu(\gamma_x, \gamma_x^{-1}) \chi_{L_x}.$$

Hence, $\chi_{L_x} \in I$, and the proof now follows as in [26]. ∎

Proposition 4.3.9. *In the situation of Proposition 4.3.7, assume that, for every nonempty compact open subset $L \subset \mathcal{G}^{(0)}$, the idempotent $\chi_L \in \mathcal{A}(\mathcal{G}, \nu)$ is infinite. Then, $\mathcal{A}(\mathcal{G}, \nu)$ is simple purely infinite.*

Proof. Let $0 \neq f \in \mathcal{A}(\mathcal{G}, \nu)$, and let $B \subset \mathcal{G}$ and $g = f \star_\nu \chi_{B^{-1}}$ be as in the proof of Proposition 4.3.6. Following the argument of the proof of [26, Theorem 3.11] and taking Lemma 4.3.5 into account, one obtains a compact open $L \subset \mathcal{G}^{(0)}$ and elements $h_1, h_2 \in \mathcal{A}(\mathcal{G}, \nu)$ such that

$$\chi_L = h_1 \star_\nu g \star_\nu h_2 = h_1 \star_\nu f \star_\nu \chi_B^{-1} \star_\nu h_2.$$

By hypothesis, χ_L is infinite. Hence, $\mathcal{A}(\mathcal{G}, \nu)$ is simple purely infinite, by [1, Proposition 3.1.7 and Definition 3.1.8]. ■

Theorem 4.3.10. *Let (G, E, ϕ_c) be a twisted EP-tuple with G countable and E countable and regular, and let ℓ be a field of characteristic 0. If $C^*(G, E, \phi)$ is simple, then $L(G, E, \phi_c)$ is simple. If, furthermore, $L(E)$ is simple purely infinite, then so is $L(G, E, \phi_c)$.*

Proof. By Proposition 4.2.2, $L(G, E, \phi_c) = \mathcal{A}(\mathcal{G}(G, E, \phi), \widetilde{\omega})$. Because G and E are countable by hypothesis, $\mathcal{G}(G, E, \phi)$ is second countable. Hence, $L(G, E, \phi_c)$ is simple, by Propositions 4.3.1 and 4.3.7 and Corollary 4.3.2. Now, suppose that $L(E)$ is simple purely infinite. Consider the groupoid $\mathcal{G}(E) = \mathcal{G}(\{1\}, E)$; this is the groupoid of [7, Example 2.1]. By [7, Example 3.2], $L(E) = \mathcal{A}(\mathcal{G}(E))$. Since $\mathcal{G}(E)$ is Hausdorff and since we are assuming that $L(E)$ is simple, $\mathcal{G}(E)$ is effective and minimal by [28, Theorem 3.5], and so, since we are further assuming that $L(E)$ is purely infinite, χ_L is infinite for every compact open subset $L \subset \mathcal{G}(E)^{(0)} = \mathcal{G}(G, E, \phi)^{(0)}$. Because of this and because by Proposition 3.4.3, $L(E)$ is a subalgebra of $L(G, E, \phi_c)$, $L(G, E, \phi_c)$ is simple purely infinite by Proposition 4.3.9. ■

5. Twisted Katsura algebras

Let E be a row-finite graph, $A = A_E \in \mathbb{N}_0^{(\mathrm{reg}(E)\times E^0)}$ its reduced incidence matrix, and $B \in \mathbb{Z}^{(\mathrm{reg}(E)\times E^0)}$ such that

$$A_{v,w} = 0 \Rightarrow B_{v,w} = 0. \tag{5.1}$$

Under the obvious identification $\mathbb{Z}/\mathbb{Z}A_{v,w} \xrightarrow{\sim} \mathbb{N}_{v,w} := \{0, \dots, A_{v,w} - 1\}$, translation by $B_{v,w}$ defines a bijection

$$\tau_{v,w} : \mathbb{N}_{v,w} \to \mathbb{N}_{v,w}, \quad n \mapsto \overline{B_{v,w} + n}$$

and therefore a $\mathbb{Z}$-action on $\mathbb{N}_{v,w}$. Here, $\bar{m}$ is the remainder of m under division by $A_{v,w}$. A 1-cocycle $\psi_{v,w} : \mathbb{Z} \times \mathbb{N}_{v,w} \to \mathbb{Z}$ for this action is determined by

$$\psi_{v,w}(1, n) = \frac{B_{v,w} + n - \tau_{v,w}(n)}{A_{v,w}}.$$

For every pair of vertices (v, w) with $A_{v,w} \neq 0$, choose a bijection

$$\mathfrak{n}_{v,w} : vE^1w \xrightarrow{\sim} \{0, \dots, A_{v,w} - 1\},$$

and let

$$\mathfrak{n} = \coprod_{v,w} \mathfrak{n}_{v,w} : E^1 = \coprod_{v,w} vE^1w \to \mathbb{N}_0.$$

Consider the $\mathbb{Z}$-action on E that fixes the vertices and is induced by the bijection $\sigma : E^1 \to E^1$ that is determined by

$$\mathfrak{n}(\sigma(e)) = \tau_{s(e),r(e)}(\mathfrak{n}(e)).$$

Write $\mathbb{Z} = \langle t \rangle$ multiplicatively, and let $\phi : \mathbb{Z} \times E^1 \to \mathbb{Z}$ be the 1-cocycle determined by

$$\phi(t,e) = t^{\psi_{s(e),r(e)}(1,\mathfrak{n}(e))}.$$

The *Katsura ℓ-algebra* associated to the pair (A, B) is

$$\mathcal{O}_{A,B} = L(\mathbb{Z}, E, \phi).$$

When E is regular so that A and B are square matrices, $\mathcal{O}_{A,B}$ is the Katsura algebra considered, for example, in [21] (see [21, Proposition 1.13]), which is itself the algebraic analog of the C^*-algebra introduced by Katsura in [23]. We will presently consider a twisted version of $\mathcal{O}_{A,B}$. For this purpose, we start with a row-finite matrix

$$C \in \mathcal{U}(\ell)^{(\mathrm{reg}(E)\times E^0)}$$

such that

$$A_{v,w} = 0 \Rightarrow C_{v,w} = 1. \tag{5.2}$$

Consider the 1-cocycle $c : \mathbb{Z} \times E^1 \to \mathcal{U}(\ell)$ determined by

$$c(t,e) = \begin{cases} (-1)^{(A_{s(e),r(e)}-1)B_{s(e),r(e)}} C_{s(e),r(e)} & \text{if } \mathfrak{n}(e) = 0, \\ 1 & \text{else.} \end{cases} \tag{5.3}$$

Write

$$\mathcal{O}^C_{A,B} = L(\mathbb{Z}, E, \phi_c)$$

for the twisted Exel–Pardo algebra associated to the twisted EP-tuple $(\mathbb{Z}, E, \phi_c)$.

Remark 5.4. Our motivation for defining the cocycle c as in (5.3) is kk-theoretic. It is defined as above so that the matrix C appears as it does in the kk-triangle of Theorem 7.3.

We say that (A, B) is *KSPI* if E is countable, (5.1) holds, and if in addition we have the following.

- For any pair $(v, w) \in E^0 \times E^0$, there exists $\alpha \in \mathcal{P}(E)$ such that $s(\alpha) = v$ and $r(\alpha) = w$.
- For every vertex v, there are at least two distinct loops based at v.
- $B_{v,v} = 1$ for all $v \in E^0$.

Katsura proved in [23, Proposition 2.10] that if (A, B) is a Katsura pair, then the C^*-algebra completion $C^*_{A,B} = \overline{\mathcal{O}_{A,B}(\mathbb{C})}$ is simple and purely infinite.

Theorem 5.5. *Let ℓ be a field of characteristic* 0*. Let E be a countable regular graph, $A = A_E$ its incidence matrix, and $B \in \mathbb{Z}^{(E^0\times E^0)}$ such that (A, B) is a KSPI pair. Let $C \in (\mathcal{U}(\ell))^{(E^0\times E^0)}$ be arbitrary. Then, the twisted Katsura algebra $\mathcal{O}^C_{A,B}$ is simple purely infinite.*

Proof. By [23, Proposition 2.10], $C^*_{A,B}$ is simple. By [1, Theorem 3.1.10], the first two Katsura conditions imply that $L(E)$ is simple purely infinite. Hence, $\mathcal{O}^C_{A,B}$ is simple purely infinite by Theorem 4.3.10. ∎

For a general field ℓ, we have the following more restrictive simplicity criterion.

Proposition 5.6. *Let E, A, and B be as in Theorem 5.5, ℓ a field, and $C \in \mathcal{U}(\ell)^{(E^0\times E^0)}$ satisfying (5.2). Further, assume the following.*

- *Whenever $A_{v,w} \neq 0 = B_{v,w}$, for each $l \geq 1$ there exist finitely many paths $\alpha = e_1,\dots,e_n \in w\mathcal{P}(E)$ such that $l\prod_{i=1}^{n-1} B_{s(e_i),r(e_i)}/A_{s(e_i),r(e_i)} \in \mathbb{Z}$.*

  Then, $\mathcal{O}^C_{A,B}$ is simple.

Proof. As mentioned above, Katsura proved that the C^*-algebra is simple whenever (A, B) is KSPI. By [18, Theorem 18.6], the additional condition of the proposition implies that (and is in fact equivalent to) $\mathcal{G}(G,E,\phi)$ being Hausdorff. Thus, by [18, Theorems 18.7, 18.8, and 18.12], $\mathcal{G}(G, E, \phi)$ is effective (essentially principal in the notation of [18]) and minimal. Hence, $\mathcal{O}^C_{A,B} = \mathcal{A}(\mathcal{G}(G, E, \phi), \widetilde{\omega})$ is simple by [4, Theorem 6.2]. ∎

6. Twisted EP-tuples and kk

6.1. Preliminaries on kk

Let $\mathfrak{C}$ be a category and $F : \mathrm{Alg}_\ell \to \mathfrak{C}$ a functor. We say that F is *homotopy invariant* if, for every algebra A, the map $F(A) \to F(A[t])$ induced by the natural inclusion $A \subset A[t]$ is an isomorphism. Let X be an infinite set and $x \in X$. Define a natural transformation $\iota_x : A \to M_X A$, $\iota_x(a) = \varepsilon_{x,x}a$. We say that F is *M_X-stable* if the natural transformation $F(\iota_x) : F \to F \circ M_X$ is an isomorphism. M_X-stability implies M_Y-stability for every set Y whose cardinality is at most that of X. We fix such an X and use the term *matricially stable* for M_X-stable. An algebra extension is a sequence of algebra homomorphisms

$$\mathcal{E} : 0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0 \tag{6.1.1}$$

which is exact as a sequence of ℓ-modules. An *excisive* functor with values in a triangulated category $\mathcal{T}$ with suspension $T \mapsto T[-1]$ is a functor $F : \mathrm{Alg}_\ell \to \mathcal{T}$ together with a family of maps $\partial^F_{\mathcal{E}} : F(C)[+1] \to F(A)$ indexed by the algebra extensions (6.1.1) such that

$$F(C)[+1] \xrightarrow{\partial^F_{\mathcal{E}}} F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(\pi)} F(C)$$

is a distinguished triangle. The maps $\partial^F_{\mathcal{E}}$ are to satisfy the compatibility conditions of [14, Section 6.6]. An excisive, homotopy invariant, and matricially stable *homology theory* of ℓ-algebras consists of a triangulated category $\mathcal{T}$ and a functor

$$F : \mathrm{Alg}_\ell \to \mathcal{T}$$

that is excisive, homotopy invariant, and matricially stable. Such homology theories form a category, where a *homomorphism* from F to another homology theory $G : \mathrm{Alg}_\ell \to \mathcal{T}'$ is a triangulated functor $H : \mathcal{T} \to \mathcal{T}'$, together with a natural isomorphism $\phi : H(-[+1]) \to H(-)[+1]$ such that the following diagram commutes for every extension (6.1.1):

$$\begin{array}{ccc} H(F(C)[+1]) & \xrightarrow{H(\partial^F_{\mathcal{E}})} & G(A) \\ {\scriptstyle\phi}\downarrow & \nearrow_{\partial^G_{\mathcal{E}}} & \\ G(C)[+1] & & \end{array}$$

It was shown in [14, Theorem 6.6.2] that the category of homotopy invariant, matricially stable, and excisive homology theories of ℓ-algebras has an initial object $j : \mathrm{Alg}_\ell \to kk$. We will also to consider homology theories for the category $G - \mathrm{Alg}_\ell$ of algebras equipped with a G-action and G-equivariant homomorphisms and $G_{\mathrm{gr}} - \mathrm{Alg}_\ell$ of G-graded algebras and homogeneous homomorphisms. A *G-set* is a set Y with a G-action $G \times Y \to Y$, $(g, y) \mapsto g(y)$. A *G-graded set* is a set Y together with a function $d : Y \to G$. If $A \in G - \mathrm{Alg}_\ell$ and Y a G-set, $M_Y A$ is a G-algebra with $g(\varepsilon_{y_1,y_2} a) = \varepsilon_{g(y_1),g(y_2)} g(a)$. If instead $A \in G_{\mathrm{gr}} - \mathrm{Alg}_\ell$ and Y is equipped with a $d : Y \to G$, then $M_Y A$ is G-graded with degree function determined by $|\varepsilon_{x,y} a| = d(x)|a|d(y)^{-1}$. Let X be the infinite set fixed above in the definition of matricial stability and $\mathcal{T}$ a triangulated category. A functor F defined on $G - \mathrm{Alg}_\ell$ is *G-stable* if for any G-algebra A and any two G sets Y, Z of cardinality at most that $G \times X$, the map $F(M_Y A \to M_{Y \sqcup Z} A)$ induced by the obvious inclusion is an isomorphism. A functor defined on $G_{\mathrm{gr}} - \mathrm{Alg}_\ell$ is G-stable if it satisfies the same condition for G-graded sets Y and Z with the same cardinality restrictions. The universal initial homotopy invariant, matricially stable, G-stable, and excisive homology theories $j_G : G - \mathrm{Alg}_\ell \to kk_G$ and $j_{G_{\mathrm{gr}}} : G_{\mathrm{gr}} - \mathrm{Alg}_\ell \to kk_{G_{\mathrm{gr}}}$ were constructed in [16]. It was shown there [16, Theorem 7.6] that the crossed-product functors $\rtimes G : G - \mathrm{Alg}_\ell \to G_{\mathrm{gr}} - \mathrm{Alg}_\ell$ and $G \hat{\ltimes} : G_{\mathrm{gr}} - \mathrm{Alg}_\ell \to G - \mathrm{Alg}_\ell$ induce inverse triangulated equivalences $kk_G \leftrightarrows kk_{G_{\mathrm{gr}}}$.

6.2. The algebra $\mathcal{K}(G, E, \phi_c)$

Lemma 6.2.1. *Let X be a G-set. Then, the G-equivariant homomorphism*

$$\iota_X : \ell^{(X)} \to M_X \ell^{(X)}, \quad \chi_x \mapsto \varepsilon_{x,x} \otimes \chi_x$$

is a kk_G-equivalence.

Proof. Let $\{\bullet\}$ be a one-point G-set; consider $X_\bullet = X \sqcup \{\bullet\}$. By G-stability, the inclusion $\mathrm{inc} : M_X \ell^{(X)} \to M_{X_\bullet} \ell^{(X)}$ is a kk_G-equivalence. Hence, it suffices to show that so is $\mathrm{inc} \circ \iota_X$. For each $x \in X$, let $\sigma_x \in M_{X_\bullet}$ be the matrix associated to the permutation that exchanges x and $\bullet$ and fixes the remaining elements. Observe that

$$R := M_{X_\bullet} \ell^{(X)} = \bigoplus_{x \in X} M_{X_\bullet} \otimes \chi_x \lhd \prod_{x \in X} \Gamma_{X_\bullet} \otimes \chi_x =: S.$$

Let $\sigma = \prod_{x\in X} \sigma_x \otimes \chi_x \in S$. One checks that σ is fixed by G; hence, the conjugation map $\mathrm{ad}(\sigma) : R \to R$ is the identity in kk_G, by the argument of [9, Proposition 2.2.6]. Moreover, $\mathrm{ad}(\sigma) \circ \mathrm{inc} \circ \iota_X = \varepsilon_{\bullet,\bullet} \otimes \mathrm{id}_{\ell(X)}$, another kk_G-equivalence. Thus, $j_G(\mathrm{inc} \circ \iota_X)$ is an isomorphism, finishing the proof. ∎

Let $X \subset E^0$ be a set of vertices closed under the G-action. Consider the G-equivariant homomorphism

$$\iota : \ell^{(X)} \to \bigoplus_{v\in X} M_{\mathcal{P}_v} \otimes \chi_v, \quad \iota(\chi_v) = \varepsilon_{v,v} \otimes \chi_v. \tag{6.2.2}$$

Proposition 6.2.3. *The map* (6.2.2) *is a* kk_G*-equivalence.*

Proof. We adapt the argument of the proof of [6, Lemma 10.3] as follows. Let $S = \bigoplus_{v\in X} M_{\mathcal{P}_v} \otimes \chi_v$; set $p = \sum_{\alpha\in\mathcal{P}_X} \varepsilon_{\alpha,\alpha} \otimes \varepsilon_{\alpha,\alpha} \otimes \chi_{r(\alpha)} \in \Gamma_{\mathcal{P}_X} S$. Then, for $T := pM_{\mathcal{P}_X} Sp$, the map

$$S \to T, \quad \varepsilon_{\alpha,\beta} \otimes \chi_{r(\alpha)} \mapsto \varepsilon_{\alpha,\beta} \otimes \varepsilon_{\alpha,\beta} \otimes \chi_{r(\alpha)} \tag{6.2.4}$$

is a G-equivariant isomorphism. Let

$$u = \sum_{\alpha\in\mathcal{P}_X} \varepsilon_{\alpha,\alpha} \otimes \varepsilon_{r(\alpha),\alpha} \otimes \chi_{r(\alpha)}, \quad u^* = \sum_{\alpha\in\mathcal{P}_X} \varepsilon_{\alpha,\alpha} \otimes \varepsilon_{\alpha,r(\alpha)} \otimes \chi_{r(\alpha)} \in \Gamma_{\mathcal{P}_X} S.$$

Observe that u and u^* are fixed by G and that $u^*u = p$. Put

$$R = \bigoplus_{v\in X} \varepsilon_{v,v} \otimes \ell\chi_v \subset S;$$

then, $\mathrm{ad}(u) : T \to M_{\mathcal{P}_X} R$, $t \mapsto utu^*$ is a G-equivariant homomorphism; by the G-equivariant version of [8, Lemma 8.11], the composite of $\mathrm{ad}(u)$ with the inclusion $M_{\mathcal{P}_X} R \subset T$ equals the identity in kk_G. Hence, the composite ζ of $\mathrm{ad}(u)$ with (6.2.4) is a split monomorphism in kk_G. Observe that (6.2.2) is the composite of the isomorphism $f : \ell^{(X)} \to R$, $f(\chi_v) = \varepsilon_{v,v} \otimes \chi_v$, with the inclusion $R \subset S$. Consider the composite $\jmath : \ell^{(X)} \to M_{\mathcal{P}_X}\ell^{(X)}$ of $\iota_X : \ell^{(X)} \to M_X\ell^{(X)}$ with the inclusion $M_X\ell^{(X)} \subset M_{\mathcal{P}_X}\ell^{(X)}$. The latter maps are kk_G-equivalences, by Lemma 6.2.1 and G-stability, and so, is $\jmath$ too. One checks that $\zeta|R \circ f = M_{\mathcal{P}_X} f \circ \jmath$. Hence, ζ is an isomorphism in kk_G and therefore so is ι, as desired. ∎

Proposition 6.2.5. *Let* (G, E, ϕ_c) *be a twisted EP-tuple. Then, the homogeneous homomorphism of* G*-graded algebras*

$$q^G : (\ell^{(\mathrm{reg}(E))}) \rtimes G \to \mathcal{K}(G, E, \phi_c), \quad q^G(\chi_v \rtimes g) = q_v g.$$

is a $kk_{G_{\mathrm{gr}}}$*-equivalence.*

Proof. Let $X = \mathrm{reg}(E)$, ι as in (6.2.2), and ι' the composite of ι with the obvious isomorphism $\bigoplus_{v\in X} M_{\mathcal{P}_v} \otimes \chi_v \cong \bigoplus_{v\in X} M_{\mathcal{P}_v}$. One checks that q^G is the composite of the maps $\iota' \rtimes G$ and υ of Proposition 3.2.5. Then, $j_G(\iota')$ is an isomorphism by Proposition 6.2.3, whence $j_{G_{\mathrm{gr}}}(\iota' \rtimes G)$ is an isomorphism. This implies that $j_{G_{\mathrm{gr}}}(q^G)$ is an isomorphism, since υ is a G-graded algebra isomorphism. ∎

6.3. The Cohn algebra $C(G, E, \phi_c)$

The purpose of this subsection is to prove the following.

Theorem 6.3.1. *Let (G, E, ϕ_c) be a twisted EP-tuple. Then, the algebra homomorphism*

$$\varphi : \ell^{(E^0)} \rtimes G \to C(G, E, \phi_c), \quad \chi_v \rtimes g \mapsto vg$$

is a kk-equivalence.

The proof of Theorem 6.3.1 will be given at the end of the subsection. We remark that, in the case when G is trivial, Theorem 6.3.1 specializes to [13, Theorem 4.2]. We will adapt the argument of [13, Theorem 4.2] to prove Theorem 6.3.1.

We abbreviate

$$C(G, E) = C(G, E, \phi_c)$$

throughout. Recall from Section 1 that $\operatorname{sink}(E), \inf(E) \subset E^0$ are the subsets of sinks and of infinite emitters. For $v \in E^0 \setminus \inf(E)$, consider the following element of $C(G, E)$:

$$m_v = \begin{cases} \sum_{s(e)=v} ee^* & \text{if } v \in \operatorname{reg}(E), \\ 0 & \text{if } v \in \operatorname{sink}(E). \end{cases} \tag{6.3.2}$$

We define $C^m(G, E)$ as the result of adding an indeterminate m_v for each $v \in \inf(E)$, subject to the relations [13, Identities 4.4] plus the additional relation

$$(g(v)g)m_v = m_{g(v)}g(v)g. \tag{6.3.3}$$

We remark that (6.3.3) is already satisfied in $C(G, E)$, by all $v \in E^0 \setminus \inf(E)$. Thus, in $C^m(G, E)$, (6.3.3) holds for all $v \in E^0$. We will abbreviate $m_v g = m_v(vg)$. The elements $q_v = v - m_v \in C^m(G, E)$ generate an ideal

$$\hat{\mathcal{K}}(G, E) = \mathcal{K}(G, E) \oplus \bigoplus_{v \in \inf(E)} \operatorname{span}\{\alpha q_v g \beta^* : \ g \in G, \ \alpha \in \mathcal{P}_v, \ \beta \in \mathcal{P}_{g^{-1}(v)}\}.$$

One checks that the isomorphism υ of Proposition 3.2.5 extends to an isomorphism

$$\hat{\upsilon} : \Big(\bigoplus_{v \in E^0} \mathcal{M}_{\mathcal{P}_v}\Big) \rtimes G \xrightarrow{\sim} \hat{\mathcal{K}}(G, E)$$

defined by the same formula as υ. We have an algebra homomorphism

$$\hat{\varphi} : \hat{\mathcal{K}}(G, E) \to M_{\mathcal{P}} C(G, E), \quad \hat{\varphi}(\alpha q_v g \beta^*) = \varepsilon_{\alpha,\beta} \otimes vg. \tag{6.3.4}$$

Next we consider an analog of the homomorphism ρ of [13, (4.7)]. The map therein goes from $C^m(E)$ to $\Gamma_{\mathcal{P}}$; the analog will be a homomorphism

$$\rho : C^m(G, E) \to \Gamma^a_{\mathcal{P}}. \tag{6.3.5}$$

We define $\rho(e)$, $\rho(e^*)$ for $e \in E^1$ and $\rho(m_v)$ for $v \in \inf(E)$ exactly as in [13, formulas (4.7)], and for $v \in E^0$ and $g \in G$, we set

$$\rho(vg) = \sum_{g(r(\alpha))=v} c(g,\alpha)\varepsilon_{g(\alpha),\alpha}. \tag{6.3.6}$$

One checks that the above prescriptions actually define an algebra homomorphism

$$\rho : C^m(G,E) \to \Gamma^a_{\mathcal{P}}.$$

The latter agrees with that of [13] when G is trivial, and is therefore injective in that case, by [13, Lemma 4.8]. For nontrivial G, ρ need not be injective (e.g., if G acts trivially on E).

Next, we need an analog of the algebra $\mathfrak{A}$ of [13, p. 131], and an action of $C^m(G,E)$ on $\mathfrak{A}$; both are defined in Lemma 6.3.8 below. First, we introduce some notation and vocabulary. Recall (e.g., from [22, Definition 1.1.1]) that the *multiplier algebra* of an algebra B is the set $\mathcal{M}(B)$ of all pairs (L,R) of ℓ-linear maps $L, R : B \to B$ such that L is a right B-module homomorphism, R is a left B-module homomorphism, and

$$R(x)y = xL(y) \quad \forall x,y \in B.$$

Following [22], elements of $\mathcal{M}(B)$ are called *double centralizers*. Addition in $\mathcal{M}(B)$ is defined pointwise and multiplication is given by composition of functions as follows:

$$(L_1,R_1)(L_2,R_2) = (L_1 \circ L_2, R_2 \circ R_1).$$

Let A be another algebra. An *A-algebra* structure on B is an algebra homomorphism $A \to \mathcal{M}(B)$, $a \mapsto (L_a, R_a)$. We write

$$a \cdot b = L_a(b), \quad b \cdot a = R_a(b) \quad (\forall a \in A, b \in B).$$

Thus, we have, for all $a \in A$ and $b,c \in B$,

$$(a \cdot b)c = a \cdot (bc), \quad (b \cdot a)c = b(a \cdot c), \quad \text{and} \quad b(c \cdot a) = (bc) \cdot a. \tag{6.3.7}$$

The *semi-direct product* $A \ltimes B$ is the algebra that results from equipping the direct sum $A \oplus B$ with the following product:

$$(a,b)(a',b') = (aa', a \cdot b' + b \cdot a' + bb').$$

Lemma 6.3.8. *Consider the following ℓ-submodule of $M_{\mathcal{P}}C(G,E)$:*

$$\mathfrak{A} := \operatorname{span}\{\varepsilon_{\gamma,\theta} \otimes \alpha g \beta^* : s(\alpha) = r(\gamma), s(\beta) = r(\theta), r(\alpha) = g(r(\beta))\}. \tag{6.3.9}$$

We have the following.

(i) *$\mathfrak{A} \subset M_{\mathcal{P}}C(G,E)$ is a subalgebra.*

(ii) $\mathfrak{A}$ *carries a* $C^m(G,E)$*-algebra structure such that for all* $x \in C^m(E)$*,* $g \in G$*,* $v \in E^0$*, and* $\varepsilon_{\gamma,\theta} \otimes z \in \mathfrak{A}$*,*

$$x \cdot (\varepsilon_{\gamma,\theta} \otimes y) = (\rho(x)\varepsilon_{\gamma,\theta}) \otimes y,$$

$$(\varepsilon_{\gamma,\theta} \otimes y) \cdot x = (\varepsilon_{\gamma,\theta})\rho(x) \otimes y, \tag{6.3.10}$$

$$vg \cdot (\varepsilon_{\gamma,\theta} \otimes y) = \delta_{v,g(s(\gamma))}\varepsilon_{g(\gamma),\theta} \otimes (g(r(\gamma))\phi_c(g,\gamma))y, \tag{6.3.11}$$

$$(\varepsilon_{\gamma,\theta} \otimes y) \cdot vg = \delta_{v,s(\theta)}\varepsilon_{\gamma,g^{-1}(\theta)} \otimes y(r(\theta)\phi_c(g,g^{-1}(\theta))). \tag{6.3.12}$$

(iii) *Let* $a \in \widehat{K}(G,E)$ *and* $\mathfrak{a} \in \mathfrak{A}$*. For* $\widehat{\varphi}$ *as in* (6.3.4)*, we have*

$$a \cdot \mathfrak{a} = \widehat{\varphi}(a)\mathfrak{a}, \quad \mathfrak{a} \cdot a = \mathfrak{a}\widehat{\varphi}(a).$$

Proof. Part (i) follows from Proposition 3.1.5 using the product formula (2.4.4) and the definition of $\mathfrak{A}$. Next, observe that because ρ is an algebra homomorphism, (6.3.10) defines a homomorphism $C^m(E) \to \mathcal{M}(\mathfrak{A})$, $x \mapsto (L_x, R_x)$, where L_x and R_x are left and right multiplication by $\rho(x) \otimes 1$. To prove (ii), one shows that this homomorphism extends to $C^m(G,E)$, by sending $vg \in E^0G$ to the pair (L_{vg}, R_{vg}) of left and right multiplication operators defined by (6.3.11) and (6.3.12). This entails first checking that, for each $vg \in E^0G$, (L_{vg}, R_{vg}) is a double centralizer, and then that the relations (3.1.3) and (3.1.4) involving vg are satisfied by (L_{vg}, R_{vg}) for $vg \in E^0G$. These verifications are long and tedious but straightforward. To prove part (iii), observe that if α, β are paths and $v \in E^0$ and $g \in G$ are such that $r(\alpha) = v = g(r(\beta))$, then

$$\widehat{\phi}(\alpha q_v g\beta^*) = \varepsilon_{\alpha,\beta} \otimes vg = (\rho(\alpha) \otimes 1)(\varepsilon_{v,v} \otimes vg)(\rho(\beta^*) \otimes 1).$$

Hence, in view of (6.3.10), it suffices to prove (iii) for $a = q_v g$ and $\mathfrak{a} = \varepsilon_{\gamma,\theta} \otimes \alpha h\beta^*$. This again is a straightforward calculation. ∎

Let A and B be algebras over our fixed ground ring ℓ. Recall (e.g., from [14, Definition 5.2.1]) that a *quasi-homomorphism* from A to B consists of an algebra R containing B as a two-sided ideal and algebra homomorphisms $\phi, \psi : A \rightrightarrows R$ such that $\phi(a) - \psi(a) \in B$ for all $a \in A$. Such a triple is denoted by $(\phi, \psi) : A \rightrightarrows R \triangleright B$. In what follows, we will use some standard properties of quasi-homomorphisms which can be found, for example, in [14, Section 5.2].

Proof of Theorem 6.3.1. We indicate how to modify the proof of [13, Theorem 4.2] to the present setting. By the argument of Proposition 6.2.5, the algebra homomorphism $\widehat{\iota} : \ell^{(E^0)} \rtimes G \to \widehat{\mathcal{K}}(G,E)$ defined by the rule $\widehat{\iota}(v \rtimes g) = q_v g$ is a $kk_{G_{\mathrm{gr}}}$-equivalence, and thus also a kk-equivalence. Let

$$\xi : C(G,E) \to C^m(G,E), \quad \xi(vg) = m_v g, \quad \xi(e) = em_{r(e)}, \quad \xi(e^*) = m_{r(e)}e^*.$$

The map ξ together with the canonical map can : $C(G,E) \to C^m(G,E)$ forms a quasi-homomorphism

$$(\mathrm{can}, \xi) : C(G,E) \rightrightarrows C^m(G,E) \triangleright \widehat{\mathcal{K}}(G,E),$$

and the argument of [13, part I of the proof of Theorem 4.2] shows that $j(\hat{\iota})^{-1} \circ j(\mathrm{can}, \xi)$ is left inverse to $j(\varphi)$. Next, define

$$\hat{\iota}_\tau : C(G,E) \to M_{\mathcal{P}} C(G,E), \quad \hat{\iota}_\tau(\alpha g \beta^*) = \varepsilon_{s(\alpha),s(\beta)} \otimes \alpha g \beta^*,$$

Remark that

$$\hat{\varphi}(q_v g) = \varepsilon_{v,g^{-1}(v)} \otimes vg = \hat{\iota}_\tau(vg).$$

It follows that the analog of [13, Diagram 4.16] commutes. Next, we show that $\hat{\iota}_\tau$ is a kk-equivalence. As in the proof of Proposition 6.2.3, we consider the G-set with one added fixed point $\mathcal{P}_\bullet = \mathcal{P} \sqcup \{\bullet\}$. Since the inclusion $\mathrm{inc} : M_{\mathcal{P}} C(G,E) \subset M_{\mathcal{P}_\bullet} C(G,E)$ is a kk-equivalence, it suffices to show that so is the composite $\mathrm{inc} \circ \hat{\iota}_\tau$. For this purpose, we consider, for each $v \in E^0$, the matrices $A_v, B_v \in M_{\mathcal{P}_\bullet} C(G,E)[t]$ defined just as in the proof of [13, Lemma 4.17], but with $\bullet$ substituted for what is called w in the cited reference. One checks that the following prescriptions define a homotopy $H : C(G,E) \to M_{\mathcal{P}_\bullet} C(G,E)[t]$ between $\mathrm{inc} \circ \hat{\iota}_\tau$ and $\iota_\bullet$:

$$\begin{aligned} H(vg) &= A_v(\varepsilon_{v,g^{-1}(v)} \otimes vg) B_{g^{-1}(v)}, \\ H(e) &= A_{s(e)}(\varepsilon_{s(e),r(e)} \otimes e) B_{r(e)}, \\ H(e^*) &= A_{r(e)}(\varepsilon_{r(e),s(e)} \otimes e^*) B_{s(e)}. \end{aligned}$$

We have thus come to [13, Proof of Theorem 4.2, part II]. By Lemma 6.3.8, we can form the semi-direct product $C^m(G,E) \ltimes \mathfrak{A}$ and consider the ℓ-submodule

$$C^m(G,E) \ltimes \mathfrak{A} \supset J := \{(x, -\hat{\varphi}(x)) : x \in \hat{\mathcal{K}}(G,E)\}.$$

Using the fact that $\hat{\varphi}$ is a $*$-homomorphism and part (iii) of Lemma 6.3.8, we obtain that J is a two-sided ideal. Write $D = C^m(G,E)/J$; it is clear that the canonical homomorphisms $\mathfrak{A} \to D \leftarrow C^m(G,E)$ are injective; write $\Upsilon : C^m(G,E) \to D$ for the latter map. We may thus regard $\mathfrak{A}$ as an ideal and $C^m(G,E)$ as a subalgebra of D. We can now proceed to [13, Proof of Theorem 4.2, part III]. We define homomorphisms $\psi_0, \psi_{1/2}, \psi_1 : C(G,E) \to D$ just as in the cited reference, with ξ, can, $\hat{\iota}_\tau$, and Υ as defined above, so that we have quasi-homomorphisms $(\psi_0, \psi_1), (\psi_0, \psi_{1/2}), (\psi_{1/2}, \psi_1) : C(G,E) \rightrightarrows D \rhd \mathfrak{A}$. Next, we need an analog of [13, Lemma 4.21], proving that $j(\psi_0, \psi_{1/2}) = 0$. One checks that the following prescriptions define an algebra homomorphism $H^+ : C(G,E) \to D[t]$:

$$\begin{aligned} H^+(vg) &= (m_v g, \varepsilon_{v,g^{-1}(v)} \otimes vg), \\ H^+(e) &= (e m_{r(e)}, (1-t^2)\varepsilon_{s(e),r(e)} \otimes e + t\varepsilon_{e,r(e)} \otimes r(e)), \\ H^+(e^*) &= (m_{r(e)} e^*, (1-t^2)\varepsilon_{r(e),s(e)} \otimes e^* + (2t - t^3)\varepsilon_{r(e),e} \otimes r(e)), \end{aligned}$$

and that $(H, \psi_{1/2}) : C(G,E) \rightrightarrows D[t] \rhd \mathfrak{A}[t]$ is a homotopy

$$(\psi_0, \psi_{1/2}) \to (\psi_{1/2}, \psi_{1/2}).$$

The rest of the proof now proceeds just as in [13]. ∎

7. Twisted Katsura algebras in kk

Let E be a row-finite graph, $A = A_E$ and $\mathfrak{n} : E^1 \to \mathbb{N}_0$ as in Section 5. Let $v, w \in E^0$ such that $A_{v,w} \neq 0$, and set

$$m_{v,w} = \sum_{e \in vE^1 w} ee^* \in C(\mathbb{Z}, E, \phi_c).$$

For $0 \le i \le A_{v,w} - 1$, let $e_i = \mathfrak{n}_{v,w}^{-1}(i) \in vE^1 w$. Consider the element

$$u_{v,w} = e_0 t e^*_{A_{v,w}-1} + \sum_{i=0}^{A_{v,w}-2} e_{i+1} e_i^* \in C(\mathbb{Z}, E, \phi_c).$$

Let E and F be graphs; write r_E, r_F, s_E, s_F for their range and source maps. We say that F is a *subgraph* of E if $F^i \subset E^i$ $(i = 0, 1)$ and r_F and s_F are the restrictions of r_E and s_E. Recall from [1, Definition 1.6.7] that a subgraph $F \subset E$ is *complete* if $s_F^{-1}(\{v\}) = s_E^{-1}(\{v\})$ for all $v \in (F^0 \setminus \operatorname{sink}(F)) \cap \operatorname{reg}(E)$.

Lemma 7.1. (i) *We have the following:*

$$u_{v,w}^{B_{v,w}} = (-1)^{(A_{v,w}-1)B_{v,w}} C_{v,w}^{-1} t \cdot (e_0 e_0^*) + \sum_{i=1}^{A_{v,w}-1} t \cdot (e_i e_i^*).$$

(ii) *Let $F \subset E$ be a finite complete subgraph containing $\{v\} \cup r(s^{-1}(\{v\}))$; set*

$$1_F = \sum_{w \in F^0} w \in C(\mathbb{Z}, E, \phi_c).$$

Then, the following identity holds in $K_1(C(\mathbb{Z}, F, \phi_c))$:

$$[1_F - m_{v,w} + u_{v,w}] = \big[1_F - w + (-1)^{(A_{v,w}-1)} wt\big].$$

Proof. For $m \in \mathbb{Z}$, let $q(m)$ and $\bar{m}$ be the quotient and the remainder under division by $A_{v,w}$. One checks that for any $n \in \mathbb{N}$

$$u_{v,w}^n = \sum_{i=0}^{A_{v,w}-1} e_{\overline{i+n}} t^{q(n+i)} e_i^*. \tag{7.2}$$

Use (7.2) at the first step and (5.3) at the second to obtain, for $\psi = \psi_{w,v}$,

$$u_{v,w}^{B_{v,w}} = \sum_{e \in vE^1 w} t(e) t^{\psi(t,e)} e^* = (-1)^{(A_{v,w}-1)B_{v,w}} C_{v,w}^{-1} t \cdot (e_0 e_0^*) + \sum_{i=1}^{A_{v,w}-1} t \cdot (e_i e_i^*).$$

This proves (i). Next, observe that left multiplication by $u_{v,w}$ defines an automorphism of the projective module $\bigoplus_{i=0}^{A_{v,w}-1} e_i e_i^* C(\mathbb{Z}, F, \phi_c)$. The latter is isomorphic to

$$P = w C(\mathbb{Z}, F, \phi_c)^{A_{v,w}}$$

under the isomorphism defined as left multiplication by e_i^* on the i-th summand. Under this isomorphism, $u_{v,w}$ corresponds to the automorphism of P represented by the following matrix with coefficients in $R = wC(\mathbb{Z}, F, \phi_c)w$:

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & tw \\ w & 0 & \cdots & 0 & 0 \\ 0 & w & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & \cdots & w & 0 \end{bmatrix} = \begin{bmatrix} tw & 0 & 0 & \dots & 0 \\ 0 & w & 0 & \dots & 0 \\ 0 & 0 & w & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & w \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & \cdots & 0 & w \\ w & 0 & \cdots & 0 & 0 \\ 0 & w & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & w & 0 \end{bmatrix}.$$

The second matrix is a cyclic permutation matrix with determinant $(-1)^{A_{v,w}-1}w$. Hence, the product above is mapped to the class of $(-1)^{A_{v,w}-1}wt$ in $K_1(R)$ and to that of $1 - w + (-1)^{A_{v,w}-1}wt$ in $K_1(C(\mathbb{Z}, F, \phi_c))$. Assertion (ii) of the lemma is immediate from this. ∎

Let E be a graph. We say that a homology theory $H : \mathrm{Alg}_\ell \to \mathcal{T}$ is *E-stable* if it is M_X-stable with respect to a set X of cardinality $\#(E^0 \amalg E^1 \amalg \mathbb{N})$. Let I be a set. We say that H is *E^0-additive* if first of all direct sums of cardinality $\leq \#I$ exist in $\mathcal{T}$ and second of all the map

$$\bigoplus_{j\in J} H(A_j) \to H\Big(\bigoplus_{j\in J} A_j\Big)$$

is an isomorphism for any family of algebras $\{A_j : j \in J\} \subset \mathrm{Alg}_\ell$ with $\#J \leq \#I$.

Theorem 7.3. *Let E be a row-finite graph, let A be its reduced incidence matrix, $B \in \mathbb{Z}^{\mathrm{reg}(E)\times E^0}$ and $C \in \mathcal{U}(\ell)^{(\mathrm{reg}(E)\times E^0)}$ such that $A_{v,w} = 0 \Rightarrow B_{v,w} = 0$, $C_{v,w} = 1$. Let $\mathcal{T}$ be a triangulated category and $H : \mathrm{Alg}_\ell \to \mathcal{T}$ an excisive, homotopy invariant, E-stable, and E^0 additive functor. Consider the matrix*

$$D = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}.$$

Set $C^* \in \mathcal{U}(\ell)^{E^0\times\mathrm{reg}(E)}$, $C^*_{v,w} = C^{-1}_{w,v}$. Put

$$D^* = \begin{bmatrix} A^t & C^* \\ 0 & B^t \end{bmatrix}.$$

Then, the Cohn extension of (1.5) *induces the following distinguished triangle in $\mathcal{T}$:*

$$H(\ell)^{(\mathrm{reg}(E))} \oplus H(\ell)[-1]^{(\mathrm{reg}(E))} \xrightarrow{I-D^*} H(\ell)^{(E^0)} \oplus H(\ell)[-1]^{(E^0)} \to H(\mathcal{O}^C_{A,B}).$$

Proof. Consider the Laurent polynomial algebra $L_1 = \ell[t, t^{-1}]$. By Proposition 6.2.3 and Theorem 6.3.1, we have a triangle in kk of the form

$$j\big(L_1^{(\mathrm{reg}(E))}\big) \xrightarrow{f} j\big(L_1^{(E^0)}\big) \to j(\mathcal{O}^C_{A,B}).$$

Observe that L_1 is the Leavitt path algebra of the graph consisting of a single loop; hence, by [13, Theorem 5.4], we have $j(L_1 \otimes R) = j(R) \oplus j(R)[-1]$ for every $R \in \mathrm{Alg}_\ell$. For each $v \in E^0$, $w \in \mathrm{reg}(E)$, and $i \in \{0,1\}$, let

$$\iota_{v,i} : j(\ell)[-i] \to j(\ell^{\mathrm{reg}(E)})[-i] \subset j(\ell^{\mathrm{reg}(E)}) \oplus j(\ell^{\mathrm{reg}(E)})[-1]$$

be the inclusion and

$$p_{v,i} : j\left(\ell^{(E^0)}\right) \oplus j\left(\ell^{(E^0)}\right)[-1] \to j(\ell)[-i]$$

the projection in/onto the (v,i)-summand. Because H is homotopy invariant, excisive, and E-stable, it factors as $H = \bar{H} \circ j$ with $\bar{H}$ a triangulated functor. Further, in view of the additivity hypothesis on H, $H(f) = \bar{H}(M)$, where M is a matrix indexed by $(E^0 \times \{0,1\}) \times (\mathrm{reg}(E) \times \{0,1\})$, with coefficients

$$M_{(w,i),(v,j)} = p_{(w,i)} \circ f \circ \iota_{(v,j)} \in kk(\ell[-j], \ell[-i]).$$

The argument of [12, Proposition 5.2] shows that

$$M_{(w,i),(v,0)} = \delta_{i,0}(\delta_{w,v} - A_{v,w}) = D_{(v,0),(w,i)}.$$

To compute the coefficients $M_{(w,i),(v,1)}$, proceed as follows. Observe that if $F \subset E$ is any finite complete subgraph containing $s^{-1}\{v\}$, then

$$1_F - q_v + q_v t = (1_F - v + vt)(1_F - m_v + m_v t)^{-1}.$$

Hence, it suffices to establish that the following identity holds in $K_1(C(\mathbb{Z}, F, \phi_c))$ for any finite subgraph $F \subset E$ as above:

$$[1_F - m_v + m_v t] = \Big[\prod_{w \in F^0} (1_F - w + wt)^{B_{v,w}} (1_F - w + C_{v,w} w)\Big]. \tag{7.4}$$

Thus, we may assume that $E = F$ is finite. Let $v, w \in E^0$ such that $A_{v,w} \neq 0$. Using the isomorphism

$$wC(\mathbb{Z}, F, \phi_c)^{A_{v,w}} \cong \bigoplus_{e \in s^{-1}\{v\}} ee^* C(\mathbb{Z}, F, \phi_c)$$

as in the proof of Lemma 7.1, we see that, with notation as in said lemma, for any $\lambda \in \mathcal{U}(\ell)$ and $1 \le i \le A_{v,w} - 1$, we have

$$[1 - w + \lambda w] = \Big[1 - m_{v,w} + \lambda e_i e_i^* + \sum_{j \neq i} e_j e_j^*\Big] \in K_1(C(\mathbb{Z}, F, \phi_c)). \tag{7.5}$$

Set $\sigma_{v,w} = 1 - w + (-1)^{(A_{v,w}-1)} w$. By Lemma 7.1,

$$\begin{aligned}
&[(1 - w + wt)^{B_{v,w}}] \\
&\quad = \left[\sigma_{v,w}^{B_{v,w}} (1 - m_{v,w} + u_{v,w})^{B_{v,w}}\right] \\
&\quad = \left[\sigma_{v,w}^{B_{v,w}} (1 - m_{v,w} + (-1)^{(A_{v,w}-1)B_{v,w}} C_{v,w}^{-1} t \cdot (e_0 e_0^*) + \sum_{i=1}^{A_{v,w}-1} t \cdot (e_i e_i^*))\right].
\end{aligned} \tag{7.6}$$

Next, use (7.5) and (7.6) to compute that, in $K_1(C(\mathbb{Z}, F, \phi_c))$, we have

$$\Big[\prod_{w\in E^0}(1-w+wt)^{B_{v,w}}(1-w+C_{v,w}w)\Big]$$
$$=\Bigg[\prod_{w\in E^0}\sigma_{v,w}^{B_{v,w}}\Bigg(1-m_{v,w}+t\cdot\Bigg((-1)^{(A_{v,w}-1)B_{v,w}}(e_0e_0^*)+\sum_{i=1}^{A_{v,w}-1}e_ie_i^*\Bigg)\Bigg)\Bigg]$$
$$=\Big[\prod_{w\in E^0}(1-m_{v,w}+t\cdot m_{v,w})\Big]=[1-m_v+t\cdot m_v]=[1-m_v+m_vt]. \qquad\blacksquare$$

Next, we introduce notation and vocabulary that are used in the next two corollaries of Theorem 7.3. If R is a commutative ring and M is an R-module, we write $R\ltimes M$ for the commutative ring with additive group $R\oplus M$ and multiplication defined by

$$(a,m)\cdot(b,n)=(ab,an+bm).$$

In particular, regarding the group $\mathcal{U}(\ell)$ of invertible elements of ℓ as a $\mathbb{Z}$-module, we obtain a commutative ring

$$\mathfrak{W}=\mathbb{Z}\ltimes\mathcal{U}(\ell). \tag{7.7}$$

Recall that the cup-product makes $K_{\ge 0}(\ell)=\bigoplus_{n\ge 0}K_n(\ell)$ into a graded commutative ring, and $K_{\ge n}(\ell)$ into an ideal for every $n\in\mathbb{N}_0$. Observe that the canonical map

$$\mathfrak{W}\to K_0(\ell)\oplus K_1(\ell)=K_{\ge 0}(\ell)/K_{\ge 2}(\ell)$$

is a ring homomorphism. Thus, for every unital $R\in\mathrm{Alg}_\ell$ and $n\in\mathbb{Z}$, the cup-product makes $K_n(R)\oplus K_{n-1}(R)$ into a graded $\mathfrak{W}$-module. We will also consider the following $\mathfrak{W}$-modules:

$$\begin{aligned}\mathfrak{BF}(A,B,C)&=\mathrm{Coker}\big(I-D^*:\mathfrak{W}^{\mathrm{reg}(E)}\to\mathfrak{W}^{E^0}\big),\\ \widetilde{\mathfrak{BF}}(A,B,C)&=\mathrm{Coker}\big(I-D:\mathfrak{W}^{E^0}\to\mathfrak{W}^{\mathrm{reg}(E)}\big).\end{aligned}\tag{7.8}$$

Corollary 7.9. *Let $R\in\mathrm{Alg}_\ell$ and $n\in\mathbb{Z}$. If R is flat over ℓ, then there is an exact sequence*

$$0\to\mathfrak{BF}(A,B,C)\otimes_{\mathfrak{W}}(KH_n(R)\oplus KH_{n-1}(R))\to KH_n(\mathcal{O}_{A,B}^C\otimes R)$$
$$\to\mathrm{Ker}((I-D^*)\otimes_{\mathfrak{W}}(KH_{n-1}(R)\oplus KH_{n-2}(R)))\to 0.$$

Proof. The homotopy K-theory spectrum defines a functor $KH:\mathrm{Alg}_\ell\to\mathrm{Ho(Spt)}$ to the homotopy category of spectra. Applying Theorem 7.3 to the functor $KH(-\otimes R)$ and taking stable homotopy groups, one gets a long exact sequence that can be expressed as the collection of the exact sequences of the corollary for $n\in\mathbb{Z}$. ■

Remark 7.10. The flatness of R is needed in Corollary 7.9 to guarantee that $\otimes R$ preserves algebra extensions, and so, it extends to a triangulated functor $kk\to kk$. One may also consider a variant of j, $j^s:\mathrm{Alg}_\ell\to kk^s$ that is excisive only with respect to those extensions that admit an ℓ-linear splitting so that $\otimes R$ extends to kk for any R. The analog of Corollary 7.9 then holds for j^s whenever the Cohn extension is ℓ-linearly split, e.g., when the conditions of Corollary 3.4.2 are satisfied.

Corollary 7.11. *Assume that E^0 is finite. Then, there is a distinguished triangle in kk:*

$$j(\ell)^{\mathrm{reg}(E)} \oplus j(\ell)[-1]^{\mathrm{reg}(E)} \xrightarrow{I-D^*} j(\ell)^{E^0} \oplus j(\ell)[-1]^{E^0} \to j(\mathcal{O}_{A,B}^C).$$

Proof. Apply Theorem 7.3 to the functor $j : \mathrm{Alg}_\ell \to kk$. ∎

Corollary 7.12. *Let A, B, C be as in Corollary 7.11, and $R \in \mathrm{Alg}_\ell$. Then, there is an exact sequence:*

$$\begin{aligned} 0 &\to \widetilde{\mathfrak{BF}}(A,B,C) \otimes_{\mathfrak{W}} (KH_2(R) \oplus KH_1(R)) \to kk(\mathcal{O}_{A,B}^C, R) \\ &\to \hom_{\mathfrak{W}}(\mathfrak{BF}(A,B,C), KH_1(R) \oplus KH_0(R)) \to 0. \end{aligned}$$

Theorem 7.13. *Let $n \geq 1$, $M, N \in M_n(\mathbb{Z})$ and $P \in M_n(\mathcal{U}(\ell))$, and let $R \in \mathrm{Alg}_\ell$ such that there is a distinguished triangle in kk*

$$j(\ell)^n \oplus j(\ell)[-1]^n \xrightarrow{\left[\begin{smallmatrix} M & P \\ 0 & N \end{smallmatrix}\right]} j(\ell)^n \oplus j(\ell)[-1]^n \longrightarrow j(R)\,. \tag{7.14}$$

Then, there exist matrices $A \in M_{2n}(\mathbb{N}_0)$, $B \in M_{2n}(\mathbb{Z})$, and $C \in M_{2n}(\mathcal{U}(\ell))$ such that (A, B) is KSPI and an isomorphism

$$j(\mathcal{O}_{A,B}^C) \cong j(R).$$

Proof. Let $I_m \in M_m(\mathbb{Z})$ be the identity matrix. The matrix

$$E = \begin{bmatrix} M & 0 & P & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & N & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}$$

defines an endomorphism of $j(\ell)^n \oplus j(\ell)[-1]^n$ in kk, and the direct sum of (7.14) with the trivial triangle

$$j(\ell)^n \oplus j(\ell)[-1]^n \xrightarrow{I_{2n}} j(\ell)^n \oplus j(\ell)[-1]^n \to 0$$

is isomorphic to a triangle

$$j(\ell)^{2n} \oplus j(\ell)^{2n} \xrightarrow{E} j(\ell)^{2n} \oplus j(\ell)^{2n} \to j(R). \tag{7.15}$$

Define matrices X and Y in $M_n(\mathbb{Z})$ as in the proof of [23, Lemma 3.1], taking into account that the matrices named A' and B' are what we call M and N here. Let

$$U = \begin{bmatrix} I_n & I_n & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_{2n} \end{bmatrix}, \quad V = \begin{bmatrix} 0 & -I_n & 0 & 0 \\ I_n & Y & 0 & 0 \\ 0 & 0 & 0 & -I_n \\ 0 & 0 & -I_n & 0 \end{bmatrix} \in M_{2n}(\mathbb{Z}).$$

Define $A, B \in M_{2n}(\mathbb{Z})$ and $C \in M_{2n}(\mathcal{U}(\ell))$ by the identity

$$\begin{bmatrix} I_{2n} - A^t & C^* \\ 0 & I_{2n} - B^t \end{bmatrix} = UEV.$$

Observe that the matrices A^t and B^t are called A and B in [23, Lemma 3.1]; in particular, (A, B) is KSPI. Moreover, the matrices U and V induce an isomorphism of triangles between (7.15) and the triangle of Corollary 7.11 associated to the matrices A, B, and C we have just defined. In particular, $j(R) \cong j(\mathcal{O}_{A,B}^C)$. ∎

Corollary 7.16. *Let ℓ be either a field or a PID, and let $R \in \mathrm{Alg}_\ell$ such that there is a distinguished triangle in kk:*

$$(j(\ell) \oplus j(\ell)[-1])^n \xrightarrow{f} (j(\ell) \oplus j(\ell)[-1])^n \to j(R).$$

Then, there exist matrices A, B, C such that (A, B) is KSPI and $j(R) \cong j(\mathcal{O}_{A,B}^C)$. If moreover ℓ is a field of characteristic zero, or if ℓ is any field and the pair (A, B) satisfies the condition of Proposition 5.6, then $\mathcal{O}_{A,B}^C$ is simple purely infinite.

Proof. If ℓ is a field or a PID, $kk(\ell,\ell) = kk(\ell[-1], \ell[-1]) = K_0(\ell) = \mathbb{Z}$, $kk(\ell[-1], \ell) = K_1(\ell) = \mathcal{U}(\ell)$ and $kk(\ell, \ell[-1]) = K_{-1}(\ell) = 0$. Thus, f must be as in Theorem 7.13. Thus, the first assertion follows from said theorem. The second assertion follows from Theorem 5.5 and Proposition 5.6. ∎

8. K-regularity via the twisted Laurent polynomial picture

Let E be a row-finite graph, and let (G, E, ϕ_c) be a twisted EP-tuple. Consider the $\mathbb{Z}$-grading $L(G, E, \phi_c) = \bigoplus_{n\in\mathbb{Z}} L(G, E, \phi_c)_n$,

$$L(G, E, \phi_c)_n = \mathrm{span}\{\alpha g \beta^* : |\alpha| - |\beta| = n\}.$$

For each $n \geq 0$, let

$$\begin{aligned} L_{0,n}(G, E, \phi_c) = {} & \mathrm{span}\{\alpha g \beta^* : r(\alpha) = g(r(\beta)) \in \mathrm{sink}(E),\ |\alpha| = |\beta| \leq n\} \\ & + \mathrm{span}\{\alpha g \beta^* : r(\alpha) = g(r(\beta)) \in \mathrm{reg}(E),\ |\alpha| = |\beta| \leq n\}. \end{aligned}$$

We have $L(G, E, \phi_c)_0 = \bigcup_{n\geq 0} L_{0,n}(G, E, \phi_c)$. Observe that setting $G = \{1\}$, we recover the usual grading of $L(E)$ and the usual filtration of $L(E)_0$. Like in the Leavitt path algebra case [2, Section 5] when E is finite without sources, we may also regard $L(G, E, \phi_c)$ as a corner skew Laurent polynomial ring in the sense of [3], as follows. Pick an edge $e_v \in r^{-1}\{v\}$ for each $v \in E^0$. Then, the elements of $L(G, E, \phi_c)$

$$t_+ = \sum_{v\in E^0} e_v \quad \text{and} \quad t_- = t_+^*$$

are homogeneous of degrees 1 and -1, respectively, and satisfy $t_-t_+ = 1$. Hence,

$$p = \psi(1) \in L(G, E, \phi_c)_0$$

is an idempotent, and the map

$$\psi : L(G, E, \phi_c)_0 \to L(G, E, \phi_c)_0, \quad \psi(x) = t_+ x t_-, \tag{8.1}$$

is an algebra monomorphism; its image is the corner $pL(G, E, \phi_c)_0 p$. Hence, by [3, Lemma 2.4], we may regard $L(G, E, \phi_c)$ as a Laurent polynomial ring twisted by the corner isomorphism ψ; using the notation of [3], we write

$$L(G, E, \phi_c) = L(G, E, \phi_c)_0[t_+, t_-; \psi]. \tag{8.2}$$

In the next lemma, we use the following notation. For a graph E, a vertex $v \in E^0$, and $m \geq 0$, we put

$$\mathcal{P}_{v,m} = \{\alpha \in \mathcal{P}(E)_v \colon |\alpha| = m\}.$$

Assumption 8.3. From this point on, in all (twisted) EP-triples we consider, the group G will be assumed to act trivially on E^0.

Notation 8.4. Let $v \in E^0$; regard v as an element of $L(G, E, \phi_c)$. By Assumption 8.3, the identity $gv = vg$ holds in $L(G, E, \phi_c)$ for all $g \in G$. Hence, the right annihilator

$$I_v = \operatorname{Ann}_{\ell[G]}(v) \subset \ell[G]$$

is a two-sided ideal. We write

$$R_v = \ell[G]/I_v$$

for the quotient ring.

Lemma 8.5. *Let (G, E, ϕ_c) be a twisted EP-tuple with E row-finite. Assume that G acts trivially on E^0. Put*

$$\mathcal{M}_n(G, E) := \left(\bigoplus_{v \in \operatorname{sink}(E)} \bigoplus_{m=0}^{n} M_{\mathcal{P}_{v,m}} \ell[G]\right) \oplus \Big(\bigoplus_{v \in \operatorname{reg}(E)} M_{\mathcal{P}_{v,n}} \ell[G]\Big).$$

(i) *There is a surjective algebra homomorphism*

$$\pi_n : \mathcal{M}_n(G, E) \twoheadrightarrow L(G, E, \phi_c)_{0,n}, \quad \pi_n(\varepsilon_{\alpha,\beta} g) = \alpha g \beta^*.$$

(ii) $I_v = 0$ *for all* $v \in \operatorname{sink}(E)$ *and*

$$\operatorname{Ker}(\pi_n) = \bigoplus_{v \in \operatorname{reg}(E)} M_{\mathcal{P}_{v,n}} I_v.$$

Hence, we have

$$L(G, E, \phi_c)_{0,n} \cong \left(\bigoplus_{v \in \operatorname{sink}(E)} \bigoplus_{m=0}^{n} M_{\mathcal{P}_{v,m}} R_v\right) \oplus \Big(\bigoplus_{v \in \operatorname{reg}(E)} M_{\mathcal{P}_{v,n}} R_v\Big).$$

(iii) *Under the isomorphism above, the inclusion* $L(G,E,\phi_c)_{0,n} \subset L(G,E,\phi_c)_{0,n+1}$ *identifies with the inclusion*

$$\bigoplus_{v\in\mathrm{sink}(E)}\bigoplus_{m=0}^{n} M_{\mathcal{P}_{v,m}}R_v \subset \bigoplus_{v\in\mathrm{sink}(E)}\bigoplus_{m=0}^{n+1} M_{\mathcal{P}_{v,m}}R_v$$

on the first summand and is induced by the map

$$\bigoplus_{v\in\mathrm{reg}(E)} M_{\mathcal{P}_{v,n}}R_v \to \Big(\bigoplus_{v\in\mathrm{sink}(E)} M_{\mathcal{P}(n+1,v)}R_v\Big)\oplus\Big(\bigoplus_{v\in\mathrm{reg}(E)} M_{\mathcal{P}(n+1,v)}R_v\Big)$$
$$\varepsilon_{\alpha,\beta}g \mapsto \sum_{s(e)=r(\alpha)} \varepsilon_{\alpha g(e),\beta e}\phi_c(g,e)$$

on the second.

Proof. It is clear that the ℓ-linear map π_n is surjective. Observe that

$$\alpha,\beta \in \Big(\bigcup_{v\in\mathrm{sink}(E),1\le m\le n}\mathcal{P}_{v,m}\cup\bigcup_{v\in\mathrm{reg}(E)}\mathcal{P}_{v,n}\Big) \Rightarrow \beta^*\alpha = \delta_{\alpha,\beta}r(\alpha). \tag{8.6}$$

It follows from this that π_n is a ring homomorphism, proving (i). Next, let

$$x = \sum_{\alpha,\beta}\varepsilon_{\alpha,\beta}x_{\alpha,\beta} \in \mathrm{Ker}(\pi_n).$$

Then,

$$\pi_n(x) = \sum_{\alpha,\beta}\alpha x_{\alpha,\beta}\beta^* = 0. \tag{8.7}$$

Taking (8.6) into account and multiplying (8.7) by α^* on the left and by β on the right, we obtain that (8.7) holds if and only if

$$x_{\alpha,\beta} \in I_{r(\alpha)}$$

holds in $L(G,E,\phi_c)$ for all α, β for which $x_{\alpha,\beta}$ is defined. Observe that if we regard a vertex $v \in E^0$ as an element of $C(G,E,\phi_c)$, then $I_v = \{x \in \ell[G]\colon vx \in \mathcal{K}(G,E,\phi_c)\}$. Recall that $\mathcal{K}(G,E,\phi_c)$ is spanned by the elements

$$\alpha q_w g\beta^* = \alpha g\beta^* - \sum_{s(e)=w}\alpha e\phi(g,g^{-1}(e))(\beta g^{-1}(e))^*$$

with $w \in \mathrm{reg}(E)$. The above is the unique expression of the element of the left as a linear combination of the basis $\mathcal{B}$ of $C(G,E,\phi_c)$ and contains no basis elements of the form vh with $v \in \mathrm{sink}(E)$ and $h \in G$. This proves that $I_v = 0$ for every $v \in \mathrm{sink}(E)$. Thus, part (ii) is proved. Part (iii) is a straightforward application of the Cuntz–Krieger relation CK2 (3.4.1). ∎

Let $v \in \operatorname{reg}(E)$ and $w \in E^0$ such that $A_{v,w} \neq 0$; set

$$X_{v,w} = \bigoplus_{e \in vE^1 w} \ell e \otimes R_w. \tag{8.8}$$

Consider the ℓ-linear map

$$X_{v,w} \to L(G, E, \phi_c), \quad e \otimes x \mapsto ex. \tag{8.9}$$

The map (8.9) is injective, because if $\sum_{e \in vE^1 w} ea_e = 0$ and $f \in vE^1 w$, then multiplying on the left by f^* we get that $a_f = 0$. Observe also that, in view of Assumption 8.3, its image N is a left $\ell[G]$-submodule, and we have $v \cdot x = x$ for all $x \in N$. In particular, $I_v N = vI_v N = 0$, and thus, N is a left R_v-module. In Lemma 8.10 below, we regard $X_{v,w}$ as a left R_v-module via the map (8.9).

Lemma 8.10. *Let E be a finite graph with incidence matrix A. Assume that G acts trivially on E^0. The following are equivalent for $n \geq 0$.*

(i) *$L(G, E, \phi_c)_{0,n+1}$ is a flat left $L(G, E, \phi_c)_{0,n}$-module.*

(ii) *For every $(v, w) \in \operatorname{reg}(E) \times E^0$ such that $A_{v,w} \neq 0$, the left R_v-module (8.8) is flat.*

Proof. If M is a left module over a unital ring R, $m \geq 1$, and $p_1, \dots, p_m \in R$ are central orthogonal idempotents such that $\sum_{i=1}^m p_i = 1$, then M is flat over R if and only if $p_i M$ is a flat Rp_i-module for all i. We apply this with $R = L_{0,n}$, $M = L_{0,n+1}$ and the orthogonal idempotents that correspond to the identity matrices of each of the matrix algebras in the direct sum decomposition of Lemma 8.5 and using the identification of the inclusion $L(G, E, \phi_c)_{0,n} \subset L(G, E, \phi_c)_{0,n+1}$ given therein. If $v \in \operatorname{sink}(E)$ and $m \leq n$, then under the identification of Lemma 8.5, $\mathrm{id}_{\mathcal{P}_{v,m}} \cdot L(G, E, \phi_c)_{0,n+1} = M_{\mathcal{P}_{v,m}} R_v$, which is flat over itself. If $v \in \operatorname{reg}(E)$, then

$$\mathrm{id}_{\mathcal{P}_{v,n}} \cdot L(G, E, \phi_c)_{0,n+1} = \bigoplus_{\{w \mid A_{v,w} \neq 0\}} \sum_{\{e \in vE^1 w, \alpha \in \mathcal{P}_{v,n}\}} \varepsilon_{\alpha e, \alpha e} M_{\mathcal{P}(w,n+1)} R_w.$$

One checks that each of the summands

$$\sum_{\{e \in vE^1 w, \alpha \in \mathcal{P}_{v,n}\}} \varepsilon_{\alpha e, \alpha e} M_{\mathcal{P}(w,n+1)} R_w \tag{8.11}$$

is a left $M_{\mathcal{P}_{v,n}} R_v$-submodule. Hence, $L(G, E, \phi_c)_{0,n} \subset L(G, E, \phi_c)_{0,n+1}$ is flat if and only if (8.11) is flat for every $w \in E^0$ such that $A_{v,w} \neq 0$. Moreover, (8.11) decomposes as direct sum, indexed by $\gamma \in \mathcal{P}(w, n+1)$, of the $\mathcal{M}_{P(n,v)} R_v$-submodules

$$X_{v,w,\gamma} = \bigoplus_{\{\alpha \in \mathcal{P}_{v,n}, e \in vE^1 w\}} \varepsilon_{\alpha e, \gamma} R_w.$$

So, again, the flatness of $L(G, E, \phi_c)_{0,n+1}$ over $L(G, E, \phi_c)_{0,n}$ boils down to that of each of the $X_{v,w,\gamma}$. Equip $\ell^{\mathcal{P}(v,n)} = \ell^{\mathcal{P}(v,n) \times \{1\}}$ with its canonical left $M_{\mathcal{P}_{v,n}}$-module structure and view $\ell^{\mathcal{P}(v,n)} \otimes X_{v,w}$ as a module over $M_{\mathcal{P}_{v,n}} R_v = M_{\mathcal{P}_{v,n}} \otimes R_v$ in the obvious way. One checks that

$$X_{v,w,\gamma} \to \ell^{\mathcal{P}_{v,n}} \otimes X_{v,w}, \quad \varepsilon_{\alpha e,\gamma} x \mapsto \alpha \otimes e \otimes x$$

is an isomorphism of left $M_{\mathcal{P}_{v,n}} R_v$-modules. Since the $M_{\mathcal{P}_{v,n}}$-module $\ell^{\mathcal{P}_{v,n}}$ is projective, whence flat, we get that $X_{v,w,\gamma}$ is flat over $M_{\mathcal{P}_{v,n}} \otimes R_v$ if and only if $X_{v,w}$ is flat over R_v. This concludes the proof. ∎

Lemma 8.12. *Let (G, E, ϕ_c) be a twisted EP-tuple satisfying the conditions of Lemma 8.10, and let $(v, w) \in \operatorname{reg}(E) \times E^0$. Assume that $I_v = I_w = 0$. Then, $X_{v,w}$ is a flat left $R_v = \ell[G]$-module if and only if $\ell[G]/\operatorname{Ann}_{\ell[G]}(e)$ is $\ell[G]$-flat for all $e \in vE^1w$.*

Proof. For $g \in G$ and $(e, h) \in vE^1w \times G$, set

$$g \cdot (e,h) = (g(e), \phi(g,e)h). \tag{8.13}$$

One checks, using that ϕ is a cocycle, that (8.13) defines a left action of G on $vE^1w \times G$, and thus a linear G-action on

$$\ell[vE^1w \times G] = X_{v,w}.$$

For the identification above, we have used our hypothesis that $I_w = 0$. The decomposition of $vE^1w \times G$ into G-orbits gives a corresponding direct sum decomposition of $R_v = \ell[G]$-modules

$$X_{v,w} = \bigoplus_{K \in (vE^1w \times G)/G} \ell[K].$$

Let $e \in vE^1w$, $h \in G$, and $K_{e,h}$ be the orbit of (e,h). Observe that right multiplication by h gives an $\ell[G]$-module isomorphism $\ell[K_{e,1}] \cong \ell[K_{e,h}]$. Moreover, we have an isomorphism of left $\ell[G]$-modules

$$\ell[G]/\operatorname{Ann}_{\ell[G]}(e) \cong \ell[K_{(e,1)}]. \tag{8.14}$$

Summing up, $X_{v,w}$ is flat if and only if (8.14) is flat for all $e \in vE^1w$, as we had to prove. ∎

Theorem 8.15. *Let (G, E, ϕ_c) be a twisted EP-tuple. Assume that E is row-finite and that G acts trivially on E^0. Further, assume that R_v is regular supercoherent for every $v \in E^0$ and that condition (ii) of Lemma 8.10 is satisfied. Then, $L(G, E, \phi_c)$ is K-regular.*

Proof. Assume first that E is finite without sources. Put $S = L(G, E, \phi_c)_0$; let $\psi : S \to S$ be the corner isomorphism of (8.1) and $B = S[\psi^{-1}]$ the colimit of the inductive system

$$S \xrightarrow{\psi} S \xrightarrow{\psi} S \xrightarrow{\psi} \cdots,$$

$\widetilde{B} = B \oplus \ell$ its unitalization, and $\widetilde{\psi} : \widetilde{B} \to \widetilde{B}$ the induced unital automorphism. Let $NK_*(S, \psi)_\pm = NK_*(\widetilde{B}, \widetilde{\psi})$, the twisted nil-$K$-theory groups. Note that because B is a filtering colimit of unital rings, it satisfies excision in K-theory, and thus these nil-K-groups are the same as those defined in [2, Notation 3.4.1]. By (8.2) and [2, Theorems 3.6 and 8.4], the comparison map $K_*(L(G, E, \phi_c)) \to KH_*(L(G, E, \phi_c))$ fits into a map of long exact sequences

$$\begin{array}{ccccccc} K_n(S) & \xrightarrow{1-\psi} & K_n(S) \oplus NK_n(S,\psi)_+ \oplus NK_n(S,\psi)_- & \to & K_n(L(G,E,\phi_c)) & \longrightarrow & K_{n-1}(S) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ KH_n(S) & \xrightarrow{1-\psi} & KH_n(S) & \longrightarrow & KH_n(L(G,E,\phi_c)) & \to & KH_{n-1}(S) \end{array} \tag{8.16}$$

Because by hypothesis, the R_v are regular supercoherent for all v, so is any finite sum of finite matrix rings over them; in particular, $S_n = L(G, E, \phi_c)_{0,n}$ is regular supercoherent for all n. By [20, Proposition 1.6], the colimit of an inductive system of regular supercoherent rings with unital flat transition maps is regular supercoherent. Hence, S is regular supercoherent by Lemma 8.10. In particular, the comparison map

$$K_n(S[t_1, \ldots, t_p]) \to KH_n(S[t_1, \ldots, t_p])$$

is an isomorphism for all n and p. Now, the argument of [2, Proposition 7.1] applies verbatim to show that $\widetilde{B}$ is regular supercoherent. Thus, $NK_*(S, \psi)_\pm = NK_*(\widetilde{B}, \widetilde{\psi})_\pm = 0$ by [2, Lemma 7.2]. It follows that the comparison map

$$K_*(L(G, E, \phi_c)) \to KH_*(L(G, E, \phi_c))$$

is an isomorphism. Substituting $\ell[t_1, \ldots, t_m]$ for ℓ, we get that

$$K_*(L(G, E, \phi_c)[t_1, \ldots, t_m]) \to KH_*(L(G, E, \phi_c)[t_1, \ldots, t_m])$$

is an isomorphism for all m. This proves that $L(G, E, \phi_c)$ is K-regular whenever E is finite without sources and the hypotheses of the theorem are satisfied. Let now E be a finite graph, $v \in E^0$ a source, and $E_{|v}$ the graph obtained from E upon removing v. Then, $1 - v$ is a full idempotent of $L(E)$ and therefore also of $L(G, E, \phi_c)$. Furthermore, the corner $C = (1 - v)L(G, E, \phi_c)(1 - v)$ is the span of elements $\alpha g \beta^*$ with $g \in G$ and $\alpha, \beta \in \mathcal{P}(E_{|v})$; it follows that $C \cong L(G, E_{|v}, \phi_c|E_{|v})$ is the algebra of the EP-tuple obtained from (G, E, ϕ_c) upon restricting ϕ and c to $G \times E^1_{|v}$. Repeating this process a finite number of times, we end up with a finite graph E' without sources such that $L(G, E', \phi_c|E')$ is isomorphic to a full corner of $L(G, E, \phi_c)$. By what we have already proved, $L(G, E', \phi_{|E'})$ is K-regular; therefore, the same is true of $L(G, E, \phi_c)$, since the latter is Morita equivalent to the former. This proves the theorem for finite E. If now E is any row-finite graph and $F \subset E$ a finite complete subgraph, then the algebra $L(G, F, \phi_c|F)$ of the restriction EP-tuple is isomorphic to the subalgebra of

$L(G, E, \phi_c)$ linearly spanned by the elements $\alpha g\beta^*$ with $\alpha, \beta \in F$ and $g \in G$. Hence, $L(G, E, \phi_c) = \operatorname{colim}_F L(G, F, \phi_c|F)$, where the colimit runs over all finite complete subgraphs. Thus, $L(G, E, \phi_c)$ is K-regular because each $L(G, F, \phi_c|F)$ is, by what we have already proved. ∎

Corollary 8.17. *Let (G, E, ϕ_c) be a twisted EP-tuple as in Theorem 8.15. If (G, E, ϕ_c) is pseudo-free, then $L(G, E, \phi_c)$ is K-regular.*

Proof. Because (G, E, ϕ_c) is pseudo-free, we have $I_v = 0$ for all $v \in E^0$, by Corollary 3.4.2. Hence, in view of Lemma 8.12, it suffices to show that $\operatorname{Ann}_{\ell[G]}(e) = 0$ for all $e \in E^1$. Let $e \in E^1$ and $K = K_{(e,1)}$ as in the proof of Lemma 8.12. It follows from Corollary 3.4.2 and the identity (3.3.8) that $x = \sum_{g\in G} \lambda_g g \in \operatorname{Ann}_{\ell[G]}(e)$ if and only if for every $(f, h) \in K$ we have

$$0 = \sum_{\{g: g(e), \phi(g,e)=(f,h)\}} \lambda_g c(g, e). \tag{8.18}$$

Next, observe that if x satisfies (8.18), then it annihilates $ee^* \in C(G, E, \phi_c)$. By Lemma 3.3.3 and pseudo-freeness, this implies that $x = 0$. Hence, $\operatorname{Ann}_{\ell[G](e)=0}$ for all $e \in E^1$, concluding the proof. ∎

Example 8.19. Let ℓ be a Noetherian domain, and let $(\mathbb{Z}, E, \phi_c)$ be a twisted EP-tuple where $\mathbb{Z}$ acts trivially on E^0. Assume that $(\mathbb{Z}, E, \phi_c)$ is partially pseudo-free but not pseudo-free. Write $L = \ell[t, t^{-1}]$; then, (3.3.8) maps injectively to a basis of $L(\mathbb{Z}, E, \phi_c)$. Since the latter basis contains vg for all $g \in G$, they are ℓ-linearly independent, so we have $R_v = L$ for all $v \in E^0$. By Lemma 3.3.3, there is an edge e and a nonzero element $x \in L$ that annihilates $ee^* \in C(\mathbb{Z}, E, \phi_c)$. Then, x also annihilates $e \in L(\mathbb{Z}, E, \phi_c)$, so $J := \operatorname{Ann}_L(e) \neq 0$. Moreover, $J \neq L$, by Proposition 3.3.7. Because we are assuming that ℓ is Noetherian, the same is true of L, and thus, L/J is flat if and only if it is projective, by Villamayor's theorem. Hence, the flatness of L/J would imply that L is a decomposable L-module, which would contradict the hypothesis that ℓ is a domain. So, L/J is not flat, and thus neither is $X_{s(e),r(e)}$, by Lemma 8.12. By Lemma 8.10, this implies that $L(\mathbb{Z}, E, \phi_c)_{0,n+1}$ is not flat over $L(\mathbb{Z}, E, \phi_c)_{0,n}$ for any $n \geq 0$.

9. K-regularity of twisted Katsura algebras

Lemma 9.1. *Let (A, B, C) and E be as in Section 5 above, and let $(v, w) \in \operatorname{reg}(E) \times E^0$ such that $A_{v,w} \neq 0$. Let $X_{v,w}$ be the R_v-module of (8.8).*

(i) *If $B_{v,w} \neq 0$, then $X_{v,w}$ is flat.*

(ii) *If $B_{v,w} = 0$ and there exists w' such that $B_{v,w'} \neq 0$, then $X_{v,w}$ is not flat.*

(iii) *Suppose $B_{v,w} = 0$ for all $w \in r(s^{-1}\{v\})$. Further, assume that $C_{v,w} - C_{v,w'}$ and $C_{v,w} - 1$ are either zero or invertible in ℓ for all $w, w' \in r(s^{-1}\{v\})$. Then, $X_{v,w}$ is flat for all $w \in r(s^{-1}\{v\})$.*

Proof. Set $a = A_{v,w}$, $b = B_{v,w}$, $c = C_{v,w}$, $X = X_{v,w}$, and $L = \ell[t, t^{-1}]$. For each $0 \leq i < a$, write e_i for the unique edge $e \in vE^1w$ with $\mathfrak{n}(e) = i$. If $b = 0$, then $te_0 \otimes x = ce_0 \otimes x$ and $te_i \otimes x = e_i \otimes x$ for all $x \in L$ and $1 \leq i < a$. Hence, the monic polynomial $f(t) = (t-1)(t-c)$ annihilates X. Because $f(t)$ is not a zero divisor in L,

$$L \xrightarrow{f(t)} L$$

is a free resolution of $L/f(t)L$. Hence, $\mathrm{Tor}_1^L(L/(t-1)(t-c), X) = X \neq 0$. In particular, X is not flat over L. In the situation of (ii), $I_v = 0$, so $R_v = L$, and thus, X is not flat over R_v. In the situation of (iii), if $b \neq 0$, let $c_1, \ldots, c_r$ be the distinct elements of the set $\{1\} \cup \{C_{v,w} : w \in r(s^{-1}(\{v\}))\}$. Then,

$$R_v \cong L/\prod_{i=1}^{r}(t-c_i)L \cong \bigoplus_{i=1}^{r} \ell[t]/(t-c_i)\ell[t],$$

and X is a direct sum of copies of some of the summands in the decomposition above. Hence, it is a projective R_v-module and in particular it is flat.

Next, assume that $b \neq 0$; note that $R_v = L$ in this case. Let $r : \mathbb{Z} \to \{0, \ldots, a-1\}$ and $q : \mathbb{Z} \to \mathbb{Z}$ be the remainder and the quotient function in the division by a. Let $L_{b,c}$ be the ℓ-module L equipped with the following $\mathbb{Z}$-action:

$$t \cdot_b x = ct^b x.$$

Then, $L_{b,c}$ is free with basis $1, t, \ldots, t^{|b|-1}$ and

$$L_{b,c} \to X \quad t^n \mapsto e_{r(n)} \otimes t^{q(n)}$$

is an isomorphism of left L-modules. Hence, X is a free left L-module. ■

Proposition 9.2. *Let ℓ be regular supercoherent and (A, B, C) a twisted Katsura triple. If either of the following holds, then $\mathcal{O}_{A,B}^C$ is K-regular.*

(i) $B_{v,w} = 0 \leftrightarrow A_{v,w} = 0$.

(ii) *ℓ is a field and if $v \in \mathrm{reg}(E)$ is such that $B_{v,w} = 0$ for some $w \in r(s^{-1}\{v\})$, then $B_{v,w'} = 0$ for all $w' \in r(s^{-1}\{v\})$.*

Proof. Immediate from Lemma 9.1 and Theorem 8.15. ■

Remark 9.3. By [18, Lemma 18.5], condition (i) of Proposition 9.2 is equivalent to the condition that the EP-tuple $(\mathbb{Z}, E, \phi)$ associated to (A, B) in Section 5 be pseudo-free. If condition (ii) of the same proposition is satisfied, then any vertex v with $B_{v,w} = 0$ for some $w \in r(s^{-1}\{v\})$ is a B-*sink* in the sense of [24, p. 2248].

Corollary 9.4. *Assume either of the conditions of Proposition 9.2 is satisfied. Then, for any $n \in \mathbb{Z}$, there is a short exact sequence*

$$0 \to \mathfrak{BF}(A, B, C) \otimes_{\mathfrak{W}} (K_n(\ell) \oplus K_{n-1}(\ell)) \to K_n(\mathcal{O}_{A,B}^C)$$
$$\to \mathrm{Ker}((I - D^*) \otimes_{\mathfrak{W}} (K_{n-1}(\ell) \oplus KH_{n-2}(\ell))) \to 0.$$

Proof. Immediate from Proposition 9.2 and Corollary 7.9. ∎

10. Ring theoretic regularity: The universal localization picture

Let (G, E, ϕ_c) be a twisted EP-tuple. In this section, we assume that E is finite and that G acts trivially on E^0. We show that $L(G, E, \phi_c)$ can be interpreted as the universal localization of an algebra $P(G, E, \phi_c)$. We use this to give sufficient conditions that guarantee that $L(G, E, \phi_c)$ is *regular* in the sense that every (right) S-module has finite projective dimension, and apply this to the case of twisted Katsura algebras.

For each $(v, w) \in \operatorname{reg}(E) \times E^0$, let $X_{v,w}$ be as in (8.8). Regard $X_{v,w}$ as an (R_v, R_w)-bimodule, with the left R_v-module structure induced via the map (8.9) and the obvious right R_w-module structure. Set

$$R = \bigoplus_{v\in E^0} R_v, \quad X = \bigoplus_{v,w} X_{v,w}. \tag{10.1}$$

Observe that X is an R-bimodule; let

$$P(G, E, \phi_c) := T_R X \tag{10.2}$$

be the tensor algebra. Since by definition R and X are a subalgebra and an R-sub-bimodule of $L(G, E, \phi_c)$, we have a canonical algebra homomorphism

$$P(G, E, \phi_c) \to L(G, E, \phi_c). \tag{10.3}$$

Lemma 10.4. *The homomorphism* (10.3) *is injective.*

Proof. By definition, R and X are included in $L(G, E, \phi_c)$. Moreover, the n-fold tensor product $T_R^n X$ is freely generated as a right R-module by the paths of length n in E. Thus, it suffices to show that if $\mathcal{F} \subset \mathcal{P}(E)$ is a finite subset and $x_\alpha \in R_{r(\alpha)}$ for all $\alpha \in \mathcal{F}$, then

$$\sum_\alpha \alpha x_\alpha = 0 \tag{10.5}$$

implies that each $x_\alpha = 0$. Choose a lift $y_\alpha \in \ell[G]$ for each $\alpha \in \mathcal{F}$; then, (10.5) implies that $\sum_{\alpha\in\mathcal{F}} \alpha y_\alpha \in \mathcal{K}(G, E, \phi_c)$. It follows from this and the fact that $\mathcal{B}$ and $\mathcal{B}'$ are ℓ-linear basis of $C(G, E, \phi_c)$ and $\mathcal{K}(G, E, \phi_c)$, that $y_\alpha \in I_{r(\alpha)}$ so that $x_{\alpha,\beta} = 0$ for each $\alpha \in \mathcal{F}$. ∎

For each regular vertex v, let

$$\sigma_v : \bigoplus_{s(e)=v} r(e)P(G, E, \phi_c) \to vP(G, E, \phi_c), \quad \sigma_v(r(e)a) = ea.$$

Let $\Sigma = \{\sigma_v : v \in \operatorname{reg}(E)\}$, and let $P(G, E, \phi_c)_\Sigma$ be the universal localization. Observe that scalar extension along (10.3) inverts the elements of Σ; hence, we have a canonical algebra homomorphism

$$P(G, E, \phi_c)_\Sigma \to L(G, E, \phi_c). \tag{10.6}$$

Lemma 10.7. *The algebra homomorphism* (10.6) *is an isomorphism.*

Proof. The algebra $P(G,E,\phi_c)_\Sigma$ is obtained from $P(G,E,\phi_c)$ upon adjoining an element y_e for each $e \in E^1$ so that $r(e)y_e s(e) = y_e$, and so, that the matrices

$$M_v = \sum_{s(e)=v} \varepsilon_{v,e} e \quad \text{and} \quad N_v = \sum_{s(e),v} \varepsilon_{e,v} y_e$$

satisfy $M_v N_v = \varepsilon_{v,v} v$ and $N_v M_v = \sum_{s(e)=v} \varepsilon_{e,e} r(e)$. The homomorphism (10.6) is the inclusion on $P(G,E,\phi_c)$ and sends $y_e \mapsto e^*$; to prove it is an isomorphism, it suffices to show that the y_e satisfy the same relations (3.1.2), (3.1.3), and (3.1.4) as the e^*. This is clear for all but the last identity of (3.1.4). Moreover, we have

$$\begin{aligned} y_e g = y_e g s(e) &= y_e g \sum_{s(f)=s(e)} f y_f = y_e \sum_{s(f)=s(e)} g(f)\phi(g,f) y_f \\ &= y_e \sum_{s(f)=s(e)} f\phi(g,g^{-1}f) y_{g^{-1}(f)} = r(e)\phi(g,g^{-1}e) y_{g^{-1}(e)} = \phi(g,g^{-1}e) y_{g^{-1}(e)}. \end{aligned}$$

Thus, the y_e satisfy all the required identities for the existence of homomorphism of $P(G,E,\phi_c)$-algebras $C(G,E,\phi_c) \to P(G,E,\phi_c)_\Sigma$ mapping $e^* \mapsto y_e$. Furthermore, the identity $M_v N_v = \varepsilon_{v,v} v$ implies that the latter induces a homomorphism $L(G,E,\phi_c) \to P(G,E,\phi_c)_\Sigma$ inverse to (10.6). ∎

Lemma 10.8. *The inclusion $P(G,E,\phi_c) \subset L(G,E,\phi_c)$ makes $L(G,E,\phi_c)$ into a flat left $P(G,E,\phi_c)$-module.*

Proof. It follows from Lemma 10.7 that $P(G,E,\phi_c) \subset L(G,E,\phi_c)$ is a ring epimorphism. Hence, by [30, Theorem 2.1], it suffices to find, for each $x \in L(G,E,\phi_c)$, a finite subset $\mathcal{F} \subset \mathcal{P}(E)$ such that

$$\forall \gamma \in \mathcal{F}, \quad x\gamma \in P(G,E,\phi_c) \tag{10.9}$$

$$\sum_{\gamma\in\mathcal{F}} \gamma P(G,E,\phi_c) = P(G,E,\phi_c). \tag{10.10}$$

Any element $x \in L(G,E,\phi_c)$ can be written as a finite linear combination

$$x = \sum_{\alpha,\beta} \alpha x_{\alpha,\beta} \beta^* = \sum_{\beta} \Big(\sum_{\alpha} \alpha x_{\alpha,\beta}\Big)\beta^* \tag{10.11}$$

with each $x_{\alpha,\beta} \in R_{r(\alpha)}$ and $r(\beta) = r(\alpha)$. Let $\mathcal{F}' \subset \mathcal{P}(E)$ be the set of all those paths β such that β^* appears in (10.11) with a nonzero coefficient. Using the CK2 relation (3.4.1), we may arrange that there is an n such all $\beta \in \mathcal{F}'$ with $r(\beta) \in \operatorname{sink}(E)$ have length $\le n$, and all those with $r(\beta) \in \operatorname{reg}(E)$ have length n. Hence, for all $\beta \in \mathcal{F}'$,

$$x\beta = \sum_{\alpha} \alpha x_{\alpha,\beta} \in P(G,E,\phi_c).$$

Let

$$\mathcal{P}(E) \supset \mathcal{F} = \{\gamma : r(\gamma) \in \operatorname{reg}(E),\ |\gamma| = n\} \cup \{\gamma : r(\gamma) \in \operatorname{sink}(E),\ |\gamma| \le n\};$$

then $\mathcal{F} \supset \mathcal{F}'$ and $x\gamma = 0$ for all $\gamma \in \mathcal{F} \setminus \mathcal{F}'$. Hence, $\mathcal{F}$ satisfies (10.9). Moreover, $\sum_{\gamma\in\mathcal{F}} \gamma\gamma^* = 1$, so (10.10) is also satisfied. ■

Corollary 10.12. *If E is finite and $P(G, E, \phi_c)$ is either right regular or right coherent, then the same is true of $L(G, E, \phi_c)$.*

Proof. By Lemmas 10.7 and 10.8, $P(G, E, \phi_c) \subset L(G, E, \phi_c)$ is a perfect right localization in the sense of [30, Definition on p. 229]. Hence, by [30, Theorem 2.1 (b)], the family $\mathfrak{F}$ of all right ideals $\mathfrak{a} \subset P(G, E, \phi_c)$ such that $\mathfrak{a}L(G, E, \phi_c) = L(G, E, \phi_c)$ is a Gabriel topology, and $L(G, E, \phi_c) = P(G, E, \phi_c)_{\mathfrak{F}}$ is the localization with respect to $\mathfrak{F}$. It then follows from [30, Corollary 1.10 of Chapter IX and Proposition 3.4 of Chapter XI] that localization of right $P(G, E, \phi_c)$-modules with respect to $\mathfrak{F}$ is exact and essentially surjective. Moreover, it preserves projectivity by [30, Proposition 1.11 of Chapter IX]. It follows that $L(G, E, \phi_c)$ is regular whenever $P(G, E, \phi_c)$ is. If $P(G, E, \phi_c)$ is right coherent, then $L(G, E, \phi_c)$ is right coherent by [30, Proposition 3.12 of Chapter XI]. ■

The following two lemmas are probably well known. I came to them together with my colleague Marco Farinati after a fruitful discussion.

Lemma 10.13. *Let S be a unital ring containing a semisimple commutative ring k, and such that S has left projective dimension d as an $S^e := S \otimes_k S^{\mathrm{op}}$-module. Then, S and S^e have (both right and left) global projective dimensions $\le d$ and $\le 2d$, respectively. In particular, both S and S^e are regular.*

Proof. Let $(S \otimes_k \bar{S}^{\otimes_k \bullet} \otimes_k S, b') \overset{\sim}{\twoheadrightarrow} S$ be the bar resolution. The hypothesis means that

$$\Omega^d S = \operatorname{Ker}\big(b' : S \otimes_k \bar{S}^{\otimes_k d-1} \otimes_k S \to S \otimes_k \bar{S}^{\otimes_k d-2} \otimes_k S\big)$$

is a projective $S \otimes_k S^{\mathrm{op}}$-module. Consider the truncated bar resolution

$$Q_m = \begin{cases} S \otimes_k \bar{S}^{\otimes_k m} \otimes_k S & \text{if } 0 \le m \le d-1, \\ \Omega^d S & \text{if } m = n, \\ 0 & \text{if } m > d. \end{cases}$$

Observe that $Q_\bullet$ is both right and left split; so tensoring it over S on either side with an S-module M yields a resolution $P \overset{\sim}{\twoheadrightarrow} M$ of length d such that each P_m is a scalar extension of a k-module, and therefore projective, as k is semisimple. Thus, both the right and the left global dimensions of S are $\le d$. Next, observe that S^e is isomorphic to its opposite ring via the flip $s \otimes t \mapsto t \otimes s$; in particular, its left and right global dimensions coincide. Let M be a left S^e-module; then, $Q_\bullet \otimes_S M \otimes_S Q_\bullet$ is the total complex of a bicomplex whose m-th row is an S^e-projective resolution of $M \otimes_S Q_m$. Hence, the composite

$$Q_\bullet \otimes_S M \otimes_S Q_\bullet \overset{\sim}{\twoheadrightarrow} M \otimes_S Q_\bullet \overset{\sim}{\twoheadrightarrow} M$$

is a quasi-isomorphism. This completes the proof that the left global dimension of S^e is $\leq 2d$, since $Q_\bullet \otimes_S M \otimes_S Q_\bullet$ has length $2d$. ∎

Lemma 10.14. *Let S and k be as in Lemma 10.13, and let M be a left $S \otimes_k S^{\mathrm{op}}$-module. If either M_S or ${}_SM$ is projective, then the tensor algebra $T = T_S(M)$ has both right and left global dimensions $\leq 2d + 1$. In particular, T is regular.*

Proof. We will assume that M is right projective. In view of Lemma 10.13, it suffices to show that the $T \otimes_k T^{\mathrm{op}}$-module T has projective dimension $\leq 2d$. Let $\Omega^\bullet$ be as in the proof of Lemma 10.13. Consider the relative cotangent sequence [15, Corollary 2.10]

$$0 \to T \otimes_S \Omega^1 S \otimes_S T \to \Omega^1 T \to T \otimes_S M \otimes_S T \to 0. \tag{10.15}$$

It suffices to show that the bimodules on the left and right of the sequence above have projective dimension $\leq 2d$. An appropriate truncation of the bar resolution provides an $S \otimes_k S^{\mathrm{op}}$-projective resolution $P_\bullet \overset{\sim}{\twoheadrightarrow} \Omega^1_S$ of length $\leq d - 1$ which is both right and left split. Hence, $P_\bullet \otimes_S T \to \Omega^1 S \otimes_S T$ is a quasi-isomorphism because $P_\bullet$ is right split, and $T \otimes_S P_\bullet \otimes_S T \to T \otimes_S \Omega^1 S \otimes_S T$ is a quasi-isomorphism because T is right projective. Moreover,

$$T \otimes_S P_\bullet \otimes_S T = T \otimes_k T^{\mathrm{op}} \otimes_{S \otimes_k S^{\mathrm{op}}} P_\bullet$$

is projective because $P_\bullet$ is. So, the first term from the left in the exact sequence (10.15) has projective dimension at most $d - 1$. Next, we consider the last term of (10.15). By Lemma 10.13, there is an S^e-projective resolution $P'_\bullet \overset{\sim}{\twoheadrightarrow} M$ of length $\leq 2d$, which is split both as a complex of right and of left modules. Then, $P'_\bullet \otimes_S T \to M \otimes_S T$ is quasi-isomorphism, and therefore so is $T \otimes_S P'_\bullet \otimes_S T \to T \otimes_S M \otimes_S T$, since T_S is projective. This concludes the proof. ∎

Proposition 10.16. *Let (G, E, ϕ_c) be a twisted EP tuple. Assume that E is finite and regular and that G acts trivially on E^0. Further, assume that, for every $v \in E^0$, R_v contains a field k_v and has finite projective dimension as a left $R_v \otimes_k R_v^{\mathrm{op}}$-module. Then, $L(G, E, \phi_c)$ is a regular ring.*

Proof. Let $k = \bigoplus_{v \in E^0} k_v$; then, k is semisimple, and the hypothesis implies that R has finite projective dimension as an $R \otimes_k R^{\mathrm{op}}$-module. Since X is right-projective, Lemma 10.14 tells us that $P(G, E, \phi_c)$ is regular. Hence, $L(G, E, \phi_c)$ is regular by Corollary 10.12. ∎

Corollary 10.17. *Let ℓ be a field, and let (A, B, C) be a twisted Katsura triple. Then, $\mathcal{O}^C_{A,B}$ is a regular ring.*

Proof. Let $v \in E^0$, and let $L = \ell[t, t^{-1}]$. As shown in the proof of Lemma 9.1, R_v is either L or a product of copies of ℓ. Thus, R_v satisfies the hypothesis of Proposition 10.16. ∎

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