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Compressible fluid limit for smooth solutions to the Landau equation

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Abstract. Although the compressible fluid limit of the Boltzmann equation with cutoff has been extensively investigated in Caflisch [Comm. Pure Appl. Math. 33 (1980), 651–666] and Guo, Jang, and Jiang [Comm. Pure Appl. Math. 63 (2010), 337–361], obtaining analogous results in the case of the angular non-cutoff or even in the grazing limit which gives the Landau equation, still remains largely open, essentially due to the velocity diffusion effect of the collision operator such that L^∞ estimates are hard to obtain without using Sobolev embeddings. In this paper, we are concerned with the compressible Euler and acoustic limits of the Landau equation for Coulomb potentials in the whole space. Specifically, over any finite time interval where the full compressible Euler system admits a smooth solution around constant states, we construct a unique solution in a high-order weighted Sobolev space for the Landau equation with suitable initial data and also show the uniform estimates independent of the small Knudsen number $\varepsilon > 0$, yielding the $O(\varepsilon)$ convergence of the Landau solution to the local Maxwellian whose fluid quantities are the given Euler solution. Moreover, the acoustic limit for smooth solutions to the Landau equation in an optimal scaling is also established. For the proof, by using the macro-micro decomposition around local Maxwellians, together with techniques for viscous compressible fluid and properties of Burnett functions, we design an ε-dependent energy functional to capture the dissipation in the compressible fluid limit with the feature that only the highest-order derivatives are most singular.

Contents

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1. Introduction

In plasma physics, the spatially inhomogeneous Landau equation is a fundamental mathematical model at the kinetic level. It is used to describe the time evolution of the unknown density distribution function $F = F(t, x, v) \geq 0$ of particles in plasma with space position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ at time $t \geq 0$, written as

$$\partial_t F + v \cdot \nabla_x F = \frac{1}{\varepsilon} Q(F, F). \tag{1.1}$$

Here, the dimensionless parameter $\varepsilon > 0$ is small and reciprocal to logarithm of the Debye shielding length, and it plays the same role as the Knudsen number in Boltzmann theory; cf. [11, 61]. We have omitted the explicit dependence of the solution $F(t, x, v)$ on ε for brevity whenever there is no confusion. The Landau collision operator $Q(\cdot, \cdot)$ is a bilinear integro-differential operator acting only on velocity variables of the form

$$Q(F_1, F_2)(v) = \nabla_v \cdot \int_{\mathbb{R}^3} \Phi(v - v_*)\big\{F_1(v_*)\nabla_v F_2(v) - \nabla_{v_*} F_1(v_*) F_2(v)\big\}\, dv_*,$$

where for the Landau collision kernel $\Phi(\xi) = [\Phi_{ij}(\xi)]$ with $\xi = v - v_*$ (cf. [28, 38]), we consider only the case of the physically most realistic Coulomb interactions throughout the paper, namely,

$$\Phi_{ij}(\xi) = \frac{1}{|\xi|}\Big(\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2}\Big), \quad 1 \leq i, j \leq 3,$$

with δ_{ij} being the Kronecker delta.

On the other hand, the hydrodynamic description for the motion of plasmas at the fluid level is also given by the compressible Euler system

$$\begin{cases} \partial_t \bar{\rho} + \nabla_x \cdot (\bar{\rho}\bar{u}) = 0, \\ \partial_t (\bar{\rho}\bar{u}) + \nabla_x \cdot (\bar{\rho}\bar{u} \otimes \bar{u}) + \nabla_x \bar{p} = 0, \\ \partial_t \Big[\bar{\rho}\Big(\bar{e} + \frac{1}{2}|\bar{u}|^2\Big)\Big] + \nabla_x \cdot \Big[\bar{\rho}\bar{u}\Big(\bar{e} + \frac{1}{2}|\bar{u}|^2\Big)\Big] + \nabla_x \cdot (\bar{p}\bar{u}) = 0, \end{cases} \tag{1.2}$$

with the equation of state $\bar{p} = R\bar{\rho}\bar{\theta}$. These are the local conservation laws of mass, momentum, and energy. Here $\bar{e} = \bar{e}(t, x) > 0$ is the internal energy which is related to the temperature $\bar{\theta} = \bar{\theta}(t, x)$ by $\bar{e} = \frac{3}{2} R\bar{\theta} = \bar{\theta}$, with the gas constant $R = \frac{2}{3}$ taken for convenience, and $\bar{\rho} = \bar{\rho}(t, x)$, $\bar{u} = \bar{u}(t, x)$ are the mass density and bulk velocity, respectively.

It is well known that the compressible Euler system (1.2) can be formally derived from the Landau equation (1.1) through the first-order approximation to the famous Hilbert expansion when the Knudsen number $\varepsilon > 0$ is close to zero, which is similar to the case of the Boltzmann equation; see [23, 27, 60]. The rigorous mathematical justification of establishing such a limit in a general setting, particularly for general initial data in three dimensions, is still a challenging subject in kinetic theory, although it has been extensively studied for the Boltzmann equation with cutoff; see for instance [7, 34, 35, 49, 58, 65] and the references therein.

In what follows we review some specific results on the compressible Euler limit of the Boltzmann equation with cutoff so as to make a comparison with the Landau case later. Note that there are few analogous results on the issue for the non-cutoff Boltzmann equation. First of all, based on the truncated Hilbert expansion, Caflisch [7] showed that, if the compressible Euler system has a smooth solution which exists up to a finite time, then there exist corresponding solutions to the Boltzmann equation with a zero initial condition for the remainder term in the same time interval such that the Boltzmann solution converges to a smooth local Maxwellian whose fluid dynamical parameters satisfy the compressible Euler system as $\varepsilon \to 0$. Due to this zero initial condition, the obtained Boltzmann solution may be negative. The result was extended later by Lachowicz [49] to the case allowing for the initial layer. Following Caflisch's strategy, together with a new L^2–L^∞ approach developed in [30], Guo–Jang–Jiang [34, 35] removed the restriction with zero initial condition so that the positivity of the Boltzmann solution can be guaranteed. Recently, Guo–Huang–Wang [31] and Jiang–Luo–Tang [44, 45] made significant progress on the topic by making use of the L^2–L^∞ interplay technique to generalize the results of [7, 34, 35] to the half-space problem with different types of boundary conditions, including specular reflection, diffuse reflection, and even the mixed Maxwell reflection. We also mention recent progress by Guo–Xiao [36] for an application of the L^2–L^∞ approach to the global Hilbert expansion to the relativistic Vlasov–Maxwell–Boltzmann system.

The compressible Euler limit of the Boltzmann equation is also studied in other contexts. Based on the abstract Cauchy–Kovalevskaya theorem and the spectral analysis of the semigroup generated by the linearized Boltzmann equation, Nishida [58] constructed the local solution of the cutoff Boltzmann equation in the analytic framework and proved that the solution tends to the solution of the Euler system as $\varepsilon \to 0$. Later, Ukai–Asano [65] improved Nishida's result by using the classical contraction mapping principle on a space with a time-dependent analytic norm, and also considered the case with the initial layer. Moreover, in the special setting of one space dimension, regarding the hydrodynamic limit of the Boltzmann equation with cutoff to the compressible Euler system, which admits specific solutions of basic wave patterns, such as rarefaction waves, contact discontinuities, and shock waves, there have been extensive studies by [40, 41, 69, 70, 72] and the references therein.

As is well known, the linearization of compressible Euler equations around constant states gives the acoustic system, see (3.23), to be specified later. For inviscid compressible fluids, the acoustic system could be the simplest one describing essentially the wave propagation, which can be formally derived from the Boltzmann equation. Under Grad's angular cutoff assumption, Bardos–Golse–Levermore [5] proved the convergence in the acoustic limit from the DiPerna–Lions [15] renormalized solutions of the Boltzmann equation with a restriction on the size of fluctuations. The restriction was relaxed later by Golse–Levermore [25] and Jiang–Levermore–Masmoudi [43], and finally removed by

Guo–Jang–Jiang [35] via the Hilbert expansion with the help of the L^2–L^∞ interplay approach as mentioned before. Concerning the limit to other fluid equations from the Boltzmann equation with cutoff, see [3, 4, 6, 16, 21, 22, 26, 29, 33, 42, 46, 51, 57] and the references cited therein. We also mention the recent work [59] on the incompressible fluid limit from the Landau equation.

However, despite a lot of great progress mentioned above, the hydrodynamic limit to the compressible Euler or acoustic system for the Landau equation has remained largely open and a similar situation occurs for the Boltzmann equation in the non-cutoff case. The main reason is that those long-range collision operators expose the velocity diffusion property so that the strategy in [7, 35, 58] cannot be directly adopted. In particular, trouble arises from the action of the transport operator on local Maxwellians, inducing large velocity growth in the L^2 framework; cf. [35]. To overcome it, a robust idea is to combine L^2 estimates with the velocity-weighted L^∞ estimates which strongly rely on the Grad's splitting of the linearized Boltzmann operator with cutoff. For either the non-cutoff Boltzmann or Landau equation, it seems still very hard to obtain any L^∞ estimate uniform in the fluid limit $\varepsilon \to 0$ without using Sobolev imbedding, although one may observe several recent important results [2, 32, 48] on the global existence of low-regularity solutions in L^∞ space around global Maxwellians for $\varepsilon = 1$. Therefore, one has to develop some new ideas to deal with the problem whenever the direct L^∞ estimates are not available.

Finally, we further mention some early and recent results on the global existence and large-time behavior for the Landau equation (1.1) with $\varepsilon = 1$, for instance, global existence of weak solutions by Lions [50] and Villani [67, 68], the grazing collision limit of the non-cutoff Boltzmann to the Landau equation by Desvillettes [14] and Alexandre–Villani [1], spectrum analysis by Degond–Lemou [13], global existence of classical solutions near global Maxwellians by Guo [28] for the torus and Hsiao–Yu [39] for the whole space, large-time behavior of classical solutions near global Maxwellians by Strain–Guo [62, 63], and see also [8–10, 17], global classical solutions near vacuum by Luk [55], regularity of solutions by Golse–Imbert–Mouhot–Vasseur [24] and Henderson–Snelson [37]. See also more recent works [18–20, 71] by the authors of this paper for the one-dimensional Landau equation around local Maxwellians with rarefaction wave and contact wave.

In this paper, we will construct a unique solution in a high-order weighted Sobolev space for the Landau equation with suitable initial data over any finite time interval where the full compressible Euler system admits a smooth solution around constant states. We will obtain uniform estimates independent of the small Knudsen number $\varepsilon > 0$, which yields $O(\varepsilon)$ convergence of the Landau solution to the local Maxwellian whose fluid quantities are the given Euler solution. In the meantime, we also will establish the acoustic limit for smooth solutions to the Landau equation in optimal scaling. We believe that the same results hold for the non-cutoff Boltzmann equation even with soft potentials. For the proof, a key point is to use the macro-micro decomposition around local Maxwellians to design the ε-dependent temporal energy functional and its corresponding energy dissipation functional, such that uniform estimates can be obtained under the smallness assumption. More details will be specified later on.

The rest of this paper is organized as follows. In Section 2 we present the macro-micro decomposition for the Landau equation in order to study the compressible fluid limit for smooth solutions to the Landau equation. In Section 3 we state two main results of this paper, namely, Theorems 3.4 and 3.9 for the compressible Euler limit and the acoustic limit, respectively. In the meantime, we also list key points for the strategy of the proof throughout the paper for the convenience of readers. To advance the proof of the main results, in Section 4 we first prepare some basic estimates. Section 5 is the main part of the proof for establishing the a priori estimates of both the fluid part and the kinetic part. The proofs of Theorems 3.4 and 3.9 are given briefly in Section 6. In Appendix A we give details of the derivation of identity (5.7) for completeness.

Notation. Throughout this paper, generic positive constants are denoted by either c or C, varying from line to line, and they are independent of other small parameters such as the Knudsen number and the amplitude of Euler solutions. The notation $\langle\cdot,\cdot\rangle$ denotes the standard L^2 inner product in $\mathbb{R}^3_v$ with its corresponding L^2-norm $|\cdot|_2$, and $(\cdot,\cdot)$ denotes the L^2 inner product in either $\mathbb{R}^3_x$ or $\mathbb{R}^3_x\times\mathbb{R}^3_v$, with its corresponding L^2-norm $\|\cdot\|$. We use the standard notation $H^k(\mathbb{R}^3_x)$ to denote the Sobolev space $W^{k,2}(\mathbb{R}^3_x)$ with its corresponding norm $\|\cdot\|_{H^k}$, and also use $\|\cdot\|_{L^p}$ to denote the L^p_x-norm with $p\in[1,+\infty]$. The norm of $\nabla^k_x f$ means the sum of the norms of functions $\partial^\alpha f$ with $|\alpha|=k$. Let α and β be multi-indices $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and $\beta=(\beta_1,\beta_2,\beta_3)$, respectively. As in [28], it is convenient to denote

$$\partial^\alpha_\beta=\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\partial^{\beta_1}_{v_1}\partial^{\beta_2}_{v_2}\partial^{\beta_3}_{v_3}.$$

If each component of β is not greater than the corresponding one of $\bar\beta$, we use the standard notation $\beta\le\bar\beta$. And $\beta<\bar\beta$ means that $\beta\le\bar\beta$ and $|\beta|<|\bar\beta|$. The constant $C^{\bar\beta}_\beta$ is the usual binomial coefficient.

2. Macro-micro decomposition

In this section we will present the macro-micro decomposition for the Landau equation (1.1) in order to study the compressible fluid limit for smooth solutions to the Landau equation.

We first recall a basic property of the Landau collision operator. It is well known that the Landau collision operator admits five collision invariants:

$$\psi_0(v)=1,\quad \psi_i(v)=v_i\ \ (i=1,2,3),\quad \psi_4(v)=\frac{1}{2}|v|^2,$$

namely, it holds that

$$\int_{\mathbb{R}^3}\psi_i(v)Q(F,F)\,dv=0,\quad \text{for } i=0,1,2,3,4. \tag{2.1}$$

The macro-micro decomposition of the solution with respect to the local Maxwellian was initiated by Liu–Yu [54] and developed by Liu–Yang–Yu [52] in the context of

Boltzmann theory, and it can be analogously carried over to the Landau equation. Indeed, associated with a solution $F(t,x,v)$ to the Landau equation (1.1), we introduce five macroscopic (fluid) quantities: the mass density $\rho(t,x)>0$, momentum $\rho(t,x)u(t,x)$, and energy density $e(t,x)+\frac{1}{2}|u(t,x)|^2$, given as

$$\begin{cases} \rho(t,x) \equiv \int_{\mathbb{R}^3} \psi_0(v) F(t,x,v)\,dv, & \\ \rho(t,x)u_i(t,x) \equiv \int_{\mathbb{R}^3} \psi_i(v) F(t,x,v)\,dv, & \text{for } i=1,2,3, \\ \rho(t,x)\Big[e(t,x)+\frac{1}{2}|u(t,x)|^2\Big] \equiv \int_{\mathbb{R}^3} \psi_4(v) F(t,x,v)\,dv, & \end{cases} \tag{2.2}$$

and the corresponding local Maxwellian,

$$M \equiv M_{[\rho,u,\theta]}(t,x,v) := \frac{\rho(t,x)}{\sqrt{(2\pi R\theta(t,x))^3}} \exp\Big\{-\frac{|v-u(t,x)|^2}{2R\theta(t,x)}\Big\}. \tag{2.3}$$

Here, $e(t,x)>0$ is the internal energy, which is related to the temperature $\theta(t,x)$ by $e=\frac{3}{2}R\theta=\theta$, and $u(t,x)=(u_1,u_2,u_3)(t,x)$ is the bulk velocity.

Note that the L^2 inner product in $v\in\mathbb{R}^3$ is denoted by

$$\langle h,g\rangle = \int_{\mathbb{R}^3} h(v)g(v)\,dv.$$

Then the macroscopic space is spanned by the following five orthonormal basis functions:

$$\begin{cases} \chi_0(v) = \frac{1}{\sqrt{\rho}} M, & \\ \chi_i(v) = \frac{v_i-u_i}{\sqrt{R\rho\theta}} M, & \text{for } i=1,2,3, \\ \chi_4(v) = \frac{1}{\sqrt{6\rho}}\Big(\frac{|v-u|^2}{R\theta}-3\Big)M, & \\ \Big\langle \chi_i, \frac{\chi_j}{M}\Big\rangle = \delta_{ij}, & \text{for } i,j=0,1,2,3,4. \end{cases} \tag{2.4}$$

In view of the orthonormal basis above, we define the macroscopic projection P_0 and microscopic projection P_1 as

$$P_0 h \equiv \sum_{i=0}^{4} \Big\langle h, \frac{\chi_i}{M}\Big\rangle \chi_i, \quad P_1 h \equiv h - P_0 h, \tag{2.5}$$

where the operators P_0 and P_1 are orthogonal projections, that is,

$$P_0P_0 = P_0, \quad P_1P_1 = P_1, \quad P_1P_0 = P_0P_1 = 0.$$

A function $h(v)$ is called microscopic or non-fluid if

$$\langle h(v), \psi_i(v)\rangle = 0, \quad \text{for } i=0,1,2,3,4. \tag{2.6}$$

Using the notation above, the solution $F = F(t, x, v)$ of the Landau equation (1.1) can be decomposed into the macroscopic (fluid) part, i.e. the local Maxwellian $M = M(t, x, v)$ defined in (2.3), and the microscopic (non-fluid) part, i.e. $G = G(t, x, v)$:

$$F = M + G, \quad P_0 F = M, \quad P_1 F = G. \tag{2.7}$$

Then the Landau equation (1.1) can be rewritten as

$$\partial_t (M + G) + v \cdot \nabla_x (M + G) = \frac{1}{\varepsilon} L_M G + \frac{1}{\varepsilon} Q(G, G), \tag{2.8}$$

due to the fact that $Q(M, M) = 0$. Here, L_M is the linearized Landau collision operator with respect to the local Maxwellian M, given by

$$L_M h := Q(h, M) + Q(M, h), \tag{2.9}$$

and its null space $\mathcal{N}$ is spanned by the macroscopic variables $\chi_i(v)$ $(i = 0, 1, 2, 3, 4)$.

Let us now decompose the Landau equation into the macroscopic system and microscopic system. Multiplying the Landau equation (2.8) by the collision invariants $\psi_i(v)$ $(i = 0, 1, 2, 3, 4)$ and integrating the resulting equations with respect to v over $\mathbb{R}^3$, one has the following macroscopic (fluid) system:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x p = -\displaystyle\int_{\mathbb{R}^3} v \otimes v \cdot \nabla_x G \, dv, \\ \partial_t \Big[\rho\Big(\theta + \frac{1}{2}|u|^2\Big)\Big] + \nabla_x \cdot \Big[\rho u \Big(\theta + \frac{1}{2}|u|^2\Big) + p u\Big] = -\displaystyle\int_{\mathbb{R}^3} \frac{1}{2}|v|^2 v \cdot \nabla_x G \, dv, \end{cases} \tag{2.10}$$

with the pressure $p = R\rho\theta = \frac{2}{3}\rho\theta$. Here we have used (2.1), (2.2), and the fact that $\partial_t G$ is microscopic by (2.6).

Since the projection of collision terms under P_0 is zero, it holds that $P_1 L_M G = L_M G$ and $P_1 Q(G, G) = Q(G, G)$. It is also clear that $P_1 \partial_t M = 0$ and $P_1 \partial_t G = \partial_t G$ in terms of (2.6) and (2.5). With these facts, one has the following microscopic (non-fluid) system:

$$\partial_t G + P_1(v \cdot \nabla_x G) + P_1(v \cdot \nabla_x M) = \frac{1}{\varepsilon} L_M G + \frac{1}{\varepsilon} Q(G, G), \tag{2.11}$$

by applying the microscopic operator P_1 to the Landau equation (2.8). Since L_M is invertible on $\mathcal{N}^{\perp}$, we can rewrite (2.11) to present G as

$$G = \varepsilon L_M^{-1}[P_1(v \cdot \nabla_x M)] + L_M^{-1}\Theta, \tag{2.12}$$

with

$$\Theta := \varepsilon \partial_t G + \varepsilon P_1(v \cdot \nabla_x G) - Q(G, G). \tag{2.13}$$

Plugging (2.12) into (2.10) and using the two identities

$$-\int_{\mathbb{R}^3} v_i v \cdot \nabla_x L_M^{-1}[P_1(v \cdot \nabla_x M)]\, dv \equiv \sum_{j=1}^{3} \partial_{x_j}[\mu(\theta) D_{ij}], \quad i = 1,2,3,$$

$$-\int_{\mathbb{R}^3} \frac{1}{2}|v|^2 v \cdot \nabla_x L_M^{-1}[P_1(v \cdot \nabla_x M)]\, dv \equiv \sum_{j=1}^{3} \partial_{x_j}(\kappa(\theta)\partial_{x_j}\theta) + \sum_{i,j=1}^{3} \partial_{x_j}[\mu(\theta) u_i D_{ij}],$$

with the viscous stress tensor $D = [D_{ij}]_{1\le i,j\le 3}$ given by

$$D_{ij} = \partial_{x_j} u_i + \partial_{x_i} u_j - \frac{2}{3}\delta_{ij} \nabla_x \cdot u, \tag{2.14}$$

we further obtain the compressible Navier–Stokes-type equations

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t(\rho u_i) + \nabla_x \cdot (\rho u_i u) + \partial_{x_i} p \\ \quad = \varepsilon \displaystyle\sum_{j=1}^{3} \partial_{x_j}[\mu(\theta) D_{ij}] - \int_{\mathbb{R}^3} v_i (v \cdot \nabla_x L_M^{-1}\Theta)\, dv, \quad i = 1,2,3, \\ \partial_t\Big[\rho\Big(\theta + \frac{1}{2}|u|^2\Big)\Big] + \nabla_x \cdot \Big[\rho u\Big(\theta + \frac{1}{2}|u|^2\Big) + pu\Big] \\ \quad = \varepsilon \displaystyle\sum_{j=1}^{3} \partial_{x_j}(\kappa(\theta)\partial_{x_j}\theta) + \varepsilon \sum_{i,j=1}^{3} \partial_{x_j}[\mu(\theta) u_i D_{ij}] \\ \qquad - \displaystyle\int_{\mathbb{R}^3} \frac{1}{2}|v|^2 v \cdot \nabla_x L_M^{-1}\Theta\, dv. \end{cases} \tag{2.15}$$

Here $\mu(\theta) > 0$ and $\kappa(\theta) > 0$ are the viscosity coefficient and heat conductivity coefficient respectively, and they are smooth functions depending only on the temperature θ, represented by

$$\mu(\theta) = -R\theta \int_{\mathbb{R}^3} \widehat{B}_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big) B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\, dv > 0, \quad i \ne j,$$

$$\kappa(\theta) = -R^2\theta \int_{\mathbb{R}^3} \hat{A}_j\Big(\frac{v-u}{\sqrt{R\theta}}\Big) A_j\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\, dv > 0,$$

where $\hat{A}_j(\cdot)$ and $\widehat{B}_{ij}(\cdot)$ are Burnett functions, cf. [3,4,20,29,66], defined by

$$\hat{A}_j(v) = \frac{|v|^2 - 5}{2} v_j \quad \text{and} \quad \widehat{B}_{ij}(v) = v_i v_j - \frac{1}{3}\delta_{ij}|v|^2, \quad \text{for } i,j = 1,2,3, \tag{2.16}$$

and $A_j(\cdot)$ and $B_{ij}(\cdot)$ satisfy $P_0 A_j(\cdot) = 0$ and $P_0 B_{ij}(\cdot) = 0$, given by

$$A_j(v) = L_M^{-1}[\hat{A}_j(v) M] \quad \text{and} \quad B_{ij}(v) = L_M^{-1}[\widehat{B}_{ij}(v) M]. \tag{2.17}$$

Some elementary properties of the Burnett functions are summarized in the following lemma; cf. [3,4,20,29,66].

Lemma 2.1. *The Burnett functions have the following properties:*

- $-\langle \hat{A}_i, A_i \rangle$ *is positive and independent of* i*;*
- $\langle \hat{A}_i, A_j \rangle = 0$ *for any* $i \neq j$*;* $\langle \hat{A}_i, B_{jk} \rangle = 0$ *for any* i, j, k*;*
- $\langle \widehat{B}_{ij}, B_{kj} \rangle = \langle \widehat{B}_{kl}, B_{ij} \rangle = \langle \widehat{B}_{ji}, B_{kj} \rangle$*, which is independent of* i, j*, for fixed* k, l*;*
- $-\langle \widehat{B}_{ij}, B_{ij} \rangle$ *is positive and independent of* i, j *when* $i \neq j$*;*
- $\langle \widehat{B}_{ii}, B_{jj} \rangle$ *is positive and independent of* i, j *when* $i \neq j$*;*
- $-\langle \widehat{B}_{ii}, B_{ii} \rangle$ *is positive and independent of* i*;*
- $\langle \widehat{B}_{ij}, B_{kl} \rangle = 0$ *unless either* $(i, j) = (k, l)$ *or* (l, k)*, or* $i = j$ *and* $k = l$*;*
- $\langle \widehat{B}_{ii}, B_{ii} \rangle - \langle \widehat{B}_{ii}, B_{jj} \rangle = 2\langle \widehat{B}_{ij}, B_{ij} \rangle$ *holds for any* $i \neq j$*.*

To make a conclusion, we have decomposed the Landau equation (1.1) as the coupling of the viscous compressible fluid-type system (2.15) and the microscopic equation (2.11), which is similar to the case of the Boltzmann equation; cf. [52]. In this way, one advantage is that the viscosity and heat conductivity coefficients can be expressed explicitly so that the energy analysis in the context of the viscous compressible fluid can be applied to capture the dissipation of the fluid part. The other advantage is that the non-linear term $Q(G, G)$ in (2.11) depends only on the microscopic part G so that the trilinear estimate is easily obtained without treating the fluid part, which is essentially different from the one in [28] with respect to a given global Maxwellian.

As pointed out in [52], when the Knudsen number ε and the microscopic part G are set to zero, the system (2.15) becomes the compressible Euler system. If only the microscopic part G is set to zero, the system (2.15) becomes the compressible Navier–Stokes system with the parameter ε. These fluid systems can also be derived from the Boltzmann (Landau) equation through the Hilbert and Chapman–Enskog expansions; cf. [12]. This means that the macro-micro decomposition (2.7) in some sense can be viewed as a unification of the classical Hilbert and Chapman–Enskog expansions up to the second-order approximation. Therefore, this decomposition gives a good framework for rigorously deriving the compressible fluid system from collisional kinetic equations.

3. Main results

In this section, we employ the macro-micro decomposition introduced in the previous section to establish our main results on the compressible fluid limit for smooth solutions to the Landau equation.

3.1. Compressible Euler limit

The first goal of the present paper is to establish the compressible Euler limit of the Landau equation. Precisely, we will construct the solution $F^\varepsilon(t, x, v)$ of the Landau equation (1.1) which converges to a local Maxwellian

$$\bar{M} \equiv M_{[\bar{\rho},\bar{u},\bar{\theta}]}(t,x,v) := \frac{\bar{\rho}(t,x)}{\sqrt{(2\pi R\bar{\theta}(t,x))^3}} \exp\Big\{-\frac{|v-\bar{u}(t,x)|^2}{2R\bar{\theta}(t,x)}\Big\}, \tag{3.1}$$

as the Knudsen number $\varepsilon > 0$ tends to zero, where the fluid parameters $(\bar{\rho},\bar{u},\bar{\theta})(t,x)$ satisfy the compressible Euler system (1.2).

3.1.1. Smooth solutions for the Euler system. To solve (1.2), we supplement it with prescribed initial data

$$(\bar{\rho},\bar{u},\bar{\theta})(0,x) = (\bar{\rho}_0,\bar{u}_0,\bar{\theta}_0)(x). \tag{3.2}$$

Then we have the following existence result.

Proposition 3.1. *Let $\tau > 0$ be a fixed finite time; then there is a sufficiently small constant $\eta_\tau > 0$ and a constant $C_\tau > 0$ such that if the initial data $(\bar{\rho}_0,\bar{u}_0,\bar{\theta}_0)(x)$ around the constant state $(1,0,3/2)$ satisfies*

$$\eta_0 := \Big\|\Big(\bar{\rho}_0(x)-1,\bar{u}_0(x),\bar{\theta}_0(x)-\frac{3}{2}\Big)\Big\|_{H^k} \le \eta_\tau$$

for $k \ge N+2$ with integer $N \ge 3$ as in (3.17), then the Cauchy problem on the compressible Euler system (1.2) and (3.2) admits a unique smooth solution $(\bar{\rho},\bar{u},\bar{\theta})(t,x)$ over $[0,\tau]\times\mathbb{R}^3$ such that

$$\inf_{t\in[0,\tau],x\in\mathbb{R}^3} \bar{\rho}(t,x) > 0, \qquad \inf_{t\in[0,\tau],x\in\mathbb{R}^3} \bar{\theta}(t,x) > 0,$$

and the following estimate holds true:

$$\sup_{t\in[0,\tau]} \Big\|\Big(\bar{\rho}(t,x)-1,\bar{u}(t,x),\bar{\theta}(t,x)-\frac{3}{2}\Big)\Big\|_{H^k} \le C_\tau\eta_0. \tag{3.3}$$

The proof of Proposition 3.1 can be obtained by a straightforward modification of the arguments as in [35, Lemmas 3.1 and 3.2], so we omit the details for brevity. We remark that for any given $\tau > 0$ we always let $\eta_\tau > 0$ be small enough such that $C_\tau\eta_0 > 0$ can be sufficiently small since the smallness of $C_\tau\eta_0$ is crucially used to close the a priori assumptions (4.1).

3.1.2. Reformulated system. Let us now define the macroscopic perturbation around the smooth solution $(\bar{\rho},\bar{u},\bar{\theta})(t,x)$ by

$$\begin{cases} \tilde{\rho}(t,x) = \rho(t,x) - \bar{\rho}(t,x), \\ \tilde{u}(t,x) = u(t,x) - \bar{u}(t,x), \\ \tilde{\theta}(t,x) = \theta(t,x) - \bar{\theta}(t,x), \end{cases} \tag{3.4}$$

where $(\rho, u, \theta)(t,x)$ satisfies (2.15), defined by (2.2). Subtracting (1.2) from system (2.15), one obtains the system for the perturbation $(\tilde{\rho}, \tilde{u}, \tilde{\theta})(t,x)$ as follows:

$$\begin{cases} \partial_t \tilde{\rho} + u \cdot \nabla_x \tilde{\rho} + \bar{\rho} \nabla_x \cdot \tilde{u} + \tilde{u} \cdot \nabla_x \bar{\rho} + \tilde{\rho} \nabla_x \cdot u = 0, \\ \partial_t \tilde{u}_i + u \cdot \nabla_x \tilde{u}_i + \frac{2\bar{\theta}}{3\bar{\rho}} \partial_{x_i} \tilde{\rho} + \frac{2}{3} \partial_{x_i} \tilde{\theta} + \tilde{u} \cdot \nabla_x \bar{u}_i \\ \quad + \frac{2}{3} \Big(\frac{\theta}{\rho} - \frac{\bar{\theta}}{\bar{\rho}} \Big) \partial_{x_i} \rho \\ \quad = \varepsilon \frac{1}{\rho} \sum_{j=1}^{3} \partial_{x_j} [\mu(\theta) D_{ij}] - \frac{1}{\rho} \int_{\mathbb{R}^3} v_i v \cdot \nabla_x L_M^{-1} \Theta \, dv, \quad i = 1,2,3, \\ \partial_t \tilde{\theta} + u \cdot \nabla_x \tilde{\theta} + \frac{2}{3} \bar{\theta} \nabla_x \cdot \tilde{u} + \tilde{u} \cdot \nabla_x \bar{\theta} + \frac{2}{3} \tilde{\theta} \nabla_x \cdot u \\ \quad = \varepsilon \frac{1}{\rho} \sum_{j=1}^{3} \partial_{x_j} (\kappa(\theta) \partial_{x_j} \theta) + \varepsilon \frac{1}{\rho} \sum_{i,j=1}^{3} \mu(\theta) \partial_{x_j} u_i D_{ij} \\ \quad - \frac{1}{\rho} \int_{\mathbb{R}^3} \frac{1}{2} |v|^2 v \cdot \nabla_x L_M^{-1} \Theta \, dv \\ \quad + \frac{1}{\rho} u \cdot \int_{\mathbb{R}^3} v \otimes v \cdot \nabla_x L_M^{-1} \Theta \, dv. \end{cases} \tag{3.5}$$

Throughout the paper we fix a normalized global Maxwellian with the fluid constant state $(1, 0, 3/2)$,

$$\mu \equiv M_{[1,0,\frac{3}{2}]}(v) := (2\pi)^{-\frac{3}{2}} \exp\Big(-\frac{|v|^2}{2} \Big) \tag{3.6}$$

as the reference equilibrium state. Then we define the microscopic perturbation $f(t,x,v)$ by

$$\sqrt{\mu} f(t,x,v) = G(t,x,v) - \overline{G}(t,x,v), \tag{3.7}$$

where the function $\overline{G}(t,x,v)$ is defined as

$$\overline{G}(t,x,v) \equiv \varepsilon L_M^{-1} P_1 \Big\{ v \cdot \Big(\frac{|v-u|^2 \nabla_x \bar{\theta}}{2R\theta^2} + \frac{(v-u) \cdot \nabla_x \bar{u}}{R\theta} \Big) M \Big\}, \tag{3.8}$$

which corresponds to the first-order correction term in the Chapman–Enskog expansion; see (2.12) and (2.15). Note that a similar correction function $\overline{G}(t,x,v)$ was first introduced by Liu–Yang–Yu–Zhao [53] for the stability of the rarefaction wave to the one-dimensional Boltzmann equation with cutoff. Some detailed comments on introducing the subtraction $G - \overline{G}$ in our case will be given later on.

To derive the equation of the microscopic perturbation $f(t,x,v)$ in (3.7), by the properties of P_1 in (2.5) and the definition of $\overline{G}$, we have from a direct computation that

$$\begin{aligned} P_1(v \cdot \nabla_x M) &= P_1 \Big\{ v \cdot (\frac{|v-u|^2 \nabla_x \theta}{2R\theta^2} + \frac{(v-u) \cdot \nabla_x u}{R\theta}) M \Big\} \\ &= P_1 \Big\{ v \cdot (\frac{|v-u|^2 \nabla_x \tilde{\theta}}{2R\theta^2} + \frac{(v-u) \cdot \nabla_x \tilde{u}}{R\theta}) M \Big\} + \frac{1}{\varepsilon} L_M \overline{G}. \end{aligned} \tag{3.9}$$

With this and the fact that $L_M G = L_M(\overline{G} + \sqrt{\mu} f)$, we can rewrite (2.11) as

$$\partial_t G + P_1(v \cdot \nabla_x G) + P_1\Big\{v \cdot \Big(\frac{|v-u|^2 \nabla_x \tilde{\theta}}{2R\theta^2} + \frac{(v-u)\cdot \nabla_x \tilde{u}}{R\theta}\Big) M\Big\}$$
$$= \frac{1}{\varepsilon} L_M(\sqrt{\mu} f) + \frac{1}{\varepsilon} Q(G,G). \tag{3.10}$$

Inspired by [28], we denote

$$\Gamma(h,g) := \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} h, \sqrt{\mu} g), \quad \mathcal{L} h := \Gamma(h, \sqrt{\mu}) + \Gamma(\sqrt{\mu}, h), \tag{3.11}$$

which together with (2.9) immediately gives rise to

$$\frac{1}{\sqrt{\mu}} L_M(\sqrt{\mu} f) = \frac{1}{\sqrt{\mu}}\{Q(M, \sqrt{\mu} f) + Q(\sqrt{\mu} f, M)\}$$
$$= \mathcal{L} f + \Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, f\Big) + \Gamma\Big(f, \frac{M-\mu}{\sqrt{\mu}}\Big). \tag{3.12}$$

With these and the fact that $G = \overline{G} + \sqrt{\mu} f$, (3.10) implies that f satisfies

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \mathcal{L} f + \frac{1}{\varepsilon} \Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, f\Big) + \frac{1}{\varepsilon} \Gamma\Big(f, \frac{M-\mu}{\sqrt{\mu}}\Big) + \frac{1}{\varepsilon} \Gamma\Big(\frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}}\Big)$$
$$+ \frac{P_0[v \cdot \nabla_x(\sqrt{\mu} f)]}{\sqrt{\mu}} - \frac{P_1(v \cdot \nabla_x \overline{G})}{\sqrt{\mu}} - \frac{\partial_t \overline{G}}{\sqrt{\mu}}$$
$$- \frac{1}{\sqrt{\mu}} P_1\Big\{v \cdot \Big(\frac{|v-u|^2 \nabla_x \tilde{\theta}}{2R\theta^2} + \frac{(v-u)\cdot \nabla_x \tilde{u}}{R\theta}\Big) M\Big\}. \tag{3.13}$$

Remark 3.2. It is well known that the linearized operator $\mathcal{L}$ in (3.13) is self-adjoint and non-positive definite, and its null space $\ker \mathcal{L}$ is spanned by the basis $\{\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}\}$; cf. [28].

Remark 3.3. It should be pointed out that the unknown $f(t,x,v)$ in (3.13) is purely microscopic since G and $\overline{G}$ are purely microscopic, namely $f(t,x,v) \in (\ker \mathcal{L})^{\perp}$. This is essentially different from the one in [28] since $f(t,x,v)$ used in [28] involves the macroscopic part. More comments will be given in Section 3.3.

Motivated by the fundamental work Guo [28], we introduce a velocity weight function as

$$w \equiv w(v) := \langle v \rangle^{-1}, \quad \langle v \rangle = \sqrt{1+|v|^2}. \tag{3.14}$$

This weight function is designed to deal with the velocity derivatives of the free streaming term $v \cdot \nabla_x f$; see (5.80). With (3.14), for $\ell \in \mathbb{R}$ we denote the weighted L^2 norms as

$$|f|_{2,\ell}^2 \equiv \int_{\mathbb{R}^3} w^{2\ell} |f|^2 \, dv, \quad \|f\|_{2,\ell}^2 \equiv \int_{\mathbb{R}^3} |f|_{2,\ell}^2 \, dx.$$

Corresponding to the reference global Maxwellian μ in (3.6), the Landau collision frequency is given by

$$\sigma^{ij}(v) := \Phi_{ij} * \mu = \int_{\mathbb{R}^3} \Phi_{ij}(v - v_*)\mu(v_*)\, dv_*, \tag{3.15}$$

where $[\sigma^{ij}(v)]_{1\le i,j\le 3}$ is a positive-definite self-adjoint matrix. In terms of linearization of the non-linear Landau operator around μ (cf. [28]), with (3.15), we define the weighted dissipation norms as

$$|f|_{\sigma,\ell}^2 \equiv \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} w^{2\ell}\Big\{\sigma^{ij}\partial_{v_i} f \partial_{v_j} f + \sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}|f|^2\Big\}\, dv, \quad \|f\|_{\sigma,\ell}^2 \equiv \int_{\mathbb{R}^3} |f|_{\sigma,\ell}^2\, dx,$$

where $|f|_\sigma = |f|_{\sigma,0}$ and $\|f\|_\sigma = \|f\|_{\sigma,0}$. Moreover, we have from [28, Corollary 1, p. 399] that

$$|f|_\sigma \approx |\langle v\rangle^{-\frac{1}{2}} f|_2 + \Big|\langle v\rangle^{-\frac{3}{2}}\nabla_v f \cdot \frac{v}{|v|}\Big|_2 + \Big|\langle v\rangle^{-\frac{1}{2}}\nabla_v f \times \frac{v}{|v|}\Big|_2. \tag{3.16}$$

The notation $A \approx B$ means that there exists $C > 1$ such that $C^{-1}B \le A \le CB$.

In order to prove the uniform-in-ε existence of smooth solutions for the Landau equation, a key point is to establish uniform energy estimates in the high-order Sobolev space. For this, we define the instant energy functional $\mathcal{E}_N(t)$ as

$$\begin{aligned}\mathcal{E}_N(t) :=& \sum_{|\alpha|\le N-1} \{\|\partial^\alpha(\tilde\rho,\tilde u,\tilde\theta)(t)\|^2 + \|\partial^\alpha f(t)\|^2\} + \sum_{\substack{|\alpha|+|\beta|\le N\\ |\beta|\ge 1}} \|\partial^\alpha_\beta f(t)\|^2_{2,|\beta|}\\ &+ \varepsilon^2 \sum_{|\alpha|=N} \{\|\partial^\alpha(\tilde\rho,\tilde u,\tilde\theta)(t)\|^2 + \|\partial^\alpha f(t)\|^2\}.\end{aligned} \tag{3.17}$$

Correspondingly, the dissipation energy functional $\mathcal{D}_N(t)$ is given by

$$\begin{aligned}\mathcal{D}_N(t) :=& \varepsilon \sum_{1\le|\alpha|\le N} \|\partial^\alpha(\tilde\rho,\tilde u,\tilde\theta)(t)\|^2 + \varepsilon \sum_{|\alpha|=N} \|\partial^\alpha f(t)\|^2_\sigma\\ &+ \frac{1}{\varepsilon}\sum_{|\alpha|\le N-1} \|\partial^\alpha f(t)\|^2_\sigma + \frac{1}{\varepsilon}\sum_{\substack{|\alpha|+|\beta|\le N\\ |\beta|\ge 1}} \|\partial^\alpha_\beta f(t)\|^2_{\sigma,|\beta|}.\end{aligned} \tag{3.18}$$

Throughout this article, we assume $N \ge 3$. A crucial feature of the above instant energy $\mathcal{E}_N(t)$ is that the highest Nth-order space derivatives are much more singular with respect to ε than those derivatives of order up to $N-1$, and it occurs similarly to the dissipation rate $\mathcal{D}_N(t)$ for the non-fluid component.

3.1.3. Main result. With the above preparation, our first result on the compressible Euler limit can be stated as follows.

Theorem 3.4. *Let $\tau > 0$ be given and $(\bar{\rho}, \bar{u}, \bar{\theta})(t, x)$ be the smooth solution of the compressible Euler system (1.2) and (3.2) given in Proposition 3.1. Construct the local Maxwellian $M_{[\bar{\rho},\bar{u},\bar{\theta}]}(t, x, v)$ from $(\bar{\rho}, \bar{u}, \bar{\theta})(t, x)$ as in (3.1). Then there exists a small constant $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, the Cauchy problem on the Landau equation (1.1) with non-negative initial data*

$$F^{\varepsilon}(0, x, v) \equiv M_{[\bar{\rho},\bar{u},\bar{\theta}]}(0, x, v) \tag{3.19}$$

admits a unique smooth solution $F^{\varepsilon}(t, x, v) \geq 0$ over $[0, \tau] \times \mathbb{R}^3 \times \mathbb{R}^3$, satisfying the estimate

$$\mathcal{E}_N(t) + \frac{1}{2}\int_0^t \mathcal{D}_N(s)\, ds \leq \frac{1}{2}\varepsilon^2, \tag{3.20}$$

for any $0 \leq t \leq \tau$. In particular, there exists a constant $C_\tau > 0$ independent of ε such that

$$\sup_{t\in[0,\tau]} \left\| \frac{F^{\varepsilon}(t, x, v) - M_{[\bar{\rho},\bar{u},\bar{\theta}]}(t, x, v)}{\sqrt{\mu}} \right\|_{L^2_x L^2_v} + \sup_{t\in[0,\tau]} \left\| \frac{F^{\varepsilon}(t, x, v) - M_{[\bar{\rho},\bar{u},\bar{\theta}]}(t, x, v)}{\sqrt{\mu}} \right\|_{L^\infty_x L^2_v} \leq C_\tau \varepsilon. \tag{3.21}$$

Remark 3.5. We point out that the energy inequality (3.20) holds true at $t = 0$ under the choice of the initial data (3.19). For this, one can claim that the initial data (3.19) automatically satisfies

$$\mathcal{E}_N(t)|_{t=0} \leq C\eta_0^2\varepsilon^2. \tag{3.22}$$

In fact, we have $M_{[\rho,u,\theta]}(0, x, v) = M_{[\bar{\rho},\bar{u},\bar{\theta}]}(0, x, v)$ and $G(0, x, v) = 0$ in terms of the decomposition $F(t, x, v) = M_{[\rho,u,\theta]}(t, x, v) + G(t, x, v)$ and (3.19). This implies that $(\rho, u, \theta)(0, x) = (\bar{\rho}, \bar{u}, \bar{\theta})(0, x)$ and $\overline{G}(0, x, v) + \sqrt{\mu} f(0, x, v) = 0$. Then it holds that

$$(\tilde{\rho}, \tilde{u}, \tilde{\theta})(0, x) = (\rho - \bar{\rho}, u - \bar{u}, \theta - \bar{\theta})(0, x) = 0 \quad \text{and} \quad f(0, x, v) = -\frac{\overline{G}(0, x, v)}{\sqrt{\mu}}.$$

With these and Lemma 4.3, we see directly that (3.22) holds. Therefore, (3.20) for $t = 0$ follows by letting $\eta_0 > 0$ be small enough.

Remark 3.6. As pointed out in the aforementioned relevant literature, all those known results [7, 31, 34, 35, 44, 45, 49, 58, 65] on the hydrodynamic limit from the Boltzmann equation to the compressible Euler system treat only the angular cutoff case, and obtaining similar results for both the non-cutoff Boltzmann equation and the Landau equation still remains largely open, essentially due to the effect of the grazing singularity of the collision operator. To the best of our knowledge, Theorem 3.4 seems the first result concerning the hydrodynamic limit for smooth solutions of the Landau equation to those of the compressible Euler system in the whole space over any finite time interval. Moreover, one may expect that a similar result should also hold for the non-cutoff Boltzmann equation.

Remark 3.7. It should be pointed out that all the results [31, 34, 35, 44, 45] are based on the Hilbert expansion and an L^2–L^∞ interplay method developed by Guo [30] for the Boltzmann equation with cutoff in a general bounded domain. Recently Kim–Guo–Hwang [48] also developed an L^2–L^∞ approach to the Landau equation around global Maxwellians in the torus domain, where initial data are required to be small in $L^\infty_{x,v}$ but additionally belong to $H^1_{x,v}$, and this result was improved later by Guo–Hwan–Jang–Ouyang [32] in a general bounded domain. However, these methods seem difficult to carry over directly to our problem on the compressible Euler limit of the Landau equation in the whole space.

Remark 3.8. Our analysis tool is mainly the combination of techniques for viscous compressible fluids, properties of Burnett functions, and the elaborate energy approach based on the macro-micro decomposition of the solution for the Landau equation with respect to the local Maxwellian that was initiated by Liu–Yu [54] and developed by Liu–Yang–Yu [52] in the Boltzmann theory. Thus, the idea we will adopt for the proof is different from the approach used by Caflisch in [7] via the truncated Hilbert expansion and by Guo–Jang–Jiang in [35] via the Hilbert expansion and the L^2–L^∞ interplay estimates.

3.2. Acoustic limit

The second goal of the present paper is to establish the acoustic limit of the Landau equation. The acoustic system is the linearization around the uniform equilibrium $\rho = R\theta = 1$ and $u = 0$ for the compressible Euler system. After a suitable choice of units to be consistent with the notation in [35], the fluid fluctuations $(\varrho, \varphi, \vartheta) = (\varrho, \varphi, \vartheta)(t, x)$ satisfy

$$\begin{cases} \partial_t \varrho + \nabla_x \cdot \varphi = 0, \\ \partial_t \varphi + \nabla_x(\varrho + \vartheta) = 0, \\ \partial_t \vartheta + \dfrac{2}{3}\nabla_x \cdot \varphi = 0. \end{cases} \tag{3.23}$$

To solve (3.23), we supplement it with prescribed initial data

$$(\varrho, \varphi, \vartheta)(0, x) = (\varrho_0, \varphi_0, \vartheta_0)(x) \in H^s(\mathbb{R}^3), \quad \text{for } s \geq 0. \tag{3.24}$$

It is well known that the Cauchy problem on the acoustic system (3.23)–(3.24) admits a unique global-in-time classical solution $(\varrho, \varphi, \vartheta)(t, x) \in C([0, +\infty); H^s(\mathbb{R}^3))$. Moreover, the solution satisfies

$$\Big\|\Big(\varrho, \varphi, \sqrt{\frac{3}{2}}\vartheta\Big)(t)\Big\|^2_{H^s} = \Big\|\Big(\varrho_0, \varphi_0, \sqrt{\frac{3}{2}}\vartheta_0\Big)\Big\|^2_{H^s}, \quad \text{for all } t \geq 0.$$

On the other hand, the acoustic system (3.23) can also be formally derived from the Landau equation (1.1) by letting

$$F^\varepsilon(t, x, v) = \mu + \delta\sqrt{\mu}\, \mathbf{f}^\varepsilon(t, x, v), \tag{3.25}$$

where μ is the global Maxwellian given by (3.6). The fluctuation amplitude δ is a function of ε satisfying

$$\delta \to 0, \quad \text{as } \varepsilon \to 0. \tag{3.26}$$

For instance, one may take

$$\delta = \varepsilon^{\sigma}, \quad \text{for any } \sigma > 0.$$

With the above scalings, $\mathbf{f}^{\varepsilon} = \mathbf{f}^{\varepsilon}(t,x,v)$ formally converges to

$$\mathbf{f} = \Big\{\varrho + v\cdot\varphi + \Big(\frac{|v|^2-3}{2}\Big)\vartheta\Big\}\sqrt{\mu}, \quad \text{as } \varepsilon \to 0, \tag{3.27}$$

where $(\varrho,\varphi,\vartheta)$ is the solution of the acoustic system (3.23). For a detailed formal derivation, see [3, 4, 25].

As in [35], to get the optimal scaling, we use δ to denote the fluctuation amplitude and assume that in addition to (3.26),

$$\frac{\varepsilon}{\delta} \to 0, \quad \text{as } \varepsilon \to 0. \tag{3.28}$$

We now state the second result on the acoustic limit.

Theorem 3.9. *Let $\tau > 0$ be given and $(\varrho_0,\varphi_0,\vartheta_0)(x) \in H^{N+2}(\mathbb{R}^3)$ with $N \geq 3$ be initial data for the acoustic system (3.23). Construct the local Maxwellian*

$$\mu^{\delta}(0,x,v) := \frac{1+\delta\varrho_0(x)}{\sqrt{[2\pi(1+\delta\vartheta_0(x))]^3}}\exp\Big\{-\frac{|v-\delta\varphi_0(x)|^2}{2(1+\delta\vartheta_0(x))}\Big\}$$

in terms of the initial datum $1+\delta\varrho_0$, $\delta\varphi_0$, and $1+\delta\vartheta_0$. Then there exist small constants $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$ and any $\delta \in (0,\delta_0)$ satisfying the restrictions (3.26) and (3.28), the Cauchy problem on the Landau equation (1.1) with non-negative initial data

$$F^{\varepsilon}(0,x,v) \equiv \mu^{\delta}(0,x,v)$$

admits a unique smooth solution $F^{\varepsilon}(t,x,v) \geq 0$ over $[0,\tau]\times\mathbb{R}^3\times\mathbb{R}^3$. Moreover, let $F^{\varepsilon}(t,x,v) = \mu + \delta\sqrt{\mu}\,\mathbf{f}^{\varepsilon}(t,x,v)$ in terms of (3.25); then the following convergence estimate holds:

$$\sup_{t\in[0,\tau]}\|\mathbf{f}^{\varepsilon}(t,x,v)-\mathbf{f}(t,x,v)\|_{L^2_xL^2_v} + \sup_{t\in[0,\tau]}\|\mathbf{f}^{\varepsilon}(t,x,v)-\mathbf{f}(t,x,v)\|_{L^\infty_xL^2_v} \leq C_\tau\Big(\frac{\varepsilon}{\delta}+\delta\Big), \tag{3.29}$$

where $\mathbf{f}(t,x,v)$ is defined in (3.27) and the constant $C_\tau > 0$ is independent of ε and δ.

Remark 3.10. The convergence rate obtained in (3.29) should be optimal, similarly to the one in [35] for the acoustic limit of the Boltzmann equation with cutoff.

3.3. Strategy of the proof

In what follows we give some key points in the proof of the main results. As mentioned above, the L^2–L^∞ framework in [35] cannot be directly carried over to our problem on the compressible fluid limit of the Landau equation in the whole space because their analysis depends crucially on Grad's splitting of the linearized Boltzmann operator with cutoff, which fails for the angular non-cutoff case or even in the grazing limit giving the Landau equation. Therefore, we have to develop new ideas instead of the L^2–L^∞ framework.

Our strategy is based on the pure high-order energy estimates framework developed in [28, 52, 54] to construct global solutions in the perturbative framework for kinetic equations. Note that the energy method in [52, 54] is based on the macro-micro decomposition with respect to the local Maxwellian M determined by the solution of kinetic equations. We can make use of such macro-micro decomposition to rewrite the Landau equation as a Navier–Stokes-type system with the non-fluid component appearing in the conservative source terms, cf. (2.15), coupled with an equation for the non-fluid component, cf. (2.11).

In order to carry out the energy estimate of the non-fluid component, we have to overcome some major difficulties. First of all, we need to subtract $\overline{G}$ in (3.8) from G so as to remove the troublesome term

$$\frac{1}{\varepsilon}L_M\overline{G} = P_1\Big\{v\cdot\Big(\frac{|v-u|^2\nabla_x\bar{\theta}}{2R\theta^2}+\frac{(v-u)\cdot\nabla_x\bar{u}}{R\theta}\Big)M\Big\}$$

when estimating the linear term $P_1(v\cdot\nabla_x M)$ in (2.11); see the identity (3.9). Otherwise, a troublesome term $\varepsilon(\|\nabla_x\bar{\theta}\|^2+\|\nabla_x\bar{u}\|^2)$ appears in the L^2 estimate. This term is bounded by $C\eta_0^2\varepsilon$ in terms of (3.3), and it is out of control by $O(\varepsilon^2)$ corresponding to (4.1) so that the energy estimate cannot be closed. To further decompose $G-\overline{G}$, if one sets $\sqrt{M}f = G-\overline{G}$, the equation for f includes the large-velocity growth term $\sqrt{M}^{-1}(\partial_t+v\cdot\nabla_x)\sqrt{M}f$, which involves the cubic power of v and this creates a key analytical difficulty in the L^2 estimate similar to the one in [35]. To avoid this difficulty, we set $\sqrt{\mu}f = G-\overline{G}$ to deduce the microscopic equation for f as (3.13) and then perform the energy estimate for f.

One of the most important points in the proof is that $f\in(\ker\mathcal{L})^\perp$ in (3.13) is purely microscopic such that the estimate

$$-\frac{1}{\varepsilon}(\mathcal{L}f,f)\geq c_1\frac{1}{\varepsilon}\|f\|_\sigma^2$$

holds true, which is again not true for the one in [28]. Indeed, [28] used the decomposition $F=\mu+\sqrt{\mu}f$ with the perturbation f involving the macroscopic part, namely $f\notin(\ker\mathcal{L})^\perp$, and as such one has to decompose $f=\mathbf{P}f+\{\mathbf{I}-\mathbf{P}\}f$ with $\mathbf{P}$ the projection on the kernel space of $\mathcal{L}$, so that one can only obtain

$$-\frac{1}{\varepsilon}(\mathcal{L}f,f)\geq c_1\frac{1}{\varepsilon}\|\{\mathbf{I}-\mathbf{P}\}f\|_\sigma^2.$$

This results in the appearance of a difficult term $\frac{1}{\varepsilon}\|\mathbf{P}f\|_\sigma^2$ that we would not be able to control since it involves a strong singularity about $\frac{1}{\varepsilon}$. The main reason is that the ε-dependent coefficient of the hydrodynamic part $\mathbf{P}f$ in the energy dissipation functional is just ε instead of $1/\varepsilon$. Therefore, the fact that $f \in (\ker \mathcal{L})^\perp$ as in (3.13) is the most important point in the whole proof. In this case, the trilinear estimate $\frac{1}{\varepsilon}(\Gamma(f,f),f)$ can be obtained easily because the difficult term $\frac{1}{\varepsilon}(\Gamma(\mathbf{P}f,\mathbf{P}f),\{\mathbf{I}-\mathbf{P}\}f)$ would no longer appear. Moreover, many known estimates of $\mathcal{L}$ and Γ in [28] can be directly employed; for instance, see Lemmas 4.4 and 4.5.

Note that the linearized Landau operator $\mathcal{L}$ in (3.13) lacks a spectral gap, which results in the very weak velocity dissipation by (4.10) and (3.16). The velocity derivatives of the free streaming term $v\cdot\nabla_x f$ in the L^2 estimate cannot be directly bounded by the dissipation of $\mathcal{L}$. So we adapt techniques in [28] based on the velocity weight function as (3.14) to overcome this difficulty. To treat the Nth-order space derivative estimate, we need to deal with a complicated term $\frac{1}{\varepsilon}(\mathcal{L}\partial^\alpha f, \frac{\partial^\alpha M}{\sqrt{\mu}}) = \frac{1}{\varepsilon}(\mathcal{L}\partial^\alpha f, \frac{I_4}{\sqrt{\mu}}) + \frac{1}{\varepsilon}(\mathcal{L}\partial^\alpha f, \frac{I_5}{\sqrt{\mu}})$ in (5.54) and (5.55). The linear term $\frac{1}{\varepsilon}(\mathcal{L}\partial^\alpha f, \frac{I_4}{\sqrt{\mu}})$ cannot be directly estimated since a significant difficulty occurs with it. The key technique for handling this term is to use the properties of the linearized operator $\mathcal{L}$ and the relation between M and μ as in (4.2), as well as the smallness of ε and η_0. In particular, to estimate the term $\frac{1}{\varepsilon}(\mathcal{L}\partial^\alpha f, \frac{I_4^{22}}{\sqrt{\mu}})$ in $\frac{1}{\varepsilon}(\mathcal{L}\partial^\alpha f, \frac{I_4}{\sqrt{\mu}})$, we must move one derivative from the Nth-order derivative $\mathcal{L}\partial^\alpha f$ to the other component of the inner product by integration by parts; see (5.56). In the end, we can obtain the estimate

$$\frac{1}{\varepsilon}\Big|\Big(\mathcal{L}\partial^\alpha f, \frac{\partial^\alpha M}{\sqrt{\mu}}\Big)\Big| \\ \leq C(\eta_0+\varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon}\Big(\|\partial^\alpha f\|_\sigma^2 + \|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2 + \frac{1}{\varepsilon^2}\|\partial^{\alpha'} f\|_\sigma^2 + \varepsilon^2\Big), \tag{3.30}$$

for $|\alpha'| = N-1$; see (5.55)–(5.60) for detailed calculations. Similar difficulties also arise elsewhere, such as $\frac{1}{\varepsilon}(\Gamma(\frac{M-\mu}{\sqrt{\mu}}, \partial^\alpha f), \frac{\partial^\alpha M}{\sqrt{\mu}})$ in (5.62). Because of the ε^{-1} factor in front of the fluid part $\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2$ in (3.30), one has to multiply the estimate (3.30) by ε^2 so that the fluid term can be controlled by $\mathcal{D}_N(t)$ in (3.18). Hence, this motivates how we include the Knudsen number ε in the highest-order derivative $|\alpha| = N$ for the energy functional $\mathcal{E}_N(t)$ in (3.17).

On the other hand, to make the energy estimate for the fluid component, we need to treat those integral terms in (5.9) involving the inverse of the linearized operator L_M around the local Maxwellian M, such as the terms

$$\int_{\mathbb{R}^3} v_i v_j L_M^{-1}\Theta\, dv, \qquad \int_{\mathbb{R}^3} v_i |v|^2 L_M^{-1}\Theta\, dv, \tag{3.31}$$

where Θ is defined in (2.13). Those terms in (3.31) are difficult to estimate in a direct way since they involve the polynomial velocity growth. To bound them, we follow the strategy developed in [18, 20] based on the Burnett functions $\hat{A}_i$ and $\widehat{B}_{ij}$ as in (2.16).

Indeed, in terms of the basic properties of the Burnett functions, the integral terms in (3.31) can be represented as the inner products of A_i and B_{ij} with Θ, where A_i and B_{ij} defined in (2.17) are the inverse of $\hat{A}_i$ and $\widehat{B}_{ij}$ under the linear operator L_M, respectively; see the identities (5.17), (5.18), and (5.19) for details. Hence, any polynomial velocity growth in Θ can be absorbed since A_i and B_{ij} enjoy fast velocity decay; see (5.20). In addition, we notice that any smooth solution of the compressible Euler system (1.2) and (3.2) given in Proposition 3.1 does not enjoy an explicit time decay rate. One then has to use the smallness of $C_\tau \eta_0$ in (3.3) to control those hard terms as in (5.10). This kind of technique will be used for the energy estimates of both the fluid-type system and the non-fluid system.

Combining the energy estimates of the non-fluid component and the fluid component, we are able to obtain the uniform a priori estimate (3.20) and then derive the convergence rate (3.21) as stated in Theorem 3.4. Following the same strategy to above, we can obtain similar arguments to (3.20) and (3.21) under the assumptions in Theorem 3.9, and then derive the convergence rate (3.29).

4. Basic estimates

In this section we first make the a priori assumption in order to perform energy analysis conveniently. Then we estimate the correction term and the complicated collision terms. Finally, we derive an estimate of the fluid quantities involved with the temporal derivatives. We should emphasize that in all estimates below, all constants $C > 0$ at different places may depend on τ but do not depend on either of the small parameters ε and η_0.

4.1. A priori assumption

Since the local existence of the solutions to the Landau equation near a global Maxwellian is well known in the torus or the whole space, cf. [28, 39], by a straightforward modification of the arguments there one can construct a unique short-time solution to the Landau equation (1.1) under the assumptions in Theorem 3.4. The details are omitted for simplicity of presentation. In order to extend the short-time solution to any finite time where Proposition 3.1 is satisfied, we only need to close the a priori assumption

$$\sup_{0 \le t \le T} \mathcal{E}_N(t) \le \varepsilon^2, \tag{4.1}$$

for an arbitrary time $T \in (0, \tau]$ with τ as in Proposition 3.1, where $\mathcal{E}_N(t)$ is given by (3.17).

Under the a priori assumption (4.1), we have from the imbedding inequality and (3.17) that

$$\sup_{t \in [0,\tau]} \|\tilde{\rho}(t, \cdot)\|_{L^\infty_x} \le C \sup_{t \in [0,\tau]} \|\tilde{\rho}(t, \cdot)\|_{H^2_x} \le C\varepsilon,$$

which together with (3.3) and (3.4) yields

$$|\rho(t,x)-1| \le |\rho(t,x)-\bar{\rho}(t,x)| + |\bar{\rho}(t,x)-1| \le C_\tau(\varepsilon+\eta_0).$$

Similar estimates also hold for $u(t,x)$ and $\theta(t,x)$. Therefore, for sufficiently small ε and η_0, it holds that

$$|\rho(t,x)-1| + |u(t,x)| + \left|\theta(t,x)-\frac{3}{2}\right| < C_\tau(\varepsilon+\eta_0), \quad 1<\theta(t,x)<2, \tag{4.2}$$

uniformly in all $(t,x)\in[0,\tau]\times\mathbb{R}^3$.

4.2. Sobolev inequalities

We list several basic inequalities frequently used throughout this paper.

Lemma 4.1. *For any function $h=h(x)\in H^1(\mathbb{R}^3)$, we have*

$$\begin{aligned}
&\|h\|_{L^6(\mathbb{R}^3)} \le C\|\nabla_x h\|,\\
&\|h\|_{L^3(\mathbb{R}^3)} \le C\|h\|^{\frac12}\|\nabla_x h\|^{\frac12},\\
&\|h\|_{L^p(\mathbb{R}^3)} \le C\|h\|_{H^1}, \quad 2\le p\le 6,
\end{aligned}$$

and for any function $h=h(x)\in H^2(\mathbb{R}^3)$, it holds that

$$\|h\|_{L^\infty(\mathbb{R}^3)} \le C\|\nabla_x h\|^{\frac12}\|\nabla_x^2 h\|^{\frac12}.$$

Here, C is a positive constant independent of $h(x)$.

4.3. Estimates of the correction term $\overline{G}$

To perform the energy estimates for the equations (3.13) and (3.5), one has to treat those integral terms involving L_M^{-1} and $\overline{G}$. For this, we need to give the estimate concerning the inverse of the linearized operator, whose proof can be found in [20, Lemma 6.1].

Lemma 4.2. *Suppose that $U(v)$ is any polynomial of $\frac{v-\hat{u}}{\sqrt{R\hat{\theta}}}$ such that $U(v)\widehat{M}\in(\ker L_{\widehat{M}})^\perp$ for any Maxwellian $\widehat{M}=M_{[\hat{\rho},\hat{u},\hat{\theta}]}(v)$ where $L_{\widehat{M}}$ is as (2.9). For any $\epsilon\in(0,1)$ and any multi-index β, there exists a constant $C_\beta>0$ such that*

$$|\partial_\beta L_{\widehat{M}}^{-1}(U(v)\widehat{M})| \le C_\beta(\hat{\rho},\hat{u},\hat{\theta})\widehat{M}^{1-\epsilon}.$$

In particular, under the condition of (4.2), there exists a constant $C_\beta>0$ such that

$$\left|\partial_\beta A_j\left(\frac{v-u}{\sqrt{R\theta}}\right)\right| + \left|\partial_\beta B_{ij}\left(\frac{v-u}{\sqrt{R\theta}}\right)\right| \le C_\beta M^{1-\epsilon}, \tag{4.3}$$

where $A_j(\cdot)$ and $B_{ij}(\cdot)$ are defined in (2.17).

Based on the properties of the Burnett functions and Lemma 4.2, we can prove the following lemma, which will be used frequently in the energy analysis later on.

Lemma 4.3. *Assume* (4.1) *and* (4.2) *hold. Let* $\bar{G}$ *be defined in* (3.8) *and* $\langle v\rangle = \sqrt{1+|v|^2}$*; then for any* $l \geq 0$, $|\beta| \geq 0$, *and* $|\alpha| \leq N$, *one has*

$$\left\| \langle v\rangle^l \partial^\alpha_\beta \Big(\frac{\bar{G}}{\sqrt{\mu}}\Big)\right\|_{2,|\beta|} + \left\| \langle v\rangle^l \partial^\alpha_\beta \Big(\frac{\bar{G}}{\sqrt{\mu}}\Big)\right\|_{\sigma,|\beta|} \leq C\eta_0\varepsilon. \tag{4.4}$$

Proof. In view of (2.16) and (2.17), the term $\bar{G}$ in (3.8) can be represented precisely as

$$\bar{G} = \varepsilon\frac{\sqrt{R}}{\sqrt{\theta}}\sum_{j=1}^{3}\frac{\partial\bar{\theta}}{\partial x_j}A_j\Big(\frac{v-u}{\sqrt{R\theta}}\Big) + \varepsilon\sum_{j=1}^{3}\sum_{i=1}^{3}\frac{\partial \bar{u}_j}{\partial x_i}B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big). \tag{4.5}$$

Then, for $k=1,2,3$, we have

$$\frac{\partial\bar{G}}{\partial v_k} = \varepsilon\frac{\sqrt{R}}{\sqrt{\theta}}\sum_{j=1}^{3}\frac{\partial\bar{\theta}}{\partial x_j}\partial_{v_k}A_j\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{1}{\sqrt{R\theta}} + \varepsilon\sum_{i,j=1}^{3}\frac{\partial \bar{u}_j}{\partial x_i}\partial_{v_k}B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{1}{\sqrt{R\theta}}, \tag{4.6}$$

and

$$\begin{aligned}\frac{\partial\bar{G}}{\partial x_k} = \varepsilon\Big\{&\frac{\sqrt{R}}{\sqrt{\theta}}\sum_{j=1}^{3}\frac{\partial^2\bar{\theta}}{\partial x_j\partial x_k}A_j\Big(\frac{v-u}{\sqrt{R\theta}}\Big) - \frac{\sqrt{R}}{2\sqrt{\theta^3}}\sum_{j=1}^{3}\frac{\partial\bar{\theta}}{\partial x_j}\frac{\partial\theta}{\partial x_k}A_j\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\\ &-\frac{\sqrt{R}}{\sqrt{\theta}}\sum_{j=1}^{3}\frac{\partial\bar{\theta}}{\partial x_j}\frac{\partial u}{\partial x_k}\cdot\nabla_v A_j\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{1}{\sqrt{R\theta}}\\ &-\frac{\sqrt{R}}{\sqrt{\theta}}\sum_{j=1}^{3}\frac{\partial\bar{\theta}}{\partial x_j}\frac{\partial\theta}{\partial x_k}\nabla_v A_j\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\cdot\frac{v-u}{2\sqrt{R\theta^3}}\\ &+\sum_{i,j=1}^{3}\frac{\partial^2\bar{u}_j}{\partial x_i\partial x_k}B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big) - \sum_{i,j=1}^{3}\frac{\partial\bar{u}_j}{\partial x_i}\frac{\partial u}{\partial x_k}\cdot\nabla_v B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{1}{\sqrt{R\theta}}\\ &-\sum_{i,j=1}^{3}\frac{\partial\bar{u}_j}{\partial x_i}\frac{\partial\theta}{\partial x_k}\nabla_v B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\cdot\frac{v-u}{2\sqrt{R\theta^3}}\Big\}.\end{aligned} \tag{4.7}$$

Likewise, $\partial_t\bar{G}$ has a similar expression to (4.7). For any $|\beta|\geq 0$, any $l \geq 0$, and sufficiently small $\epsilon > 0$, we can deduce from (4.2), (3.14), and (3.16) that

$$|\langle v\rangle^l w^{|\beta|}\mu^{-\frac{1}{2}}M^{1-\epsilon}|_2 + |\langle v\rangle^l w^{|\beta|}\mu^{-\frac{1}{2}}M^{1-\epsilon}|_\sigma \leq C. \tag{4.8}$$

For any $|\beta| \geq 0$, we get by (4.3), (4.5), and similar arguments to (4.6), (4.8), and (3.3) that

$$\left\| \langle v\rangle^l \partial_\beta \Big(\frac{\bar{G}}{\sqrt{\mu}}\Big)\right\|^2_{2,|\beta|} + \left\| \langle v\rangle^l \partial_\beta \Big(\frac{\bar{G}}{\sqrt{\mu}}\Big)\right\|^2_{\sigma,|\beta|} \leq C\varepsilon^2(\|\nabla_x\bar{u}\|^2 + \|\nabla_x\bar{\theta}\|^2) \leq C\eta_0^2\varepsilon^2.$$

For $|\alpha| = 1$, we use (4.7), (4.8), (4.3), (4.1), and (3.3) to get

$$\begin{aligned}&\Big\|\langle v\rangle^l \partial^\alpha_\beta\Big(\frac{\bar{G}}{\sqrt{\mu}}\Big)\Big\|^2_{2,|\beta|} + \Big\|\langle v\rangle^l \partial^\alpha_\beta\Big(\frac{\bar{G}}{\sqrt{\mu}}\Big)\Big\|^2_{\sigma,|\beta|}\\ &\quad\le C\varepsilon^2\int_{\mathbb{R}^3}(|\partial^\alpha\nabla_x\bar{u}|^2+|\partial^\alpha\nabla_x\bar{\theta}|^2)+(|\nabla_x\bar{u}|^2+|\nabla_x\bar{\theta}|^2)(|\partial^\alpha u|^2+|\partial^\alpha\theta|^2)\,dx\\ &\quad\le C\varepsilon^2(\|(\partial^\alpha\nabla_x\bar{u},\partial^\alpha\nabla_x\bar{\theta})\|^2+\|(\nabla_x\bar{u},\nabla_x\bar{\theta})\|^2_{L^\infty}\|(\partial^\alpha u,\partial^\alpha\theta)\|^2)\\ &\quad\le C\eta_0^2\varepsilon^2(1+\eta_0^2+\varepsilon^2)\le C\eta_0^2\varepsilon^2.\end{aligned}$$

Similar arguments also hold for the cases $2 \le |\alpha| \le N$. Therefore, we can prove that the desired estimate (4.4) holds true. This ends the proof of Lemma 4.3. ∎

4.4. Estimates of collision terms $\mathcal{L}$ and Γ

Next we summarize some refined estimates for the linearized Landau operator $\mathcal{L}$ and the non-linear collision terms $\Gamma(g_1, g_2)$ defined in (3.11). We start by collecting some known basic estimates. The following two lemmas can be found in [28, Lemma 6] and [64, Proposition 1], respectively.

Lemma 4.4. *Let $|\alpha| \ge 0$ and $|\beta| > 0$; then for w defined in (3.14) and any small $\eta > 0$, there exist $c_0 > 0$ and $C_\eta > 0$ such that*

$$-\langle\partial^\alpha_\beta\mathcal{L}g, w^{2|\beta|}\partial^\alpha_\beta g\rangle \ge c_0|\partial^\alpha_\beta g|^2_{\sigma,|\beta|} - \eta\sum_{|\beta_1|\le|\beta|}|\partial^\alpha_{\beta_1}g|^2_{\sigma,|\beta_1|} - C_\eta|\partial^\alpha g|^2_\sigma. \tag{4.9}$$

If $|\beta| = 0$, then for any $g \in (\ker\mathcal{L})^\perp$, there exists a generic constant $c_1 > 0$ such that

$$-\langle\mathcal{L}\partial^\alpha g, \partial^\alpha g\rangle \ge c_1|\partial^\alpha g|^2_\sigma. \tag{4.10}$$

Lemma 4.5. *Let $|\alpha| \ge 0$ and $|\beta| \ge 0$; then for arbitrarily large constant $b > 0$, one has*

$$|\langle\partial^\alpha\Gamma(g_1,g_2),\partial^\alpha g_3\rangle| \le C\sum_{\alpha_1\le\alpha}|\langle v\rangle^{-b}\partial^{\alpha_1}g_1|_2|\partial^{\alpha-\alpha_1}g_2|_\sigma|\partial^\alpha g_3|_\sigma. \tag{4.11}$$

Moreover, for w defined in (3.14) and $l \ge 0$, one has

$$|\langle\partial^\alpha_\beta\Gamma(g_1,g_2),w^{2l}\partial^\alpha_\beta g_3\rangle| \le C\sum_{\alpha_1\le\alpha}\sum_{\beta'\le\beta_1\le\beta}|\langle v\rangle^{-b}\partial^{\alpha_1}_{\beta'}g_1|_2|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\sigma,l}|\partial^\alpha_\beta g_3|_{\sigma,l}. \tag{4.12}$$

With Lemma 4.5, we now prove some non-linear energy estimates. We first consider estimates of linear collision terms $\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, f\Big)$ and $\Gamma\Big(f, \frac{M-\mu}{\sqrt{\mu}}\Big)$ in (3.13), which will be used in Section 5.

Lemma 4.6. *Let $|\alpha| + |\beta| \le N$ with $|\beta| \ge 1$ and $w = (1 + |v|^2)^{-1/2}$, as in (3.14). Let $F = M + \overline{G} + \sqrt{\mu}f$ be the solution to the Landau equation (1.1) and (3.19), and assume*

(4.1) *and* (4.2) *hold. If we choose* $\eta_0 > 0$ *in* (3.3) *and* $\varepsilon > 0$ *in* (4.1) *small enough, then for any* $\eta > 0$*, one has*

$$
\begin{aligned}
&\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha_\beta\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, f\Big), w^{2|\beta|}\partial^\alpha_\beta f\Big)\Big| + \frac{1}{\varepsilon}\Big|\Big(\partial^\alpha_\beta\Gamma\Big(f, \frac{M-\mu}{\sqrt{\mu}}\Big), w^{2|\beta|}\partial^\alpha_\beta f\Big)\Big| \\
&\quad\le C\eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|} + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\mathcal{D}_N(t).
\end{aligned} \tag{4.13}
$$

Moreover, for $|\beta| = 0$ *and* $|\alpha| \le N-1$*, one has*

$$
\begin{aligned}
&\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, f\Big), \partial^\alpha f\Big)\Big| + \frac{1}{\varepsilon}\Big|\Big(\partial^\alpha\Gamma\Big(f, \frac{M-\mu}{\sqrt{\mu}}\Big), \partial^\alpha f\Big)\Big| \\
&\quad\le C\eta\frac{1}{\varepsilon}\|\partial^\alpha f\|^2_{\sigma} + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\mathcal{D}_N(t).
\end{aligned} \tag{4.14}
$$

Here, $\mathcal{D}_N(t)$ *is defined by* (3.18).

Proof. For the first term on the left-hand side of (4.13), since $w^{2|\beta|} \le w^{2|\beta-\beta_1|}$ for $|\beta - \beta_1| \le |\beta|$, we have from this and (4.12) that

$$
\begin{aligned}
&\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha_\beta\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, f\Big), w^{2|\beta|}\partial^\alpha_\beta f\Big)\Big| \\
&\le C\frac{1}{\varepsilon}\sum_{\alpha_1\le\alpha}\sum_{\beta'\le\beta_1\le\beta}\int_{\mathbb{R}^3}\Big|\langle v\rangle^{-b}\partial^{\alpha_1}_{\beta'}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|_2|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f|_{\sigma,|\beta-\beta_1|}|\partial^\alpha_\beta f|_{\sigma,|\beta|}\,dx.
\end{aligned} \tag{4.15}
$$

In order to further compute (4.15), for any $|\bar\beta| \ge 0$ and $l \ge 0$ we claim that

$$
\Big|\langle v\rangle^l\partial_{\bar\beta}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|_\sigma + \Big|\langle v\rangle^l\partial_{\bar\beta}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|_2 \le C(\eta_0+\varepsilon). \tag{4.16}
$$

In fact, for any $|\bar\beta| \ge 0$ and $l \ge 0$, using (3.16), we know that there exists a small constant $\epsilon_1 > 0$ such that

$$
\begin{aligned}
&\Big|\langle v\rangle^l\partial_{\bar\beta}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|^2_\sigma + \Big|\langle v\rangle^l\partial_{\bar\beta}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|^2_2 \\
&\quad\le C_l\sum_{|\bar\beta|\le|\beta'|\le|\bar\beta|+1}\int_{\mathbb{R}^3}\mu^{-\epsilon_1}\Big|\partial_{\beta'}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|^2\,dv.
\end{aligned}
$$

Thanks to (4.2), we can choose a suitably large constant $R > 0$ such that

$$
\int_{|v|\ge R}\mu^{-\epsilon_1}\Big|\partial_{\beta'}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|^2\,dv \le C(\eta_0+\varepsilon)^2,
$$

and

$$
\int_{|v|\le R}\mu^{-\epsilon_1}\Big|\partial_{\beta'}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|^2\,dv \le C\Big(|\rho-1|+|u-0|+|\theta-\frac{3}{2}|\Big)^2 \le C(\eta_0+\varepsilon)^2.
$$

By these estimates, we can get the desired estimate (4.16), and thus ends the proof of (4.16).

First, from direct calculation we have

$$\partial_{x_i} M = M\Big(\frac{\partial_{x_i}\rho}{\rho} + \frac{(v-u)\cdot\partial_{x_i}u}{R\theta} + \Big(\frac{|v-u|^2}{2R\theta} - \frac{3}{2}\Big)\frac{\partial_{x_i}\theta}{\theta}\Big). \tag{4.17}$$

Then for $|\alpha| \geq 2$ and $\partial^\alpha = \partial^{\alpha'}\partial_{x_i}$, it holds that

$$\begin{aligned}\partial^\alpha M &= M\Big(\frac{\partial^\alpha\rho}{\rho} + \frac{(v-u)\cdot\partial^\alpha u}{R\theta} + \Big(\frac{|v-u|^2}{2R\theta} - \frac{3}{2}\Big)\frac{\partial^\alpha\theta}{\theta}\Big)\\ &\quad + \sum_{1\leq\alpha_1\leq\alpha'} C_{\alpha'}^{\alpha_1}\Big(\partial^{\alpha_1}\Big(M\frac{1}{\rho}\Big)\partial^{\alpha'-\alpha_1}\partial_{x_i}\rho + \partial^{\alpha_1}\Big(M\frac{v-u}{R\theta}\Big)\cdot\partial^{\alpha'-\alpha_1}\partial_{x_i}u\\ &\qquad + \partial^{\alpha_1}\Big(M\frac{|v-u|^2}{2R\theta^2} - M\frac{3}{2\theta}\Big)\partial^{\alpha'-\alpha_1}\partial_{x_i}\theta\Big).\end{aligned} \tag{4.18}$$

We now turn to computing (4.15). Note that $|\alpha_1| \leq |\alpha| \leq N-1$ since we only consider the case $|\alpha| + |\beta| \leq N$ and $|\beta| \geq 1$. If $|\alpha_1| = 0$, we get from (4.16), the Cauchy–Schwarz inequality, and (3.18) that

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\Big|\langle v\rangle^{-b}\partial_{\beta'}^{\alpha_1}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|_2 |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f|_{\sigma,|\beta-\beta_1|}|\partial_\beta^\alpha f|_{\sigma,|\beta|}\,dx\\ &\quad\leq \eta\frac{1}{\varepsilon}\|\partial_\beta^\alpha f\|_{\sigma,|\beta|}^2 + C_\eta(\eta_0+\varepsilon)^2\frac{1}{\varepsilon}\|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f\|_{\sigma,|\beta-\beta_1|}^2\\ &\quad\leq \eta\frac{1}{\varepsilon}\|\partial_\beta^\alpha f\|_{\sigma,|\beta|}^2 + C_\eta(\eta_0+\varepsilon)^2\mathcal{D}_N(t).\end{aligned}$$

If $1 \leq |\alpha_1| \leq |\alpha| \leq N-1$, then $|\alpha-\alpha_1| \leq N-2$ and $|\alpha-\alpha_1| + |\beta-\beta_1| \leq |\alpha| + |\beta| - 1$. Taking the L^6–L^3–L^2 Hölder inequality and using (4.2) and the Cauchy–Schwarz and Sobolev inequalities, we get

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\Big|\langle v\rangle^{-b}\partial_{\beta'}^{\alpha_1}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|_2 |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f|_{\sigma,|\beta-\beta_1|}|\partial_\beta^\alpha f|_{\sigma,|\beta|}\,dx\\ &\quad\leq C\frac{1}{\varepsilon}\Big\|\Big|\langle v\rangle^{-b}\partial_{\beta'}^{\alpha_1}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|_2\Big\|_{L^3}\|\,|\partial_{\beta-\beta_1}^{\alpha-\alpha_1} f|_{\sigma,|\beta-\beta_1|}\|_{L^6}\|\,|\partial_\beta^\alpha f|_{\sigma,|\beta|}\|_{L^2}\\ &\quad\leq C\frac{1}{\varepsilon}(\eta_0+\varepsilon^{\frac{1}{2}})\|\partial_{\beta-\beta_1}^{\alpha-\alpha_1}\nabla_x f\|_{\sigma,|\beta-\beta_1|}\|\partial_\beta^\alpha f\|_{\sigma,|\beta|}\\ &\quad\leq \eta\frac{1}{\varepsilon}\|\partial_\beta^\alpha f\|_{\sigma,|\beta|}^2 + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})^2\mathcal{D}_N(t),\end{aligned}$$

where we have used (3.3), (4.1), (4.17), (4.18), and the fact

$$\Big\|\Big|\langle v\rangle^{-b}\partial_{\beta'}^{\alpha_1}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|_2\Big\|_{L^3} \leq C(\eta_0+\varepsilon^{\frac{1}{2}}),$$

since one has to deal with $|\alpha_1| = N - 1$,

$$\begin{aligned}\|\partial^{\alpha_1}(\rho, u, \theta)\|_{L^3} &\le C\|\partial^{\alpha_1}(\bar\rho, \bar u, \bar\theta)\|_{L^3} + C\|\partial^{\alpha_1}(\tilde\rho, \tilde u, \tilde\theta)\|_{L^3} \\ &\le C\eta_0 + C\|\partial^{\alpha_1}(\tilde\rho, \tilde u, \tilde\theta)\|^{\frac12}\|\nabla_x\partial^{\alpha_1}(\tilde\rho, \tilde u, \tilde\theta)\|^{\frac12} \\ &\le C(\eta_0 + \varepsilon^{\frac12}).\end{aligned}$$

Therefore, substituting the above estimates into (4.15) and using the smallness of η_0 and ε, we obtain

$$\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha_\beta\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, f\Big), w^{2|\beta|}\partial^\alpha_\beta f\Big)\Big| \le C\eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|} + C_\eta(\eta_0 + \varepsilon^{\frac12})\mathcal{D}_N(t). \quad (4.19)$$

For the second term on the left-hand side of (4.13), it is straightforward to see by (4.12) that

$$\begin{aligned}&\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha_\beta\Gamma\Big(f, \frac{M-\mu}{\sqrt{\mu}}\Big), w^{2|\beta|}\partial^\alpha_\beta f\Big)\Big| \\ &\quad\le C\frac{1}{\varepsilon}\sum_{\alpha_1\le\alpha}\sum_{\beta'\le\beta_1\le\beta}\int_{\mathbb{R}^3}|\langle v\rangle^{-b}\partial^{\alpha_1}_{\beta'} f|_2\Big|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|_{\sigma,|\beta|}|\partial^\alpha_\beta f|_{\sigma,|\beta|}\,dx. \quad (4.20)\end{aligned}$$

Estimate (4.20) can be treated in a similar way to (4.15). First note that $|\alpha_1| \le |\alpha| \le N-1$ due to $|\alpha| + |\beta| \le N$ and $|\beta| \ge 1$. If $|\alpha - \alpha_1| = 0$, we use (3.16), (4.16), the Cauchy–Schwarz inequality, and (3.18) again, to obtain

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{\mathbb{R}^3}|\langle v\rangle^{-b}\partial^{\alpha_1}_{\beta'} f|_2\Big|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|_{\sigma,|\beta|}|\partial^\alpha_\beta f|_{\sigma,|\beta|}\,dx \\ &\quad\le C(\eta_0+\varepsilon)\frac{1}{\varepsilon}\|\langle v\rangle^{-b}\partial^{\alpha_1}_{\beta'} f\|\,\|\partial^\alpha_\beta f\|_{\sigma,|\beta|} \\ &\quad\le C(\eta_0+\varepsilon)\frac{1}{\varepsilon}\|\partial^{\alpha_1}_{\beta'} f\|_{\sigma,|\beta'|}\|\partial^\alpha_\beta f\|_{\sigma,|\beta|} \\ &\quad\le \eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|} + C_\eta(\eta_0+\varepsilon)^2\mathcal{D}_N(t).\end{aligned}$$

Here we have used $\langle v\rangle^{-b} \le \langle v\rangle^{-\frac12}\langle v\rangle^{-|\beta'|}$ by choosing $b \ge N + 1/2 \ge |\beta'| + 1/2$. If $|\alpha - \alpha_1| \ne 0$, that is $|\alpha_1| < |\alpha| \le N-1$, then $|\alpha_1| + |\beta'| \le |\alpha_1| + |\beta| \le |\alpha| - 1 + |\beta| \le N - 1$, and it holds that

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{\mathbb{R}^3}|\langle v\rangle^{-b}\partial^{\alpha_1}_{\beta'} f|_2\Big|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|_{\sigma,|\beta|}|\partial^\alpha_\beta f|_{\sigma,|\beta|}\,dx \\ &\quad\le C\frac{1}{\varepsilon}\|\,|\langle v\rangle^{-b}\partial^{\alpha_1}_{\beta'} f|_2\|_{L^6}\Big\|\,\Big|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|_{\sigma,|\beta|}\Big\|_{L^3}\|\partial^\alpha_\beta f\|_{\sigma,|\beta|} \\ &\quad\le \eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|} + C_\eta(\eta_0+\varepsilon^{\frac12})^2\mathcal{D}_N(t).\end{aligned}$$

All in all, plugging the above estimates into (4.20) yields

$$\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha_\beta\Gamma\Big(f,\frac{M-\mu}{\sqrt{\mu}}\Big),w^{2|\beta|}\partial^\alpha_\beta f\Big)\Big|\le C\eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}+C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\mathcal{D}_N(t). \quad (4.21)$$

In summary, the desired estimate (4.13) follows from (4.19) and (4.21). This concludes the proof of (4.13). By (4.11) and similar arguments to (4.19) and (4.21), we can prove that (4.14) holds, and details are omitted for brevity. This ends the proof of Lemma 4.6. ∎

Next we consider estimates of the non-linear collision term $\Gamma\big(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}\big)$, which will be used in Section 5.

Lemma 4.7. *Let $|\alpha|+|\beta|\le N$ with $|\beta|\ge 1$ and the conditions of Lemma 4.6 be satisfied; then for any $\eta>0$, one has*

$$\begin{aligned}&\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha_\beta\Gamma\Big(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}\Big),w^{2|\beta|}\partial^\alpha_\beta f\Big)\Big|\\&\quad\le C\eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}+C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\mathcal{D}_N(t)+C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2. \end{aligned}\quad (4.22)$$

Moreover, for $|\beta|=0$ and $|\alpha|\le N-1$, it holds that

$$\begin{aligned}&\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha\Gamma\Big(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}\Big),\partial^\alpha f\Big)\Big|\\&\quad\le C\eta\frac{1}{\varepsilon}\|\partial^\alpha f\|^2_{\sigma}+C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\mathcal{D}_N(t)+C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2. \end{aligned}\quad (4.23)$$

Proof. We only prove estimate (4.22) since estimate (4.23) can be handled in the same way. Let $|\alpha|+|\beta|\le N$ with $|\beta|\ge 1$; we use $G=\overline{G}+\sqrt{\mu}f$ to show

$$\partial^\alpha_\beta\Gamma\Big(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}\Big)=\partial^\alpha_\beta\Gamma\Big(\frac{\overline{G}}{\sqrt{\mu}},\frac{\overline{G}}{\sqrt{\mu}}\Big)+\partial^\alpha_\beta\Gamma\Big(\frac{\overline{G}}{\sqrt{\mu}},f\Big)+\partial^\alpha_\beta\Gamma\Big(f,\frac{\overline{G}}{\sqrt{\mu}}\Big)+\partial^\alpha_\beta\Gamma(f,f).$$

We take the inner product of the above equality with $\frac{1}{\varepsilon}w^{2|\beta|}\partial^\alpha_\beta f$ and then compute each term. In view of (4.12), Lemma 4.3, the Cauchy–Schwarz and Sobolev inequalities, (3.3), and (4.1), we arrive at

$$\begin{aligned}&\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha_\beta\Gamma\Big(\frac{\overline{G}}{\sqrt{\mu}},\frac{\overline{G}}{\sqrt{\mu}}\Big),w^{2|\beta|}\partial^\alpha_\beta f\Big)\Big|\\&\quad\le C\frac{1}{\varepsilon}\sum_{\alpha_1\le\alpha}\sum_{\beta'\le\beta_1\le\beta}\int_{\mathbb{R}^3}\Big|\langle v\rangle^{-b}\partial^{\alpha_1}_{\beta'}\Big(\frac{\overline{G}}{\sqrt{\mu}}\Big)\Big|_2\Big|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\Big(\frac{\overline{G}}{\sqrt{\mu}}\Big)\Big|_{\sigma,|\beta-\beta_1|}|\partial^\alpha_\beta f|_{\sigma,|\beta|}\,dx\\&\quad\le\eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}+C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2. \end{aligned}\quad (4.24)$$

Using (4.12) again gives

$$
\begin{aligned}
&\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha_\beta\Gamma\Big(\frac{\bar G}{\sqrt{\mu}},f\Big),w^{2|\beta|}\partial^\alpha_\beta f\Big)\Big|\\
&\le C\sum_{\alpha_1\le\alpha}\sum_{\beta'\le\beta_1\le\beta}\underbrace{\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\Big|\langle v\rangle^{-b}\partial^{\alpha_1}_{\beta'}\Big(\frac{\bar G}{\sqrt{\mu}}\Big)\Big|_2|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f|_{\sigma,|\beta-\beta_1|}|\partial^\alpha_\beta f|_{\sigma,|\beta|}\,dx}_{J_1}\,. \qquad (4.25)
\end{aligned}
$$

Note that for $|\alpha|+|\beta|\le N$ with $|\beta|\ge 1$, one has $|\alpha|\le N-1$. To estimate the term J_1, we consider the following two cases. If $|\alpha-\alpha_1|+|\beta-\beta_1|\le\frac{|\alpha|+|\beta|}{2}$, we derive from the Cauchy–Schwarz and Sobolev inequalities, (4.4), (3.3), (4.1), and (3.18) that

$$
\begin{aligned}
J_1&\le C\frac{1}{\varepsilon}\Big\|\Big|\langle v\rangle^{-b}\partial^{\alpha_1}_{\beta'}\Big(\frac{\bar G}{\sqrt{\mu}}\Big)\Big|_2\Big\|_{L^2}\|\,|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f|_{\sigma,|\beta-\beta_1|}\|_{L^\infty}\|\,|\partial^\alpha_\beta f|_{\sigma,|\beta|}\|_{L^2}\\
&\le\eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}+C_\eta\eta_0\varepsilon\|\nabla_x\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f\|_{\sigma,|\beta-\beta_1|}\|\nabla_x^2\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f\|_{\sigma,|\beta-\beta_1|}\\
&\le\eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}+C_\eta(\eta_0+\varepsilon^{\frac12})\mathcal{D}_N(t).
\end{aligned}
$$

If $\frac{|\alpha|+|\beta|}{2}<|\alpha-\alpha_1|+|\beta-\beta_1|\le|\alpha|+|\beta|$, then it holds that

$$
\begin{aligned}
J_1&\le C\frac{1}{\varepsilon}\Big\|\Big|\langle v\rangle^{-b}\partial^{\alpha_1}_{\beta'}\Big(\frac{\bar G}{\sqrt{\mu}}\Big)\Big|_2\Big\|_{L^\infty}\|\,|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f|_{\sigma,|\beta-\beta_1|}\|_{L^2}\|\,|\partial^\alpha_\beta f|_{\sigma,|\beta|}\|_{L^2}\\
&\le\eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}+C_\eta(\eta_0+\varepsilon^{\frac12})\mathcal{D}_N(t).
\end{aligned}
$$

Consequently, plugging the above estimates into (4.25) implies

$$
\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha_\beta\Gamma\Big(\frac{\bar G}{\sqrt{\mu}},f\Big),w^{2|\beta|}\partial^\alpha_\beta f\Big)\Big|\le C\eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}+C_\eta(\eta_0+\varepsilon^{\frac12})\mathcal{D}_N(t). \qquad (4.26)
$$

Carrying out similar calculations to (4.26), one has the following same bound:

$$
\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha_\beta\Gamma\Big(f,\frac{\bar G}{\sqrt{\mu}}\Big),w^{2|\beta|}\partial^\alpha_\beta f\Big)\Big|\le C\eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}+C_\eta(\eta_0+\varepsilon^{\frac12})\mathcal{D}_N(t). \qquad (4.27)
$$

Similarly, one has from (4.12) that

$$
\begin{aligned}
&\frac{1}{\varepsilon}|(\partial^\alpha_\beta\Gamma(f,f),w^{2|\beta|}\partial^\alpha_\beta f)|\\
&\qquad\le C\sum_{\alpha_1\le\alpha}\sum_{\beta'\le\beta_1\le\beta}\underbrace{\frac{1}{\varepsilon}\int_{\mathbb{R}^3}|\langle v\rangle^{-b}\partial^{\alpha_1}_{\beta'}f|_2|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f|_{\sigma,|\beta-\beta_1|}|\partial^\alpha_\beta f|_{\sigma,|\beta|}\,dx}_{J_2}\,.
\end{aligned}
$$

The term J_2 can be treated in a similar way to the term J_1. If $|\alpha-\alpha_1|+|\beta-\beta_1|\le \frac{|\alpha|+|\beta|}{2}$, using the Cauchy–Schwarz and Sobolev inequalities, (4.1), and (3.18), we get

$$\begin{aligned}J_2 &\le C\frac{1}{\varepsilon}\| \,|\langle v\rangle^{-b}\partial^{\alpha_1}_{\beta'}f|_2\|_{L^2}\| \,|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f|_{\sigma,|\beta-\beta_1|}\|_{L^\infty}\| \,|\partial^\alpha_\beta f|_{\sigma,|\beta|}\|_{L^2}\\ &\le \eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}+C_\eta\frac{1}{\varepsilon}\|\partial^{\alpha_1}_{\beta'}f\|^2_{2,|\beta'|}\|\nabla_x\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f\|_{\sigma,|\beta-\beta_1|}\|\nabla^2_x\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f\|_{\sigma,|\beta-\beta_1|}\\ &\le \eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}+C_\eta\varepsilon^{\frac12}\mathcal{D}_N(t).\end{aligned}$$

Here we have used $\langle v\rangle^{-b}\le\langle v\rangle^{-|\beta'|}$ by choosing $b\ge N\ge|\beta'|$. If $\frac{|\alpha|+|\beta|}{2}<|\alpha-\alpha_1|+|\beta-\beta_1|\le|\alpha|+|\beta|$, we also have

$$\begin{aligned}J_2 &\le C\frac{1}{\varepsilon}\| \,|\langle v\rangle^{-b}\partial^{\alpha_1}_{\beta'}f|_2\|_{L^\infty}\| \,|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f|_{\sigma,|\beta-\beta_1|}\|_{L^2}\| \,|\partial^\alpha_\beta f|_{\sigma,|\beta|}\|_{L^2}\\ &\le \eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}+C_\eta\varepsilon^{\frac12}\mathcal{D}_N(t).\end{aligned}$$

Therefore, with these estimates in hand, it follows that

$$\frac{1}{\varepsilon}|(\partial^\alpha_\beta\Gamma(f,f),w^{2|\beta|}\partial^\alpha_\beta f)|\le C\eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}+C_\eta\varepsilon^{\frac12}\mathcal{D}_N(t). \tag{4.28}$$

In summary, the desired estimate (4.22) follows from (4.24), (4.26), (4.27), and (4.28). By similar arguments, we can prove that (4.23) holds and we omit the details for brevity. This completes the proof of Lemma 4.7. ∎

4.5. Estimates of fluid quantities

In what follows we give the estimates of the fluid quantities involving the temporal derivatives, which will be used in Section 5.

Lemma 4.8. *For $|\alpha|\le N-1$, one has*

$$\|\partial^\alpha\partial_t(\tilde\rho,\tilde u,\tilde\theta)\|^2\le C\|\partial^\alpha\nabla_x(\tilde\rho,\tilde u,\tilde\theta)\|^2+C\|\langle v\rangle^{-\frac12}\partial^\alpha\nabla_x f\|^2+C(\eta_0+\varepsilon^{\frac12})\varepsilon^2. \tag{4.29}$$

For $|\alpha|\le N-2$, it also holds that

$$\|\partial^\alpha\partial_t(\tilde\rho,\tilde u,\tilde\theta)\|^2\le C\varepsilon^2. \tag{4.30}$$

Proof. Subtracting (1.2) from system (2.10) yields

$$\begin{cases}\partial_t\tilde\rho+u\cdot\nabla_x\tilde\rho+\bar\rho\nabla_x\cdot\tilde u+\tilde u\cdot\nabla_x\bar\rho+\tilde\rho\nabla_x\cdot u=0,\\ \partial_t\tilde u+u\cdot\nabla_x\tilde u+\dfrac{2\bar\theta}{3\bar\rho}\nabla_x\tilde\rho+\dfrac{2}{3}\nabla_x\tilde\theta+\tilde u\cdot\nabla_x\bar u+\dfrac{2}{3}\Big(\dfrac{\theta}{\rho}-\dfrac{\bar\theta}{\bar\rho}\Big)\nabla_x\rho\\ \quad=-\dfrac{1}{\rho}\displaystyle\int_{\mathbb{R}^3}v\otimes v\cdot\nabla_x G\,dv,\\ \partial_t\tilde\theta+u\cdot\nabla_x\tilde\theta+\dfrac{2}{3}\bar\theta\nabla_x\cdot\tilde u+\tilde u\cdot\nabla_x\bar\theta+\dfrac{2}{3}\tilde\theta\nabla_x\cdot u\\ \quad=-\dfrac{1}{\rho}\displaystyle\int_{\mathbb{R}^3}\dfrac12|v|^2v\cdot\nabla_x G\,dv+\dfrac{1}{\rho}u\cdot\displaystyle\int_{\mathbb{R}^3}v\otimes v\cdot\nabla_x G\,dv.\end{cases} \tag{4.31}$$

Applying ∂^α with $|\alpha| \le N-1$ to the second equation of (4.31) and taking the inner product of the resulting equation with $\partial^\alpha\partial_t\tilde{u}$, we arrive at

$$\begin{aligned}(\partial^\alpha\partial_t\tilde{u}, \partial^\alpha\partial_t\tilde{u}) &= -\Big(\partial^\alpha\Big[u\cdot\nabla_x\tilde{u} + \frac{2\bar{\theta}}{3\bar{\rho}}\nabla_x\tilde{\rho} + \frac{2}{3}\nabla_x\tilde{\theta} + \tilde{u}\cdot\nabla_x\bar{u} \\ &\quad + \frac{2}{3}\Big(\frac{\theta}{\rho} - \frac{\bar{\theta}}{\bar{\rho}}\Big)\nabla_x\rho\Big], \partial^\alpha\partial_t\tilde{u}\Big) \\ &\quad - \Big(\partial^\alpha\Big(\frac{1}{\rho}\int_{\mathbb{R}^3} v\otimes v\cdot\nabla_x G\,dv\Big), \partial^\alpha\partial_t\tilde{u}\Big) \\ &\le C\eta\|\partial^\alpha\partial_t\tilde{u}\|^2 + C_\eta\|\partial^\alpha(\nabla_x\tilde{\rho}, \nabla_x\tilde{u}, \nabla_x\tilde{\theta})\|^2 \\ &\quad + C_\eta\|\langle v\rangle^{-\frac{1}{2}}\partial^\alpha\nabla_x f\|^2 + C_\eta(\eta_0 + \varepsilon^{\frac{1}{2}})\varepsilon^2.\end{aligned}$$

Similar estimates also hold for $\partial^\alpha\partial_t\tilde{\rho}$ and $\partial^\alpha\partial_t\tilde{\theta}$. Therefore, choosing sufficiently small $\eta > 0$, we can obtain the desired estimate (4.29). If $|\alpha| \le N-2$, we can deduce from (4.1) that

$$\|\partial^\alpha(\nabla_x\tilde{\rho}, \nabla_x\tilde{u}, \nabla_x\tilde{\theta})\|^2 + \|\langle v\rangle^{-\frac{1}{2}}\partial^\alpha\nabla_x f\|^2 \le C\varepsilon^2.$$

This and (4.29) together give (4.30) by using the smallness of η_0 and ε. Thus the proof of Lemma 4.8 is completed. ∎

5. A priori estimates

This section is a core part in preparation for the proofs of the main results. We will obtain the desired a priori estimate (3.20) of the solution step by step in a series of lemmas in order to close the a priori assumption (4.1). In all the lemmas below, $F = M + \overline{G} + \sqrt{\mu} f$ is assumed to be the smooth solution to the Landau equation (1.1) and (3.19) for $t \in [0, T]$ with $T \in (0, \tau]$, and all derived estimates are satisfied for any $0 \le t \le T$. In the meantime, we assume that (4.1) and (4.2) are valid.

5.1. Zeroth-order estimate of the fluid part

We start from the zeroth-order energy estimates of the fluid part $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ by the entropy and entropy flux motivated in [52]. We show that the energy and energy dissipation for the fluid part are bounded by the dissipation up to first order for the non-fluid part. The proof also makes use of the dissipation mechanism for the Navier–Stokes-type equations (2.15), as well as the Euler-type equations (2.10).

Lemma 5.1. *It holds that*

$$\begin{aligned}&\|(\tilde{\rho}, \tilde{u}, \tilde{\theta})(t)\|^2 + c\varepsilon\int_0^t \|\nabla_x(\tilde{\rho}, \tilde{u}, \tilde{\theta})(s)\|^2\,ds \\ &\quad \le C\varepsilon\int_0^t (\|f(s)\|_\sigma^2 + \|\nabla_x f(s)\|_\sigma^2)\,ds + C(1+t)(\eta_0 + \varepsilon)\varepsilon^2. \qquad (5.1)\end{aligned}$$

Proof. As in [52], we define the macroscopic entropy S by

$$-\frac{3}{2}\rho S := \int_{\mathbb{R}^3} M \ln M \, dv.$$

By plugging in (2.3) and integrating, it follows that

$$S = -\frac{2}{3}\ln\rho + \ln(2\pi R\theta) + 1, \quad p = R\rho\theta = \frac{1}{2\pi e}\rho^{\frac{5}{3}}\exp(S), \quad R = \frac{2}{3}. \tag{5.2}$$

Multiplying (2.8) by $\ln M$ and integrating over v, direct computations give

$$-\frac{3}{2}\partial_t(\rho S) - \frac{3}{2}\nabla_x\cdot(\rho u S) + \nabla_x \cdot \int_{\mathbb{R}^3} vG\ln M\, dv = \int_{\mathbb{R}^3} \frac{Gv\cdot\nabla_x M}{M}\, dv.$$

We denote

$$\begin{aligned} m &:= (m_0, m_1, m_2, m_3, m_4)^{\mathsf{T}} = \Big(\rho, \rho u_1, \rho u_2, \rho u_3, \rho\Big(\theta + \frac{|u|^2}{2}\Big)\Big)^{\mathsf{T}}, \\ n &:= (n_0, n_1, n_2, n_3, n_4)^{\mathsf{T}} \\ &= \Big(\rho u, \rho u u_1 + p\mathbb{I}_1, \rho u u_2 + p\mathbb{I}_2, \rho u u_3 + p\mathbb{I}_3, \rho u\Big(\theta + \frac{|u|^2}{2}\Big) + pu\Big)^{\mathsf{T}}, \end{aligned}$$

where $\mathbb{I}_1 = (1,0,0)^{\mathsf{T}}$, $\mathbb{I}_2 = (0,1,0)^{\mathsf{T}}$, $\mathbb{I}_3 = (0,0,1)^{\mathsf{T}}$, and $(\cdot,\cdot,\cdot)^{\mathsf{T}}$ is the transpose of a row vector. Then the conservation law (2.15) can be rewritten as

$$\partial_t m + \nabla_x\cdot n = \begin{pmatrix} 0 \\ \varepsilon\sum_{j=1}^3 \partial_{x_j}[\mu(\theta)D_{1j}] - \int_{\mathbb{R}^3} v_1(v\cdot\nabla_x L_M^{-1}\Theta)\, dv \\ \varepsilon\sum_{j=1}^3 \partial_{x_j}[\mu(\theta)D_{2j}] - \int_{\mathbb{R}^3} v_2(v\cdot\nabla_x L_M^{-1}\Theta)\, dv \\ \varepsilon\sum_{j=1}^3 \partial_{x_j}[\mu(\theta)D_{3j}] - \int_{\mathbb{R}^3} v_3(v\cdot\nabla_x L_M^{-1}\Theta)\, dv \\ \varepsilon\nabla_x\cdot(\kappa(\theta)\nabla_x\theta) + \varepsilon\nabla_x\cdot[\mu(\theta)u\cdot D] - \int_{\mathbb{R}^3}\frac{1}{2}|v|^2 v\cdot\nabla_x L_M^{-1}\Theta\, dv \end{pmatrix},$$

where $D = [D_{ij}]_{1\le i,j\le 3}$ is given in (2.14). Define a relative entropy-entropy flux pair $(\eta, q)(t,x)$ around the local Maxwellian $\bar{M} = M_{[\bar\rho,\bar u,\bar S]}$ with $\bar S = -\frac{2}{3}\ln\bar\rho + \ln(2\pi R\bar\theta) + 1$ as

$$\begin{cases} \eta(t,x) = \bar\theta\Big\{-\frac{3}{2}\rho S + \frac{3}{2}\bar\rho\bar S + \frac{3}{2}\nabla_m(\rho S)\Big|_{m=\bar m}(m-\bar m)\Big\}, \\ q_j(t,x) = \bar\theta\Big\{-\frac{3}{2}\rho u_j S + \frac{3}{2}\bar\rho\bar u_j\bar S + \frac{3}{2}\nabla_m(\rho S)\Big|_{m=\bar m}\cdot(n_j - \bar n_j)\Big\}, \quad j = 1,2,3. \end{cases} \tag{5.3}$$

Here, $\bar m = (\bar\rho, \bar\rho\bar u_1, \bar\rho\bar u_2, \bar\rho\bar u_3, \bar\rho(\bar\theta + \frac{1}{2}|\bar u|^2))^{\mathsf{T}}$. Since

$$\partial_{m_0}(\rho S) = S + \frac{|u|^2}{2\theta} - \frac{5}{3}, \quad \partial_{m_i}(\rho S) = -\frac{u_i}{\theta}, \ i = 1,2,3, \quad \partial_{m_4}(\rho S) = \frac{1}{\theta}, \tag{5.4}$$

an elementary calculation leads to

$$\begin{cases} \eta(t,x) = \frac{3}{2}\Big\{\rho\theta - \bar\theta\rho S + \rho\Big[\Big(\bar S - \frac{5}{3}\Big)\bar\theta + \frac{|u-\bar u|^2}{2}\Big] + \frac{2}{3}\bar\rho\bar\theta\Big\} \\ \qquad\quad = \rho\bar\theta\Psi\Big(\frac{\bar\rho}{\rho}\Big) + \frac{3}{2}\rho\bar\theta\Psi\Big(\frac{\theta}{\bar\theta}\Big) + \frac{3}{4}\rho|u-\bar u|^2, \\ q_j(t,x) = u_j\eta(t,x) + (u_j - \bar u_j)(\rho\theta - \bar\rho\bar\theta), \qquad j = 1,2,3, \end{cases}$$

where $\Psi(s) = s - \ln s - 1$ is a strictly convex function around $s = 1$. By these facts and (3.4), for $\mathbf{X}$ in any closed bounded region in $\sum = \{\mathbf{X} : \rho > 0,\ \ \theta > 0\}$, there exists a constant $C > 1$ such that

$$C^{-1}|(\tilde{\rho}, \tilde{u}, \tilde{\theta})|^2 \leq \eta(t, x) \leq C|(\tilde{\rho}, \tilde{u}, \tilde{\theta})|^2. \tag{5.5}$$

Using (5.3) and making a direct calculation, it holds that

$$\begin{aligned}
&\partial_t \eta(t, x) + \nabla_x \cdot q(t, x) \\
&\quad = \nabla_{[\bar{\rho},\bar{u},\bar{S}]}\eta(t, x) \cdot \partial_t(\bar{\rho}, \bar{u}, \bar{S}) + \sum_{j=1}^{3} \nabla_{[\bar{\rho},\bar{u},\bar{S}]} q_j(t, x) \cdot \partial_{x_j}(\bar{\rho}, \bar{u}, \bar{S}) \\
&\qquad + \bar{\theta}\Big\{-\frac{3}{2}\partial_t(\rho S) - \frac{3}{2}\nabla_x \cdot (\rho u S)\Big\} + \frac{3}{2}\bar{\theta}\nabla_m(\rho S)\Big|_{m=\bar{m}}(\partial_t m + \nabla_x \cdot n).
\end{aligned} \tag{5.6}$$

In order to further estimate (5.6), we first claim that the following identity holds:

$$\begin{aligned}
&\nabla_{[\bar{\rho},\bar{u},\bar{S}]}\eta(t, x) \cdot \partial_t(\bar{\rho}, \bar{u}, \bar{S}) + \sum_{j=1}^{3} \nabla_{[\bar{\rho},\bar{u},\bar{S}]} q_j(t, x) \cdot \partial_{x_j}(\bar{\rho}, \bar{u}, \bar{S}) \\
&\quad = -\frac{3}{2}\rho\tilde{u} \cdot (\tilde{u} \cdot \nabla_x \bar{u}) - \frac{2}{3}\rho\bar{\theta}(\nabla_x \cdot \bar{u})\Psi\Big(\frac{\bar{\rho}}{\rho}\Big) - \rho\bar{\theta}(\nabla_x \cdot \bar{u})\Psi\Big(\frac{\theta}{\bar{\theta}}\Big) \\
&\qquad - \frac{3}{2}\rho\nabla_x\bar{\theta} \cdot \tilde{u}\Big(\frac{2}{3}\ln\frac{\bar{\rho}}{\rho} + \ln\frac{\theta}{\bar{\theta}}\Big),
\end{aligned} \tag{5.7}$$

whose proof is given in the appendix. Note that

$$\partial_t S = -\frac{2}{3}\frac{1}{\rho}\partial_t\rho + \frac{1}{\theta}\partial_t\theta, \quad \nabla_x S = -\frac{2}{3}\frac{1}{\rho}\nabla_x\rho + \frac{1}{\theta}\nabla_x\theta,$$

due to (5.2), then we deduce from this and (2.15) that

$$\begin{aligned}
&\bar{\theta}\Big\{-\frac{3}{2}\partial_t(\rho S) - \frac{3}{2}\nabla_x \cdot (\rho u S)\Big\} = \bar{\theta}\Big\{-\frac{3}{2}\frac{\rho}{\theta}(\partial_t\theta + u \cdot \nabla_x\theta) - \rho\nabla_x \cdot u\Big\} \\
&= -\frac{3}{2}\frac{\bar{\theta}}{\theta}\varepsilon\Big\{\sum_{j=1}^{3}\partial_{x_j}(\kappa(\theta)\partial_{x_j}\theta) + \sum_{i,j=1}^{3}(\mu(\theta)\partial_{x_j}u_i D_{ij})\Big\} \\
&\quad + \frac{3}{2}\frac{\bar{\theta}}{\theta}\Big\{\int_{\mathbb{R}^3}\frac{1}{2}|v|^2 v \cdot \nabla_x L_M^{-1}\Theta\, dv - \sum_{i=1}^{3} u_i\int_{\mathbb{R}^3} v_i v \cdot \nabla_x L_M^{-1}\Theta\, dv\Big\}.
\end{aligned}$$

In view of the conservation law and similar arguments to (5.4), one has

$$\begin{aligned}
&\frac{3}{2}\bar{\theta}\nabla_m(\rho S)|_{m=\bar{m}}(\partial_t m + \nabla_x \cdot n) \\
&\quad = \frac{3}{2}\sum_{i=1}^{3}\bar{u}_i\int_{\mathbb{R}^3} v_i v \cdot \nabla_x L_M^{-1}\Theta\, dv - \frac{3}{2}\int_{\mathbb{R}^3}\frac{1}{2}|v|^2 v \cdot \nabla_x L_M^{-1}\Theta\, dv + \frac{3}{2}\varepsilon\nabla_x \cdot (\kappa(\theta)\nabla_x\theta) \\
&\qquad - \frac{3}{2}\varepsilon\sum_{i,j=1}^{3}\bar{u}_i\partial_{x_j}[\mu(\theta)D_{ij}] + \frac{3}{2}\varepsilon\nabla_x \cdot [\mu(\theta)u \cdot D].
\end{aligned}$$

Therefore, plugging these estimates into (5.6) and making a direct computation, we get

$$
\begin{aligned}
&\partial_t \eta(t,x) + \nabla_x \cdot q(t,x) + \varepsilon \frac{3\bar{\theta}}{2\theta}\mu(\theta) \sum_{i,j=1}^{3} \partial_{x_j}\tilde{u}_i \Big(\partial_{x_j}\tilde{u}_i + \partial_{x_i}\tilde{u}_j - \frac{2}{3}\delta_{ij}\nabla_x \cdot \tilde{u}\Big) \\
&\quad + \varepsilon \frac{3\bar{\theta}}{2\theta^2}\kappa(\theta)|\nabla_x \tilde{\theta}|^2 \\
&= \frac{3}{2}\varepsilon \nabla_x \cdot [\mu(\theta)\tilde{u}\cdot D] + \frac{3}{2}\varepsilon \nabla_x \cdot \Big(\frac{\tilde{\theta}}{\theta}\kappa(\theta)\nabla_x \theta\Big) - \frac{3}{2}\rho\tilde{u}\cdot(\tilde{u}\cdot\nabla_x \bar{u}) \\
&\quad - \frac{2}{3}\rho\bar{\theta}(\nabla_x \cdot \bar{u})\Psi\Big(\frac{\bar{\rho}}{\rho}\Big) - \rho\bar{\theta}(\nabla_x \cdot \bar{u})\Psi\Big(\frac{\theta}{\bar{\theta}}\Big) - \frac{3}{2}\rho \nabla_x \bar{\theta}\cdot \tilde{u}\Big(\frac{2}{3}\ln\frac{\bar{\rho}}{\rho} + \ln\frac{\theta}{\bar{\theta}}\Big) \\
&\quad - \varepsilon \frac{3\bar{\theta}}{2\theta^2}\kappa(\theta)\nabla_x \tilde{\theta}\cdot \nabla_x \bar{\theta} + \varepsilon \frac{3\tilde{\theta}}{2\theta^2}\kappa(\theta)\nabla_x \bar{\theta}\cdot \nabla_x \theta + \varepsilon \frac{3\tilde{\theta}}{2\theta}\sum_{i,j=1}^{3}[\partial_{x_j}\bar{u}_i \mu(\theta) D_{ij}] \\
&\quad - \varepsilon \frac{3\bar{\theta}}{2\theta}\sum_{i,j=1}^{3}\mu(\theta)\partial_{x_j}\tilde{u}_i\Big(\partial_{x_j}\bar{u}_i + \partial_{x_i}\bar{u}_j - \frac{2}{3}\delta_{ij}\nabla_x \cdot \bar{u}\Big) + I_1. \qquad (5.8)
\end{aligned}
$$

Here I_1 is given as

$$
\begin{aligned}
I_1 &= -\frac{3}{2}\nabla_x \cdot \Big(\frac{\tilde{\theta}}{\theta}\int_{\mathbb{R}^3}\Big(\frac{1}{2}|v|^2 - u\cdot v\Big) v L_M^{-1}\Theta \, dv\Big) - \frac{3}{2}\nabla_x \cdot \Big(\sum_{i=1}^{3}\tilde{u}_i \int_{\mathbb{R}^3} v_i v L_M^{-1}\Theta \, dv\Big) \\
&\quad + \frac{3}{2}\nabla_x \Big(\frac{\tilde{\theta}}{\theta}\Big)\cdot \int_{\mathbb{R}^3}\Big(\frac{1}{2}|v|^2 - v\cdot u\Big) v L_M^{-1}\Theta \, dv + \frac{3}{2}\frac{\bar{\theta}}{\theta}\sum_{i=1}^{3}\nabla_x \tilde{u}_i \cdot \int_{\mathbb{R}^3} v_i v L_M^{-1}\Theta \, dv \\
&\quad - \frac{3}{2}\frac{\tilde{\theta}}{\theta}\sum_{i=1}^{3}\nabla_x \bar{u}_i \cdot \int_{\mathbb{R}^3} v_i v L_M^{-1}\Theta \, dv. \qquad (5.9)
\end{aligned}
$$

To complete the estimate of (5.8), we integrate it with respect to x over $\mathbb{R}^3$ and then compute the resulting equation for each term. Thanks to (3.4) and (4.1), one has

$$
\Psi\Big(\frac{\bar{\rho}}{\rho}\Big) \approx |\tilde{\rho}|^2, \quad \Psi\Big(\frac{\theta}{\bar{\theta}}\Big) \approx |\tilde{\theta}|^2, \quad \Big|\ln\frac{\bar{\rho}}{\rho}| \approx |\tilde{\rho}\Big|, \quad \Big|\ln\frac{\theta}{\bar{\theta}}| \approx |\tilde{\theta}\Big|.
$$

By taking the L^2–L^2–L^∞ estimate, we get from this, (4.2), (3.3), (4.1), and the imbedding inequality that

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\Big|\frac{3}{2}\rho\tilde{u}\cdot(\tilde{u}\cdot\nabla_x \bar{u}) + \frac{2}{3}\rho\bar{\theta}(\nabla_x \cdot \bar{u})\Psi\Big(\frac{\bar{\rho}}{\rho}\Big) + \rho\bar{\theta}(\nabla_x \cdot \bar{u})\Psi\Big(\frac{\theta}{\bar{\theta}}\Big) \\
&\qquad + \frac{3}{2}\rho\nabla_x \bar{\theta}\cdot \tilde{u}\Big(\frac{2}{3}\ln\frac{\bar{\rho}}{\rho} + \ln\frac{\theta}{\bar{\theta}}\Big)\Big| \, dx \\
&\qquad \le C(\|\nabla_x \bar{u}\|_{L^\infty} + \|\nabla_x \bar{\theta}\|_{L^\infty})\|(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2 \le C\eta_0 \varepsilon^2. \qquad (5.10)
\end{aligned}
$$

Since both $\mu(\theta)$ and $\kappa(\theta)$ are smooth functions of θ, there exists a constant $C > 1$ such that $\mu(\theta), \kappa(\theta) \in [C^{-1}, C]$. It follows from this, (4.2), (3.3), and (4.1) that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left(\left| \varepsilon \frac{3\bar{\theta}}{2\theta^2} \kappa(\theta) \nabla_x \tilde{\theta} \cdot \nabla_x \bar{\theta} \right| + \left| \varepsilon \frac{3\tilde{\theta}}{2\theta^2} \kappa(\theta) \nabla_x \bar{\theta} \cdot \nabla_x \theta \right| \right) dx \\
&\quad \le C\varepsilon \|\nabla_x \tilde{\theta}\| \, \|\nabla_x \bar{\theta}\| + C\varepsilon \|\nabla_x \bar{\theta}\|_{L^\infty} \|\tilde{\theta}\| \, \|\nabla_x \theta\| \le C(\eta_0 + \varepsilon)\varepsilon^2.
\end{aligned}$$

Recall D_{ij} in (2.14); then similar calculations to above lead us to

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left| \varepsilon \frac{3\tilde{\theta}}{2\theta} \sum_{i,j=1}^3 [\partial_{x_j} \bar{u}_i \mu(\theta) D_{ij}] \right| dx \\
&\qquad + \int_{\mathbb{R}^3} \left| \varepsilon \frac{3\bar{\theta}}{2\theta} \sum_{i,j=1}^3 \mu(\theta) \partial_{x_j} \tilde{u}_i \left(\partial_{x_j} \bar{u}_i + \partial_{x_i} \bar{u}_j - \frac{2}{3} \delta_{ij} \nabla_x \cdot \bar{u} \right) \right| dx \\
&\le C(\eta_0 + \varepsilon)\varepsilon^2.
\end{aligned}$$

Applying the above estimates together with (5.15), whose proof will be postponed to Lemma 5.3 later, we have from (5.8) that

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^3} \eta(t,x)\, dx - \frac{d}{dt} E(t) + c\varepsilon (\|\nabla_x \tilde{u}\|^2 + \|\nabla_x \tilde{\theta}\|^2) \\
&\qquad \le C\varepsilon \|f\|_\sigma^2 + C(\eta_0 + \varepsilon)\varepsilon^2.
\end{aligned} \tag{5.11}$$

Here, the following crucial estimate has been used:

$$\begin{aligned}
&\varepsilon \int_{\mathbb{R}^3} \frac{3\bar{\theta}}{2\theta} \mu(\theta) \sum_{i,j=1}^3 \partial_{x_j} \tilde{u}_i \left(\partial_{x_j} \tilde{u}_i + \partial_{x_i} \tilde{u}_j - \frac{2}{3} \delta_{ij} \nabla_x \cdot \tilde{u} \right) dx \\
&\quad = \varepsilon \sum_{i,j=1}^3 \int_{\mathbb{R}^3} \frac{3\bar{\theta}}{2\theta} \mu(\theta) (\partial_{x_j} \tilde{u}_i)^2 \, dx + \varepsilon \int_{\mathbb{R}^3} \frac{3\bar{\theta}}{2\theta} \mu(\theta) \frac{1}{3} (\nabla_x \cdot \tilde{u})^2 \, dx \\
&\qquad - \varepsilon \sum_{i,j=1}^3 \int_{\mathbb{R}^3} \partial_{x_i} \left(\frac{3\bar{\theta}}{2\theta} \mu(\theta) \right) \partial_{x_j} \tilde{u}_i \tilde{u}_j \, dx + \varepsilon \sum_{i,j=1}^3 \int_{\mathbb{R}^3} \partial_{x_j} \left(\frac{3\bar{\theta}}{2\theta} \mu(\theta) \right) \partial_{x_i} \tilde{u}_i \tilde{u}_j \, dx \\
&\quad \ge c\varepsilon \|\nabla_x \tilde{u}\|^2 - C(\eta_0 + \varepsilon)\varepsilon^2.
\end{aligned}$$

Recall $E(t)$ in (5.16). Then we employ (5.20), (3.3), and (4.1) to obtain

$$\begin{aligned}
|E(t)| &\le C\varepsilon \left\{ \left\| \nabla_x \left(\frac{\tilde{\theta}}{\theta} \right) \right\| \|f\| + \|\nabla_x \tilde{u}\| \, \|f\| + \|\nabla_x \bar{u}\|_{L^\infty} \|\tilde{\theta}\| \, \|f\| \right\} \\
&\le C(\eta_0 + \varepsilon)\varepsilon^2.
\end{aligned}$$

Integrating (5.11) with respect to t and using (3.22), (5.5), and the above estimate of $E(t)$, we have

$$\|(\tilde{\rho},\tilde{u},\tilde{\theta})(t)\|^2 + c\varepsilon\int_0^t \|\nabla_x(\tilde{u},\tilde{\theta})(s)\|^2\,ds \leq C\varepsilon\int_0^t \|f(s)\|_\sigma^2\,ds + C(1+t)(\eta_0+\varepsilon)\varepsilon^2. \tag{5.12}$$

Remark 5.2. Instead of the above proof based on the entropy-entropy flux method, one can also simply use the usual L^2 energy method similar to the one in the space derivative estimate of the fluid part in Lemma 5.6 in order to get the same estimate as (5.12), since the estimate is restricted to only the finite time interval $[0,\tau]$.

It should be noted that there are no dissipation terms for the density function in (5.12). For this, we turn to the Euler-type equations (2.10). Recall that the difference system (4.31) is derived from subtraction of (2.10) and (1.2). We then take the inner product of the second equation of (4.31) with $\nabla_x\tilde{\rho}$ to get

$$\begin{aligned}\Big(\frac{2\bar{\theta}}{3\bar{\rho}}\nabla_x\tilde{\rho},\nabla_x\tilde{\rho}\Big) &= -(\partial_t\tilde{u},\nabla_x\tilde{\rho}) - \Big(u\cdot\nabla_x\tilde{u} + \frac{2}{3}\nabla_x\tilde{\theta},\nabla_x\tilde{\rho}\Big)\\ &\quad - \Big(\tilde{u}\cdot\nabla_x\bar{u} + \frac{2}{3}\Big(\frac{\theta}{\rho}-\frac{\bar{\theta}}{\bar{\rho}}\Big)\nabla_x\rho,\nabla_x\tilde{\rho}\Big)\\ &\quad - \Big(\frac{1}{\rho}\int_{\mathbb{R}^3} v\otimes v\cdot\nabla_x G\,dv,\nabla_x\tilde{\rho}\Big).\end{aligned}$$

Using integration by parts and the first equation of (4.31), one gets

$$\begin{aligned}-(\partial_t\tilde{u},\nabla_x\tilde{\rho}) &= -\frac{d}{dt}(\tilde{u},\nabla_x\tilde{\rho}) - (\nabla_x\tilde{u},\partial_t\tilde{\rho})\\ &= -\frac{d}{dt}(\tilde{u},\nabla_x\tilde{\rho}) + (\nabla_x\tilde{u}, u\cdot\nabla_x\tilde{\rho} + \bar{\rho}\nabla_x\cdot\tilde{u} + \tilde{u}\cdot\nabla_x\bar{\rho} + \tilde{\rho}\nabla_x\cdot u)\\ &\leq -\frac{d}{dt}(\tilde{u},\nabla_x\tilde{\rho}) + C\eta\|\nabla_x\tilde{\rho}\|^2 + C_\eta\|\nabla_x\tilde{u}\|^2 + C(\eta_0+\varepsilon)\varepsilon^2,\end{aligned}$$

where in the last inequality we have used the Cauchy–Schwarz inequality, (4.2), (3.3), and (4.1). Likewise, we have

$$\begin{aligned}&\Big|\Big(u\cdot\nabla_x\tilde{u} + \frac{2}{3}\nabla_x\tilde{\theta},\nabla_x\tilde{\rho}\Big)\Big| + \Big|\Big(\tilde{u}\cdot\nabla_x\bar{u} + \frac{2}{3}\Big(\frac{\theta}{\rho}-\frac{\bar{\theta}}{\bar{\rho}}\Big)\nabla_x\rho,\nabla_x\tilde{\rho}\Big)\Big|\\ &\quad\leq C\eta\|\nabla_x\tilde{\rho}\|^2 + C_\eta(\|\nabla_x\tilde{u}\|^2 + \|\nabla_x\tilde{\theta}\|^2) + C_\eta(\eta_0+\varepsilon)\varepsilon^2.\end{aligned}$$

On the other hand, by the fact that $G=\overline{G}+\sqrt{\mu}f$ and (4.4), it holds that

$$\Big|\Big(\frac{1}{\rho}\int_{\mathbb{R}^3} v\otimes v\cdot\nabla_x G\,dv,\nabla_x\tilde{\rho}\Big)\Big| \leq C\eta\|\nabla_x\tilde{\rho}\|^2 + C_\eta\|\nabla_x f\|_\sigma^2 + C_\eta(\eta_0+\varepsilon)\varepsilon^2.$$

Collecting the above estimates and taking a small constant $\eta > 0$, one has

$$\frac{d}{dt}\varepsilon(\tilde{u}, \nabla_x \tilde{\rho}) + c\varepsilon\|\nabla_x \tilde{\rho}\|^2 \leq C\varepsilon(\|\nabla_x \tilde{u}\|^2 + \|\nabla_x \tilde{\theta}\|^2 + \|\nabla_x f\|_\sigma^2) + C(\eta_0 + \varepsilon)\varepsilon^3. \quad (5.13)$$

Integrating (5.13) with respect to t and using the estimate

$$\varepsilon|(\tilde{u}, \nabla_x \tilde{\rho})| \leq C\varepsilon\|\tilde{u}\| \, \|\nabla_x \tilde{\rho}\| \leq C\varepsilon^3,$$

as well as (3.22), we have

$$\varepsilon \int_0^t \|\nabla_x \tilde{\rho}(s)\|^2 \, ds \leq C\varepsilon \int_0^t \{\|\nabla_x(\tilde{u}, \tilde{\theta})(s)\|^2 + \|\nabla_x f\|_\sigma^2\} \, ds + C(1+t)(\eta_0 + \varepsilon)\varepsilon^2. \quad (5.14)$$

In summary, a suitable linear combination of (5.12) and (5.14) gives the desired estimate (5.1). This completes the proof of Lemma 5.1 for the zeroth-order energy estimates of the macroscopic component $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$. ∎

To deduce (5.11) in the proof of Lemma 5.1 above, we have used the following estimate on the basis of the Burnett functions.

Lemma 5.3. *Recall* (5.9) *for* I_1. *It holds that*

$$\int_{\mathbb{R}^3} I_1 \, dx \leq \frac{d}{dt} E(t) + C\varepsilon\|f\|_\sigma^2 + C(\eta_0 + \varepsilon)\varepsilon^2. \quad (5.15)$$

Here, $E(t)$ *is given by*

$$\begin{aligned} E(t) = {} & \frac{3}{2}\sum_{i=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_{x_i}\Big(\frac{\tilde{\theta}}{\theta}\Big)(R\theta)^{\frac{3}{2}} A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M} f \, dv \, dx \\ & + \frac{3}{2}\sum_{i,j=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_{x_j}\tilde{u}_i R\bar{\theta} B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M} f \, dv \, dx \\ & - \frac{3}{2}\sum_{i,j=1}^{3} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_{x_j}\bar{u}_i R\tilde{\theta} B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M} f \, dv \, dx. \end{aligned} \quad (5.16)$$

Proof. In order to estimate $\int_{\mathbb{R}^3} I_1 \, dx$, we only need to estimate the last three terms in (5.9) since other terms vanish after integration. First note that the following identities hold:

$$\begin{aligned} \int_{\mathbb{R}^3} \Big(\frac{1}{2}v_i|v|^2 - v_i u \cdot v\Big) L_M^{-1}\Theta \, dv &= \int_{\mathbb{R}^3} L_M^{-1}\Big\{P_1\Big(\frac{1}{2}v_i|v|^2 - v_i u \cdot v\Big)M\Big\}\frac{\Theta}{M} \, dv \\ &= \int_{\mathbb{R}^3} L_M^{-1}\Big\{(R\theta)^{\frac{3}{2}}\hat{A}_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)M\Big\}\frac{\Theta}{M} \, dv \\ &= (R\theta)^{\frac{3}{2}} \int_{\mathbb{R}^3} A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\Theta}{M} \, dv \end{aligned} \quad (5.17)$$

and

$$\begin{aligned}\int_{\mathbb{R}^3} v_i v_j L_M^{-1}\Theta\, dv &= \int_{\mathbb{R}^3} L_M^{-1}\{P_1(v_i v_j M)\}\frac{\Theta}{M}\, dv\\ &= \int_{\mathbb{R}^3} L_M^{-1}\Big\{R\theta \widehat{B}_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)M\Big\}\frac{\Theta}{M}\, dv\\ &= R\theta\int_{\mathbb{R}^3} B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\Theta}{M}\, dv,\end{aligned} \tag{5.18}$$

for $i, j = 1, 2, 3$, in terms of the self-adjoint property of L_M^{-1}, (2.5), (2.16), and (2.17). Hence, the third term in (5.9) can be rewritten as

$$\begin{aligned}&\int_{\mathbb{R}^3} \frac{3}{2}\nabla_x\Big(\frac{\tilde{\theta}}{\theta}\Big)\cdot\int_{\mathbb{R}^3}\Big(\frac{1}{2}|v|^2 - v\cdot u\Big)vL_M^{-1}\Theta\, dv\, dx\\ &\quad= \sum_{i=1}^{3}\int_{\mathbb{R}^3}\Big\{\frac{3}{2}\partial_{x_i}\Big(\frac{\tilde{\theta}}{\theta}\Big)(R\theta)^{\frac{3}{2}}\int_{\mathbb{R}^3} A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\Theta}{M}\, dv\Big\}\, dx.\end{aligned} \tag{5.19}$$

Before computing (5.19), we give the following desired estimate that, for any multi-index β and $k \geq 0$,

$$\int_{\mathbb{R}^3} \frac{|\langle v\rangle^k\sqrt{\mu}\partial_\beta A_i(\frac{v-u}{\sqrt{R\theta}})|^2}{M^2}\, dv + \int_{\mathbb{R}^3} \frac{|\langle v\rangle^k\sqrt{\mu}\partial_\beta B_{ij}(\frac{v-u}{\sqrt{R\theta}})|^2}{M^2}\, dv \leq C, \tag{5.20}$$

in terms of (4.3) and (4.2). Recall that Θ in (2.13) is given by

$$\Theta := \varepsilon\partial_t G + \varepsilon P_1(v\cdot\nabla_x G) - Q(G, G). \tag{5.21}$$

We now estimate (5.19) associated with (5.21). For the first term of (5.21), noting that $G = \overline{G} + \sqrt{\mu} f$, we use (5.20) and similar arguments to (4.4) to obtain

$$\begin{aligned}&\int_{\mathbb{R}^3}\Big\{\frac{3}{2}\partial_{x_i}\Big(\frac{\tilde{\theta}}{\theta}\Big)(R\theta)^{\frac{3}{2}}\int_{\mathbb{R}^3} A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\partial_t\overline{G}}{M}\, dv\Big\}\, dx\\ &\quad\leq C\varepsilon^2\int_{\mathbb{R}^3}\Big|\partial_{x_i}\Big(\frac{\tilde{\theta}}{\theta}\Big)\Big|\{|(\nabla_x\partial_t\bar{u}, \nabla_x\partial_t\bar{\theta})| + |(\nabla_x\bar{u}, \nabla_x\bar{\theta})|\cdot|(\partial_t u, \partial_t\theta)|\}\, dx\\ &\quad\leq C\varepsilon^2(\|\partial_{x_i}\tilde{\theta}\| + \|\tilde{\theta}\partial_{x_i}\theta\|)\{\|(\nabla_x\partial_t\bar{u}, \nabla_x\partial_t\bar{\theta})\| + \|(\nabla_x\bar{u}, \nabla_x\bar{\theta})\|_{L^\infty}\|(\partial_t u, \partial_t\theta)\|\}\\ &\quad\leq C\eta_0\varepsilon^3,\end{aligned}$$

where in the last inequality we have used (4.1) and the fact that

$$\|\partial_t(u, \theta)\| \leq C, \quad \|\partial_t(\bar{\rho}, \bar{u}, \bar{\theta})\|_{H^k} \leq C\eta_0, \quad \text{for } k \geq 3, \tag{5.22}$$

due to (4.30), (3.3), and (1.2). We use an integration by parts about t to get

$$\begin{aligned}&\int_{\mathbb{R}^3}\Big\{\frac{3}{2}\partial_{x_i}\Big(\frac{\tilde{\theta}}{\theta}\Big)(R\theta)^{\frac{3}{2}}\int_{\mathbb{R}^3} A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}\partial_t f}{M}\, dv\Big\}\, dx\\ &\quad= \frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{3}{2}\partial_{x_i}\Big(\frac{\tilde{\theta}}{\theta}\Big)(R\theta)^{\frac{3}{2}} A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M} f\, dv\, dx\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial_t\Big\{\frac{3}{2}\partial_{x_i}\Big(\frac{\tilde{\theta}}{\theta}\Big)(R\theta)^{\frac{3}{2}}A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M}\Big\}f\,dv\,dx\\
&\le\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big\{\frac{3}{2}\partial_{x_i}\Big(\frac{\tilde{\theta}}{\theta}\Big)(R\theta)^{\frac{3}{2}}A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M}f\Big\}\,dv\,dx+C(\eta_0+\varepsilon)\varepsilon^2,
\end{aligned}
$$

with the help of the following estimates:

$$
\begin{aligned}
&-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial_t\Big\{\frac{3}{2}\partial_{x_i}\Big(\frac{\tilde{\theta}}{\theta}\Big)(R\theta)^{\frac{3}{2}}A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M}\Big\}f\,dv\,dx\\
&\quad\le C\varepsilon\Big\|\partial_{x_i}\partial_t\Big(\frac{\tilde{\theta}}{\theta}\Big)\Big\|\|f\|+C\varepsilon\Big\|\partial_{x_i}\Big(\frac{\tilde{\theta}}{\theta}\Big)\Big\|\|\,|f|_2\|_{L^\infty_x}\|\partial_t(\rho,u,\theta)\|\\
&\quad\le C(\eta_0+\varepsilon)\varepsilon^2.
\end{aligned}
$$

Collecting the above estimates, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{3}\int_{\mathbb{R}^3}\Big\{\frac{3}{2}\partial_{x_i}\Big(\frac{\tilde{\theta}}{\theta}\Big)(R\theta)^{\frac{3}{2}}\int_{\mathbb{R}^3}A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\partial_t G}{M}\,dv\Big\}\,dx\\
&\quad\le\frac{d}{dt}\sum_{i=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big\{\frac{3}{2}\partial_{x_i}\Big(\frac{\tilde{\theta}}{\theta}\Big)(R\theta)^{\frac{3}{2}}A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M}f\Big\}\,dv\,dx+C(\eta_0+\varepsilon)\varepsilon^2.
\end{aligned}
$$

Similarly, it is straightforward to check that

$$
\begin{aligned}
&\sum_{i=1}^{3}\int_{\mathbb{R}^3}\Big\{\frac{3}{2}\partial_{x_i}\Big(\frac{\tilde{\theta}}{\theta}\Big)(R\theta)^{\frac{3}{2}}\int_{\mathbb{R}^3}A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon P_1(v\cdot\nabla_x G)}{M}\,dv\Big\}\,dx\\
&\quad=\sum_{i=1}^{3}\int_{\mathbb{R}^3}\Big\{\frac{3}{2}\partial_{x_i}\Big(\frac{\tilde{\theta}}{\theta}\Big)(R\theta)^{\frac{3}{2}}\int_{\mathbb{R}^3}A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon[v\cdot\nabla_x G-P_0(v\cdot\nabla_x G)]}{M}\,dv\Big\}\,dx\\
&\quad\le C(\eta_0+\varepsilon)\varepsilon^2.
\end{aligned}
$$

For the third term of (5.21), applying (3.11), (4.11), and (5.20) yields that

$$
\begin{aligned}
&\sum_{i=1}^{3}\Big|\int_{\mathbb{R}^3}\Big\{\frac{3}{2}\partial_{x_i}\Big(\frac{\tilde{\theta}}{\theta}\Big)(R\theta)^{\frac{3}{2}}\int_{\mathbb{R}^3}A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{Q(G,G)}{M}\,dv\Big\}\,dx\Big|\\
&\quad=\sum_{i=1}^{3}\Big|\int_{\mathbb{R}^3}\Big\{\frac{3}{2}\partial_{x_i}\Big(\frac{\tilde{\theta}}{\theta}\Big)(R\theta)^{\frac{3}{2}}\int_{\mathbb{R}^3}A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\sqrt{\mu}}{M}\Gamma\Big(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}\Big)\,dv\Big\}\,dx\Big|\\
&\quad\le C\int_{\mathbb{R}^3}\Big|\frac{G}{\sqrt{\mu}}\Big|_2\Big|\frac{G}{\sqrt{\mu}}\Big|_\sigma\Big|\nabla_x\Big(\frac{\tilde{\theta}}{\theta}\Big)\Big|\,dx\\
&\quad\le C\sup_{x\in\mathbb{R}^3}\Big(\Big|\frac{\overline{G}}{\sqrt{\mu}}\Big|_2+|f|_2\Big)\Big(\Big\|\frac{\overline{G}}{\sqrt{\mu}}\Big\|_\sigma+\|f\|_\sigma\Big)\Big\|\nabla_x\Big(\frac{\tilde{\theta}}{\theta}\Big)\Big\|\\
&\quad\le C\varepsilon\|f\|_\sigma^2+C(\eta_0+\varepsilon)\varepsilon^2,
\end{aligned}
$$

where in the last two inequalities we have used $G = \overline{G} + \sqrt{\mu} f$, Lemma 4.3, the Cauchy–Schwarz inequality, (3.3), and (4.1). Recalling (5.21) and plugging the estimates above into (5.19) leads to

$$\begin{aligned}
&\frac{3}{2}\int_{\mathbb{R}^3} \nabla_x\Big(\frac{\tilde{\theta}}{\theta}\Big)\cdot\int_{\mathbb{R}^3}\Big(\frac{1}{2}|v|^2 - v\cdot u\Big) v L_M^{-1}\Theta\, dv\, dx \\
&\quad\le \frac{d}{dt}\frac{3}{2}\sum_{i=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial_{x_i}\Big(\frac{\tilde{\theta}}{\theta}\Big)(R\theta)^{\frac{3}{2}} A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M} f\, dv\, dx \\
&\quad\quad + C\varepsilon\|f\|_\sigma^2 + C(\eta_0+\varepsilon)\varepsilon^2.
\end{aligned} \tag{5.23}$$

The estimation of the last two terms in (5.9) can be done similarly to (5.23) because they have the same structure. It follows that

$$\begin{aligned}
&\frac{3}{2}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\frac{\bar{\theta}}{\theta}\partial_{x_j}\tilde{u}_i\int_{\mathbb{R}^3} v_i v_j L_M^{-1}\Theta\, dv\, dx \\
&\quad = \frac{3}{2}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\bar{\theta}}{\theta}\partial_{x_j}\tilde{u}_i R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\Theta}{M}\, dv\, dx \\
&\quad \le \frac{d}{dt}\frac{3}{2}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial_{x_j}\tilde{u}_i R\bar{\theta} B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M} f\, dv\, dx \\
&\quad\quad + C\varepsilon\|f\|_\sigma^2 + C(\eta_0+\varepsilon)\varepsilon^2,
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{3}{2}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\frac{\tilde{\theta}}{\theta}\partial_{x_j}\bar{u}_i\int_{\mathbb{R}^3} v_i v_j L_M^{-1}\Theta\, dv\, dx \\
&\quad = -\frac{3}{2}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\tilde{\theta}}{\theta}\partial_{x_j}\bar{u}_i R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\Theta}{M}\, dv\, dx \\
&\quad \le -\frac{d}{dt}\frac{3}{2}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial_{x_j}\bar{u}_i R\tilde{\theta} B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M} f\, dv\, dx \\
&\quad\quad + C\varepsilon\|f\|_\sigma^2 + C(\eta_0+\varepsilon)\varepsilon^2.
\end{aligned}$$

With the above two estimates and (5.23) in hand, the desired estimate (5.15) follows. This then completes the proof of Lemma 5.3. ∎

5.2. Zeroth-order estimate of the non-fluid part

Next we make use of the microscopic equation (3.13) to derive the zeroth-order energy estimates of the non-fluid part f by using the properties of the linearized operator. We should emphasize that the fact that $f \in (\ker \mathcal{L})^{\perp}$ is crucial in these estimates.

Lemma 5.4. *It holds that*

$$\|f(t)\|^2 + c_1\frac{1}{\varepsilon}\int_0^t \|f(s)\|_\sigma^2\,ds \le C\varepsilon\int_0^t (\|(\nabla_x\tilde{u},\nabla_x\tilde{\theta})(s)\|^2 + \|\nabla_x f(s)\|_\sigma^2)\,ds + C(1+t)(\eta_0+\varepsilon)\varepsilon^2. \quad (5.24)$$

Proof. Recall the microscopic equation (3.13) together with Remarks 3.2 and 3.3. Taking the inner product of (3.13) with f over $\mathbb{R}^3\times\mathbb{R}^3$ and using (4.10), one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|f\|^2 + c_1\frac{1}{\varepsilon}\|f\|_\sigma^2\\
&\quad\le \frac{1}{\varepsilon}\Big(\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},f\Big)+\Gamma\Big(f,\frac{M-\mu}{\sqrt{\mu}}\Big),f\Big)+\frac{1}{\varepsilon}\Big(\Gamma\Big(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}\Big),f\Big)\\
&\qquad-\Big(\frac{P_1(v\cdot\nabla_x\bar{G})}{\sqrt{\mu}},f\Big)-\Big(\frac{\partial_t\bar{G}}{\sqrt{\mu}},f\Big)+\Big(\frac{P_0[v\cdot\nabla_x(\sqrt{\mu}f)]}{\sqrt{\mu}},f\Big)\\
&\qquad-\Big(\frac{1}{\sqrt{\mu}}P_1\Big\{v\cdot\Big(\frac{|v-u|^2\nabla_x\tilde{\theta}}{2R\theta^2}+\frac{(v-u)\cdot\nabla_x\tilde{u}}{R\theta}\Big)M\Big\},f\Big). \qquad (5.25)
\end{aligned}$$

We will deal with each term in (5.25). For the first term on the right-hand side of (5.25), in view of (4.11), (4.16) and (3.16), we get

$$\begin{aligned}
&\frac{1}{\varepsilon}|(\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},f\Big),f)|+\frac{1}{\varepsilon}|(\Gamma\Big(f,\frac{M-\mu}{\sqrt{\mu}}\Big),f)|\\
&\le C\frac{1}{\varepsilon}\int_{\mathbb{R}^3}|\langle v\rangle^{-1}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)|_2|f|_\sigma^2\,dx + C\frac{1}{\varepsilon}\int_{\mathbb{R}^3}|\langle v\rangle^{-1}f|_2|\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)|_\sigma|f|_\sigma\,dx\\
&\le C(\varepsilon+\eta_0)\frac{1}{\varepsilon}\|f\|_\sigma^2.
\end{aligned}$$

For the second term on the right-hand side of (5.25), we first note that

$$\Gamma\Big(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}\Big)=\Gamma\Big(\frac{\bar{G}}{\sqrt{\mu}},\frac{\bar{G}}{\sqrt{\mu}}\Big)+\Gamma\Big(\frac{\bar{G}}{\sqrt{\mu}},f\Big)+\Gamma\Big(f,\frac{\bar{G}}{\sqrt{\mu}}\Big)+\Gamma(f,f).$$

Then, using (4.11), Lemma 4.3, the Cauchy–Schwarz and Sobolev inequalities, (3.3), (4.1), and (4.2), we get

$$\begin{aligned}
\frac{1}{\varepsilon}\Big|\Big(\Gamma\Big(\frac{\bar{G}}{\sqrt{\mu}},\frac{\bar{G}}{\sqrt{\mu}}\Big),f\Big)\Big| &\le C\varepsilon\int_{\mathbb{R}^3}|(\nabla_x\bar{u},\nabla_x\bar{\theta})|^2|f|_\sigma\,dx \le C\eta_0\frac{1}{\varepsilon}\|f\|_\sigma^2 + C\eta_0\varepsilon^3,\\
\frac{1}{\varepsilon}\Big|\Big(\Gamma\Big(\frac{\bar{G}}{\sqrt{\mu}},f\Big),f\Big)\Big|+\frac{1}{\varepsilon}\Big|\Big(\Gamma\Big(f,\frac{\bar{G}}{\sqrt{\mu}}\Big),f\Big)\Big| &\le C\int_{\mathbb{R}^3}|(\nabla_x\bar{u},\nabla_x\bar{\theta})|\,|f|_\sigma^2\,dx\\
&\le C\eta_0\|f\|_\sigma^2,
\end{aligned}$$

and

$$\frac{1}{\varepsilon}|(\Gamma(f,f),f)| \le C\frac{1}{\varepsilon}\|\,|f|_2\|_{L_x^\infty}\|f\|_\sigma^2 \le C\|f\|_\sigma^2.$$

With these, it follows that

$$\frac{1}{\varepsilon}\Big|\Big(\Gamma\Big(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}\Big),f\Big)\Big|\le C(\eta_0+\varepsilon)\frac{1}{\varepsilon}\|f\|_\sigma^2+C\eta_0\varepsilon^3.$$

For the fourth term on the right-hand side of (5.25), applying similar arguments to (4.4), (3.16), (5.22), and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\Big|\Big(\frac{\partial_t\overline{G}}{\sqrt{\mu}},f\Big)\Big|&\le \eta\frac{1}{\varepsilon}\|\langle v\rangle^{-\frac12}f\|^2+C_\eta\varepsilon\Big\|\langle v\rangle^{\frac12}\frac{\partial_t\overline{G}}{\sqrt{\mu}}\Big\|^2\\&\le C\eta\frac{1}{\varepsilon}\|f\|_\sigma^2+C_\eta\varepsilon^3\int_{\mathbb{R}^3}\{|\partial_t(\nabla_x\bar{u},\nabla_x\bar{\theta})|+|(\nabla_x\bar{u},\nabla_x\bar{\theta})|\,|\partial_t(u,\theta)|\}^2\,dx\\&\le C\eta\frac{1}{\varepsilon}\|f\|_\sigma^2+C_\eta(\eta_0+\varepsilon)\varepsilon^2.\end{aligned}$$

The third term on the right-hand side of (5.25) shares a similar bound to

$$\begin{aligned}\Big|\Big(\frac{P_1(v\cdot\nabla_x\overline{G})}{\sqrt{\mu}},f\Big)\Big|&=\Big|\Big(\frac{v\cdot\nabla_x\overline{G}}{\sqrt{\mu}},f\Big)-\Big(\frac{P_0(v\cdot\nabla_x\overline{G})}{\sqrt{\mu}},f\Big)\Big|\\&\le C\eta\frac{1}{\varepsilon}\|f\|_\sigma^2+C_\eta(\eta_0+\varepsilon)\varepsilon^2.\end{aligned}$$

For the fifth term on the right-hand side of (5.25), we deduce from (2.5), (2.4), (4.2), (3.16), and the Cauchy–Schwarz inequality that

$$\begin{aligned}\Big|\Big(\frac{P_0[v\cdot\nabla_x(\sqrt{\mu}f)]}{\sqrt{\mu}},f\Big)\Big|&=\Big|\Big(\frac{1}{\sqrt{\mu}}\sum_{i=0}^4\Big\langle v\sqrt{\mu}\cdot\nabla_xf,\frac{\chi_i}{M}\Big\rangle\chi_i,f\Big)\Big|\\&\le\eta\frac{1}{\varepsilon}\|\langle v\rangle^{-\frac12}f\|^2+C_\eta\varepsilon\Big\|\langle v\rangle^{\frac12}\frac{1}{\sqrt{\mu}}\sum_{i=0}^4\Big\langle v\sqrt{\mu}\cdot\nabla_xf,\frac{\chi_i}{M}\Big\rangle\chi_i\Big\|^2\\&\le C\eta\frac{1}{\varepsilon}\|f\|_\sigma^2+C_\eta\varepsilon\|\nabla_xf\|_\sigma^2,\end{aligned}$$

where we have used the fact that $|\langle v\rangle^l\mu^{-\frac12}M|_2\le C$ for any $l\ge 0$ by (4.2). The last term of (5.25) can be handled in the same manner and it is bounded by

$$\begin{aligned}&\Big|\Big(\frac{1}{\sqrt{\mu}}P_1\Big\{v\cdot\Big(\frac{|v-u|^2\nabla_x\tilde{\theta}}{2R\theta^2}+\frac{(v-u)\cdot\nabla_x\tilde{u}}{R\theta}\Big)M\Big\},f\Big)\Big|\\&\quad\le C\eta\frac{1}{\varepsilon}\|f\|_\sigma^2+C_\eta\varepsilon\|(\nabla_x\tilde{u},\nabla_x\tilde{\theta})\|^2.\end{aligned}$$

In summary, we substitute the above estimates into (5.25) and choose $\eta>0$, $\eta_0>0$, and $\varepsilon>0$ suitably small to get

$$\frac12\frac{d}{dt}\|f\|^2+\frac{c_1}{2}\frac{1}{\varepsilon}\|f\|_\sigma^2\le C\varepsilon(\|(\nabla_x\tilde{u},\nabla_x\tilde{\theta})\|^2+\|\nabla_xf\|_\sigma^2)+C(\eta_0+\varepsilon)\varepsilon^2.\qquad(5.26)$$

Integrating (5.26) with respect to t and using (3.22) yields the desired estimate (5.24). This then completes the proof of Lemma 5.4. ■

Combining Lemma 5.4 with Lemma 5.1 immediately implies the following result, which gives the estimates of the zeroth-order energy norm for both the fluid and non-fluid parts.

Lemma 5.5. *It holds that*

$$\|(\tilde{\rho},\tilde{u},\tilde{\theta})(t)\|^2+\|f(t)\|^2+c\varepsilon\int_0^t\|\nabla_x(\tilde{\rho},\tilde{u},\tilde{\theta})(s)\|^2\,ds+c\frac{1}{\varepsilon}\int_0^t\|f(s)\|_\sigma^2\,ds$$
$$\leq C\varepsilon\int_0^t\|\nabla_x f(s)\|_\sigma^2\,ds+C(1+t)(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2. \tag{5.27}$$

Proof. Multiplying (5.1) by a large constant $C>0$ and then adding the resultant equation to (5.24), one can obtain (5.27) by using the smallness of ε. This completes the proof of Lemma 5.5. ∎

5.3. Space derivative estimate of the fluid part up to $(N-1)$th order

This subsection is devoted to deriving the space derivative estimate up to the $(N-1)$th order for the fluid part $(\tilde{\rho},\tilde{u},\tilde{\theta})$. As in Section 5.1, the proof is based on the fluid-type systems (3.5) and (4.31).

Lemma 5.6. *It holds that*

$$\sum_{1\leq|\alpha|\leq N-1}\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})(t)\|^2+c\varepsilon\sum_{2\leq|\alpha|\leq N}\int_0^t\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})(s)\|^2\,ds$$
$$\leq C\varepsilon^2\sum_{|\alpha|=N}(\|\partial^\alpha\tilde{\rho}(t)\|^2+\|\partial^\alpha f(t)\|^2)+C\varepsilon\sum_{1\leq|\alpha|\leq N}\int_0^t\|\partial^\alpha f(s)\|_\sigma^2\,ds$$
$$+C(\eta_0+\varepsilon^{\frac{1}{2}})\int_0^t\mathcal{D}_N(s)\,ds+C(1+t)(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2. \tag{5.28}$$

Proof. It is divided into six steps as follows. In the first three steps we make the direct energy estimates of $\tilde{\rho}$, $\tilde{u}$, and $\tilde{\theta}$ in terms of the Navier–Stokes-type system (3.5) and then obtain the combined estimate in Step 4. In Step 5 we use the Euler-type system (4.31) to obtain the energy dissipation of $\tilde{\rho}$ as in Section 5.1 for the zeroth-order estimate. In the last step we combine those results to deduce the desired estimate (5.28).

Step 1. Applying ∂^α with $1\leq|\alpha|\leq N-1$ to the first equation of (3.5) and taking the inner product of the resulting equation with $\frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}$, one has

$$\frac{1}{2}\frac{d}{dt}\Big(\partial^\alpha\tilde{\rho},\frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\Big)-\frac{1}{2}\Big(\partial^\alpha\tilde{\rho},\partial_t\Big(\frac{2\bar{\theta}}{3\bar{\rho}^2}\Big)\partial^\alpha\tilde{\rho}\Big)+\Big(\bar{\rho}\nabla_x\cdot\partial^\alpha\tilde{u},\frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\Big)$$
$$+\sum_{1\leq\alpha_1\leq\alpha}C_\alpha^{\alpha_1}\Big(\partial^{\alpha_1}\bar{\rho}\nabla_x\cdot\partial^{\alpha-\alpha_1}\tilde{u},\frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\Big)$$

$$
\begin{aligned}
&= -\Big(\partial^\alpha(u\cdot\nabla_x\tilde{\rho}), \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\Big) - \Big(\partial^\alpha(\tilde{u}\cdot\nabla_x\bar{\rho}), \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\Big)\\
&\quad - \Big(\partial^\alpha(\tilde{\rho}\nabla_x\cdot u), \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\Big).
\end{aligned}
\tag{5.29}
$$

Let us now deal with (5.29) term by term. By the Sobolev inequality, (5.22), and (4.1), one has

$$
\Big|\Big(\partial^\alpha\tilde{\rho}, \partial_t\Big(\frac{2\bar{\theta}}{3\bar{\rho}^2}\Big)\partial^\alpha\tilde{\rho}\Big)\Big| \le C(\|\partial_t\bar{\rho}\|_{L^\infty} + \|\partial_t\bar{\theta}\|_{L^\infty})\|\partial^\alpha\tilde{\rho}\|^2 \le C\eta_0\|\partial^\alpha\tilde{\rho}\|^2 \le C\eta_0\varepsilon^2.
$$

Performing a similar calculation to the above estimate implies

$$
\sum_{1\le\alpha_1\le\alpha} C_\alpha^{\alpha_1}\Big|\Big(\partial^{\alpha_1}\bar{\rho}\nabla_x\cdot\partial^{\alpha-\alpha_1}\tilde{u}, \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\Big)\Big| \le C\eta_0\varepsilon^2.
$$

The first term on the right-hand side of (5.29) can be written as

$$
\begin{aligned}
-\Big(\partial^\alpha(u\cdot\nabla_x\tilde{\rho}), \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\Big) &= -\Big(u\cdot\nabla_x\partial^\alpha\tilde{\rho}, \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\Big)\\
&\quad - \sum_{1\le\alpha_1\le\alpha} C_\alpha^{\alpha_1}\Big(\partial^{\alpha_1}u\cdot\nabla_x\partial^{\alpha-\alpha_1}\tilde{\rho}, \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\Big).
\end{aligned}
$$

By virtue of integration by parts, the Sobolev inequality, (3.3), and (4.1), we obtain

$$
\begin{aligned}
\Big|\Big(u\cdot\nabla_x\partial^\alpha\tilde{\rho}, \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\Big)\Big| &= \Big|\frac{1}{2}\Big(\partial^\alpha\tilde{\rho}, \nabla_x\cdot\Big(u\frac{2\bar{\theta}}{3\bar{\rho}^2}\Big)\partial^\alpha\tilde{\rho}\Big)\Big|\\
&\le C(\|\nabla_x(\bar{\rho},\bar{u},\bar{\theta})\|_{L^\infty} + \|\nabla_x\tilde{u}\|_{L^\infty})\|\partial^\alpha\tilde{\rho}\|^2 \le C(\eta_0 + \varepsilon^{\frac{1}{2}})\varepsilon^2,
\end{aligned}
$$

where we have used the fact that

$$
\|\nabla_x\tilde{u}\|_{L^\infty} \le C\|\nabla_x^2\tilde{u}\|^{\frac{1}{2}}\|\nabla_x^3\tilde{u}\|^{\frac{1}{2}} \le C\varepsilon^{\frac{1}{2}}.
\tag{5.30}
$$

For $1 \le |\alpha_1| \le |\alpha|$, it is clear to see that

$$
\begin{aligned}
\Big(\partial^{\alpha_1}u\cdot\nabla_x\partial^{\alpha-\alpha_1}\tilde{\rho}, \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\Big) &= \Big(\partial^{\alpha_1}\tilde{u}\cdot\nabla_x\partial^{\alpha-\alpha_1}\tilde{\rho}, \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\Big)\\
&\quad + \Big(\partial^{\alpha_1}\bar{u}\cdot\nabla_x\partial^{\alpha-\alpha_1}\tilde{\rho}, \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\Big).
\end{aligned}
$$

If $1 \le |\alpha_1| \le |\alpha|/2$, we use similar arguments to (5.30) and (4.1) to get

$$
\Big|\Big(\partial^{\alpha_1}\tilde{u}\cdot\nabla_x\partial^{\alpha-\alpha_1}\tilde{\rho}, \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\Big)\Big| \le \|\partial^{\alpha_1}\tilde{u}\|_{L^\infty}\|\nabla_x\partial^{\alpha-\alpha_1}\tilde{\rho}\|\,\|\partial^\alpha\tilde{\rho}\| \le C\varepsilon^{\frac{1}{2}}\varepsilon^2.
$$

If $|\alpha|/2 < |\alpha_1| \le |\alpha|$, we have the same bound as

$$\left|\left(\partial^{\alpha_1}\tilde{u}\cdot\nabla_x\partial^{\alpha-\alpha_1}\tilde{\rho}, \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\right)\right| \le C\|\partial^{\alpha_1}\tilde{u}\|\,\|\nabla_x\partial^{\alpha-\alpha_1}\tilde{\rho}\|_{L^\infty}\|\partial^\alpha\tilde{\rho}\| \le C\varepsilon^{\frac{1}{2}}\varepsilon^2.$$

Thanks to these estimates, it follows that

$$\sum_{1\le\alpha_1\le\alpha} C_\alpha^{\alpha_1}\left|\left(\partial^{\alpha_1}\tilde{u}\cdot\nabla_x\partial^{\alpha-\alpha_1}\tilde{\rho}, \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\right)\right| \le C\varepsilon^{\frac{1}{2}}\varepsilon^2.$$

On the other hand, we can obtain

$$\sum_{1\le\alpha_1\le\alpha} C_\alpha^{\alpha_1}\left|\left(\partial^{\alpha_1}\bar{u}\cdot\nabla_x\partial^{\alpha-\alpha_1}\tilde{\rho}, \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\right)\right| \le C\eta_0\varepsilon^2.$$

Collecting the above estimates, we thereby obtain

$$\left|\left(\partial^\alpha(u\cdot\nabla_x\tilde{\rho}), \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\right)\right| \le C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2. \tag{5.31}$$

The second term on the right-hand side of (5.29) is relatively easy and it is bounded by

$$\left|\left(\partial^\alpha(\tilde{u}\cdot\nabla_x\bar{\rho}), \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\right)\right| \le C\sum_{\alpha_1\le\alpha}\|\nabla_x\partial^{\alpha-\alpha_1}\bar{\rho}\|_{L^\infty}\|\partial^{\alpha_1}\tilde{u}\|\,\|\partial^\alpha\tilde{\rho}\| \le C\eta_0\varepsilon^2. \tag{5.32}$$

We divide the last term of (5.29) into three parts:

$$\begin{aligned}-\left(\partial^\alpha(\tilde{\rho}\nabla_x\cdot u), \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\right) &= -\left(\tilde{\rho}\nabla_x\cdot\partial^\alpha\tilde{u}, \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\right) - \left(\partial^\alpha(\tilde{\rho}\nabla_x\cdot\bar{u}), \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\right)\\ &\quad - \sum_{1\le\alpha_1\le\alpha} C_\alpha^{\alpha_1}\left(\partial^{\alpha_1}\tilde{\rho}\nabla_x\cdot\partial^{\alpha-\alpha_1}\tilde{u}, \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\right).\end{aligned}$$

The last two terms of the above equality can be treated in the same way as (5.31) and (5.32), so that we obtain

$$\sum_{1\le\alpha_1\le\alpha} C_\alpha^{\alpha_1}\left|\left(\partial^{\alpha_1}\tilde{\rho}\nabla_x\cdot\partial^{\alpha-\alpha_1}\tilde{u}, \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\right)\right| + \left|\left(\partial^\alpha(\tilde{\rho}\nabla_x\cdot\bar{u}), \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\right)\right| \le C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2.$$

Applying the Cauchy–Schwarz and Sobolev inequalities together with (4.1) gives

$$\left|\left(\tilde{\rho}\nabla_x\cdot\partial^\alpha\tilde{u}, \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\right)\right| \le \eta\varepsilon\|\nabla_x\cdot\partial^\alpha\tilde{u}\|^2 + C_\eta\frac{1}{\varepsilon}\|\tilde{\rho}\|_{L^\infty}^2\|\partial^\alpha\tilde{\rho}\|^2 \le \eta\varepsilon\|\nabla_x\cdot\partial^\alpha\tilde{u}\|^2 + C_\eta\varepsilon^3.$$

It follows from the above estimates that

$$\left|\left(\partial^\alpha(\tilde{\rho}\nabla_x\cdot u), \frac{2\bar{\theta}}{3\bar{\rho}^2}\partial^\alpha\tilde{\rho}\right)\right| \le C\eta\varepsilon\|\nabla_x\cdot\partial^\alpha\tilde{u}\|^2 + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2. \tag{5.33}$$

For $1 \le |\alpha| \le N-1$ and any small $\eta > 0$, we substitute the above estimates into (5.29) to get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\frac{2\bar{\theta}}{3\bar{\rho}^2}|\partial^\alpha\tilde{\rho}|^2\,dx + \Big(\nabla_x\cdot\partial^\alpha\tilde{u}, \frac{2\bar{\theta}}{3\bar{\rho}}\partial^\alpha\tilde{\rho}\Big) \le C\eta\varepsilon\|\nabla_x\cdot\partial^\alpha\tilde{u}\|^2 + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2. \tag{5.34}$$

Step 2. Next we concentrate on the second equation of (3.5). Applying ∂^α with $1 \le |\alpha| \le N-1$ to the second equation of (3.5) and taking the inner product of the resulting equation with $\partial^\alpha\tilde{u}_i$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\tilde{u}_i\|^2 + \Big(\frac{2\bar{\theta}}{3\bar{\rho}}\partial^\alpha\partial_{x_i}\tilde{\rho}, \partial^\alpha\tilde{u}_i\Big) + \sum_{1\le\alpha_1\le\alpha} C_\alpha^{\alpha_1}\Big(\partial^{\alpha_1}\Big(\frac{2\bar{\theta}}{3\bar{\rho}}\Big)\partial^{\alpha-\alpha_1}\partial_{x_i}\tilde{\rho}, \partial^\alpha\tilde{u}_i\Big)\\
&\quad + \Big(\frac{2}{3}\partial^\alpha\partial_{x_i}\tilde{\theta}, \partial^\alpha\tilde{u}_i\Big) + (\partial^\alpha(u\cdot\nabla_x\tilde{u}_i) + \partial^\alpha(\tilde{u}\cdot\nabla_x\bar{u}_i), \partial^\alpha\tilde{u}_i)\\
&\quad + \Big(\partial^\alpha\Big[\frac{2}{3}\Big(\frac{\theta}{\rho}-\frac{\bar{\theta}}{\bar{\rho}}\Big)\partial_{x_i}\rho\Big], \partial^\alpha\tilde{u}_i\Big)\\
&\quad = \varepsilon\sum_{j=1}^{3}\Big(\partial^\alpha\Big(\frac{1}{\rho}\partial_{x_j}[\mu(\theta)D_{ij}]\Big), \partial^\alpha\tilde{u}_i\Big)\\
&\quad - \Big(\partial^\alpha\Big(\frac{1}{\rho}\int_{\mathbb{R}^3} v_i v\cdot\nabla_x L_M^{-1}\Theta\,dv\Big), \partial^\alpha\tilde{u}_i\Big).
\end{aligned} \tag{5.35}$$

We will estimate each term for (5.35). Following the same method as in (5.31) and (5.32), we get

$$\begin{aligned}
&\sum_{1\le\alpha_1\le\alpha} C_\alpha^{\alpha_1}\Big|\Big(\partial^{\alpha_1}\Big(\frac{2\bar{\theta}}{3\bar{\rho}}\Big)\partial^{\alpha-\alpha_1}\partial_{x_i}\tilde{\rho}, \partial^\alpha\tilde{u}_i\Big)\Big| + |(\partial^\alpha(u\cdot\nabla_x\tilde{u}_i) + \partial^\alpha(\tilde{u}\cdot\nabla_x\bar{u}_i), \partial^\alpha\tilde{u}_i)|\\
&\quad \le C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2.
\end{aligned}$$

Carrying out similar calculations to (5.33), one can arrive at

$$\Big|\Big(\partial^\alpha\Big[\frac{2}{3}\Big(\frac{\theta}{\rho}-\frac{\bar{\theta}}{\bar{\rho}}\Big)\partial_{x_i}\rho\Big], \partial^\alpha\tilde{u}_i\Big)\Big| \le C\eta\varepsilon\|\partial_{x_i}\partial^\alpha\tilde{u}_i\|^2 + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2.$$

We will carefully deal with the first term on the right-hand side of (5.35). By the definition of D_{ij} in (2.14), we first write

$$\begin{aligned}
&\varepsilon\sum_{j=1}^{3}\Big(\partial^\alpha\Big(\frac{1}{\rho}\partial_{x_j}[\mu(\theta)D_{ij}]\Big), \partial^\alpha\tilde{u}_i\Big)\\
&\quad = \varepsilon\sum_{j=1}^{3}\Big(\partial^\alpha\Big(\frac{1}{\rho}\partial_{x_j}\Big[\mu(\theta)\Big(\partial_{x_j}\tilde{u}_i + \partial_{x_i}\tilde{u}_j - \frac{2}{3}\delta_{ij}\nabla_x\cdot\tilde{u}\Big)\Big]\Big), \partial^\alpha\tilde{u}_i\Big)\\
&\qquad + \varepsilon\sum_{j=1}^{3}\Big(\partial^\alpha\Big(\frac{1}{\rho}\partial_{x_j}\Big[\mu(\theta)\Big(\partial_{x_j}\bar{u}_i + \partial_{x_i}\bar{u}_j - \frac{2}{3}\delta_{ij}\nabla_x\cdot\bar{u}\Big)\Big]\Big), \partial^\alpha\tilde{u}_i\Big) := I_2 + I_3.
\end{aligned}$$

By integration by parts, the term I_2 can be reduced to

$$\begin{aligned} I_2 &= -\varepsilon\sum_{j=1}^{3}\Big(\partial^\alpha\Big[\frac{1}{\rho}\mu(\theta)\Big(\partial_{x_j}\tilde{u}_i+\partial_{x_i}\tilde{u}_j-\frac{2}{3}\delta_{ij}\nabla_x\cdot\tilde{u}\Big)\Big],\partial^\alpha\partial_{x_j}\tilde{u}_i\Big) \\ &\quad -\varepsilon\sum_{j=1}^{3}\Big(\partial^\alpha\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)\mu(\theta)(\partial_{x_j}\tilde{u}_i+\partial_{x_i}\tilde{u}_j-\frac{2}{3}\delta_{ij}\nabla_x\cdot\tilde{u})\Big],\partial^\alpha\tilde{u}_i\Big) \\ &:= I_2^1+I_2^2. \end{aligned}$$

To compute the term I_2, it only suffices to estimate I_2^1 and I_2^2. Note that

$$\begin{aligned} I_2^1 &= -\varepsilon\sum_{j=1}^{3}\Big(\frac{1}{\rho}\mu(\theta)\Big(\partial^\alpha\partial_{x_j}\tilde{u}_i+\partial^\alpha\partial_{x_i}\tilde{u}_j-\frac{2}{3}\delta_{ij}\nabla_x\cdot\partial^\alpha\tilde{u}\Big),\partial^\alpha\partial_{x_j}\tilde{u}_i\Big) \\ &\quad -\varepsilon\sum_{j=1}^{3}\sum_{1\le\alpha_1\le\alpha}C_\alpha^{\alpha_1}\Big(\partial^{\alpha_1}\Big[\frac{1}{\rho}\mu(\theta)\Big]\partial^{\alpha-\alpha_1}\Big(\partial_{x_j}\tilde{u}_i+\partial_{x_i}\tilde{u}_j-\frac{2}{3}\delta_{ij}\nabla_x\cdot\tilde{u}\Big),\partial^\alpha\partial_{x_j}\tilde{u}_i\Big). \end{aligned}$$

Applying the Sobolev inequality, (3.3), and (4.1), we have, with $1\le|\alpha_1|\le|\alpha|/2$,

$$\begin{aligned} &\varepsilon\Big|\Big(\partial^{\alpha_1}\Big[\frac{1}{\rho}\mu(\theta)\Big]\partial^{\alpha-\alpha_1}\Big(\partial_{x_j}\tilde{u}_i+\partial_{x_i}\tilde{u}_j-\frac{2}{3}\delta_{ij}\nabla_x\cdot\tilde{u}\Big),\partial^\alpha\partial_{x_j}\tilde{u}_i\Big)\Big| \\ &\quad\le C\varepsilon\Big\|\partial^{\alpha_1}\Big[\frac{1}{\rho}\mu(\theta)\Big]\Big\|_{L^\infty}\|\nabla_x\partial^{\alpha-\alpha_1}\tilde{u}\|\,\|\partial^\alpha\partial_{x_j}\tilde{u}_i\| \\ &\quad\le C\|\{|\partial^{\alpha_1}(\rho,\theta)|+\cdots+|\nabla_x(\rho,\theta)|^{|\alpha_1|}\}\|_{L^\infty}\varepsilon^2 \\ &\quad\le C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2. \end{aligned}$$

Likewise, the case $|\alpha|/2<|\alpha_1|\le|\alpha|$ has the same bound:

$$\begin{aligned} &\varepsilon\Big|\Big(\partial^{\alpha_1}\Big[\frac{1}{\rho}\mu(\theta)\Big]\partial^{\alpha-\alpha_1}(\partial_{x_j}\tilde{u}_i+\partial_{x_i}\tilde{u}_j-\frac{2}{3}\delta_{ij}\nabla_x\cdot\tilde{u}),\partial^\alpha\partial_{x_j}\tilde{u}_i\Big)\Big| \\ &\quad\le C\varepsilon\Big\|\partial^{\alpha_1}\Big[\frac{1}{\rho}\mu(\theta)\Big]\Big\|_{L^3}\|\nabla_x\partial^{\alpha-\alpha_1}\tilde{u}\|_{L^6}\|\partial^\alpha\partial_{x_j}\tilde{u}_i\|_{L^2} \\ &\quad\le C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2. \end{aligned}$$

By the three estimates above, we can obtain the estimation for I_2^1 as

$$I_2^1\le-\varepsilon\sum_{j=1}^{3}\Big(\frac{1}{\rho}\mu(\theta)\Big(\partial^\alpha\partial_{x_j}\tilde{u}_i+\partial^\alpha\partial_{x_i}\tilde{u}_j-\frac{2}{3}\delta_{ij}\nabla_x\cdot\partial^\alpha\tilde{u}\Big),\partial^\alpha\partial_{x_j}\tilde{u}_i\Big)+C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2.$$

The term I_2^2 can be handled in a similar manner and it can be controlled by $C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2$. We thereby obtain

$$I_2\le-\varepsilon\sum_{j=1}^{3}\Big(\frac{1}{\rho}\mu(\theta)\Big(\partial^\alpha\partial_{x_j}\tilde{u}_i+\partial^\alpha\partial_{x_i}\tilde{u}_j-\frac{2}{3}\delta_{ij}\nabla_x\cdot\partial^\alpha\tilde{u}\Big),\partial^\alpha\partial_{x_j}\tilde{u}_i\Big)+C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2.$$

The estimation for I_3 is easier and it is dominated by $C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2$. Hence, we can conclude from the above estimates of I_2 and I_3 that

$$\begin{aligned}&\varepsilon\sum_{j=1}^{3}\Big(\partial^\alpha\Big(\frac{1}{\rho}\partial_{x_j}[\mu(\theta)D_{ij}]\Big),\partial^\alpha\tilde{u}_i\Big)\\&\qquad\le-\varepsilon\sum_{j=1}^{3}\Big(\frac{1}{\rho}\mu(\theta)\Big(\partial^\alpha\partial_{x_j}\tilde{u}_i+\partial^\alpha\partial_{x_i}\tilde{u}_j-\frac{2}{3}\delta_{ij}\nabla_x\cdot\partial^\alpha\tilde{u}\Big),\partial^\alpha\partial_{x_j}\tilde{u}_i\Big)\\&\qquad\quad+C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2.\end{aligned}\tag{5.36}$$

We still need to estimate the last term of (5.35). In light of (5.18) and the integration by parts, we write

$$\begin{aligned}&-\Big(\partial^\alpha\Big[\frac{1}{\rho}\int_{\mathbb{R}^3}v_i v\cdot\nabla_x L_M^{-1}\Theta\,dv\Big],\partial^\alpha\tilde{u}_i\Big)\\&\quad=-\sum_{j=1}^{3}\Big(\partial^\alpha\Big[\frac{1}{\rho}\partial_{x_j}\Big(\int_{\mathbb{R}^3}R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\Theta}{M}\,dv\Big)\Big],\partial^\alpha\tilde{u}_i\Big)\\&\quad=\sum_{j=1}^{3}\Big(\partial^\alpha\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)\int_{\mathbb{R}^3}R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\Theta}{M}\,dv\Big],\partial^\alpha\tilde{u}_i\Big)\\&\qquad+\sum_{j=1}^{3}\Big(\partial^\alpha\Big[\frac{1}{\rho}\int_{\mathbb{R}^3}R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\Theta}{M}\,dv\Big],\partial^\alpha\partial_{x_j}\tilde{u}_i\Big).\end{aligned}\tag{5.37}$$

Consider the first term on the right-hand side of (5.37) associated with $\Theta:=\varepsilon\partial_t G+\varepsilon P_1(v\cdot\nabla_x G)-Q(G,G)$. Thanks to $1\le|\alpha|\le N-1$, applying (5.20), similar arguments to (4.4), Lemma 4.8, (5.22), (3.3), (4.1), as well as the Cauchy–Schwarz and Sobolev inequalities, we arrive at

$$\sum_{j=1}^{3}\int_{\mathbb{R}^3}\Big\{\partial^\alpha\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)\int_{\mathbb{R}^3}R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\partial_t\overline{G}}{M}\,dv\Big]\partial^\alpha\tilde{u}_i\Big\}\,dx\le C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2.$$

We use an integration by parts about t to get

$$\begin{aligned}&\sum_{j=1}^{3}\int_{\mathbb{R}^3}\Big\{\partial^\alpha\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)\int_{\mathbb{R}^3}R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}\partial_t f}{M}\,dv\Big]\partial^\alpha\tilde{u}_i\Big\}\,dx\\&\quad=\sum_{j=1}^{3}\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial^\alpha\Big\{\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M}\Big]f\Big\}\partial^\alpha\tilde{u}_i\,dv\,dx\\&\qquad-\sum_{j=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial^\alpha\Big\{\partial_t\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M}\Big]f\Big\}\partial^\alpha\tilde{u}_i\,dv\,dx\\&\qquad-\sum_{j=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial^\alpha\Big\{\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M}\Big]f\Big\}\partial^\alpha\partial_t\tilde{u}_i\,dv\,dx.\end{aligned}$$

Denote $\partial^{e_i} = \partial_{x_i}$ with $|e_i| = 1$. We have from this and the integration by parts that

$$
\begin{aligned}
&-\sum_{j=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial^\alpha\Big\{\partial_t\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M}\Big]f\Big\}\partial^\alpha\tilde{u}_i\,dv\,dx\\
&\quad=\sum_{j=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial^{\alpha-e_i}\Big\{\partial_t\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M}\Big]f\Big\}\partial^{\alpha+e_i}\tilde{u}_i\,dv\,dx\\
&\quad\le\eta\varepsilon\|\partial^{\alpha+e_i}\tilde{u}_i\|^2\\
&\qquad+C_\eta\varepsilon\sum_{j=1}^{3}\int_{\mathbb{R}^3}\Big|\int_{\mathbb{R}^3}\partial^{\alpha-e_i}\Big\{\partial_t\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\sqrt{\mu}}{M}\Big]f\Big\}\,dv\Big|^2dx\\
&\quad\le C\eta\varepsilon\|\partial^\alpha\nabla_x\tilde{u}_i\|^2+C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2,
\end{aligned}
$$

according to (5.20), Lemma 4.8, (5.22), (3.3), and (4.1). Similarly, it also holds that

$$
\begin{aligned}
&-\sum_{j=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial^\alpha\Big\{\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M}\Big]f\Big\}\partial^\alpha\partial_t\tilde{u}_i\,dv\,dx\\
&\quad\le\eta\varepsilon\|\partial^\alpha\partial_t\tilde{u}_i\|^2\\
&\qquad+C_\eta\varepsilon\sum_{j=1}^{3}\int_{\mathbb{R}^3}\Big|\int_{\mathbb{R}^3}\partial^{\alpha}\Big\{\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\sqrt{\mu}}{M}\Big]f\Big\}\,dv\Big|^2dx\\
&\quad\le C\eta\varepsilon(\|\partial^\alpha\nabla_x(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2+\|\partial^\alpha\nabla_x f\|_\sigma^2)+C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2.
\end{aligned}
$$

With these estimates, it is clear to see that

$$
\begin{aligned}
&\sum_{j=1}^{3}\int_{\mathbb{R}^3}\Big\{\partial^\alpha\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)\int_{\mathbb{R}^3}R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}\partial_t f}{M}\,dv\Big]\partial^\alpha\tilde{u}_i\Big\}\,dx\\
&\quad\le\sum_{j=1}^{3}\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial^\alpha\Big\{\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M}\Big]f\Big\}\partial^\alpha\tilde{u}_i\,dv\,dx\\
&\qquad+C\eta\varepsilon(\|\partial^\alpha\nabla_x(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2+\|\partial^\alpha\nabla_x f\|_\sigma^2)+C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2.
\end{aligned}
$$

We therefore can conclude from the above estimates and $G=\overline{G}+\sqrt{\mu}f$ that

$$
\begin{aligned}
&\sum_{j=1}^{3}\int_{\mathbb{R}^3}\Big\{\partial^\alpha\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)\int_{\mathbb{R}^3}R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\partial_t G}{M}\,dv\Big]\partial^\alpha\tilde{u}_i\Big\}\,dx\\
&\quad\le\sum_{j=1}^{3}\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial^\alpha\Big\{\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M}\Big]f\Big\}\partial^\alpha\tilde{u}_i\,dv\,dx\\
&\qquad+C\eta\varepsilon(\|\partial^\alpha\nabla_x(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2+\|\partial^\alpha\nabla_x f\|_\sigma^2)+C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2.
\end{aligned}
$$

The second term in Θ can be handled in a similar manner and it can be controlled by

$$\sum_{j=1}^{3}\int_{\mathbb{R}^3}\Big\{\partial^\alpha\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)\int_{\mathbb{R}^3}R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon P_1(v\cdot\nabla_x G)}{M}\,dv\Big]\partial^\alpha\tilde{u}_i\Big\}\,dx\le C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2.$$

For the last term in Θ, it holds by using (3.11), (4.11), (5.20), Lemma 4.3, (3.3), and (4.1) that

$$\begin{aligned}&-\sum_{j=1}^{3}\int_{\mathbb{R}^3}\Big\{\partial^\alpha\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)\int_{\mathbb{R}^3}R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{Q(G,G)}{M}\,dv\Big]\partial^\alpha\tilde{u}_i\Big\}\,dx\\&\quad=-\sum_{j=1}^{3}\int_{\mathbb{R}^3}\Big\{\partial^\alpha\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)\int_{\mathbb{R}^3}R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\sqrt{\mu}}{M}\Gamma\Big(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}\Big)\,dv\Big]\partial^\alpha\tilde{u}_i\Big\}\,dx\\&\quad\le C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2+C(\eta_0+\varepsilon^{\frac{1}{2}})\mathcal{D}_N(t).\end{aligned}$$

Collecting the above estimates, we can obtain with any small $\eta>0$ that

$$\begin{aligned}&\sum_{j=1}^{3}\Big(\partial^\alpha\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)\int_{\mathbb{R}^3}R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\Theta}{M}\,dv\Big],\partial^\alpha\tilde{u}_i\Big)\\&\qquad\le\sum_{j=1}^{3}\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial^\alpha\Big\{\Big[\partial_{x_j}\Big(\frac{1}{\rho}\Big)R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}}{M}\Big]f\Big\}\partial^\alpha\tilde{u}_i\,dv\,dx\\&\qquad\quad+C\eta\varepsilon(\|\partial^\alpha\nabla_x(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2+\|\partial^\alpha\nabla_x f\|_\sigma^2)\\&\qquad\quad+C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2+C(\eta_0+\varepsilon^{\frac{1}{2}})\mathcal{D}_N(t).\end{aligned}\tag{5.38}$$

The second term of (5.37) has the same structure as the first term and it shares the same bound. For brevity, we give the following computations directly:

$$\begin{aligned}&\sum_{j=1}^{3}\Big(\partial^\alpha\Big[\frac{1}{\rho}\int_{\mathbb{R}^3}R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\partial_t G}{M}\,dv\Big],\partial^\alpha\partial_{x_j}\tilde{u}_i\Big)\\&\qquad\le-\sum_{j=1}^{3}\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big\{\partial^\alpha\partial_{x_j}\Big[\frac{1}{\rho}R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}f}{M}\Big]\partial^\alpha\tilde{u}_i\Big\}\,dv\,dx\\&\qquad\quad+C\eta\varepsilon\|\partial^\alpha\nabla_x(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2+C_\eta\varepsilon\|\partial^\alpha\nabla_x f\|_\sigma^2+C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=1}^{3}\Big|\Big(\partial^\alpha\Big[\frac{1}{\rho}\int_{\mathbb{R}^3}R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon P_1(v\cdot\nabla_x G)}{M}\,dv\Big],\partial^\alpha\partial_{x_j}\tilde{u}_i\Big)\Big|\\&\qquad\le C\eta\varepsilon\|\partial^\alpha\nabla_x\tilde{u}\|^2+C_\eta\varepsilon\|\partial^\alpha\nabla_x f\|_\sigma^2+C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2,\end{aligned}$$

as well as

$$\sum_{j=1}^{3}\left|\left(\partial^\alpha\left[\frac{1}{\rho}\int_{\mathbb{R}^3} R\theta B_{ij}\left(\frac{v-u}{\sqrt{R\theta}}\right)\frac{Q(G,G)}{M}\,dv\right],\partial^\alpha\partial_{x_j}\tilde{u}_i\right)\right|$$
$$\le C\eta\varepsilon\|\partial^\alpha\nabla_x\tilde{u}\|^2 + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2 + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\mathcal{D}_N(t).$$

Then, for any small $\eta>0$, the second term of (5.37) is bounded by

$$\sum_{j=1}^{3}\left(\partial^\alpha\left[\frac{1}{\rho}\int_{\mathbb{R}^3} R\theta B_{ij}\left(\frac{v-u}{\sqrt{R\theta}}\right)\frac{\Theta}{M}\,dv\right],\partial^\alpha\partial_{x_j}\tilde{u}_i\right)$$
$$\le -\sum_{j=1}^{3}\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left\{\partial^\alpha\partial_{x_j}\left[\frac{1}{\rho}R\theta B_{ij}\left(\frac{v-u}{\sqrt{R\theta}}\right)\frac{\varepsilon\sqrt{\mu}f}{M}\right]\partial^\alpha\tilde{u}_i\right\}dv\,dx$$
$$+ C\eta\varepsilon\|\partial^\alpha\nabla_x(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2 + C_\eta\varepsilon\|\partial^\alpha\nabla_x f\|_\sigma^2$$
$$+ C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2 + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\mathcal{D}_N(t).$$

This estimate, combined with (5.38), as well as (5.37), gives

$$-\left(\partial^\alpha\left[\frac{1}{\rho}\int_{\mathbb{R}^3} v_i v\cdot\nabla_x L_M^{-1}\Theta\,dv\right],\partial^\alpha\tilde{u}_i\right)$$
$$\le -\sum_{j=1}^{3}\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left\{\partial^\alpha\left[\frac{1}{\rho}\partial_{x_j}\left(R\theta B_{ij}\left(\frac{v-u}{\sqrt{R\theta}}\right)\frac{\varepsilon\sqrt{\mu}f}{M}\right)\right]\partial^\alpha\tilde{u}_i\right\}dv\,dx$$
$$+ C\eta\varepsilon\|\partial^\alpha\nabla_x(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2 + C_\eta\varepsilon\|\partial^\alpha\nabla_x f\|_\sigma^2$$
$$+ C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2 + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\mathcal{D}_N(t).$$

For $1\le|\alpha|\le N-1$ and any small $\eta>0$, plugging all the estimates above into (5.35) and summing i from 1 to 3, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\tilde{u}\|^2 + \left(\frac{2\bar{\theta}}{3\bar{\rho}}\partial^\alpha\nabla_x\tilde{\rho},\partial^\alpha\tilde{u}\right) + \left(\frac{2}{3}\partial^\alpha\nabla_x\tilde{\theta},\partial^\alpha\tilde{u}\right) + c\varepsilon\|\partial^\alpha\nabla_x\tilde{u}\|^2$$
$$+ \sum_{i,j=1}^{3}\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left\{\partial^\alpha\left[\frac{1}{\rho}\partial_{x_j}\left(R\theta B_{ij}\left(\frac{v-u}{\sqrt{R\theta}}\right)\frac{\varepsilon\sqrt{\mu}f}{M}\right)\right]\partial^\alpha\tilde{u}_i\right\}dv\,dx$$
$$\le C\eta\varepsilon\|\partial^\alpha\nabla_x(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2 + C_\eta\varepsilon\|\partial^\alpha\nabla_x f\|_\sigma^2$$
$$+ C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2 + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\mathcal{D}_N(t). \tag{5.39}$$

Here, the following crucial inequality has been used:

$$\varepsilon\sum_{i,j=1}^{3}\left(\frac{1}{\rho}\mu(\theta)\left(\partial^\alpha\partial_{x_j}\tilde{u}_i + \partial^\alpha\partial_{x_i}\tilde{u}_j - \frac{2}{3}\delta_{ij}\nabla_x\cdot\partial^\alpha\tilde{u}\right),\partial^\alpha\partial_{x_j}\tilde{u}_i\right)$$
$$= \varepsilon\sum_{i,j=1}^{3}\left(\frac{\mu(\theta)}{\rho}\partial^\alpha\partial_{x_j}\tilde{u}_i,\partial^\alpha\partial_{x_j}\tilde{u}_i\right) + \frac{1}{3}\varepsilon\sum_{i,j=1}^{3} l\left(\frac{\mu(\theta)}{\rho}\partial^\alpha\partial_{x_j}\tilde{u}_j,\partial^\alpha\partial_{x_i}\tilde{u}_i\right)$$

$$
\begin{aligned}
&-\varepsilon\sum_{i,j=1}^{3}\Big(\partial_{x_j}\Big[\frac{\mu(\theta)}{\rho}\Big]\partial^\alpha\partial_{x_i}\tilde{u}_j,\partial^\alpha\tilde{u}_i\Big)+\varepsilon\sum_{i,j=1}^{3}\Big(\partial_{x_i}\Big[\frac{\mu(\theta)}{\rho}\Big]\partial^\alpha\partial_{x_j}\tilde{u}_j,\partial^\alpha\tilde{u}_i\Big)\\
&\geq c\varepsilon\|\partial^\alpha\nabla_x\tilde{u}\|^2-C(\eta_0+\varepsilon^{\frac12})\varepsilon^2.
\end{aligned}
$$

Step 3. Let us now turn to considering the third equation of (3.5). Applying ∂^α with $1\leq|\alpha|\leq N-1$ to the third equation of (3.5) and taking the inner product of the resulting equation with $\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}$ yields

$$
\begin{aligned}
&\Big(\partial_t\partial^\alpha\tilde{\theta},\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big)+\Big(\frac23\nabla_x\cdot\partial^\alpha\tilde{u},\partial^\alpha\tilde{\theta}\Big)+\sum_{1\leq\alpha_1\leq\alpha}C_\alpha^{\alpha_1}\Big(\frac23\partial^{\alpha_1}\bar{\theta}\nabla_x\cdot\partial^{\alpha-\alpha_1}\tilde{u},\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big)\\
&=-\Big(\partial^\alpha(u\cdot\nabla_x\tilde{\theta}),\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big)-\Big(\partial^\alpha(\tilde{u}\cdot\nabla_x\bar{\theta}),\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big)-\Big(\frac23\partial^\alpha(\tilde{\theta}\nabla_x\cdot u),\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big)\\
&\quad+\varepsilon\sum_{j=1}^{3}\Big(\partial^\alpha\Big[\frac{1}{\rho}\partial_{x_j}(\kappa(\theta)\partial_{x_j}\theta)\Big],\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big)\\
&\quad+\varepsilon\sum_{i,j=1}^{3}\Big(\partial^\alpha\Big[\frac{1}{\rho}\mu(\theta)\partial_{x_j}u_iD_{ij}\Big],\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big)\\
&\quad-\Big(\partial^\alpha\Big[\frac{1}{\rho}\int_{\mathbb{R}^3}\frac12|v|^2v\cdot\nabla_xL_M^{-1}\Theta\,dv\Big],\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big)\\
&\quad+\Big(\partial^\alpha\Big[\frac{1}{\rho}u\cdot\int_{\mathbb{R}^3}v\otimes v\cdot\nabla_xL_M^{-1}\Theta\,dv\Big],\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big).
\end{aligned}\tag{5.40}
$$

We will estimate each term for (5.40). We use the Sobolev inequality, (5.22), and (4.1) to get

$$
\begin{aligned}
\Big(\partial_t\partial^\alpha\tilde{\theta},\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big)&=\frac12\frac{d}{dt}\Big(\partial^\alpha\tilde{\theta},\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big)-\frac12\Big(\partial^\alpha\tilde{\theta},\partial_t\Big(\frac{1}{\bar{\theta}}\Big)\partial^\alpha\tilde{\theta}\Big)\\
&\geq\frac12\frac{d}{dt}\Big(\partial^\alpha\tilde{\theta},\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big)-C\|\partial_t\bar{\theta}\|_\infty\|\partial^\alpha\tilde{\theta}\|^2\\
&\geq\frac12\frac{d}{dt}\Big(\partial^\alpha\tilde{\theta},\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big)-C\eta_0\varepsilon^2.
\end{aligned}
$$

Performing calculations similar to those for (5.31), (5.32), and (5.33), we get

$$
\begin{aligned}
&\sum_{1\leq\alpha_1\leq\alpha}C_\alpha^{\alpha_1}\Big|\Big(\frac23\partial^{\alpha_1}\bar{\theta}\nabla_x\cdot\partial^{\alpha-\alpha_1}\tilde{u},\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big)\Big|+\Big|\Big(\partial^\alpha(u\cdot\nabla_x\tilde{\theta}),\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big)\Big|\\
&\quad+\Big|\Big(\partial^\alpha(\tilde{u}\cdot\nabla_x\bar{\theta}),\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big)\Big|+\Big|\Big(\frac23\partial^\alpha(\tilde{\theta}\nabla_x\cdot u),\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big)\Big|\\
&\leq C\eta\varepsilon\|\nabla_x\cdot\partial^\alpha\tilde{u}\|^2+C_\eta(\eta_0+\varepsilon^{\frac12})\varepsilon^2.
\end{aligned}
$$

The fourth term on the right-hand side of (5.40) can be handled in the same way as in (5.36); it follows that

$$
\begin{aligned}
&\varepsilon\sum_{j=1}^{3}\Big(\partial^\alpha\Big[\frac{1}{\rho}\partial_{x_j}(\kappa(\theta)\partial_{x_j}\theta)\Big],\frac{1}{\bar\theta}\partial^\alpha\tilde\theta\Big)\\
&\qquad\le-\varepsilon\sum_{j=1}^{3}\Big(\frac{1}{\rho}\kappa(\theta)\partial^\alpha\partial_{x_j}\tilde\theta,\frac{1}{\bar\theta}\partial^\alpha\partial_{x_j}\tilde\theta\Big)+C(\eta_0+\varepsilon^{\frac12})\varepsilon^2.
\end{aligned}
$$

The fifth term on the right-hand side of (5.40) is controlled by $C(\eta_0+\varepsilon^{\frac12})\varepsilon^2$. Since the structure of the last two terms in (5.40) is almost the same as (5.37), we thus arrive at

$$
\begin{aligned}
&-\Big(\partial^\alpha\Big[\frac{1}{\rho}\int_{\mathbb{R}^3}\frac12|v|^2v\cdot\nabla_xL_M^{-1}\Theta\,dv\Big],\frac{1}{\bar\theta}\partial^\alpha\tilde\theta\Big)\\
&\qquad+\Big(\partial^\alpha\Big[\frac{1}{\rho}u\cdot\int_{\mathbb{R}^3}v\otimes v\cdot\nabla_xL_M^{-1}\Theta\,dv\Big],\frac{1}{\bar\theta}\partial^\alpha\tilde\theta\Big)\\
&\quad=-\sum_{i=1}^{3}\Big(\partial^\alpha\Big[\frac{1}{\rho}\partial_{x_i}\Big(\int_{\mathbb{R}^3}\Big(\frac12|v|^2v_i-u\cdot vv_i\Big)L_M^{-1}\Theta\,dv\Big)\Big],\frac{1}{\bar\theta}\partial^\alpha\tilde\theta\Big)\\
&\qquad-\sum_{i=1}^{3}\Big(\partial^\alpha\Big[\frac{1}{\rho}\int_{\mathbb{R}^3}\partial_{x_i}u\cdot vv_iL_M^{-1}\Theta\,dv\Big],\frac{1}{\bar\theta}\partial^\alpha\tilde\theta\Big)\\
&\quad=-\sum_{i=1}^{3}\Big(\partial^\alpha\Big[\frac{1}{\rho}\partial_{x_i}\Big((R\theta)^{\frac32}\int_{\mathbb{R}^3}A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\Theta}{M}\,dv\Big)\Big],\frac{1}{\bar\theta}\partial^\alpha\tilde\theta\Big)\\
&\qquad-\sum_{i,j=1}^{3}\Big(\partial^\alpha\Big[\frac{1}{\rho}\partial_{x_i}u_jR\theta\int_{\mathbb{R}^3}B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\Theta}{M}\,dv\Big],\frac{1}{\bar\theta}\partial^\alpha\tilde\theta\Big),
\end{aligned}
$$

which can be further bounded by

$$
\begin{aligned}
&-\sum_{i=1}^{3}\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big\{\partial^\alpha\Big[\frac{1}{\rho}\partial_{x_i}\Big((R\theta)^{\frac32}A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}f}{M}\Big)\Big]\frac{1}{\bar\theta}\partial^\alpha\tilde\theta\Big\}\,dv\,dx\\
&\qquad-\sum_{i,j=1}^{3}\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big\{\partial^\alpha\Big[\frac{1}{\rho}\partial_{x_i}u_jR\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}f}{M}\Big]\frac{1}{\bar\theta}\partial^\alpha\tilde\theta\Big\}\,dv\,dx\\
&\qquad+C\eta\varepsilon\|\partial^\alpha\nabla_x(\tilde\rho,\tilde u,\tilde\theta)\|^2+C_\eta\varepsilon\|\partial^\alpha\nabla_xf\|_\sigma^2\\
&\qquad+C_\eta(\eta_0+\varepsilon^{\frac12})\varepsilon^2+C_\eta(\eta_0+\varepsilon^{\frac12})\mathcal{D}_N(t).
\end{aligned}
$$

Hence, substituting the above estimates into (5.40), we have established, for $1 \le |\alpha| \le N-1$ and any small $\eta > 0$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\partial^\alpha\tilde{\theta}, \frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big) + \Big(\frac{2}{3}\nabla_x\cdot\partial^\alpha\tilde{u}, \partial^\alpha\tilde{\theta}\Big) + \varepsilon\Big(\frac{1}{\rho}\kappa(\theta)\partial^\alpha\nabla_x\tilde{\theta}, \frac{1}{\bar{\theta}}\partial^\alpha\nabla_x\tilde{\theta}\Big)\\
&\qquad + \sum_{i=1}^{3}\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big\{\partial^\alpha\Big[\frac{1}{\rho}\partial_{x_i}\Big((R\theta)^{\frac{3}{2}}A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}f}{M}\Big)\Big]\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big\}\,dv\,dx\\
&\qquad + \sum_{i,j=1}^{3}\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big\{\partial^\alpha\Big[\frac{1}{\rho}\partial_{x_i}u_j R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}f}{M}\Big]\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big\}\,dv\,dx\\
&\quad \le C\eta\varepsilon\|\partial^\alpha\nabla_x(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2 + C_\eta\varepsilon\|\partial^\alpha\nabla_x f\|_\sigma^2\\
&\qquad + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2 + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\mathcal{D}_N(t).
\end{aligned} \tag{5.41}$$

Step 4. In summary, for any $1 \le |\alpha| \le N-1$ and any small $\eta > 0$, adding (5.34), (5.39), and (5.41) together, and combining with the estimates

$$\begin{aligned}
&\Big|\Big(\nabla_x\cdot\partial^\alpha\tilde{u}, \frac{2\bar{\theta}}{3\bar{\rho}}\partial^\alpha\tilde{\rho}\Big) + \Big(\frac{2\bar{\theta}}{3\bar{\rho}}\partial^\alpha\nabla_x\tilde{\rho}, \partial^\alpha\tilde{u}\Big) + \Big(\frac{2}{3}\partial^\alpha\nabla_x\tilde{\theta}, \partial^\alpha\tilde{u}\Big) + \Big(\frac{2}{3}\nabla_x\cdot\partial^\alpha\tilde{u}, \partial^\alpha\tilde{\theta}\Big)\Big|\\
&\quad = \Big|\Big(\partial^\alpha\tilde{u}, \nabla_x\Big(\frac{2\bar{\theta}}{3\bar{\rho}}\Big)\partial^\alpha\tilde{\rho}\Big)\Big| \le C\|\nabla_x(\bar{\rho},\bar{\theta})\|_{L^\infty}\|\partial^\alpha\tilde{u}\|\,\|\partial^\alpha\tilde{\rho}\| \le C\eta_0\varepsilon^2,
\end{aligned}$$

the summation of the resulting equation over $|\alpha|$ through a suitable linear combination gives

$$\begin{aligned}
&\frac{1}{2}\sum_{1\le|\alpha|\le N-1}\frac{d}{dt}\int_{\mathbb{R}^3}\Big(\frac{2\bar{\theta}}{3\bar{\rho}^2}|\partial^\alpha\tilde{\rho}|^2 + |\partial^\alpha\tilde{u}|^2 + \frac{1}{\bar{\theta}}|\partial^\alpha\tilde{\theta}|^2\Big)\,dx\\
&\qquad + \frac{d}{dt}E_1(t) + c\varepsilon\sum_{2\le|\alpha|\le N}\|\partial^\alpha(\tilde{u},\tilde{\theta})\|^2\\
&\quad \le C\eta\varepsilon\sum_{2\le|\alpha|\le N}\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2 + C_\eta\varepsilon\sum_{2\le|\alpha|\le N}\|\partial^\alpha f\|_\sigma^2\\
&\qquad + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2 + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\mathcal{D}_N(t).
\end{aligned} \tag{5.42}$$

Here we have denoted

$$\begin{aligned}
E_1(t) = &\sum_{1\le|\alpha|\le N-1}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big\{\partial^\alpha\Big[\frac{1}{\rho}\partial_{x_j}\Big(R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}f}{M}\Big)\Big]\partial^\alpha\tilde{u}_i\Big\}\,dv\,dx\\
&+ \sum_{1\le|\alpha|\le N-1}\sum_{i=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big\{\partial^\alpha\Big[\frac{1}{\rho}\partial_{x_i}\Big((R\theta)^{\frac{3}{2}}A_i\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}f}{M}\Big)\Big]\frac{1}{\bar{\theta}}\partial^\alpha\tilde{\theta}\Big\}\,dv\,dx
\end{aligned}$$

$$
+\sum_{1\le|\alpha|\le N-1}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big\{\partial^\alpha\Big[\frac{1}{\rho}\partial_{x_i}u_j R\theta B_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)\frac{\varepsilon\sqrt{\mu}f}{M}\Big]\times\frac{1}{\bar\theta}\partial^\alpha\tilde\theta\Big\}\,dv\,dx. \tag{5.43}
$$

Step 5. As before, to get the dissipation rate for the density function, we apply ∂^α to the second equation of (4.31) with $1\le|\alpha|\le N-1$ and then take the inner product of the resulting equation with $\nabla_x\partial^\alpha\tilde\rho$ to obtain

$$
\begin{aligned}
&(\partial^\alpha\partial_t\tilde u,\nabla_x\partial^\alpha\tilde\rho)+\Big(\frac{2\bar\theta}{3\bar\rho}\nabla_x\partial^\alpha\tilde\rho,\nabla_x\partial^\alpha\tilde\rho\Big)\\
&\quad=-\sum_{1\le\alpha_1\le\alpha}C_\alpha^{\alpha_1}\Big(\partial^{\alpha_1}\Big(\frac{2\bar\theta}{3\bar\rho}\Big)\nabla_x\partial^{\alpha-\alpha_1}\tilde\rho,\nabla_x\partial^\alpha\tilde\rho\Big)\\
&\qquad-\Big(\partial^\alpha(u\cdot\nabla_x\tilde u)+\frac{2}{3}\nabla_x\partial^\alpha\tilde\theta+\partial^\alpha(\tilde u\cdot\nabla_x\bar u)+\partial^\alpha\Big[\frac{2}{3}\Big(\frac{\theta}{\rho}-\frac{\bar\theta}{\bar\rho}\Big)\nabla_x\rho\Big],\nabla_x\partial^\alpha\tilde\rho\Big)\\
&\qquad-\Big(\partial^\alpha\Big(\frac{1}{\rho}\int_{\mathbb{R}^3}v\otimes v\cdot\nabla_x G\,dv\Big),\nabla_x\partial^\alpha\tilde\rho\Big).
\end{aligned}
$$

Thanks to $1\le|\alpha|\le N-1$, we follow a similar strategy to (5.13) to claim that

$$
\begin{aligned}
&\varepsilon\frac{d}{dt}(\partial^\alpha\tilde u,\nabla_x\partial^\alpha\tilde\rho)+c\varepsilon\|\nabla_x\partial^\alpha\tilde\rho\|^2\\
&\quad\le C\varepsilon(\|\nabla_x\partial^\alpha\tilde u\|^2+\|\nabla_x\partial^\alpha\tilde\theta\|^2+\|\nabla_x\partial^\alpha f\|_\sigma^2)+C(\eta_0+\varepsilon^{\frac12})\varepsilon^2.
\end{aligned}\tag{5.44}
$$

For any $1\le|\alpha|\le N-1$, the summation of (5.44) over $|\alpha|$ through a suitable linear combination gives

$$
\begin{aligned}
&\varepsilon\sum_{1\le|\alpha|\le N-1}\frac{d}{dt}(\partial^\alpha\tilde u,\nabla_x\partial^\alpha\tilde\rho)+c\varepsilon\sum_{2\le|\alpha|\le N}\|\partial^\alpha\tilde\rho\|^2\\
&\quad\le C\varepsilon\sum_{2\le|\alpha|\le N}(\|\partial^\alpha\tilde u\|^2+\|\partial^\alpha\tilde\theta\|^2+\|\partial^\alpha f\|_\sigma^2)+C(\eta_0+\varepsilon^{\frac12})\varepsilon^2.
\end{aligned}\tag{5.45}
$$

Step 6. Multiplying (5.42) by a large constant $C_2>1$ and adding the resulting equation to (5.45), by choosing $\eta>0$ small enough we have

$$
\begin{aligned}
&\frac12 C_2\sum_{1\le|\alpha|\le N-1}\frac{d}{dt}\int_{\mathbb{R}^3}\Big(\frac{2\bar\theta}{3\bar\rho^2}|\partial^\alpha\tilde\rho|^2+|\partial^\alpha\tilde u|^2+\frac{1}{\bar\theta}|\partial^\alpha\tilde\theta|^2\Big)\,dx+C_2\frac{d}{dt}E_1(t)\\
&\qquad+\varepsilon\sum_{1\le|\alpha|\le N-1}\frac{d}{dt}(\partial^\alpha\tilde u,\nabla_x\partial^\alpha\tilde\rho)+c\varepsilon\sum_{2\le|\alpha|\le N}\|\partial^\alpha(\tilde\rho,\tilde u,\tilde\theta)\|^2\\
&\quad\le C\varepsilon\sum_{1\le|\alpha|\le N}\|\partial^\alpha f\|_\sigma^2+C(\eta_0+\varepsilon^{\frac12})\varepsilon^2+C(\eta_0+\varepsilon^{\frac12})\mathcal{D}_N(t).
\end{aligned}\tag{5.46}
$$

Recall $E_1(t)$ in (5.43). We then employ (5.20), (3.3), and (4.1) to get

$$|E_1(t)| \le C\eta \sum_{1\le|\alpha|\le N-1} \|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2 + C_\eta\varepsilon^2 \sum_{|\alpha|=N} \|\partial^\alpha f\|^2 + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2.$$

By integrating (5.46) with respect to t, we deduce the desired estimate (5.28) from the above estimate of $E_1(t)$ with a small $\eta > 0$ and (3.22), as well as (4.1). This then completes the proof of Lemma 5.6. ∎

5.4. Space derivative estimate of the non-fluid part up to $(N-1)$th order

Next we deduce the space derivative estimate up to $(N-1)$th order for the non-fluid part $f(t,x,v)$. As before, the proof is based on the microscopic equation (3.13).

Lemma 5.7. *It holds that*

$$\begin{aligned}\sum_{1\le|\alpha|\le N-1}&\Big(\|\partial^\alpha f(t)\|^2 + c\frac{1}{\varepsilon}\int_0^t \|\partial^\alpha f(s)\|_\sigma^2\,ds\Big)\\ &\le C\varepsilon \sum_{2\le|\alpha|\le N}\int_0^t \|(\partial^\alpha\tilde{u},\partial^\alpha\tilde{\theta})(s)\|^2\,ds + C\varepsilon\sum_{|\alpha|=N}\int_0^t \|\partial^\alpha f(s)\|_\sigma^2\,ds\\ &\quad + C(\eta_0+\varepsilon^{\frac{1}{2}})\int_0^t \mathcal{D}_N(s)\,ds + C(1+t)(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2. \qquad (5.47)\end{aligned}$$

Proof. Applying ∂^α to (3.13) with $1\le|\alpha|\le N-1$ and taking the inner product of the resulting equation with $\partial^\alpha f$ over $\mathbb{R}^3\times\mathbb{R}^3$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f\|^2 + c_1\frac{1}{\varepsilon}\|\partial^\alpha f\|_\sigma^2\\ &\quad\le \frac{1}{\varepsilon}\Big(\partial^\alpha\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},f\Big)+\partial^\alpha\Gamma\Big(f,\frac{M-\mu}{\sqrt{\mu}}\Big),\partial^\alpha f\Big) + \frac{1}{\varepsilon}\Big(\partial^\alpha\Gamma\Big(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}\Big),\partial^\alpha f\Big)\\ &\qquad + \Big(\frac{\partial^\alpha P_0(v\sqrt{\mu}\cdot\nabla_x f)}{\sqrt{\mu}},\partial^\alpha f\Big) - \Big(\frac{\partial^\alpha P_1(v\cdot\nabla_x\overline{G})}{\sqrt{\mu}},\partial^\alpha f\Big) - \Big(\frac{\partial^\alpha\partial_t\overline{G}}{\sqrt{\mu}},\partial^\alpha f\Big)\\ &\qquad - \Big(\frac{1}{\sqrt{\mu}}\partial^\alpha P_1\Big\{v\cdot\Big(\frac{|v-u|^2\nabla_x\tilde{\theta}}{2R\theta^2}+\frac{(v-u)\cdot\nabla_x\tilde{u}}{R\theta}\Big)M\Big\},\partial^\alpha f\Big). \qquad (5.48)\end{aligned}$$

We now compute (5.48) term by term. With Lemmas 4.6 and 4.7 in hand, it is clear to see that

$$\frac{1}{\varepsilon}\Big(\partial^\alpha\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},f\Big)+\partial^\alpha\Gamma\Big(f,\frac{M-\mu}{\sqrt{\mu}}\Big),\partial^\alpha f\Big) \le C\eta\frac{1}{\varepsilon}\|\partial^\alpha f\|_\sigma^2 + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\mathcal{D}_N(t),$$

and

$$\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha\Gamma\Big(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}\Big),\partial^\alpha f\Big)\Big| \le C\eta\frac{1}{\varepsilon}\|\partial^\alpha f\|_\sigma^2 + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\mathcal{D}_N(t) + C_\eta(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2.$$

In view of (2.5), (2.4), (3.16), (3.3), (4.1), and the Cauchy–Schwarz and Sobolev inequalities, we arrive at

$$
\begin{aligned}
&\Big|\Big(\frac{1}{\sqrt{\mu}}\partial^\alpha P_0(v\sqrt{\mu}\cdot\nabla_x f),\partial^\alpha f\Big)\Big| \\
&\quad\le \eta\frac{1}{\varepsilon}\|\langle v\rangle^{-\frac12}\partial^\alpha f\|^2 + C_\eta\varepsilon\Big\|\langle v\rangle^{\frac12}\frac{1}{\sqrt{\mu}}\partial^\alpha P_0(v\sqrt{\mu}\cdot\nabla_x f)\Big\|^2 \\
&\quad\le C\eta\frac{1}{\varepsilon}\|\partial^\alpha f\|_\sigma^2 + C_\eta\varepsilon\|\partial^\alpha\nabla_x f\|_\sigma^2 + C_\eta(\eta_0+\varepsilon^{\frac12})\varepsilon^2.
\end{aligned}
$$

To deal with the term involving $\bar{G}$, we use Lemmas 4.3 and 4.8, (2.5), (3.3), (3.16), and (4.1) to get

$$
\begin{aligned}
&\Big|\Big(\frac{\partial^\alpha P_1(v\cdot\nabla_x\bar{G})}{\sqrt{\mu}},\partial^\alpha f\Big)\Big| + \Big|\Big(\frac{\partial^\alpha\partial_t\bar{G}}{\sqrt{\mu}},\partial^\alpha f\Big)\Big| \\
&\quad\le \eta\frac{1}{\varepsilon}\|\langle v\rangle^{-\frac12}\partial^\alpha f\|^2 + C_\eta\varepsilon\Big(\Big\|\langle v\rangle^{\frac12}\frac{\partial^\alpha P_1(v\cdot\nabla_x\bar{G})}{\sqrt{\mu}}\Big\|^2 + \Big\|\langle v\rangle^{\frac12}\frac{\partial^\alpha\partial_t\bar{G}}{\sqrt{\mu}}\Big\|^2\Big) \\
&\quad\le C\eta\frac{1}{\varepsilon}\|\partial^\alpha f\|_\sigma^2 + C_\eta(\eta_0+\varepsilon^{\frac12})\varepsilon^2.
\end{aligned}
$$

Thanks to (2.5) and (2.16), the following identity holds true:

$$
\begin{aligned}
&P_1 v\cdot\Big\{\frac{|v-u|^2\nabla_x\tilde{\theta}}{2R\theta^2} + \frac{(v-u)\cdot\nabla_x\tilde{u}}{R\theta}\Big\}M \\
&\quad= \frac{\sqrt{R}}{\sqrt{\theta}}\sum_{j=1}^{3}\frac{\partial\tilde{\theta}}{\partial x_j}\hat{A}_j\Big(\frac{v-u}{\sqrt{R\theta}}\Big)M + \sum_{j=1}^{3}\sum_{i=1}^{3}\frac{\partial\tilde{u}_j}{\partial x_i}\hat{B}_{ij}\Big(\frac{v-u}{\sqrt{R\theta}}\Big)M.
\end{aligned}
$$

We then see that the last term of (5.48) can be bounded by

$$
\begin{aligned}
&\Big|\Big(\frac{1}{\sqrt{\mu}}\partial^\alpha P_1\Big\{v\cdot\Big(\frac{|v-u|^2\nabla_x\tilde{\theta}}{2R\theta^2} + \frac{(v-u)\cdot\nabla_x\tilde{u}}{R\theta}\Big)M\Big\},\partial^\alpha f\Big)\Big| \\
&\quad\le \eta\frac{1}{\varepsilon}\|\langle v\rangle^{-\frac12}\partial^\alpha f\|^2 \\
&\qquad + C_\eta\varepsilon\Big\|\langle v\rangle^{\frac12}\frac{1}{\sqrt{\mu}}\partial^\alpha P_1\Big\{v\cdot\Big(\frac{|v-u|^2\nabla_x\tilde{\theta}}{2R\theta^2} + \frac{(v-u)\cdot\nabla_x\tilde{u}}{R\theta}\Big)M\Big\}\Big\|^2 \\
&\quad\le C\eta\frac{1}{\varepsilon}\|\partial^\alpha f\|_\sigma^2 + C_\eta\varepsilon\|\partial^\alpha(\nabla_x\tilde{u},\nabla_x\tilde{\theta})\|^2 + C_\eta(\eta_0+\varepsilon^{\frac12})\varepsilon^2.
\end{aligned}
$$

Therefore, substituting all the above estimates into (5.48) and choosing a small $\eta>0$, we get

$$
\begin{aligned}
\frac{d}{dt}\|\partial^\alpha f\|^2 + \frac{c_1}{2}\frac{1}{\varepsilon}\|\partial^\alpha f\|_\sigma^2 &\le C\varepsilon(\|\partial^\alpha(\nabla_x\tilde{u},\nabla_x\tilde{\theta})\|^2 + \|\partial^\alpha\nabla_x f\|_\sigma^2) \\
&\quad + C(\eta_0+\varepsilon^{\frac12})\mathfrak{D}_N(t) + C(\eta_0+\varepsilon^{\frac12})\varepsilon^2. \qquad (5.49)
\end{aligned}
$$

Integrating (5.49) with respect to t and using (3.22), the summation of the resulting equation over $|\alpha|$ with $1 \leq |\alpha| \leq N-1$ with a suitable linear combination gives the desired estimate (5.47). This completes the proof of Lemma 5.7. ■

From Lemmas 5.7 and 5.6, we immediately have the full energy estimate of the space derivatives up to $(N-1)$th order for both fluid and non-fluid parts.

Lemma 5.8. *It holds that*

$$\sum_{1\leq|\alpha|\leq N-1}\Big(\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})(t)\|^2+\|\partial^\alpha f(t)\|^2+c\frac{1}{\varepsilon}\int_0^t\|\partial^\alpha f(s)\|_\sigma^2\,ds\Big)$$
$$+c\varepsilon\sum_{2\leq|\alpha|\leq N}\int_0^t\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})(s)\|^2\,ds$$
$$\leq C\varepsilon^2\sum_{|\alpha|=N}(\|\partial^\alpha\tilde{\rho}(t)\|^2+\|\partial^\alpha f(t)\|^2)+C\varepsilon\sum_{|\alpha|=N}\int_0^t\|\partial^\alpha f(s)\|_\sigma^2\,ds$$
$$+C(\eta_0+\varepsilon^{\frac{1}{2}})\int_0^t\mathcal{D}_N(s)\,ds+C(1+t)(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2. \tag{5.50}$$

Proof. Taking the summation of (5.47) with (5.28) multiplied by a large constant $C_3>0$ and using the smallness of ε gives (5.50) directly. ■

5.5. Nth-order space derivative estimate

To complete the estimate of the space derivatives of all orders, we need to treat the highest Nth-order space derivatives more carefully. Note that for $|\alpha|=N$, we cannot directly obtain the dissipation of $\|\partial^\alpha f\|_\sigma^2$ in terms of (3.13) since the estimate of the transport term $(\frac{1}{\sqrt{\mu}}\partial^\alpha P_0[v\cdot\nabla_x(\sqrt{\mu}f)],\partial^\alpha f)$ induces $(N+1)$th-order derivatives so that the estimates cannot be closed. For this, we must use the original equation (1.1) to deduce the Nth-order energy estimates. In fact, in view of (3.11) and (3.12), the original equation (1.1) can be equivalently rewritten as

$$\partial_t\Big(\frac{F}{\sqrt{\mu}}\Big)+v\cdot\nabla_x\Big(\frac{F}{\sqrt{\mu}}\Big)-\frac{1}{\varepsilon}\mathcal{L}f$$
$$=\frac{1}{\varepsilon}\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},f\Big)+\frac{1}{\varepsilon}\Gamma\Big(f,\frac{M-\mu}{\sqrt{\mu}}\Big)+\frac{1}{\varepsilon}\Gamma\Big(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}\Big)$$
$$+\frac{1}{\sqrt{\mu}}P_1\Big\{v\cdot\Big(\frac{|v-u|^2\nabla_x\bar{\theta}}{2R\theta^2}+\frac{(v-u)\cdot\nabla_x\bar{u}}{R\theta}\Big)M\Big\}. \tag{5.51}$$

Using the above formulation, the corresponding linear transport term induces the inner product $(\frac{1}{\sqrt{\mu}}v\cdot\nabla_x\partial^\alpha F,\frac{1}{\sqrt{\mu}}\partial^\alpha F)$, which vanishes by integration by parts. Extra effort has to be made to estimate all other terms, such as $\frac{1}{\varepsilon}\mathcal{L}f$, $\frac{1}{\varepsilon}\Gamma(\frac{M-\mu}{\sqrt{\mu}},f)$, and $\partial_t(\frac{F}{\sqrt{\mu}})$, because of the singularity of the highest-order space derivatives. In particular, we need to develop delicate estimates of the inner products $-\frac{1}{\varepsilon}(\mathcal{L}\partial^\alpha f,\frac{\partial^\alpha F}{\sqrt{\mu}})$ and $\frac{1}{\varepsilon}(\Gamma(\frac{M-\mu}{\sqrt{\mu}},\partial^\alpha f),\frac{\partial^\alpha F}{\sqrt{\mu}})$ for

$|\alpha| = N$. Thus, we first obtain the following lemma, and the characterization of a lower bound for the energy norm $\|\frac{\partial^\alpha F(t)}{\sqrt{\mu}}\|^2$ in terms of $\|\partial^\alpha(\tilde{\rho}, \tilde{u}, \tilde{\theta})\|^2 + \|\partial^\alpha f\|^2$ will be treated in Lemma 5.10 later on.

Lemma 5.9. *It holds that*

$$\varepsilon^2 \sum_{|\alpha|=N} (\|\partial^\alpha(\tilde{\rho}, \tilde{u}, \tilde{\theta})(t)\|^2 + \|\partial^\alpha f(t)\|^2) + c\varepsilon \sum_{|\alpha|=N} \int_0^t \|\partial^\alpha f(s)\|_\sigma^2 \, ds$$
$$\leq C(\eta_0 + \varepsilon^{\frac{1}{2}}) \int_0^t \mathcal{D}_N(s) \, ds + C(1+t)(\eta_0 + \varepsilon^{\frac{1}{2}})\varepsilon^2. \tag{5.52}$$

Proof. Applying ∂^α to (5.51) with $|\alpha| = N$ and taking the inner product of the resulting equation with $\frac{\partial^\alpha F}{\sqrt{\mu}}$ over $\mathbb{R}^3 \times \mathbb{R}^3$, we obtain

$$\frac{1}{2}\frac{d}{dt}\Big\|\frac{\partial^\alpha F}{\sqrt{\mu}}\Big\|^2 - \frac{1}{\varepsilon}\Big(\mathcal{L}\partial^\alpha f, \frac{\partial^\alpha F}{\sqrt{\mu}}\Big)$$
$$= \frac{1}{\varepsilon}\Big(\partial^\alpha \Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, f\Big) + \partial^\alpha \Gamma\Big(f, \frac{M-\mu}{\sqrt{\mu}}\Big), \frac{\partial^\alpha F}{\sqrt{\mu}}\Big) + \frac{1}{\varepsilon}\Big(\partial^\alpha \Gamma\Big(\frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}}\Big), \frac{\partial^\alpha F}{\sqrt{\mu}}\Big)$$
$$+ \Big(\frac{1}{\sqrt{\mu}}\partial^\alpha P_1\Big\{v \cdot \Big(\frac{|v-u|^2 \nabla_x \bar{\theta}}{2R\theta^2} + \frac{(v-u)\cdot \nabla_x \bar{u}}{R\theta}\Big) M\Big\}, \frac{\partial^\alpha F}{\sqrt{\mu}}\Big). \tag{5.53}$$

We now compute (5.53) term by term. By the fact that $F = M + \overline{G} + \sqrt{\mu} f$, the second term on the left-hand side of (5.53) becomes

$$-\frac{1}{\varepsilon}\Big(\mathcal{L}\partial^\alpha f, \frac{\partial^\alpha F}{\sqrt{\mu}}\Big) = -\frac{1}{\varepsilon}\Big(\mathcal{L}\partial^\alpha f, \frac{\partial^\alpha M}{\sqrt{\mu}}\Big) - \frac{1}{\varepsilon}(\mathcal{L}\partial^\alpha f, \partial^\alpha f) - \frac{1}{\varepsilon}\Big(\mathcal{L}\partial^\alpha f, \frac{\partial^\alpha \overline{G}}{\sqrt{\mu}}\Big). \tag{5.54}$$

We first present the calculations for the last two terms of (5.54), since the first term of (5.54) is more complex and is thus left to the end. Thanks to $f \in (\ker \mathcal{L})^\perp$, one has from (4.10) that

$$-\frac{1}{\varepsilon}(\mathcal{L}\partial^\alpha f, \partial^\alpha f) \geq c_1 \frac{1}{\varepsilon}\|\partial^\alpha f\|_\sigma^2.$$

By the definition $\mathcal{L}f = \Gamma(\sqrt{\mu}, f) + \Gamma(f, \sqrt{\mu})$ in (3.11), one has from (4.11), Lemma 4.3, (3.16), (3.3), and (4.1) that

$$\frac{1}{\varepsilon}\Big|\Big(\mathcal{L}\partial^\alpha f, \frac{\partial^\alpha \overline{G}}{\sqrt{\mu}}\Big)\Big| \leq C\frac{1}{\varepsilon}(\|\partial^\alpha f\|_\sigma + \|\langle v\rangle^{-1}\partial^\alpha f\|)\Big\|\frac{\partial^\alpha \overline{G}}{\sqrt{\mu}}\Big\|_\sigma$$
$$\leq \eta_0 \frac{1}{\varepsilon}\|\partial^\alpha f\|_\sigma^2 + C\eta_0 \varepsilon.$$

Now we estimate the first term of (5.54). Recalling (4.17), one has

$$\partial_{x_i} M = M\Big(\frac{\partial_{x_i}\rho}{\rho} + \frac{(v-u)\cdot \partial_{x_i} u}{R\theta} + \Big(\frac{|v-u|^2}{2R\theta} - \frac{3}{2}\Big)\frac{\partial_{x_i}\theta}{\theta}\Big).$$

Let $\partial^\alpha = \partial^{\alpha'}\partial_{x_i}$ with $|\alpha'| = N-1$ due to $|\alpha| = N$; then

$$\begin{aligned}\partial^\alpha M &= M\Big(\frac{\partial^{\alpha'}\partial_{x_i}\rho}{\rho} + \frac{(v-u)\cdot\partial^{\alpha'}\partial_{x_i}u}{R\theta} + \Big(\frac{|v-u|^2}{2R\theta} - \frac{3}{2}\Big)\frac{\partial^{\alpha'}\partial_{x_i}\theta}{\theta}\Big)\\ &\quad + \sum_{1\leq\alpha_1\leq\alpha'} C_{\alpha'}^{\alpha_1}\Big(\partial^{\alpha_1}\Big(M\frac{1}{\rho}\Big)\partial^{\alpha'-\alpha_1}\partial_{x_i}\rho + \partial^{\alpha_1}\Big(M\frac{v-u}{R\theta}\Big)\cdot\partial^{\alpha'-\alpha_1}\partial_{x_i}u\\ &\qquad + \partial^{\alpha_1}(M\frac{|v-u|^2}{2R\theta^2} - M\frac{3}{2\theta})\partial^{\alpha'-\alpha_1}\partial_{x_i}\theta\Big)\\ &:= I_4 + I_5. \end{aligned} \tag{5.55}$$

Note that the linear term $\frac{1}{\varepsilon}(\mathcal{L}\partial^\alpha f, \frac{I_4}{\sqrt{\mu}})$ presents a significant difficulty and cannot be estimated directly. The key technique to handle this term is to use the properties of the linearized operator $\mathcal{L}$ and the smallness of $M-\mu$. For this, we denote

$$I_4 = (\mu + (M-\mu))\Big(\frac{\partial^\alpha\rho}{\rho} + \frac{(v-u)\cdot\partial^\alpha u}{R\theta} + \Big(\frac{|v-u|^2}{2R\theta} - \frac{3}{2}\Big)\frac{\partial^\alpha\theta}{\theta}\Big) := I_4^1 + I_4^2.$$

Since $\frac{I_4^1}{\sqrt{\mu}} \in \ker\mathcal{L}$, it follows that $(\mathcal{L}f, \frac{I_4^1}{\sqrt{\mu}}) = 0$. As for I_4^2, we further decompose it as

$$\begin{aligned} I_4^2 &= (M-\mu)\Big(\frac{\partial^\alpha\tilde{\rho}}{\rho} + \frac{(v-u)\cdot\partial^\alpha\tilde{u}}{R\theta} + \Big(\frac{|v-u|^2}{2R\theta} - \frac{3}{2}\Big)\frac{\partial^\alpha\tilde{\theta}}{\theta}\Big)\\ &\quad + (M-\mu)\Big(\frac{\partial^\alpha\bar{\rho}}{\rho} + \frac{(v-u)\cdot\partial^\alpha\bar{u}}{R\theta} + \Big(\frac{|v-u|^2}{2R\theta} - \frac{3}{2}\Big)\frac{\partial^\alpha\bar{\theta}}{\theta}\Big) := I_4^{21} + I_4^{22}. \end{aligned}$$

Thanks to $\mathcal{L}f = \Gamma(\sqrt{\mu}, f) + \Gamma(f, \sqrt{\mu})$, we deduce from (4.11), (3.16), and (4.16) that

$$\begin{aligned} \frac{1}{\varepsilon}\Big|\Big(\mathcal{L}\partial^\alpha f, \frac{I_4^{21}}{\sqrt{\mu}}\Big)\Big| &\leq C(\eta_0+\varepsilon)\frac{1}{\varepsilon}(\|\partial^\alpha f\|_\sigma + \|\langle v\rangle^{-1}\partial^\alpha f\|)\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})\|\\ &\leq C(\eta_0+\varepsilon)\frac{1}{\varepsilon}(\|\partial^\alpha f\|_\sigma^2 + \|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2). \end{aligned}$$

Let $\partial^\alpha = \partial^{\alpha'}\partial_{x_i}$ with $|\alpha'| = N-1$ due to $|\alpha| = N$; we have from integration by parts, (4.11), (3.16), (3.3), and (4.1) that

$$\begin{aligned} &\frac{1}{\varepsilon}\Big|\Big(\mathcal{L}\partial^\alpha f, \frac{I_4^{22}}{\sqrt{\mu}}\Big)\Big|\\ &\quad = \frac{1}{\varepsilon}\Big|\Big(\partial^{\alpha'}\partial_{x_i}\mathcal{L}f, \frac{M-\mu}{\sqrt{\mu}}\Big\{\frac{\partial^\alpha\bar{\rho}}{\rho} + \frac{(v-u)\cdot\partial^\alpha\bar{u}}{R\theta} + \Big(\frac{|v-u|^2}{2R\theta} - \frac{3}{2}\Big)\frac{\partial^\alpha\bar{\theta}}{\theta}\Big\}\Big)\Big|\\ &\quad = \frac{1}{\varepsilon}\Big|\Big(\partial^{\alpha'}\mathcal{L}f, \partial_{x_i}\Big[\frac{M-\mu}{\sqrt{\mu}}\Big\{\frac{\partial^\alpha\bar{\rho}}{\rho} + \frac{(v-u)\cdot\partial^\alpha\bar{u}}{R\theta} + \Big(\frac{|v-u|^2}{2R\theta} - \frac{3}{2}\Big)\frac{\partial^\alpha\bar{\theta}}{\theta}\Big\}\Big]\Big)\Big| \end{aligned}$$

$$
\begin{aligned}
&\le C\frac{1}{\varepsilon}\int_{\mathbb{R}^3}(|\partial^{\alpha'} f|_\sigma + |\langle v\rangle^{-1}\partial^{\alpha'} f|_2)\big\{(|\partial_{x_i}\partial^\alpha\bar\rho| + |\partial_{x_i}\partial^\alpha\bar u| + |\partial_{x_i}\partial^\alpha\bar\theta|)\\
&\qquad\qquad + (|\partial^\alpha\bar\rho| + |\partial^\alpha\bar u| + |\partial^\alpha\bar\theta|)\\
&\qquad\qquad \times (|\partial_{x_i}\rho| + |\partial_{x_i}u| + |\partial_{x_i}\theta|)\big\}\,dx\\
&\le C(\eta_0+\varepsilon)\frac{1}{\varepsilon}\|\partial^{\alpha'} f\|_\sigma \le C(\eta_0+\varepsilon)\frac{1}{\varepsilon}\Big(\frac{1}{\varepsilon^2}\|\partial^{\alpha'} f\|_\sigma^2 + \varepsilon^2\Big).
\end{aligned}\tag{5.56}
$$

Combining the above estimates of I_4^1 and I_4^2, we obtain that for $|\alpha'| = N-1$,

$$
\begin{aligned}
\frac{1}{\varepsilon}\Big|\Big(\mathcal{L}\partial^\alpha f, \frac{I_4}{\sqrt{\mu}}\Big)\Big| &= \frac{1}{\varepsilon}\Big|\Big(\mathcal{L}\partial^\alpha f, \frac{I_4^2}{\sqrt{\mu}}\Big)\Big|\\
&\le C(\eta_0+\varepsilon)\frac{1}{\varepsilon}\Big(\|\partial^\alpha f\|_\sigma^2 + \|\partial^\alpha(\tilde\rho,\tilde u,\tilde\theta)\|^2 + \frac{1}{\varepsilon^2}\|\partial^{\alpha'} f\|_\sigma^2 + \varepsilon^2\Big).
\end{aligned}\tag{5.57}
$$

In order to get the estimates of I_5, we first write $I_5 = I_5^1 + I_5^2$ with

$$
\begin{aligned}
I_5^1 = \sum_{1\le\alpha_1\le\alpha'} C_{\alpha'}^{\alpha_1}\Big(&\partial^{\alpha_1}\Big(M\frac{1}{\rho}\Big)\partial^{\alpha'-\alpha_1}\partial_{x_i}\tilde\rho + \partial^{\alpha_1}\Big(M\frac{v-u}{R\theta}\Big)\cdot\partial^{\alpha'-\alpha_1}\partial_{x_i}\tilde u\\
&+ \partial^{\alpha_1}\Big(M\frac{|v-u|^2}{2R\theta^2} - M\frac{3}{2\theta}\Big)\partial^{\alpha'-\alpha_1}\partial_{x_i}\tilde\theta\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
I_5^2 = \sum_{1\le\alpha_1\le\alpha'} C_{\alpha'}^{\alpha_1}\Big(&\partial^{\alpha_1}\Big(M\frac{1}{\rho}\Big)\partial^{\alpha'-\alpha_1}\partial_{x_i}\bar\rho + \partial^{\alpha_1}\Big(M\frac{v-u}{R\theta}\Big)\cdot\partial^{\alpha'-\alpha_1}\partial_{x_i}\bar u\\
&+ \partial^{\alpha_1}\Big(M\frac{|v-u|^2}{2R\theta^2} - M\frac{3}{2\theta}\Big)\partial^{\alpha'-\alpha_1}\partial_{x_i}\bar\theta\Big).
\end{aligned}
$$

For I_5^1, using $\mathcal{L}f = \Gamma(\sqrt{\mu}, f) + \Gamma(f, \sqrt{\mu})$, (4.11), (3.16), and (4.2) again, we see that $\frac{1}{\varepsilon}|(\mathcal{L}\partial^\alpha f, \frac{I_5^1}{\sqrt{\mu}})|$ is bounded as

$$
C\sum_{1\le\alpha_1\le\alpha'}\frac{1}{\varepsilon}\int_{\mathbb{R}^3}|\partial^\alpha f|_\sigma|\partial^{\alpha'-\alpha_1}\partial_{x_i}(\tilde\rho,\tilde u,\tilde\theta)|(|\partial^{\alpha_1}(\rho,u,\theta)| + \cdots + |\nabla_x(\rho,u,\theta)|^{|\alpha_1|})\,dx.
$$

If $|\alpha_1| = |\alpha'| = N-1$, then $|\alpha'-\alpha_1| = 0$; we take the L^6–L^3–L^2 Hölder inequality and use Lemma 4.1, (3.3), and (4.1), as well as the Cauchy–Schwarz and Sobolev inequalities, to get

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_{\mathbb{R}^3}|\partial^\alpha f|_\sigma|\partial^{\alpha'-\alpha_1}\partial_{x_i}(\tilde\rho,\tilde u,\tilde\theta)|(|\partial^{\alpha_1}(\rho,u,\theta)| + \cdots + |\nabla_x(\rho,u,\theta)|^{|\alpha_1|})\,dx\\
&\quad\le C\frac{1}{\varepsilon}\|\,|\partial^\alpha f|_\sigma\|_{L^2}\|\partial_{x_i}(\tilde\rho,\tilde u,\tilde\theta)\|_{L^6}\|(|\partial^{\alpha_1}(\rho,u,\theta)| + \cdots + |\nabla_x(\rho,u,\theta)|^{|\alpha_1|})\|_{L^3}\\
&\quad\le C(\eta_0+\varepsilon^{\frac12})\frac{1}{\varepsilon}\|\partial^\alpha f\|_\sigma\|\partial_{x_i}(\tilde\rho,\tilde u,\tilde\theta)\|_{H^1} \le C(\eta_0+\varepsilon^{\frac12})\frac{1}{\varepsilon}(\|\partial^\alpha f\|_\sigma^2 + \varepsilon^2).
\end{aligned}
$$

If $1 \le |\alpha_1| < |\alpha'| = N-1$, we also have

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\mathbb{R}^3} |\partial^\alpha f|_\sigma |\partial^{\alpha'-\alpha_1}\partial_{x_i}(\tilde{\rho},\tilde{u},\tilde{\theta})|(|\partial^{\alpha_1}(\rho,u,\theta)| + \cdots + |\nabla_x(\rho,u,\theta)|^{|\alpha_1|})\,dx \\
&\quad\le C\frac{1}{\varepsilon}\|\,|\partial^\alpha f|_\sigma\|_{L^2}\|\partial^{\alpha'-\alpha_1}\partial_{x_i}(\tilde{\rho},\tilde{u},\tilde{\theta})\|_{L^2} \\
&\qquad\times\|(|\partial^{\alpha_1}(\rho,u,\theta)| + \cdots + |\nabla_x(\rho,u,\theta)|^{|\alpha_1|})\|_{L^\infty} \\
&\quad\le C(\eta_0+\varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon}(\|\partial^\alpha f\|_\sigma^2+\varepsilon^2).
\end{aligned}$$

It follows from the above three estimates that

$$\frac{1}{\varepsilon}\Big|\Big(\mathcal{L}\partial^\alpha f, \frac{I_5^1}{\sqrt{\mu}}\Big)\Big| \le C(\eta_0+\varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon}(\|\partial^\alpha f\|_\sigma^2+\varepsilon^2). \tag{5.58}$$

The term I_5^2 can be treated in a similar way to (5.56) and it holds that

$$\frac{1}{\varepsilon}\Big|\Big(\mathcal{L}\partial^\alpha f, \frac{I_5^2}{\sqrt{\mu}}\Big)\Big| \le C(\eta_0+\varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon}\Big(\frac{1}{\varepsilon^2}\|\partial^{\alpha'} f\|_\sigma^2+\varepsilon^2\Big).$$

This estimate together with (5.58) yields

$$\frac{1}{\varepsilon}\Big|\Big(\mathcal{L}\partial^\alpha f, \frac{I_5}{\sqrt{\mu}}\Big)\Big| \le C(\eta_0+\varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon}\Big(\|\partial^\alpha f\|_\sigma^2+\frac{1}{\varepsilon^2}\|\partial^{\alpha'} f\|_\sigma^2+\varepsilon^2\Big). \tag{5.59}$$

Combining (5.55), (5.57), and (5.59), we get

$$\begin{aligned}
\frac{1}{\varepsilon}\Big|\Big(\mathcal{L}\partial^\alpha f, \frac{\partial^\alpha M}{\sqrt{\mu}}\Big)\Big| &\le C(\eta_0+\varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon}\Big(\|\partial^\alpha f\|_\sigma^2+\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2+\frac{1}{\varepsilon^2}\|\partial^{\alpha'} f\|_\sigma^2+\varepsilon^2\Big) \\
&\le C(\eta_0+\varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon^2}\mathcal{D}_N(t)+C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon.
\end{aligned} \tag{5.60}$$

Recalling (5.54) and collecting all estimates above, we thereby obtain

$$-\frac{1}{\varepsilon}\Big(\mathcal{L}\partial^\alpha f, \frac{\partial^\alpha F}{\sqrt{\mu}}\Big) \ge \frac{c_1}{2}\frac{1}{\varepsilon}\|\partial^\alpha f\|_\sigma^2 - C(\eta_0+\varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon^2}\mathcal{D}_N(t) - C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon. \tag{5.61}$$

Let us now consider the terms on the right-hand of (5.53). First note that

$$\partial^\alpha\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, f\Big) = \Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, \partial^\alpha f\Big) + \sum_{1\le\alpha_1\le\alpha} C_\alpha^{\alpha_1}\Gamma\Big(\frac{\partial^{\alpha_1}(M-\mu)}{\sqrt{\mu}}, \partial^{\alpha-\alpha_1} f\Big).$$

By the fact that $F = M + \overline{G} + \sqrt{\mu} f$, one writes

$$\frac{1}{\varepsilon}\Big(\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, \partial^\alpha f\Big), \frac{\partial^\alpha F}{\sqrt{\mu}}\Big) = \frac{1}{\varepsilon}\Big(\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, \partial^\alpha f\Big), \frac{\partial^\alpha M}{\sqrt{\mu}} + \frac{\partial^\alpha \overline{G}}{\sqrt{\mu}} + \partial^\alpha f\Big). \tag{5.62}$$

Recalling (5.55), the first term of (5.62) can be denoted as

$$\frac{1}{\varepsilon}\Big(\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},\partial^\alpha f\Big),\frac{\partial^\alpha M}{\sqrt{\mu}}\Big)=\frac{1}{\varepsilon}\Big(\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},\partial^\alpha f\Big),\frac{I_4+I_5}{\sqrt{\mu}}\Big).$$

Using the definition of I_4 in (5.55) again, one has

$$\begin{aligned}&\frac{1}{\varepsilon}\Big(\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},\partial^\alpha f\Big),\frac{I_4}{\sqrt{\mu}}\Big)\\&=\frac{1}{\varepsilon}\Big(\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},\partial^\alpha f\Big),\frac{M}{\sqrt{\mu}}\Big\{\frac{\partial^\alpha\tilde{\rho}}{\rho}+\frac{(v-u)\cdot\partial^\alpha\tilde{u}}{R\theta}+\Big(\frac{|v-u|^2}{2R\theta}-\frac{3}{2}\Big)\frac{\partial^\alpha\tilde{\theta}}{\theta}\Big\}\Big)\\&\quad+\frac{1}{\varepsilon}\Big(\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},\partial^\alpha f\Big),\frac{M}{\sqrt{\mu}}\Big\{\frac{\partial^\alpha\bar{\rho}}{\rho}+\frac{(v-u)\cdot\partial^\alpha\bar{u}}{R\theta}+\Big(\frac{|v-u|^2}{2R\theta}-\frac{3}{2}\Big)\frac{\partial^\alpha\bar{\theta}}{\theta}\Big\}\Big)\\&\le C(\eta_0+\varepsilon)\frac{1}{\varepsilon}\Big(\|\partial^\alpha f\|_\sigma^2+\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2+\frac{1}{\varepsilon^2}\|\partial^{\alpha'}f\|_\sigma^2+\varepsilon^2\Big),\end{aligned}$$

where we have used similar arguments to (5.57) and $\partial^\alpha=\partial^{\alpha'}\partial_{x_i}$ with $|\alpha'|=N-1$. On the other hand, performing similar calculations to (5.59) gives

$$\frac{1}{\varepsilon}\Big(\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},\partial^\alpha f\Big),\frac{I_5}{\sqrt{\mu}}\Big)\le C(\eta_0+\varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon}\Big(\|\partial^\alpha f\|_\sigma^2+\frac{1}{\varepsilon^2}\|\partial^{\alpha'}f\|_\sigma^2+\varepsilon^2\Big).$$

With the help of the above two estimates, the first term of (5.62) is bounded by

$$\begin{aligned}&\frac{1}{\varepsilon}\Big|\Big(\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},\partial^\alpha f\Big),\frac{\partial^\alpha M}{\sqrt{\mu}}\Big)\Big|\\&\le C(\eta_0+\varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon}\Big(\|\partial^\alpha f\|_\sigma^2+\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2+\frac{1}{\varepsilon^2}\|\partial^{\alpha'}f\|_\sigma^2+\varepsilon^2\Big),\end{aligned}$$

for $|\alpha'|=N-1$. The estimates of the last two terms of (5.62) will be much easier and they are controlled by

$$\begin{aligned}&\frac{1}{\varepsilon}\Big(\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},\partial^\alpha f\Big),\frac{\partial^\alpha\overline{G}}{\sqrt{\mu}}+\partial^\alpha f\Big)\\&\le C\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\Big|\frac{M-\mu}{\sqrt{\mu}}\Big|_2|\partial^\alpha f|_\sigma\Big(\Big|\frac{\partial^\alpha\overline{G}}{\sqrt{\mu}}\Big|_\sigma+|\partial^\alpha f|_\sigma\Big)\,dx\\&\le C(\eta_0+\varepsilon)\frac{1}{\varepsilon}(\|\partial^\alpha f\|_\sigma^2+\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2+\varepsilon^2),\end{aligned}$$

according to (4.11), (4.16), Lemma 4.3, (3.3), and (4.1). Plugging the above estimates into (5.62) gives

$$\begin{aligned}&\frac{1}{\varepsilon}\Big|\Big(\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},\partial^\alpha f\Big),\frac{\partial^\alpha F}{\sqrt{\mu}}\Big)\Big|\\&\le C(\eta_0+\varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon}\Big(\|\partial^\alpha f\|_\sigma^2+\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2+\frac{1}{\varepsilon^2}\|\partial^{\alpha'}f\|_\sigma^2+\varepsilon^2\Big)\\&\le C(\eta_0+\varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon^2}\mathcal{D}_N(t)+C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon.\end{aligned}\tag{5.63}$$

Using (4.11) again, it is straightforward to see that

$$\frac{1}{\varepsilon}\sum_{1\le\alpha_1\le\alpha} C_\alpha^{\alpha_1}\Big(\Gamma\Big(\partial^{\alpha_1}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big),\partial^{\alpha-\alpha_1}f\Big),\frac{\partial^\alpha F}{\sqrt{\mu}}\Big)$$
$$\le C\sum_{1\le\alpha_1\le\alpha}\underbrace{\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\Big|\langle v\rangle^{-1}\partial^{\alpha_1}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|_2|\partial^{\alpha-\alpha_1}f|_\sigma\Big|\frac{\partial^\alpha F}{\sqrt{\mu}}\Big|_\sigma\,dx}_{J_3}.$$

Let us carefully deal with the term J_3. If $1\le|\alpha_1|\le|\alpha|/2$, we use the Cauchy–Schwarz and Sobolev inequalities, (4.17), and (4.18), as well as (3.3) and (4.1), to get

$$\begin{aligned}
J_3 &\le C\frac{1}{\varepsilon}\Big\|\Big|\partial^{\alpha_1}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|_2\Big\|_{L^\infty}\|\partial^{\alpha-\alpha_1}f\|_\sigma\Big\|\frac{\partial^\alpha F}{\sqrt{\mu}}\Big\|_\sigma\\
&\le C\frac{1}{\varepsilon}\|\,|(|\partial^{\alpha_1}(\rho,u,\theta)|+\cdots+|\nabla_x(\rho,u,\theta)|^{|\alpha_1|})|\|_{L^\infty}\|\partial^{\alpha-\alpha_1}f\|_\sigma\Big\|\frac{\partial^\alpha F}{\sqrt{\mu}}\Big\|_\sigma\\
&\le C(\eta_0+\varepsilon^{\frac12})\frac{1}{\varepsilon}\|\partial^{\alpha-\alpha_1}f\|_\sigma(\|\partial^\alpha f\|_\sigma+\|\partial^\alpha(\tilde\rho,\tilde u,\tilde\theta)\|+\eta_0+\varepsilon)\\
&\le C(\eta_0+\varepsilon^{\frac12})\frac{1}{\varepsilon}\Big\{\|\partial^\alpha f\|_\sigma^2+\|\partial^\alpha(\tilde\rho,\tilde u,\tilde\theta)\|^2+\frac{1}{\varepsilon^2}\|\partial^{\alpha-\alpha_1}f\|_\sigma^2+\varepsilon^2(\eta_0+\varepsilon)\Big\}\\
&\le C(\eta_0+\varepsilon^{\frac12})\frac{1}{\varepsilon^2}\mathcal{D}_N(t)+C(\eta_0+\varepsilon^{\frac12})\varepsilon,
\end{aligned}$$

where we have used the smallness of ε and η_0, as well as the following estimate for $|\alpha|=N$:

$$\begin{aligned}
\Big\|\frac{\partial^\alpha F}{\sqrt{\mu}}\Big\|_\sigma &\le \Big\|\frac{\partial^\alpha\sqrt{\mu}f}{\sqrt{\mu}}\Big\|_\sigma+\Big\|\frac{\partial^\alpha\overline{G}}{\sqrt{\mu}}\Big\|_\sigma+\Big\|\frac{\partial^\alpha M}{\sqrt{\mu}}\Big\|_\sigma\\
&\le C(\|\partial^\alpha f\|_\sigma+\|\partial^\alpha(\tilde\rho,\tilde u,\tilde\theta)\|+\eta_0+\varepsilon).
\end{aligned}\tag{5.64}$$

If $|\alpha|/2<|\alpha_1|\le|\alpha|-1$, one takes the L^6–L^3–L^2 Hölder inequality to claim

$$\begin{aligned}
J_3 &\le C\frac{1}{\varepsilon}\Big\|\Big|\partial^{\alpha_1}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big|_2\Big\|_{L^3}\|\,|\partial^{\alpha-\alpha_1}f|_\sigma\|_{L^6}\Big\|\frac{\partial^\alpha F}{\sqrt{\mu}}\Big\|_\sigma\\
&\le C\frac{1}{\varepsilon}(\eta_0+\varepsilon^{\frac12})\|\nabla_x\partial^{\alpha-\alpha_1}f\|_\sigma(\|\partial^\alpha f\|_\sigma+\|\partial^\alpha(\tilde\rho,\tilde u,\tilde\theta)\|+\eta_0+\varepsilon)\\
&\le C(\eta_0+\varepsilon^{\frac12})\frac{1}{\varepsilon^2}\mathcal{D}_N(t)+C(\eta_0+\varepsilon^{\frac12})\varepsilon.
\end{aligned}$$

If $|\alpha_1|=|\alpha|$, then $|\alpha-\alpha_1|=0$ and it is easy to check that

$$\begin{aligned}
J_3 &\le C\frac{1}{\varepsilon}\Big\|\partial^{\alpha_1}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\Big\|\,\|\,|f|_\sigma\|_{L^\infty}\Big\|\frac{\partial^\alpha F}{\sqrt{\mu}}\Big\|_\sigma\\
&\le C\frac{1}{\varepsilon}\Big(\varepsilon^{-\frac32}\|\nabla_x f\|_\sigma\|\nabla_x^2 f\|_\sigma+\varepsilon^{\frac32}\Big\|\frac{\partial^\alpha F}{\sqrt{\mu}}\Big\|_\sigma^2\Big)\\
&\le C\varepsilon^{\frac12}\frac{1}{\varepsilon^2}\mathcal{D}_N(t)+C(\eta_0+\varepsilon)\varepsilon^{\frac12}.
\end{aligned}$$

We thus conclude from the above estimates of J_3 that

$$\frac{1}{\varepsilon}\sum_{1\le\alpha_1\le\alpha} C_\alpha^{\alpha_1}\Big(\Gamma\Big(\frac{\partial^{\alpha_1}(M-\mu)}{\sqrt{\mu}},\partial^{\alpha-\alpha_1}f\Big),\frac{\partial^\alpha F}{\sqrt{\mu}}\Big)$$
$$\le C(\eta_0+\varepsilon^{\frac12})\frac{1}{\varepsilon^2}\mathcal{D}_N(t)+C(\eta_0+\varepsilon^{\frac12})\varepsilon^{\frac12}. \tag{5.65}$$

With (5.63) and (5.65) in hand, we get

$$\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},f\Big),\frac{\partial^\alpha F}{\sqrt{\mu}}\Big)\Big|\le C(\eta_0+\varepsilon^{\frac12})\frac{1}{\varepsilon^2}\mathcal{D}_N(t)+C(\eta_0+\varepsilon^{\frac12})\varepsilon^{\frac12}. \tag{5.66}$$

The second term on the right-hand side of (5.53) can be treated in the same way as (5.66). It follows that

$$\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha\Gamma\Big(f,\frac{M-\mu}{\sqrt{\mu}}\Big),\frac{\partial^\alpha F}{\sqrt{\mu}}\Big)\Big|\le C(\eta_0+\varepsilon^{\frac12})\frac{1}{\varepsilon^2}\mathcal{D}_N(t)+C(\eta_0+\varepsilon^{\frac12})\varepsilon^{\frac12}. \tag{5.67}$$

Next we will concentrate on the third term on the right-hand side of (5.53). By $G=\overline{G}+\sqrt{\mu}f$, we see that

$$\frac{1}{\varepsilon}\Big(\partial^\alpha\Gamma\Big(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}\Big),\frac{\partial^\alpha F}{\sqrt{\mu}}\Big)=\frac{1}{\varepsilon}\Big(\partial^\alpha\Gamma\Big(\frac{\overline{G}}{\sqrt{\mu}},\frac{\overline{G}}{\sqrt{\mu}}\Big)+\partial^\alpha\Gamma\Big(\frac{\overline{G}}{\sqrt{\mu}},f\Big)$$
$$+\partial^\alpha\Gamma\Big(f,\frac{\overline{G}}{\sqrt{\mu}}\Big)+\partial^\alpha\Gamma(f,f),\frac{\partial^\alpha F}{\sqrt{\mu}}\Big). \tag{5.68}$$

We compute (5.68) term by term. By (4.11), Lemma 4.3, the Cauchy–Schwarz and Sobolev inequalities, (5.64), (3.3), and (4.1), one has

$$\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha\Gamma\Big(\frac{\overline{G}}{\sqrt{\mu}},\frac{\overline{G}}{\sqrt{\mu}}\Big),\frac{\partial^\alpha F}{\sqrt{\mu}}\Big)\Big|$$
$$\le C\frac{1}{\varepsilon}\sum_{\alpha_1\le\alpha}\int_{\mathbb{R}^3}\Big|\langle v\rangle^{-1}\partial^{\alpha_1}\Big(\frac{\overline{G}}{\sqrt{\mu}}\Big)\Big|_2\Big|\partial^{\alpha-\alpha_1}\Big(\frac{\overline{G}}{\sqrt{\mu}}\Big)\Big|_\sigma\Big|\frac{\partial^\alpha F}{\sqrt{\mu}}\Big|_\sigma\,dx$$
$$\le C(\eta_0+\varepsilon^{\frac12})\frac{1}{\varepsilon^2}\mathcal{D}_N(t)+C(\eta_0+\varepsilon^{\frac12})\varepsilon^{\frac12}.$$

Using (4.11) again, the second term on the right-hand side of (5.68) is bounded by

$$\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha\Gamma\Big(\frac{\overline{G}}{\sqrt{\mu}},f\Big),\frac{\partial^\alpha F}{\sqrt{\mu}}\Big)\Big|$$
$$\le C\sum_{\alpha_1\le\alpha}\underbrace{\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\Big|\langle v\rangle^{-1}\partial^{\alpha_1}\Big(\frac{\overline{G}}{\sqrt{\mu}}\Big)\Big|_2|\partial^{\alpha-\alpha_1}f|_\sigma\Big|\frac{\partial^\alpha F}{\sqrt{\mu}}\Big|_\sigma\,dx}_{J_4}.$$

If $|\alpha_1| \le |\alpha|/2$, we can deduce from Lemma 4.3, the Cauchy–Schwarz and Sobolev inequalities, (3.3), (4.1), and (5.64) that

$$\begin{aligned} J_4 &\le C\frac{1}{\varepsilon}\Big\| \Big|\partial^{\alpha_1}\Big(\frac{\overline{G}}{\sqrt{\mu}}\Big)\Big|_2\Big\|_{L^\infty}\|\partial^{\alpha-\alpha_1} f\|_\sigma \Big\|\frac{\partial^\alpha F}{\sqrt{\mu}}\Big\|_\sigma \\ &\le C\eta_0\|\partial^{\alpha-\alpha_1} f\|_\sigma(\|\partial^\alpha f\|_\sigma + \|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})\| + \eta_0 + \varepsilon) \\ &\le C(\eta_0 + \varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon^2}\mathcal{D}_N(t) + C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^{\frac{1}{2}}. \end{aligned}$$

If $|\alpha|/2 < |\alpha_1| \le |\alpha|$ then it holds that

$$\begin{aligned} J_4 &\le C\frac{1}{\varepsilon}\Big\|\partial^{\alpha_1}\Big(\frac{\overline{G}}{\sqrt{\mu}}\Big)\Big\| \, \| \, |\partial^{\alpha-\alpha_1} f|_\sigma\|_{L^\infty}\Big\|\frac{\partial^\alpha F}{\sqrt{\mu}}\Big\|_\sigma \\ &\le C(\eta_0 + \varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon^2}\mathcal{D}_N(t) + C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^{\frac{1}{2}}. \end{aligned}$$

It follows from the above estimate of J_4 that

$$\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha\Gamma\Big(\frac{\overline{G}}{\sqrt{\mu}}, f\Big), \frac{\partial^\alpha F}{\sqrt{\mu}}\Big)\Big| \le C(\eta_0 + \varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon^2}\mathcal{D}_N(t) + C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^{\frac{1}{2}}.$$

The third term on the right-hand side of (5.68) can be handled in the same manner and it shares the same bound. We still compute the last term of (5.68). It is clear to see by (4.11) that

$$\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha\Gamma(f,f), \frac{\partial^\alpha F}{\sqrt{\mu}}\Big)\Big| \le C\sum_{\alpha_1\le\alpha}\underbrace{\frac{1}{\varepsilon}\int_{\mathbb{R}^3}|\langle v\rangle^{-1}\partial^{\alpha_1} f|_2|\partial^{\alpha-\alpha_1} f|_\sigma\Big|\frac{\partial^\alpha F}{\sqrt{\mu}}\Big|_\sigma\,dx}_{J_5}.$$

The term J_5 is treated in the same way as the term J_4. For $|\alpha_1| \le |\alpha|/2$, we have

$$J_5 \le C\frac{1}{\varepsilon}\|\partial^{\alpha_1} f\|_{L^\infty}\|\partial^{\alpha-\alpha_1} f\|_\sigma\Big\|\frac{\partial^\alpha F}{\sqrt{\mu}}\Big\|_\sigma \le C(\eta_0 + \varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon^2}\mathcal{D}_N(t) + C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^{\frac{1}{2}}.$$

In case of $|\alpha|/2 < |\alpha_1| \le |\alpha|$, it follows that

$$J_5 \le C\frac{1}{\varepsilon}\|\partial^{\alpha_1} f\| \, \| \, |\partial^{\alpha-\alpha_1} f|_\sigma\|_{L^\infty}\Big\|\frac{\partial^\alpha F}{\sqrt{\mu}}\Big\|_\sigma \le C(\eta_0 + \varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon^2}\mathcal{D}_N(t) + C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^{\frac{1}{2}}.$$

With these estimates, we can obtain

$$\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha\Gamma(f,f), \frac{\partial^\alpha F}{\sqrt{\mu}}\Big)\Big| \le C(\eta_0 + \varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon^2}\mathcal{D}_N(t) + C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^{\frac{1}{2}}.$$

Consequently, plugging the above estimates into (5.68) leads us to

$$\frac{1}{\varepsilon}\Big|\partial^\alpha\Big(\Gamma\Big(\frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}}\Big), \frac{\partial^\alpha F}{\sqrt{\mu}}\Big)\Big| \le C(\eta_0 + \varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon^2}\mathcal{D}_N(t) + C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^{\frac{1}{2}}. \tag{5.69}$$

The last term of (5.53) can be controlled by

$$
\begin{aligned}
&C\Big\|\langle v\rangle^{\frac{1}{2}}\frac{1}{\sqrt{\mu}}\partial^\alpha P_1\Big\{v\cdot\Big(\frac{|v-u|^2\nabla_x\bar{\theta}}{2R\theta^2}+\frac{(v-u)\cdot\nabla_x\bar{u}}{R\theta}\Big)M\Big\}\Big\|^2+C\Big\|\langle v\rangle^{-\frac{1}{2}}\frac{\partial^\alpha F}{\sqrt{\mu}}\Big\|^2\\
&\quad\le C(\|\partial^\alpha f\|_\sigma^2+\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2+\eta_0+\varepsilon)\\
&\quad\le C(\eta_0+\varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon^2}\mathcal{D}_N(t)+C(\eta_0+\varepsilon^{\frac{1}{2}}),
\end{aligned}
\tag{5.70}
$$

where the Cauchy–Schwarz and Sobolev inequalities, (3.16), (3.3), (4.1), and (5.64) have been used.

As a consequence, substituting (5.61), (5.66), (5.67), (5.69), and (5.70) into (5.53), and using the smallness of ε and η_0, we get

$$
\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=N}\Big\|\frac{\partial^\alpha F}{\sqrt{\mu}}\Big\|^2+c\frac{1}{\varepsilon}\sum_{|\alpha|=N}\|\partial^\alpha f\|_\sigma^2\le C(\eta_0+\varepsilon^{\frac{1}{2}})\frac{1}{\varepsilon^2}\mathcal{D}_N(t)+C(\eta_0+\varepsilon^{\frac{1}{2}}). \tag{5.71}
$$

To further estimate (5.71), using (5.73), whose proof will be postponed to Lemma 5.10 later on, one has

$$
\varepsilon^2\sum_{|\alpha|=N}\Big\|\frac{\partial^\alpha F(t)}{\sqrt{\mu}}\Big\|^2\ge c_3\varepsilon^2\sum_{|\alpha|=N}(\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2+\|\partial^\alpha f\|^2)-C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2. \tag{5.72}
$$

Therefore, by integrating (5.71) with respect to t and multiplying the resulting equation by ε^2, and then using (5.72), (3.22), and the estimate

$$
\begin{aligned}
\varepsilon^2\sum_{|\alpha|=N}\Big\|\frac{\partial^\alpha F(0)}{\sqrt{\mu}}\Big\|^2&\le C\varepsilon^2\sum_{|\alpha|=N}\Big(\Big\|\frac{\sqrt{\mu}\partial^\alpha f(0)}{\sqrt{\mu}}\Big\|^2+\Big\|\frac{\partial^\alpha\overline{G}(0)}{\sqrt{\mu}}\Big\|^2+\Big\|\frac{\partial^\alpha M(0)}{\sqrt{\mu}}\Big\|^2\Big)\\
&\le C(\eta_0+\varepsilon^{\frac{1}{2}})\varepsilon^2,
\end{aligned}
$$

we thus arrive at the desired estimate (5.52). This completes the proof of Lemma 5.9. ∎

To apply (5.72) in the proof of Lemma 5.9 above, we need the following result.

Lemma 5.10. *There is a constant $c_2>0$ such that*

$$
\Big\|\frac{\partial^\alpha F(t)}{\sqrt{\mu}}\Big\|^2\ge c_2(\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})\|^2+\|\partial^\alpha f\|^2)-C(\eta_0+\varepsilon^{\frac{1}{2}}), \tag{5.73}
$$

for any α with $|\alpha|=N$.

Proof. Let $|\alpha|=N$. In view of the macro-micro decomposition $F=M+G$, it holds that

$$
\Big\|\frac{\partial^\alpha F(t)}{\sqrt{\mu}}\Big\|^2=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\partial^\alpha M)^2+(\partial^\alpha G)^2+2\partial^\alpha G\partial^\alpha M}{\mu}\,dv\,dx. \tag{5.74}
$$

First of all, we write

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\partial^\alpha M)^2}{\mu}\,dv\,dx=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\partial^\alpha M)^2}{M}+\Big(\frac{1}{\mu}-\frac{1}{M}\Big)(\partial^\alpha M)^2\,dv\,dx. \tag{5.75}$$

Since $\partial^\alpha M = I_4 + I_5$ holds in terms of (5.55), one has

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\partial^\alpha M)^2}{M}\,dv\,dx=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(I_4)^2+(I_5)^2+2I_4I_5}{M}\,dv\,dx.$$

Recalling the definition of I_4 in (5.55), we can deduce from (2.4) and (3.3) that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(I_4)^2}{M}\,dv\,dx\\
&\quad=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}M\Big\{\frac{\partial^\alpha\rho}{\rho}+\frac{(v-u)\cdot\partial^\alpha u}{R\theta}+\Big(\frac{|v-u|^2}{2R\theta}-\frac{3}{2}\Big)\frac{\partial^\alpha\theta}{\theta}\Big\}^2\,dv\,dx\\
&\quad=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}M\Big\{\Big(\frac{\partial^\alpha\rho}{\rho}\Big)^2+\Big(\frac{(v-u)\cdot\partial^\alpha u}{R\theta}\Big)^2+\Big(\Big(\frac{|v-u|^2}{2R\theta}-\frac{3}{2}\Big)\frac{\partial^\alpha\theta}{\theta}\Big)^2\Big\}\,dv\,dx,
\end{aligned}$$

which is further bounded from below as

$$c\|\partial^\alpha(\rho,u,\theta)\|^2\geq c\|\partial^\alpha(\tilde\rho,\tilde u,\tilde\theta)\|^2-C\|\partial^\alpha(\bar\rho,\bar u,\bar\theta)\|^2\geq c\|\partial^\alpha(\tilde\rho,\tilde u,\tilde\theta)\|^2-C\eta_0.$$

By the definition of I_5 in (5.55), we see that $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(I_5)^2}{M}\,dv\,dx$ is given by

$$\begin{aligned}
\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{M}\Big\{\sum_{1\leq\alpha_1\leq\alpha'}C_{\alpha'}^{\alpha_1}\Big[\partial^{\alpha_1}\Big(M\frac{1}{\rho}\Big)\partial^{\alpha'-\alpha_1}\partial_{x_i}\rho+\partial^{\alpha_1}\Big(M\frac{v-u}{R\theta}\Big)\cdot\partial^{\alpha'-\alpha_1}\partial_{x_i}u\\
+\partial^{\alpha_1}\Big(M\frac{|v-u|^2}{2R\theta^2}-M\frac{3}{2\theta}\Big)\partial^{\alpha'-\alpha_1}\partial_{x_i}\theta\Big]\Big\}^2\,dv\,dx,
\end{aligned}$$

which can be further bounded by $C(\eta_0+\varepsilon^{\frac{1}{2}})$. Similarly, using the definitions of I_4 and I_5 in (5.55) again, we claim that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{2I_4I_5}{M}\,dv\,dx\leq C(\eta_0+\varepsilon^{\frac{1}{2}}).$$

It follows from the above estimates that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\partial^\alpha M)^2}{M}\,dv\,dx\geq c\|\partial^\alpha(\tilde\rho,\tilde u,\tilde\theta)\|^2-C(\eta_0+\varepsilon^{\frac{1}{2}}).$$

On the other hand, it holds that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big(\frac{1}{\mu}-\frac{1}{M}\Big)(\partial^\alpha M)^2\,dv\,dx\leq C(\eta_0+\varepsilon^{\frac{1}{2}}).$$

Plugging the above two estimates into (5.75), we obtain

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(\partial^\alpha M)^2}{\mu}\,dv\,dx\geq c\|\partial^\alpha(\tilde\rho,\tilde u,\tilde\theta)\|^2-C(\eta_0+\varepsilon^{\frac{1}{2}}).$$

Thanks to $G = \overline{G} + \sqrt{\mu} f$, one has

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(\partial^\alpha G)^2}{\mu}\, dv\, dx \\
&\quad = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(\sqrt{\mu}\partial^\alpha f + \partial^\alpha \overline{G})^2}{\mu}\, dv\, dx \\
&\quad = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (\partial^\alpha f)^2\, dv\, dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(\partial^\alpha \overline{G})^2 + 2\sqrt{\mu}\partial^\alpha f \partial^\alpha \overline{G}}{\mu}\, dv\, dx \\
&\quad \geq \frac{1}{2}\|\partial^\alpha f\|^2 - C(\eta_0 + \varepsilon),
\end{aligned}$$

where in the last inequality we used Lemma 4.3, (3.3), (4.1), and the smallness of ε and η_0. For the last term of (5.74), we see that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{2\partial^\alpha G \partial^\alpha M}{\mu}\, dv\, dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{2\partial^\alpha G \partial^\alpha M}{M} + \Big(\frac{1}{\mu} - \frac{1}{M}\Big) 2\partial^\alpha G \partial^\alpha M\, dv\, dx.$$

First note that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{2\partial^\alpha G I_4}{M}\, dv\, dx \\
&\quad = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} 2\partial^\alpha G \Big(\frac{\partial^\alpha \rho}{\rho} + \frac{(v-u)\cdot \partial^\alpha u}{R\theta} + \Big(\frac{|v-u|^2}{2R\theta} - \frac{3}{2}\Big)\frac{\partial^\alpha \theta}{\theta}\Big)\, dv\, dx = 0,
\end{aligned}$$

due to (5.55) and (2.6). Then it holds by this and the definition of I_5 in (5.55) that

$$\begin{aligned}
\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{2\partial^\alpha G \partial^\alpha M}{M}\, dv\, dx &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{2\partial^\alpha G (I_4 + I_5)}{M}\, dv\, dx \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{2\partial^\alpha G I_5}{M}\, dv\, dx,
\end{aligned}$$

which can be bounded by $C(\eta_0 + \varepsilon^{\frac{1}{2}})$. On the other hand, it is easy to check that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Big(\frac{1}{\mu} - \frac{1}{M}\Big) 2\partial^\alpha G \partial^\alpha M\, dv\, dx \leq C(\eta_0 + \varepsilon^{\frac{1}{2}}).$$

It follows that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{2\partial^\alpha G \partial^\alpha M}{\mu}\, dv\, dx \leq C(\eta_0 + \varepsilon^{\frac{1}{2}}).$$

Consequently, substituting all the above estimates into (5.74), the desired estimate (5.73) follows. This then completes the proof of Lemma 5.10. ■

Combining the estimates in Lemmas 5.9 and 5.8, we are able to obtain all the space derivative estimates for both the fluid and non-fluid parts.

Lemma 5.11. *It holds that*

$$\begin{aligned}
&\sum_{1\le|\alpha|\le N-1}\left\{\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})(t)\|^2+\|\partial^\alpha f(t)\|^2+c\frac{1}{\varepsilon}\int_0^t\|\partial^\alpha f(s)\|_\sigma^2\,ds\right\}\\
&\qquad+c\varepsilon\sum_{2\le|\alpha|\le N}\int_0^t\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})(s)\|^2\,ds\\
&\qquad+C\varepsilon^2\sum_{|\alpha|=N}\left\{\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})(t)\|^2+\|\partial^\alpha f(t)\|^2+c\frac{1}{\varepsilon}\int_0^t\|\partial^\alpha f(s)\|_\sigma^2\,ds\right\}\\
&\quad\le C(\eta_0+\varepsilon^{\frac12})\int_0^t\mathcal{D}_N(s)\,ds+C(1+t)(\eta_0+\varepsilon^{\frac12})\varepsilon^2.
\end{aligned}\tag{5.76}$$

Notice that Lemma 5.11 does not include the zeroth-order energy estimate. Therefore, adding (5.76) to (5.27), we conclude the energy estimate of solutions without velocity derivatives.

Lemma 5.12. *It holds that*

$$\begin{aligned}
&\sum_{|\alpha|\le N-1}(\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})(t)\|^2+\|\partial^\alpha f(t)\|^2)+\varepsilon^2\sum_{|\alpha|=N}(\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})(t)\|^2+\|\partial^\alpha f(t)\|^2)\\
&\qquad+c\varepsilon\sum_{1\le|\alpha|\le N}\int_0^t\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})(s)\|^2\,ds\\
&\qquad+c\frac{1}{\varepsilon}\sum_{|\alpha|\le N-1}\int_0^t\|\partial^\alpha f(s)\|_\sigma^2\,ds+c\varepsilon\sum_{|\alpha|=N}\int_0^t\|\partial^\alpha f(s)\|_\sigma^2\,ds\\
&\quad\le C(\eta_0+\varepsilon^{\frac12})\int_0^t\mathcal{D}_N(s)\,ds+C(1+t)(\eta_0+\varepsilon^{\frac12})\varepsilon^2.
\end{aligned}\tag{5.77}$$

5.6. Mixed derivative estimate

This subsection is devoted to deriving the mixed derivative estimate of the microscopic component f. We follow the iteration technique for velocity derivatives as in [28].

Lemma 5.13. *It holds that*

$$\begin{aligned}
&\sum_{\substack{|\alpha|+|\beta|\le N\\|\beta|\ge1}}\left\{\|\partial_\beta^\alpha f(t)\|_{2,|\beta|}^2+c\frac{1}{\varepsilon}\int_0^t\|\partial_\beta^\alpha f(s)\|_{\sigma,|\beta|}^2\,ds\right\}\\
&\quad\le C\frac{1}{\varepsilon}\sum_{|\alpha|\le N-1}\int_0^t\|\partial^\alpha f(s)\|_\sigma^2\,ds\\
&\qquad+C\varepsilon\sum_{1\le|\alpha|\le N}\int_0^t\{\|\partial^\alpha f(s)\|_\sigma^2+\|\partial^\alpha(\tilde{\rho},\tilde{u},\tilde{\theta})(s)\|^2\}\,ds\\
&\qquad+C(\eta_0+\varepsilon^{\frac12})\int_0^t\mathcal{D}_N(s)\,ds+C(1+t)(\eta_0+\varepsilon^{\frac12})\varepsilon^2.
\end{aligned}\tag{5.78}$$

Proof. Let $|\alpha| + |\beta| \le N$ with $|\beta| \ge 1$ and w be defined in (3.14); then we apply ∂^α_β to (3.13) and take the inner product of the resulting equation with $w^{2|\beta|}\partial^\alpha_\beta f$ over $\mathbb{R}^3 \times \mathbb{R}^3$ to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha_\beta f\|^2_{2,|\beta|} + (v\cdot\nabla_x\partial^\alpha_\beta f, w^{2|\beta|}\partial^\alpha_\beta f) + (C^{\beta-e_i}_\beta \delta^{e_i}_\beta \partial^{\alpha+e_i}_{\beta-e_i} f, w^{2|\beta|}\partial^\alpha_\beta f)\\
&\qquad - \frac{1}{\varepsilon}(\partial^\alpha_\beta \mathcal{L} f, w^{2|\beta|}\partial^\alpha_\beta f)\\
&\quad = \frac{1}{\varepsilon}\Big(\partial^\alpha_\beta\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, f\Big) + \partial^\alpha_\beta\Gamma\Big(f, \frac{M-\mu}{\sqrt{\mu}}\Big), w^{2|\beta|}\partial^\alpha_\beta f\Big)\\
&\qquad + \frac{1}{\varepsilon}\Big(\partial^\alpha_\beta\Gamma\Big(\frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}}\Big), w^{2|\beta|}\partial^\alpha_\beta f\Big)\\
&\qquad + \Big(\partial^\alpha_\beta\Big[\frac{P_0(v\sqrt{\mu}\cdot\nabla_x f)}{\sqrt{\mu}}\Big], w^{2|\beta|}\partial^\alpha_\beta f\Big) - \Big(\partial^\alpha_\beta\Big[\frac{P_1(v\cdot\nabla_x\bar{G})}{\sqrt{\mu}}\Big], w^{2|\beta|}\partial^\alpha_\beta f\Big)\\
&\qquad - \Big(\partial^\alpha_\beta\Big(\frac{\partial_t\bar{G}}{\sqrt{\mu}}\Big), w^{2|\beta|}\partial^\alpha_\beta f\Big)\\
&\qquad - \Big(\partial^\alpha_\beta\Big\{\frac{1}{\sqrt{\mu}}P_1\Big[v\cdot\Big(\frac{|v-u|^2\nabla_x\tilde{\theta}}{2R\theta^2} + \frac{(v-u)\cdot\nabla_x\tilde{u}}{R\theta}\Big)M\Big]\Big\}, w^{2|\beta|}\partial^\alpha_\beta f\Big). \qquad (5.79)
\end{aligned}
$$

Here, $\delta^{e_i}_\beta = 1$ if $e_i \le \beta$ or $\delta^{e_i}_\beta = 0$ otherwise.

We now compute (5.79) term by term. The second term on the left-hand side of (5.79) vanishes by integration by parts. Thanks to $|\beta - e_i| = |\beta| - 1$ and $\|w^{\frac{1}{2}} w^{|\beta|}\partial^\alpha_\beta f\| \le C\|\partial^\alpha_\beta f\|_{\sigma,|\beta|}$ for $w = \langle v\rangle^{-1}$, the third term on the left-hand side of (5.79) can be estimated as

$$
\begin{aligned}
|(C^{\beta-e_i}_\beta \delta^{e_i}_\beta \partial^{\alpha+e_i}_{\beta-e_i} f, w^{2|\beta|}\partial^\alpha_\beta f)| &\le C\|w^{\frac{1}{2}+(|\beta|-1)}\partial^{\alpha+e_i}_{\beta-e_i} f\|\,\|w^{|\beta|+\frac{1}{2}}\partial^\alpha_\beta f\|\\
&= C\|w^{\frac{1}{2}} w^{|\beta-e_i|}\partial^{\alpha+e_i}_{\beta-e_i} f\|\,\|w^{\frac{1}{2}} w^{|\beta|}\partial^\alpha_\beta f\|\\
&\le \eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|} + C_\eta\varepsilon\|\partial^{\alpha+e_i}_{\beta-e_i} f\|^2_{\sigma,|\beta-e_i|}. \qquad (5.80)
\end{aligned}
$$

In view of (4.9), it is easy to see that

$$
-\frac{1}{\varepsilon}(\partial^\alpha_\beta\mathcal{L} f, w^{2|\beta|}\partial^\alpha_\beta f) \ge c\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|} - \eta\frac{1}{\varepsilon}\sum_{|\beta_1|\le|\beta|}\|\partial^\alpha_{\beta_1} f\|^2_{\sigma,|\beta_1|} - C_\eta\frac{1}{\varepsilon}\|\partial^\alpha f\|^2_\sigma.
$$

For the first two terms on the right-hand side of (5.79), we use (4.13) and (4.22) respectively, to get

$$
\begin{aligned}
&\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha_\beta\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, f\Big) + \partial^\alpha_\beta\Gamma\Big(f, \frac{M-\mu}{\sqrt{\mu}}\Big), w^{2|\beta|}\partial^\alpha_\beta f\Big)\Big|\\
&\qquad \le C\eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|} + C_\eta(\eta_0 + \varepsilon^{\frac{1}{2}})\mathcal{D}_N(t)
\end{aligned}
$$

and

$$\frac{1}{\varepsilon}\Big|\Big(\partial^\alpha_\beta\Gamma\Big(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}\Big),w^{2|\beta|}\partial^\alpha_\beta f\Big)\Big|$$
$$\le C\eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}+C_\eta(\eta_0+\varepsilon^{\frac12})\mathcal{D}_N(t)+C_\eta(\eta_0+\varepsilon^{\frac12})\varepsilon^2.$$

The term involving $\overline{G}$ in (5.79) can be estimated, by Lemma 4.3, (3.16), Lemma 4.8, the Cauchy–Schwarz and Sobolev inequalities, (3.3), and (4.1), as

$$\Big|\Big(\partial^\alpha_\beta\Big[\frac{P_1(v\cdot\nabla_x\overline{G})}{\sqrt{\mu}}\Big],w^{2|\beta|}\partial^\alpha_\beta f\Big)\Big|+\Big|\Big(\partial^\alpha_\beta\Big(\frac{\partial_t\overline{G}}{\sqrt{\mu}}\Big),w^{2|\beta|}\partial^\alpha_\beta f\Big)\Big|$$
$$\le C\Big(\Big\|\langle v\rangle^{\frac12}w^{|\beta|}\partial^\alpha_\beta\Big[\frac{P_1(v\cdot\nabla_x\overline{G})}{\sqrt{\mu}}\Big]\Big\|+\Big\|\langle v\rangle^{\frac12}w^{|\beta|}\partial^\alpha_\beta\Big(\frac{\partial_t\overline{G}}{\sqrt{\mu}}\Big)\Big\|\Big)\|\langle v\rangle^{-\frac12}w^{|\beta|}\partial^\alpha_\beta f\|$$
$$\le C\eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}+C_\eta(\eta_0+\varepsilon^{\frac12})\varepsilon^2.$$

For the third term on the right-hand side of (5.79), we can deduce from (2.5), (3.16), the Cauchy–Schwarz and Sobolev inequalities, (3.3), and (4.1) that

$$\Big|\Big(\partial^\alpha_\beta\Big[\frac{1}{\sqrt{\mu}}P_0(v\sqrt{\mu}\cdot\nabla_x f)\Big],w^{2|\beta|}\partial^\alpha_\beta f\Big)\Big|$$
$$=\Big|\sum_{j=0}^4\Big(\langle v\rangle^{\frac12}w^{|\beta|}\partial^\alpha_\beta\Big[\Big\langle v\sqrt{\mu}\cdot\nabla_x f,\frac{\chi_j}{M}\Big\rangle\frac{\chi_j}{\sqrt{\mu}}\Big],\langle v\rangle^{-\frac12}w^{|\beta|}\partial^\alpha_\beta f\Big)\Big|$$
$$\le C\eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}+C_\eta\varepsilon\|\nabla_x\partial^\alpha f\|^2_\sigma+C_\eta(\eta_0+\varepsilon^{\frac12})\varepsilon^2,$$

where we have used the fact that $|\langle v\rangle^l\mu^{-\frac12}\partial_\beta M|_2\le C$ for any $l\ge 0$ and $|\beta|\ge 0$ by (4.2). Likewise, the last term of (5.79) is dominated by

$$C\eta\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}+C_\eta\varepsilon\|(\nabla_x\partial^\alpha\tilde{u},\nabla_x\partial^\alpha\tilde{\theta})\|^2+C_\eta(\eta_0+\varepsilon^{\frac12})\varepsilon^2.$$

Hence, for $|\alpha|+|\beta|\le N$ with $|\beta|\ge 1$ and any small $\eta>0$, we can deduce from the above estimates that

$$\frac12\frac{d}{dt}\|\partial^\alpha_\beta f\|^2_{2,|\beta|}+c\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}$$
$$\le C\eta\frac{1}{\varepsilon}\sum_{|\beta_1|\le|\beta|}\|\partial^\alpha_{\beta_1}f\|^2_{\sigma,|\beta_1|}+C_\eta\varepsilon\|\partial^{\alpha+e_i}_{\beta-e_i}f\|^2_{\sigma,|\beta-e_i|}$$
$$+C_\eta\frac{1}{\varepsilon}\|\partial^\alpha f\|^2_\sigma+C_\eta\varepsilon(\|(\nabla_x\partial^\alpha\tilde{u},\nabla_x\partial^\alpha\tilde{\theta})\|^2+\|\nabla_x\partial^\alpha f\|^2_\sigma)$$
$$+C_\eta(\eta_0+\varepsilon^{\frac12})\mathcal{D}_N(t)+C_\eta(\eta_0+\varepsilon^{\frac12})\varepsilon^2. \tag{5.81}$$

We will use induction on $|\beta|$ to cancel the first and second terms on the right-hand side of (5.81). By a suitable linear combination of (5.81) for all cases such that $|\alpha| + |\beta| \le N$ with $|\beta| \ge 1$ and then taking $\eta > 0$ and $\varepsilon > 0$ small enough, we see that there exist constants $C_{\alpha,\beta} > 0$ such that

$$\sum_{\substack{|\alpha|+|\beta|\le N\\|\beta|\ge 1}} \left\{\frac{d}{dt} C_{\alpha,\beta} \|\partial^\alpha_\beta f\|^2_{2,|\beta|} + c\frac{1}{\varepsilon}\|\partial^\alpha_\beta f\|^2_{\sigma,|\beta|}\right\}$$
$$\le C\varepsilon \sum_{1\le|\alpha|\le N} (\|\partial^\alpha f\|^2_\sigma + \|\partial^\alpha(\tilde\rho,\tilde u,\tilde\theta)\|^2) + C\frac{1}{\varepsilon}\sum_{|\alpha|\le N-1}\|\partial^\alpha f\|^2_\sigma$$
$$+ C(\eta_0 + \varepsilon^{\frac12})\mathcal{D}_N(t) + C(\eta_0 + \varepsilon^{\frac12})\varepsilon^2. \tag{5.82}$$

Integrating (5.82) with respect to t and using (3.22), we can obtain the desired estimate (5.78). This then completes the proof of Lemma 5.13. ∎

6. Proofs of the main results

In this section we will prove our main results, Theorems 3.4 and 3.9, by the energy estimates derived in the previous section. First of all, we prove Theorem 3.4 on the compressible Euler limit for the Landau equation.

6.1. Proof of Theorem 3.4

Multiplying (5.77) by a large positive constant C and then adding the resultant inequality to (5.78), we get, by letting $\eta_0 > 0$ and $\varepsilon > 0$ be small enough, that

$$\mathcal{E}_N(t) + \frac12\int_0^t \mathcal{D}_N(s)\,ds \le \frac12\varepsilon^2, \tag{6.1}$$

for $t \in (0,T]$ with $T \in (0,\tau]$. Here, $\mathcal{E}_N(t)$ and $\mathcal{D}_N(t)$ are defined in (3.17) and (3.18), respectively.

Note that the a priori assumption (4.1) can be closed since the estimate (6.1) is strictly stronger than (4.1). Therefore, by the uniform a priori estimates, the local existence of the solution, and the standard continuity argument, we can immediately derive the existence and uniqueness of smooth solutions to the Landau equation (1.1) with initial data (3.19) as stated in Theorem 3.4. Moreover, the desired estimate (3.20) holds true in terms of (6.1).

To finish the proof of Theorem 3.4, we still need to prove the uniform convergence rate in ε as in (3.21). Note from (4.2) that (ρ,u,θ) and $(\bar\rho,\bar u,\bar\theta)$ are close enough to the state $(1,0,3/2)$; we can deduce from this, (6.1), and (3.17) that

$$\left\|\frac{(M_{[\rho,u,\theta]} - M_{[\bar\rho,\bar u,\bar\theta]})(t)}{\sqrt{\mu}}\right\|_{L^2_xL^2_v} + \left\|\frac{(M_{[\rho,u,\theta]} - M_{[\bar\rho,\bar u,\bar\theta]})(t)}{\sqrt{\mu}}\right\|_{L^\infty_xL^2_v}$$
$$\le C_\tau(\|(\tilde\rho,\tilde u,\tilde\theta)(t)\| + \|(\tilde\rho,\tilde u,\tilde\theta)(t)\|_{L^\infty_x}) \le C_\tau\varepsilon,$$

for any $t \in [0, \tau]$. Similarly, it holds that

$$\sup_{t\in[0,\tau]} (\|f(t)\|_{L^2_x L^2_v} + \|f(t)\|_{L^\infty_x L^2_v}) \le C_\tau \varepsilon.$$

With Lemma 4.3, it is easy to check that

$$\sup_{t\in[0,\tau]} \Big(\Big\| \frac{\bar{G}(t)}{\sqrt{\mu}} \Big\|_{L^2_x L^2_v} + \Big\| \frac{\bar{G}(t)}{\sqrt{\mu}} \Big\|_{L^\infty_x L^2_v} \Big) \le C_\tau \eta_0 \varepsilon.$$

Therefore, by these facts and $F = M + \bar{G} + \sqrt{\mu} f$, we immediately get

$$\Big\| \frac{(F - M_{[\bar{\rho},\bar{u},\bar{\theta}]})(t)}{\sqrt{\mu}} \Big\|_{L^2_x L^2_v} + \Big\| \frac{(F - M_{[\bar{\rho},\bar{u},\bar{\theta}]})(t)}{\sqrt{\mu}} \Big\|_{L^\infty_x L^2_v} \le C_\tau \varepsilon,$$

for any $t \in [0, \tau]$. This, combined with the fact $F = F^\varepsilon(t, x, v)$, gives (3.21) and hence ends the proof of Theorem 3.4. ∎

6.2. Proof of Theorem 3.9

We are now in a position to prove Theorem 3.9. We first give the following lemma on the existence result of the compressible Euler system (1.2) and the initial data associated with (3.24) and δ in (3.26). It can be found in [35, Lemma 3.1]. Readers may also refer to [47, 56] and the references cited therein.

Lemma 6.1. *Consider the compressible Euler system* (1.2) *with initial data*

$$(\bar{\rho}, \bar{u}, \bar{\theta})(0, x) = \Big(1 + \delta\varrho_0, \delta\varphi_0, \frac{3}{2} + \frac{3}{2}\delta\vartheta_0\Big)(x) \tag{6.2}$$

for any given $(\varrho_0, \varphi_0, \vartheta_0)(x) \in H^k(\mathbb{R}^3)$ with integer $k \ge 3$, where $\delta > 0$ is a parameter. Choose a suitable constant $\delta_1 > 0$ so that for any $\delta \in (0, \delta_1]$, the positivity of $1 + \delta\varrho_0$ and $\frac{3}{2} + \frac{3}{2}\delta\vartheta_0$ is guaranteed. Then for each $\delta \in (0, \delta_1]$, there exists a family of classical solutions $(\bar{\rho}^\delta, \bar{u}^\delta, \bar{\theta}^\delta)(t, x) \in C([0, \tau^\delta]; H^k) \cap C^1([0, \tau^\delta]; H^{k-1})$ of the compressible Euler system (1.2) *and* (6.2) *such that the following conditions hold true: $\bar{\rho}^\delta(t, x) > 0$, $\bar{\theta}^\delta(t, x) > 0$, and*

$$\Big\| \Big(\bar{\rho}^\delta(t, x) - 1, \bar{u}^\delta(t, x), \bar{\theta}^\delta(t, x) - \frac{3}{2}\Big) \Big\|_{C([0,\tau^\delta];H^k)\cap C^1([0,\tau^\delta];H^{k-1})} \le C_0.$$

Moreover, the life span τ^δ has the following lower bound:

$$\tau^\delta > \frac{C_1}{\delta}.$$

Here, the positive constants C_0 and C_1 are independent of δ, depending only on the H^k-norm of $(\varrho_0, \varphi_0, \vartheta_0)(x)$.

In what follows we give a refined estimate of two solutions to compressible Euler and acoustic systems. Let $(\bar\rho^\delta, \bar u^\delta, \bar\theta^\delta)(t,x)$ be the compressible Euler solutions as obtained in Lemma 6.1 and $(\varrho, \varphi, \vartheta)(t,x)$ be the solutions of acoustic system (3.23)–(3.24). Then we define

$$\varrho_d^\delta = \frac{1}{\delta^2}(\bar\rho^\delta - 1 - \delta\varrho), \quad \varphi_d^\delta = \frac{1}{\delta^2}(\bar u^\delta - \delta\varphi), \quad \vartheta_d^\delta = \frac{2}{3}\frac{1}{\delta^2}\Big(\bar\theta^\delta - \frac{3}{2} - \frac{3}{2}\delta\vartheta\Big).$$

Following the same strategy as [35, Lemma 3.2], we know that $(\varrho_d^\delta, \varphi_d^\delta, \vartheta_d^\delta)(t)$ satisfies

$$\sup_{t\in[0,\tau]} \|(\varrho_d^\delta, \varphi_d^\delta, \vartheta_d^\delta)(t)\|_{H^k}^2 \le C, \tag{6.3}$$

for $k \ge 3$, where the constant $C > 0$ depends only on τ and on the H^{k+1}-norm of $(\varrho_0, \varphi_0, \vartheta_0)(x)$.

In terms of $(\bar\rho^\delta, \bar u^\delta, \bar\theta^\delta)(t,x)$ as obtained in Lemma 6.1, we denote the local Maxwellian by

$$\bar M^\delta \equiv M_{[\bar\rho^\delta, \bar u^\delta, \bar\theta^\delta]}(t,x,v) := \frac{\bar\rho^\delta(t,x)}{\sqrt{[2\pi R\bar\theta^\delta(t,x)]^3}} \exp\Big\{-\frac{|v-\bar u^\delta(t,x)|^2}{2R\bar\theta^\delta(t,x)}\Big\}.$$

Using (6.3), we can choose a sufficiently small constant $\delta_0 > 0$ such that $(\bar\rho^\delta, \bar u^\delta, \bar\theta^\delta)(t,x)$ with any $0 < \delta \le \delta_0$ satisfies (3.3). With these facts, by using the same arguments as in Theorem 3.4 we can prove the existence and uniqueness of smooth solutions to the Landau equation (1.1) under the assumptions in Theorem 3.9. The details are omitted for brevity of presentation. Therefore, similarly to Theorem 3.4, there exists a small constant $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, the following holds:

$$\sup_{t\in[0,\tau]} \Big\|\frac{F^\varepsilon(t) - \bar M^\delta(t)}{\sqrt\mu}\Big\|_{L_x^2 L_v^2} + \sup_{t\in[0,\tau]} \Big\|\frac{F^\varepsilon(t) - \bar M^\delta(t)}{\sqrt\mu}\Big\|_{L_x^\infty L_v^2} \le C_\tau \varepsilon. \tag{6.4}$$

Following the same method used in [35, Lemma 3.3], it holds that

$$\sup_{t\in[0,\tau]} \Big\|\frac{\bar M^\delta(t) - \mu - \delta\sqrt\mu \mathbf{f}(t)}{\sqrt\mu}\Big\|_{L_x^2 L_v^2} + \sup_{t\in[0,\tau]} \Big\|\frac{\bar M^\delta(t) - \mu - \delta\sqrt\mu \mathbf{f}(t)}{\sqrt\mu}\Big\|_{L_x^\infty L_v^2} \le C_\tau \delta^2, \tag{6.5}$$

where $\mathbf{f}(t)$ is given in (3.27). Hence, we can deduce from (3.25), (6.4), and (6.5) that

$$\begin{aligned}
&\sup_{t\in[0,\tau]} \|\mathbf{f}^\varepsilon(t) - \mathbf{f}(t)\|_{L_x^\infty L_v^2} \\
&= \sup_{t\in[0,\tau]} \Big\|\frac{F^\varepsilon(t) - \mu - \delta\sqrt\mu \mathbf{f}(t)}{\delta\sqrt\mu}\Big\|_{L_x^\infty L_v^2} \\
&\le \sup_{t\in[0,\tau]} \Big\|\frac{F^\varepsilon(t) - \bar M^\delta(t)}{\delta\sqrt\mu}\Big\|_{L_x^\infty L_v^2} + \sup_{t\in[0,\tau]} \Big\|\frac{\bar M^\delta(t) - \mu - \delta\sqrt\mu \mathbf{f}(t)}{\delta\sqrt\mu}\Big\|_{L_x^\infty L_v^2} \\
&\le C_\tau\Big(\frac{\varepsilon}{\delta} + \delta\Big).
\end{aligned}$$

Similar estimates also hold for $\sup_{t\in[0,\tau]}\|\mathbf{f}^{\varepsilon}(t)-\mathbf{f}(t)\|_{L^2_xL^2_v}$, and then the desired estimate (3.29) holds. We consequently finish the proof of Theorem 3.9. ■

A. An estimate of the relative entropy

In this appendix, for completeness, we give the details of the derivation of estimate (5.7).

Proof of (5.7). Note that $\theta=\frac{3}{2}\frac{1}{2\pi e}\rho^{\frac{2}{3}}\exp(S)$ due to (5.2); then we have

$$\theta_\rho:=\partial_\rho\theta=\frac{1}{2\pi e}\rho^{-\frac{1}{3}}\exp(S)=\frac{2}{3}\frac{\theta}{\rho},\quad \theta_S=\frac{3}{2}\frac{1}{2\pi e}\rho^{\frac{2}{3}}\exp(S)=\theta.$$

Using this together with (5.3) and (5.4), direct computations give

$$\eta_{\bar\rho}=-\frac{\rho}{\bar\rho}\bar\theta(S-\bar S)-\frac{5}{3\bar\rho}\bar\theta(\rho-\bar\rho),\quad \eta_{\bar u}=-\frac{3}{2}\rho(u-\bar u),$$
$$\eta_{\bar S}=-\frac{3}{2}\rho\bar\theta(S-\bar S)-\bar\theta(\rho-\bar\rho).$$

Similarly, it also holds that

$$\begin{aligned}
q_{\bar\rho}&=-u\frac{\rho}{\bar\rho}\bar\theta(S-\bar S)-\frac{5u}{3\bar\rho}\bar\theta(\rho-\bar\rho)-\frac{5}{3}\bar\theta(u-\bar u),\\
q_{j\bar u_j}&=-\frac{3}{2}u_j\rho(u_j-\bar u_j)-\rho\theta+\bar\rho\bar\theta,\\
q_{\bar S}&=-\frac{3}{2}u\rho\bar\theta(S-\bar S)+\bar u\bar\theta\bar\rho-u\rho\bar\theta.
\end{aligned}$$

On the other hand, one gets from (1.2) that

$$\begin{cases}
\partial_t\bar\rho=-\bar u\cdot\nabla_x\bar\rho-\bar\rho\nabla_x\cdot\bar u,\\
\partial_t\bar u=-\bar u\cdot\nabla_x\bar u-\frac{2}{3}\frac{\bar\theta}{\bar\rho}\nabla_x\bar\rho-\frac{2}{3}\nabla_x\bar\theta,\\
\partial_t\bar\theta=-\bar u\cdot\nabla_x\bar\theta-\frac{2}{3}\bar\theta\nabla_x\cdot\bar u.
\end{cases}$$

In view of these facts, we have from straightforward computations that

$$\begin{aligned}
&\nabla_{[\bar\rho,\bar u,\bar S]}\eta(t,x)\cdot\partial_t(\bar\rho,\bar u,\bar S)=\eta_{\bar\rho}\partial_t\bar\rho+\eta_{\bar u}\partial_t\bar u+\eta_{\bar S}\partial_t\bar S\\
&=\frac{\bar\theta}{\bar\rho}(\bar\rho-\rho)\partial_t\bar\rho-\frac{3}{2}\rho(u-\bar u)\cdot\partial_t\bar u-\Big\{\frac{3}{2}\rho(S-\bar S)+(\rho-\bar\rho)\Big\}\partial_t\bar\theta\\
&=\frac{5}{3}(\rho-\bar\rho)\bar\theta\nabla_x\cdot\bar u+\frac{\bar\theta}{\bar\rho}\rho u\cdot\nabla_x\bar\rho+\rho u\cdot\nabla_x\bar\theta-\bar u\bar\theta\cdot\nabla_x\bar\rho\\
&\quad+\frac{3}{2}\rho(u-\bar u)\cdot(\bar u\cdot\nabla_x\bar u)-\bar\rho\bar u\cdot\nabla_x\bar\theta-\frac{3}{2}\rho(\bar S-S)\Big(\bar u\cdot\nabla_x\bar\theta+\frac{2}{3}\bar\theta\nabla_x\cdot\bar u\Big).
\end{aligned}\tag{A.1}$$

Likewise, it also holds that

$$\begin{aligned}\sum_{j=1}^{3}\nabla_{[\bar\rho,\bar u,\bar S]}q_j(t,x)\cdot\partial_{x_j}(\bar\rho,\bar u,\bar S) &= q_{\bar\rho}\cdot\nabla_x\bar\rho+\sum_{j=1}^{3}q_{j\bar u_j}\partial_{x_j}\bar u_j+q_{\bar S}\cdot\nabla_x\bar S\\ &=\bar\rho\bar\theta\nabla_x\cdot\bar u-\rho\theta\nabla_x\cdot\bar u-\frac{\bar\theta}{\bar\rho}\rho u\cdot\nabla_x\bar\rho-\rho u\cdot\nabla_x\bar\theta+\bar\theta\bar u\cdot\nabla_x\bar\rho\\ &\quad-\frac{3}{2}\rho(u-\bar u)\cdot(u\cdot\nabla_x\bar u)+\bar\rho\bar u\cdot\nabla_x\bar\theta+\frac{3}{2}\rho u(\bar S-S)\nabla_x\bar\theta.\end{aligned}\tag{A.2}$$

Combining (A.1) and (A.2) yields

$$\begin{aligned}&\nabla_{[\bar\rho,\bar u,\bar S]}\eta(t,x)\cdot\partial_t(\bar\rho,\bar u,\bar S)+\sum_{j=1}^{3}\nabla_{[\bar\rho,\bar u,\bar S]}q_j(t,x)\cdot\partial_{x_j}(\bar\rho,\bar u,\bar S)\\ &=\frac{5}{3}\rho\bar\theta\nabla_x\cdot\bar u-\frac{2}{3}\bar\rho\bar\theta\nabla_x\cdot\bar u-\frac{3}{2}\rho(u-\bar u)\tilde u\cdot\nabla_x\bar u\\ &\quad-\frac{3}{2}\rho(\bar S-S)\Big(\bar u\cdot\nabla_x\bar\theta+\frac{2}{3}\bar\theta\nabla_x\cdot\bar u\Big)+\frac{3}{2}\rho u(\bar S-S)\nabla_x\bar\theta-\rho\theta\nabla_x\cdot\bar u\\ &=-\frac{3}{2}\rho\tilde u\cdot(\tilde u\cdot\nabla_x\bar u)-\frac{2}{3}\rho\bar\theta\nabla_x\cdot\bar u\Psi\Big(\frac{\bar\rho}{\rho}\Big)-\rho\bar\theta\nabla_x\cdot\bar u\Psi\Big(\frac{\theta}{\bar\theta}\Big)\\ &\quad-\frac{3}{2}\rho\nabla_x\bar\theta\cdot\tilde u\Big(\frac{2}{3}\ln\frac{\bar\rho}{\rho}+\ln\frac{\theta}{\bar\theta}\Big),\end{aligned}\tag{A.3}$$

where we have used $\Psi(s)=s-\ln s-1$ and $\bar S-S=-\frac{2}{3}\ln\frac{\bar\rho}{\rho}-\ln\frac{\theta}{\bar\theta}$. Hence the desired estimate (5.7) follows from (A.3) and then the proof is completed. ■

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