Ann. Inst. H. Poincaré C
Anal. Non Linéaire 42 (2025), 1455–1508
DOI 10.4171/AIHPC/143

Stability of the composite wave of two planar viscous shock waves for the compressible Navier–Stokes system

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Abstract. We are concerned with the nonlinear stability of the composite wave consisting of two planar viscous shock waves to the three-dimensional compressible Navier–Stokes system. It is shown that if the shock strengths are suitably small but not necessarily of the same order of magnitude, and the initial perturbations are suitably small without the zero-mass conditions, then there exists a unique, globally strong solution in time to the compressible Navier–Stokes system, which asymptotically approaches the corresponding composite wave up to time-dependent shifts in L^∞ norm. The proof employs the weighted relative entropy method, an L^2-contraction technique with time-dependent shifts to the shocks developed by Kang and Vasseur [J. Eur. Math. Soc. (JEMS) 23 (2021), 585–638; Invent. Math. 224 (2021), 55–146]. We perform the stability analysis within the original H^2-perturbation framework instead of using the anti-derivative technique. Compared with the previous work of the author and Wang [J. Eur. Math. Soc. (JEMS) (2023), DOI 10.4171/JEMS/1486] for the single shock case, a major difficulty is the construction of shifts to ensure that the two shock waves are well separated.

1. Introduction and main result

1.1. Introduction

We are concerned with the following isentropic compressible Navier–Stokes system in Eulerian coordinates:

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho u) = 0, \quad (t,x) \in \mathbb{R}^+ \times \Omega, \\ \partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u) + \nabla_x p(\rho) = \mu \Delta_x u + (\mu + \lambda)\nabla_x \operatorname{div}_x u. \end{cases} \tag{1.1}$$

Here, $\rho = \rho(t,x)\colon \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$, $u = u(t,x) = (u_1,u_2,u_3)^\intercal(t,x)\colon \mathbb{R}^+ \times \Omega \to \mathbb{R}^3$ represent the mass density and the velocity of a fluid in $\Omega \subset \mathbb{R}^3$ respectively, and $p(\rho) = b\rho^\gamma$ $(b > 0,\ \gamma > 1)$ stands for the classical γ-law pressure. The real numbers μ and λ designate the shear and bulk viscosity coefficients, respectively, and satisfy the physical constraints $\mu > 0$ and $2\mu + 3\lambda \geq 0$. Without loss of generality, we assume $b = 1$ from now on.

Mathematics Subject Classification 2020: 76E19 (primary); 76N06, 76L05, 76E30, 76N30 (secondary).
Keywords: stability, composite wave patterns, planar viscous shock, compressible Navier–Stokes system, wave interaction.

In this paper, we consider the spatial coordinates $x = (x_1, x')^{\top} \in \Omega := \mathbb{R} \times \mathbb{T}^2$, where $x_1 \in \mathbb{R}$ is the real line and $x' = (x_2, x_3) \in \mathbb{T}^2$ is a unit torus. The system is supplemented with the initial data:

$$(\rho, u)|_{t=0} = (\rho_0, u_0) \to (\rho_\pm, u_\pm), \quad \text{as} \quad x_1 \to \pm\infty, \tag{1.2}$$

where $\rho_\pm > 0$ and $u_\pm = (u_{1\pm}, 0, 0)^{\top}$ are given constants describing the far-field conditions on the x_1-direction, and the periodic boundary conditions are imposed on $(x_2, x_3) \in \mathbb{T}^2$ for the solution (ρ, u).

The large-time asymptotic behavior of the solution to the three-dimensional compressible Navier–Stokes system (1.1)–(1.2) is expected to be closely related to the planar Riemann problem of the corresponding three-dimensional Euler system

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u) + \nabla_x p(\rho) = 0, \\ (\rho, u)(0, x) = \begin{cases} (\rho_-, u_-), & x_1 < 0, \\ (\rho_+, u_+), & x_1 > 0. \end{cases} \end{cases} \tag{1.3}$$

The solution to the Riemann problem (1.3) in general contains two nonlinear waves, i.e., shock and rarefaction waves.

The stability of the viscous shock waves for the Navier–Stokes system is a very important issue from both mathematical and physical standpoints. The conjecture of the time-asymptotic stability of the Riemann solution has been well established in one-dimensional case. In 1960, Il'in and Oleĭnik [13] first proved the stability of shock and rarefaction waves to the one-dimensional scalar Burgers equation. Then Matsumura and Nishihara [25] proved the stability of the viscous shock wave to the one-dimensional compressible Navier–Stokes system with physical viscosity, provided that $|v_+ - v_-| \le C(\gamma - 1)^{-1}$ under the zero-mass condition. Independently, Goodman [2] proved the same result for a general system with "artificial" diffusions. On the one hand, to remove the zero-mass condition in [2, 25], Liu [22], Szepessy and Xin [28], Liu and Zeng [23] introduced the constant shift on the viscous shock and the diffusion waves and the coupled diffusion waves in the transverse characteristic fields. On the other hand, to relax the restriction of the shock wave strength, Kawashima and Matsumura [20] improved the condition in [25] to $|v_+ - v_-| \le C(\gamma - 1)^{-2}$. When the viscosity depends on the density, i.e., $\mu = \mu(v) = v^\alpha$ ($\alpha > 0$, $v = 1/\rho$), Matsumura and Wang [26] showed the shock wave is asymptotically stable through a weighted energy method provided $\alpha > (\gamma - 1)/2$ under small initial perturbations with integral zero, but with no restriction on the shock strength. Then Vasseur and Yao [29] removed the condition $\alpha > (\gamma - 1)/2$ by introducing a new effective velocity. He and Huang [4] extended the result of [29] to general pressure $p(v)$ and viscosity $\mu(v)$, where $\mu(v)$ could be any positive smooth function.

Another interesting and important direction is to investigate the spectral stability of viscous shock profiles. The works [7,24,35] revealed that the nonlinear stability of viscous shocks can be implied by the spectral stability, where the latter is somehow equivalent to the linearized stability with respect to zero-mass perturbations; see [34]. In particular, for the compressible Navier–Stokes equations, [10] and [1, 11] verified the spectral stability of strong viscous shocks through the large-amplitude limit and numeric proof, respectively, which, together with the previous results, can yield the full nonlinear stability of strong viscous shocks.

For the stability of the composite wave patterns, to the best of our knowledge, all the previous results for the Navier–Stokes require a smallness condition on initial perturbations and wave strengths. We refer to [9] for two viscous shocks for one-dimensional full compressible Navier–Stokes equations, and to [8] for the superposition of viscous contact wave and two rarefaction waves. Note that all the above stability results concerned with shocks are based on the classical anti-derivative technique. Recently, Kang, Vasseur, and Wang [18], using a contraction method, demonstrated that the perturbed solution of Navier–Stokes converges to the composite wave consisting of a viscous shock with time-dependent shift and rarefaction wave. Furthermore, they showed [19] the solution tends to the generic Riemann solution composed of a viscous shock with time-dependent shift, a viscous contact wave, and a rarefaction wave.

The viscous shock theory for general multi-dimensional viscous conservation laws is unsatisfactory compared with that of the one-dimensional case. Nevertheless, Goodman [3] first proved the time-asymptotic stability of a weak planar viscous shock for the scalar viscous equation by anti-derivative techniques with the shift function depending on both time and spatially transverse directions, and then Hoff and Zumbrun [5, 6] extended Goodman's result to the large amplitude shock case. Kang, Vasseur, and Wang [17] proved L^2-contraction of large viscous shock up to a shift function depending on both time and spatial variables.

Results are very few on the nonlinearly time-asymptotic stability of planar viscous shocks to the multi-dimensional Navier–Stokes system (1.1) due to the substantial difficulties in the high-dimensional propagation of shocks and the nonlinearities of the system. Humpherys, Lyng, and Zumbrun [12] proved the spectral stability of planar viscous Navier–Stokes shocks under the spectral assumptions with the numerical Evans-function methods, and for the related results one can refer to the survey paper by Zumbrun [34] and the references therein. Recently, the author and Wang [31] proved the nonlinear stability of a planar viscous shock wave up to a time-dependent shift for the three-dimensional compressible Navier–Stokes system (1.1) by using the weighted relative entropy method under generic H^2-perturbations without the zero-mass conditions. Under the assumption that the initial perturbation is periodic at spatial infinity, Yuan [32, 33] established the nonlinear time-asymptotic stability of planar shock profiles for the system (1.1) in three dimensions.

The purpose of this paper is to extend the result of [31] to the composite wave consisting of two planar viscous shock waves, i.e., we aim to show the nonlinear stability of two planar Navier–Stokes shocks up to time-dependent shifts for system (1.1) in three dimensions under the generic H^2-perturbations without the zero-mass conditions.

We now introduce some preliminary notation and give some background materials before stating the main theorem. Consider the one-dimensional compressible Euler system

$$\begin{cases} \partial_t \rho + \partial_{x_1}(\rho u_1) = 0, \\ \partial_t(\rho u_1) + \partial_{x_1}(\rho u_1^2 + p(\rho)) = 0, \end{cases} \tag{1.4}$$

with Riemann initial data

$$(\rho, u_1)(0, x_1) = (\rho_0, u_{10})(x_1) = \begin{cases} (\rho_-, u_{1-}), & x_1 < 0, \\ (\rho_+, u_{1+}), & x_1 > 0. \end{cases} \tag{1.5}$$

We call (1.4) and (1.5) the Riemann problem, which admits rich wave phenomena such as shock waves, rarefaction waves, and their linear combinations. It is known that system (1.4) has two eigenvalues: $\lambda_1 = u - \sqrt{p'(\rho)}$ and $\lambda_2 = u + \sqrt{p'(\rho)}$. In the present paper, we focus our attention on the situation where the Riemann solution of (1.4) and (1.5) consists of two shock waves (and two constant states). That is, there exists an intermediate state (ρ_m, u_{1m}) such that (ρ_-, u_{1-}) connects with (ρ_m, u_{1m}) by the 1-shock wave with the shock speed s_1,

$$(\rho, u_1)(t, x_1) = \begin{cases} (\rho_-, u_{1-}), & x_1 < s_1 t, \\ (\rho_m, u_{1m}), & x_1 > s_1 t. \end{cases}$$

The shock speed s_1 is determined by the Rankine–Hugoniot condition

$$\begin{cases} -s_1(\rho_m - \rho_-) + (\rho_m u_{1m} - \rho_- u_{1-}) = 0, \\ -s_1(\rho_m u_{1m} - \rho_- u_{1-}) + (\rho_m u_{1m}^2 - \rho_- u_{1-}^2) + (p(\rho_m) - p(\rho_-)) = 0, \end{cases}$$

and satisfies the Lax entropy condition $\lambda_1(\rho_m, u_{1m}) < s_1 < \lambda_1(\rho_-, u_{1-})$, while (ρ_m, u_{1m}) connects with (ρ_+, u_{1+}) by the 2-shock wave with the shock speed s_2,

$$(\rho, u_1)(t, x_1) = \begin{cases} (\rho_m, u_{1m}), & x_1 < s_2 t, \\ (\rho_+, u_{1+}), & x_1 > s_2 t. \end{cases}$$

The shock speed s_2 is determined by the Rankine–Hugoniot condition

$$\begin{cases} -s_2(\rho_+ - \rho_m) + (\rho_+ u_{1+} - \rho_m u_{1m}) = 0, \\ -s_2(\rho_+ u_{1+} - \rho_m u_{1m}) + (\rho_+ u_{1+}^2 - \rho_m u_{1m}^2) + (p(\rho_+) - p(\rho_m)) = 0, \end{cases}$$

and satisfies the Lax entropy condition $\lambda_2(\rho_+, u_{1+}) < s_2 < \lambda_2(\rho_m, u_{1m})$.

Next we recall the definitions of viscous shock waves of (1.1) which correspond to the above shock waves. We see that the 1-viscous shock wave with the formula $(\rho^{s_1}, u_1^{s_1})(x_1 - s_1 t)$ corresponding to the 1-shock wave, is a traveling wave solution of (1.1) determined by

$$\begin{cases} -s_1(\rho^{s_1})' + (\rho^{s_1} u_1^{s_1})' = 0, \\ -s_1(\rho^{s_1} u_1^{s_1})' + (\rho^{s_1}(u_1^{s_1})^2)' + p(\rho^{s_1})' = (2\mu + \lambda)(u_1^{s_1})'', \\ (\rho^{s_1}, u_1^{s_1})(-\infty) = (\rho_-, u_{1-}), \quad (\rho^{s_1}, u_1^{s_1})(+\infty) = (\rho_m, u_{1m}), \end{cases} \tag{1.6}$$

where $' := \frac{\mathrm{d}}{\mathrm{d}\xi_1}$, $\xi_1 = x_1 - s_1 t$. Similarly, the 2-viscous shock wave $(\rho^{s_2}, u_1^{s_2})(x_1 - s_2 t)$ is defined by

$$\begin{cases} -s_2(\rho^{s_2})' + (\rho^{s_2}u_1^{s_2})' = 0, \\ -s_2(\rho^{s_2}u_1^{s_2})' + (\rho^{s_2}(u_1^{s_2})^2)' + p(\rho^{s_2})' = (2\mu+\lambda)(u_1^{s_2})'', \\ (\rho^{s_2}, u_1^{s_2})(-\infty) = (\rho_m, u_{1m}), \quad (\rho^{s_2}, u_1^{s_2})(+\infty) = (\rho_+, u_{1+}), \end{cases} \tag{1.7}$$

where $' := \frac{\mathrm{d}}{\mathrm{d}\xi_2}$, $\xi_2 = x_1 - s_2 t$.

To describe the strengths of the shock waves for later use, we set

$$\delta_1 = p(\rho_m) - p(\rho_-), \quad \delta_2 = p(\rho_m) - p(\rho_+), \quad \delta := \delta_1 + \delta_2. \tag{1.8}$$

1.2. Main result

We state our main result on the nonlinear stability of two composite planar viscous shocks to the three-dimensional compressible Navier–Stokes system (1.1) under the generic H^2-perturbations without zero-mass conditions.

Theorem 1.1. *For each (ρ_-, u_{1-}), suppose (ρ_+, u_{1+}) belongs to a neighborhood of (ρ_-, u_{1-}), and let $(\rho^{s_i}, u^{s_i})(x_1 - s_i t)$ $(i = 1, 2)$ be the planar i-viscous shock waves defined in (1.6), (1.7) with $u^{s_i} := (u_1^{s_i}, 0, 0)^\mathsf{T}$ and $u_m = (u_{1m}, 0, 0)^\mathsf{T}$. Then there exist positive constants δ_0, ε_0 such that if the shock wave strength $\delta \le \delta_0$, and the initial data (ρ_0, u_0) satisfies*

$$\left\| \left(\rho_0(x) - \left(\sum_{i=1}^2 \rho^{s_i}(x_1) - \rho_m \right), u_0(x) - \left(\sum_{i=1}^2 u^{s_i}(x_1) - u_m \right) \right) \right\|_{H^2(\mathbb{R}\times\mathbb{T}^2)} \le \varepsilon_0,$$

then the Cauchy problem (1.1), (1.2) of the three-dimensional compressible Navier–Stokes system admits a unique global-in-time solution $(\rho, u)(t, x)$, and there exists an absolutely continuous shift $\mathbf{X}_i(t)$ $(i = 1, 2)$ such that

$$\rho(t,x) - \left(\sum_{i=1}^2 \rho^{s_i}(x_1 - s_i t - \mathbf{X}_i(t)) - \rho_m \right) \in C([0,+\infty); H^2(\mathbb{R}\times\mathbb{T}^2)),$$

$$u(t,x) - \left(\sum_{i=1}^2 u^{s_i}(x_1 - s_i t - \mathbf{X}_i(t)) - u_m \right) \in C([0,+\infty); H^2(\mathbb{R}\times\mathbb{T}^2)),$$

$$\nabla_x \left(\rho(t,x) - \left(\sum_{i=1}^2 \rho^{s_i}(x_1 - s_i t - \mathbf{X}_i(t)) - \rho_m \right) \right) \in L^2(0,+\infty; H^1(\mathbb{R}\times\mathbb{T}^2)),$$

$$\nabla_x \left(u(t,x) - \left(\sum_{i=1}^2 u^{s_i}(x_1 - s_i t - \mathbf{X}_i(t)) - u_m \right) \right) \in L^2(0,+\infty; H^2(\mathbb{R}\times\mathbb{T}^2)).$$

Furthermore, the large-time behavior

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}\times\mathbb{T}^2} \left| (\rho, u)(t,x) - \left(\sum_{i=1}^2 (\rho^{s_i}, u^{s_i})(x_1 - s_i t - \mathbf{X}_i(t)) - (\rho_m, u_m) \right) \right| = 0 \tag{1.9}$$

holds, and the shift function $\mathbf{X}_i(t)$ *($i = 1, 2$) satisfies the time-asymptotic behavior*

$$\lim_{t\to\infty} |\dot{\mathbf{X}}_i(t)| = 0. \tag{1.10}$$

Remark 1.2. Theorem 1.1 states that if the two far-fields states $(\rho_\pm, u_\pm)$ in (1.2) are connected by the two shock waves, then the solution to the three-dimensional compressible Navier–Stokes system (1.1) tends to the corresponding two planar viscous shocks with the time-dependent shift $\mathbf{X}_i(t)$ ($i = 1, 2$) under the generic H^2-perturbation without zero-mass conditions. In the present paper, we remove the condition of wave strengths that "small with the same order" needed in [9].

Remark 1.3. The shift function $\mathbf{X}_i(t)$ ($i = 1, 2$) is proved to satisfy the time-asymptotic behavior (1.10), which implies

$$\lim_{t\to+\infty} \frac{\mathbf{X}_i(t)}{t} = 0.$$

That is, the shift function $\mathbf{X}_i(t)$ grows at most sub-linearly with respect to the time t, and therefore, the shifted $(\rho^{s_i}, u^{s_i})(x_1 - s_i t - \mathbf{X}_i(t))$ keeps the original traveling wave profile time-asymptotically.

Our proof is motivated by Kang and Vasseur [14–16] for L^2-contraction of viscous shock in the one-dimensional case and a recent work [31] on the stability of the single viscous shock in the three-dimensional case. We introduce the time-dependent shifts $\mathbf{X}_i(t)$ ($i = 1, 2$) to the two viscous shocks so that a three-dimensional weighted Poincaré inequality (2.5) can be applied to overcome the difficulty arising from the "bad" sign of the derivative of viscous shock velocity, and the anti-derivative technique is not needed. However, compared with the single Navier–Stokes shock in [31], there are several difficulties and differences:

- A major difficulty stems from the two Navier–Stokes shocks interacting at far distance, not being independent anymore. Indeed, the separation of waves is far more complicated at the level of the Navier–Stokes equations; in particular, only a rough a priori control on the artificial shifts can be imposed through this method. Therefore, we introduce cut-off functions (4.30) and (4.31), to show that shifts associated with the two different families of shocks cannot produce artificial collisions (see Lemma 4.4), thereby, we can ensure that a 1-shock will not be pushed through the artificial shift, so much that it would not collide with a 2-shock from the left.
- Another difference lies in its not being necessary to introduce an effect velocity to transform the regularization of the system from the velocity u to the specific volume v $(= 1/\rho)$ as in [31] (see also [14–16] for the one-dimensional case). In fact, this technique is really effective in dealing with the stability of general large perturbations of the viscous shock wave, since the hyperbolic part of the system is linear in the new effective velocity (also velocity) but nonlinear in the specific volume (via the pressure). However, as far as the small perturbations are concerned, the way we approach it is

by decomposing the "bad" term $\mathbf{B}_{2,i}(t)$ $(i = 1, 2)$ in terms of the velocity u, and calculating $\mathbf{B}_{2,i}(t) + \mathbf{B}_{3,i}(t)$ $(i = 1, 2)$ with the help of cut-off functions (4.30) and (4.31) (see Lemma 4.6).

- Finally, we derive a uniform estimate in the new variable on the variable transformation together with the three-dimensional weighted Poincaré inequality (2.5) (see Lemmas 4.7, 4.8).

The rest of the paper is organized as follows. In Section 2 we introduce the planar viscous shock and some useful functional inequalities. In Section 3 we first construct the shift functions $\mathbf{X}_i(t)$ $(i = 1, 2)$, and prove the stability result. In Sections 4 and 5 we present the basic energy estimate and higher-order derivative estimates, respectively.

Notation. Throughout this paper, several positive generic constants are denoted by C if there is no risk of confusion. We define

$$x' := (x_2, x_3)^{\mathsf{T}}, \quad \mathrm{d}x' := \mathrm{d}x_2\, \mathrm{d}x_3.$$

For $1 \le r \le \infty$, we denote $L^r := L^r(\Omega) = L^r(\mathbb{R} \times \mathbb{T}^2)$, and use the notation $\|\cdot\| := \|\cdot\|_{L^2}$. For a nonnegative integer m, the space $H^m(\Omega)$ denotes the mth-order Sobolev space over Ω in the L^2 sense with the norm

$$\|f\|_{H^m} := \Big(\sum_{l=0}^{m} \|\nabla^l f\|^2\Big)^{1/2}, \quad \|f\| := \Big(\int_{\mathbb{T}^2}\int_{\mathbb{R}} |f|^2\, \mathrm{d}x_1\, \mathrm{d}x'\Big)^{1/2}.$$

Also, we denote

$$\|(f, g)\|_{H^m} = \|f\|_{H^m} + \|g\|_{H^m}.$$

2. Preliminaries

In this section we present the planar viscous shocks and sone useful inequalities for later use.

2.1. Viscous shock wave

First we show the existence and properties of the planar viscous shocks. It is convenient to introduce the volume function $v^{s_i} := 1/\rho^{s_i}$ $(i = 1, 2)$ to describe the existence result. Then the ODE systems (1.6) and (1.7) are transformed to

$$\begin{cases} \rho^{s_i}(-s_i(v^{s_i})' + u_1^{s_i}(v^{s_i})') = (u_1^{s_i})', \quad i = 1, 2, \\ \rho^{s_i}(-s_i(u_1^{s_i})' + u_1^{s_i}(u_1^{s_i})') + p(v^{s_i})' = (2\mu + \lambda)(u_1^{s_i})'', \end{cases}$$

where $p(v^{s_i}) = (v^{s_i})^{-\gamma}$ and $' := \frac{\mathrm{d}}{\mathrm{d}\xi_i}$.

For $i = 1$, integrating $(1.6)_1$ from $(\xi_1, +\infty)$, it holds that

$$-s_1 \rho^{s_1} + \rho^{s_1} u_1^{s_1} = -s_1 \rho_m + \rho_m u_{1m} =: -\sigma_1. \tag{2.1}$$

For $i = 2$, integrating $(1.7)_1$ from $(-\infty, \xi_2)$, it holds that

$$-s_2 \rho^{s_2} + \rho^{s_2} u_1^{s_2} = -s_2 \rho_m + \rho_m u_{1m} =: -\sigma_2. \tag{2.2}$$

Therefore, the systems (1.6) and (1.7) can be rewritten as

$$\begin{cases} -\sigma_1 (v^{s_1})' = (u_1^{s_1})', \\ -\sigma_1 (u_1^{s_1})' + p(v^{s_1})' = (2\mu + \lambda)(u_1^{s_1})'', \\ (v^{s_1}, u_1^{s_1})(-\infty) = (v_-, u_{1-}), \quad (v^{s_1}, u_1^{s_1})(+\infty) = (v_m, u_{1m}), \end{cases} \tag{2.3}$$

and

$$\begin{cases} -\sigma_2 (v^{s_2})' = (u_1^{s_2})', \\ -\sigma_2 (u_1^{s_2})' + p(v^{s_2})' = (2\mu + \lambda)(u_1^{s_2})'', \\ (v^{s_2}, u_1^{s_2})(-\infty) = (v_m, u_{1m}), \quad (v^{s_2}, u_1^{s_2})(+\infty) = (v_+, u_{1+}), \end{cases} \tag{2.4}$$

where $v_m = 1/\rho_m$ and $v_\pm = 1/\rho_\pm$. Integrating (2.3) from $(-\infty, +\infty)$, we have

$$\begin{cases} -\sigma_1 (v_m - v_-) = u_{1m} - u_{1-}, \\ -\sigma_1 (u_{1m} - u_{1-}) + p(v_m) - p(v_-) = 0. \end{cases}$$

Integrating (2.4) from $(-\infty, +\infty)$, we have

$$\begin{cases} -\sigma_2 (v_+ - v_m) = u_{1+} - u_{1m}, \\ -\sigma_2 (u_{1+} - u_{1m}) + p(v_+) - p(v_m) = 0. \end{cases}$$

Therefore, we have

$$\sigma_1 = -\sqrt{-\frac{p(v_m) - p(v_-)}{v_m - v_-}} < 0$$

for 1-viscous shock wave and

$$\sigma_2 = \sqrt{-\frac{p(v_m) - p(v_+)}{v_m - v_+}} > 0$$

for 2-viscous shock wave, respectively.

The existence and property of the i-viscous shock wave $(v^{s_i}, u_1^{s_i})(x_1 - s_i t)$ $(i = 1, 2)$ can be summarized in the following lemma, while its proof can be found in [25].

Lemma 2.1. *Fix the left state (v_-, u_{1-}), suppose (v_+, u_{1+}) belongs to the neighborhood of (v_-, u_{1-}), and the Riemann solution of (1.4) and (1.5) consists of two shock waves. Then there exists a unique (up to a constant shift) solution $(v^{s_1}, u_1^{s_1})(x_1 - s_1 t)$ to ODE* *system* (2.3) *and a unique (up to a constant shift) solution* $(v^{s_2}, u_1^{s_2})(x_1 - s_2 t)$ *to ODE system* (2.4)*, respectively. Moreover, there exist constants c and C which depend only on* (v_-, u_{1-}) *such that, for* $i = 1, 2$*, it holds that*

$$\partial_{x_1} u_1^{s_i} = -\sigma_i \partial_{x_1} v^{s_i} < 0, \quad \partial_{x_1} v^{s_1} < 0, \quad \partial_{x_1} v^{s_2} > 0, \quad x_1 \in \mathbb{R},\ t \geq 0,$$

and

$$\begin{aligned} |v^{s_1}(x_1 - s_1 t) - v_m| &\leq C\delta_1 e^{-C\delta_1 |x_1 - s_1 t|}, && x_1 - s_1 t > 0,\ t \geq 0, \\ |v^{s_2}(x_1 - s_2 t) - v_m| &\leq C\delta_2 e^{-C\delta_2 |x_1 - s_2 t|}, && x_1 - s_2 t < 0,\ t \geq 0, \\ |(\partial_{x_1} v^{s_i}, \partial_{x_1} u_1^{s_i})| &\leq C\delta_i^2 e^{-C\delta_i |x_1 - s_i t|}, && x_1 \in \mathbb{R}, \\ |(\partial_{x_1}^k v^{s_i}, \partial_{x_1}^k u_1^{s_i})| &\leq C\delta_i^{k-1} |(\partial_{x_1} v^{s_i}, \partial_{x_1} u_1^{s_i})|, && x_1 \in \mathbb{R},\ k \geq 2. \end{aligned}$$

Remark 2.2. Indeed, Lemma 2.1 can be proved through ODE (1.6) for $(\rho^{s_1}, u_1^{s_1})(x_1 - s_1 t)$, and ODE (1.7) for $(\rho^{s_2}, u_1^{s_2})(x_1 - s_2 t)$, then can be transferred into $(v^{s_1}, u_1^{s_1})(x_1 - s_1 t)$ and $(v^{s_2}, u_1^{s_2})(x_1 - s_2 t)$, not directly from ODE systems (2.3), (2.4).

2.2. Some functional inequalities

First, we give an estimate involving the inverse of the pressure function $p(v) = v^{-\gamma}$, while its proof can be found in [14].

Lemma 2.3. *Fix* $v_- > 0$*. Then there exist* $\delta_0 > 0$ *and* $C > 0$ *such that for any* $v_+ > 0$*, such that* $0 < \delta_* = p(v_-) - p(v_+) \leq \delta_0$*,* $v_- \leq v \leq v_+$*, we have*

$$\left| \frac{v - v_-}{p(v) - p(v_-)} + \frac{v - v_+}{p(v_+) - p(v)} + \frac{1}{2} \frac{p''(v_-)}{p'(v_-)^2} (v_- - v_+) \right| \leq C\delta_*^2.$$

Next we present a three-dimensional weighted sharp Poincaré-type inequality, which plays a very important role in our stability analysis. The proof can be found in [31].

Lemma 2.4. *For any* $f : [0,1] \times \mathbb{T}^2 \to \mathbb{R}$*, it holds that*

$$\begin{aligned} \int_{\mathbb{T}^2} \int_0^1 |f - \bar{f}|^2 \, \mathrm{d}y_1 \, \mathrm{d}y' &\leq \frac{1}{2} \int_{\mathbb{T}^2} \int_0^1 y_1(1 - y_1) |\partial_{y_1} f|^2 \, \mathrm{d}y_1 \, \mathrm{d}y' \\ &\quad + \frac{1}{16\pi^2} \int_{\mathbb{T}^2} \int_0^1 \frac{|\nabla_{y'} f|^2}{y_1(1 - y_1)} \, \mathrm{d}y_1 \, \mathrm{d}y', \end{aligned} \tag{2.5}$$

where $\bar{f} = \int_{\mathbb{T}^2} \int_0^1 f \, \mathrm{d}y_1 \, \mathrm{d}y'$*,* $y' = (y_2, y_3)$*, and* $\mathrm{d}y' = \mathrm{d}y_2 \, \mathrm{d}y_3$*.*

Finally, we list a three-dimensional Gagliardo–Nirenberg inequality in the domain $\Omega := \mathbb{R} \times \mathbb{T}^2$, whose proof can be found in [21,30].

Lemma 2.5. *It holds for* $g(x) \in H^2(\Omega)$ *with* $x = (x_1, x_2, x_3)^\top \in \Omega := \mathbb{R} \times \mathbb{T}^2$ *that*

$$\|g\|_{L^\infty(\Omega)} \leq \sqrt{2} \|g\|_{L^2(\Omega)}^{\frac{1}{2}} \|\partial_{x_1} g\|_{L^2(\Omega)}^{\frac{1}{2}} + C \|\nabla_x g\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla_x^2 g\|_{L^2(\Omega)}^{\frac{1}{2}}, \tag{2.6}$$

where $C > 0$ *is a positive constant.*

3. Proof of main result

3.1. Construction of shift functions

First we set the ansatz as

$$\begin{aligned}\bar\rho(t,x_1) &= \rho^{s_1}(x_1-s_1t)+\rho^{s_2}(x_1-s_2t)-\rho_m,\\ \bar u(t,x_1) &= u^{s_1}(x_1-s_1t)+u^{s_2}(x_1-s_2t)-u_m,\end{aligned}\tag{3.1}$$

where $(\rho^{s_i},u^{s_i})(x_1-s_it)$ $(i=1,2)$ denotes the planar i-viscous shock wave to the three-dimensional compressible Navier–Stokes system (1.1), which can be defined by $\rho^{s_i}(x_1-s_it)>0$, $u^{s_i}(x_1-s_it)=(u_1^{s_i}(x_1-s_it),0,0)^{\mathsf T}$, and $u_m=(u_{1m},0,0)^{\mathsf T}$. For notational simplification, we denote

$$\begin{aligned}&((\rho^{s_i})^{-\mathbf X_i},(u^{s_i})^{-\mathbf X_i})(t,x_1):=(\rho^{s_i},u^{s_i})(x_1-s_it-\mathbf X_i(t)),\quad i=1,2,\\ &\bar\rho^{-\mathbf X}(t,x_1):=\rho^{s_1}(x_1-s_1t-\mathbf X_1(t))+\rho^{s_2}(x_1-s_2t-\mathbf X_2(t))-\rho_m,\\ &\bar u^{-\mathbf X}(t,x_1):=u^{s_1}(x_1-s_1t-\mathbf X_1(t))+u^{s_2}(x_1-s_2t-\mathbf X_2(t))-u_m\end{aligned}\tag{3.2}$$

with the shift function $\mathbf X_i(t)$ $(i=1,2)$ to be defined in (4.17) later, and also set

$$(\phi,\psi)(t,x)=(\rho-\bar\rho^{-\mathbf X},u-\bar u^{-\mathbf X})(t,x).$$

In order to prove Theorem 1.1, we will combine a local existence result together with a priori estimates by continuation arguments.

Proposition 3.1 (Local existence). *Let $(\bar\rho,\bar u)(t,x_1)$ be the composite planar viscous shock waves defined in (3.1). For any $\Xi>0$, suppose the initial data (ρ_0,u_0) and the wave strength δ satisfy*

$$\|(\rho_0(x)-\bar\rho(x_1),u_0(x)-\bar u(x_1))\|_{H^2(\mathbb R\times\mathbb T^2)}+\delta\le\Xi.$$

Then there exists a positive constant T_0 depending on Ξ such that the Cauchy problem (1.1), (1.2) of the three-dimensional compressible Navier–Stokes system has a unique solution (ρ,u) on $(0,T_0)$ satisfying

$$\begin{aligned}&\rho-\bar\rho\in C([0,T_0];H^2(\mathbb R\times\mathbb T^2)),\quad \nabla_x(\rho-\bar\rho)\in L^2(0,T_0;H^1(\mathbb R\times\mathbb T^2)),\\ &u-\bar u\in C([0,T_0];H^2(\mathbb R\times\mathbb T^2)),\quad \nabla_x(u-\bar u)\in L^2(0,T_0;H^2(\mathbb R\times\mathbb T^2)),\end{aligned}$$

and for $t\in[0,T_0]$, it holds that

$$\begin{aligned}&\|(\rho-\bar\rho,u-\bar u)(t)\|_{H^2}^2+\int_0^t(\|\rho-\bar\rho\|_{H^2}^2+\|u-\bar u\|_{H^3}^2)\,\mathrm d\tau\\ &\quad\le 4\big(\|(\rho_0-\bar\rho(0,\cdot),u_0-\bar u(0,\cdot))\|_{H^2}^2+\delta^{\frac32}\big).\end{aligned}$$

Proposition 3.1 can be proved by a standard way (see [27]), so we omit it.

Proposition 3.2 (A priori estimates). *Suppose that* (ρ, u) *is the solution to the Cauchy problem* (1.1), (1.2) *on* $[0, T]$ *for some* $T > 0$, *and* $((\rho^{s_i})^{-\mathbf{X}_i}, (u_1^{s_i})^{-\mathbf{X}_i})$ *is the solution of* (1.6), (1.7) *with the shift function* $\mathbf{X}_i = \mathbf{X}_i(t)$ $(i = 1, 2)$, *which is an absolutely continuous solution to* (4.17). *Then there exist positive constants* $\delta_0 \leqq 1$, $\chi_0 \leqq 1$, *and* C_0 *independent of* T, *such that if the shock wave strength* $\delta := \delta_1 + \delta_2 < \delta_0$ *and*

$$\phi \in C([0, T]; H^2(\mathbb{R} \times \mathbb{T}^2)), \quad \nabla_x \phi \in L^2(0, T; H^1(\mathbb{R} \times \mathbb{T}^2)),$$
$$\psi \in C([0, T]; H^2(\mathbb{R} \times \mathbb{T}^2)), \quad \nabla_x \psi \in L^2(0, T; H^2(\mathbb{R} \times \mathbb{T}^2)),$$

with

$$\chi := \sup_{0 \le t \le T} \|(\phi, \psi)(t, \cdot)\|_{H^2} < \chi_0, \tag{3.3}$$

then the estimate

$$\sup_{0 \le t \le T} \|(\phi, \psi)(t, \cdot)\|_{H^2}^2 + \int_0^T (\|\nabla_x \phi\|_{H^1}^2 + \|\nabla_x \psi\|_{H^2}^2)\, \mathrm{d}t$$
$$+ \sum_{i=1}^2 \int_0^T \left(\delta_i |\dot{\mathbf{X}}_i(t)|^2 + \left\| \sqrt{|\partial_{x_1} (\rho^{s_i})^{-\mathbf{X}_i}|}\,\psi \right\|^2\right) \mathrm{d}t \le C(\|(\phi_0, \psi_0)\|^2 + \delta^{\frac{3}{2}}) \tag{3.4}$$

follows. In addition, by (4.17), *we have*

$$|\dot{\mathbf{X}}_i(t)| \le C_0 \|\psi_1(t, \cdot)\|_{L^\infty}, \quad \forall t \le T. \tag{3.5}$$

Proposition 3.2 will be proved in Sections 4 and 5.

3.2. Proof of Theorem 1.1

We can demonstrate Theorem 1.1 by the continuation arguments based on Propositions 3.1 and 3.2; the detailed proof can be found in [31]. In the following, we only prove the large-time behaviors (1.9) and (1.10). Set

$$g(t) := \|\nabla_x \phi(t)\|^2 + \|\nabla_x \psi(t)\|^2.$$

The aim is to show

$$\int_0^\infty (|g(t)| + |g'(t)|)\, \mathrm{d}t < \infty,$$

which implies

$$\lim_{t \to \infty} g(t) = \lim_{t \to \infty} (\|\nabla_x \phi(t)\|^2 + \|\nabla_x \psi(t)\|^2) = 0. \tag{3.6}$$

First, we can deduce from (3.4) that $\int_0^{+\infty} |g(t)|\, \mathrm{d}t < \infty$. It follows from (5.2) that

$$\int_0^t \|\nabla_x \partial_t \phi\|^2\, \mathrm{d}\tau \le C \int_0^t \|\nabla_x^2 (\phi, \psi)\|^2\, \mathrm{d}\tau + C(\chi + \delta) \int_0^t \|\nabla_x (\phi, \psi)\|^2\, \mathrm{d}\tau$$
$$+ C\delta \sum_{i=1}^2 \int_0^t (\delta_i |\dot{\mathbf{X}}_i(\tau)|^2 + G_{3,i}(\tau) + G^{s_i}(\tau))\, \mathrm{d}\tau \le C.$$

Meanwhile, we apply ∇_x to equation $(4.4)_2$ to get

$$\int_0^t \|\nabla_x\partial_t\psi\|^2\,d\tau \le C\int_0^t (\|\nabla_x^2(\phi,\psi)\|^2 + \|\nabla_x^3\psi\|^2)\,d\tau + C(\chi+\delta)\int_0^t \|\nabla_x(\phi,\psi)\|^2\,d\tau$$
$$+ C\delta\sum_{i=1}^{2}\int_0^t (\delta_i|\dot{\mathbf{X}}_i(\tau)|^2 + G_{3,i}(\tau) + G^{s_i}(\tau))\,d\tau \le C.$$

Thus, using the above two facts and the Cauchy inequality, we have

$$\begin{aligned}\int_0^{+\infty}|g'(t)|\,dt &= \int_0^{+\infty}\int_{\mathbb{T}^2}\int_{\mathbb{R}}(2|\nabla_x\phi||\nabla_x\partial_t\phi| + 2|\nabla_x\psi||\nabla_x\partial_t\psi|)\,dx_1\,dx'\,dt\\ &\le 2\int_0^{+\infty}(\|\nabla_x\phi\|\|\nabla_x\partial_t\phi\| + \|\nabla_x\psi\|\|\nabla_x\partial_t\psi\|)\,dt\\ &\le \int_0^{+\infty}(\|\nabla_x(\phi,\psi)\|^2 + \|\nabla_x\partial_t(\phi,\psi)\|^2)\,dt < \infty.\end{aligned}$$

By the Gagliardo–Nirenberg inequality in Lemma 2.5 and (3.6), we have

$$\lim_{t\to+\infty}\|(\phi,\psi)(t)\|_{L^\infty} \le \lim_{t\to+\infty}(\sqrt{2}\|(\phi,\psi)(t)\|^{\frac12}\|\partial_{x_1}(\phi,\psi)(t)\|^{\frac12}$$
$$+ C\|\nabla_x(\phi,\psi)(t)\|^{\frac12}\|\nabla_x^2(\phi,\psi)(t)\|^{\frac12}) = 0,$$

which proves (1.9). In addition, by (3.5) and the large-time behavior (1.9), it holds that

$$|\dot{\mathbf{X}}_i(t)| \le C\|\psi_1(t,\cdot)\|_{L^\infty} \to 0, \quad \text{as } t\to+\infty,$$

which proves (1.10). Thus the proof of Theorem 1.1 is completed.

4. Proof of Proposition 3.2: Basic energy estimate

First, we give the following key basic energy estimate.

Proposition 4.1. *Under the hypotheses of Proposition 3.2, there exists a constant $C > 0$ independent of δ_i $(i = 1, 2)$, χ, and T, such that for all $t \in [0, T]$, it holds that*

$$\|(\phi,\psi)(t)\|^2 + \sum_{i=1}^{2}\int_0^t\left(\delta_i|\dot{\mathbf{X}}_i(\tau)|^2 + \sum_{j=2}^{3}G_{j,i}(\tau) + G^{s_i}(\tau)\right)d\tau + \int_0^t\|\nabla_x\psi\|^2\,d\tau$$
$$\le C\left(\|(\phi_0,\psi_0)\|^2 + \max\{\delta_1,\delta_2\}\sum_{i=1}^{2}\sqrt{\delta_i}\right)$$
$$+ C\sum_{i=1}^{2}\int_1^t \delta_i\exp(-C\delta_i\tau)\|\psi_1\|_{H^2}^2\,d\tau, \tag{4.1}$$

where for $i = 1, 2$, *we denote*

$$
\begin{aligned}
G_{2,i}(t) &:= \frac{1}{\sqrt{\delta_i}} \int_{\mathbb{T}^2} \int_{\mathbb{R}} |\partial_{x_1} (\rho^{s_i})^{-\mathbf{X}_i}| (u_2^2 + u_3^2) \, \mathrm{d}x_1 \, \mathrm{d}x', \\
G_{3,i}(t) &:= \frac{1}{\sqrt{\delta_i}} \int_{\mathbb{T}^2} \int_{\mathbb{R}} |\partial_{x_1} (\rho^{s_i})^{-\mathbf{X}_i}| \Big(\phi - \frac{(\bar{\rho}^{-\mathbf{X}})^2}{\sigma_i} \psi_1 \Big)^2 \, \mathrm{d}x_1 \, \mathrm{d}x', \\
G^{s_i}(t) &:= \int_{\mathbb{T}^2} \int_{\mathbb{R}} |\partial_{x_1} (\rho^{s_i})^{-\mathbf{X}_i}| \psi_1^2 \, \mathrm{d}x_1 \, \mathrm{d}x'.
\end{aligned}
\tag{4.2}
$$

The proof of Proposition 4.1 consists of the following lemmas; we will prove it at the end of this section.

4.1. Reformulation of the problem

From (1.6), (1.7), and the definition (3.2), we can see the i-viscous shock $((\rho^{s_i})^{-\mathbf{X}_i}, (u_1^{s_i})^{-\mathbf{X}_i})$ $(i = 1, 2)$ satisfies

$$
\begin{cases}
\partial_t (\rho^{s_i})^{-\mathbf{X}_i} + \dot{\mathbf{X}}_i(t) \partial_{x_1} (\rho^{s_i})^{-\mathbf{X}_i} + (u_1^{s_i})^{-\mathbf{X}_i} \partial_{x_1} (\rho^{s_i})^{-\mathbf{X}_i} + (\rho^{s_i})^{-\mathbf{X}_i} \partial_{x_1} (u_1^{s_i})^{-\mathbf{X}_i} = 0, \\
(\rho^{s_i})^{-\mathbf{X}_i} (\partial_t (u_1^{s_i})^{-\mathbf{X}_i} + \dot{\mathbf{X}}_i(t) \partial_{x_1} (u_1^{s_i})^{-\mathbf{X}_i} + (u_1^{s_i})^{-\mathbf{X}_i} \partial_{x_1} (u_1^{s_i})^{-\mathbf{X}_i}) + \partial_{x_1} p((\rho^{s_i})^{-\mathbf{X}_i}) \\
\qquad = (2\mu + \lambda) \partial_{x_1}^2 (u_1^{s_i})^{-\mathbf{X}_i}.
\end{cases}
$$

Then, by the definition of $(\bar{\rho}^{-\mathbf{X}}, \bar{u}_1^{-\mathbf{X}})$, it holds that

$$
\begin{cases}
\partial_t \bar{\rho}^{-\mathbf{X}} + \bar{u}_1^{-\mathbf{X}} \partial_{x_1} \bar{\rho}^{-\mathbf{X}} + \bar{\rho}^{-\mathbf{X}} \partial_{x_1} \bar{u}_1^{-\mathbf{X}} + \displaystyle\sum_{i=1}^2 \dot{\mathbf{X}}_i(t) \partial_{x_1} (\rho^{s_i})^{-\mathbf{X}_i} = R_1, \\
\bar{\rho}^{-\mathbf{X}} (\partial_t \bar{u}_1^{-\mathbf{X}} + \bar{u}_1^{-\mathbf{X}} \partial_{x_1} \bar{u}_1^{-\mathbf{X}}) + \partial_{x_1} p(\bar{\rho}^{-\mathbf{X}}) + \bar{\rho}^{-\mathbf{X}} \displaystyle\sum_{i=1}^2 \dot{\mathbf{X}}_i(t) \partial_{x_1} (u_1^{s_i})^{-\mathbf{X}_i} \\
\qquad = (2\mu + \lambda) \partial_{x_1}^2 \bar{u}_1^{-\mathbf{X}} + R_2,
\end{cases}
\tag{4.3}
$$

where

$$
\begin{cases}
R_1 = \displaystyle\sum_{i=1}^2 (\bar{u}_1^{-\mathbf{X}} - (u_1^{s_i})^{-\mathbf{X}_i}) \partial_{x_1} (\rho^{s_i})^{-\mathbf{X}_i}, \\
R_2 = \displaystyle\sum_{i=1}^2 [-s_i (\bar{\rho}^{-\mathbf{X}} - (\rho^{s_i})^{-\mathbf{X}_i}) + (\bar{\rho}\bar{u}_1)^{-\mathbf{X}} - (\rho^{s_i} u_1^{s_i})^{-\mathbf{X}_i}] \partial_{x_1} (u_1^{s_i})^{-\mathbf{X}_i} \\
\qquad + \displaystyle\sum_{i=1}^2 [p'(\bar{\rho}^{-\mathbf{X}}) - p'((\rho^{s_i})^{-\mathbf{X}_i})] \partial_{x_1} (\rho^{s_i})^{-\mathbf{X}_i}.
\end{cases}
$$

Therefore, it follows from (1.1) and (4.3) that,

$$\begin{cases} \partial_t\phi + u\cdot\nabla_x\phi + \rho\,\mathrm{div}_x\,\psi + \psi_1\partial_{x_1}\bar{\rho}^{-\mathbf{X}} + \phi\partial_{x_1}\bar{u}_1^{-\mathbf{X}} \\ \qquad -\displaystyle\sum_{i=1}^{2}\dot{\mathbf{X}}_i(t)\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i} = -R_1, \\ \rho(\partial_t\psi + u\cdot\nabla_x\psi) + \nabla_x(p(\rho) - p(\bar{\rho}^{-\mathbf{X}})) + \rho\psi_1\partial_{x_1}\bar{u}^{-\mathbf{X}} \\ \qquad -\dfrac{\nabla_x p(\bar{\rho}^{-\mathbf{X}})}{\bar{\rho}^{-\mathbf{X}}}\phi - \rho\displaystyle\sum_{i=1}^{2}\dot{\mathbf{X}}_i(t)\partial_{x_1}(u^{s_i})^{-\mathbf{X}_i} \\ = \mu\Delta_x\psi + (\mu+\lambda)\nabla_x\,\mathrm{div}_x\,\psi - \dfrac{2\mu+\lambda}{\bar{\rho}^{-\mathbf{X}}}\phi\partial_{x_1}^2\bar{u}^{-\mathbf{X}} - \dfrac{\rho}{\bar{\rho}^{-\mathbf{X}}}(R_2,0,0)^{\mathsf{T}}. \end{cases} \tag{4.4}$$

We define the weight function $a(t,x_1)$ by

$$a(t,x_1) := a_1(x_1 - s_1t) + a_2(x_1 - s_2t), \tag{4.5}$$

with

$$a_i(x_1 - s_it) = \frac{1}{2} + \frac{1}{\sqrt{\delta_i}}\big(p(\rho_m) - p(\rho^{s_i}(x_1 - s_it))\big), \quad i = 1,2. \tag{4.6}$$

By the definition of the wave strengths (1.8), it holds that

$$\frac{1}{2} \le a_i(x_1 - s_it) \le \frac{1}{2} + \sqrt{\delta_i}, \quad 1 \le a(t,x_1) \le 1 + \sqrt{\delta_1} + \sqrt{\delta_2} \le 1 + \sqrt{2\delta}, \tag{4.7}$$

and

$$\begin{aligned} \partial_{x_1}a_1(x_1 - s_1t) &= -\frac{1}{\sqrt{\delta_1}}p'(\rho^{s_1})\partial_{x_1}\rho^{s_1} < 0, \\ \partial_{x_1}a_2(x_1 - s_2t) &= -\frac{1}{\sqrt{\delta_2}}p'(\rho^{s_2})\partial_{x_1}\rho^{s_2} > 0. \end{aligned} \tag{4.8}$$

In order to obtain the basic energy estimate, we need to use a weighted relative entropy method to derive the following lemma.

Lemma 4.1. *Let $a(t,x_1)$ be the weighted function defined by (4.5), and we set*

$$a^{-\mathbf{X}} := a_1^{-\mathbf{X}_1} + a_2^{-\mathbf{X}_2} = a_1(x_1 - s_1t - \mathbf{X}_1(t)) + a_2(x_1 - s_2t - \mathbf{X}_2(t)).$$

Then it holds that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho\Big(\Phi(\rho,\bar{\rho}^{-\mathbf{X}}) + \frac{|\psi|^2}{2}\Big)\,\mathrm{d}x_1\,\mathrm{d}x' \\ &\qquad = \sum_{i=1}^{2}\Big(\dot{\mathbf{X}}_i(t)\mathbf{Y}_i(t) + \sum_{j=1}^{10}\mathbf{B}_{j,i}(t) - \sum_{j=1}^{3}\mathbf{G}_{j,i}(t)\Big) - \mathbf{D}(t), \end{aligned} \tag{4.9}$$

where

$$\Phi(\rho,\bar\rho^{-\mathbf{X}}) = \int_{\bar\rho^{-\mathbf{X}}}^{\rho} \frac{p(s)-p(\bar\rho^{-\mathbf{X}})}{s^2}\,\mathrm{d}s = \frac{1}{\gamma-1}\frac{1}{\rho}\big(p(\rho)-p(\bar\rho^{-\mathbf{X}})-p'(\bar\rho^{-\mathbf{X}})(\rho-\bar\rho^{-\mathbf{X}})\big),$$

$$\begin{aligned}
\mathbf{Y}_i(t) :=& \int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i} a^{-\mathbf{X}}\rho\psi_1\,\mathrm{d}x_1\,\mathrm{d}x' \\
&+ \int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i} a^{-\mathbf{X}} \frac{p'(\bar\rho^{-\mathbf{X}})}{\bar\rho^{-\mathbf{X}}}\phi\,\mathrm{d}x_1\,\mathrm{d}x' \\
&- \int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{x_1} a_i^{-\mathbf{X}_i}\rho\Big(\Phi(\rho,\bar\rho^{-\mathbf{X}}) + \frac{|\psi|^2}{2}\Big)\,\mathrm{d}x_1\,\mathrm{d}x', \quad i=1,2,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{B}_{1,i}(t) &:= \int_{\mathbb{T}^2}\int_{\mathbb{R}} \frac{\partial_{x_1} a_i^{-\mathbf{X}_i}}{\sigma_i}\frac{\gamma}{2}(\bar\rho^{-\mathbf{X}})^{\gamma+1}\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x', \\
\mathbf{B}_{2,i}(t) &:= -\frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i} a^{-\mathbf{X}} p''(\bar\rho^{-\mathbf{X}})\phi^2\,\mathrm{d}x_1\,\mathrm{d}x', \\
\mathbf{B}_{3,i}(t) &:= -\int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i} a^{-\mathbf{X}}\rho\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x', \\
\mathbf{B}_{4,i}(t) &:= \int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{x_1} a_i^{-\mathbf{X}_i}(-s_i\phi + \rho\psi_1 + \bar u_1^{-\mathbf{X}}\phi)\Big(\Phi(\rho,\bar\rho^{-\mathbf{X}}) + \frac{|\psi|^2}{2}\Big)\,\mathrm{d}x_1\,\mathrm{d}x', \\
\mathbf{B}_{5,i}(t) &:= \int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{x_1} a_i^{-\mathbf{X}_i}[-s_i(\bar\rho^{-\mathbf{X}} - (\rho^{s_i})^{-\mathbf{X}_i}) + (\bar\rho\bar u_1)^{-\mathbf{X}} - (\rho^{s_i}u_1^{s_i})^{-\mathbf{X}_i}] \\
&\qquad\qquad \cdot\Big(\Phi(\rho,\bar\rho^{-\mathbf{X}}) + \frac{|\psi|^2}{2}\Big)\,\mathrm{d}x_1\,\mathrm{d}x', \\
\mathbf{B}_{6,i}(t) &:= O(1)\int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{x_1} a_i^{-\mathbf{X}_i}|(\phi,\psi_1)|\,\phi^2\,\mathrm{d}x_1\,\mathrm{d}x', \\
\mathbf{B}_{7,i}(t) &:= O(1)\int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i} a^{-\mathbf{X}}|\phi|^3\,\mathrm{d}x_1\,\mathrm{d}x', \\
\mathbf{B}_{8,i}(t) &:= -\int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{x_1} a_i^{-\mathbf{X}_i}[\mu\partial_{x_1}\psi\cdot\psi + (\mu+\lambda)\psi_1\,\mathrm{div}_x\,\psi]\,\mathrm{d}x_1\,\mathrm{d}x', \\
\mathbf{B}_{9,i}(t) &:= -\int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{x_1}^2(u_1^{s_i})^{-\mathbf{X}_i} a^{-\mathbf{X}}\frac{2\mu+\lambda}{\bar\rho^{-\mathbf{X}}}\phi\psi_1\,\mathrm{d}x_1\,\mathrm{d}x', \\
\mathbf{B}_{10,i}(t) &:= -\int_{\mathbb{T}^2}\int_{\mathbb{R}} a_i^{-\mathbf{X}_i}\Big(\frac{p'(\bar\rho^{-\mathbf{X}})}{\bar\rho^{-\mathbf{X}}}\phi R_1 + \frac{\rho}{\bar\rho^{-\mathbf{X}}}\psi_1 R_2\Big)\,\mathrm{d}x_1\,\mathrm{d}x',
\end{aligned}$$

and

$$\mathbf{G}_{1,i}(t) := \int_{\mathbb{T}^2}\int_{\mathbb{R}} \sigma_i\partial_{x_1} a_i^{-\mathbf{X}_i}\frac{\psi_1^2}{2}\,\mathrm{d}x_1\,\mathrm{d}x', \quad \mathbf{G}_{2,i}(t) := \int_{\mathbb{T}^2}\int_{\mathbb{R}} \sigma_i\partial_{x_1} a_i^{-\mathbf{X}_i}\frac{|\psi'|^2}{2}\,\mathrm{d}x_1\,\mathrm{d}x',$$

$$\mathbf{G}_{3,i}(t) := \int_{\mathbb{T}^2}\int_{\mathbb{R}} \sigma_i\partial_{x_1} a_i^{-\mathbf{X}_i}\frac{\gamma}{2}(\bar\rho^{-\mathbf{X}})^{\gamma-3}\Big(\phi - \frac{(\bar\rho^{-\mathbf{X}})^2}{\sigma_i}\psi_1\Big)^2\,\mathrm{d}x_1\,\mathrm{d}x',$$

$$\mathbf{D}(t) := \mu\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}|\nabla_x\psi|^2\,\mathrm{d}x_1\,\mathrm{d}x' + (\mu+\lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}(\mathrm{div}_x\,\psi)^2\,\mathrm{d}x_1\,\mathrm{d}x'.$$

Remark 4.2. Since $\sigma_i \partial_{x_1} a_i^{-\mathbf{X}_i} > 0$ $(i = 1, 2)$, hence $\mathbf{G}_{j,i}(t)$ consists of terms with good sign, while $\mathbf{B}_{j,i}(t)$ consists of bad terms.

Proof of Lemma 4.1. It follows from (4.7) that $|a^{-\mathbf{X}} - 1| \le \sqrt{\delta_1} + \sqrt{\delta_2}$ and

$$
\begin{aligned}
\partial_t a^{-\mathbf{X}} = \sum_{i=1}^{2} \partial_t a_i^{-\mathbf{X}_i} &= \sum_{i=1}^{2} (-s_i - \dot{\mathbf{X}}_i(t)) \partial_{x_1} a_i^{-\mathbf{X}_i} \\
&= -\sum_{i=1}^{2} s_i \partial_{x_1} a_i^{-\mathbf{X}_i} - \sum_{i=1}^{2} \dot{\mathbf{X}}_i(t) \partial_{x_1} a_i^{-\mathbf{X}_i}.
\end{aligned}
$$

First, we multiply $(4.4)_1$ by $a^{-\mathbf{X}} \frac{p(\rho) - p(\bar{\rho}^{-\mathbf{X}})}{\rho}$ to have

$$
\begin{aligned}
&\partial_t (a^{-\mathbf{X}} \rho \Phi(\rho, \bar{\rho}^{-\mathbf{X}})) + \operatorname{div}_x (a^{-\mathbf{X}} \rho u \Phi(\rho, \bar{\rho}^{-\mathbf{X}})) + a^{-\mathbf{X}} (p(\rho) - p(\bar{\rho}^{-\mathbf{X}})) \operatorname{div}_x \psi \\
&\quad = -\sum_{i=1}^{2} \dot{\mathbf{X}}_i(t) \partial_{x_1} a_i^{-\mathbf{X}_i} \rho \Phi(\rho, \bar{\rho}^{-\mathbf{X}}) + \sum_{i=1}^{2} \dot{\mathbf{X}}_i(t) \partial_{x_1} (\rho^{s_i})^{-\mathbf{X}_i} a^{-\mathbf{X}} \frac{p'(\bar{\rho}^{-\mathbf{X}})}{\bar{\rho}^{-\mathbf{X}}} \phi \\
&\qquad + \sum_{i=1}^{2} (-s_i \rho + \rho u_1) \partial_{x_1} a_i^{-\mathbf{X}_i} \Phi(\rho, \bar{\rho}^{-\mathbf{X}}) \\
&\qquad - a^{-\mathbf{X}} (p(\rho) - p(\bar{\rho}^{-\mathbf{X}}) - p'(\bar{\rho}^{-\mathbf{X}}) \phi) \partial_{x_1} \bar{u}_1^{-\mathbf{X}} \\
&\qquad - a^{-\mathbf{X}} \frac{\partial_{x_1} p(\bar{\rho}^{-\mathbf{X}})}{\bar{\rho}^{-\mathbf{X}}} \phi \psi_1 - a^{-\mathbf{X}} \frac{p'(\bar{\rho}^{-\mathbf{X}})}{\bar{\rho}^{-\mathbf{X}}} \phi R_1.
\end{aligned}
\tag{4.10}
$$

Next we multiply $(4.4)_2$ by $a^{-\mathbf{X}} \psi$ to have

$$
\begin{aligned}
&\partial_t \Big(a^{-\mathbf{X}} \rho \frac{|\psi|^2}{2} \Big) + \operatorname{div}_x \Big(a^{-\mathbf{X}} \rho u \frac{|\psi|^2}{2} \Big) + \operatorname{div}_x \big(a^{-\mathbf{X}} (p(\rho) - p(\bar{\rho}^{-\mathbf{X}})) \psi \big) \\
&\quad = a^{-\mathbf{X}} (p(\rho) - p(\bar{\rho}^{-\mathbf{X}})) \operatorname{div}_x \psi - \sum_{i=1}^{2} \dot{\mathbf{X}}_i(t) \partial_{x_1} a_i^{-\mathbf{X}_i} \rho \frac{|\psi|^2}{2} \\
&\qquad + \sum_{i=1}^{2} \dot{\mathbf{X}}_i(t) \partial_{x_1} (u_1^{s_i})^{-\mathbf{X}_i} a^{-\mathbf{X}} \rho \psi_1 + \partial_{x_1} a^{-\mathbf{X}} \psi_1 (p(\rho) - p(\bar{\rho}^{-\mathbf{X}})) \\
&\qquad + \sum_{i=1}^{2} (-s_i \rho + \rho u_1) \partial_{x_1} a_i^{-\mathbf{X}_i} \frac{|\psi|^2}{2} + a^{-\mathbf{X}} \frac{\partial_{x_1} p(\bar{\rho}^{-\mathbf{X}})}{\bar{\rho}^{-\mathbf{X}}} \phi \psi_1 \\
&\qquad - a^{-\mathbf{X}} \rho \psi_1^2 \partial_{x_1} \bar{u}_1^{-\mathbf{X}} + \mu \operatorname{div}_x (a^{-\mathbf{X}} \nabla_x \psi \cdot \psi) + (\mu + \lambda)(a^{-\mathbf{X}} \psi \operatorname{div}_x \psi) \\
&\qquad - \mu a^{-\mathbf{X}} |\nabla_x \psi|^2 - (\mu + \lambda) a^{-\mathbf{X}} (\operatorname{div}_x \psi)^2 \\
&\qquad - \mu \partial_{x_1} a^{-\mathbf{X}} \partial_{x_1} \psi \cdot \psi - (\mu + \lambda) \partial_{x_1} a^{-\mathbf{X}} \psi_1 \operatorname{div}_x \psi \\
&\qquad - a^{-\mathbf{X}} \frac{2\mu + \lambda}{\bar{\rho}^{-\mathbf{X}}} \partial_{x_1}^2 \bar{u}_1^{-\mathbf{X}} \phi \psi_1 - a^{-\mathbf{X}} \frac{\rho}{\bar{\rho}^{-\mathbf{X}}} \psi_1 R_2.
\end{aligned}
\tag{4.11}
$$

For $i = 1, 2$, direct calculations yield

$$-s_i\rho + \rho u_1 = -\sigma_i - s_i\phi + \rho\psi_1 + \bar{u}_1^{-\mathbf{X}}\phi - s_i(\bar{\rho}^{-\mathbf{X}} - (\rho^{s_i})^{-\mathbf{X}_i}) + (\bar{\rho}\bar{u}_1)^{-\mathbf{X}} - (\rho^{s_i}u_1^{s_i})^{-\mathbf{X}_i}. \tag{4.12}$$

Adding (4.10) and (4.11) together, integrating the resultant equation by parts over $\mathbb{R}\times\mathbb{T}^2$, using (4.12), it holds that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho\Big(\Phi(\rho,\bar{\rho}^{-\mathbf{X}}) + \frac{|\psi|^2}{2}\Big)\,\mathrm{d}x_1\,\mathrm{d}x' \\
&\quad = \sum_{i=1}^{2}\dot{\mathbf{X}}_i(t)\mathbf{Y}_i(t) + \sum_{i=1}^{2}\sum_{j=3}^{5}\mathbf{B}_{j,i}(t) + \sum_{i=1}^{2}\sum_{j=8}^{10}\mathbf{B}_{j,i}(t) - \sum_{i=1}^{2}\sum_{j=1}^{2}\mathbf{G}_{j,i}(t) - \mathbf{D}(t) \\
&\qquad + \sum_{i=1}^{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\partial_{x_1}a_i^{-\mathbf{X}_i}[-\sigma_i\Phi(\rho,\bar{\rho}^{-\mathbf{X}}) + \psi_1(p(\rho) - p(\bar{\rho}^{-\mathbf{X}}))]\,\mathrm{d}x_1\,\mathrm{d}x' \\
&\qquad - \int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}(p(\rho) - p(\bar{\rho}^{-\mathbf{X}}) - p'(\bar{\rho}^{-\mathbf{X}})\phi)\partial_{x_1}\bar{u}_1^{-\mathbf{X}}\,\mathrm{d}x_1\,\mathrm{d}x'.
\end{aligned} \tag{4.13}$$

Then we deal with the last two terms on the right-hand side of (4.13) by using a Taylor expansion at $\bar{\rho}^{-\mathbf{X}}$ for the integrand. It holds that

$$\begin{aligned}
\Phi(\rho,\bar{\rho}^{-\mathbf{X}}) &= \frac{1}{\gamma-1}\frac{1}{\rho}\Big(\frac{p''(\bar{\rho}^{-\mathbf{X}})}{2!}\phi^2 + \frac{p'''(\zeta_1)}{3!}\phi^3\Big) \\
&= \frac{\gamma}{2}(\bar{\rho}^{-\mathbf{X}})^{\gamma-3}\phi^2 + O(1)|\phi|^3
\end{aligned} \tag{4.14}$$

and

$$p(\rho) - p(\bar{\rho}^{-\mathbf{X}}) = p'(\bar{\rho}^{-\mathbf{X}})\phi + \frac{p''(\zeta_2)}{2}\phi^2 = \gamma(\bar{\rho}^{-\mathbf{X}})^{\gamma-1}\phi + O(1)\phi^2,$$

where ζ_1, ζ_2 exist between ρ and $\bar{\rho}^{-\mathbf{X}}$. Therefore, we have

$$\begin{aligned}
&\partial_{x_i}a_i^{-\mathbf{X}_i}[-\sigma_i\Phi(\rho,\bar{\rho}^{-\mathbf{X}}) + \psi_1(p(\rho) - p(\bar{\rho}^{-\mathbf{X}}))] \\
&\quad = \partial_{x_i}a_i^{-\mathbf{X}_i}\Big[-\sigma_i\frac{\gamma}{2}(\bar{\rho}^{-\mathbf{X}})^{\gamma-3}\Big(\phi - \frac{(\bar{\rho}^{-\mathbf{X}})^2}{\sigma_i}\psi_1\Big)^2 + \frac{\gamma}{2\sigma_i}(\bar{\rho}^{-\mathbf{X}})^{\gamma+1}\psi_1^2 \\
&\qquad + O(1)|(\phi,\psi_1)|\phi^2\Big].
\end{aligned}$$

Thus, it holds that

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\partial_{x_i}a_i^{-\mathbf{X}_i}[-\sigma_i\Phi(\rho,\bar{\rho}^{-\mathbf{X}}) + \psi_1(p(\rho) - p(\bar{\rho}^{-\mathbf{X}}))]\,\mathrm{d}x_1\,\mathrm{d}x' \\
&\quad = \sum_{i=1}^{2}(-\mathbf{G}_{3,i}(t) + \mathbf{B}_{1,i}(t) + \mathbf{B}_{6,i}(t))
\end{aligned} \tag{4.15}$$

and

$$-\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}(p(\rho)-p(\bar{\rho}^{-\mathbf{X}})-p'(\bar{\rho}^{-\mathbf{X}})\phi)\partial_{x_1}\bar{u}_1^{-\mathbf{X}}\,\mathrm{d}x_1\,\mathrm{d}x' \\ = \sum_{i=1}^{2}(\mathbf{B}_{2,i}(t)+\mathbf{B}_{7,i}(t)). \tag{4.16}$$

In order to derive the a-contraction property of the viscous shock waves, we decompose the functional $\mathbf{Y}_i(t)$ $(i=1,2)$ as

$$\begin{aligned}\mathbf{Y}_i(t) = &\int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}a^{-\mathbf{X}}\rho\psi_1\,\mathrm{d}x_1\,\mathrm{d}x' \\ &+\int_{\mathbb{T}^2}\int_{\mathbb{R}} \frac{\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}}{\sigma_i}a^{-\mathbf{X}}\bar{\rho}^{-\mathbf{X}}p'(\bar{\rho}^{-\mathbf{X}})\psi_1\,\mathrm{d}x_1\,\mathrm{d}x' \\ &+\int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}a^{-\mathbf{X}}\frac{p'(\bar{\rho}^{-\mathbf{X}})}{\bar{\rho}^{-\mathbf{X}}}\Big(\phi-\frac{(\bar{\rho}^{-\mathbf{X}})^2}{\sigma_i}\psi_1\Big)\,\mathrm{d}x_1\,\mathrm{d}x' \\ &-\int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{x_1}a_i^{-\mathbf{X}_i}\frac{\gamma}{2}\rho(\bar{\rho}^{-\mathbf{X}})^{\gamma-3}\Big(\phi+\frac{(\bar{\rho}^{-\mathbf{X}})^2}{\sigma_i}\psi_1\Big)\Big(\phi-\frac{(\bar{\rho}^{-\mathbf{X}})^2}{\sigma_i}\psi_1\Big)\,\mathrm{d}x_1\,\mathrm{d}x' \\ &-\int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{x_1}a_i^{-\mathbf{X}_i}\rho\Big(\frac{\gamma}{2\sigma_i^2}(\bar{\rho}^{-\mathbf{X}})^{\gamma+1}\psi_1^2+\frac{|\psi|^2}{2}+O(1)|\phi|^3\Big)\,\mathrm{d}x_1\,\mathrm{d}x' \\ =: &\sum_{j=1}^{5}\mathbf{Y}_{j,i}(t),\end{aligned}$$

we have used (4.14) to derive $\mathbf{Y}_{5,i}(t)$.

The a priori estimates depend on the shift functions, and for this reason, we give their definition right now. The definition of the shift functions depends on the weight function $a:\mathbb{R}\to\mathbb{R}$ defined in (4.5). For now we will only assume the fact that $\|a\|_{C^1(\mathbb{R})}\le 2$. Then we can define the shift $\mathbf{X}_i(t)$ $(i=1,2)$ as a solution to the ODE

$$\begin{cases}\dot{\mathbf{X}}_i(t) = -\dfrac{M}{\delta_i}(\mathbf{Y}_{1,i}(t)+\mathbf{Y}_{2,i}(t)), \\ \mathbf{X}_i(0)=0,\end{cases} \tag{4.17}$$

where the constant $M := (\gamma+1)\frac{8}{3}\frac{\sigma_m}{\rho_m}$ with $\sigma_m := \sqrt{-p'(v_m)}$. Thus, it holds that

$$\dot{\mathbf{X}}_i(t)\mathbf{Y}_i(t) = -\frac{\delta_i}{M}|\dot{\mathbf{X}}_i(t)|^2+\dot{\mathbf{X}}_i(t)\sum_{j=3}^{5}\mathbf{Y}_{j,i}(t). \tag{4.18}$$

The system (4.17) has a unique absolutely continuous solution $(\mathbf{X}_1(t),\mathbf{X}_2(t))^{\mathsf{T}}$ on $[0,T]$; see [16, Appendix C].

Substituting (4.15), (4.16) and (4.18) into (4.13) yields (4.9). The proof of Lemma 4.1 is completed. ∎

4.2. Estimates for waves interaction

In the following, we will apply the a-contraction property to each wave ρ^{s_1} and ρ^{s_2} respectively. Our idea is to estimate the separate waves. Thus, we have to control the wave interaction. The following lemma provides inequalities on the interaction of waves, which is useful to obtain the basic energy estimate.

Lemma 4.3. *For given $\rho_- > 0$ and $u_{1-} \in \mathbb{R}$, there exist positive constants δ_0, C such that for any $\delta_1, \delta_2 \in (0, \delta_0)$ and $i = 1, 2$, the following estimates hold:*

$$\int_{\mathbb{R}} |\bar{\rho}^{-\mathbf{X}} - (\rho^{s_i})^{-\mathbf{X}_i}||\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}|\,\mathrm{d}x_1 \le C\delta_1\delta_2 \exp(-C\min\{\delta_1, \delta_2\}t), \tag{4.19}$$

$$\int_{\mathbb{R}} |\bar{\rho}^{-\mathbf{X}} - (\rho^{s_i})^{-\mathbf{X}_i}|^2|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}|^2\,\mathrm{d}x_1 \le C\delta_i(\delta_1\delta_2)^2 \exp(-C\min\{\delta_1, \delta_2\}t), \tag{4.20}$$

$$\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}||\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}|\,\mathrm{d}x_1 \le C\delta_1\delta_2 \sum_{i=1}^{2} \delta_i \exp(-C\delta_i t). \tag{4.21}$$

Proof. First, by (2.1) and (2.2), we have

$$s_2 - s_1 = \frac{\sigma_2 - \sigma_1}{\rho_m} > 0. \tag{4.22}$$

Then, by (4.17) and assumption (3.3), we have

$$|\dot{\mathbf{X}}_i(t)| \le \frac{C}{\delta_i}\|\psi_1\|_{L^\infty} \int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}|\,\mathrm{d}x_1 \le C\chi, \quad i = 1, 2,\ t > 0,$$

which together with $\mathbf{X}_i(0) = 0$ yields

$$\mathbf{X}_1(t) \le C\chi t < \frac{s_2 - s_1}{4}t, \quad \mathbf{X}_2(t) \ge -C\chi t > -\frac{s_2 - s_1}{4}t, \quad t > 0, \tag{4.23}$$

for suitably small χ such that $C\chi < \frac{s_2 - s_1}{4}$. Therefore, it holds that

$$\begin{cases} x_1 - s_1 t - \mathbf{X}_1(t) > \frac{s_2 - s_1}{2}t - \frac{s_2 - s_1}{4}t = \frac{s_2 - s_1}{4}t > 0, & x_1 \ge \frac{s_1 + s_2}{2}t, \\ x_1 - s_2 t - \mathbf{X}_2(t) < -\frac{s_2 - s_1}{2}t + \frac{s_2 - s_1}{4}t = -\frac{s_2 - s_1}{4}t < 0, & x_1 \le \frac{s_1 + s_2}{2}t. \end{cases} \tag{4.24}$$

To prove (4.19), by Lemma 2.1, there exists a constant $C > 0$ such that for $i = 1, 2$, it holds that

$$\begin{aligned} |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}| &= |\partial_{x_1}\rho^{s_i}(x_1 - s_i t - \mathbf{X}_i(t))| \\ &\le C\delta_i^2 \exp(-C\delta_i|x_1 - s_i t - \mathbf{X}_i(t)|), \quad \forall x_1 \in \mathbb{R},\ t > 0. \end{aligned}$$

Since $\bar{\rho}^{-\mathbf{X}} = (\rho^{s_1})^{-\mathbf{X}_1} + (\rho^{s_2})^{-\mathbf{X}_2} - \rho_m$ (by (3.2)), it follows from Lemma 2.1 that

$$\begin{aligned} |\bar{\rho}^{-\mathbf{X}} - (\rho^{s_1})^{-\mathbf{X}_1}| &= |(\rho^{s_2})^{-\mathbf{X}_2} - \rho_m| \\ &\le \begin{cases} C\delta_2 \exp(-C\delta_2|x_1 - s_2 t - \mathbf{X}_2(t)|), & x_1 \le s_2 t + \mathbf{X}_2(t), \\ C\delta_2, & x_1 \ge s_2 t + \mathbf{X}_2(t), \end{cases} \end{aligned}$$

and

$$
\begin{aligned}
|\bar\rho^{-\mathbf{X}} - (\rho^{s_2})^{-\mathbf{X}_2}| &= |(\rho^{s_1})^{-\mathbf{X}_1} - \rho_m| \\
&\le \begin{cases} C\delta_1, & x_1 \le s_1 t + \mathbf{X}_1(t), \\ C\delta_1 \exp(-C\delta_1 |x_1 - s_1 t - \mathbf{X}_1(t)|), & x_1 \ge s_1 t + \mathbf{X}_1(t). \end{cases}
\end{aligned}
$$

Thus, using the above estimates together with (4.24), we find

$$
\begin{aligned}
&|\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}||\bar\rho^{-\mathbf{X}} - (\rho^{s_1})^{-\mathbf{X}_1}| \\
&\le \begin{cases} C\delta_2 \exp\Big(-C\delta_2 \dfrac{|x_1 - s_2 t - \mathbf{X}_2(t)|}{2}\Big) & \\ \qquad \times \exp\Big(-C\delta_2 \dfrac{s_2 - s_1}{8} t\Big)|\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}|, & x_1 \le \dfrac{s_1 + s_2}{2} t, \\ C\delta_1^2 \delta_2 \exp\Big(-C\delta_1 \dfrac{|x_1 - s_1 t - \mathbf{X}_1(t)|}{2}\Big) & \\ \qquad \times \exp\Big(-C\delta_1 \dfrac{s_2 - s_1}{8} t\Big), & x_1 \ge \dfrac{s_1 + s_2}{2} t, \end{cases}
\end{aligned}
\tag{4.25}
$$

and

$$
\begin{aligned}
&|\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}||\bar\rho^{-\mathbf{X}} - (\rho^{s_2})^{-\mathbf{X}_2}| \\
&\le \begin{cases} C\delta_1 \delta_2^2 \exp\Big(-C\delta_2 \dfrac{|x_1 - s_2 t - \mathbf{X}_2(t)|}{2}\Big) & \\ \qquad \times \exp\Big(-C\delta_2 \dfrac{s_2 - s_1}{8} t\Big), & x_1 \le \dfrac{s_1 + s_2}{2} t, \\ C\delta_1 \exp\Big(-C\delta_1 \dfrac{|x_1 - s_1 t - \mathbf{X}_1(t)|}{2}\Big) & \\ \qquad \times \exp\Big(-C\delta_1 \dfrac{s_2 - s_1}{8} t\Big)|\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}|, & x_1 \ge \dfrac{s_1 + s_2}{2} t. \end{cases}
\end{aligned}
\tag{4.26}
$$

Integrating (4.25) by parts with respect to x_1 over $\mathbb{R}$ leads to

$$
\begin{aligned}
&\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}||\bar\rho^{-\mathbf{X}} - (\rho^{s_1})^{-\mathbf{X}_1}| \,\mathrm{d}x_1 \\
&\quad \le C\delta_2 \exp(-C\delta_2 t) \int_{-\infty}^{\frac{s_1+s_2}{2} t} |\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}| \,\mathrm{d}x_1 \\
&\qquad + C\delta_1^2 \delta_2 \exp(-C\delta_1 t) \int_{\frac{s_1+s_2}{2} t}^{+\infty} \exp\big(-C\delta_1 (x_1 - s_1 t - \mathbf{X}_1(t))\big) \,\mathrm{d}x_1 \\
&\quad \le C\delta_1 \delta_2 \exp(-C\delta_2 t) + C\delta_1 \delta_2 \exp(-C\delta_1 t) \\
&\quad \le C\delta_1 \delta_2 \exp(-C \min\{\delta_1, \delta_2\} t).
\end{aligned}
\tag{4.27}
$$

Likewise, integrating (4.26) by parts with respect to x_1 over $\mathbb{R}$ gives

$$\begin{aligned}
&\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}||\bar{\rho}^{-\mathbf{X}} - (\rho^{s_2})^{-\mathbf{X}_2}|\,\mathrm{d}x_1 \\
&\quad \le C\delta_1 \exp(-C\delta_1 t)\int_{\frac{s_1+s_2}{2}t}^{+\infty} |\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}|\,\mathrm{d}x_1 \\
&\qquad + C\delta_1\delta_2^2 \exp(-C\delta_2 t)\int_{-\infty}^{\frac{s_1+s_2}{2}t} \exp\big(C\delta_2(x_1 - s_2 t - \mathbf{X}_2(t))\big)\,\mathrm{d}x_1 \\
&\quad \le C\delta_1\delta_2 \exp(-C\delta_1 t) + C\delta_1\delta_2 \exp(-C\delta_2 t) \\
&\quad \le C\delta_1\delta_2 \exp(-C\min\{\delta_1,\delta_2\}t).
\end{aligned} \tag{4.28}$$

Thus, we combine (4.27) and (4.28) to have (4.19) for $i = 1, 2$.

Likewise, using (4.27) and (4.28), it holds that

$$\begin{aligned}
&\int_{\mathbb{R}} |\bar{\rho}^{-\mathbf{X}} - (\rho^{s_i})^{-\mathbf{X}_i}|^2 |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}|^2\,\mathrm{d}x_1 \\
&\quad \le C\|\bar{\rho}^{-\mathbf{X}} - (\rho^{s_i})^{-\mathbf{X}_i}\|_{L^\infty(\mathbb{R})}\|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}\|_{L^\infty(\mathbb{R})} \\
&\qquad \times \int_{\mathbb{R}} |\bar{\rho}^{-\mathbf{X}} - (\rho^{s_i})^{-\mathbf{X}_i}||\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}|\,\mathrm{d}x_1 \\
&\quad \le C\|\bar{\rho}^{-\mathbf{X}} - (\rho^{s_i})^{-\mathbf{X}_i}\|_{L^\infty(\mathbb{R})}\delta_i^2\delta_1\delta_2 \exp(-C\min\{\delta_1,\delta_2\}t) \\
&\quad \le C\delta_i(\delta_1\delta_2)^2 \exp(-C\min\{\delta_1,\delta_2\}t),
\end{aligned}$$

which gives (4.20) for $i = 1, 2$.

Similarly to (4.25) and (4.26), it holds that

$$\begin{aligned}
&|\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}||\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}| \\
&\le \begin{cases}
C\delta_2^2 \exp\Big(-C\delta_2 \dfrac{|x_1 - s_2 t - \mathbf{X}_2(t)|}{2}\Big) \\
\qquad \times \exp\Big(-C\delta_2 \dfrac{s_2 - s_1}{8} t\Big)|\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}|, \quad x_1 \le \dfrac{s_1+s_2}{2}t, \\
C\delta_1^2 \exp\Big(-C\delta_1 \dfrac{|x_1 - s_1 t - \mathbf{X}_1(t)|}{2}\Big) \\
\qquad \times \exp\Big(-C\delta_1 \dfrac{s_2 - s_1}{8} t\Big)|\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}|, \quad x_1 \ge \dfrac{s_1+s_2}{2}t.
\end{cases}
\end{aligned} \tag{4.29}$$

Integrating (4.29) by parts with respect to x_1 over $\mathbb{R}$ yields

$$\begin{aligned}
&\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}||\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}|\,\mathrm{d}x_1 \\
&\quad \le C\delta_2^2 \exp(-C\delta_2 t)\int_{-\infty}^{\frac{s_1+s_2}{2}t} |\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}|\,\mathrm{d}x_1 \\
&\qquad + C\delta_1^2 \exp(-C\delta_1 t)\int_{\frac{s_1+s_2}{2}t}^{+\infty} |\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}|\,\mathrm{d}x_1 \\
&\quad \le C\delta_1\delta_2^2 \exp(-C\delta_2 t) + C\delta_1^2\delta_2 \exp(-C\delta_1 t).
\end{aligned}$$

Thus, we have (4.21). The proof of Lemma 4.3 is completed. ∎

4.3. Estimates for separation of waves

We see from the expression for $\mathbf{B}_{j,i}(t)$ ($j = 2, 3$, $i = 1, 2$) that there exist $\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}$ ($i = 1, 2$). In order to extract the dissipation from the flux term along the shock wave propagation direction, we introduce the cut-off functions to eliminate the effect from wave propagation along the transverse directions normal to x_1 and their interactions with the viscous shock.

We define two nonnegative Lipschitz monotonic functions $\eta_i(x_1)$ ($i = 1, 2$) on $\mathbb{R}$ such that $\eta_1(x_1) + \eta_2(x_1) = 1$ as follows: for any fixed $t > 0$,

$$\eta_1(x_1) = \begin{cases} 1, & x_1 < \dfrac{3s_2 + 5s_1}{8}t, \\ \text{linear}, & \dfrac{3s_2 + 5s_1}{8}t \le x_1 \le \dfrac{5s_2 + 3s_1}{8}t, \\ 0, & x_1 > \dfrac{5s_2 + 3s_1}{8}t, \end{cases} \tag{4.30}$$

and

$$\eta_2(x_1) = \begin{cases} 0, & x_1 < \dfrac{3s_2 + 5s_1}{8}t, \\ \text{linear}, & \dfrac{3s_2 + 5s_1}{8}t \le x_1 \le \dfrac{5s_2 + 3s_1}{8}t, \\ 1, & x_1 > \dfrac{5s_2 + 3s_1}{8}t. \end{cases} \tag{4.31}$$

Lemma 4.4. *Let $\eta_1(x_1)$ and $\eta_2(x_1)$ be the nonnegative Lipschitz monotonic functions defined as in (4.30) and in (4.31). For any $\rho_- > 0$ and $u_{1-} \in \mathbb{R}$, there exist positive constants δ_0, C such that for any $\delta_1, \delta_2 \in (0, \delta_0)$ and for all $t > 0$,*

$$\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}|(1 - \eta_i)\,\mathrm{d}x_1 \le C\delta_i \exp(-C\delta_i t), \quad i = 1, 2.$$

Proof. The proof is similar to Lemma 4.3. For $i = 1$, by (4.22) and (4.23), it holds that

$$x_1 - s_1 t - \mathbf{X}_1(t) \ge \frac{3s_2 + 5s_1}{8}t - s_1 t - \frac{s_2 - s_1}{4}t = \frac{s_2 - s_1}{8}t > 0, \quad \text{if } x_1 \ge \frac{3s_2 + 5s_1}{8}t.$$

Thus, we have

$$\begin{aligned} |\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}|(1 - \eta_1) &= |\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}|^{\frac{1}{2}}|\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}|^{\frac{1}{2}}\eta_2 \\ &\le C|\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}|^{\frac{1}{2}}\delta_1 \exp\Big(-C\delta_1 \frac{|x_1 - s_1 t - \mathbf{X}_1(t)|}{2}\Big) \\ &\quad \times \exp\Big(-C\delta_1 \frac{s_2 - s_1}{16}t\Big)\eta_2 \mathbf{1}_{\{x_1 \ge \frac{3s_2 + 5s_1}{8}t\}} \\ &\le C\delta_1 |\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}|^{\frac{1}{2}} \exp(-C\delta_1 t). \end{aligned}$$

Integrating the above inequality with respect to x_1 over $\mathbb{R}$ yields

$$\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}|(1-\eta_1)\,\mathrm{d}x_1 \le C\delta_1 \exp(-C\delta_1 t)\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}|^{\frac{1}{2}}\,\mathrm{d}x_1 \\ \le C\delta_1\exp(-C\delta_1 t). \tag{4.32}$$

Likewise, for $i=2$, by (4.22) and (4.23), we have

$$x_1 - s_2 t - \mathbf{X}_2(t) \le \frac{5s_2+3s_1}{8}t - s_2 t + \frac{s_2-s_1}{4}t \\ = -\frac{s_2-s_1}{8}t < 0 \quad \text{if } x_1 \le \frac{5s_2+3s_1}{8}t.$$

Thus, it holds that

$$\begin{aligned}|\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}|(1-\eta_2) &= |\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}|^{\frac{1}{2}}|\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}|^{\frac{1}{2}}\eta_1 \\ &\le C|\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}|^{\frac{1}{2}}\delta_2\exp\Big(-C\delta_2\frac{|x_1-s_2t-\mathbf{X}_2(t)|}{2}\Big) \\ &\quad\times\exp\Big(-C\delta_2\frac{s_2-s_1}{16}t\Big)\eta_1\mathbf{1}_{\{x_1\le\frac{5s_2+3s_1}{8}t\}} \\ &\le C\delta_2|\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}|^{\frac{1}{2}}\exp(-C\delta_2 t).\end{aligned}$$

Integrating the above inequality with respect to x_1 over $\mathbb{R}$ gives

$$\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}|(1-\eta_2)\,\mathrm{d}x_1 \le C\delta_2 \exp(-C\delta_2 t)\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}|^{\frac{1}{2}}\,\mathrm{d}x_1 \\ \le C\delta_2\exp(-C\delta_2 t). \tag{4.33}$$

Therefore, we complete the proof of Lemma 4.4 by combining (4.32) and (4.33). ■

4.4. Proof of Proposition 4.1

In order to use the sharp Poincaré inequality (2.5), we should introduce a new variable y_1, such that the variable $x_1 \in \mathbb{R}$ is mapped onto a new variable $y_1 \in [0,1]$. Therefore, for $i=1$, we set

$$y_1 = \frac{p\big(\rho^{s_1}(x_1-s_1t-\mathbf{X}_1(t))\big)-p(\rho_-)}{\delta_1} = \frac{p((\rho^{s_1})^{-\mathbf{X}_1})-p(\rho_-)}{\delta_1}, \\ y' = (y_2,y_3) = (x_2,x_3) = x'. \tag{4.34}$$

For $i=2$, we set

$$y_1 = \frac{p(\rho_m)-p\big(\rho^{s_2}(x_1-s_2t-\mathbf{X}(t))\big)}{\delta_2} = \frac{p(\rho_m)-p((\rho^{s_2})^{-\mathbf{X}_2})}{\delta_2}, \\ y' = (y_2,y_3) = (x_2,x_3) = x'. \tag{4.35}$$

Then it follows from (4.6) that

$$\frac{\mathrm{d}y_1}{\mathrm{d}x_1} = \frac{(-1)^{i+1}}{\delta_i} p'((\rho^{s_i})^{-\mathbf{X}_i})\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i} = \frac{(-1)^i}{\sqrt{\delta_i}}\partial_{x_1} a_i^{-\mathbf{X}_i} > 0, \quad i = 1, 2. \tag{4.36}$$

To perform the sharp estimates, we will consider the $O(1)$-constants,

$$\sigma_m := \sqrt{-p'(v_m)} = \sqrt{\gamma v_m^{-\gamma-1}} = \sqrt{\gamma \rho_m^{\gamma+1}} \quad \text{with } v_m = 1/\rho_m,$$

and

$$\alpha_m := \frac{\gamma+1}{2}\frac{\rho_m}{\sigma_m},$$

which are indeed independent of the small constant δ_i $(i = 1, 2)$. Note that

$$|(-1)^i \sigma_i - \sigma_m| \le C\delta_i, \quad i = 1, 2, \tag{4.37}$$

which together with $\sigma_m^2 = -p'(v_m) = \gamma v_m^{-\gamma-1} = \gamma \rho_m^{\gamma+1}$ implies

$$|\sigma_m^2 - \gamma((\rho^{s_i})^{-\mathbf{X}_i})^{\gamma+1}| \le C\delta_i, \quad \left|\frac{1}{\sigma_m^2} - \frac{1}{\gamma((\rho^{s_i})^{-\mathbf{X}_i})^{\gamma+1}}\right| \le C\delta_i \quad i = 1, 2. \tag{4.38}$$

In the following lemmas, we use the superscript $(\cdot)^c$ to denote the cut-off part, the superscript $(\cdot)^e$ to denote the error terms, and the superscript $(\cdot)^r$ to denote the remaining parts. We also set

$$w_i := \eta_i(u_1 - (u_1^{s_i})^{-\mathbf{X}_i}), \quad i = 1, 2,$$

where η_i $(i = 1, 2)$ are cut-off functions defined in (4.30) and (4.31).

Lemma 4.5. *Let $\mathbf{X}_i(t)$, $\mathbf{Y}_i(t)$ $(i = 1, 2)$ be defined in Lemma 4.1. It holds that*

$$\begin{aligned}\sum_{i=1}^{2}\dot{\mathbf{X}}_i(t)\mathbf{Y}_i(t) \le \sum_{i=1}^{2}\Bigg[& -\delta_i \frac{M}{4}\frac{\rho_m^2}{\sigma_m^2}\left(\int_{\mathbb{T}^2}\int_0^1 w_i \,\mathrm{d}y_1\,\mathrm{d}y'\right)^2 \\ & + C\delta_i(\chi + \sqrt{\delta})^2 \int_{\mathbb{T}^2}\int_0^1 w_i^2 \,\mathrm{d}y_1\,\mathrm{d}y'\Bigg] \\ & + \sum_{i=1}^{2}[C\delta_i\delta_1\delta_2 \exp(-C\min\{\delta_1,\delta_2\}t) + C\delta_i \exp(-C\delta_i t)\|\psi_1\|_{H^2}^2] \\ & + \sum_{i=1}^{2}\left[-\frac{\delta_i}{4M}|\dot{\mathbf{X}}_i(t)|^2 + \frac{C}{\delta_i}\sum_{j=3}^{5}|\mathbf{Y}_{j,i}(t)|^2\right],\end{aligned} \tag{4.39}$$

Proof. In order to control the separate waves, we decompose $\mathbf{Y}_{j,i}(t)$ $(j = 1, 2, i = 1, 2)$ into

$$\mathbf{Y}_{j,i}(t) := \mathbf{Y}_{j,i}^c(t) + \mathbf{Y}_{j,i}^e(t) + \mathbf{Y}_{j,i}^r(t), \quad j = 1, 2, \ i = 1, 2,$$

$$\begin{cases} \mathbf{Y}^c_{1,i}(t) := \int_{\mathbb{T}^2}\int_{\mathbb{R}} w_i \partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i} a^{-\mathbf{X}} \rho \, dx_1\, dx', \\ \mathbf{Y}^e_{1,i}(t) := \int_{\mathbb{T}^2}\int_{\mathbb{R}} ((u_1^{s_i})^{-\mathbf{X}_i} - \bar{u}_1^{-\mathbf{X}})\eta_i \partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i} a^{-\mathbf{X}} \rho \, dx_1\, dx', \\ \mathbf{Y}^r_{1,i}(t) := \int_{\mathbb{T}^2}\int_{\mathbb{R}} (1-\eta_i)\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i} a^{-\mathbf{X}} \rho \psi_1 \, dx_1\, dx', \end{cases}$$

and

$$\begin{cases} \mathbf{Y}^c_{2,i}(t) := \int_{\mathbb{T}^2}\int_{\mathbb{R}} w_i \frac{\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}}{\sigma_i} a^{-\mathbf{X}} \bar{\rho}^{-\mathbf{X}} p'(\bar{\rho}^{-\mathbf{X}}) \, dx_1\, dx', \\ \mathbf{Y}^e_{2,i}(t) := \int_{\mathbb{T}^2}\int_{\mathbb{R}} ((u_1^{s_i})^{-\mathbf{X}_i} - \bar{u}_1^{-\mathbf{X}})\eta_i \frac{\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}}{\sigma_i} a^{-\mathbf{X}} \bar{\rho}^{-\mathbf{X}} p'(\bar{\rho}^{-\mathbf{X}}) \, dx_1\, dx', \\ \mathbf{Y}^r_{2,i}(t) := \int_{\mathbb{T}^2}\int_{\mathbb{R}} (1-\eta_i)\frac{\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}}{\sigma_i} a^{-\mathbf{X}} \bar{\rho}^{-\mathbf{X}} p'(\bar{\rho}^{-\mathbf{X}}) \psi_1 \, dx_1\, dx'. \end{cases}$$

We define

$$\dot{\mathbf{X}}^c_i(t) = -\frac{M}{\delta_i}(\mathbf{Y}^c_{1,i}(t) + \mathbf{Y}^c_{2,i}(t)). \tag{4.40}$$

• *Estimate on* $-\frac{\delta_i}{8M}|\dot{\mathbf{X}}^c_i(t)|^2$. To do this, we will control $\mathbf{Y}^c_{i,1}(t)$ and $\mathbf{Y}^c_{i,2}(t)$ due to (4.40). From (2.3), (2.4), and (4.8), it holds that

$$\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i} = -\sigma_i \partial_{x_1}(v^{s_i})^{-\mathbf{X}_i} = -\sqrt{\delta_i}\frac{\sigma_i}{\gamma((\rho^{s_i})^{-\mathbf{X}_i})^{\gamma+1}}\partial_{x_1} a_i^{-\mathbf{X}_i}, \quad i = 1, 2. \tag{4.41}$$

Then, we use (4.37), (4.38), (4.41), and the new variable (4.36) to have

$$\begin{aligned} \mathbf{Y}^c_{1,i}(t) &= -\int_{\mathbb{T}^2}\int_{\mathbb{R}} w_i \sqrt{\delta_i}\frac{\sigma_i}{\gamma((\rho^{s_i})^{-\mathbf{X}_i})^{\gamma+1}}\partial_{x_1} a_i^{-\mathbf{X}_i} a^{-\mathbf{X}}\rho \, dx_1\, dx' \\ &= -\frac{\sqrt{\delta_i}}{\sigma_m}\int_{\mathbb{T}^2}\int_{\mathbb{R}} w_i (-1)^i \partial_{x_1} a_i^{-\mathbf{X}_i} a^{-\mathbf{X}}\rho \, dx_1\, dx' \\ &\quad - \sqrt{\delta_i}\int_{\mathbb{T}^2}\int_{\mathbb{R}} w_i \Big(\frac{\sigma_i}{\gamma((\rho^{s_i})^{-\mathbf{X}_i})^{\gamma+1}} - \frac{(-1)^i\sigma_m}{\sigma_m^2}\Big)\partial_{x_1} a_i^{-\mathbf{X}_i} a^{-\mathbf{X}}\rho \, dx_1\, dx' \\ &\le -\frac{\delta_i}{\sigma_m}\int_{\mathbb{T}^2}\int_0^1 w_i a^{-\mathbf{X}}\rho \, dy_1\, dy' + C\delta_i^2 \int_{\mathbb{T}^2}\int_0^1 |w_i| \, dy_1\, dy', \end{aligned}$$

which together with $|a^{-\mathbf{X}} - 1| \le \sqrt{\delta_1} + \sqrt{\delta_2} \le \sqrt{2\delta}$ yields

$$\left|\mathbf{Y}^c_{1,i}(t) + \delta_i \frac{\rho_m}{\sigma_m}\int_{\mathbb{T}^2}\int_0^1 w_i \, dy_1\, dy'\right| \le C\delta_i(\chi + \sqrt{\delta})\int_{\mathbb{T}^2}\int_0^1 |w_i| \, dy_1\, dy'. \tag{4.42}$$

From (4.8), it holds that

$$\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i} = -\frac{\sqrt{\delta_i}}{p'((\rho^{s_i})^{-\mathbf{X}_i})}\partial_{x_1} a_i^{-\mathbf{X}_i}, \quad i = 1, 2. \tag{4.43}$$

Likewise, for $\mathbf{Y}^c_{2,i}(t)$, we use (4.43) and the change of variable (4.36) to have

$$\begin{aligned}\mathbf{Y}^c_{2,i}(t) &= -\sqrt{\delta_i}\int_{\mathbb{T}^2}\int_{\mathbb{R}} \frac{\partial_{x_1} a_i^{-\mathbf{X}_i}}{\sigma_i} w_i \frac{p'(\bar\rho^{-\mathbf{X}})}{p'((\rho^{s_i})^{-\mathbf{X}_i})} a^{-\mathbf{X}} \bar\rho^{-\mathbf{X}}\,\mathrm{d}x_1\,\mathrm{d}x' \\ &= -\sqrt{\delta_i}\frac{1}{\sigma_m}\int_{\mathbb{T}^2}\int_{\mathbb{R}} (-1)^i \partial_{x_1} a_i^{-\mathbf{X}_i} w_i a^{-\mathbf{X}}\bar\rho^{-\mathbf{X}}\,\mathrm{d}x_1\,\mathrm{d}x' \\ &\quad - \sqrt{\delta_i}\int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{x_1} a_i^{-\mathbf{X}_i} w_i \Big(\frac{p'(\bar\rho^{-\mathbf{X}})}{p'((\rho^{s_i})^{-\mathbf{X}_i})}\frac{1}{\sigma_i} - \frac{(-1)^i}{\sigma_m}\Big) a^{-\mathbf{X}}\bar\rho^{-\mathbf{X}}\,\mathrm{d}x_1\,\mathrm{d}x' \\ &\le -\frac{\delta_i}{\sigma_m}\int_{\mathbb{T}^2}\int_0^1 w_i a^{-\mathbf{X}}\bar\rho^{-\mathbf{X}}\,\mathrm{d}y_1\,\mathrm{d}y' + C\delta_i\delta\int_{\mathbb{T}^2}\int_0^1 |w_i|\,\mathrm{d}y_1\,\mathrm{d}y',\end{aligned}$$

which together with $|a^{-\mathbf{X}} - 1| \le \sqrt{2\delta}$ implies

$$\left|\mathbf{Y}^c_{2,i}(t) + \delta_i\frac{\rho_m}{\sigma_m}\int_{\mathbb{T}^2}\int_0^1 w_i\,\mathrm{d}y_1\,\mathrm{d}y'\right| \le C\delta_i\sqrt{\delta}\int_{\mathbb{T}^2}\int_0^1 |w_i|\,\mathrm{d}y_1\,\mathrm{d}y'. \tag{4.44}$$

By (4.40), (4.42), and (4.44), we have

$$\begin{aligned}\left|\dot{\mathbf{X}}^c_i(t) - 2M\frac{\rho_m}{\sigma_m}\int_{\mathbb{T}^2}\int_0^1 w_i\,\mathrm{d}y_1\,\mathrm{d}y'\right| &= \left|\frac{M}{\delta_i}\sum_{j=1}^2\Big(\mathbf{Y}^c_{j,i}(t) + \delta_i\frac{\rho_m}{\sigma_m}\int_{\mathbb{T}^2}\int_0^1 w_i\,\mathrm{d}y_1\,\mathrm{d}y'\Big)\right| \\ &\le C(\chi + \sqrt{\delta})\int_{\mathbb{T}^2}\int_0^1 |w_i|\,\mathrm{d}y_1\,\mathrm{d}y',\end{aligned}$$

which yields

$$\begin{aligned}\left(\left|2M\frac{\rho_m}{\sigma_m}\int_{\mathbb{T}^2}\int_0^1 w_i\,\mathrm{d}y_1\,\mathrm{d}y'\right| - |\dot{\mathbf{X}}^c_i(t)|\right)^2 &\le C(\chi + \sqrt{\delta})^2\Big(\int_{\mathbb{T}^2}\int_0^1 |w_i|\,\mathrm{d}y_1\,\mathrm{d}y'\Big)^2 \\ &\le C(\chi + \sqrt{\delta})^2\int_{\mathbb{T}^2}\int_0^1 w_i^2\,\mathrm{d}y_1\,\mathrm{d}y',\end{aligned}$$

which together with the algebraic inequality $\frac{p^2}{2} - q^2 \le (p-q)^2$ for all $p, q \ge 0$ indicates

$$2M^2\frac{\rho_m^2}{\sigma_m^2}\Big(\int_{\mathbb{T}^2}\int_0^1 w_i\,\mathrm{d}y_1\,\mathrm{d}y'\Big)^2 - |\dot{\mathbf{X}}^c_i(t)|^2 \le C(\chi + \sqrt{\delta})^2\int_{\mathbb{T}^2}\int_0^1 w_i^2\,\mathrm{d}y_1\,\mathrm{d}y'.$$

Thus, we can get

$$\begin{aligned}-\frac{\delta_i}{8M}|\dot{\mathbf{X}}^c_i(t)|^2 &\le -\delta_i\frac{M}{4}\frac{\rho_m^2}{\sigma_m^2}\Big(\int_{\mathbb{T}^2}\int_0^1 w_i\,\mathrm{d}y_1\,\mathrm{d}y'\Big)^2 \\ &\quad + C\delta_i(\chi + \sqrt{\delta})^2\int_{\mathbb{T}^2}\int_0^1 w_i^2\,\mathrm{d}y_1\,\mathrm{d}y'.\end{aligned} \tag{4.45}$$

• *Estimate on* $|\mathbf{Y}^e_{j,i}(t)|^2 + |\mathbf{Y}^r_{j,i}(t)|^2$. For $j = 1, 2$, $i = 1, 2$, using (4.19), it holds that

$$\frac{C}{\delta_i}|\mathbf{Y}^e_{j,i}(t)|^2 \le \frac{C}{\delta_i}\left(\int_{\mathbb{R}} |(u_1^{s_i})^{-\mathbf{X}_i} - \bar{u}_1^{-\mathbf{X}}||\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}|\,\mathrm{d}x_1\right)^2 \\ \le \frac{C}{\delta_i}(\delta_1\delta_2)^2\exp(-C\min\{\delta_1,\delta_2\}t). \tag{4.46}$$

We use Lemma 4.4 to have

$$\frac{C}{\delta_i}|\mathbf{Y}^r_{j,i}(t)|^2 \le \frac{C}{\delta_i}\left(\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\psi_1|(1-\eta_i)\,|\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}|\,\mathrm{d}x_1\,\mathrm{d}x'\right)^2 \\ \le \frac{C}{\delta_i}\|\psi_1\|^2_{L^\infty}\left(\int_{\mathbb{R}}(1-\eta_i)\,|\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}|\,\mathrm{d}x_1\right)^2 \\ \le C\delta_i\exp(-C\delta_i t)\|\psi_1\|^2_{H^2}. \tag{4.47}$$

By (4.18) and the Cauchy inequality, it holds that

$$\dot{\mathbf{X}}_i(t)\mathbf{Y}_i(t) = -\frac{\delta_i}{M}|\dot{\mathbf{X}}_i(t)|^2 + \dot{\mathbf{X}}_i(t)\sum_{j=3}^{5}\mathbf{Y}_{j,i}(t) \le -\frac{\delta_i}{2M}|\dot{\mathbf{X}}_i(t)|^2 + \frac{C}{\delta_i}\sum_{j=3}^{5}|\mathbf{Y}_{j,i}(t)|^2 \\ = -\frac{\delta_i}{4M}|\dot{\mathbf{X}}_i(t)|^2 - \frac{\delta_i}{4M}|\dot{\mathbf{X}}^c_i(t) + \dot{\mathbf{X}}_i(t) - \dot{\mathbf{X}}^c_i(t)|^2 + \frac{C}{\delta_i}\sum_{j=3}^{5}|\mathbf{Y}_{j,i}(t)|^2 \\ \le -\frac{\delta_i}{4M}|\dot{\mathbf{X}}_i(t)|^2 - \frac{\delta_i}{8M}|\dot{\mathbf{X}}^c_i(t)|^2 + C\delta_i|\dot{\mathbf{X}}_i(t) - \dot{\mathbf{X}}^c_i(t)|^2 + \frac{C}{\delta_i}\sum_{j=3}^{5}|\mathbf{Y}_{j,i}(t)|^2 \\ \le -\frac{\delta_i}{4M}|\dot{\mathbf{X}}_i(t)|^2 - \frac{\delta_i}{8M}|\dot{\mathbf{X}}^c_i(t)|^2 + \frac{C}{\delta_i}\sum_{j=1}^{2}(|\mathbf{Y}^e_{j,i}(t)|^2 + |\mathbf{Y}^r_{j,i}(t)|^2) \\ + \frac{C}{\delta_i}\sum_{j=3}^{5}|\mathbf{Y}_{j,i}(t)|^2. \tag{4.48}$$

Then we sum the above inequality with respect to i from 1 to 2, and use (4.45), (4.46), (4.47), and (4.48) to have the desired inequality (4.39). We complete the proof of Lemma 4.5. ■

Lemma 4.6. *Let* $\mathbf{B}_{j,i}(t)$ *(* $j = 2, 3$, $i = 1, 2$*) be defined in Lemma 4.1. It holds that*

$$\sum_{i=1}^{2}\sum_{j=2}^{3}\mathbf{B}_{j,i}(t) \le \sum_{i=1}^{2}\Bigg[\alpha_m\delta_i\int_{\mathbb{T}^2}\int_0^1 w_i^2\,\mathrm{d}y_1\,\mathrm{d}y' \\ + C(\kappa + \chi + \sqrt{\delta})\delta_i\int_{\mathbb{T}^2}\int_0^1 w_i^2\,\mathrm{d}y_1\,\mathrm{d}y'$$

$$+\frac{C}{\kappa}\delta_i\delta_1\delta_2\exp(-C\min\{\delta_1,\delta_2\}t)+C\delta_i\exp(-C\delta_i t)\|\psi_1\|_{H^2}^2$$
$$\left.+\frac{C\sqrt{\delta_i}}{\sqrt{\kappa}}\mathbf{G}_{3,i}(t)\right], \tag{4.49}$$

for some small positive constant κ.

Proof. For $\mathbf{B}_{2,i}(t)$ $(i=1,2)$, using the algebraic inequality $(a+b)^2\le\sqrt{1+\kappa}a^2+(1+\frac{1}{\sqrt{1+\kappa}-1})b^2$ for some small κ to be determined later, we decompose $\mathbf{B}_{2,i}(t)$ into

$$\begin{aligned}
\mathbf{B}_{2,i}(t)&\le-\frac{\sqrt{1+\kappa}}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}}a^{-\mathbf{X}}p''(\bar\rho^{-\mathbf{X}})\frac{(\bar\rho^{-\mathbf{X}})^4}{\sigma_i^2}\psi_1^2\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}\,\mathrm{d}x_1\,\mathrm{d}x'\\
&\quad-\frac{C}{\sqrt{\kappa}}\int_{\mathbb{T}^2}\int_{\mathbb{R}}a^{-\mathbf{X}}p''(\bar\rho^{-\mathbf{X}})\Big(\phi-\frac{(\bar\rho^{-\mathbf{X}})^2}{\sigma_i}\psi_1\Big)^2\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}\,\mathrm{d}x_1\,\mathrm{d}x'\\
&=\sqrt{1+\kappa}\frac{\gamma(\gamma-1)}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}}-a^{-\mathbf{X}}(\bar\rho^{-\mathbf{X}})^{\gamma+2}\frac{\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}}{\sigma_i^2}\eta_i^2\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x'\\
&\quad+\sqrt{1+\kappa}\underbrace{\frac{\gamma(\gamma-1)}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}}-a^{-\mathbf{X}}(\bar\rho^{-\mathbf{X}})^{\gamma+2}\frac{\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}}{\sigma_i^2}(1-\eta_i^2)\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x'}_{=:\mathbf{B}^r_{2,i}(t)}\\
&\quad+\frac{C}{\sqrt{\kappa}}\int_{\mathbb{T}^2}\int_{\mathbb{R}}-a^{-\mathbf{X}}p''(\bar\rho^{-\mathbf{X}})\Big(\phi-\frac{(\bar\rho^{-\mathbf{X}})^2}{\sigma_i}\psi_1\Big)^2\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}\,\mathrm{d}x_1\,\mathrm{d}x'.
\end{aligned}$$

For the first term on the right-hand side in the above inequality, we use the algebraic inequality $(a+b)^2\le\sqrt{1+\kappa}a^2+(1+\frac{1}{\sqrt{1+\kappa}-1})b^2$ for some small κ again to have

$$\begin{aligned}
&\sqrt{1+\kappa}\frac{\gamma(\gamma-1)}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}}-a^{-\mathbf{X}}(\bar\rho^{-\mathbf{X}})^{\gamma+2}\frac{\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}}{\sigma_i^2}\eta_i^2\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x'\\
&\quad\le(1+\kappa)\underbrace{\frac{\gamma(\gamma-1)}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}}-a^{-\mathbf{X}}(\bar\rho^{-\mathbf{X}})^{\gamma+2}\frac{\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}}{\sigma_i^2}w_i^2\,\mathrm{d}x_1\,\mathrm{d}x'}_{=:\mathbf{B}^c_{2,i}(t)}\\
&\quad+\frac{C}{\sqrt{\kappa}}\underbrace{\int_{\mathbb{T}^2}\int_{\mathbb{R}}-a^{-\mathbf{X}}(\bar\rho^{-\mathbf{X}})^{\gamma+2}\frac{\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}}{\sigma_i^2}\eta_i^2((u_1^{s_i})^{-\mathbf{X}_i}-\bar u_1^{-\mathbf{X}})^2\,\mathrm{d}x_1\,\mathrm{d}x'}_{=:\mathbf{B}^e_{2,i}(t)}.
\end{aligned}$$

Combining the above two inequalities yields

$$\mathbf{B}_{2,i}(t)\le(1+\kappa)\mathbf{B}^c_{2,i}(t)+\frac{C}{\sqrt{\kappa}}\mathbf{B}^e_{2,i}(t)+\sqrt{1+\kappa}\mathbf{B}^r_{2,i}(t)+\frac{C\sqrt{\delta_i}}{\sqrt{\kappa}}\mathbf{G}_{3,i}(t), \tag{4.50}$$

where we use (4.41) to have

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}}-a^{-\mathbf{X}}p''(\bar\rho^{-\mathbf{X}})\Big(\phi-\frac{(\bar\rho^{-\mathbf{X}})^2}{\sigma_i}\psi_1\Big)^2\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}\,\mathrm{d}x_1\,\mathrm{d}x'\le C\sqrt{\delta_i}\mathbf{G}_{3,i}(t).$$

Similarly to (4.50), it holds that

$$\mathbf{B}_{3,i}(t) \leq (1+\kappa)\mathbf{B}^c_{3,i}(t) + \frac{C}{\kappa}\mathbf{B}^e_{3,i}(t) + \mathbf{B}^r_{3,i}(t), \tag{4.51}$$

where

$$\begin{aligned}
\mathbf{B}^c_{3,i}(t) &:= -\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho w_i^2 \partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}\, \mathrm{d}x_1\, \mathrm{d}x',\\
\mathbf{B}^e_{3,i}(t) &:= -\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho \eta_i^2 ((u_1^{s_i})^{-\mathbf{X}_i} - \bar{u}_1^{-\mathbf{X}})^2 \partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}\, \mathrm{d}x_1\, \mathrm{d}x',\\
\mathbf{B}^r_{3,i}(t) &:= -\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho (1-\eta_i^2)\psi_1^2 \partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}\, \mathrm{d}x_1\, \mathrm{d}x'.
\end{aligned}$$

• *Estimate on* $|\mathbf{B}^c_{j,i}(t)|^2$ *($j = 2, 3$).* For $\mathbf{B}^c_{2,i}(t)$, using (4.41) and the change of variable (4.36), it holds that

$$\begin{aligned}
\mathbf{B}^c_{2,i}(t) &= \frac{\gamma-1}{2}\sqrt{\delta_i}\int_{\mathbb{T}^2}\int_{\mathbb{R}} \frac{\partial_{x_1} a_i^{-\mathbf{X}_i}}{\sigma_i} w_i^2 a^{-\mathbf{X}} \frac{\gamma(\bar{\rho}^{-\mathbf{X}})^{\gamma+2}}{\gamma((\rho^{s_i})^{-\mathbf{X}_i})^{\gamma+1}}\, \mathrm{d}x_1\, \mathrm{d}x'\\
&= \sqrt{\delta_i}\frac{\gamma-1}{2}\frac{\rho_m}{\sigma_m}\int_{\mathbb{T}^2}\int_{\mathbb{R}} (-1)^i \partial_{x_1} a_i^{-\mathbf{X}_i} w_i^2\, \mathrm{d}x_1\, \mathrm{d}x'\\
&\quad + \sqrt{\delta_i}\frac{\gamma-1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{x_1} a_i^{-\mathbf{X}_i} w_i^2 \left(a^{-\mathbf{X}} \frac{\gamma(\bar{\rho}^{-\mathbf{X}})^{\gamma+1}}{\gamma((\rho^{s_i})^{-\mathbf{X}_i})^{\gamma+1}} \frac{\bar{\rho}^{-\mathbf{X}}}{\sigma_i} - (-1)^i \frac{\rho_m}{\sigma_m}\right) \mathrm{d}x_1\, \mathrm{d}x'\\
&\leq \delta_i \frac{\gamma-1}{2}\frac{\rho_m}{\sigma_m}\int_{\mathbb{T}^2}\int_0^1 w_i^2\, \mathrm{d}y_1\, \mathrm{d}y' + C\delta_i\sqrt{\delta}\int_{\mathbb{T}^2}\int_0^1 w_i^2\, \mathrm{d}y_1\, \mathrm{d}y',
\end{aligned}$$

where in the last inequality we have used (4.37) and (4.38). For $\mathbf{B}^c_{3,i}(t)$, using (4.41) and the change of variable (4.36), it holds that

$$\begin{aligned}
\mathbf{B}^c_{3,i}(t) &= \sqrt{\delta_i}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho w_i^2 \partial_{x_1} a_i^{-\mathbf{X}_i} \frac{\sigma_i}{\gamma((\rho^{s_i})^{-\mathbf{X}_i})^{\gamma+1}}\, \mathrm{d}x_1\, \mathrm{d}x'\\
&= \sqrt{\delta_i}\frac{\rho_m}{\sigma_m}\int_{\mathbb{T}^2}\int_{\mathbb{R}} w_i^2 (-1)^i \partial_{x_1} a_i^{-\mathbf{X}_i}\, \mathrm{d}x_1\, \mathrm{d}x'\\
&\quad + \sqrt{\delta_i}\int_{\mathbb{T}^2}\int_{\mathbb{R}} w_i^2 \partial_{x_1} a_i^{-\mathbf{X}_i}\left(a^{-\mathbf{X}}\rho\frac{\sigma_i}{\gamma((\rho^{s_i})^{-\mathbf{X}_i})^{\gamma+1}} - \rho_m \frac{(-1)^i \sigma_m}{\sigma_m^2}\right) \mathrm{d}x_1\, \mathrm{d}x'\\
&\leq \delta_i \frac{\rho_m}{\sigma_m}\int_{\mathbb{T}^2}\int_0^1 w_i^2\, \mathrm{d}y_1\, \mathrm{d}y' + C\delta_i(\chi + \sqrt{\delta})\int_{\mathbb{T}^2}\int_0^1 w_i^2\, \mathrm{d}y_1\, \mathrm{d}y',
\end{aligned}$$

where in the last inequality we have used (4.37) and (4.38).

We add the above two inequalities to have

$$\begin{aligned}
\sum_{j=2}^{3} |\mathbf{B}^c_{j,i}(t)| &\leq \delta_i \frac{\gamma+1}{2}\frac{\rho_m}{\sigma_m}\int_{\mathbb{T}^2}\int_0^1 w_i^2\, \mathrm{d}y_1\, \mathrm{d}y' + C\delta_i(\chi+\sqrt{\delta})\int_{\mathbb{T}^2}\int_0^1 w_i^2\, \mathrm{d}y_1\, \mathrm{d}y'\\
&= \delta_i \alpha_m \int_{\mathbb{T}^2}\int_0^1 w_i^2\, \mathrm{d}y_1\, \mathrm{d}y' + C\delta_i(\chi+\sqrt{\delta})\int_{\mathbb{T}^2}\int_0^1 w_i^2\, \mathrm{d}y_1\, \mathrm{d}y'.
\end{aligned} \tag{4.52}$$

• *Estimate on* $|\mathbf{B}^e_{j,i}(t)|$ $(j=2,3)$. By the definition of $\mathbf{B}^e_{j,i}(t)$ $(j=2,3)$, we use (4.19) to have

$$\begin{aligned}\sum_{j=2}^{3}|\mathbf{B}^e_{j,i}(t)| &\le C\int_{\mathbb{R}}|\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}||(u_1^{s_i})^{-\mathbf{X}_i}-\bar{u}_1^{-\mathbf{X}}|^2\,\mathrm{d}x_1\\ &\le C\|(u_1^{s_i})^{-\mathbf{X}_i}-\bar{u}_1^{-\mathbf{X}}\|_{L^\infty(\mathbb{R})}\delta_1\delta_2\exp(-C\min\{\delta_1,\delta_2\}t)\\ &\le \begin{cases} C\delta_1\delta_2^2\exp(-C\min\{\delta_1,\delta_2\}t), & i=1,\\ C\delta_1^2\delta_2\exp(-C\min\{\delta_1,\delta_2\}t), & i=2.\end{cases}\end{aligned}\tag{4.53}$$

• *Estimate on* $|\mathbf{B}^r_{j,i}(t)|$ $(j=2,3)$. By the definition of $\mathbf{B}^r_{j,i}(t)$ $(j=2,3)$, we use Lemma 4.4 to have

$$\begin{aligned}\sum_{j=2}^{3}|\mathbf{B}^r_{j,i}(t)| &\le C\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}|(1-\eta_i)\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x'\\ &\le C\|\psi_1\|^2_{L^\infty}\int_{\mathbb{R}}|\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}|(1-\eta_i)\,\mathrm{d}x_1\\ &\le C\|\psi_1\|^2_{H^2}\delta_i\exp(-C\delta_i t).\end{aligned}\tag{4.54}$$

Finally, we use (4.50) and (4.51), and combine (4.52), (4.53), and (4.54) to have the desired inequality (4.49). We complete the proof of Lemma 4.6. ∎

Lemma 4.7. *Let* $\mathbf{D}(t)$ *be defined in Lemma 4.1, it holds that*

$$\begin{aligned}\mathbf{D}(t) \ge\ & 2\alpha_m\frac{1-C\delta}{1+\kappa}\sum_{i=1}^{2}\delta_i\int_{\mathbb{T}^2}\int_0^1 w_i^2\,\mathrm{d}y_1\,\mathrm{d}y'\\ &-2\alpha_m\frac{1-C\delta}{1+\kappa}\sum_{i=1}^{2}\delta_i\Big(\int_{\mathbb{T}^2}\int_0^1 w_i\,\mathrm{d}y_1\,\mathrm{d}y'\Big)^2\\ &-\frac{C}{\sqrt{\kappa}}\sum_{i=1}^{2}\Big(\frac{1}{t^2}\delta_i\exp(-C\delta_i t)+\delta_i^3\exp(-C\delta_i t)\Big)\\ &-\frac{C}{\sqrt{\kappa}}\frac{1}{t^2}\int_{\mathbb{T}^2}\int_{\mathbb{R}}a^{-\mathbf{X}}\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x',\end{aligned}\tag{4.55}$$

for some small positive constant κ.

Proof. • *Change of variable for* $\mathbf{D}(t)$. First, we rewrite $\mathbf{D}(t)$ as

$$\begin{aligned}\mathbf{D}(t) :=\ & \underbrace{(2\mu+\lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}}a^{-\mathbf{X}}(\partial_{x_1}\psi_1)^2\,\mathrm{d}x_1\,\mathrm{d}x'}_{=:\mathbf{D}_1(t)}+\underbrace{\mu\int_{\mathbb{T}^2}\int_{\mathbb{R}}a^{-\mathbf{X}}|\nabla_{x'}\psi_1|^2\,\mathrm{d}x_1\,\mathrm{d}x'}_{=:\mathbf{D}_2(t)}\\ &+\mu\int_{\mathbb{T}^2}\int_{\mathbb{R}}a^{-\mathbf{X}}|\nabla_x\psi'|^2\,\mathrm{d}x_1\,\mathrm{d}x'\\ &+(\mu+\lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}}a^{-\mathbf{X}}|\nabla_{x'}\cdot\psi'|^2\,\mathrm{d}x_1\,\mathrm{d}x'.\end{aligned}$$

Then recalling the cut-off functions η_i $(i = 1, 2)$ defined in (4.30) and (4.31), which satisfy $\eta_1 + \eta_2 = 1$ and $1 \geq \eta_i \geq 0$ for any $i = 1, 2$, we can separate $\mathbf{D}_1(t)$ into

$$\begin{aligned}\mathbf{D}_1(t) &= (2\mu+\lambda)\sum_{i=1}^{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\eta_i(\partial_{x_1}\psi_1)^2\,\mathrm{d}x_1\,\mathrm{d}x' \\ &\geq (2\mu+\lambda)\sum_{i=1}^{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\eta_i^2(\partial_{x_1}\psi_1)^2\,\mathrm{d}x_1\,\mathrm{d}x'.\end{aligned}$$

Using the algebraic inequality $(a+b)^2 \leq \sqrt{1+\kappa}a^2 + (1+\frac{1}{\sqrt{1+\kappa}-1})b^2$ for some small κ, it holds that

$$\begin{aligned}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}|\partial_{x_1}(\eta_i\psi_1)|^2\,\mathrm{d}x_1\,\mathrm{d}x' &= \int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}(\eta_i\partial_{x_1}\psi_1 + \partial_{x_1}\eta_i\psi_1)^2\,\mathrm{d}x_1\,\mathrm{d}x' \\ &\leq \sqrt{1+\kappa}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\eta_i^2(\partial_{x_1}\psi_1)^2\,\mathrm{d}x_1\,\mathrm{d}x' \\ &\quad + \frac{C}{\sqrt{\kappa}}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}(\partial_{x_1}\eta_i\psi_1)^2\,\mathrm{d}x_1\,\mathrm{d}x'.\end{aligned}$$

Thus, it follows from the above two inequalities that

$$\begin{aligned}\mathbf{D}_1(t) \geq{}& \frac{2\mu+\lambda}{\sqrt{1+\kappa}}\sum_{i=1}^{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}|\partial_{x_1}(\eta_i\psi_1)|^2\,\mathrm{d}x_1\,\mathrm{d}x' \\ &- \frac{C}{\sqrt{\kappa}}\sum_{i=1}^{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}(\partial_{x_1}\eta_i)^2\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x'.\end{aligned}$$

Note that since (4.22) gives

$$\frac{5s_2+3s_1}{8}t - \frac{3s_2+5s_1}{8}t = \frac{s_2-s_1}{4}t = \frac{\sigma_2-\sigma_1}{4\rho_m}t > 0,$$

we can deduce from (4.30) and (4.31) that for each $i = 1, 2$,

$$|\partial_{x_1}\eta_i| \leq \frac{4}{s_2-s_1}\frac{1}{t}, \quad \text{a.e. } x_1 \in \mathbb{R}, t > 0,$$

which yields

$$\begin{aligned}\mathbf{D}_1(t) \geq{}& \frac{2\mu+\lambda}{\sqrt{1+\kappa}}\sum_{i=1}^{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}|\partial_{x_1}(\eta_i\psi_1)|^2\,\mathrm{d}x_1\,\mathrm{d}x' \\ &- \frac{C}{\sqrt{\kappa}}\frac{1}{t^2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x'. \qquad (4.56)\end{aligned}$$

Using the algebraic inequality $(a+b)^2 \leq \sqrt{1+\kappa}a^2 + (1+\frac{1}{\sqrt{1+\kappa}-1})b^2$ for some small κ again, it holds that

$$\begin{aligned}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}(\partial_{x_1} w_i)^2 \,\mathrm{d}x_1\,\mathrm{d}x' &= \int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}|\partial_{x_1}(\eta_i(u_1-(u_1^{s_i})^{-\mathbf{X}_i}))|^2 \,\mathrm{d}x_1\,\mathrm{d}x' \\ &\leq \sqrt{1+\kappa}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}|\partial_{x_1}(\eta_i\psi_1)|^2 \,\mathrm{d}x_1\,\mathrm{d}x' \\ &\quad + \frac{C}{\sqrt{\kappa}}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}|\partial_{x_1}(\eta_i(\bar{u}_1^{-\mathbf{X}}-(u_1^{s_i})^{-\mathbf{X}_i}))|^2 \,\mathrm{d}x_1\,\mathrm{d}x',\end{aligned}$$

which implies

$$\begin{aligned}&\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}|\partial_{x_1}(\eta_i\psi_1)|^2 \,\mathrm{d}x_1\,\mathrm{d}x' \\ &\quad\geq \frac{1}{\sqrt{1+\kappa}}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}(\partial_{x_1} w_i)^2 \,\mathrm{d}x_1\,\mathrm{d}x' \\ &\quad\quad - \frac{C}{\sqrt{\kappa}}\underbrace{\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}|\partial_{x_1}(\eta_i(\bar{u}_1^{-\mathbf{X}}-(u_1^{s_i})^{-\mathbf{X}_i}))|^2 \,\mathrm{d}x_1\,\mathrm{d}x'}_{=:\mathbf{D}_i^e(t)}. \end{aligned} \tag{4.57}$$

For $i=1$, we have

$$\begin{aligned}\mathbf{D}_1^e(t) &\leq 2\underbrace{\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}(\partial_{x_1}\eta_1)^2\,|\bar{u}_1^{-\mathbf{X}}-(u_1^{s_1})^{-\mathbf{X}_1}|^2 \,\mathrm{d}x_1\,\mathrm{d}x'}_{=:\mathbf{D}_{1,1}^e(t)} \\ &\quad + 2\underbrace{\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\eta_1^2|\partial_{x_1}(u_1^{s_2})^{-\mathbf{X}_2}|^2 \,\mathrm{d}x_1\,\mathrm{d}x'}_{=:\mathbf{D}_{1,2}^e(t)}.\end{aligned}$$

Since (4.30) and (4.31) imply that for each $i=1,2$,

$$\partial_{x_1}\eta_i(x_1) = \partial_{x_1}\eta_i(x_1)\mathbf{1}_{\{\frac{3s_2+5s_1}{8}t\leq x_1\leq\frac{5s_2+3s_1}{8}t\}},$$

then, using

$$x_1 - s_2t - \mathbf{X}_2(t) \leq \frac{5s_2+3s_1}{8}t - s_2t + \frac{s_2-s_1}{4}t = -\frac{s_2-s_1}{8}t < 0, \quad x_1 \leq \frac{5s_2+3s_1}{8}t,$$

we have

$$\begin{aligned}|\partial_{x_1}\eta_1||\bar{u}_1^{-\mathbf{X}}-(u_1^{s_1})^{-\mathbf{X}_1}| &= |\partial_{x_1}\eta_1||(u_1^{s_2})^{-\mathbf{X}_2}-u_{1m}| \\ &\leq \frac{C}{t}\mathbf{1}_{\{\frac{3s_2+5s_1}{8}t\leq x_1\leq\frac{5s_2+3s_1}{8}t\}}\delta_2\exp(-C\delta_2|x_1-s_2t-\mathbf{X}_2(t)|)\exp(-C\delta_2 t),\end{aligned}$$

which implies

$$\begin{aligned}\mathbf{D}^e_{1,1}(t) &\le \frac{C}{t^2}\delta_2^2\exp(-C\delta_2 t)\int_{\mathbb{R}}\mathbf{1}_{\{\frac{3s_2+5s_1}{8}t\le x_1\le\frac{5s_2+3s_1}{8}t\}}\exp(-C\delta_2|x_1-s_2t-\mathbf{X}_2(t)|)\,\mathrm{d}x_1\\ &\le \frac{C}{t^2}\delta_2\exp(-C\delta_2 t).\end{aligned}$$

Using Lemma 4.4, it holds that

$$\mathbf{D}^e_{1,2}(t) \le C\|\partial_{x_1}(u_1^{s_2})^{-\mathbf{X}_2}\|_{L^\infty}\int_{\mathbb{R}}\eta_1\,|\partial_{x_1}(u_1^{s_2})^{-\mathbf{X}_2}|\,\mathrm{d}x_1 \le C\delta_2^3\exp(-C\delta_2 t).$$

Therefore, we have

$$\mathbf{D}^e_1(t) \le \frac{C}{t^2}\delta_2\exp(-C\delta_2 t) + C\delta_2^3\exp(-C\delta_2 t).$$

Likewise, we can get

$$\mathbf{D}^e_2(t) \le \frac{C}{t^2}\delta_1\exp(-C\delta_1 t) + C\delta_1^3\exp(-C\delta_1 t).$$

Combining the above two estimations and together with (4.56) and (4.57) yields

$$\begin{aligned}\mathbf{D}_1(t) \ge{}& \frac{2\mu+\lambda}{1+\kappa}\sum_{i=1}^{2}\int_{\mathbb{T}^2}\int_0^1(\partial_{y_1}w_i)^2\frac{\mathrm{d}y_1}{\mathrm{d}x_1}\,\mathrm{d}y_1\,\mathrm{d}y' - \frac{C}{\sqrt{\kappa}}\frac{1}{t^2}\int_{\mathbb{T}^2}\int_{\mathbb{R}}a^{-\mathbf{X}}\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x'\\ &- \frac{C}{\sqrt{\kappa}}\sum_{i=1}^{2}\Big(\frac{1}{t^2}\delta_i\exp(-C\delta_i t) + \delta_i^3\exp(-C\delta_i t)\Big),\end{aligned} \tag{4.58}$$

where we have used $a^{-\mathbf{X}} > 1$ and the new variable y_1.

In the following, in order to deal with the Jacobi transformation $\mathrm{d}y_1/\mathrm{d}x_1$, it is convenient to use the variable v^{s_i} $(= 1/\rho^{s_i})$ $(i = 1, 2)$ for the two viscous shock waves.

For $i = 1$, we integrate (2.3) with respect to ξ_1 over $(-\infty, \xi_1]$ to have

$$(2\mu+\lambda)\partial_{x_1}(v^{s_1})^{-\mathbf{X}_1} = \frac{-1}{\sigma_1}[\sigma_1^2((v^{s_1})^{-\mathbf{X}_1} - v_-) + p((v^{s_1})^{-\mathbf{X}_1}) - p(v_-)].$$

On the other hand, by (4.34), it holds that

$$\partial_{x_1}(v^{s_1})^{-\mathbf{X}_1} = \frac{\partial_{x_1}p((v^{s_1})^{-\mathbf{X}_1})}{p'((v^{s_1})^{-\mathbf{X}_1})} = \frac{\delta_1}{p'((v^{s_1})^{-\mathbf{X}_1})}\frac{\mathrm{d}y_1}{\mathrm{d}x_1}.$$

Hence, we have

$$(2\mu+\lambda)\frac{\delta_1}{p'((v^{s_1})^{-\mathbf{X}_1})}\frac{\mathrm{d}y_1}{\mathrm{d}x_1} = \frac{-1}{\sigma_1}[\sigma_1^2((v^{s_1})^{-\mathbf{X}_1} - v_-) + p((v^{s_1})^{-\mathbf{X}_1}) - p(v_-)],$$

which together with $\sigma_1^2 = -\frac{p(v_-)-p(v_m)}{v_--v_m}$ leads to

$$(2\mu+\lambda)\frac{\delta_1}{p'((v^{s_1})^{-\mathbf{X}_1})}\frac{\mathrm{d}y_1}{\mathrm{d}x_1} = \frac{-1}{\sigma_1}\frac{1}{v_- - v_m}\Big[\big(p(v_m) - p((v^{s_1})^{-\mathbf{X}_1})\big)((v^{s_1})^{-\mathbf{X}_1} - v_-) + \big(p((v^{s_1})^{-\mathbf{X}_1}) - p(v_-)\big)((v^{s_1})^{-\mathbf{X}_1} - v_m)\Big].$$

We rewrite the new variable (4.34) in the specific volume function as

$$y_1 = \frac{p((v^{s_1})^{-\mathbf{X}_1}) - p(v_-)}{\delta_1}, \quad 1 - y_1 = \frac{p(v_m) - p((v^{s_1})^{-\mathbf{X}_1})}{\delta_1}.$$

Thus, it holds that

$$\frac{1}{y_1(1-y_1)}\frac{2\mu+\lambda}{p'((v^{s_1})^{-\mathbf{X}_1})}\frac{\mathrm{d}y_1}{\mathrm{d}x_1} = \frac{-1}{\sigma_1}\frac{\delta_1}{v_- - v_m}\Big(\frac{(v^{s_1})^{-\mathbf{X}_1} - v_-}{p((v^{s_1})^{-\mathbf{X}_1}) - p(v_-)} + \frac{(v^{s_1})^{-\mathbf{X}_1} - v_m}{p(v_m) - p((v^{s_1})^{-\mathbf{X}_1})}\Big).$$

Then we get

$$\begin{aligned}
&\Big|\frac{1}{y_1(1-y_1)}\frac{2\mu+\lambda}{p'((v^{s_1})^{-\mathbf{X}_1})}\frac{\mathrm{d}y_1}{\mathrm{d}x_1} + \frac{\delta_1 p''(v_-)}{2\sigma_m(p'(v_-))^2}\Big| \\
&\quad\le \Big|\frac{1}{y_1(1-y_1)}\frac{2\mu+\lambda}{p'((v^{s_1})^{-\mathbf{X}_1})}\frac{\mathrm{d}y_1}{\mathrm{d}x_1} - \frac{\delta_1 p''(v_-)}{2\sigma_1(p'(v_-))^2}\Big| + \frac{\delta_1 p''(v_-)}{2(p'(v_-))^2}\Big|\frac{1}{\sigma_1} + \frac{1}{\sigma_m}\Big| \\
&\quad=: I_1 + I_2.
\end{aligned}$$

Using Lemma 2.3, we have

$$I_1 = \Big|\frac{-1}{\sigma_1}\frac{\delta_1}{v_- - v_m}\Big(\frac{(v^{s_1})^{-\mathbf{X}_1} - v_-}{p((v^{s_1})^{-\mathbf{X}_1}) - p(v_-)} + \frac{(v^{s_1})^{-\mathbf{X}_1} - v_m}{p(v_m) - p((v^{s_1})^{-\mathbf{X}_1})} + \frac{1}{2}\frac{p''(v_-)}{(p'(v_-))^2}(v_- - v_m)\Big)\Big| \le C\delta_1^2.$$

Since it follows from (4.37) that $I_2 \le C\delta_1^2$, therefore we can obtain

$$\Big|\frac{1}{y_1(1-y_1)}\frac{2\mu+\lambda}{p'((v^{s_1})^{-\mathbf{X}_1})}\frac{\mathrm{d}y_1}{\mathrm{d}x_1} + \frac{\delta_1 p''(v_-)}{2\sigma_m(p'(v_-))^2}\Big| \le C\delta_1^2.$$

Furthermore, using the above inequality and (4.37), we have

$$\begin{aligned}
&\Big|\frac{2\mu+\lambda}{y_1(1-y_1)}\frac{\mathrm{d}y_1}{\mathrm{d}x_1} + \frac{\delta_1}{2\sigma_m}\frac{p''(v_m)}{p'(v_m)}\Big| \\
&\quad\le \Big|\frac{2\mu+\lambda}{y_1(1-y_1)}\frac{\mathrm{d}y_1}{\mathrm{d}x_1} + \frac{\delta_1 p''(v_-)}{2\sigma_m(p'(v_-))^2}p'((v^{s_1})^{-\mathbf{X}_1})\Big| \\
&\qquad + \frac{\delta_1}{2\sigma_m}\Big|\frac{p''(v_-)}{(p'(v_-))^2}p'((v^{s_1})^{-\mathbf{X}_1}) - \frac{p''(v_m)}{p'(v_m)}\Big| \\
&\quad\le C\delta_1^2 + \frac{\delta_1}{2\sigma_m}\Big|\frac{p''(v_-)}{p'(v_-)} - \frac{p''(v_m)}{p'(v_m)}\Big| + \frac{\delta_1}{2\sigma_m}\frac{p''(v_m)}{|p'(v_m)|}\Big|\frac{p'((v^{s_1})^{-\mathbf{X}_1})}{p'(v_-)} - 1\Big| \\
&\quad\le C\delta_1^2.
\end{aligned} \tag{4.59}$$

Likewise, for $i = 2$, we integrate (2.4) with respect to ξ_2 over $(-\infty, \xi_2]$ to have

$$(2\mu + \lambda)\partial_{x_1}(v^{s_2})^{-\mathbf{X}_2} = \frac{-1}{\sigma_2}[\sigma_2^2((v^{s_2})^{-\mathbf{X}_2} - v_m) + p((v^{s_2})^{-\mathbf{X}_2}) - p(v_m)].$$

On the other hand, by (4.35), it holds that

$$\partial_{x_1}(v^{s_2})^{-\mathbf{X}_2} = \frac{\partial_{x_1} p((v^{s_2})^{-\mathbf{X}_2})}{p'((v^{s_2})^{-\mathbf{X}_2})} = \frac{-\delta_2}{p'((v^{s_2})^{-\mathbf{X}_2})}\frac{\mathrm{d}y_1}{\mathrm{d}x_1}.$$

Hence, we have

$$\begin{aligned}(2\mu+\lambda)\frac{\delta_2}{p'((v^{s_2})^{-\mathbf{X}_2})}\frac{\mathrm{d}y_1}{\mathrm{d}x_1} = \frac{1}{\sigma_2}\frac{1}{v_m - v_+}\Big[\big(p(v_+) - p((v^{s_2})^{-\mathbf{X}_2})\big)((v^{s_2})^{-\mathbf{X}_2} - v_m)\\ + \big(p((v^{s_2})^{-\mathbf{X}_2}) - p(v_m)\big)((v^{s_2})^{-\mathbf{X}_2} - v_+)\Big],\end{aligned}$$

with $\sigma_2^2 = -\frac{p(v_m)-p(v_+)}{v_m-v_+}$. We rewrite the new variable (4.35) in the specific volume function as

$$y_1 = \frac{p(v_m) - p((v^{s_2})^{-\mathbf{X}_2})}{\delta_2}, \quad 1 - y_1 = \frac{p((v^{s_2})^{-\mathbf{X}_2}) - p(v_+)}{\delta_2}.$$

Thus, it holds that

$$\begin{aligned}&\frac{1}{y_1(1-y_1)}\frac{2\mu+\lambda}{p'((v^{s_2})^{-\mathbf{X}_2})}\frac{\mathrm{d}y_1}{\mathrm{d}x_1}\\ &\quad= \frac{1}{\sigma_2}\frac{\delta_2}{v_m - v_+}\Big(\frac{(v^{s_2})^{-\mathbf{X}_2} - v_m}{p((v^{s_2})^{-\mathbf{X}_2}) - p(v_m)} + \frac{(v^{s_2})^{-\mathbf{X}_2} - v_+}{p(v_+) - p((v^{s_2})^{-\mathbf{X}_2})}\Big).\end{aligned}$$

Then we get

$$\begin{aligned}&\Big|\frac{1}{y_1(1-y_1)}\frac{2\mu+\lambda}{p'((v^{s_2})^{-\mathbf{X}_2})}\frac{\mathrm{d}y_1}{\mathrm{d}x_1} + \frac{\delta_2 p''(v_m)}{2\sigma_m(p'(v_m))^2}\Big|\\ &\quad\le \Big|\frac{1}{y_1(1-y_1)}\frac{2\mu+\lambda}{p'((v^{s_2})^{-\mathbf{X}_2})}\frac{\mathrm{d}y_1}{\mathrm{d}x_1} + \frac{\delta_2 p''(v_m)}{2\sigma_2(p'(v_m))^2}\Big| + \frac{\delta_2 p''(v_m)}{2(p'(v_m))^2}\Big|\frac{1}{\sigma_2} - \frac{1}{\sigma_m}\Big|\\ &\quad =: I_3 + I_4.\end{aligned}$$

Using Lemma 2.3, we have

$$\begin{aligned}I_3 = \Big|\frac{1}{\sigma_2}\frac{\delta_2}{v_m - v_+}\Big(&\frac{(v^{s_2})^{-\mathbf{X}_2} - v_m}{p((v^{s_2})^{-\mathbf{X}_2}) - p(v_m)} + \frac{(v^{s_2})^{-\mathbf{X}_2} - v_+}{p(v_+) - p((v^{s_2})^{-\mathbf{X}_2})}\\ &+ \frac{1}{2}\frac{p''(v_m)}{(p'(v_m))^2}(v_m - v_+)\Big)\Big| \le C\delta_2^2.\end{aligned}$$

Since it follows from (4.37) that $I_4 \le C\delta_2^2$, it holds that

$$\Big|\frac{1}{y_1(1-y_1)}\frac{2\mu+\lambda}{p'((v^{s_2})^{-\mathbf{X}_2})}\frac{\mathrm{d}y_1}{\mathrm{d}x_1} + \frac{\delta_2 p''(v_m)}{2\sigma_m(p'(v_m))^2}\Big| \le C\delta_2^2.$$

Furthermore, using the above inequality and (4.37), we have

$$\begin{aligned}&\Big|\frac{2\mu+\lambda}{y_1(1-y_1)}\frac{\mathrm{d}y_1}{\mathrm{d}x_1}+\frac{\delta_2}{2\sigma_m}\frac{p''(v_m)}{p'(v_m)}\Big|\\&\quad\le\Big|\frac{2\mu+\lambda}{y_1(1-y_1)}\frac{\mathrm{d}y_1}{\mathrm{d}x_1}+\frac{\delta_2 p''(v_m)}{2\sigma_m(p'(v_m))^2}p'((v^{s_2})^{-\mathbf{X}_2})\Big|\\&\qquad+\frac{\delta_2}{2\sigma_m}\frac{p''(v_m)}{|p'(v_m)|}\Big|\frac{p'((v^{s_2})^{-\mathbf{X}_2})}{p'(v_m)}-1\Big|\le C\delta_2^2.\end{aligned}\tag{4.60}$$

Since

$$\frac{1}{2\sigma_m}\frac{p''(v_m)}{-p'(v_m)}=\frac{\gamma+1}{2}\frac{v_m^{-1}}{\sigma_m}=\frac{\gamma+1}{2}\frac{\rho_m}{\sigma_m}=:\alpha_m,$$

hence, a combination of (4.59) and (4.60) yields

$$\Big|\frac{2\mu+\lambda}{y_1(1-y_1)}\frac{\mathrm{d}y_1}{\mathrm{d}x_1}-\delta_i\alpha_m\Big|\le C\delta_i^2\le C\delta_i\delta,\quad i=1,2,\tag{4.61}$$

which together with the three-dimensional weighted sharp Poincaré inequality (2.5) implies

$$\begin{aligned}&(2\mu+\lambda)\int_{\mathbb{T}^2}\int_0^1(\partial_{y_1}w_i)^2\frac{\mathrm{d}y_1}{\mathrm{d}x_1}\,\mathrm{d}y_1\,\mathrm{d}y'\\&\quad\ge\alpha_m\delta_i(1-C\delta)\int_{\mathbb{T}^2}\int_0^1 y_1(1-y_1)(\partial_{y_1}w_i)^2\,\mathrm{d}y_1\,\mathrm{d}y'\\&\quad\ge 2\alpha_m\delta_i(1-C\delta)\int_{\mathbb{T}^2}\int_0^1 w_i^2\,\mathrm{d}y_1\,\mathrm{d}y'-2\alpha_m\delta_i(1-C\delta)\Big(\int_{\mathbb{T}^2}\int_0^1 w_i\,\mathrm{d}y_1\,\mathrm{d}y'\Big)^2\\&\qquad-\frac{\alpha_m}{8\pi^2}\delta_i(1-C\delta)\int_{\mathbb{T}^2}\int_0^1\frac{|\nabla_{y'}w_i|^2}{y_1(1-y_1)}\,\mathrm{d}y_1\,\mathrm{d}y',\end{aligned}$$

where in the last inequality we have used the fact

$$\begin{aligned}\int_{\mathbb{T}^2}\int_0^1|w_i-\bar{w}_i|^2\,\mathrm{d}y_1\,\mathrm{d}y'&=\int_{\mathbb{T}^2}\int_0^1 w_i^2\,\mathrm{d}y_1\,\mathrm{d}y'-(\bar{w}_i)^2\\&=\int_{\mathbb{T}^2}\int_0^1 w_i^2\,\mathrm{d}y_1\,\mathrm{d}y'-\Big(\int_{\mathbb{T}^2}\int_0^1 w_i\,\mathrm{d}y_1\,\mathrm{d}y'\Big)^2.\end{aligned}$$

Thus, we can deduce from (4.58) that

$$\begin{aligned}\mathbf{D}_1(t)\ge{}&2\alpha_m\frac{1-C\delta}{1+\kappa}\sum_{i=1}^2\delta_i\int_{\mathbb{T}^2}\int_0^1 w_i^2\,\mathrm{d}y_1\,\mathrm{d}y'\\&-2\alpha_m\frac{1-C\delta}{1+\kappa}\sum_{i=1}^2\delta_i\Big(\int_{\mathbb{T}^2}\int_0^1 w_i\,\mathrm{d}y_1\,\mathrm{d}y'\Big)\end{aligned}$$

$$
\begin{aligned}
&- \frac{1-C\delta}{1+\kappa}\frac{\alpha_m}{8\pi^2}\sum_{i=1}^2 \delta_i \int_{\mathbb{T}^2}\int_0^1 \frac{|\nabla_{y'} w_i|^2}{y_1(1-y_1)}\,\mathrm{d}y_1\,\mathrm{d}y' \\
&- \frac{C}{\sqrt{\kappa}}\frac{1}{t^2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x' \\
&- \frac{C}{\sqrt{\kappa}}\sum_{i=1}^2\Big(\frac{1}{t^2}\delta_i\exp(-C\delta_i t) + \delta_i^3\exp(-C\delta_i t)\Big).
\end{aligned}
$$

On the other hand, it follows from (4.61) that

$$
y_1(1-y_1)\frac{\mathrm{d}x_1}{\mathrm{d}y_1} \ge \frac{2\mu+\lambda}{\delta_i\alpha_m + C\delta_i\delta} \ge \frac{2\mu+\lambda}{2\delta_i\alpha_m}.
$$

Hence, we use the fact that $\eta_1+\eta_2=1$ and $1\ge\eta_i\ge 0$ for any $i=1,2$, and the new variable y_1, so that $\mathbf{D}_2(t)$ can be transformed into

$$
\begin{aligned}
\mathbf{D}_2(t) &\ge \mu\sum_{i=1}^2\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\eta_i^2|\nabla_{x'}\psi_1|^2\,\mathrm{d}x_1\,\mathrm{d}x' = \mu\sum_{i=1}^2\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}|\nabla_{x'} w_i|^2\,\mathrm{d}x_1\,\mathrm{d}x' \\
&= \mu\sum_{i=1}^2\int_{\mathbb{T}^2}\int_0^1 a^{-\mathbf{X}}|\nabla_{y'} w_i|^2\frac{\mathrm{d}x_1}{\mathrm{d}y_1}\,\mathrm{d}y_1\,\mathrm{d}y' \\
&\ge \frac{\mu(2\mu+\lambda)}{2\alpha_m}\sum_{i=1}^2\frac{1}{\delta_i}\int_{\mathbb{T}^2}\int_0^1\frac{|\nabla_{y'} w_i|^2}{y_1(1-y_1)}\,\mathrm{d}y_1\,\mathrm{d}y'.
\end{aligned}
$$

Combining the estimates on $\mathbf{D}_1(t)$ and $\mathbf{D}_2(t)$, we have

$$
\begin{aligned}
\mathbf{D}(t) &\ge \mathbf{D}_1(t) + \mathbf{D}_2(t) \\
&\ge 2\alpha_m\frac{1-C\delta}{1+\kappa}\sum_{i=1}^2\delta_i\int_{\mathbb{T}^2}\int_0^1 w_i^2\,\mathrm{d}y_1\,\mathrm{d}y' \\
&\quad - 2\alpha_m\frac{1-C\delta}{1+\kappa}\sum_{i=1}^2\delta_i\Big(\int_{\mathbb{T}^2}\int_0^1 w_i\,\mathrm{d}y_1\,\mathrm{d}y'\Big)^2 \\
&\quad + \sum_{i=1}^2\Big(\frac{\mu(2\mu+\lambda)}{2\alpha_m\delta_i} - \frac{1-C\delta}{1+\kappa}\frac{\alpha_m\delta_i}{8\pi^2}\Big)\int_{\mathbb{T}^2}\int_0^1\frac{|\nabla_{y'} w_i|^2}{y_1(1-y_1)}\,\mathrm{d}y_1\,\mathrm{d}y' \\
&\quad - \frac{C}{\sqrt{\kappa}}\sum_{i=1}^2\Big(\frac{1}{t^2}\delta_i\exp(-C\delta_i t) + \delta_i^3\exp(-C\delta_i t)\Big) \\
&\quad - \frac{C}{\sqrt{\kappa}}\frac{1}{t^2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x',
\end{aligned}
$$

which implies (4.55) for suitably small δ_i, and the proof of Lemma 4.7 is completed. ■

We combine Lemmas 4.5, 4.6 and 4.7 to have the following lemma.

Lemma 4.8. *For $i = 1, 2$, let $\mathbf{X}_i(t)$, $\mathbf{Y}_i(t)$, $\mathbf{B}_{j,i}(t)$ ($j = 1, \ldots, 10$), $\mathbf{Y}_{j,i}(t)$ ($j = 1, \ldots, 5$), $\mathbf{G}^{s_i}(t)$, and $\mathbf{D}(t)$ be as defined in Lemma 4.1. Then it holds that*

$$
\begin{aligned}
&\sum_{i=1}^{2}\Big(\dot{\mathbf{X}}_i(t)\mathbf{Y}_i(t) + \sum_{j=2}^{3}\mathbf{B}_{j,i}(t)\Big) - \frac{3}{4}\mathbf{D}(t) \\
&\quad \le \sum_{i=1}^{2}\Big[-C_1\mathbf{G}^{s_i}(t) - \frac{\delta_i}{4M}|\dot{\mathbf{X}}_i(t)|^2 + \frac{C}{\delta_i}\sum_{j=3}^{5}|\mathbf{Y}_{j,i}(t)|^2\Big] \\
&\qquad + C\sum_{i=1}^{2}[\delta_i\delta_1\delta_2\exp(-C\min\{\delta_1,\delta_2\}t) + \delta_i\exp(-C\delta_i t)\|\psi_1\|_{H^2}^2 + \sqrt{\delta_i}\mathbf{G}_{3,i}(t)] \\
&\qquad + C\sum_{i=1}^{2}\Big(\frac{1}{t^2}\delta_i\exp(-C\delta_i t) + \delta_i^3\exp(-C\delta_i t)\Big) \\
&\qquad + \frac{C}{t^2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x,
\end{aligned}
\tag{4.62}
$$

where $C_1 = \frac{\alpha_m}{8}\min\{p'(\rho_-), p'(\rho_+)\}$ and $\mathbf{G}^{s_i}(t) = \int_{\mathbb{T}^2}\int_{\mathbb{R}}|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}|\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x'$.

Proof. We choose κ, χ, δ suitably small in Lemmas 4.5–4.7 such that

$$
\frac{1 - C\delta}{1+\kappa} \le \frac{8}{9}, \qquad C(\kappa + \chi + \sqrt{\delta}) + C(\chi + \sqrt{\delta})^2 \le \frac{\alpha_m}{12},
$$

and set

$$
M = \frac{16}{3}\frac{\sigma_m^2}{\rho_m^2}\alpha_m = (\gamma + 1)\frac{8}{3}\frac{\sigma_m}{\rho_m}.
$$

Then a combination of (4.39), (4.49), and (4.55) yields

$$
\begin{aligned}
&\sum_{i=1}^{2}\Big(\dot{\mathbf{X}}_i(t)\mathbf{Y}_i(t) + \sum_{j=2}^{3}\mathbf{B}_{j,i}(t)\Big) - \frac{3}{4}\mathbf{D}(t) \\
&\quad \le \sum_{i=1}^{2}\Big[-\frac{\alpha_m}{4}\delta_i\int_{\mathbb{T}^2}\int_0^1 w_i^2\,\mathrm{d}y_1\,\mathrm{d}y' - \frac{\delta_i}{4M}|\dot{\mathbf{X}}_i(t)|^2 + \frac{C}{\delta_i}\sum_{j=3}^{5}|\mathbf{Y}_{j,i}(t)|^2\Big] \\
&\qquad + C\sum_{i=1}^{2}[\delta_i\delta_1\delta_2\exp(-C\min\{\delta_1,\delta_2\}t) + \delta_i\exp(-C\delta_i t)\|\psi_1\|_{H^2}^2 + \sqrt{\delta_i}\mathbf{G}_{3,i}(t)] \\
&\qquad + C\sum_{i=1}^{2}\Big(\frac{1}{t^2}\delta_i\exp(-C\delta_i t) + \delta_i^3\exp(-C\delta_i t)\Big) \\
&\qquad + \frac{C}{t^2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x.
\end{aligned}
\tag{4.63}
$$

Using the change of variable (4.36), it holds that

$$\delta_i \int_{\mathbb{T}^2}\int_0^1 w_i^2 \,\mathrm{d}y_1\,\mathrm{d}y' = \int_{\mathbb{T}^2}\int_{\mathbb{R}} p'((\rho^{s_i})^{-\mathbf{X}_i})|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}| w_i^2 \,\mathrm{d}x_1\,\mathrm{d}x'$$
$$\geq \min\{p'(\rho_-), p'(\rho_+)\}\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}| w_i^2 \,\mathrm{d}x_1\,\mathrm{d}x'.$$

Note that

$$\begin{aligned}
&\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}| \psi_1^2 \,\mathrm{d}x_1\,\mathrm{d}x' \\
&\quad= \int_{\mathbb{T}^2}\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}| \eta_i^2\psi_1^2 \,\mathrm{d}x_1\,\mathrm{d}x' + \int_{\mathbb{T}^2}\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}| (1-\eta_i^2)\psi_1^2 \,\mathrm{d}x_1\,\mathrm{d}x' \\
&\quad\leq 2\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}| w_i^2 \,\mathrm{d}x_1\,\mathrm{d}x' \\
&\qquad + \underbrace{2\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}|\, \eta_i^2 |((u_1^{s_i})^{-\mathbf{X}_i} - \bar{u}_1^{-\mathbf{X}})|^2 \,\mathrm{d}x_1\,\mathrm{d}x'}_{=:\mathbf{G}^e(t)} \\
&\qquad + \underbrace{\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}| (1-\eta_i^2)\psi_1^2 \,\mathrm{d}x_1\,\mathrm{d}x'}_{=:\mathbf{G}^r(t)},
\end{aligned}$$

which yields

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}| w_i^2 \,\mathrm{d}x_1\,\mathrm{d}x' \geq \frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}| \psi_1^2 \,\mathrm{d}x_1\,\mathrm{d}x' - \mathbf{G}^e(t) - \frac{1}{2}\mathbf{G}^r(t).$$

Similarly to (4.53), we use (4.19) to obtain

$$|\mathbf{G}^e(t)| \leq \begin{cases} C\delta_1\delta_2^2\exp(-C\min\{\delta_1,\delta_2\}t), & i=1, \\ C\delta_1^2\delta_2\exp(-C\min\{\delta_1,\delta_2\}t), & i=2. \end{cases}$$

Similarly to (4.54), we use Lemma 4.4 to obtain

$$|\mathbf{G}^r(t)| \leq C\|\psi_1\|_{H^2}^2\delta_i\exp(-C\delta_i t).$$

Therefore, we can obtain

$$\begin{aligned}
&-\frac{\alpha_m}{4}\sum_{i=1}^2\delta_i\int_{\mathbb{T}^2}\int_0^1 w_i^2\,\mathrm{d}y_1\,\mathrm{d}y' \\
&\qquad\leq -\frac{\alpha_m}{4}\frac{\min\{p'(\rho_-),p'(\rho_+)\}}{2}\sum_{i=1}^2\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}| \psi_1^2 \,\mathrm{d}x_1\,\mathrm{d}x' \\
&\qquad\quad + C\sum_{i=1}^2\big(\delta_i\delta_1\delta_2\exp(-C\min\{\delta_1,\delta_2\})t + \delta_i\exp(-C\delta_i t)\|\psi_1\|_{H^2}^2\big),
\end{aligned}$$

which together with (4.63) yields the desired inequality (4.62). We complete the proof of Lemma 4.8. ∎

Proof of Proposition 4.1. First of all, we use (4.9) and Lemma 4.8 to have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho\Big(\Phi(\rho,\bar\rho^{-\mathbf{X}})+\frac{|\psi|^2}{2}\Big)\,\mathrm{d}x_1\,\mathrm{d}x' \\
&\quad=\sum_{i=1}^{2}\Big(\dot{\mathbf{X}}_i(t)\mathbf{Y}_i(t)+\sum_{j=2}^{3}\mathbf{B}_{j,i}(t)\Big)-\frac{3}{4}\mathbf{D}(t) \\
&\qquad+\sum_{i=1}^{2}\Big(\mathbf{B}_{1,i}(t)-\mathbf{G}_{1,i}(t)+\sum_{j=4}^{10}\mathbf{B}_{j,i}(t)-\sum_{j=2}^{3}\mathbf{G}_{j,i}(t)\Big)-\frac{1}{4}\mathbf{D}(t).
\end{aligned}\tag{4.64}
$$

By Lemma 4.8, in what follows, we will use the good terms $\mathbf{G}_{j,i}(t)$ $(j=2,3)$, $\mathbf{D}(t)$, and $\mathbf{G}^{s_i}(t)$ to control the bad terms $\mathbf{B}_{j,i}(t)$ $(j=1,4,\dots,10,\ i=1,2)$ and $\mathbf{Y}_{j,i}(t)$ $(j=3,4,5,\ i=1,2)$.

• *Estimates for* $\mathbf{B}_{j,i}(t)$ *(*$j=1,4,\dots,10$*).* Using (4.37), (4.38), and the change of variable (4.36), we have

$$
\begin{aligned}
\mathbf{B}_{1,i}(t)&=\frac{\sigma_m}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}}(-1)^i\partial_{x_1}a_i^{-\mathbf{X}_i}\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x' \\
&\quad+\frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\Big(\frac{\gamma(\bar\rho^{-\mathbf{X}})^{\gamma+1}}{\sigma_i}-(-1)^i\frac{\sigma_m^2}{\sigma_m}\Big)\partial_{x_1}a_i^{-\mathbf{X}_i}\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x' \\
&\le\sqrt{\delta_i}\frac{\sigma_m}{2}\int_{\mathbb{T}^2}\int_0^1\psi_1^2\,\mathrm{d}y_1\,\mathrm{d}y'+C(\delta_1+\delta_2)\sqrt{\delta_i}\int_{\mathbb{T}^2}\int_0^1\psi_1^2\,\mathrm{d}y_1\,\mathrm{d}y'.
\end{aligned}
$$

Furthermore, we use the change of variable (4.36) and (4.37) to obtain

$$
\begin{aligned}
\mathbf{G}_{1,i}(t)&=\frac{\sigma_m}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}}(-1)^i\partial_{x_1}a_i^{-\mathbf{X}_i}\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x' \\
&\quad+\frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}}(\sigma_i-(-1)^i\sigma_m)\partial_{x_1}a_i^{-\mathbf{X}_i}\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x' \\
&\ge\sqrt{\delta_i}\frac{\sigma_m}{2}\int_{\mathbb{T}^2}\int_0^1\psi_1^2\,\mathrm{d}y_1\,\mathrm{d}y'-C\delta_i\sqrt{\delta_i}\int_{\mathbb{T}^2}\int_0^1\psi_1^2\,\mathrm{d}y_1\,\mathrm{d}y',
\end{aligned}
$$

which together with the above inequality and the change of variable (4.36) yields

$$
\begin{aligned}
\mathbf{B}_{1,i}(t)-\mathbf{G}_{1,i}(t)&\le C(\delta_1+\delta_2)\sqrt{\delta_i}\int_{\mathbb{T}^2}\int_0^1\psi_1^2\,\mathrm{d}y_1\,\mathrm{d}y' \\
&\le C\sqrt{\delta}\sum_{i=1}^{2}\delta_i\int_{\mathbb{T}^2}\int_0^1\psi_1^2\,\mathrm{d}y_1\,\mathrm{d}y' \\
&\le C\sqrt{\delta}\sum_{i=1}^{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}}p'((\rho^{s_i})^{-\mathbf{X}_i})|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}|\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x' \\
&\le C\sqrt{\delta}\sum_{i=1}^{2}\mathbf{G}^{s_i}(t).
\end{aligned}\tag{4.65}
$$

For $\mathbf{B}_{4,i}(t)$ $(i=1,2)$, it holds that

$$
\begin{aligned}
|\mathbf{B}_{4,i}(t)| &\le C\int_{\mathbb{T}^2}\int_{\mathbb{R}}|(\phi,\psi_1)||\partial_{x_1}a_i^{-\mathbf{X}_i}|\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x' + C\|(\phi,\psi_1)\|_{L^\infty}(\mathbf{G}_{2,i}(t)+\mathbf{G}_{3,i}(t))\\
&\le C\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\partial_{x_1}a_i^{-\mathbf{X}_i}||\psi_1|^3\,\mathrm{d}x_1\,\mathrm{d}x'\\
&\quad + C\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\partial_{x_1}a_i^{-\mathbf{X}_i}|\left|\phi-\frac{(\bar\rho^{-\mathbf{X}})^2}{\sigma_i}\psi_1\right|\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x' + C\chi(\mathbf{G}_{2,i}(t)+\mathbf{G}_{3,i}(t)),
\end{aligned}
$$

where using the three-dimensional Gagliardo–Nirenberg inequality (2.6) and (4.43), we obtain

$$
\begin{aligned}
C\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\partial_{x_1}a_i^{-\mathbf{X}_i}||\psi_1|^3\,\mathrm{d}x_1\,\mathrm{d}x' &\le C\|\psi_1\|_{L^\infty}^2\frac{1}{\sqrt{\delta_i}}\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}||\psi_1|\,\mathrm{d}x_1\,\mathrm{d}x'\\
&\le C(\|\psi_1\|\|\partial_{x_1}\psi_1\| + \|\nabla_x\psi_1\|\|\nabla_x^2\psi_1\|)\frac{1}{\sqrt{\delta_i}}\sqrt{\mathbf{G}^{s_i}(t)}\left(\int_{\mathbb{R}}|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}|\,\mathrm{d}x_1\right)^{\frac12}\\
&\le C\frac{1}{\sqrt{\delta_i}}(\|\psi_1\|+\|\nabla_x^2\psi_1\|)\|\nabla_x\psi_1\|\sqrt{\mathbf{G}^{s_i}(t)} \le C\chi(\|\nabla_x\psi_1\|^2+\mathbf{G}^{s_i}(t)).
\end{aligned}
$$

Likewise, it holds that

$$
\begin{aligned}
&C\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\partial_{x_1}a_i^{-\mathbf{X}_i}|\left|\phi-\frac{(\bar\rho^{-\mathbf{X}})^2}{\sigma_i}\psi_1\right|\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x'\\
&\quad\le C\|\psi_1\|_{L^\infty}^2\sqrt{\mathbf{G}_{3,i}(t)}\left(\int_{\mathbb{R}}|\partial_{x_1}a_i^{-\mathbf{X}_i}|\,\mathrm{d}x_1\right)^{\frac12}\\
&\quad\le C(\|\psi_1\|\|\partial_{x_1}\psi_1\|+\|\nabla_x\psi_1\|\|\nabla_x^2\psi_1\|)\delta_i^{\frac14}\sqrt{\mathbf{G}_{3,i}(t)} \le C\chi(\|\nabla_x\psi_1\|^2+\mathbf{G}_{3,i}(t)).
\end{aligned}
$$

Hence, we have

$$
|\mathbf{B}_{4,i}(t)| \le C\chi(\|\nabla_x\psi_1\|^2+\mathbf{G}^{s_i}(t)+\mathbf{G}_{2,i}(t)+\mathbf{G}_{3,i}(t)).
$$

For $\mathbf{B}_{5,i}(t)$, it holds that

$$
\begin{aligned}
|\mathbf{B}_{5,i}(t)| &\le C\int_{\mathbb{T}^2}\int_{\mathbb{R}}(|\bar\rho^{-\mathbf{X}}-(\rho^{s_i})^{-\mathbf{X}_i}| + |(\bar\rho\bar u_1)^{-\mathbf{X}}-(\rho^{s_i}u_1^{s_i})^{-\mathbf{X}_i}|)\\
&\qquad\qquad\times|\partial_{x_1}a_i^{-\mathbf{X}_i}|(\phi^2+|\psi|^2)\,\mathrm{d}x_1\,\mathrm{d}x'\\
&\le C\delta\int_{\mathbb{T}^2}\int_{\mathbb{R}}(-1)^i\partial_{x_1}a_i^{-\mathbf{X}_i}\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x' + C\delta(\mathbf{G}_{2,i}(t)+\mathbf{G}_{3,i}(t)),
\end{aligned}
$$

where the first term in the last inequality can be treated similarly to (4.65):

$$
\begin{aligned}
&C\delta\int_{\mathbb{T}^2}\int_{\mathbb{R}}(-1)^i\partial_{x_1}a_i^{-\mathbf{X}_i}\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x'\\
&\quad= C\delta\sqrt{\delta_i}\int_{\mathbb{T}^2}\int_0^1\psi_1^2\,\mathrm{d}y_1\,\mathrm{d}y' \le C\sqrt{\delta}\sum_{i=1}^2\delta_i\int_{\mathbb{T}^2}\int_0^1\psi_1^2\,\mathrm{d}y_1\,\mathrm{d}y'
\end{aligned}
$$

$$
\begin{aligned}
&= C\sqrt{\delta}\sum_{i=1}^{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} p'((\rho^{s_i})^{-\mathbf{X}_i})|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}|\psi_1^2 \,\mathrm{d}x_1\,\mathrm{d}x' \\
&\le C\sqrt{\delta}\sum_{i=1}^{2}\mathbf{G}^{s_i}(t).
\end{aligned}
$$

Hence, it holds that

$$
|\mathbf{B}_{5,i}(t)| \le C\sqrt{\delta}\sum_{i=1}^{2}\mathbf{G}^{s_i}(t) + C\delta(\mathbf{G}_{2,i}(t) + \mathbf{G}_{3,i}(t)).
$$

Similarly to $\mathbf{B}_{4,i}(t)$, it holds that

$$
|\mathbf{B}_{6,i}(t)| + |\mathbf{B}_{7,i}(t)| \le C\chi(\|\nabla_x\psi_1\|^2 + \mathbf{G}^{s_i}(t) + \mathbf{G}_{3,i}(t)).
$$

By (4.43) and the Cauchy inequality, it holds that

$$
\begin{aligned}
|\mathbf{B}_{8,i}(t)| &\le \frac{C}{\sqrt{\delta_i}}\|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}}\|\nabla_x\psi\|\sqrt{\mathbf{G}^{s_i}(t)} \\
&\quad + C\|\partial_{x_1}a_i^{-\mathbf{X}_i}\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}}\|\nabla_x\psi\|\sqrt{\mathbf{G}_{2,i}(t)} \\
&\le C\sqrt{\delta_i}\|\nabla_x\psi\|\sqrt{\mathbf{G}^{s_i}(t)} + C\delta_i^{\frac{3}{4}}\|\nabla_x\psi\|\sqrt{\mathbf{G}_{2,i}(t)} \\
&\le C\sqrt{\delta_i}(\|\nabla_x\psi\|^2 + \mathbf{G}^{s_i}(t) + \mathbf{G}_{2,i}(t)).
\end{aligned}
$$

Using the Cauchy inequality and (4.41), we have

$$
\begin{aligned}
|\mathbf{B}_{9,i}(t)| &\le C\delta_i\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}|\left|\phi - \frac{(\bar{\rho}^{-\mathbf{X}})^2}{\sigma_i}\psi_1\right||\psi_1|\,\mathrm{d}x_1\,\mathrm{d}x' \\
&\quad + C\delta_i\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}|\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x' \\
&\le C\delta_i\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}|\left|\phi - \frac{(\bar{\rho}^{-\mathbf{X}})^2}{\sigma_i}\psi_1\right|^2\mathrm{d}x_1\,\mathrm{d}x' + C\delta_i\mathbf{G}^{s_i}(t) \\
&\le C\delta_i(\mathbf{G}_{3,i}(t) + \mathbf{G}^{s_i}(t)).
\end{aligned}
$$

Using (4.20) and the Hölder inequality, we can get

$$
\begin{aligned}
|\mathbf{B}_{10,i}(t)| &\le C\|(\phi,\psi_1)\|\left(\int_{\mathbb{R}}|\bar{\rho}^{-\mathbf{X}} - (\rho^{s_i})^{-\mathbf{X}_i}|^2|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}|^2\,\mathrm{d}x_1\right)^{\frac{1}{2}} \\
&\le C\chi(\delta_i(\delta_1\delta_2)^2\exp(-C\min\{\delta_1,\delta_2\}t))^{\frac{1}{2}} \\
&\le C\sqrt{\delta_i}\delta_1\delta_2\exp(-C\min\{\delta_1,\delta_2\}t).
\end{aligned}
$$

Thus, we combine the above estimates and choose χ and δ_i $(i = 1, 2)$ suitably small such that

$$\sum_{i=1}^{2}\Big(\mathbf{B}_{1,i}(t) - \mathbf{G}_{1,i}(t) + \sum_{j=4}^{10}\mathbf{B}_{j,i}(t)\Big) \le \frac{1}{16}\sum_{i=1}^{2}(\mathbf{G}^{s_i}(t) + \mathbf{G}_{2,i}(t) + \mathbf{G}_{3,i}(t)) + \frac{1}{16}\mathbf{D}(t) + C\delta_1\delta_2\exp(-C\min\{\delta_1,\delta_2\}t)\sum_{i=1}^{2}\sqrt{\delta_i}. \tag{4.66}$$

• *Estimates for* $\mathbf{Y}_{j,i}(t)$ $(j = 3, 4, 5)$. For $\mathbf{Y}_{3,i}(t)$, using (4.43), it holds that

$$\frac{C}{\delta_i}|\mathbf{Y}_{3,i}(t)|^2 \le \frac{C}{\delta_i}\delta_i\Big(\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\partial_{x_1}a_i^{-\mathbf{X}_i}|\Big|\phi - \frac{(\bar\rho^{-\mathbf{X}})^2}{\sigma_i}\psi_1\Big|\,\mathrm{d}x_1\,\mathrm{d}x'\Big)^2 \le C\Big(\int_{\mathbb{R}}|\partial_{x_1}a_i^{-\mathbf{X}_i}|\,\mathrm{d}x_1\Big)\mathbf{G}_{3,i}(t) \le C\sqrt{\delta_i}\mathbf{G}_{3,i}(t).$$

For $\mathbf{Y}_{4,i}(t)$, it holds that

$$\frac{C}{\delta_i}|\mathbf{Y}_{4,i}(t)| \le \frac{C}{\delta_i}\mathbf{G}_{3,i}(t)\|\partial_{x_1}a_i^{-\mathbf{X}_i}\|_{L^\infty(\mathbb{R})}\|(\phi,\psi_1)\|^2 \le C\chi^2\sqrt{\delta_i}\mathbf{G}_{3,i}(t).$$

For $\mathbf{Y}_{5,i}(t)$, it holds that

$$\frac{C}{\delta_i}|\mathbf{Y}_{5,i}(t)|^2 \le \frac{C}{\delta_i^2}\|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}\|_{L^\infty(\mathbb{R})}\|\psi_1\|^2\mathbf{G}^{s_i}(t) + \frac{C}{\delta_i}\|\partial_{x_1}a_i^{-\mathbf{X}_i}\|_{L^\infty(\mathbb{R})}\|\psi'\|^2\mathbf{G}_{2,i}(t) + \frac{C}{\delta_i}\chi^2\|\partial_{x_1}a_i^{-\mathbf{X}_i}\|_{L^\infty(\mathbb{R})}\|(\phi,\psi_1)\|^2\mathbf{G}_{3,i}(t) \le C\chi^2(\mathbf{G}^{s_i}(t) + \mathbf{G}_{2,i}(t) + \mathbf{G}_{3,i}(t)).$$

Thus, we combine the above estimates and choose χ and δ_i $(i = 1, 2)$ suitably small to obtain

$$\sum_{i=1}^{2}\sum_{j=3}^{5}\frac{C}{\delta_i}|\mathbf{Y}_{j,i}(t)|^2 \le \frac{1}{16}\sum_{i=1}^{2}(\mathbf{G}^{s_i}(t) + \mathbf{G}_{2,i}(t) + \mathbf{G}_{3,i}(t)). \tag{4.67}$$

• *Conclusion.* Finally, substituting (4.62), (4.66), and (4.67) into (4.64) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho\Big(\Phi(\rho,\bar\rho^{-\mathbf{X}}) + \frac{|\psi|^2}{2}\Big)\,\mathrm{d}x_1\,\mathrm{d}x' \le \sum_{i=1}^{2}\Big(-\frac{\delta_i}{4M}|\dot{\mathbf{X}}_i(t)|^2 - \frac{1}{2}C_1\mathbf{G}^{s_i}(t) - \frac{1}{2}\sum_{j=2}^{3}\mathbf{G}_{j,i}(t)\Big) - \frac{1}{8}\mathbf{D}(t)$$

$$
+ C\sum_{i=1}^{2}[\sqrt{\delta_i}\delta_1\delta_2\exp(-C\min\{\delta_1,\delta_2\}t) + \delta_i\exp(-C\delta_i t)\|\psi_1\|_{H^2}^2]
+ C\sum_{i=1}^{2}\Big(\frac{1}{t^2}\delta_i\exp(-C\delta_i t) + \delta_i^3\exp(-C\delta_i t)\Big)
+ \frac{C}{t^2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x' \tag{4.68}
$$

for all time $t \in [0,T]$.

On one hand, for $t \in (0,1)$, we substitute the first line of (4.48), (4.66), (4.67) into (4.9) to obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho\Big(Q(\rho|\bar{\rho}^{-\mathbf{X}}) + \frac{|\psi|^2}{2}\Big)\,\mathrm{d}x_1\,\mathrm{d}x' + \sum_{i=1}^{2}\mathbf{G}^{s_i}(t)\\
&\le \sum_{i=1}^{2}\Big(-\frac{\delta_i}{2M}|\dot{\mathbf{X}}_i(t)|^2 + \frac{C}{\delta_i}\sum_{j=3}^{5}|\mathbf{Y}_{j,i}(t)|^2 + \sum_{j=1}^{10}\mathbf{B}_{j,i}(t) - \sum_{j=1}^{3}\mathbf{G}_{j,i}(t)\Big)\\
&\quad - \mathbf{D}(t) + \sum_{i=1}^{2}\mathbf{G}^{s_i}(t)\\
&\le \sum_{i=1}^{2}\Big(-\frac{\delta_i}{2M}|\dot{\mathbf{X}}_i(t)|^2 - \frac{7}{8}\sum_{j=1}^{3}\mathbf{G}_{j,i}(t)\Big) - \frac{15}{16}\mathbf{D}(t)\\
&\quad + C\delta_1\delta_2\exp(-C\min\{\delta_1,\delta_2\}t)\sum_{i=1}^{2}\sqrt{\delta_i} + C\|\psi_1\|^2,
\end{aligned} \tag{4.69}
$$

where we have used the facts that for suitably small δ_i $(i = 1,2)$, it holds that

$$
\begin{aligned}
\mathbf{G}^{s_i}(t) &= \int_{\mathbb{T}^2}\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}|\psi_1^2\,\mathrm{d}x_1\,\mathrm{d}x' \le \|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}\|_{L^\infty(\mathbb{R})}\|\psi_1\|^2\\
&\le C\delta_i^2\|\psi_1\|^2 \le C\|\psi_1\|^2,\\
\mathbf{B}_{1,i}(t) &\le C\|\partial_{x_1}a_i^{-\mathbf{X}_i}\|_{L^\infty(\mathbb{R})}\|\psi_1\|^2 \le C\delta_i^{\frac{3}{2}}\|\psi_1\|^2 \le C\|\psi_1\|^2,\\
\mathbf{B}_{2,i}(t) &\le C\sqrt{\delta_i}\mathbf{G}_{3,i}(t) + C\mathbf{G}^{s_i}(t) \le C\sqrt{\delta_i}\mathbf{G}_{3,i}(t) + C\|\psi_1\|^2,\\
\mathbf{B}_{3,i}(t) &\le \|\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}\|_{L^\infty(\mathbb{R})}\|\psi_1\|^2 \le C\delta_i^2\|\psi_1\|^2 \le C\|\psi_1\|^2.
\end{aligned}
$$

Note that by (4.7), (4.8), and the uniform lower and upper bounds of the density function ρ, we have

$$
\Phi(\rho,\bar{\rho}^{-\mathbf{X}}) \sim |\rho-\bar{\rho}^{-\mathbf{X}}|^2,\quad \mathbf{G}_{2,i}(t)\sim G_{2,i}(t),\quad \mathbf{G}_{3,i}(t)\sim G_{3,i}(t),\quad \mathbf{D}(t)\ge\|\nabla_x\psi(t)\|^2.
$$

Thus, we first integrate (4.69) with respect to t over $(0,1)$ and use Gronwall's inequality to obtain

$$\sup_{0\le t\le 1}\|(\phi,\psi)(t)\|^2+\sum_{i=1}^{2}\int_0^1\Big(\delta_i|\dot{\mathbf{X}}_i(\tau)|^2+G^{s_i}(\tau)+\sum_{j=1}^{3}G_{j,i}(\tau)\Big)\,\mathrm{d}\tau+\int_0^1\|\nabla_x\psi\|^2\,\mathrm{d}\tau$$
$$\le C\|(\phi_0,\psi_0)\|^2+C\max\{\delta_1,\delta_2\}\sum_{i=1}^{2}\sqrt{\delta_i}.$$

Then we integrate (4.68) with respect to t over $(1,t)$, and use Gronwall's inequality to obtain

$$\|(\phi,\psi)(t)\|^2+\sum_{i=1}^{2}\int_1^t\Big(\delta_i|\dot{\mathbf{X}}_i(\tau)|^2+G^{s_i}(\tau)+\sum_{j=2}^{3}G_{j,i}(\tau)\Big)\,\mathrm{d}\tau+\int_1^t\|\nabla_x\psi\|^2\,\mathrm{d}\tau$$
$$\le C\|(\phi,\psi)(1)\|^2+C\max\{\delta_1,\delta_2\}\sum_{i=1}^{2}\sqrt{\delta_i}+C\sum_{i=1}^{2}\int_1^t\delta_i\exp(-C\delta_i\tau)\|\psi_1\|_{H^2}^2\,\mathrm{d}\tau.$$

Therefore, we combine the above two inequalities to obtain the desired inequality (4.1) with the new notation (4.2). The proof of Proposition 4.1 is completed. ∎

5. Proof of Proposition 3.2: Higher-order derivative estimates

In this section we give the higher-order derivative estimates of the a priori estimates.

5.1. First-order derivative estimates

Proposition 5.1. *Under the hypotheses of Proposition 3.2, there exists a constant $C>0$ independent of δ_i $(i=1,2)$, χ, and T, such that for all $t\in[0,T]$, it holds that*

$$\|\nabla_x(\phi,\psi)(t)\|^2+\int_0^t(\|\nabla_x\phi\|^2+\|\nabla_x^2\psi(t)\|^2)\,\mathrm{d}\tau$$
$$\le C(\|\nabla_x\phi_0\|^2+\|\psi_0\|_{H^1}^2)+C\|\psi(t)\|^2+C\int_0^t\|\nabla_x\psi\|^2\,\mathrm{d}\tau$$
$$+C\chi\int_0^t\|\nabla_x^3\psi\|^2\,\mathrm{d}\tau$$
$$+C\sum_{i=1}^{2}\int_0^t\Big(\delta_i|\dot{\mathbf{X}}_i(\tau)|^2+G^{s_i}(\tau)+\sum_{j=2}^{3}G_{2,i}(\tau)\Big)\,\mathrm{d}\tau$$
$$+C\max\{\delta_1,\delta_2\}\sum_{i=1}^{2}\delta_i^{\frac{3}{2}}+C\delta_1\delta_2\sum_{i=1}^{2}\delta_i^{\frac{1}{2}}.$$

Proposition 5.1 is a direct combination of Lemmas 5.1 and 5.2.

Lemma 5.1. *Under the hypotheses of Proposition 3.2, there exists a constant $C > 0$ independent of δ_i $(i = 1, 2)$, χ, and T, such that for all $t \in [0, T]$, it holds that*

$$\begin{aligned}
&\|\nabla_x \phi(t)\|^2 + \int_0^t \|\nabla_x \phi\|^2 \,\mathrm{d}\tau \\
&\quad \le C(\|\nabla_x \phi_0\|^2 + \|\psi_0\|^2) + C\|\psi(t)\|^2 + C\int_0^t \|\nabla_x \psi\|^2 \,\mathrm{d}\tau + C\chi \int_0^t \|\nabla_x^3 \psi\|^2 \,\mathrm{d}\tau \\
&\qquad + C\sum_{i=1}^{2} \int_0^t \Big(\delta_i |\dot{\mathbf{X}}_i(\tau)|^2 + G^{s_i}(\tau) + \sum_{j=2}^{3} G_{2,i}(\tau)\Big) \,\mathrm{d}\tau \\
&\qquad + C\max\{\delta_1, \delta_2\} \sum_{i=1}^{2} \delta_i^{\frac{3}{2}} + C\delta_1\delta_2 \sum_{i=1}^{2} \delta_i^{\frac{1}{2}}.
\end{aligned} \tag{5.1}$$

Proof. Applying ∇_x to equation $(4.4)_1$, it holds that

$$\begin{aligned}
&\partial_t \nabla_x \phi + u \cdot \nabla_x(\nabla_x \phi) + \rho \nabla_x \operatorname{div}_x \psi + \nabla_x u \cdot \nabla_x \phi + \nabla_x \rho \operatorname{div}_x \psi + \nabla_x \psi_1 \partial_{x_1} \bar{\rho}^{-\mathbf{X}} \\
&\qquad + \nabla_x \phi \partial_{x_1} \bar{u}_1^{-\mathbf{X}} + \Big(\psi_1 \partial_{x_1}^2 \bar{\rho}^{-\mathbf{X}} + \phi \partial_{x_1}^2 \bar{u}_1^{-\mathbf{X}} - \sum_{i=1}^{2} \dot{\mathbf{X}}_i(t) \partial_{x_1}^2 (\rho^{s_i})^{-\mathbf{X}_i}, 0, 0\Big)^{\mathsf{T}} \\
&\quad = (-\partial_{x_1} R_1, 0, 0)^{\mathsf{T}}.
\end{aligned} \tag{5.2}$$

We multiply (5.2) by $\frac{2\mu+\lambda}{\rho} \nabla_x \phi$ and integrate the resultant equation by parts over $\mathbb{R} \times \mathbb{T}^2$ to obtain

$$\begin{aligned}
&\frac{2\mu+\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^2} \int_{\mathbb{R}} \frac{|\nabla_x \phi|^2}{\rho} \,\mathrm{d}x_1 \,\mathrm{d}x' + (2\mu+\lambda) \int_{\mathbb{T}^2} \int_{\mathbb{R}} \nabla_x \phi \cdot \nabla_x \operatorname{div}_x \psi \,\mathrm{d}x_1 \,\mathrm{d}x' \\
&\quad = (2\mu+\lambda) \int_{\mathbb{T}^2} \int_{\mathbb{R}} \frac{\operatorname{div}_x u}{\rho} |\nabla_x \phi|^2 \,\mathrm{d}x_1 \,\mathrm{d}x' \\
&\qquad - \int_{\mathbb{T}^2} \int_{\mathbb{R}} \frac{2\mu+\lambda}{\rho} \nabla_x \phi \cdot (\nabla_x u \cdot \nabla_x \phi + \nabla_x \rho \operatorname{div}_x \psi) \,\mathrm{d}x_1 \,\mathrm{d}x' \\
&\qquad - \int_{\mathbb{T}^2} \int_{\mathbb{R}} \frac{2\mu+\lambda}{\rho} \nabla_x \phi \cdot (\nabla_x \psi_1 \partial_{x_1} \bar{\rho}^{-\mathbf{X}} + \nabla_x \phi \partial_{x_1} \bar{u}_1^{-\mathbf{X}}) \,\mathrm{d}x_1 \,\mathrm{d}x' \\
&\qquad - \int_{\mathbb{T}^2} \int_{\mathbb{R}} \frac{2\mu+\lambda}{\rho} \partial_{x_1} \phi (\psi_1 \partial_{x_1}^2 \bar{\rho}^{-\mathbf{X}} + \phi \partial_{x_1}^2 \bar{u}_1^{-\mathbf{X}}) \,\mathrm{d}x_1 \,\mathrm{d}x' \\
&\qquad + \sum_{i=1}^{2} \dot{\mathbf{X}}_i(t) \int_{\mathbb{T}^2} \int_{\mathbb{R}} \frac{2\mu+\lambda}{\rho} \partial_{x_1} \phi \partial_{x_1}^2 (\rho^{s_i})^{-\mathbf{X}_i} \,\mathrm{d}x_1 \,\mathrm{d}x' \\
&\qquad - \int_{\mathbb{T}^2} \int_{\mathbb{R}} \frac{2\mu+\lambda}{\rho} \partial_{x_1} \phi \partial_{x_1} R_1 \,\mathrm{d}x_1 \,\mathrm{d}x' =: \sum_{i=1}^{6} J_i(t).
\end{aligned} \tag{5.3}$$

We multiply (4.4)$_2$ by $\nabla_x\phi$ and integrate the resultant equation by parts over $\mathbb{R}\times\mathbb{T}^2$ to obtain

$$
\begin{aligned}
&\int_{\mathbb{T}^2}\int_{\mathbb{R}} p'(\rho)|\nabla_x\phi|^2\,\mathrm{d}x_1\,\mathrm{d}x' - (2\mu+\lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}} \nabla_x\phi\cdot\nabla_x\,\mathrm{div}_x\,\psi\,\mathrm{d}x_1\,\mathrm{d}x'\\
&\quad = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\psi\cdot\nabla_x\phi\,\mathrm{d}x_1\,\mathrm{d}x' + \int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\psi\cdot[\partial_t\nabla_x\phi + u\cdot\nabla_x(\nabla_x\phi)]\,\mathrm{d}x_1\,\mathrm{d}x'\\
&\qquad - \int_{\mathbb{T}^2}\int_{\mathbb{R}}\Big[p'(\rho) - \frac{\rho}{\bar\rho^{-\mathbf{X}}}p'(\bar\rho^{-\mathbf{X}})\partial_{x_1}\bar\rho^{-\mathbf{X}} + \rho\psi_1\partial_{x_1}\bar u_1^{-\mathbf{X}}\Big]\partial_{x_1}\phi\,\mathrm{d}x_1\,\mathrm{d}x'\\
&\qquad - \int_{\mathbb{T}^2}\int_{\mathbb{R}}\frac{2\mu+\lambda}{\bar\rho^{-\mathbf{X}}}\partial_{x_1}^2\bar u_1^{-\mathbf{X}}\phi\,\partial_{x_1}\phi\,\mathrm{d}x_1\,\mathrm{d}x'\\
&\qquad + \sum_{i=1}^{2}\dot{\mathbf{X}}_i(t)\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\partial_{x_1}\phi\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}\,\mathrm{d}x_1\,\mathrm{d}x'\\
&\qquad - \int_{\mathbb{T}^2}\int_{\mathbb{R}}\frac{\rho}{\bar\rho^{-\mathbf{X}}}R_2\partial_{x_1}\phi\,\mathrm{d}x_1\,\mathrm{d}x'\\
&\quad =: -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\psi\cdot\nabla_x\phi\,\mathrm{d}x_1\,\mathrm{d}x' + \sum_{i=7}^{11}J_i(t),
\end{aligned}\tag{5.4}
$$

We add (5.3) and (5.4) together to obtain

$$
\begin{aligned}
&\frac{2\mu+\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\frac{|\nabla_x\phi|^2}{\rho}\,\mathrm{d}x_1\,\mathrm{d}x' + \int_{\mathbb{T}^2}\int_{\mathbb{R}}p'(\rho)|\nabla_x\phi|^2\,\mathrm{d}x_1\,\mathrm{d}x'\\
&\quad = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\psi\cdot\nabla_x\phi\,\mathrm{d}x_1\,\mathrm{d}x' + \sum_{i=1}^{11}J_i(t).
\end{aligned}\tag{5.5}
$$

In the following, we control the terms on the right-hand side of (5.5) one by one. Recalling $\delta=\delta_1+\delta_2$, and using the Cauchy inequality and Sobolev inequality, it holds that

$$
\begin{aligned}
|J_1(t)|+|J_2(t)| &\le C\|\nabla_x\psi\|_{L^\infty}\|\nabla_x\phi\|^2\\
&\quad + C\|\partial_{x_1}(\bar\rho^{-\mathbf{X}},\bar u_1^{-\mathbf{X}})\|_{L^\infty(\mathbb{R})}(\|\nabla_x\phi\|^2 + \|\partial_{x_1}\phi\|\|\nabla_x\psi\|)\\
&\le C\|\nabla_x\psi\|_{H^2}\|\nabla_x\phi\|^2 + C\delta^2(\|\nabla_x\phi\|^2+\|\nabla_x\psi\|^2)\\
&\le C(\chi+\delta^2)\|\nabla_x\phi\|^2 + C\delta^2\|\nabla_x\psi\|^2 + C\chi\|\nabla_x^3\psi\|^2,
\end{aligned}
$$

where in the last inequality we have used the assumption (3.3) and the fact

$$
\begin{aligned}
\|\nabla_x\psi\|_{H^2}\|\nabla_x\phi\|^2 &\le \|\nabla_x\psi\|_{H^1}\|\nabla_x\phi\|^2 + \|\nabla_x^3\psi\|\|\nabla_x\phi\|^2\\
&\le \chi\|\nabla_x\phi\|^2 + \chi\|\nabla_x^3\psi\|\|\nabla_x\phi\| \le \chi(\|\nabla_x\phi\|^2+\|\nabla_x^3\psi\|^2).
\end{aligned}
$$

Using the Cauchy inequality, it holds that

$$
\begin{aligned}
|J_3(t)| &\le C\|\partial_{x_1}(\bar\rho^{-\mathbf{X}},\bar u_1^{-\mathbf{X}})\|_{L^\infty(\mathbb{R})}\|\nabla_x\phi\|(\|\nabla_x\psi_1\|+\|\nabla_x\phi\|)\\
&\le C\delta^2(\|\nabla_x\phi\|^2+\|\nabla_x\psi_1\|^2).
\end{aligned}
$$

Using the Cauchy inequality and (4.43), we have

$$
\begin{aligned}
&|J_4(t)| + |J_8(t)| + |J_9(t)| \\
&\le C \sum_{i=1}^{2} \int_{\mathbb{T}^2} \int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}| |\partial_{x_1}\phi| \Big| \Big(\phi - \frac{(\bar\rho^{-\mathbf{X}})^2}{\sigma_i}\psi_1, \psi_1\Big)\Big| \, \mathrm{d}x_1 \, \mathrm{d}x' \\
&\le C \|\partial_{x_1}\phi\| \sum_{i=1}^{2} \|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}\|_{L^\infty(\mathbb{R})}^{\frac12} \big(\delta_i^{\frac14}\sqrt{G_{3,i}(t)} + \sqrt{G^{s_i}(t)}\big) \\
&\le C\delta \|\partial_{x_1}\phi\|^2 + C\delta \sum_{i=1}^{2} (G_{3,i}(t) + G^{s_i}(t)).
\end{aligned}
$$

By the Cauchy inequality and Lemma 2.1, it holds that

$$
\begin{aligned}
|J_5(t)| + |J_{10}(t)| &\le C \|\partial_{x_1}\phi\| \sum_{i=1}^{2} |\dot{\mathbf{X}}_i(t)| \, \|\partial_{x_1}(u_1^{s_i})^{-\mathbf{X}_i}\|_{L^2(\mathbb{R})} \\
&\le C\delta \|\partial_{x_1}\phi\|^2 + C\delta \sum_{i=1}^{2} \delta_i |\dot{\mathbf{X}}_i(t)|^2.
\end{aligned}
$$

Similarly to $\mathbf{B}_{10,i}(t)$, using (4.20), we have

$$
\begin{aligned}
|J_6(t)| + |J_{11}(t)| &\le C \|\partial_{x_1}\phi\| \sum_{i=1}^{2} \Big(\int_{\mathbb{R}} |\bar\rho^{-\mathbf{X}} - (\rho^{s_i})^{-\mathbf{X}_i}|^2 |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}|^2 \, \mathrm{d}x_1 \Big)^{\frac12} \\
&\le \frac18 \|\sqrt{p'(\rho)}\partial_{x_1}\phi\|^2 + C \sum_{i=1}^{2} \delta_i (\delta_1\delta_2)^2 \exp(-C\min\{\delta_1, \delta_2\} t).
\end{aligned}
$$

We rewrite $J_7(t)$ by using equation (5.2) as

$$
\begin{aligned}
J_7(t) = &- \int_{\mathbb{T}^2} \int_{\mathbb{R}} \rho^2 \psi \cdot \nabla_x \operatorname{div}_x \psi \, \mathrm{d}x_1 \, \mathrm{d}x' \\
&- \int_{\mathbb{T}^2} \int_{\mathbb{R}} \rho\psi \cdot (\nabla_x u \cdot \nabla_x \phi + \nabla_x \rho \operatorname{div}_x \psi) \, \mathrm{d}x_1 \, \mathrm{d}x' \\
&- \int_{\mathbb{T}^2} \int_{\mathbb{R}} \rho\psi \cdot (\nabla_x \psi_1 \partial_{x_1} \bar\rho^{-\mathbf{X}} + \nabla_x \phi \partial_{x_1} \bar u_1^{-\mathbf{X}}) \, \mathrm{d}x_1 \, \mathrm{d}x' \\
&- \int_{\mathbb{T}^2} \int_{\mathbb{R}} \rho\psi_1 (\psi_1 \partial_{x_1}^2 \bar\rho^{-\mathbf{X}} + \phi \partial_{x_1}^2 \bar u_1^{-\mathbf{X}}) \, \mathrm{d}x_1 \, \mathrm{d}x' \\
&+ \sum_{i=1}^{2} \dot{\mathbf{X}}_i(t) \int_{\mathbb{T}^2} \int_{\mathbb{R}} \rho\psi_1 \partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i} \, \mathrm{d}x_1 \, \mathrm{d}x' \\
&- \int_{\mathbb{T}^2} \int_{\mathbb{R}} \rho\psi_1 \partial_{x_1} R_1 \, \mathrm{d}x_1 \, \mathrm{d}x' =: \sum_{i=1}^{6} J_{7,i}(t).
\end{aligned}
$$

Integration by parts yields

$$J_{7,1}(t) = \int_{\mathbb{T}^2}\int_{\mathbb{R}} \rho^2(\operatorname{div}_x \psi)^2 \, dx_1 \, dx' + 2\int_{\mathbb{T}^2}\int_{\mathbb{R}} \rho\psi \cdot \nabla_x \rho \operatorname{div}_x \psi \, dx_1 \, dx',$$

which together with $J_{7,2}(t)$ leads to

$$\begin{aligned} J_{7,1}(t) + J_{7,2}(t) &= \int_{\mathbb{T}^2}\int_{\mathbb{R}} \rho^2(\operatorname{div}_x \psi)^2 \, dx_1 \, dx' \\ &\quad + \int_{\mathbb{T}^2}\int_{\mathbb{R}} \rho\psi \cdot (\nabla_x \rho \operatorname{div}_x \psi - \nabla_x u \cdot \nabla_x \phi) \, dx_1 \, dx', \end{aligned}$$

where the last term on the right-hand side can be treated as

$$\begin{aligned} &\int_{\mathbb{T}^2}\int_{\mathbb{R}} \rho\psi \cdot (\nabla_x \rho \operatorname{div}_x \psi - \nabla_x u \cdot \nabla_x \phi) \, dx_1 \, dx' \\ &\leq C\|\psi\|_{L^\infty}\|\nabla_x\phi\|\,\|\nabla_x\psi\| + C\sum_{i=1}^{2}\|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}}\sqrt{G^{s_i}(t)}\|\nabla_x(\phi,\psi)\| \\ &\leq C(\chi+\delta)\|\nabla_x(\phi,\psi)\|^2 + C\delta G^{s_i}(t). \end{aligned}$$

Thus, we have

$$|J_{7,1}(t)| + |J_{7,2}(t)| \leq C\|\nabla_x\psi\|^2 + C(\chi+\delta)\|\nabla_x\phi\|^2 + C\delta G^{s_i}(t).$$

Using the Cauchy inequality and (4.43), it holds that

$$\begin{aligned} |J_{7,3}(t)| &\leq C\|\nabla_x(\phi,\psi_1)\|\sum_{i=1}^{2}\|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}}\Big(\sqrt{G^{s_i}(t)} + \delta_i^{\frac{1}{4}}\sqrt{G_{2,i}(t)}\Big) \\ &\leq C\delta\|\nabla_x(\phi,\psi_1)\|^2 + C\delta\sum_{i=1}^{2}(G^{s_i}(t) + G_{2,i}(t)). \end{aligned}$$

Similarly to $J_4(t)$, we have

$$\begin{aligned} |J_{7,4}(t)| &\leq C\sum_{i=1}^{2}\delta_i\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}|\,|\psi_1|\Big|\Big(\phi - \frac{(\bar{\rho}^{-\mathbf{X}})^2}{\sigma_i}\psi_1, \psi_1\Big)\Big| \, dx_1 \, dx' \\ &\leq C\delta\sum_{i=1}^{2}(G_{3,i}(t) + G^{s_i}(t)). \end{aligned}$$

Similarly to $J_5(t)$, it holds that

$$\begin{aligned} |J_{7,5}(t)| &\leq C\sum_{i=1}^{2}|\dot{\mathbf{X}}_i(t)|\Big(\int_{\mathbb{R}}|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}| \, dx_1\Big)^{\frac{1}{2}}\sqrt{G^{s_i}(t)} \\ &\leq C\sum_{i=1}^{2}(\delta_i|\dot{\mathbf{X}}_i(t)|^2 + G^{s_i}(t)). \end{aligned}$$

Using (4.20) and (4.21), we have

$$
\begin{aligned}
|J_{7,6}(t)| &\le C\|\psi_1\|\sum_{i=1}^{2}\delta_i\left(\int_{\mathbb{R}}|\bar{u}_1^{-\mathbf{X}}-(u_1^{s_i})^{-\mathbf{X}_i}|^2|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}|^2\,\mathrm{d}x_1\right)^{\frac{1}{2}}\\
&\quad + C\|\psi_1\|\|\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}}\|\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}}\\
&\quad \times\left(\int_{\mathbb{R}}|\partial_{x_1}(\rho^{s_1})^{-\mathbf{X}_1}||\partial_{x_1}(\rho^{s_2})^{-\mathbf{X}_2}|\,\mathrm{d}x_1\right)^{\frac{1}{2}}\\
&\le C\delta_1\delta_2\exp(-C\min\{\delta_1,\delta_2\}t)\sum_{i=1}^{2}\delta_i^{\frac{3}{2}}+C(\delta_1\delta_2)^{\frac{3}{2}}\sum_{i=1}^{2}\delta_i^{\frac{1}{2}}\exp(-C\delta_i t).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
|J_7(t)| &\le C(\chi+\delta)\|\nabla_x\phi\|^2+C\|\nabla_x\psi\|^2\\
&\quad + C\sum_{i=1}^{2}\left(G^{s_i}(t)+\sum_{j=2}^{3}G_{j,i}(t)+\delta_i|\dot{\mathbf{X}}_i(t)|^2\right)\\
&\quad + C\delta_1\delta_2\exp(-C\min\{\delta_1,\delta_2\}t)\sum_{i=1}^{2}\delta_i^{\frac{3}{2}}+C(\delta_1\delta_2)^{\frac{3}{2}}\sum_{i=1}^{2}\delta_i^{\frac{1}{2}}\exp(-C\delta_i t).
\end{aligned}
$$

We substitute estimates $J_i(t)$ $(i=1,\ldots,11)$ into (5.5), integrate the resultant inequality over $(0,t)$, choose χ, δ suitably small, to obtain (5.1). The proof of Lemma 5.1 is completed. ■

Lemma 5.2. *Under the hypotheses of Proposition 3.2, there exists a constant $C>0$ independent of δ_i $(i=1,2)$, χ, and T, such that for all $t\in[0,T]$, it holds that*

$$
\begin{aligned}
\|\nabla_x\psi(t)\|^2+\int_0^t\|\nabla_x^2\psi\|^2\,\mathrm{d}\tau &\le C\|\nabla_x\psi_0\|^2+C\int_0^t\|\nabla_x(\phi,\psi)\|^2\,\mathrm{d}\tau\\
+C\delta\sum_{i=1}^{2}\int_0^t(\delta_i|\dot{\mathbf{X}}_i(\tau)|^2+G_{3,i}(\tau)&+G^{s_i}(\tau))\,\mathrm{d}\tau+C\delta_1\delta_2\max\{\delta_1,\delta_2\}\sum_{i=1}^{2}\delta_i.
\end{aligned}
$$

The proof is similar to [31, Lemma 4.6], and we omit it for brevity.

5.2. Second-order derivative estimates

Proposition 5.2. *Under the hypotheses of Proposition 3.2, there exists a constant $C>0$ independent of δ_i $(i=1,2)$, χ, and T, such that for all $t\in[0,T]$, it holds that*

$$
\begin{aligned}
&\|\nabla_x^2(\phi,\psi)(t)\|^2+\int_0^t(\|\nabla_x^2\phi\|^2+\|\nabla_x^3\psi\|^2)\,\mathrm{d}\tau\\
&\quad\le C(\|\nabla_x^2\phi_0\|^2+\|\nabla_x\psi_0\|_{H^1}^2)+C\|\nabla_x\psi(t)\|^2
\end{aligned}
$$

$$+ C\int_0^t \|\nabla_x^2\psi\|^2\,\mathrm{d}\tau + C\delta\sum_{i=1}^{2}\int_0^t (\delta_i|\dot{\mathbf{X}}_i(\tau)|^2 + G^{s_i}(\tau) + G_{3,i}(\tau))\,\mathrm{d}\tau$$

$$+ C(\chi+\delta)\int_0^t \|\nabla_x(\phi,\psi)\|^2\,\mathrm{d}\tau + C\delta_1\delta_2\max\{\delta_1,\delta_2\}\sum_{i=1}^{2}\delta_i^2.$$

Proposition 5.2 is evidently the combination of Lemmas 5.3 and 5.4.

Lemma 5.3. *Under the hypotheses of Proposition 3.2, there exists a constant $C > 0$ independent of δ_i $(i = 1, 2)$, χ, and T, such that for all $t \in [0, T]$, it holds that*

$$\begin{aligned}
&\|\nabla_x^2\phi(t)\|^2 + \int_0^t \|\nabla_x^2\phi\|^2\,\mathrm{d}\tau \\
&\quad\le C(\|\nabla_x^2\phi_0\|^2 + \|\nabla_x\psi_0\|^2) + C\|\nabla_x\psi(t)\|^2 \\
&\qquad + C\int_0^t \|\nabla_x^2\psi\|^2\,\mathrm{d}\tau \\
&\qquad + C\delta\sum_{i=1}^{2}\int_0^t (\delta_i|\dot{\mathbf{X}}_i(\tau)|^2 + G^{s_i}(\tau) + G_{3,i}(\tau))\,\mathrm{d}\tau \\
&\qquad + C(\chi+\delta)\int_0^t (\|\nabla_x(\phi,\psi)\|^2 + \|\nabla_x^3\psi\|^2)\,\mathrm{d}\tau \\
&\qquad + C\delta_1\delta_2\max\{\delta_1,\delta_2\}\sum_{i=1}^{2}\delta_i^2.
\end{aligned}$$

The proof is similar to [31, Lemma 4.7], and we omit it for brevity.

Lemma 5.4. *Under the hypotheses of Proposition 3.2, there exists a constant $C > 0$ independent of δ_i $(i = 1, 2)$, χ, and T, such that for all $t \in [0, T]$, it holds that*

$$\begin{aligned}
&\|\nabla_x^2\psi(t)\|^2 + \int_0^t \|\nabla_x^3\psi\|^2\,\mathrm{d}\tau \\
&\quad\le C\|\nabla_x^2\psi_0\|^2 + C\int_0^t \|\nabla_x^2(\phi,\psi)\|^2\,\mathrm{d}\tau \\
&\qquad + C\delta^2\sum_{i=1}^{2}\int_0^t (\delta_i|\dot{\mathbf{X}}_i(\tau)|^2 + G^{s_i}(\tau) + G_{3,i}(\tau))\,\mathrm{d}\tau \\
&\qquad + C(\chi+\delta)\int_0^t \|\nabla_x(\phi,\psi)\|^2\,\mathrm{d}\tau + C\delta_1\delta_2\max\{\delta_1,\delta_2\}\sum_{i=1}^{2}\delta_i^3.
\end{aligned}$$

The proof is similar to [31, Lemma 4.8], and we omit it for brevity.

5.3. Proof of Proposition 3.2

Proof. Combining Propositions 4.1, 5.1, and 5.2 yields

$$\begin{aligned}&\|(\phi,\psi)(t)\|_{H^2}^2+\int_0^t(\|\nabla_x\phi\|_{H^1}^2+\|\nabla_x\psi\|_{H^2}^2)\,d\tau\\&\quad+\sum_{i=1}^2\int_0^t\Big(\delta_i|\dot{\mathbf{X}}_i(\tau)|^2+\sum_{j=2}^3+G_{j,i}(\tau)+G^{s_i}(\tau)\Big)\,d\tau\\&\quad\le C(\|(\phi_0,\psi_0)\|^2+\delta)+C\int_1^t\Big(\sum_{i=1}^2\delta_i\exp(-C\delta_i\tau)+\frac{1}{\tau^2}\Big)\|\psi\|_{H^2}^2\,d\tau,\end{aligned}$$

which using Gronwall's inequality implies the desired inequality (3.4). In addition, using (4.17), and together with Lemma 2.1 and the assumption (3.3), we have

$$\sum_{i=1}^2|\dot{\mathbf{X}}_i(t)|\le\sum_{i=1}^2\frac{C}{\delta_i}\|\psi_1(t,\cdot)\|_{L^\infty}\int_{\mathbb{R}}|\partial_{x_1}(\rho^{s_i})^{-\mathbf{X}_i}|\,dx_1\le C\|\psi_1(t,\cdot)\|_{L^\infty},$$

which implies (3.5). The proof of Proposition 3.2 is completed. ∎

Acknowledgments. The author would like to thank the anonymous referees for their valuable suggestions that helped improve the manuscript.

Funding. This work is partially supported by the National Key R&D Program of China (No. 2022YFA1007700) and the NSFC (No. 12371215).

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Received 23 October 2023; revised 31 August 2024; accepted 19 September 2024.

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