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Logarithmically refined Gagliardo–Nirenberg interpolation and application to blow-up exclusion in a singular chemotaxis–consumption system

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Abstract. A family of interpolation inequalities is derived, which differ from estimates of classical Gagliardo–Nirenberg type through the appearance of certain logarithmic deviations from standard Lebesgue norms in zero-order expressions. Optimality of the obtained inequalities is shown. A subsequent application reveals that when posed under homogeneous Neumann boundary conditions in smoothly bounded planar domains and with suitably regular initial data, for any choice of $\alpha > 0$ the Keller–Segel-type migration–consumption system $u_t = \Delta(uv^{-\alpha})$, $v_t = \Delta v - uv$, admits a global classical solution.

1. Introduction

Objective #1: Optimal interpolation involving $L^q \log^\beta L$ spaces. In interpolation inequalities of Gagliardo–Nirenberg type ([7, 17, 18, 39]), logarithmic refinements play important roles in various contexts of nonlinear PDE analysis. In typical applications, the presence of structural properties such as energies implies bounds for some solution components in Orlicz classes differing from classical Lebesgue spaces, and an appropriate exploitation of this a priori information is sought in order to derive further regularity features; well-known examples include evolution systems in which expressions of the form $\int u \ln u$ constitute a part of associated Lyapunov functionals, such as large classes of Fokker–Planck equations, or also cross-diffusion systems of Keller–Segel and related types ([5, 6, 19, 24, 27, 38, 40, 41]; cf. also the discussion in [43]).

A classical result concerned with such a situation, going back to [6], states that in any smoothly bounded planar domain Ω, given $\varepsilon > 0$ one can find $C(\varepsilon) > 0$ with the property that

$$\|\varphi\|_{L^3(\Omega)}^3 \le \varepsilon \|\varphi\|_{W^{1,2}(\Omega)}^2 \|\varphi \ln|\varphi|\|_{L^1(\Omega)} + C(\varepsilon)\|\varphi\|_{L^1(\Omega)} \quad \text{for all } \varphi \in W^{1,2}(\Omega). \quad (1.1)$$

In particular, the appearance of an arbitrarily small multiple of the first-order expression herein can be viewed as reflecting a certain added value of presupposed knowledge of

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$L \log L$ bounds in comparison to the neighboring standard Gagliardo–Nirenberg inequality, which exclusively involves genuine L^1 norms on the zero-order part of its right-hand side, and according to which for some positive but not necessarily small $C > 0$ we have

$$\|\varphi\|_{L^3(\Omega)}^3 \le C\|\varphi\|_{W^{1,2}(\Omega)}^2 \|\varphi\|_{L^1(\Omega)} \quad \text{for all } \varphi \in W^{1,2}(\Omega);$$

we refer to [6, p. 1199] for a derivation of (1.1), to [47, Appendix] for extensions to more general summability powers and to domains of arbitrary dimension, to [43, Lemma 11.1] for localized variants and, e.g., to [31, 38, 42, 44–46] for some applications which make substantial use of such improved knowledge.

The first objective of the present study consists in further specifying this type of advantage to a quantitatively optimal extent. In anticipation of the particular application context to be subsequently addressed, we will consider this in a slightly more general framework involving a second and widely arbitrary function which can be viewed as a weight, and the influence of which can actually be eliminated on the first reading by simply setting $\psi \equiv 1$ in the following. In fact, in Section 2 an approach based on resorting directly to Sobolev inequalities, rather than to Gagliardo–Nirenberg such as done e.g. in [6], will reveal that when combined with estimates in suitable classical Lebesgue norms, bounds in $L \log^\beta L$ actually allow for a control of sizes in certain smaller Orlicz spaces.

Proposition 1.1. *Let $n \ge 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $p > 0$, $\alpha > 0$ and $\beta \ge 0$. Then there exists $C = C(p, \alpha, \beta) > 0$ such that for any $\varphi \in C^1(\bar{\Omega})$ and $\psi \in C^1(\bar{\Omega})$ fulfilling $\varphi > 0$ and $\psi > 0$ in $\bar{\Omega}$,*

$$\begin{aligned}\int_\Omega \varphi^{p+\frac{2}{n}} \ln^{\frac{2\beta}{n}}(\varphi+e) \le{}& C \cdot \left\{\int_\Omega \varphi \ln^\beta(\varphi+e)\right\}^{\frac{2}{n}} \cdot \|\psi\|_{L^\infty(\Omega)}^\alpha \cdot \int_\Omega |\nabla(\varphi^{\frac{p}{2}}\psi^{-\frac{\alpha}{2}})|^2 \\ &+ C \cdot \left\{\int_\Omega \varphi \ln^\beta(\varphi+e)\right\}^{\frac{2}{n}} \cdot \int_\Omega \varphi^p \psi^{-2}|\nabla\psi|^2 \\ &+ C \cdot \left\{\int_\Omega \varphi\right\}^p \cdot \left\{\int_\Omega \varphi \ln^\beta(\varphi+e)\right\}^{\frac{2}{n}}. \qquad (1.2)\end{aligned}$$

Indeed, upon letting $\psi \equiv 1$ here we immediately obtain, as a by-product, the following interpolation inequality that exclusively involves one function only on its right-hand side.

Corollary 1.2. *If $n \ge 1$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, then for each $p > 0$ and $\beta \ge 0$ there exists $C = C(p, \beta) > 0$ with the property that whenever $\varphi \in C^1(\bar{\Omega})$ is positive in $\bar{\Omega}$,*

$$\begin{aligned}\int_\Omega \varphi^{p+\frac{2}{n}} \ln^{\frac{2\beta}{n}}(\varphi+e) \le{}& C \cdot \left\{\int_\Omega \varphi \ln^\beta(\varphi+e)\right\}^{\frac{2}{n}} \cdot \int_\Omega |\nabla\varphi^{\frac{p}{2}}|^2 \\ &+ C \cdot \left\{\int_\Omega \varphi\right\}^p \cdot \left\{\int_\Omega \varphi \ln^\beta(\varphi+e)\right\}^{\frac{2}{n}}. \qquad (1.3)\end{aligned}$$

Remark. Let us briefly comment on two special cases of the above:

(i) In the special case when $n = 1$, $p = 1$ and $\beta = 1$, (1.3) reduces to the inequality

$$\int_\Omega \varphi^3 \ln^2(\varphi + e) \le C \cdot \left\{ \int_\Omega \varphi \ln(\varphi + e) \right\}^2 \cdot \int_\Omega \frac{\varphi_x^2}{\varphi} + C \cdot \left\{ \int_\Omega \varphi \right\} \cdot \left\{ \int_\Omega \varphi \ln(\varphi + e) \right\}^2,$$

which indeed is sharper than a preceding statement from [48, Lemma 7.5] (cf. also [48, Corollary 7.6]) in this regard, according to which there exists $C > 0$ such that whenever $0 < \varphi \in C^1(\overline{\Omega})$,

$$\int_\Omega \varphi^3 \ln(\varphi + e) \le C \cdot \left\{ \int_\Omega \varphi \ln(\varphi + e) \right\}^2 \cdot \int_\Omega \frac{\varphi_x^2}{\varphi} + C \cdot \left\{ \int_\Omega \varphi \ln(\varphi + e) \right\}^3.$$

(ii) Also for $n = 2$ and general $p > 0$ and $\beta > 0$, the inequality

$$\int_\Omega \varphi^{p+1} \ln^\beta(\varphi + e) \le C \cdot \left\{ \int_\Omega \varphi \ln^\beta(\varphi + e) \right\} \cdot \int_\Omega |\nabla \varphi^{\frac{p}{2}}|^2 + C \cdot \left\{ \int_\Omega \varphi \right\}^p \cdot \int_\Omega \varphi \ln^\beta(\varphi + e),$$

as accordingly asserted by Corollary 1.2, extends the corresponding outcome of [48, Lemma 7.5], which only for $\gamma \in [0, \beta)$ has provided $C = C(p, \beta, \gamma) > 0$ fulfilling

$$\int_\Omega \varphi^{p+1} \ln^\gamma(\varphi + e) \le C \cdot \left\{ \int_\Omega \varphi \ln^\beta(\varphi + e) \right\} \cdot \int_\Omega |\nabla \varphi^{\frac{p}{2}}|^2 + C \cdot \left\{ \int_\Omega \varphi \ln^\beta(\varphi + e) \right\}^{p+1}$$

for any such φ.

In order to indicate the appropriateness of the approach chosen here, we can finally make sure that the outcome of Corollary 1.2 is essentially optimal with regard to the expression controlled on its left-hand side, and that hence Proposition 1.1 also cannot be substantially improved; as (1.3) trivially holds when $n \ge 3$ and $p \in (0, \frac{n-2}{n}]$, we may confine ourselves to the case when $p > \frac{(n-2)_+}{n}$ in the following.

Proposition 1.3. *Let $n \ge 1$ and $\Omega \subset \mathbb{R}^n$ be any bounded domain, let $p > \frac{(n-2)_+}{n}$ and $\beta \ge 0$, and suppose that the functions $h \in C^0((0,\infty))$, $\mathcal{F}_1 \in C^0((0,\infty)^2)$ and $\mathcal{F}_2 \in C^0((0,\infty)^2)$ are nonnegative and nondecreasing with respect to each of their arguments, and such that*

$$\int_\Omega \varphi^{p+\frac{2}{n}} \ln^{\frac{2\beta}{n}}(\varphi + e) h(\varphi) \le \mathcal{F}_1\left(\int_\Omega \varphi, \int_\Omega \varphi \ln^\beta(\varphi + e) \right) \cdot \int_\Omega |\nabla \varphi^{\frac{p}{2}}|^2 + \mathcal{F}_2\left(\int_\Omega \varphi, \int_\Omega \varphi \ln^\beta(\varphi + e) \right) \tag{1.4}$$

for all $\varphi \in C^1(\overline{\Omega})$ fulfilling $\varphi > 0$ in $\overline{\Omega}$. Then there exists $C > 0$ such that

$$h(\xi) \le C \quad \textit{for all } \xi > 0.$$

Objective #2: Suppressing blow-up in a two-dimensional migration-consumption system. Our second focus will thereafter be on the parabolic model

$$\begin{cases} u_t = \Delta(uv^{-\alpha}), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

for coupled migration-consumption processes positively influenced by small concentrations v of a directing cue, such as typically found in contexts of starvation-driven motion of organisms ([10, 13, 32]). Here, splitting the nonlinear second-order operator into the diffusive part $\nabla \cdot (v^{-\alpha}\nabla u)$ and a cross-diffusive contribution $-\alpha\nabla \cdot (uv^{-\alpha-1}\nabla v)$ shows that (1.5) can be viewed as a special representative within a large class of Keller–Segel–consumption systems ([25, 26]), with one of its core features consisting of a precise link between the signal-dependent rates of random diffusion and taxis.

Resulting characteristic mathematical features of this particular structure, as becoming manifest in a priori bounds of the form

$$\int_0^T \int_\Omega u^2 v^{-\alpha} \le C(T), \quad T > 0, \tag{1.6}$$

have played essential roles in existence and qualitative theories not only for (1.5) throughout various ranges of $\alpha \in \mathbb{R}$ ([28, 30, 49, 53]) but also for several relatives accounting for signal production mechanisms (see [8, 11, 12, 15, 16, 23, 29] for a small selection of recent developments on systems of this form, and also [1, 14, 20, 22, 33–35, 54] as well as [21, 36, 50] for studies concerned with further simplifications and extensions). Especially in the case when in line with the modeling hypotheses in [10] and [13], the key parameter α is assumed to be positive; however, to date it seems unclear how far basic regularity information in the style of (1.6) can be used to appropriately control the destabilizing potential of the singularly enhanced cross-diffusion mechanism in (1.5). Accordingly, global classical solvability could so far be asserted only in spatially one-dimensional versions of (1.5) when $\alpha > 0$, while in higher-dimensional domains, only certain very weak-strong solutions seem to have been constructed for arbitrary positive α up to now ([49]).

As the second goal of this manuscript, we intend to develop a refined interpolation-based approach toward an analysis of (1.5), through which the occurrence of taxis-driven blow-up phenomena can be ruled out for arbitrary positive α at least in planar domains. To accomplish this, we will rely on the outcome of Proposition 1.1 in two essential steps related to the crucial aim to establish pointwise lower bounds for the component v, and to thereby exclude the effective appearance of singular migration rates in (1.5).

Indeed, analyzing the evolution of

$$\int_\Omega u \ln(uv^{-\alpha} + e)$$

will show that whenever $\alpha > 0$, a rigorous version of (1.6) (Lemma 3.2) together with a consequence thereof on regularity of ∇v (Lemma 3.3) implies bounds of the form

$$\sup_{t\in(0,T)} \int_\Omega u \ln(u+e) + \int_0^T \int_\Omega |\nabla(uv^{-\alpha})|^2 + \int_0^T \int_\Omega uv^{-2}|\nabla v|^2 \le C(T), \quad T > 0$$

(Lemma 3.6), whence a first application of Proposition 1.1 will assert estimates of type

$$\int_0^T \int_\Omega u^2 \ln(u+e) \le C(T), \quad T > 0, \tag{1.7}$$

(Lemma 3.7). In a second stage, this in turn will enable us to suitably control ill-signed contributions to the evolution of

$$\int_\Omega u \ln^\gamma(uv^{-\alpha}+e) \quad \text{for some } \gamma > 1,$$

and to thus, again on the basis of Proposition 1.1, improve (1.7) so as to become

$$\int_0^T \int_\Omega u^2 \ln^\gamma(u+e) \le C(T), \quad T > 0$$

(Lemmas 3.9 and 3.10). Thanks to the strict inequality $\gamma > 1$, a pointwise upper bound for $\ln \frac{1}{v}$ will directly result from this due to a general result on L^∞ estimates of solutions to inhomogeneous linear heat equations with sources bounded in spatio-temporal $L^2 \log^\gamma L$ norms in the considered two-dimensional setting (Lemmata 3.4 and 3.11).

Supplemented by a straightforward derivation of higher regularity features, this will lead to the following consequence of our interpolation results from Proposition 1.1 in the absence of finite-time singularity formation in the two-dimensional version of (1.5) for any such α.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $\alpha > 0$. Then for any choice of initial data which are such that*

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative in } \Omega \text{ with } u_0 \not\equiv 0 \quad \text{and} \\ v_0 \in W^{1,\infty}(\Omega) \text{ is positive in } \bar{\Omega}, \end{cases} \tag{1.8}$$

one can find uniquely determined functions

$$\begin{cases} u \in C^0(\bar{\Omega}\times[0,\infty)) \cap C^{2,1}(\bar{\Omega}\times(0,\infty)) \quad \text{and} \\ v \in \bigcap_{q>2} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\bar{\Omega}\times(0,\infty)) \end{cases}$$

such that $u > 0$ in $\bar{\Omega}\times(0,\infty)$ and $v > 0$ in $\bar{\Omega}\times[0,\infty)$, and that (u, v) forms a classical solution of (1.5).

2. Interpolation results. Proofs of Propositions 1.1 and 1.3

To begin with, let us suitably exploit the Sobolev inequality to establish the announced interpolation inequality in its general form.

Proof of Proposition 1.1. We first consider the case when $n \ge 2$, in which we fix $c_1 = c_1(p) > 0$ such that in accordance with the Sobolev inequality on Ω we have

$$\|\zeta\|_{L^2(\Omega)}^2 \le c_1 \|\nabla\zeta\|_{L^{\frac{2n}{n+2}}(\Omega)}^2 + c_1 \|\zeta\|_{L^{\frac{2n}{np+2}}(\Omega)}^2 \qquad \text{for all } \zeta \in C^1(\bar{\Omega}), \tag{2.1}$$

and given $\varphi \in C^1(\bar{\Omega})$ and $\psi \in C^1(\bar{\Omega})$ such that $\varphi > 0$ and $\psi > 0$ in $\bar{\Omega}$, we abbreviate

$$\rho := \varphi^{\frac{p}{2}} \psi^{-\frac{\alpha}{2}} \tag{2.2}$$

and apply (2.1) to

$$\zeta := \rho^{\frac{np+2}{np}} \psi^{\frac{(np+2)\alpha}{2np}} \ln^{\frac{\beta}{n}}(\rho^{\frac{2}{p}} \psi^{\frac{\alpha}{p}} + e). \tag{2.3}$$

Here we observe that

$$\begin{aligned}
\nabla\zeta &= \frac{np+2}{np} \rho^{\frac{2}{np}} \psi^{\frac{(np+2)\alpha}{2np}} \ln^{\frac{\beta}{n}}(\rho^{\frac{2}{p}} \psi^{\frac{\alpha}{p}} + e)\nabla\rho \\
&\quad + \frac{(np+2)\alpha}{2np} \rho^{\frac{np+2}{np}} \psi^{\frac{(np+2)\alpha-2np}{2np}} \ln^{\frac{\beta}{n}}(\rho^{\frac{2}{p}} \psi^{\frac{\alpha}{p}} + e)\nabla\psi \\
&\quad + \rho^{\frac{np+2}{np}} \psi^{\frac{(np+2)\alpha}{2np}} \cdot \frac{\beta}{n} \frac{\ln^{\frac{\beta-n}{n}}(\rho^{\frac{2}{p}} \psi^{\frac{\alpha}{p}} + e)}{\rho^{\frac{2}{p}} \psi^{\frac{\alpha}{p}} + e} \cdot \Big\{ \frac{2}{p} \rho^{\frac{2-p}{p}} \psi^{\frac{\alpha}{p}} \nabla\rho + \frac{\alpha}{p} \rho^{\frac{2}{p}} \psi^{\frac{\alpha-p}{p}} \nabla\psi \Big\} \\
&= \rho^{\frac{2}{np}} \psi^{\frac{(np+2)\alpha}{2np}} \ln^{\frac{\beta}{n}}(\rho^{\frac{2}{p}} \psi^{\frac{\alpha}{p}} + e) \cdot \Big\{ \frac{np+2}{np} + \frac{2\beta}{np} \cdot \frac{\rho^{\frac{2}{p}} \psi^{\frac{\alpha}{p}}}{(\rho^{\frac{2}{p}} \psi^{\frac{\alpha}{p}} + e)\ln(\rho^{\frac{2}{p}} \psi^{\frac{\alpha}{p}} + e)} \Big\} \nabla\rho \\
&\quad + \rho^{\frac{np+2}{np}} \psi^{\frac{(np+2)\alpha-2np}{2np}} \ln^{\frac{\beta}{n}}(\rho^{\frac{2}{p}} \psi^{\frac{\alpha}{p}} + e) \\
&\qquad \cdot \Big\{ \frac{(np+2)\alpha}{2np} + \frac{\alpha\beta}{np} \cdot \frac{\rho^{\frac{2}{p}} \psi^{\frac{\alpha}{p}}}{(\rho^{\frac{2}{p}} \psi^{\frac{\alpha}{p}} + e)\ln(\rho^{\frac{2}{p}} \psi^{\frac{\alpha}{p}} + e)} \Big\} \nabla\psi
\end{aligned}$$

in Ω, and that thus, by (2.2),

$$\begin{aligned}
\nabla\zeta &= \varphi^{\frac{1}{n}} \psi^{\frac{\alpha}{2}} \ln^{\frac{\beta}{n}}(\varphi + e) \cdot \Big\{ \frac{np+2}{np} + \frac{2\beta}{np} \cdot \frac{\varphi}{(\varphi+e)\ln(\varphi+e)} \Big\} \nabla\rho \\
&\quad + \varphi^{\frac{np+2}{2n}} \psi^{-1} \ln^{\frac{\beta}{n}}(\varphi + e) \cdot \Big\{ \frac{(np+2)\alpha}{2np} + \frac{\alpha\beta}{np} \cdot \frac{\varphi}{(\varphi+e)\ln(\varphi+e)} \Big\} \nabla\psi \quad \text{in } \Omega.
\end{aligned}$$

Since

$$0 \le \frac{\varphi}{(\varphi+e)\ln(\varphi+e)} \le 1 \quad \text{in } \Omega,$$

this implies that if we let $c_2 \equiv c_2(p,\beta) := \frac{np+2}{np} + \frac{2\beta}{np}$ and $c_3 \equiv c_3(p,\alpha,\beta) := \frac{(np+2)\alpha}{2np} + \frac{\alpha\beta}{np}$, then

$$|\nabla\zeta| \le c_2 \varphi^{\frac{1}{n}} \psi^{\frac{\alpha}{2}} \ln^{\frac{\beta}{n}}(\varphi+e)|\nabla\rho| + c_3 \varphi^{\frac{np+2}{2n}} \psi^{-1} \ln^{\frac{\beta}{n}}(\varphi+e)|\nabla\psi|$$

in Ω, so that

$$\begin{aligned} \|\nabla\zeta\|^2_{L^{\frac{2n}{n+2}}(\Omega)} &\le 2c_2^2 \|\varphi^{\frac{1}{n}} \psi^{\frac{\alpha}{2}} \ln^{\frac{\beta}{n}}(\varphi+e)\nabla\rho\|^2_{L^{\frac{2n}{n+2}}(\Omega)} \\ &\quad + 2c_3^2 \|\varphi^{\frac{np+2}{2n}} \psi^{-1} \ln^{\frac{\beta}{n}}(\varphi+e)\nabla\psi\|^2_{L^{\frac{2n}{n+2}}(\Omega)}. \end{aligned} \tag{2.4}$$

Here, the Hölder inequality implies that

$$\begin{aligned} &\|\varphi^{\frac{1}{n}} \psi^{\frac{\alpha}{2}} \ln^{\frac{\beta}{n}}(\varphi+e)\nabla\rho\|^2_{L^{\frac{2n}{n+2}}(\Omega)} \\ &\quad = \left\{ \int_\Omega \varphi^{\frac{2}{n+2}} \psi^{\frac{n\alpha}{n+2}} \ln^{\frac{2\beta}{n+2}}(\varphi+e)|\nabla\rho|^{\frac{2n}{n+2}} \right\}^{\frac{n+2}{n}} \\ &\quad \le \left\{ \int_\Omega |\nabla\rho|^2 \right\} \cdot \left\{ \int_\Omega \varphi \psi^{\frac{n\alpha}{2}} \ln^\beta(\varphi+e) \right\}^{\frac{2}{n}} \\ &\quad \le \|\psi\|^\alpha_{L^\infty(\Omega)} \cdot \left\{ \int_\Omega |\nabla\rho|^2 \right\} \cdot \left\{ \int_\Omega \varphi \ln^\beta(\varphi+e) \right\}^{\frac{2}{n}}, \end{aligned} \tag{2.5}$$

and that

$$\begin{aligned} &\|\varphi^{\frac{np+2}{2n}} \psi^{-1} \ln^{\frac{\beta}{n}}(\varphi+e)\nabla\psi\|^2_{L^{\frac{2n}{n+2}}(\Omega)} \\ &\quad = \left\{ \int_\Omega \varphi^{\frac{np+2}{n+2}} \psi^{-\frac{2n}{n+2}} \ln^{\frac{2\beta}{n+2}}(\varphi+e)|\nabla\psi|^{\frac{2n}{n+2}} \right\}^{\frac{n+2}{n}} \\ &\quad = \left\{ \int_\Omega \{\varphi^p \psi^{-2}|\nabla\psi|^2\}^{\frac{n}{n+2}} \cdot \varphi^{\frac{2}{n+2}} \ln^{\frac{2\beta}{n+2}}(\varphi+e) \right\}^{\frac{n+2}{n}} \\ &\quad \le \left\{ \int_\Omega \varphi^p \psi^{-2}|\nabla\psi|^2 \right\} \cdot \left\{ \int_\Omega \varphi \ln^\beta(\varphi+e) \right\}^{\frac{2}{n}}. \end{aligned} \tag{2.6}$$

Since, finally, again using the Hölder inequality, we infer from (2.3) and (2.2) that

$$\begin{aligned} \|\zeta\|^2_{L^{\frac{2n}{np+2}}(\Omega)} &= \left\{ \int_\Omega \rho^{\frac{2}{p}} \psi^{\frac{\alpha}{p}} \ln^{\frac{2\beta}{np+2}}(\rho^{\frac{2}{p}} \psi^{\frac{\alpha}{p}} + e) \right\}^{\frac{np+2}{n}} \\ &= \left\{ \int_\Omega \varphi \ln^{\frac{2\beta}{np+2}}(\varphi+e) \right\}^{\frac{np+2}{n}} \\ &= \left\{ \int_\Omega \{\varphi \ln^\beta(\varphi+e)\}^{\frac{2}{np+2}} \cdot \varphi^{\frac{np}{np+2}} \right\}^{\frac{np+2}{n}} \\ &\le \left\{ \int_\Omega \varphi \ln^\beta(\varphi+e) \right\}^{\frac{2}{n}} \cdot \left\{ \int_\Omega \varphi \right\}^p, \end{aligned}$$

a combination of (2.1) with (2.4), (2.5) and (2.6) yields (1.2) upon an evident choice of $C(p,\alpha,\beta)$ whenever $n \ge 2$.

If $\Omega \subset \mathbb{R}$, however, then for $0 < \varphi \in C^1(\bar{\Omega})$ and $0 < \psi \in C^1(\bar{\Omega})$ we can estimate

$$I := \int_\Omega \varphi^{p+2} \ln^{2\beta}(\varphi + e)$$

in an elementary manner: Again letting $\rho := \varphi^{\frac{p}{2}} \psi^{-\frac{\alpha}{2}}$, we first observe that similarly to the above,

$$\begin{aligned}
&|\partial_x\{\varphi^{p+1}\ln^\beta(\varphi+e)\}| \\
&= |\partial_x\{\rho^{\frac{2(p+1)}{p}} \psi^{\frac{(p+1)\alpha}{p}} \ln^\beta(\rho^{\frac{2}{p}}\psi^{\frac{\alpha}{p}} + e)\}| \\
&= \Big| \frac{2(p+1)}{p} \rho^{\frac{p+2}{p}} \psi^{\frac{(p+1)\alpha}{p}} \ln^\beta(\rho^{\frac{2}{p}}\psi^{\frac{\alpha}{p}} + e)\rho_x \\
&\quad + \frac{(p+1)\alpha}{p} \rho^{\frac{2(p+1)}{p}} \psi^{\frac{(p+1)\alpha-p}{p}} \ln^\beta(\rho^{\frac{2}{p}}\psi^{\frac{\alpha}{p}} + e)\psi_x \\
&\quad + \rho^{\frac{2(p+1)}{p}} \psi^{\frac{(p+1)\alpha}{p}} \cdot \beta \frac{\ln^{\beta-1}(\rho^{\frac{2}{p}}\psi^{\frac{\alpha}{p}} + e)}{\rho^{\frac{2}{p}}\psi^{\frac{\alpha}{p}} + e} \cdot \Big\{ \frac{2}{p} \rho^{\frac{2-p}{p}} \psi^{\frac{\alpha}{p}} \rho_x + \frac{\alpha}{p} \rho^{\frac{2}{p}} \psi^{\frac{\alpha-p}{p}} \psi_x \Big\} \Big| \\
&= \Big| \rho^{\frac{p+2}{p}} \psi^{\frac{(p+1)\alpha}{p}} \ln^\beta(\rho^{\frac{2}{p}}\psi^{\frac{\alpha}{p}} + e) \cdot \Big\{ \frac{2(p+1)}{p} + \frac{2\beta}{p} \cdot \frac{\rho^{\frac{2}{p}}\psi^{\frac{\alpha}{p}}}{(\rho^{\frac{2}{p}}\psi^{\frac{\alpha}{p}} + e)\ln(\rho^{\frac{2}{p}}\psi^{\frac{\alpha}{p}} + e)} \Big\} \rho_x \\
&\quad + \rho^{\frac{2(p+1)}{p}} \psi^{\frac{(p+1)\alpha-p}{p}} \ln^\beta(\rho^{\frac{2}{p}}\psi^{\frac{\alpha}{p}} + e) \\
&\qquad \cdot \Big\{ \frac{(p+1)\alpha}{p} + \frac{\alpha\beta}{p} \cdot \frac{\rho^{\frac{2}{p}}\psi^{\frac{\alpha}{p}}}{(\rho^{\frac{2}{p}}\psi^{\frac{\alpha}{p}} + e)\ln(\rho^{\frac{2}{p}}\psi^{\frac{\alpha}{p}} + e)} \Big\} \psi_x \Big| \\
&= \Big| \varphi^{\frac{p+2}{2}} \psi^{\frac{\alpha}{2}} \ln^\beta(\varphi+e) \cdot \Big\{ \frac{2(p+1)}{p} + \frac{2\beta}{p} \cdot \frac{\varphi}{(\varphi+e)\ln(\varphi+e)} \Big\} \rho_x \\
&\quad + \varphi^{p+1}\psi^{-1} \ln^\beta(\varphi+e) \cdot \Big\{ \frac{(p+1)\alpha}{p} + \frac{\alpha\beta}{p} \cdot \frac{\varphi}{(\varphi+e)\ln(\varphi+e)} \Big\} \cdot \psi_x \Big| \\
&\le c_4 \varphi^{\frac{p+2}{2}} \psi^{\frac{\alpha}{2}} \ln^\beta(\varphi+e)|\rho_x| + c_5 \varphi^{p+1}\psi^{-1}\ln^\beta(\varphi+e)|\psi_x| \quad \text{in } \Omega,
\end{aligned}$$

with $c_4 \equiv c_4(p,\beta) := \frac{2(p+1)}{p} + \frac{2\beta}{p}$ and $c_5 \equiv c_5(p,\alpha,\beta) := \frac{(p+1)\alpha}{p} + \frac{\alpha\beta}{p}$. To make suitable use of this, we fix $x_0 \in \bar{\Omega}$ such that $\varphi(x_0) = \inf_{x\in\Omega} \varphi(x)$, and then infer from the monotonicity of $0 \le \xi \mapsto \xi \ln^\beta(\xi + e)$ that

$$\begin{aligned}
\varphi^{p+1}(x_0)\ln^\beta(\varphi(x_0)+e) &= \{\varphi(x_0)\}^p \cdot \{\varphi(x_0)\ln^\beta(\varphi(x_0)+e)\} \\
&\le \Big\{ \frac{1}{|\Omega|} \int_\Omega \varphi \Big\}^p \cdot \Big\{ \frac{1}{|\Omega|} \int_\Omega \varphi \ln^\beta(\varphi+e) \Big\} \\
&= |\Omega|^{-p-1} \cdot \Big\{ \int_\Omega \varphi \Big\}^p \cdot \int_\Omega \varphi \ln^\beta(\varphi+e).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varphi^{p+1}(x)\ln^\beta(\varphi(x)+e) &= \varphi^{p+1}(x_0)\ln^\beta(\varphi(x_0)+e) + \int_{x_0}^x \partial_x\{\varphi^{p+1}\ln^\beta(\varphi+e)\}(y)dy \\
&\le |\Omega|^{-p-1}\cdot\Big\{\int_\Omega\varphi\Big\}^p\cdot\int_\Omega\varphi\ln^\beta(\varphi+e) \\
&\quad + c_4\int_\Omega\varphi^{\frac{p+2}{2}}\psi^{\frac{\alpha}{2}}\ln^\beta(\varphi+e)|\rho_x| \\
&\quad + c_5\int_\Omega\varphi^{p+1}\psi^{-1}\ln^\beta(\varphi+e)|\psi_x| \quad \text{for all } x\in\Omega,
\end{aligned}$$

and thus, by Young's inequality,

$$\begin{aligned}
I \equiv \int_\Omega\varphi^{p+2}\ln^{2\beta}(\varphi+e) &\le \|\varphi^{p+1}\ln^\beta(\varphi+e)\|_{L^\infty(\Omega)}\int_\Omega\varphi\ln^\beta(\varphi+e) \\
&\le |\Omega|^{-p-1}\cdot\Big\{\int_\Omega\varphi\Big\}^p\cdot\Big\{\int_\Omega\varphi\ln^\beta(\varphi+e)\Big\}^2 \\
&\quad + c_4\cdot\Big\{\int_\Omega\varphi^{\frac{p+2}{2}}\psi^{\frac{\alpha}{2}}\ln^\beta(\varphi+e)|\rho_x|\Big\}\cdot\int_\Omega\varphi\ln^\beta(\varphi+e) \\
&\quad + c_5\cdot\Big\{\int_\Omega\varphi^{p+1}\psi^{-1}\ln^\beta(\varphi+e)|\psi_x|\Big\}\cdot\int_\Omega\varphi\ln^\beta(\varphi+e) \\
&\le |\Omega|^{-p-1}\cdot\Big\{\int_\Omega\varphi\Big\}^p\cdot\Big\{\int_\Omega\varphi\ln^\beta(\varphi+e)\Big\}^2 \\
&\quad + \frac{1}{4}I + c_4^2\cdot\Big\{\int_\Omega\psi^\alpha\rho_x^2\Big\}\cdot\Big\{\int_\Omega\varphi\ln^\beta(\varphi+e)\Big\}^2 \\
&\quad + \frac{1}{4}I + c_5^2\cdot\Big\{\int_\Omega\varphi^p\psi^{-2}\psi_x^2\Big\}\cdot\Big\{\int_\Omega\varphi\ln^\beta(\varphi+e)\Big\}^2.
\end{aligned}$$

As $\int_\Omega\psi^\alpha\rho_x^2 \le \|\psi\|_{L^\infty(\Omega)}^\alpha\int_\Omega\rho_x^2$, this implies (1.2) whenever $C(p,\alpha,\beta)\ge\max\{2c_4^2, 2c_5^2, 2|\Omega|^{-p-1}\}$. ∎

Our argument revealing optimality of the statement from Corollary 1.2, and hence also of Proposition 1.1, involves families of functions with essentially self-similar structure.

Proof of Proposition 1.3. Without loss of generality assuming that $B_{R_0}(0)\subset\Omega\subset B_R(0)$ with some $R_0>0$ and $R>R_0$, we use that $\xi^{-\frac{1}{2}}\ln^\beta\xi\to 0$ as $\xi\to\infty$ to choose $\delta_0\in(0,1)$ small enough such that besides

$$\frac{\delta_0}{2}\le R_0, \tag{2.7}$$

we have

$$\delta^{\frac{n}{2}}\ln^\beta(\delta^{-n})\le 1 \quad \text{for all } \delta\in(0,\delta_0). \tag{2.8}$$

For $\delta \in (0, \delta_0)$, we then abbreviate

$$a_\delta := \ln^{-\beta}(\delta^{-n}), \tag{2.9}$$

and fixing a nonincreasing $\chi \in C^\infty([0,\infty))$ such that $\chi \equiv 1$ in $[0, \frac{1}{2}]$ and $\chi \equiv 0$ in $[1, \infty)$, we define

$$\varphi_\delta(x) := a_\delta \cdot \Big\{1 + \delta^{-\frac{np}{2}} \chi\Big(\frac{|x|}{\delta}\Big)\Big\}^{\frac{2}{p}}, \qquad x \in \overline{\Omega},\ \delta \in (0, \delta_0). \tag{2.10}$$

Then φ_δ belongs to $C^\infty(\overline{\Omega})$ and is positive in $\overline{\Omega}$ for any such δ, and writing $\omega_n := n|B_1(0)|$ we can employ Young's inequality to estimate

$$\begin{aligned}
\int_\Omega \varphi_\delta &\le \omega_n a_\delta \int_0^R r^{n-1} \cdot \Big\{1 + \delta^{-\frac{np}{2}} \chi\Big(\frac{r}{\delta}\Big)\Big\}^{\frac{2}{p}} dr \\
&\le 2^{\frac{2}{p}} \omega_n a_\delta \int_0^R r^{n-1} \cdot \Big\{1 + \delta^{-n} \chi^{\frac{2}{p}}\Big(\frac{r}{\delta}\Big)\Big\} dr \\
&= 2^{\frac{2}{p}} \omega_n a_\delta \cdot \frac{R^n}{n} + 2^{\frac{2}{p}} \omega_n a_\delta \int_0^{\frac{R}{\delta}} \xi^{n-1} \chi^{\frac{2}{p}}(\xi)\, d\xi \quad \text{for all } \delta \in (0, \delta_0),
\end{aligned}$$

so that

$$\int_\Omega \varphi_\delta \le c_1 a_\delta \quad \text{for all } \delta \in (0, \delta_0), \tag{2.11}$$

with $c_1 := \frac{2^{\frac{2}{p}} \omega_n R^n}{n} + 2^{\frac{2}{p}} \omega_n \int_0^\infty \xi^{n-1} \chi^{\frac{2}{p}}(\xi)\, d\xi$ being finite due to the compactness of supp χ. Noting that the inequalities $\delta_0 < 1$ and $\beta \ge 0$ warrant finiteness also of

$$c_2 := \sup_{\delta \in (0, \delta_0)} a_\delta \equiv \ln^{-\beta}(\delta_0^{-n}), \tag{2.12}$$

this firstly entails that with the finite positive constant $c_3 := c_1 c_2$ we have

$$\int_\Omega \varphi_\delta \le c_3 \quad \text{for all } \delta \in (0, \delta_0). \tag{2.13}$$

To make appropriate use of (2.11) for a second time, we simply estimate $\chi \le 1$ and $\delta_0 \le 1$ in verifying that once more thanks to (2.12),

$$\begin{aligned}
\varphi_\delta + e &\le a_\delta \cdot \{1 + \delta^{-\frac{np}{2}}\}^{\frac{2}{p}} + e \\
&\le a_\delta \cdot (2\delta^{-\frac{np}{2}})^{\frac{2}{p}} + e\delta^{-n} \\
&= (2^{\frac{2}{p}} a_\delta + e)\delta^{-n} \\
&\le c_4 \delta^{-n} \quad \text{in } \Omega, \text{ for all } \delta \in (0, \delta_0)
\end{aligned}$$

with $c_4 := 2^{\frac{2}{p}} c_2 + e$. Therefore, namely, (2.11) together with (2.9) and again (2.12) shows that

$$\begin{aligned}
\int_\Omega \varphi_\delta \ln^\beta(\varphi_\delta + e) &\le \ln^\beta(c_4 \delta^{-n}) \int_\Omega \varphi_\delta \\
&\le c_1 a_\delta \ln^\beta(c_4 \delta^{-n})
\end{aligned}$$

$$
\begin{aligned}
&\le 2^\beta c_1 a_\delta \ln^\beta(\delta^{-n}) + 2^\beta c_1 a_\delta \ln^\beta c_4 \\
&= 2^\beta c_1 + 2^\beta c_1 a_\delta \ln^\beta c_4 \\
&\le c_5 \quad \text{for all } \delta \in (0,\delta_0) \qquad (2.14)
\end{aligned}
$$

if we let $c_5 := 2^\beta c_1 + 2^\beta c_1 c_2 \ln^\beta c_4$.

We next use that $\chi \equiv 1$ in $[0, \frac{1}{2}]$ to see that whenever $\delta \in (0, \delta_0)$ and $x \in B_{\frac{\delta}{2}}(0)$,

$$
\varphi_\delta(x) \ge a_\delta \cdot \Big\{\delta^{-\frac{np}{2}} \chi\Big(\frac{|x|}{\delta}\Big)\Big\}^{\frac{2}{p}} \ge a_\delta \cdot \delta^{-n},
$$

so that due to (2.7) and the monotonicity of h,

$$
\begin{aligned}
&\int_\Omega \varphi_\delta^{p+\frac{2}{n}} \ln^{\frac{2\beta}{n}}(\varphi_\delta + e) h(\varphi_\delta) \\
&\ge \int_{B_{\frac{\delta}{2}}(0)} (a_\delta \cdot \delta^{-n})^{p+\frac{2}{n}} \cdot \ln^{\frac{2\beta}{n}}(a_\delta \cdot \delta^{-n}) \cdot h(a_\delta \cdot \delta^{-n}) \\
&= \frac{\omega_n \cdot (\frac{\delta}{2})^n}{n} \cdot (a_\delta \cdot \delta^{-n})^{p+\frac{2}{n}} \cdot \ln^{\frac{2\beta}{n}}(a_\delta \cdot \delta^{-n}) \cdot h(a_\delta \cdot \delta^{-n}) \\
&= \frac{\omega_n}{n \cdot 2^n} \cdot a_\delta^p \delta^{n-np-2} \cdot a_\delta^{\frac{2}{n}} \ln^{\frac{2\beta}{n}}(a_\delta \cdot \delta^{-n}) \cdot h(a_\delta \cdot \delta^{-n}) \qquad (2.15)
\end{aligned}
$$

for all $\delta \in (0, \delta_0)$. Here, our restriction in (2.8) applies so as to guarantee that

$$
a_\delta \cdot \delta^{-n} = \delta^{-n} \ln^{-\beta}(\delta^{-n}) = \delta^{-\frac{n}{2}} \cdot \{\delta^{\frac{n}{2}} \ln^\beta(\delta^{-n})\}^{-1} \ge \delta^{-\frac{n}{2}} \quad \text{for all } \delta \in (0,\delta_0)
$$

and thus

$$
a_\delta^{\frac{2}{n}} \ln^{\frac{2\beta}{n}}(a_\delta \cdot \delta^{-n}) = \Big\{\frac{\ln(a_\delta \cdot \delta^{-n})}{\ln(\delta^{-n})}\Big\}^{\frac{2\beta}{n}} \ge \Big\{\frac{\ln(\delta^{-\frac{n}{2}})}{\ln(\delta^{-n})}\Big\}^{\frac{2\beta}{n}} = 2^{-\frac{2\beta}{n}} \quad \text{for all } \delta \in (0,\delta_0),
$$

whence (2.15) implies that letting $c_5 := \frac{\omega_n}{n \cdot 2^{n+\frac{2\beta}{n}}}$ we have

$$
\int_\Omega \varphi_\delta^{p+\frac{2}{n}} \ln^{\frac{2\beta}{n}}(\varphi_\delta + e) h(\varphi_\delta) \ge c_5 a_\delta^p \delta^{n-np-2} h(a_\delta \cdot \delta^{-n}) \quad \text{for all } \delta \in (0,\delta_0). \qquad (2.16)
$$

Finally, a differentiation in (2.10) reveals that

$$
|\nabla \varphi_\delta^{\frac{p}{2}}(x)| = a_\delta^{\frac{p}{2}} \delta^{-\frac{np}{2}-1} \Big|\chi'\Big(\frac{|x|}{\delta}\Big)\Big| \quad \text{for all } x \in \bar\Omega \text{ and } \delta \in (0,\delta_0),
$$

so that upon recalling that $\chi' \equiv 0$ in $[1, \infty)$ we obtain that

$$
\begin{aligned}
\int_\Omega |\nabla \varphi_\delta^{\frac{p}{2}}(x)|^2 &\le \int_{B_\delta(0)} |\nabla \varphi_\delta^{\frac{p}{2}}(x)|^2 \\
&\le \|\chi'\|^2_{L^\infty((0,\infty))} a_\delta^p \delta^{-np-2} |B_\delta(0)| \\
&= c_6 a_\delta^p \delta^{n-np-2} \quad \text{for all } \delta \in (0,\delta_0), \qquad (2.17)
\end{aligned}
$$

with $c_6 := \frac{\omega_n \|\chi'\|^2_{L^\infty((0,\infty))}}{n}$.

In summary, from (2.13), (2.14), (2.16) and (2.17) we conclude on the basis of our hypothesis (1.4) that

$$c_5 a_\delta^p \delta^{n-np-2} h(a_\delta \cdot \delta^{-n}) \le \mathcal{F}_1(c_3, c_5) \cdot c_6 a_\delta^p \delta^{n-np-2} + \mathcal{F}_2(c_3, c_5) \quad \text{for all } \delta \in (0, \delta_0),$$

that is,

$$c_5 h(\delta^{-n} \ln^{-\beta}(\delta^{-n})) \le \mathcal{F}_1(c_3, c_5) \cdot c_6 + \mathcal{F}_2(c_3, c_5) \cdot \delta^{np-n+2} \ln^{p\beta}(\delta^{-n}) \quad \text{for all } \delta \in (0, \delta_0)$$

according to (2.9). But since our assumption on p entails that $np - n + 2$ is positive, and that thus

$$c_7 := \mathcal{F}_1(c_3, c_5) \cdot c_6 + \mathcal{F}_2(c_3, c_5) \cdot \sup_{\delta \in (0, \delta_0)} \{\delta^{np-n+2} \ln^{p\beta}(\delta^{-n})\}$$

is finite, and since clearly

$$\delta^{-n} \ln^{-\beta}(\delta^{-n}) \to +\infty \quad \text{as } \delta \searrow 0,$$

this implies the claim, because

$$c_5 \cdot \sup_{\xi > 0} h(\xi) = c_5 \cdot \limsup_{\delta \searrow 0} h(\delta^{-n} \ln^{-\beta}(\delta^{-n})) \le c_7$$

thanks to the monotonicity of h. ∎

3. Precluding blow-up in (1.5). Proof of Theorem 1.4

Next addressing the evolution problem (1.5), we begin our analysis in this regard by recalling the standard parabolic theory developed in [4] in stating the following basic result on local existence and extensibility.

Lemma 3.1. *Let $\alpha > 0$, and assume (1.8). Then there exists $T_{\max} \in (0, \infty]$ as well as*

$$\begin{cases} u \in C^0(\bar\Omega \times [0, T_{\max})) \cap C^{2,1}(\bar\Omega \times (0, T_{\max})) \quad \text{and} \\ v \in \bigcap_{q>2} C^0([0, T_{\max}); W^{1,q}(\Omega)) \cap C^{2,1}(\bar\Omega \times (0, T_{\max})) \end{cases}$$

such that $u > 0$ in $\bar\Omega \times (0, T_{\max})$ and $v > 0$ in $\bar\Omega \times [0, T_{\max})$, that (u, v) solves (1.5) classically in $\Omega \times (0, T_{\max})$, and that if $T_{\max} < \infty$, then

$$\limsup_{t \nearrow T_{\max}} \left\{ \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)} + \left\| \frac{1}{v(\cdot, t)} \right\|_{L^\infty(\Omega)} \right\} = \infty \text{ for all } q > 2.$$

Moreover,

$$\int_\Omega u(\cdot, t) = \int_\Omega u_0 \quad \text{for all } t \in (0, T_{\max}) \tag{3.1}$$

and

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \le \|v_0\|_{L^\infty(\Omega)} \quad \text{for all } t \in (0, T_{\max}). \tag{3.2}$$

3.1. A duality-based argument and immediate consequences

A first piece of regularity information beyond that from (3.1) and (3.2) can be gained by suitably adapting a standard duality-based reasoning to the present situation (cf. also [49]).

Lemma 3.2. *Let $\alpha > 0$, and suppose that $T_{\max} < \infty$. Then there exists $C > 0$ such that*

$$\int_0^t \int_\Omega u^2 v^{-\alpha} \le C \quad \textit{for all } t \in (0, T_{\max}) \tag{3.3}$$

and

$$\int_\Omega v^{-\alpha}(\cdot, t) \le C \quad \textit{for all } t \in (0, T_{\max}). \tag{3.4}$$

Proof. Letting A denote the self-adjoint invertible operator in $L^2_\perp(\Omega) := \{\varphi \in L^2(\Omega) \mid \int_\Omega \varphi = 0\}$ given by $A\varphi := -\Delta\varphi$ for $\varphi \in D(A) := \{W^{2,2}(\Omega) \cap L^2_\perp(\Omega) \mid \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega\}$, we rewrite the first equation in the lifted version $\partial_t A^{-1}(u - \bar{u}_0) = -\{uv^{-\alpha} - \overline{uv^{-\alpha}}\}$, where $\bar{\varphi} := \frac{1}{|\Omega|}\int_\Omega \varphi$ for $\varphi \in L^1(\Omega)$. A multiplication by $u - \bar{u}_0$, followed by an integration, shows that due to (3.1),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |A^{-\frac{1}{2}}(u - \bar{u}_0)|^2 &= -\int_\Omega \{uv^{-\alpha} - \overline{uv^{-\alpha}}\} \cdot (u - \bar{u}_0) \\ &= -\int_\Omega u^2 v^{-\alpha} + \bar{u}_0 \int_\Omega uv^{-\alpha} \quad \text{for all} \in (0, T_{\max}),\end{aligned}$$

and in order to appropriately compensate the rightmost summand herein, we use the second equation in (1.5) to see that

$$\begin{aligned}\frac{d}{dt}\int_\Omega v^{-\alpha} &= -\alpha(\alpha+1)\int_\Omega v^{-\alpha-2}|\nabla v|^2 + \alpha\int_\Omega uv^{-\alpha} \\ &\le \alpha \int_\Omega uv^{-\alpha} \quad \text{for all } t \in (0, T_{\max}).\end{aligned}$$

Thanks to Young's inequality, we therefore obtain that for

$$y(t) := \frac{1}{2}\int_\Omega |A^{-\frac{1}{2}}(u(\cdot,t) - \bar{u}_0)|^2 + \int_\Omega v^{-\alpha}(\cdot, t), \quad t \in [0, T_{\max}),$$

we have

$$\begin{aligned}y'(t) + \frac{1}{2}\int_\Omega u^2 v^{-\alpha} &\le -\frac{1}{2}\int_\Omega u^2 v^{-\alpha} + (\bar{u}_0 + \alpha)\int_\Omega uv^{-\alpha} \\ &\le c_1 y(t) \quad \text{for all } t \in (0, T_{\max})\end{aligned} \tag{3.5}$$

with $c_1 := \frac{(\bar{u}_0+\alpha)^2}{2}$, so that $y(t) \le c_2 := y(0)e^{c_1 T_{\max}}$ for all $t \in [0, T_{\max})$, with c_2 being finite according to our hypothesis that $T_{\max} < \infty$. While this directly implies (3.4), an integration in (3.5) thereafter shows that

$$\frac{1}{2}\int_0^t \int_\Omega u^2 v^{-\alpha} \le y(0) + c_1 \int_0^t y(s)ds \le c_2 + c_1 c_2 T_{\max} \quad \text{for all } t \in (0, T_{\max})$$

and hence also establishes (3.3). ∎

When utilized in the course of a standard testing procedure applied to the second equation in (1.5), the estimate in (3.3) quite immediately entails an integral bound for ∇v, containing a weight which becomes singular near $v = 0$.

Lemma 3.3. *Let $\alpha > 0$, and assume that $T_{max} < \infty$. Then there exists $C > 0$ such that*

$$\int_0^t \int_\Omega \frac{|\nabla v|^4}{v^3} \le C \quad \textit{for all } t \in (0, T_{max})$$

and that

$$\int_0^t \int_\Omega \frac{u}{v}|\nabla v|^2 \le C \quad \textit{for all } t \in (0, T_{max}).$$

Proof. In a fairly straightforward manner (cf. [51, Lemma 3.2]), from the second equation in (1.5) we can derive the identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla v|^2}{v} = -\int_\Omega v|D^2 \ln v|^2 + \frac{1}{2}\int_{\partial\Omega} \frac{1}{v}\cdot\frac{\partial|\nabla v|^2}{\partial\nu} + \int_\Omega u\Delta v - \frac{1}{2}\int_\Omega \frac{u}{v}|\nabla v|^2 \tag{3.6}$$

for all $t \in (0, T_{max})$, where a combination of known functional inequalities and boundary trace embedding estimates ([51, Lemma 3.3], [37, Lemma 4.2], [3, A6.6]) readily yields positive constants c_1, c_2, c_3 and c_4 such that

$$\frac{1}{2}\int_\Omega v|D^2 \ln v|^2 \ge c_1 \int_\Omega \frac{1}{v}|D^2 v|^2 + c_1 \int_\Omega \frac{|\nabla v|^4}{v^3} \quad \text{for all } t \in (0, T_{max})$$

and

$$\begin{aligned}\frac{1}{2}\int_{\partial\Omega} \frac{1}{v}\cdot\frac{\partial|\nabla v|^2}{\partial\nu} &\le c_2 \int_{\partial\Omega} \frac{|\nabla v|^2}{v} \le c_3 \int_\Omega \Big|\nabla\Big(\frac{1}{v}|\nabla v|^2\Big)\Big| + c_3\int_\Omega \frac{|\nabla v|^2}{v} \\ &\le \frac{c_1}{2}\int_\Omega \frac{1}{v}|D^2 v|^2 + \frac{c_1}{2}\int_\Omega \frac{|\nabla v|^4}{v^3} + c_4 \int_\Omega \frac{|\nabla v|^2}{v} \quad \text{for all } t \in (0, T_{max}).\end{aligned}$$

Moreover, using Young's inequality along with (3.2) we find $c_5 > 0$ such that

$$\int_\Omega u\Delta v \le \frac{c_1}{2}\int_\Omega \frac{1}{v}|D^2 v|^2 + c_5 \int_\Omega u^2 v^{-\alpha} \quad \text{for all } t \in (0, T_{max}),$$

whence (3.6) implies that for $y(t) := \int_\Omega \frac{|\nabla v(\cdot,t)|^2}{v(\cdot,t)}$, $t \in [0, T_{max})$, we have

$$\begin{aligned} y'(t) + c_1 \int_\Omega \frac{|\nabla v|^4}{v^3} + \frac{1}{2}\int_\Omega \frac{u}{v}|\nabla v|^2 \\ \le 2c_4 y(t) + 2c_5 \int_\Omega u^2 v^{-\alpha} \quad \text{for all } t \in (0, T_{max}).\end{aligned} \tag{3.7}$$

As a consequence of Lemma 3.2, we thus obtain $c_6 > 0$ such that $y(t) \le c_6$ for all $t \in [0, T_{\max})$, whereupon integrating in (3.7) we infer that

$$c_1 \int_0^t \int_\Omega \frac{|\nabla v|^4}{v^3} + \frac{1}{2} \int_0^t \int_\Omega \frac{u}{v} |\nabla v|^2 \le c_6 + 2c_4 c_6 T_{\max} + 2c_5 \int_0^t \int_\Omega u^2 v^{-\alpha} \quad \text{for all } t \in (0, T_{\max})$$

and conclude as intended by again relying on Lemma 3.2. ∎

3.2. Space-time $L^2 \log^\gamma L$ bounds for u. Application of Proposition 1.1

We now approach the core of our analysis concerned with (1.5), culminating in two applications of Proposition 1.1, in Lemmas 3.7 and 3.10, which in turn facilitate our derivation of a pointwise lower bound for v. In fact, in Lemma 3.11 the latter will be achieved by utilizing, after performing a Hopf–Cole-type transformation, the following general result from parabolic regularity theory ([52]; cf. also [9, 55] together with [2]), which we state here in order to specify the particular purpose of our subsequent efforts.

Lemma 3.4. *Let $L \in C^0([0,\infty))$ be strictly increasing and positive with $L(\xi) \to +\infty$ as $\xi \to \infty$ and*

$$\int_1^\infty \frac{d\xi}{\xi L(\xi)} < \infty.$$

Then for each $K > 0$ and any $T > 0$ there exists $C(K,T) > 0$ with the property that whenever $w \in \bigcap_{q>2} C^0([0,T]; W^{1,q}(\Omega)) \cap C^{2,1}(\bar\Omega \times (0,T))$ and $f \in C^0(\bar\Omega \times [0,T))$ are such that

$$\|w(\cdot,0)\|_{W^{1,\infty}(\Omega)} \le K$$

and

$$\int_0^t \int_\Omega f^2 L(|f|) \le K \quad \textit{for all } t \in (0,T),$$

as well as

$$\begin{cases} w_t = \Delta w + f(x,t), & x \in \Omega,\ t \in (0,T), \\ \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (0,T), \end{cases}$$

we have

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \le C(K) \quad \textit{for all } t \in (0,T).$$

Proof. This is a particular consequence of [52, Theorem 1.1]. ∎

In preparation for our application of this, we record the following observation which reflects a second structural feature of (1.5), beyond that underlying our argument in Lemma 3.2. It will be of crucial importance for our subsequent reasoning that the class of functions ℓ admissible below not only includes the choice $\ell(\xi) = \ln \xi$, as previously used in classical detections of energy structures in related problems ([15, Lemma 6.1]), but also some relatives exhibiting stronger logarithmic-type growth (see Lemma 3.6 and 3.9).

Lemma 3.5. *Let $\alpha > 0$, and suppose that $\ell \in C^2((0,\infty))$ is such that*

$$\xi\ell''(\xi) + 2\ell'(\xi) \geq 0 \quad \textit{for all } \xi > 0. \tag{3.8}$$

Then writing

$$z := uv^{-\alpha}, \tag{3.9}$$

we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u\ell(z) + \frac{1}{2}\int_\Omega \{z\ell''(z) + 2\ell'(z)\}|\nabla z|^2 \\ &\quad \leq \alpha(\alpha-1)\int_\Omega uv^{-2}\cdot z\ell'(z)|\nabla v|^2 + \frac{\alpha^2}{2}\int_\Omega uv^{\alpha-2}\cdot\{z^2\ell''(z) + 2z\ell'(z)\}\cdot|\nabla v|^2 \\ &\qquad + \alpha\int_\Omega u^2\cdot z\ell'(z) \quad \textit{for all } t\in(0,T_{\max}). \end{aligned} \tag{3.10}$$

Proof. Using (3.9) and (1.5), we compute

$$\begin{aligned} \frac{d}{dt}\int_\Omega u\ell(z) &= \int_\Omega \ell(z)u_t + \int_\Omega u\ell'(z)\cdot\{v^{-\alpha}u_t - \alpha uv^{-\alpha-1}v_t\} \\ &= \int_\Omega \ell(z)\Delta z + \int_\Omega z\ell'(z)\Delta z \\ &\quad - \alpha\int_\Omega u^2v^{-\alpha-1}\ell'(z)\cdot\{\Delta v - uv\} \quad \text{for all } t\in(0,T_{\max}), \end{aligned} \tag{3.11}$$

where two integrations by parts show that

$$\int_\Omega \ell(z)\Delta z = -\int_\Omega \ell'(z)|\nabla z|^2 \quad \text{for all } t\in(0,T_{\max})$$

and

$$\int_\Omega z\ell'(z)\Delta z = -\int_\Omega \{z\ell''(z) + \ell'(z)\}|\nabla z|^2 \quad \text{for all } t\in(0,T_{\max}),$$

so that

$$\begin{aligned} &\int_\Omega \ell(z)\Delta z + \int_\Omega z\ell'(z)\Delta z \\ &\quad = -\int_\Omega \{z\ell''(z) + 2\ell'(z)\}|\nabla z|^2 \quad \text{for all } t\in(0,T_{\max}). \end{aligned} \tag{3.12}$$

Moreover, integrating by parts once again we see that

$$\begin{aligned} &-\alpha\int_\Omega u^2v^{-\alpha-1}\ell'(z)\Delta v \\ &\quad = -\alpha\int_\Omega v^{\alpha-1}z^2\ell'(z)\Delta v \\ &\quad = \alpha(\alpha-1)\int_\Omega v^{\alpha-2}z^2\ell'(z)|\nabla v|^2 \\ &\qquad + \alpha\int_\Omega v^{\alpha-1}\cdot\{z^2\ell''(z) + 2z\ell'(z)\}\nabla v\cdot\nabla z \quad \text{for all } t\in(0,T_{\max}), \end{aligned} \tag{3.13}$$

where relying on (3.8) we may use Young's inequality to estimate

$$\alpha \int_\Omega v^{\alpha-1} \cdot \{z^2\ell''(z) + 2z\ell'(z)\}\nabla v \cdot \nabla z$$
$$\le \frac{1}{2}\int_\Omega \{z\ell''(z) + 2\ell'(z)\}|\nabla z|^2 + \frac{\alpha^2}{2}\int_\Omega v^{2\alpha-2}z^2 \cdot \{z\ell''(z) + 2\ell'(z)\}|\nabla v|^2 \quad (3.14)$$

for all $t \in (0, T_{\max})$. Since, apart from that, on the right-hand side of (3.11) we have

$$-\alpha \int_\Omega u^2 v^{-\alpha-1}\ell'(z) \cdot (-uv) = \alpha \int_\Omega u^2 z\ell'(z) \quad \text{for all } t \in (0, T_{\max}),$$

from (3.11)–(3.14) and (3.9) we readily obtain (3.10). ∎

In conjunction with the outcomes of Lemmas 3.2 and 3.3, a first application of this reveals a quasi-energy property of $\int_\Omega u \ln(uv^{-\alpha})$, which supplements our knowledge on regularity not only of u but also of certain first-order expressions.

Lemma 3.6. *Suppose that $\alpha > 0$, and that $T_{\max} < \infty$. Then there exists $C > 0$ such that*

$$\int_\Omega u(\cdot,t)\ln\{u(\cdot,t) + e\} \le C \quad \textit{for all } t \in (0, T_{\max}) \quad (3.15)$$

and

$$\int_0^t \int_\Omega |\nabla (uv^{-\alpha})^{\frac{1}{2}}|^2 \le C \quad \textit{for all } t \in (0, T_{\max}), \quad (3.16)$$

as well as

$$\int_0^t \int_\Omega uv^{-2}|\nabla v|^2 \le C \quad \textit{for all } t \in (0, T_{\max}). \quad (3.17)$$

Proof. We let $\ell(\xi) := \ln \xi$, $\xi > 0$, and note that then

$$\xi\ell''(\xi) + 2\ell'(\xi) = \xi \cdot \frac{-1}{\xi^2} + \frac{2}{\xi} = \frac{1}{\xi} \quad \text{for all } \xi > 0$$

as well as

$$\xi\ell'(\xi) = 1 \quad \text{and} \quad \xi^2\ell''(\xi) + 2\xi\ell'(\xi) = 1 \quad \text{for all } \xi > 0,$$

so that we may draw on Lemma 3.5 to see that with z as in (3.9),

$$\frac{d}{dt}\int_\Omega u \ln z + \frac{1}{2}\int_\Omega \frac{|\nabla z|^2}{z} \le \alpha(\alpha-1)\int_\Omega uv^{-2}|\nabla v|^2 + \frac{\alpha^2}{2}\int_\Omega uv^{\alpha-2}|\nabla v|^2$$
$$+ \alpha \int_\Omega u^2 \quad \text{for all } t \in (0, T_{\max}). \quad (3.18)$$

Here, when $\alpha \in (0, 1)$, we can pick $\delta > 0$ small enough such that

$$\frac{\alpha^2}{2}\delta^\alpha \le \frac{\alpha(1-\alpha)}{2},$$

and estimate

$$\begin{aligned}
&\frac{\alpha^2}{2}\int_\Omega uv^{\alpha-2}|\nabla v|^2\\
&\quad=\frac{\alpha^2}{2}\int_{\{v\le\delta\}} uv^{\alpha-2}|\nabla v|^2+\frac{\alpha^2}{2}\int_{\{v>\delta\}} uv^{\alpha-2}|\nabla v|^2\\
&\quad\le\frac{\alpha^2}{2}\delta^\alpha\int_\Omega uv^{-2}|\nabla v|^2+\frac{\alpha^2}{2}\delta^{\alpha-1}\int_\Omega \frac{u}{v}|\nabla v|^2\\
&\quad\le\frac{\alpha(1-\alpha)}{2}\int_\Omega uv^{-2}|\nabla v|^2+\frac{\alpha^2}{2}\delta^{\alpha-1}\int_\Omega \frac{u}{v}|\nabla v|^2 \quad \text{for all } t\in(0,T_{\max}),
\end{aligned}$$

whence in this case we obtain from (3.18) and (3.2) that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u\ln z+\frac{1}{2}\int_\Omega\frac{|\nabla z|^2}{z}+c_1\int_\Omega uv^{-2}|\nabla v|^2\\
&\quad\le c_2\int_\Omega\frac{u}{v}|\nabla v|^2+c_2\int_\Omega u^2v^{-\alpha}\quad\text{for all } t\in(0,T_{\max}),
\end{aligned}\tag{3.19}$$

with $c_1:=\frac{\alpha(1-\alpha)}{2}$ and $c_2:=\max\{\frac{\alpha^2}{2}\delta^{\alpha-1},\alpha\|v_0\|^\alpha_{L^\infty(\Omega)}\}$ both being positive. Therefore, given any such α we obtain that

$$\begin{aligned}
&\int_\Omega u(\cdot,t)\ln z(\cdot,t)+\frac{1}{2}\int_0^t\int_\Omega\frac{|\nabla z|^2}{z}+c_1\int_0^t\int_\Omega uv^{-2}|\nabla v|^2\\
&\quad\le\int_\Omega u_0\ln(u_0v_0^{-\alpha})+c_2\int_0^t\int_\Omega\frac{u}{v}|\nabla v|^2+c_2\int_0^t\int_\Omega u^2v^{-\alpha}\quad\text{for all } t\in(0,T_{\max}),
\end{aligned}$$

so that using Lemmas 3.3 and 3.2 together with (1.8) we then find $c_3>0$ such that

$$\int_\Omega u(\cdot,t)\ln z(\cdot,t)+\frac{1}{2}\int_0^t\int_\Omega\frac{|\nabla z|^2}{z}+c_1\int_0^t\int_\Omega uv^{-2}|\nabla v|^2\le c_3\quad\text{for all } t\in(0,T_{\max}).$$

Observing that

$$\begin{aligned}
\int_\Omega u\ln z&=\int_\Omega u\ln(u+e)-\int_\Omega u\ln\Big(1+\frac{e}{u}\Big)-\alpha\int_\Omega u\ln v\\
&\ge\int_\Omega u\ln(u+e)-e|\Omega|-\alpha\ln\|v_0\|_{L^\infty(\Omega)}\int_\Omega u_0\quad\text{for all } t\in(0,T_{\max})
\end{aligned}$$

thanks to (3.2), (3.1) and the fact that $\xi\ln(1+\frac{e}{\xi})\le\xi\cdot\frac{e}{\xi}=e$ for all $\xi>0$, from this we conclude that (3.15)–(3.17) hold with some suitably large $C>0$ in this case, because

$$\frac{|\nabla z|^2}{z}=4|\nabla(uv^{-\alpha})^{\frac{1}{2}}|^2\quad\text{in }\Omega\times(0,T_{\max}).$$

If, conversely, $\alpha \ge 1$, then by (3.2) and Young's inequality, the first two integrals on the right of (3.18) can both be estimated in modulus according to

$$\begin{aligned}\int_\Omega uv^{-2}|\nabla v|^2 + \int_\Omega uv^{\alpha-2}|\nabla v|^2 \\ &\le \{1 + \|v_0\|^{\alpha}_{L^\infty(\Omega)}\} \cdot \int_\Omega uv^{-2}|\nabla v|^2 \\ &\le \int_\Omega \frac{|\nabla v|^4}{v^3} + \frac{1}{4} \cdot \{1 + \|v_0\|^{\alpha}_{L^\infty(\Omega)}\}^2 \int_\Omega u^2 v^{-1} \quad \text{for all } t \in (0, T_{\max}),\end{aligned}$$

where in this case we know from (3.2) that

$$\int_\Omega u^2 v^{-1} \le \|v_0\|^{\alpha-1}_{L^\infty(\Omega)} \int_\Omega u^2 v^{-\alpha} \quad \text{for all } t \in (0, T_{\max}).$$

We therefore see that (3.18) implies an inequality of the form in (3.19) also within this range of larger α, so that we may conclude as before. ■

Now, together with (3.1) and (3.2), the three estimates in (3.15)–(3.17) quite precisely pave the way for the first of the announced two applications of Proposition 1.1:

Lemma 3.7. *Let $\alpha > 0$, and assume that $T_{\max} < \infty$. Then there exists $C > 0$ such that*

$$\int_0^t \int_\Omega u^2 \ln(u + e) \le C \quad \textit{for all } t \in (0, T_{\max}). \tag{3.20}$$

Proof. In view of (3.15)–(3.17), this is a consequence of Proposition 1.1 when applied to $n := 2$, $p := 1$, $\beta := 1$, $\varphi := u$ and $\psi := v$. ■

Since $\int_1^\infty \frac{d\xi}{\xi \ln^\gamma(\xi+e)}$ is finite if and only if $\gamma > 1$, the information obtained through Lemma 3.7 seems yet insufficient to allow for a successful application of Lemma 3.4 in the intended flavor. Accordingly, Lemmas 3.9 and 3.10 will be concerned with the extension of (3.20) to a corresponding integral estimate for $u^2 \ln^\gamma(u + e)$ with some $\gamma > 1$. This will be achieved by again resorting to the basic evolution feature noted in Lemma 3.5, and by using Lemma 3.7, again together with Lemmas 3.2 and 3.3, as a starting point.

A technical preparation of an elementary nature is provided by the following.

Lemma 3.8. *Let $\alpha > 0$, $\kappa > 0$ and $\eta > 0$. Then there exists $C(\alpha, \kappa, \eta) > 0$ such that for the function z in (3.9) we have*

$$\ln^\kappa(z + e) \le C(\alpha, \kappa, \eta) \cdot \{\ln^\kappa(u + e) + v^{-\eta}\} \quad \textit{in } \Omega \times (0, T_{\max}). \tag{3.21}$$

Proof. Given $\kappa > 0$ and $\eta > 0$, we pick $c_1 = c_1(\kappa, \eta) > 0$ such that

$$\ln \xi \le c_1 \xi^{\frac{\eta}{\kappa}} \quad \text{for all } \xi > 0, \tag{3.22}$$

and to make appropriate use of this, we first note that if $(x, t) \in \Omega \times (0, T_{\max})$ is such that $z(x, t) \le e$, then $\ln^\kappa(z(x, t) + e) \le \ln^\kappa(2e)$, so that (3.21) holds whenever $C(\alpha, \kappa, \eta) \ge \ln^\kappa(2e)$, because trivially $\ln^\kappa(u(x, t) + e) \ge 1$.

Thus, left with the case when $(x,t) \in \Omega \times (0, T_{max})$ is such that $z(x,t) > e$, we observe that then, by (3.22), and again by the fact that $\ln(\xi + e) \geq 1$ for all $\xi > 0$,

$$\begin{aligned}
\ln^\kappa(z+e) \leq \ln^\kappa(2z) &= \Big\{\ln u + \alpha \ln \frac{1}{v} + \ln 2\Big\}^\kappa \\
&\leq \Big\{\ln(u+e) + c_1\alpha \cdot \Big(\frac{1}{v}\Big)^{\frac{\eta}{\kappa}} + \ln 2\Big\}^\kappa \\
&\leq \{(1+\ln 2)\ln(u+e) + c_1\alpha v^{-\frac{\eta}{\kappa}}\}^\kappa \\
&\leq 2^\kappa \cdot (1+\ln 2)^\kappa \cdot \ln^\kappa(u+e) + 2^\kappa \cdot (c_1\alpha)^\kappa \cdot v^{-\eta},
\end{aligned}$$

meaning that for suitably large $C(\alpha, \kappa, \eta) > 0$, (3.21) also holds at this point. ∎

By means of a second and now more subtle exploitation of Lemma 3.5 we can indeed improve (3.15) as follows.

Lemma 3.9. *Let $\alpha > 0$, and suppose that $T_{max} < \infty$. Then there exist $\gamma > 1$ and $C > 0$ such that*

$$\int_\Omega u(\cdot,t)\ln^\gamma\{u(\cdot,t)+e\} \leq C \quad \text{for all } t \in (0, T_{max}). \tag{3.23}$$

Proof. We take $\gamma > 1$ in such a way that

$$\begin{cases} \gamma < \dfrac{3}{2} & \text{if } \alpha \leq 1, \\ \gamma \leq \dfrac{3\alpha - 1}{2\alpha} & \text{if } \alpha > 1, \end{cases} \tag{3.24}$$

and we thereupon let

$$\ell(\xi) := \ln^\gamma(\xi + e), \xi > 0,$$

observing that

$$\begin{aligned}
\ell'(\xi) &= \frac{\gamma \ln^{\gamma-1}(\xi+e)}{\xi+e} \\
\ell''(\xi) &= \frac{-\gamma \ln^{\gamma-1}(\xi+e)}{(\xi+e)^2} + \frac{\gamma(\gamma-1)\ln^{\gamma-2}(\xi+e)}{(\xi+e)^2} \quad \text{for all } \xi > 0,
\end{aligned}$$

and that thus

$$0 \leq \xi\ell'(\xi) \leq \gamma \ln^{\gamma-1}(\xi+e) \quad \text{for all } \xi > 0, \tag{3.25}$$

as well as

$$\begin{aligned}
\xi\ell''(\xi) + 2\ell'(\xi) = &\frac{-\gamma\xi \ln^{\gamma-1}(\xi+e)}{(\xi+e)^2} + \frac{\gamma(\gamma-1)\xi \ln^{\gamma-2}(\xi+e)}{(\xi+e)^2} \\
&+ \frac{2\gamma \ln^{\gamma-1}(\xi+e)}{\xi+e} \quad \text{for all } \xi > 0,
\end{aligned}$$

whence in particular

$$\begin{aligned}\xi\ell''(\xi)+2\ell'(\xi) &\ge \frac{-\gamma\xi\ln^{\gamma-1}(\xi+e)}{(\xi+e)^2}+\frac{2\gamma\ln^{\gamma-1}(\xi+e)}{\xi+e}\\ &\ge \frac{\gamma\ln^{\gamma-1}(\xi+e)}{\xi+e}\quad \text{for all } \xi>0 \end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\xi\ell''(\xi)+2\ell'(\xi) &\le \frac{\gamma(\gamma-1)\xi\ln^{\gamma-2}(\xi+e)}{(\xi+e)^2}+\frac{2\gamma\ln^{\gamma-1}(\xi+e)}{\xi+e}\\ &\le \frac{\gamma(\gamma-1)\ln^{\gamma-1}(\xi+e)}{\xi+e}+\frac{2\gamma\ln^{\gamma-1}(\xi+e)}{\xi+e}\\ &= \frac{\gamma(\gamma+1)\ln^{\gamma-1}(\xi+e)}{\xi+e}\quad \text{for all } \xi>0. \end{aligned} \tag{3.27}$$

Now (3.26) enables us to employ Lemma 3.5, and to thereby conclude using (3.27) and (3.25) that with z taken from (3.9) we have

$$\frac{d}{dt}\int_\Omega u\ln^\gamma(z+e)\le I_1+I_2+I_3\quad \text{for all } t\in(0,T_{\max}), \tag{3.28}$$

where

$$\begin{aligned} I_1 &:= \alpha(\alpha-1)\gamma\int_\Omega uv^{-2}\cdot\frac{z\ln^{\gamma-1}(z+e)}{z+e}\cdot|\nabla v|^2,\\ I_2 &:= \frac{\alpha^2\gamma(\gamma+1)}{2}\int_\Omega uv^{\alpha-2}\cdot\frac{z\ln^{\gamma-1}(z+e)}{z+e}\cdot|\nabla v|^2, \end{aligned} \tag{3.29}$$

and

$$I_3 := \alpha\gamma\int_\Omega u^2\cdot\frac{z\ln^{\gamma-1}(z+e)}{z+e} \tag{3.30}$$

for $t\in(0,T_{\max})$. Here, since $\alpha>0$, we may apply Lemma 3.8 to $\eta=\alpha$ to see that with some $c_1>0$,

$$\begin{aligned} I_3 &\le \alpha\gamma\int_\Omega u^2\ln^{\gamma-1}(z+e)\\ &\le c_1\int_\Omega u^2\ln^{\gamma-1}(u+e)+c_1\int_\Omega u^2v^{-\alpha}\\ &\le c_1\int_\Omega u^2\ln(u+e)+c_1\int_\Omega u^2v^{-\alpha}\quad \text{for all } t\in(0,T_{\max}), \end{aligned} \tag{3.31}$$

because $\gamma-1\le 1$.

In the case when $\alpha<1$, to make use of the nonpositivity of I_1 we pick $\delta>0$ small enough such that

$$\frac{\alpha^2\gamma(\gamma+1)}{2}\delta^\alpha\le\alpha(1-\alpha)\gamma,$$

and then obtain that, indeed,

$$\begin{aligned}
I_1 + I_2 &= -\alpha(1-\alpha)\gamma \int_\Omega uv^{-2} \cdot \frac{z\ln^{\gamma-1}(z+e)}{z+e} \cdot |\nabla v|^2 \\
&\quad + \frac{\alpha^2\gamma(\gamma+1)}{2} \int_{\{v\le\delta\}} uv^{\alpha-2} \cdot \frac{z\ln^{\gamma-1}(z+e)}{z+e} \cdot |\nabla v|^2 \\
&\quad + \frac{\alpha^2\gamma(\gamma+1)}{2} \int_{\{v>\delta\}} uv^{\alpha-2} \cdot \frac{z\ln^{\gamma-1}(z+e)}{z+e} \cdot |\nabla v|^2 \\
&\le -\alpha(1-\alpha)\gamma \int_\Omega uv^{-2} \cdot \frac{z\ln^{\gamma-1}(z+e)}{z+e} \cdot |\nabla v|^2 \\
&\quad + \frac{\alpha^2\gamma(\gamma+1)}{2}\delta^\alpha \int_\Omega uv^{-2} \cdot \frac{z\ln^{\gamma-1}(z+e)}{z+e} \cdot |\nabla v|^2 \\
&\quad + \frac{\alpha^2\gamma(\gamma+1)}{2} \int_{\{v>\delta\}} uv^{\alpha-2} \cdot \frac{z\ln^{\gamma-1}(z+e)}{z+e} \cdot |\nabla v|^2 \\
&\le \frac{\alpha^2\gamma(\gamma+1)}{2} \int_{\{v>\delta\}} uv^{\alpha-2} \cdot \frac{z\ln^{\gamma-1}(z+e)}{z+e} \cdot |\nabla v|^2 \quad \text{for all } t \in (0, T_{\max}).
\end{aligned}$$

Again by Lemma 3.8, this time applied to $\eta = 2\alpha$, utilizing Young's inequality and (3.2) we find $c_2 > 0$, $c_3 > 0$ and $c_4 > 0$ fulfilling

$$\begin{aligned}
I_1 + I_2 &\le c_2 \int_\Omega \frac{|\nabla v|^4}{v^3} + c_2 \int_{\{v>\delta\}} u^2 v^{2\alpha-1} \ln^{2\gamma-2}(z+e) \\
&\le c_2 \int_\Omega \frac{|\nabla v|^4}{v^3} + c_3 \int_{\{v>\delta\}} u^2 v^{2\alpha-1} \ln^{2\gamma-2}(u+e) + c_3 \int_{\{v>\delta\}} u^2 v^{-1} \\
&\le c_2 \int_\Omega \frac{|\nabla v|^4}{v^3} + c_4 \int_\Omega u^2 \ln(u+e) \\
&\quad + c_3 \int_\Omega u^2 v^{-\alpha} \quad \text{for all } t \in (0, T_{\max}),
\end{aligned} \tag{3.32}$$

where we have used that $2\gamma - 2 \le 1$ due to the fact that $\gamma \le \frac{3}{2}$. In this case $\alpha < 1$, collecting (3.28)–(3.32) we thus obtain that

$$\begin{aligned}
\frac{d}{dt} \int_\Omega u \ln^\gamma(z+e) &\le c_2 \int_\Omega \frac{|\nabla v|^4}{v^3} + (c_1 + c_4) \int_\Omega u^2 \ln(u+e) \\
&\quad + (c_1 + c_3) \int_\Omega u^2 v^{-\alpha} \quad \text{for all } t \in (0, T_{\max}),
\end{aligned}$$

which upon an integration using Lemmas 3.3, 3.7 and 3.2 implies that if $T_{\max}$ is finite, then with some $c_5 > 0$ we have

$$\int_\Omega u \ln^\gamma(z+e) \le c_5 \quad \text{for all } t \in (0, T_{\max}). \tag{3.33}$$

Here we note that abbreviating $c_6 := \min\{1, \|v_0\|_{L^\infty(\Omega)}^{-\alpha}\}$ we know from (3.9) and (3.2) that

$$\int_\Omega u \ln^\gamma(z+e) = \int_\Omega u \ln^\gamma(uv^{-\alpha}+e)$$
$$\geq \int_\Omega u \ln^\gamma(c_6 u + e) \quad \text{for all } t \in (0, T_{\max}).$$

Since $c_6 \leq 1$, we may thus draw on the concavity of $0 < \xi \mapsto \ln \xi$ to infer that

$$\begin{aligned} \int_\Omega u \ln^\gamma(z+e) &\geq \int_\Omega u \cdot \{\ln\{c_6(u+e) + (1-c_6)e\}\}^\gamma \\ &\geq \int_\Omega u \cdot \{c_6 \ln(u+e) + (1-c_6)\ln e\}^\gamma \\ &\geq c_6^\gamma \int_\Omega u \ln^\gamma(u+e) \quad \text{for all } t \in (0, T_{\max}), \end{aligned} \tag{3.34}$$

so that (3.33) entails (3.23) with some appropriately large $C > 0$ if $\alpha < 1$.

In order to simultaneously consider the cases $\alpha = 1$ and $\alpha > 1$, let us set $\iota := \alpha$ if $\alpha = 1$, and $\iota := 0$ if $\alpha > 1$, noting that then

$$\frac{2\iota - 1}{3 - 2\gamma} \geq -\alpha \tag{3.35}$$

due to our restrictions in (3.24). Using that $I_1 = 0$ when $\alpha = 1$, by a combination of (3.2) with Young's inequality and Lemma 3.8, now applied to $\eta = \alpha - 2\iota + 1 > 0$, we then obtain that whenever $\alpha \geq 1$ we can find $c_8 > 0$ and $c_9 > 0$ such that

$$\begin{aligned} I_1 + I_2 &\leq c_8 \int_\Omega uv^{\iota-2} \ln^{\gamma-1}(z+e)|\nabla v|^2 \\ &\leq c_8 \int_\Omega \frac{|\nabla v|^4}{v^3} + c_8 \int_\Omega u^2 v^{2\iota-1} \ln^{2\gamma-2}(z+e) \\ &\leq c_8 \int_\Omega \frac{|\nabla v|^4}{v^3} + c_9 \int_\Omega u^2 v^{2\iota-1} \ln^{2\gamma-2}(u+e) \\ &\quad + c_9 \int_\Omega u^2 v^{-\alpha} \quad \text{for all } t \in (0, T_{\max}), \end{aligned}$$

where by Young's inequality, (3.35) and (3.2), with some $c_{10} > 0$ we have

$$\begin{aligned} \int_\Omega u^2 v^{2\iota-1} \ln^{2\gamma-2}(u+e) &= \int_\Omega \{u^2 \ln(u+e)\}^{2\gamma-2} \cdot u^{6-4\gamma} v^{2\iota-1} \\ &\leq \int_\Omega u^2 \ln(u+e) + \int_\Omega u^2 v^{\frac{2\iota-1}{3-2\gamma}} \\ &\leq \int_\Omega u^2 \ln(u+e) + c_{10} \int_\Omega u^2 v^{-\alpha} \quad \text{for all } t \in (0, T_{\max}). \end{aligned}$$

Therefore, (3.28)–(3.31) in this case show that for all $t \in (0, T_{\max})$,

$$\frac{d}{dt}\int_\Omega u\ln^\gamma(z+e) \le c_8\int_\Omega \frac{|\nabla v|^4}{v^3} + (c_1+c_9)\int_\Omega u^2\ln(u+e) + (c_1+c_9+c_9c_{10})\int_\Omega u^2 v^{-\alpha},$$

whence arguing as before we can confirm the validity of (3.23) for $\alpha \ge 1$ as well. ∎

Once more thanks to our general interpolation result, this enables us to strengthen Lemma 3.7 in the following sense.

Lemma 3.10. *If $\alpha > 0$ and $T_{\max} < \infty$, then there exist $\gamma > 1$ and $C > 0$ such that*

$$\int_0^t\int_\Omega u^2\ln^\gamma(u+e) \le C \quad \textit{for all } t \in (0, T_{\max}).$$

Proof. We only need to take $\gamma > 1$ as in Lemma 3.9, and apply Proposition 1.1 to $p := 1$ and $\beta := \gamma$, and once more to $\varphi := u$ and $\psi := v$, using (3.23) together with, again, (3.16) and (3.17). ∎

Relying on the fact that the inequality $\gamma > 1$ ensures finiteness of $\int_1^\infty \frac{d\xi}{\xi\ln^\gamma(\xi+e)}$, we may now draw on Lemma 3.4 and a simple comparison argument to bound v from below.

Lemma 3.11. *Suppose that $\alpha > 0$, and that $T_{\max} < \infty$. Then there exists $C > 0$ such that*

$$v(x,t) \ge C \quad \textit{for all } x \in \Omega \textit{ and } t \in (0, T_{\max}).$$

Proof. We note that according to (1.5) and Lemma 3.1, the function $\underline{w} := \ln\frac{1}{v}$ belongs to $\bigcap_{q>2} C^0([0,T_{\max}); W^{1,q}(\Omega)) \cap C^{2,1}(\bar\Omega\times(0,T_{\max}))$ and solves

$$\begin{cases} \underline{w}_t = \Delta\underline{w} - |\nabla\underline{w}|^2 + u, & x\in\Omega,\ t\in(0,T_{\max}),\\ \dfrac{\partial\underline{w}}{\partial\nu} = 0, & x\in\partial\Omega,\ t\in(0,T_{\max}),\\ \underline{w}(x,0) = \ln\dfrac{1}{v_0(x)}, & x\in\Omega. \end{cases}$$

Therefore, a comparison argument shows that

$$\underline{w}(x,t) \le w(x,t) \quad \text{for all } x\in\Omega \text{ and } t\in(0,T_{\max}), \tag{3.36}$$

where $w \in \bigcap_{q>2} C^0([0,T_{\max}); W^{1,q}(\Omega)) \cap C^{2,1}(\bar\Omega\times(0,T_{\max}))$ denotes the classical solution of

$$\begin{cases} w_t = \Delta w + u, & x\in\Omega,\ t\in(0,T_{\max}),\\ \dfrac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega,\ t\in(0,T_{\max}),\\ w(x,0) = \ln\dfrac{1}{v_0(x)}, & x\in\Omega. \end{cases} \tag{3.37}$$

Since our assumptions together with Lemma 3.10 ensure that with γ as provided there we have $\int_0^{T_{max}} \int_\Omega u^2 \ln^\gamma(u+e) < \infty$, and since

$$\int_1^\infty \frac{d\sigma}{\sigma \ln^\gamma(\sigma+e)} < \infty$$

due to the inequality $\gamma > 1$, an application of Lemma 3.4 to (3.37) yields $c_1 > 0$ fulfilling

$$w(x,t) \le c_1 \quad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max}),$$

so that thanks to (3.36), the claim results if we let $C := e^{-c_1}$. ∎

3.3. Bounds in $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ via bootstrapping. Proof of Theorem 1.4

With the singularity in (1.5) being favorably under control now, we can proceed in quite a standard manner to derive higher regularity features. Indeed, L^p bounds for u can be obtained by combining Lemma 3.11 with (3.2) and Lemma 3.3.

Lemma 3.12. *Suppose that $\alpha > 0$, and that $T_{max} < \infty$. Then for all $p > 1$ there exists $C(p) > 0$ such that*

$$\int_\Omega u^p(\cdot,t) \le C(p) \quad \textit{for all } t \in (0, T_{max}). \tag{3.38}$$

Proof. We use u^{p-1} as a test function in the first equation from (1.5) to see that due to Young's inequality,

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p &= -(p-1)\int_\Omega u^{p-2}\nabla u \cdot \{v^{-\alpha}\nabla u - \alpha u v^{-\alpha-1}\nabla v\} \\
&= -(p-1)\int_\Omega u^{p-2}v^{-\alpha}|\nabla u|^2 + (p-1)\alpha \int_\Omega u^{p-1}v^{-\alpha-1}\nabla u \cdot \nabla v \\
&\le -\frac{p-1}{2}\int_\Omega u^{p-2}v^{-\alpha}|\nabla u|^2 \\
&\quad + \frac{(p-1)\alpha^2}{2}\int_\Omega u^p v^{-\alpha-2}|\nabla v|^2 \quad \text{for all } t \in (0, T_{max}).
\end{aligned}$$

In view of the two-sided positive pointwise bounds for v provided by Lemma 3.11 and (3.2), this means that with some $c_1 = c_1(p) > 0$ and $c_2 = c_2(p) < 0$ we have

$$\frac{d}{dt}\int_\Omega u^p + c_1 \int_\Omega |\nabla u^{\frac{p}{2}}|^2 \le c_2 \int_\Omega u^p |\nabla v^{\frac{1}{4}}|^2 \quad \text{for all } t \in (0, T_{max}),$$

where by the Cauchy–Schwarz inequality, the Gagliardo–Nirenberg inequality and the Young inequality, we can find $c_i = c_i(p) > 0$, $i \in \{3,4,5\}$, such that

$$\begin{aligned}
c_2 \int_\Omega u^p |\nabla v^{\frac{1}{4}}|^2 &\le c_2 \|\nabla v^{\frac{1}{4}}\|^2_{L^4(\Omega)} \|u^{\frac{p}{2}}\|^2_{L^4(\Omega)} \\
&\le c_3 \|\nabla v^{\frac{1}{4}}\|^2_{L^4(\Omega)} \|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)} \|u^{\frac{p}{2}}\|_{L^2(\Omega)} + c_3 \|\nabla v^{\frac{1}{4}}\|^2_{L^4(\Omega)} \|u^{\frac{p}{2}}\|^2_{L^2(\Omega)}
\end{aligned}$$

$$\leq c_1 \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_4 \|\nabla v^{\frac{1}{4}}\|_{L^4(\Omega)}^4 \|u^{\frac{p}{2}}\|_{L^2(\Omega)}^2 + c_3 \|\nabla v^{\frac{1}{4}}\|_{L^4(\Omega)}^2 \|u^{\frac{p}{2}}\|_{L^2(\Omega)}^2$$
$$\leq c_1 \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_5 \cdot \left\{1 + \int_\Omega \frac{|\nabla v|^4}{v^3}\right\} \cdot \int_\Omega u^p \quad \text{for all } t \in (0, T_{\max}).$$

Therefore, writing $h(t) := c_5 \cdot \{1 + \int_\Omega \frac{|\nabla v(\cdot,t)|^4}{v^3(\cdot,t)}\}$, $t \in (0, T_{\max})$, we obtain that

$$\frac{d}{dt} \int_\Omega u^p \leq h(t) \int_\Omega u^p \quad \text{for all } t \in (0, T_{\max}),$$

so that (3.38) follows upon an integration using that $\int_0^{T_{\max}} h(t)\,dt$ is finite according to Lemma 3.3. ■

Standard parabolic regularity theory directly turns the above into the following.

Lemma 3.13. *Assuming that $\alpha > 0$ and that $T_{\max} < \infty$, we can find $C > 0$ such that*

$$\|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \quad \textit{for all } t \in (0, T_{\max}).$$

Proof. In view of (3.2), this can readily be verified by applying Lemma 3.12 to any $p > 2$, and by employing well-known smoothing properties of the Neumann heat semigroup on Ω. ■

By means of a Moser-type recursion, we can finally establish an L^∞ estimate for u.

Lemma 3.14. *If $\alpha > 0$ and $T_{\max} < \infty$, then there exists $C > 0$ satisfying*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \textit{for all } t \in (0, T_{\max}).$$

Proof. Again based on the pointwise control of v from above and below, as asserted by (3.2) and Lemma 3.11, this can be derived from Lemmas 3.12 and 3.13 through a Moser-type iterative argument (cf. [49, Proof of Proposition 1.3] for similar reasoning in a one-dimensional counterpart). ■

Our main result on blow-up exclusion in (1.5) has thus been established.

Proof of Theorem 1.4. The claim is a direct consequence of Lemmas 3.14 and 3.13 when combined with Lemmas 3.11 and 3.1. ■

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