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Global-in-time well-posedness of the compressible Navier–Stokes equations with striated density

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Abstract. We first show local-in-time well-posedness of the compressible Navier–Stokes equations, assuming striated regularity while no other smoothness or smallness conditions on the initial density. With these local-in-time solutions served as blocks, for *less* regular initial data where the vacuum is permitted, the global-in-time well-posedness follows from the energy estimates and the propagated striated regularity of the density function, if the bulk viscosity coefficient is large enough in the two-dimensional case. The global-in-time well-posedness holds also true in the three-dimensional case, provided with large bulk viscosity coefficient together with small initial energy. This solves the density-patch problem in the exterior domain for the compressible model with $W^{2,p}$-interfaces. Finally, the singular incompressible limit toward the inhomogeneous incompressible model when the bulk viscosity coefficient tends to infinity is obtained.

1. Introduction

In this paper, we establish the existence and uniqueness of global-in-time weak solutions of compressible viscous flows, and at the same time, we investigate the dynamics of density interface in dimension $d \in \{2,3\}$. More precisely, we consider the following compressible Navier–Stokes equations describing the motion of compressible viscous fluids:

$$
\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \mu \Delta u + (\mu + \lambda)\nabla \operatorname{div} u. \end{cases} \tag{1.1}
$$

Here $\mu > 0$ represents the dynamic viscosity, and $\lambda > 0$ stands for the kinetic viscosity. In the present paper, μ is some fixed positive constant, while the constant λ may become very large. For notational simplicity, we introduce the so-called bulk viscosity coefficient,

$$
\nu = 2\mu + \lambda,
$$

which tends to infinity when $\lambda \to \infty$.

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We always assume that our fluids are (strictly) viscous:

$$\nu \geqslant \underline{\nu} > 0,$$

where $\underline{\nu}$ is a fixed positive constant.

In the above, $t \geq 0$ and $x \in \mathbb{R}^d$, $d = 2, 3$, denote the time and space variables, respectively. The notations $\rho = \rho(t,x) \geqslant 0$ and $u = u(t,x) \in \mathbb{R}^d$ represent, respectively, the density and velocity of the compressible fluid, which serve as the unknowns in the problem. Meanwhile, $P = P(\rho)$ is a given smooth function (in this paper, we assume $P \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$).

The system (1.1) is supplemented with initial data

$$(\rho, \rho u)_{|t=0} = (\rho_0, \rho_0 u_0), \tag{1.2}$$

which satisfy

$$\rho_0 \geqslant 0, \quad \rho_0 \in L^\infty(\mathbb{R}^d; [0,\infty)), \quad \rho_0 - \tilde{\rho} \in L^2(\mathbb{R}^d; \mathbb{R}), \quad u_0 \in H^1(\mathbb{R}^d; \mathbb{R}^d), \tag{1.3}$$

where $\tilde{\rho} > 0$ is some given positive equilibrium state of the density.

1.1. Striated regularity

We assume further striated regularity with respect to a given nondegenerate family of vector fields for the initial density ρ_0 in this paragraph.

We first introduce some notations, based on [10]. For some $p \in (d, \infty)$, $\mathbb{L}^{\infty,p}(\mathbb{R}^d; \mathbb{R}^d)$ denotes the vector space of bounded vector fields with gradients in $L^p(\mathbb{R}^d; \mathbb{R}^{d\times d})$. From now on, we denote the Lebesgue spaces $L^p(\mathbb{R}^d; \mathbb{R}^n)$ and the Sobolev spaces $H^s(\mathbb{R}^d; \mathbb{R}^n)$ with $p \in [1, \infty]$, $s \in \mathbb{R}$ and $n \in \mathbb{N}^*$, simply by $L^p(\mathbb{R}^d)$ and $H^s(\mathbb{R}^d)$, or L^p and H^s, with an abuse of notations. We have defined

$$\mathbb{L}^{\infty,p}(\mathbb{R}^d) = \big\{ Y \in L^\infty(\mathbb{R}^d) \mid \|Y\|_{\mathbb{L}^{\infty,p}(\mathbb{R}^d)} := \|Y\|_{L^\infty(\mathbb{R}^d)} + \|\nabla Y\|_{L^p(\mathbb{R}^d)} < \infty \big\}.$$

For a family of vector fields $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_m) \subset \mathbb{L}^{\infty,p}(\mathbb{R}^d)$, $m \in \mathbb{N}$, we define the norm $\|\cdot\|_{\mathbb{L}^{\infty,p}}$ as

$$\|\mathcal{Y}\|_{\mathbb{L}^{\infty,p}(\mathbb{R}^d)} := \sup_{1\leqslant \upsilon \leqslant m} \|Y_\upsilon\|_{\mathbb{L}^{\infty,p}(\mathbb{R}^d)}.$$

Definition 1.1 (Nondegeneracy). Let $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_m) \subset \mathbb{L}^{\infty,p}(\mathbb{R}^d)$ be a family of m vector fields with $m \geqslant d-1$ and $p \in (d, \infty)$. We say that $\mathcal{Y}$ is *nondegenerate* if it satisfies the following property:

$$I(\mathcal{Y}) := \inf_{x\in\mathbb{R}^d} \sup_{\Upsilon \in \Upsilon^m_{d-1}} \Big| \bigwedge^{d-1} Y_\Upsilon(x) \Big|^{1/(d-1)} > 0.$$

Above, $\Upsilon \in \Upsilon^m_{d-1}$ means that $\Upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_{d-1})$ with each $\upsilon_i \in \{1, \ldots, m\}$ and $\upsilon_i < \upsilon_j$ for $i < j$, $Y_\Upsilon := (Y_{\upsilon_1}, Y_{\upsilon_2}, \ldots, Y_{\upsilon_{d-1}})$, while the symbol $\bigwedge^{d-1} Y_\Upsilon$ stands for the unique element of $\mathbb{R}^d$ such that

$$\Big(\bigwedge^{d-1} Y_\Upsilon \Big) \cdot Z = \det(Y_{\upsilon_1}, Y_{\upsilon_2}, \ldots, Y_{\upsilon_{d-1}}, Z), \quad \forall Z \in \mathbb{R}^d.$$

Definition 1.2 (Striated regularity with respect to a nondegenerate family of vector fields). Let $Y \in \mathbb{L}^{\infty,p}(\mathbb{R}^d)$, $p \in (d,\infty)$, be a (single) vector field, and let $\mathcal{Y} = (Y_1, Y_2, \dots, Y_m) \subset \mathbb{L}^{\infty,p}(\mathbb{R}^d)$ be a *nondegenerate* family of vector fields with $m \geqslant d-1$.

(a) A function $g \in L^\infty(\mathbb{R}^d)$ is said to be of class $L^p(\mathbb{R}^d)$ along Y if

$$g \in \mathbb{L}^p_Y(\mathbb{R}^d) := \{h \in L^\infty(\mathbb{R}^d) \mid \operatorname{div}(hY) \in L^p(\mathbb{R}^d)\}.$$

We define the derivative of the function g along Y as follows:

$$\partial_Y g := \operatorname{div}(gY) - g \operatorname{div} Y,$$

and hence we can equivalently define

$$\mathbb{L}^p_Y(\mathbb{R}^d) = \{h \in L^\infty(\mathbb{R}^d) \mid \partial_Y h \in L^p(\mathbb{R}^d)\}.$$

(b) A function $g \in L^\infty(\mathbb{R}^d)$ is said to be of class $L^p(\mathbb{R}^d)$ along the family $\mathcal{Y}$ if

$$g \in \mathbb{L}^p_{\mathcal{Y}}(\mathbb{R}^d) := \bigcap_{1 \leqslant \upsilon \leqslant m} \mathbb{L}^p_{Y_\upsilon}(\mathbb{R}^d),$$

and we equip the space $\mathbb{L}^p_{\mathcal{Y}}(\mathbb{R}^d)$ with the following norm:

$$\|g\|_{\mathbb{L}^p_{\mathcal{Y}}(\mathbb{R}^d)} := \frac{1}{I(\mathcal{Y})} \sup_{1 \leqslant \upsilon \leqslant m} \big[\|g\|_{L^\infty(\mathbb{R}^d)} \|Y_\upsilon\|_{\mathbb{L}^{\infty,p}(\mathbb{R}^d)} + \|\operatorname{div}(gY_\upsilon)\|_{L^p(\mathbb{R}^d)}\big],$$

which is equivalent to the norm with $\operatorname{div}(gY_\upsilon)$ above replaced by $\partial_{Y_\upsilon} g$.

We now continue with the assumption of the initial density ρ_0 given in (1.2)–(1.3) associated with the compressible Navier–Stokes equations (1.1). We assume further that

$$\rho_0 \in \mathbb{L}^p_{\mathcal{X}_0}(\mathbb{R}^d), \tag{1.4}$$

where $\mathcal{X}_0 = (X_{0,1}, \dots, X_{0,m}) \subset \mathbb{L}^{\infty,p}(\mathbb{R}^d)$ is a given *nondegenerate* family of vector fields for some $m \geqslant d-1$ and $p \in (d,\infty)$.

Remark 1.3 (Initial density of density-patch type). It is interesting to notice that the initial density of the form

$$\rho_0 = \alpha \mathbb{1}_{D_0} + \tilde{\rho}\, \mathbb{1}_{D_0^c}, \quad \alpha \geqslant 0, \tag{1.5}$$

satisfies the assumptions for ρ_0 in (1.3)–(1.4) if D_0 is a $W^{2,p}(\mathbb{R}^d)$ (with $p > d$) bounded, simply connected domain in $\mathbb{R}^d$. Indeed, (1.4) holds for a nondegenerate (divergence-free) family of vector fields $\mathcal{X}_0 = (X_{0,1}, \dots, X_{0,m}) \subset \mathbb{L}^{\infty,p}(\mathbb{R}^d)$ which is[1] tangent to ∂D_0,

[1] Indeed, for $d = 2$ the existence of such a nondegenerate family of tangential vector fields is obvious since we can take $X_{0,1} = \begin{pmatrix} \partial_{x_2} f \\ -\partial_{x_1} f \end{pmatrix} =: \nabla^\perp f$ to be the tangent vector field close to ∂D_0 with $f \in W^{2,p}(\mathbb{R}^2)$ and $f|_{\partial D_0} = 0$ and $\nabla f|_{\partial D_0} \neq 0$, while $X_{0,2} = \nabla^\perp(\chi x_1)$ to be a non-zero vector field with χ a smooth cutoff function away from the boundary, see, e.g., equation (1.10) in [43] (with $m = 3$). The existence result for $d = 3$ with $m = 5$ follows from the similar idea, see, e.g., Proposition 3.2 in [24], where $X_{0,1} = \begin{pmatrix} 0 \\ -\partial_{x_3} f \\ \partial_{x_2} f \end{pmatrix}$, $X_{0,2} = \begin{pmatrix} \partial_{x_3} f \\ 0 \\ -\partial_{x_1} f \end{pmatrix}$ and $X_{0,3} = \begin{pmatrix} -\partial_{x_2} f \\ \partial_{x_1} f \\ 0 \end{pmatrix}$ are generated by the function $f \in W^{2,p}(\mathbb{R}^3)$ with $f|_{\partial D_0} = 0$ and $\nabla f|_{\partial D_0} \neq 0$, while $X_{0,4} = \begin{pmatrix} \partial_{x_3}(\chi x_3) \\ 0 \\ -\partial_{x_1}(\chi x_3) \end{pmatrix}$ and $X_{0,5} = \begin{pmatrix} -\partial_{x_2}(\chi x_1) \\ \partial_{x_1}(\chi x_1) \\ 0 \end{pmatrix}$ form a nondegenerate family away from ∂D_0 with χ a smooth cutoff function away from the boundary ∂D_0.

and this means that the initial density given by (1.5) possesses tangential regularity with respect to the boundary ∂D_0.

1.2. Statement of the main results

The purpose of this paper is threefold.

(1) We establish the local-in-time well-posedness of the system (1.1) for *positive* density function with striated regularity, under some compatibility condition. We thus remove the smallness condition required on the density fluctuation in Danchin, Fanelli, Paicu's paper [10].

(2) We prove that these local-in-time solutions become global-in-time unique solutions of the Cauchy problem (1.1)–(1.2)–(1.3)–(1.4), for general non-decreasing pressure law (see (1.10) below), if

- $d = 2$, and the bulk viscosity coefficient is large enough, $\nu \geq \nu_0$, with ν_0 depending on the norms of the initial data given in (1.2)–(1.3). This result is inspired by the work by Danchin and Mucha [16].
- $d = 3$, the initial energy is small, and the bulk viscosity coefficient $\nu \geq \nu_0$ is large enough. Here although $\|\rho_0 - \tilde{\rho}\|_{L^2(\mathbb{R}^3)}$ is assumed to be small, ρ_0 may have large variation in $L^\infty(\mathbb{R}^3)$. This result supplements the local-in-time well-posedness work [10] with global-in-time well-posedness result and the work by Shibata and Zhang [46] with less regular initial data.

(3) Additionally, by letting the bulk viscosity tend to infinity $\nu \to \infty$, we establish a singular limit toward the incompressible inhomogeneous model *on the whole space*, in the spirit of Danchin and Mucha's work [16], where the considered domain has finite measure.

1.2.1. Local-in-time well-posedness and continuation criterion. We begin by providing the statement of the local-in-time result, which technically further assumes the strict positivity of the initial density function and the compatibility condition on the initial data.

Theorem 1.4 (Local-in-time well-posedness and continuation criterion). *We consider the Cauchy problem of the compressible Navier–Stokes equations* (1.1) *supplemented with the initial data* (1.2) *satisfying* (1.3) *and* (1.4). *We further assume the strict positivity of the initial density and the compatibility condition as follows*:

$$0 < \underline{\rho} \leq \rho_0(x) \quad \text{and} \quad \mu\Delta u_0 + (\mu+\lambda)\nabla \operatorname{div} u_0 - \nabla P(\rho_0) \in L^2(\mathbb{R}^d). \tag{1.6}$$

Then, there exist a time $T > 0$ and a unique solution (ρ, u) to the Cauchy problem (1.1)–(1.2)*, satisfying the following properties*:

(1) (*Energy bounds*). *We have the following*:

$$u \in \mathcal{C}([0,T], H^1(\mathbb{R}^d)), \quad \dot{u} \in \mathcal{C}([0,T], L^2(\mathbb{R}^d)),$$
$$\sqrt{\sigma}\,\nabla\dot{u}, \sigma\ddot{u} \in L^\infty((0,T), L^2(\mathbb{R}^d)), \quad \text{and} \quad \nabla\dot{u}, \sqrt{\sigma}\,\ddot{u}, \sigma\nabla\ddot{u} \in L^2((0,T)\times\mathbb{R}^d).$$

Here and in what follows, we use the notations

$$\sigma = \sigma(t) := \min\{1, t\}, \quad \dot{v} := (\partial_t + u\cdot\nabla)v \quad \text{and} \quad \ddot{v} := (\partial_t + u\cdot\nabla)\dot{v}. \tag{1.7}$$

(2) (*Striated regularity*). *For all* $0 < t < T$, *we have* $\rho(t) \in \mathbb{L}^p_{\mathcal{X}(t)}(\mathbb{R}^d)$, *where* $\mathcal{X}(t) = (X_\upsilon(t))_{1\leqslant\upsilon\leqslant m} \subset \mathbb{L}^{\infty,p}(\mathbb{R}^d)$ *is the nondegenerate family of vector fields transported by the fluid flow, in the sense that each vector field* $X_\upsilon(t)$, $1 \leqslant \upsilon \leqslant m$, *solves uniquely the following Cauchy problem*:

$$\begin{cases} \partial_t X_\upsilon + u \cdot \nabla X_\upsilon = \partial_{X_\upsilon} u, \\ X_{\upsilon|t=0} = X_{0,\upsilon}. \end{cases} \tag{1.8}$$

Here, the directional derivative was given in Definition 1.2:

$$\partial_{X_\upsilon} u^j = \operatorname{div}(u^j X_\upsilon) - u^j \operatorname{div} X_\upsilon, \quad \text{for } 1 \leqslant j \leqslant d.$$

The velocity field is Lipschitz continuous when integrated in time, that is, $\nabla u \in L^1((0,T), L^\infty(\mathbb{R}^d))$, *and enjoys further the striated regularity for positive times*:

- *for* $d = 2$ *or for* $d = 3$ *and* $3 < p \leqslant 6$, $\nabla u \in L^2((0,T), \mathbb{L}^p_{\mathcal{X}}(\mathbb{R}^d))$;
- *for* $d = 3$ *and* $6 < p < \infty$, $\sigma^{3/4-1/r-3/(2p)} \nabla u \in L^r((0,T), \mathbb{L}^p_{\mathcal{X}}(\mathbb{R}^3))$ *and* $\sigma^{3/4-1/r} \nabla \dot{u} \in L^r((0,T), L^3(\mathbb{R}^3))$, *for any* $2 \leqslant r \leqslant \infty$.

(3) (*Continuation criterion*). *If* (ρ, u) *is the solution defined up to a maximal time* $T^* > 0$, *with* $T^* < \infty$, *then*

$$\limsup_{t\to T^*} \Big\{ \|\mathcal{X}(t)\|_{\mathbb{L}^{\infty,p}(\mathbb{R}^d)} + \frac{1}{I(\mathcal{X}(t))} + \Big\|\frac{1}{\rho(t)}\Big\|_{L^\infty(\mathbb{R}^d)} + \|\rho(t)\|_{L^\infty(\mathbb{R}^d)}$$
$$+ \|\partial_{\mathcal{X}(t)}\rho(t)\|_{L^p(\mathbb{R}^d)} + \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} + \|\dot{u}(t)\|_{L^2(\mathbb{R}^d)} \Big\} = \infty. \tag{1.9}$$

The solution of Theorem 1.4 is constructed in the spirit of a recent contribution of the second author [49], which deals with the more involved case of density-dependent viscosity coefficient. Thus, we only present a sketch of the proof of Theorem 1.4 in Appendix B.

While we do not pursue optimal local-in-time well-posedness in this paper, we instead employ the approximation argument to the local theory established above to prove our main result concerning global-in-time well-posedness in Theorem 1.6 below. We just mention here that the strict positivity requirement in assumption (1.6) can be relaxed through standard approximation techniques, with the estimates and results being corrected correspondingly (e.g., with density weights as in Proposition B.1).

Remark 1.5. (a) This result supplements the contribution [10] by Danchin, Fanelli and Paicu by removing the smallness condition on the density deviation. Unlike the maximum regularity argument used in [10], which requires a critical regularity for one part of the initial velocity, our method relies on the change into Lagrangian coordinates along with energy estimation methods.

(b) The compatibility condition $\mu\Delta u_0 + (\mu+\lambda)\nabla\operatorname{div} u_0 - \nabla P(\rho_0) \in L^2(\mathbb{R}^d)$ given in (1.6) expresses the continuity of the normal component of the stress tensor, and does not require (explicitly) smoothness of the density. The parabolic effect of the momentum equations ensures that this condition holds true at positive times even for less regular initial data, see [27].

(c) The velocity field possesses indeed further regularity properties which are stated in Corollary B.2 below, thanks to the decomposition of the velocity gradient (B.13).

1.2.2. Definitions of energy functionals. The global-in-time well-posedness result will follow from the above local-in-time well-posedness, the continuation criterion and a series of energy estimates. We define in this subsection the relevant energy functionals. In the following, we assume the general non-decreasing pressure law $P = P(\rho)$ as follows:

$$P \in \mathcal{C}^2(\mathbb{R},\mathbb{R}) \quad \text{and} \quad P'(\rho) \geqslant 0 \text{ for } \rho \geqslant 0. \tag{1.10}$$

Recall the positive density equilibrium $\tilde{\rho}$ in (1.3). We define first the pressure equilibrium

$$\widetilde{P} := P(\tilde{\rho}),$$

the ρ-dependent functions

$$H_l(\rho) = \rho \int_{\tilde{\rho}}^{\rho} s^{-2} |P(s) - \widetilde{P}|^{l-1} (P(s) - \widetilde{P})\, ds, \quad \text{for } l \in [1,\infty), \tag{1.11}$$

the pressure deviation

$$G(t,x) := (P(\rho))(t,x) - \widetilde{P}, \tag{1.12}$$

and the effective flux

$$F(t,x) = \nu(\operatorname{div} u)(t,x) - G(t,x). \tag{1.13}$$

We remark that due to the monotonicity property of the pressure law $P = P(\rho)$ in (1.10), the function $H_l(\rho)$ is always nonnegative for nonnegative ρ.

We also define the associated energy function of the compressible Navier–Stokes equations (1.1)–(1.10):

$$\begin{aligned} E(t) &= \int_{\mathbb{R}^d} \Big(\rho \frac{|u|^2}{2} + H_1(\rho)\Big)(t,x)\, dx \\ &\quad + \int_0^t \Big(\mu \|\nabla u(t')\|^2_{L^2(\mathbb{R}^d)} + (\mu+\lambda) \|\operatorname{div} u(t')\|^2_{L^2(\mathbb{R}^d)}\Big)\, dt, \end{aligned} \tag{1.14}$$

which consists of the kinetic energy

$$\frac{1}{2} \|\sqrt{\rho}\, u(t)\|^2_{L^2(\mathbb{R}^d)};$$

the potential energy

$$\int_{\mathbb{R}^d} \big(H_1(\rho)\big)(t,x)\, dx,$$

with $H_1(\rho)$ defined in (1.11):

$$H_1(\rho) = \rho \int_{\tilde{\rho}}^{\rho} \frac{P(s) - \widetilde{P}}{s^2}\, ds;$$

and the energy dissipation

$$\mu \|\nabla u\|^2_{L^2((0,t)\times\mathbb{R}^d)} + (\mu+\lambda) \|\operatorname{div} u\|^2_{L^2((0,t)\times\mathbb{R}^d)}.$$

The energy $E(t)$ is conserved for regular enough solutions of (1.1).

Recalling the notations in (1.7), we introduce two energy functionals of higher order:

$$\mathcal{A}_1(t) = \frac{\mu}{2}\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2 + \frac{\mu+\lambda}{2}\|\operatorname{div} u(t)\|_{L^2(\mathbb{R}^d)}^2 + \int_0^t \|\sqrt{\rho}\dot{u}(t')\|_{L^2(\mathbb{R}^d)}^2\,dt',$$

$$\begin{aligned}\mathcal{A}_2(t) &= \sigma(t)\|\sqrt{\rho}\,\dot{u}(t)\|_{L^2(\mathbb{R}^d)}^2 \\ &\quad + \int_0^t \sigma(t')\Big(\mu\|\nabla\dot{u}(t')\|_{L^2(\mathbb{R}^d)}^2 + \frac{\mu+\lambda}{\nu^2}\|\dot{F}(t')\|_{L^2(\mathbb{R}^d)}^2\Big)\,dt.\end{aligned} \tag{1.15}$$

The hierarchy of energy functionals $E(t)$, $\mathcal{A}_1(t)$ and $\mathcal{A}_2(t)$ encode the $L^2(\mathbb{R}^d)$-norm, the $\dot{H}^1(\mathbb{R}^d)$-norm for $u(t)$ and the (time-weighted) $L^2(\mathbb{R}^d)$-norm for the material derivative $\dot{u}(t)$, respectively. Although trivially $|\operatorname{div} u| \leqslant d|\nabla u|$, we will make efforts to get the (large) viscosity coefficient λ before $\operatorname{div} u$ in the definition of E and $\mathcal{A}_1$, such that intuitively $\operatorname{div} u \to 0$ as $\lambda \to \infty$ if E and $\mathcal{A}_1$ are bounded uniformly in time. A review of the history of these energy functionals can be found in Section 1.3 below.

Recalling the initial data (1.2), we denote $G_0(x) = G(0,x) = (P(\rho_0))(x) - \widetilde{P}$. For notational simplicity, we denote the first initial energy

$$E_0 := E(0) = \int_{\mathbb{R}^d}\Big(\rho_0\frac{|u_0|^2}{2} + H_1(\rho_0)\Big)(x)\,dx,$$

the total initial energy

$$E_0^\nu := E_0 + \mu\|\nabla u_0\|_{L^2(\mathbb{R}^d)}^2 + \nu\|\operatorname{div} u_0\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{\nu}\|G_0\|_{L^2(\mathbb{R}^d)}^2, \tag{1.16}$$

and the upper bound of the initial density,

$$\rho_0^* := \sup_{x\in\mathbb{R}^d}\rho_0(x).$$

We observe that for initial data given in (1.2)–(1.3),

$$\begin{aligned}E_0 &\le C(\rho_0^*)\,\|(\rho_0-\tilde{\rho}, u_0)\|_{L^2(\mathbb{R}^d)}^2 < +\infty,\\ E_0^\nu &\le C(\mu,\underline{\nu},\rho_0^*)(E_0 + \|\nabla u_0\|_{L^2(\mathbb{R}^d)}^2) + \nu\|\operatorname{div} u_0\|_{L^2(\mathbb{R}^d)}^2 < +\infty.\end{aligned}$$

We aim to bound $\mathcal{A}_1(t)$ and $\mathcal{A}_2(t)$ globally in time, in terms of E_0^ν and ρ_0^*, if $\nu \ge \nu_0$ is large enough (and if the initial energy is small enough for $d = 3$). The following quantity captures the striated regularity of the density function along the family of vector fields $\mathcal{X}(t) = (X_\upsilon(t))_{1\leqslant\upsilon\leqslant m}$ transported by the flow as in (1.8):

$$\mathcal{A}_3(t) = \|\mathcal{X}(t)\|_{\mathbb{L}^{\infty,p}(\mathbb{R}^d)} + \sup_{1\leqslant\upsilon\leqslant m}\|(\partial_{X_\upsilon}\rho)(t)\|_{L^p(\mathbb{R}^d)}. \tag{1.17}$$

It is straightforward to see that $\mathcal{A}_3(t)$ grows exponentially in $\|\nabla u\|_{L^1_tL^\infty}$. We aim to show that the striated regularity encoded in $\|\log\mathcal{A}_3\|_{L^1(0,t)}$, together with the energy functionals $\mathcal{A}_1(t)$ and $\mathcal{A}_2(t)$, control $\|\nabla u\|_{L^1_tL^\infty}$. Grönwall's inequality hence implies the exponential-in-time control of $\|\nabla u\|_{L^1_tL^\infty}$.

1.2.3. Global-in-time well-posedness. We now state our global-in-time result for less regular initial data on which the assumption (1.6) is not assumed.

Theorem 1.6. *Assume the Cauchy problem* (1.1)–(1.10) *with initial data* (1.2)–(1.3)–(1.4)*, and the following conditions*:

$$\text{either } d = 2 \text{ and } \nu \geqslant \nu_0, \tag{1.18}$$

$$\text{or } d = 3, \quad p \in (3,6), \quad E_0^\nu E_0 \leqslant c \text{ and } \nu \geqslant \nu_0, \tag{1.19}$$

where c is a fixed constant depending only on μ and $\underline{\nu}$, while ν_0 is a constant depending additionally on the initial norms: E_0, $\|\nabla u_0\|_{L^2(\mathbb{R}^d)}$ and ρ_0^*. Then the Cauchy problem has a unique global-in-time solution (ρ, u) satisfying

(1) (*Energy bounds*). *For all $t \geqslant 0$, we have*

$$\begin{cases} E(t) + \mathcal{A}_1(t) + \mathcal{A}_2(t) \leqslant C_0^\nu, \\ \|\rho(t) - \tilde{\rho}\|^2_{L^\infty(\mathbb{R}^d)} \leqslant \|\rho_0 - \tilde{\rho}\|^2_{L^\infty(\mathbb{R}^d)} + C_0^\nu, \end{cases} \tag{1.20}$$

where the constant C_0^ν depends on μ, $\underline{\nu}$, ρ_0^*, and (*superlinearly*) *on E_0^ν.*

(2) (*Striated regularity*). *For all $t \geqslant 0$, $\rho(t) \in \mathbb{L}^p_{\mathcal{X}(t)}(\mathbb{R}^d)$, where $\mathcal{X}(t) = (X_\upsilon(t))_{1\leqslant \upsilon\leqslant m} \subset \mathbb{L}^{\infty,p}(\mathbb{R}^d)$ is a nondegenerate family of vector fields defined to solve the Cauchy problem* (1.8)*. Moreover, $\nabla u \in L^1_{\text{loc}}([0,\infty), L^\infty(\mathbb{R}^d))$ with the following estimates*:

$$\begin{cases} \mathcal{A}_3(t) \leqslant \mathcal{A}_3(0) \exp\left[C_0 \int_0^t \left(1 + \sqrt{t} + \|\nabla u(t')\|_{L^\infty(\mathbb{R}^d)}\right) dt'\right], \\ \int_0^t \|\nabla u(t')\|_{L^\infty(\mathbb{R}^d)}\, dt' \leqslant C_0 \left(1 + \dfrac{\mathcal{A}_3(0)}{I(\mathcal{X}_0)}\right) \exp(C_0 t), \\ \int_0^t \|\partial_{\mathcal{X}(t')} \nabla u(t')\|_{L^p(\mathbb{R}^d)}\, dt' \leqslant C_0(1 + t + t\mathcal{A}_3(t))\mathcal{A}_3(t), \end{cases} \tag{1.21}$$

where C_0 depends on μ, $\underline{\nu}$, m, d, p, ρ_0^* and E_0^ν.

Remark 1.7 (Bounds for $\operatorname{div} u$). We have assumed some uniform bounds (with respect to ν) for $\operatorname{div} u_0$ implicitly: the conditions in (1.18) and (1.19) imply that

$$\nu\, \|\operatorname{div} u_0\|^2_{L^2(\mathbb{R}^d)} \leqslant \begin{cases} E_0^\nu < \infty, & \text{if } d = 2, \\ E_0^\nu \min\{1, c/E_0\} < \infty, & \text{if } d = 3. \end{cases}$$

This boundedness is propagated over time:

$$\nu\, \|\operatorname{div} u(t)\|^2_{L^2(\mathbb{R}^d)} \leqslant C_0^\nu.$$

Theorem 1.6 and Remark 1.3 imply immediately

Corollary 1.8 (Density patch problem in the exterior domain). *The Cauchy problem given in* (1.1)–(1.10) *with initial data* (1.2) *of density-patch type* (1.5) *and $u_0 \in H^1(\mathbb{R}^d)$, under the assumption* (1.18) *or* (1.19)*, has a unique global-in-time solution (ρ, u), with $\rho(t)$ enjoying tangential regularity with respect to the boundary ∂D_t, which is transported by the flow of u and keeps its $W^{2,p}(\mathbb{R}^d)$-boundary regularity.*

Remark 1.9. If $\alpha > 0$ in (1.5), deriving a uniformly positive lower bound for the density is straightforward (see Step 6 in Section 2.1 of [17]). This results in an exponential-in-time decay of the jump in the density $\rho(t)$ across ∂D_t, as observed in [29, 30].

Intuitively, thanks to the uniform bound $\mathcal{A}_1(t) \leqslant C_0^\nu$ in (1.20), letting $\nu \to \infty$ yields a couple (ϱ, v) that satisfies the incompressible inhomogeneous model

$$
\begin{cases}
\partial_t \varrho + \operatorname{div}(\varrho v) = 0, \\
\partial_t(\varrho v) + \operatorname{div}(\varrho v \otimes v) + \nabla \Pi - \mu \Delta v = 0, \\
\operatorname{div} v = 0.
\end{cases}
\tag{1.22}
$$

Corollary 1.10 (Incompressible limit). *Let (ρ_0, u_0) be the initial data given in (1.2) satisfying (1.3), (1.4) and $\operatorname{div} u_0 = 0$. Let $(\rho^{(\nu)}, u^{(\nu)})$ be the corresponding unique solution constructed in Theorem 1.6, under the assumption (1.18) or (1.19).*

Then the solution $(\rho^{(\nu)}, u^{(\nu)})_\nu$ converges weakly- to (ϱ, v) in $L^\infty((0,\infty)\times\mathbb{R}^d) \times L^\infty((0,\infty), H^1(\mathbb{R}^d))$ as ν goes to infinity, and (ϱ, v) solves (uniquely) the inhomogeneous, incompressible model (1.22) with initial data (ρ_0, u_0) in the distribution sense. Moreover, we have*

$$
\begin{cases}
\operatorname{div} u^{(\nu)} = \mathcal{O}(\nu^{-1/2}) & \text{in } L^2 \cap L^\infty((0,\infty), L^2(\mathbb{R}^d)), \\
\partial_t(\rho^{(\nu)} u^{(\nu)}) + \operatorname{div}(\rho^{(\nu)} u^{(\nu)} \otimes u^{(\nu)}) & \\
\qquad -\nabla F^{(\nu)} - \mu \Delta u^{(\nu)} = \mathcal{O}(\nu^{-1/2}) & \text{in } L^\infty((0,\infty), \dot{H}^{-1}(\mathbb{R}^d)),
\end{cases}
\tag{1.23}
$$

where $F^{(\nu)} = \nu \operatorname{div} u^{(\nu)} - G^{(\nu)}$ with $G^{(\nu)} = P(\rho^{(\nu)}) - \widetilde{P}$.

The proofs of Theorem 1.6 and Corollary 1.10 are presented in Section 2.3, based on the a priori estimates of Section 2.1 and their proofs in Section 2.2.

Remark 1.11. This result in Corollary 1.10 is a partial continuation of the work by Danchin and Mucha [13, 14, 16], and Danchin and Wang [17], and stands, as far as we know, as the first one dealing with *discontinuous* initial data *in the whole space*. We notice that, except for the work [13] dealing with the whole space case and initial data in the critical Besov space, the other studies rely heavily on the assumption that the domain has finite measure. The extension to the whole space, especially for $d = 2$, is not obvious, and it requires some refined computations, e.g., the compensated result by Coifman, Lions, Meyer and Semmes in [8].

1.3. Review of known results

Classical solutions for the Navier–Stokes equations (1.1) with regular initial data are known to exist, since the works by Nash [42], Itaya [33, 34], Solonnikov [47], Tani [48], just to cite a few examples. These solutions are defined up to a positive time which depends on the (norms of) initial data. The first result addressing the global-in-time well-posedness of classical solutions is provided by Matsumura and Nishida [41] for small initial data in $L^1(\mathbb{R}^3) \cap H^3(\mathbb{R}^3)$. Nowadays, global-in-time classical solutions are known to exist under smallness assumption on the initial data in critical Besov space [4, 7, 26].

Weak solutions and estimates for $E(t)$ and G. Similar to the solutions constructed by Leray [35] for the *incompressible* Navier–Stokes equations, there are well-established results that investigate the existence of global-in-time *weak* solutions for the compressible Navier–Stokes equations (1.1), with finite initial energy. The first result was obtained by P.-L. Lions [40], followed by Feireisl, Novotný, Petzeltová [23], for pressure laws of the form $P(\rho) = a\rho^\gamma$, $a > 0$, with some limitations on γ. These weak solutions satisfy the following classical energy inequality:

$$E(t) \leqslant E(0) = E_0, \tag{1.24}$$

where the functional E has been given in (1.14).

The introduction of the (generalized) specific energy $H_l(\rho)$, $l \in [1, \infty)$, in (1.11) helps (technically) to estimate the pressure deviation G. As observed in, e.g., [3], the so-defined $H_l(\rho)$ is nonnegative: $H_l(\rho) \geqslant 0$, since the pressure $P(\rho)$ is an non-decreasing function of the density (1.10).

For the classical case $l = 1$, $H_1(\rho)$ appears in the definition of $E(t)$, which is integrable in space uniformly in time due to (1.24):

$$\int_{\mathbb{R}^d} \big(H_1(\rho)\big)(t,x)\,dx \leqslant E(t) = E_0.$$

Consequently, under the a priori assumption

$$\rho(t,x) \leqslant \rho^*,$$

we bound $G(t,x) := P(\rho)(t,x) - \tilde{P}$ uniformly in time by the energy E_0 as follows:

$$\sup_{[0,t]} \|G\|^q_{L^q(\mathbb{R}^d)} \leqslant C^* \sup_{t'\in[0,t]} \int_{\mathbb{R}^d} H_1(\rho(t',x))\,dx \leqslant C^* E_0, \quad \text{with} \quad q \in [2,\infty), \tag{1.25}$$

where the constant C^* depends only on ρ^* and q. In the above, the first inequality follows from the definition of $H_1(\rho)$ in (1.11).

General $H_l(\rho)$, $l \geqslant 1$, as a function of ρ, satisfies the following ordinary differential equation:

$$\rho H_l'(\rho) - H_l(\rho) = |P(\rho) - \widetilde{P}|^{l-1}(P(\rho) - \widetilde{P}),$$

and hence, by virtue of the mass equation $(1.1)_1$, the function $(H_l(\rho))(t,x)$ satisfies the following time evolutionary equation:

$$\partial_t H_l(\rho) + \operatorname{div}(H_l(\rho)u) + |P(\rho) - \widetilde{P}|^{l-1}(P(\rho) - \widetilde{P}) \operatorname{div} u = 0,$$

which is in the same spirit of the renormalized continuity equation appearing in, e.g., [40]. In view of the definitions (1.12) and (1.13), this is equivalent to

$$\partial_t H_l(\rho) + \operatorname{div}(H_l(\rho)u) + \frac{1}{\nu}|G|^{l+1} = -\frac{1}{\nu}|G|^{l-1}GF.$$

Consequently, integrating the above in space yields, after Hölder's inequality, the following:

$$\frac{d}{dt}\|H_l(\rho)\|_{L^1(\mathbb{R}^d)} + \frac{1}{\nu}\|G\|^{l+1}_{L^{l+1}(\mathbb{R}^d)} \leqslant \frac{1}{\nu}\|G\|^l_{L^{l+1}(\mathbb{R}^d)}\|F\|_{L^{l+1}(\mathbb{R}^d)}, \tag{1.26}$$

and hence by Young's inequality and integration in time, G can be controlled by F in the following way:

$$\frac{1}{\nu}\|G\|_{L^{l+1}((0,t)\times\mathbb{R}^d)}^{l+1} \leqslant 2\|H_l(\rho_0)\|_{L^1(\mathbb{R}^d)} + \frac{C}{\nu}\|F\|_{L^{l+1}((0,t)\times\mathbb{R}^d)}^{l+1}, \quad \forall t\in(0,\infty). \tag{1.27}$$

Density patch problem. In the last three decades, there has been growing interest in exploring the properties of weak solutions to models arising from fluid mechanics that enable tracking down discontinuities of some quantities such as density or vorticity. We refer to the density patch problem for incompressible models stated in [39]: *Consider the incompressible model* (1.22) *in two dimension with initial density as the characteristic function* $\rho_0 = \mathbb{1}_{D_0}$ *of some regular domain* $D_0 \in \mathbb{R}^2$. *The density-patch problem asks whether or not, at positive times, the density is still some characteristic function* $\mathbb{1}_{D(t)}$ *with the domain* $D(t) \subset \mathbb{R}^2$ *preserving the initial regularity of* D_0. This problem is almost solved for incompressible models, even for density-dependent viscosity (under some smallness assumption) or higher Sobolev regularity of D_0, see [11, 12, 15, 18–20, 25, 36–38].

However, for a similar problem in the context of compressible fluids, there are not so many results. On one hand, the global classical solutions constructed by Matsumura and Nishida, or in critical Besov space, are too strong in a way that they do not allow for discontinuous solutions. On the other hand, the weak solutions constructed by P.-L. Lions or Feireisl, Novotný, Petzeltová only require that the initial energy is finite, allowing for discontinuous density. However, the regularity of the velocity is relatively weak, with $\nabla u \in L^2((0,\infty)\times\mathbb{R}^d)$, and this is insufficient to track down discontinuities in the density. A natural idea is to construct weak solutions in a class that allows for tracking down the discontinuity of the interface. The first result addressing this issue is, as far as we know, [29] by Hoff, where the author considered an initial density with Hölder regularity on both sides of a suitable curve, allowing for jumps across this curve. The initial curve is transported by the flow of the velocity into a curve that maintains its initial regularity. The density also remains Hölder continuous on both sides of the transported curve, and moreover, its jump through the latter decays exponentially over time. This result pertains only to the case of linear pressure law and small bulk viscosity. Recently, these restrictions were removed in [10, 50], where the later reference treated even more challenging case of density-dependent viscosity. Theorem 1.6 is thus added to this list, in the constant viscosity setting, with domains having Sobolev regularity, and the density can be large in $L^\infty(\mathbb{R}^d)$, unlike the cited results.

1.3.1. Hoff's strategy. We review briefly some key concepts in Hoff's works [27–30].

Energy functionals. In [27], Hoff introduced the following energy functionals, which can be compared with our definitions in (1.15):

$$\begin{cases} \mathcal{A}_1^H(t) = \sup\limits_{[0,t]} \sigma\|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + \int_0^t \sigma(t')\|\sqrt{\rho}\,\dot u(t')\|_{L^2(\mathbb{R}^d)}^2\,dt', \\ \mathcal{A}_2^H(t) = \sup\limits_{[0,t]} \sigma^d\|\sqrt{\rho}\,\dot u\|_{L^2(\mathbb{R}^d)}^2 + \int_0^t \sigma^d(t')\|\nabla\dot u(t')\|_{L^2(\mathbb{R}^d)}^2\,dt', \\ \mathcal{B}(t) = \sup\limits_{[0,t]} \|\rho-\tilde\rho\|_{L^\infty(\mathbb{R}^d)}^2, \end{cases}$$

where the time weight σ and the material derivative $\dot{u}$ are defined as in (1.7). He provides bounds for these functionals by requiring that the initial velocity is small in $L^2(\mathbb{R}^d)$ but can be large in $L^{2^d}(\mathbb{R}^d)$. Additionally, he requires that the initial density is bounded away from zero and bounded from above, along with some technical assumptions.

Effective flux and vorticity. Hoff's computations, mainly while propagating the lower and upper bounds of the density, rely strongly on the *effective viscous flux* $F = \nu \operatorname{div} u - G$ given in (1.13). It plays a crucial role by connecting the momentum equations and the mass equation, as was discovered by Hoff and Smoller in [31]. It is also useful in the study of the propagation of oscillations in [45], and in the constructions of weak solutions in [23, 28, 40].

In fact, recall the momentum equations $(1.1)_2$, which can be written by virtue of the mass conservation law $(1.1)_1$ as

$$\rho\dot{u} - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \nabla(P(\rho) - \widetilde{P}) = 0.$$

Applying the divergence operator, we obtain the Poisson equation for F as follows:

$$\Delta F = \operatorname{div}(\rho\dot{u}). \tag{1.28}$$

Similarly, we can apply the curl operator to the momentum equations to derive the Poisson equation for the vorticity, $\operatorname{curl} u$, as follows:

$$\mu\Delta \operatorname{curl} u = \operatorname{curl}(\rho\dot{u}). \tag{1.29}$$

Consequently, the regularity of the material derivative of the velocity $\dot{u}$, as provided by the functionals $\mathcal{A}_1^H$ and $\mathcal{A}_2^H$, allows the effective flux F and the vorticity $\operatorname{curl} u$ to be regular at positive time, even for rough density. This means that there is some cancellation between the divergence of the velocity and the pressure at positive times. In particular, the fact that $F \in L^{8/3}((1,\infty), L^\infty(\mathbb{R}^d))$ allows Hoff to propagate the $L^\infty(\mathbb{R}^d)$ estimate for the density.

Thanks to this observation, under a smallness condition on the initial data, Hoff proved the existence of global weak solutions for the system (1.1) with a linear pressure law in a first paper [27]. He later considered pressure laws of the form $P(\rho) = a\rho^\gamma$, with $\gamma > 1$, in a second paper [28], in which, again, the effective flux played a crucial role in proving compactness for the density.

Velocity gradient expression involving Riesz operators. In order to study the dynamics of discontinuous surfaces, Hoff in [29] used the following decomposition of the velocity gradient:

$$\mu\nabla u = -(-\Delta)^{-1}\nabla(\rho\dot{u}) + \frac{\mu+\lambda}{\nu}\mathcal{R}\mathcal{R}F + \frac{\mu}{\nu}\mathcal{R}\mathcal{R}G =: \mu\nabla\tilde{u} + \mu\nabla u_G; \tag{1.30}$$

this is just a rewriting of the above momentum equations, where $\mathcal{R}_j = (\frac{1}{i}\partial_j)(-\Delta)^{-1/2}$, $1 \leqslant j \leqslant d$, are the Riesz operators.

By assuming more regularity on the velocity $u_0 \in H^\beta(\mathbb{R}^2)$, he reduces the singularity of time weights in the definitions of functionals $\mathcal{A}_1^H$ and $\mathcal{A}_2^H$. Namely, in dimension two, the time weights σ and σ^2 are replaced, respectively, by $\sigma^{1-\beta}$ and $\sigma^{2-\beta}$. Thus, $\nabla\tilde{u}$

and the effective flux F belong to $L^1_{\mathrm{loc}}([0,\infty), \mathcal{C}^\alpha(\mathbb{R}^2))$ for all $0 < \alpha < \beta$. With the help of the regularity of F, Hoff propagated the *piecewise* Hölder regularity of the density, resulting *piecewise* Hölder continuity of ∇u_G on both sides of a time-dependent curve. This time-dependent curve is the transport of an initial suitable curve with some geometric assumptions, and only provided with bounded velocity gradient can the structure of the density and of the curve be propagated.

However, since Riesz operators *fail* to be continuous on $L^\infty(\mathbb{R}^d)$, additional regularity must be assumed on the density to obtain $\mathcal{R}\mathcal{R}G \in L^\infty(\mathbb{R}^d)$. In [30], Hoff and Santos observed that in the configuration of the previous works (see [27, 28]), the rough part of the velocity gradient ∇u_G belongs to $L^\infty((0,\infty), \mathrm{BMO}(\mathbb{R}^d))$. In this case, the initial interface $\gamma_0 \in \mathcal{C}^\alpha$, $\alpha > 0$, is transported to an interface $\gamma_t \in \mathcal{C}^{\alpha_t}$ at time $t > 0$, with α_t decaying exponentially to zero in time.

Hence to propagate interface regularity (more than continuity) requires a Lipschitz velocity. For the incompressible model with constant viscosity, this regularity is directly obtained from energy computations and interpolations. In contrast, for the compressible case with discontinuous density, the problem is more delicate, and the issue is to find an appropriate functional space, which can be mapped by even-order Riesz operators into $L^\infty(\mathbb{R}^d)$.

1.3.2. The strategy by use of tangential regularity. Apart from the tools used in [29,50] to handle the rough part of the velocity gradient, there exists another framework that allows for the same. It is referred as tangential/striated regularity space, and goes back to Chemin's study (see, e.g., [5, 6]) of the vortex patch problem for the ideal incompressible model. See also [2] for another interesting geometric proof for the persistence of regularity in the vortex patch problem. Chemin's idea has been further developed to higher dimensional cases in [9, 24], to the inhomogeneous case in [22], as well as to the density-patch problem for the inhomogeneous incompressible Navier–Stokes model in, e.g., [36–38,43]. However, there are very few results in this direction for the compressible case. To the best of our knowledge, the only work in the literature is [10], by Danchin, Fanelli, and Paicu. They establish the local-in-time well-posedness of the compressible equations (1.1) with a striated initial density, and we now delve into a brief discussion of their methods. From the momentum equations $(1.1)_2$, they express the velocity as:

$$u = w - \nabla(I_d - \Delta)^{-1} G, \tag{1.31}$$

where w solves a parabolic equation with source term belonging to some suitable space $L^r((0,T), L^p(\mathbb{R}^d))$. They employ maximal regularity tools to establish Lipschitz bounds for w. Meanwhile, Lipschitz bound for the second term of the velocity's expression (1.31), associated with the pressure, is obtained through tangential regularity estimates. The maximal regularity argument requires smallness assumption on the density in $L^\infty(\mathbb{R}^d)$, and the global-in-time result is still missing. Toward this, we establish local-in-time well-posedness of the system (1.1) without imposing any smallness condition on the initial data (see Theorem 1.4), and global-in-time well-posedness (see Theorem 1.6) without any smallness assumption of the initial density fluctuation in $L^\infty(\mathbb{R}^d)$, and the vacuum is allowed. This is accomplished through a coupling mechanism that involves the effective flux. By achieving this objective, we propagate the Sobolev regularity of interfaces over time.

Incompressible limit. We aim also to establish an incompressible limit in the spirit of the work of Danchin and Mucha [16]. Let us briefly look at this question. The work by Matsumura and Nishida [41] paved the way for attempts to relax the assumptions on the initial data. Despite reducing the regularity assumption to critical Besov space or even Lebesgue space, the condition of smallness is frequently encountered in the literature. In their work [13], Danchin and Mucha introduced a new framework that enables them to bypass the smallness condition on the initial data, namely, replacing the smallness in the initial data by large enough bulk viscosity coefficient. In particular, as the bulk viscosity $\nu \to \infty$, the solution converges to a limit that satisfies the incompressible model. This has been done for initial data in critical Besov space. For less regular initial data, they work on the torus in [14, 16], where they rely technically on the finite-measure of the domain, particularly on the logarithmic interpolation inequality, which proves to be crucial in handling vacuum states in [16]. We also refer to the work by Danchin and Wang [17], where exponential decay rate of the solutions of the compressible model on torus has been investigated. However, the exponential decay does not generally hold in the whole space. For instance, the work by Hu and Wu [32] provides lower bound for the norms of solutions in certain cases. We obtain similar results as those in [16] in the presence of vacuum on the whole space (see Corollary 1.10). Specifically in the two-dimensional case, we do some algebraic computations and succeed in applying Coifman, Lions, Meyer, and Semmes' compensated integrability result in [8] to achieve uniformly in ν estimates for the energy functionals.

Outline of the paper. The rest of the paper is structured as follows. In Section 2, we give the proofs of Theorem 1.6 and Corollary 1.10, by use of the results in Theorem 1.4, whose proof is postponed to Appendix B. A useful density-weighted interpolation inequality is established in Appendix A.

2. Proof of the main results

This section is devoted to the proofs of Theorem 1.6 and Corollary 1.10, which go from a priori estimates for solutions of the Navier–Stokes equations (1.1) to the proof of the compactness of approximate solutions. It is divided into three parts. In the first one, Section 2.1, we summarize all key ideas with brief explanations and give the a priori estimates in a series of lemmas. Technical details and the proofs of these lemmas are presented in the second part, Section 2.2. As we will see in the final part of the proof in Section 2.3, the existence of a local-in-time solution (without any smallness condition in the density) is by no means obvious, and it is the purpose of Section B. The regularity of the (local-in-time) solution is sufficient in order to use u and $\dot{u}$ as test functions in the subsequent computations to get energy estimates.

2.1. Proof ideas and statements of lemmas

In this section, we give the main ideas of the proof of Theorem 1.6. We state the energy estimates for the solutions of the compressible Navier–Stokes equations (1.1)–(1.10) with initial data (1.2) satisfying (1.3). The tangential regularity (1.4) is assumed when we show the boundedness of the Lipschitz-norm of the velocity vector field as a second step. Recall the definitions of the energy functionals

$$E(t),\ \mathcal{A}_1(t),\ \mathcal{A}_2(t) \text{ and } \mathcal{A}_3(t),$$

together with the notations $\widetilde{P}$, $H_l(\rho)$, G, F and E_0, E_0^ν, ρ_0^*, given in Section 1.2.2.

In the literature (see, e.g., [40] where finally only the *energy inequality* (1.24) was established for weak solutions), the following *a priori energy equality* for $E(t)$ was shown for strong solutions:

$$E(t) = E(0) =: E_0. \tag{2.1}$$

More precisely, it follows from taking the scalar product of the momentum equation $(1.1)_2$ with the velocity u and then integrating in time and space. This energy balance (2.1) is going to be used freely in the proof, and we aim to show the estimates for $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$.

In the following, we state step by step:

- In Section 2.1.1: Energy estimates for $\mathcal{A}_1$ and $\mathcal{A}_2$, together with the boundedness of the density deviation $\|\rho - \tilde{\rho}\|_{L^\infty_{t,x}}$.

  Under the assumption that the density is a priori from above bounded,

  $$0 \leqslant \rho(t,x) \leqslant \rho^*, \tag{2.2}$$

  for some $\rho^* > 0$, we show first (local-in-time) a priori *energy estimates* for $\mathcal{A}_1$ and $\mathcal{A}_2$ (see Lemmas 2.1 and 2.2) and then a *boundedness of the density* in terms of $\mathcal{A}_1$ and $\mathcal{A}_2$ (see Lemma 2.3) for solutions of the Cauchy problem (1.1)–(1.2)–(1.3). Under the assumption (1.18) or (1.19), that is, in the case of either *large bulk viscosity coefficient for $d = 2$* or with *small initial energy and large bulk viscosity coefficient for $d = 3$*, a bootstrap argument implies the *global-in-time* a priori energy estimates for $\mathcal{A}_1$ and $\mathcal{A}_2$, and density bound estimate (see Lemma 2.4).
- In Section 2.1.2: The striated regularity estimate for $\mathcal{A}_3$. together with the boundedness of the velocity gradient $\|\nabla u\|_{L^1_t L^\infty_x}$.

  With the estimates in Section 2.1.1 at hand, we turn to the striated regularity for the density function $\mathcal{A}_3(t)$ for solutions of the Cauchy problem (1.1)–(1.2)–(1.3)–(1.4), which finally implies the Lipschitz-continuity of the velocity field (see Lemma 2.6), thanks to the L^∞-estimates for the double Riesz-operators provided with extra striated regularity (see Proposition 2.5).

2.1.1. A priori estimates for $\mathcal{A}_1$, $\mathcal{A}_2$ and $\|\rho - \tilde{\rho}\|_{L^\infty_{t,x}}$. In order to derive higher-order energy estimates for the velocity u and its material derivative

$$\dot{u} := (\partial_t + u\cdot\nabla)u,$$

we use first $\dot{u}$ as a test function in the weak formulation of the momentum equation $(1.1)_2$ to establish bounds for $\mathcal{A}_1$. The functional $\mathcal{A}_2$ emerges when, first rewriting the momentum equation $(1.1)_2$ with the effective flux F, and then applying the operator $\partial_t \cdot + \operatorname{div}(\cdot u)$ to the resulting equation before testing it with $\dot{u}$.

In dimension two, the following estimates are valid for these functionals $\mathcal{A}_1$ and $\mathcal{A}_2$.

Lemma 2.1 (Preliminary energy estimates for $d = 2$). *Assume that $d = 2$ and (2.2). Then the following a priori bounds hold true for the functionals $\mathcal{A}_1$ and $\mathcal{A}_2$:*

$$\mathcal{A}_1(t) \leqslant C^*\Big(E_0^\nu + \frac{1}{\nu^{3/2}}\,\mathcal{A}_1(t)(E_0 + \mathcal{A}_1(t))\Big)\exp(C^* E_0), \tag{2.3}$$

$$\mathcal{A}_2(t) \leqslant C^*\Big(\Big(E_0 + \frac{1}{\nu^4}\,E_0^2\Big) + (1 + E_0 + \mathcal{A}_1(t))\,\mathcal{A}_1(t)\Big), \tag{2.4}$$

where the constant C^* depends on $\mu, \underline{\nu}$ and (increasingly) on ρ^*.

The proof of Lemma 2.1 is established through refined computations, and the compensated result by Coifman et al. [8] turns out to be crucial for achieving a uniform bound with respect to ν. We refer to Section 2.2.2 below for the detailed proof.

For $d = 3$, the following estimates hold true for the functionals $\mathcal{A}_1$ and $\mathcal{A}_2$.

Lemma 2.2 (Preliminary energy estimates for $d = 3$). *Assume that $d = 3$ and (2.2). Then, the following estimates hold true for the functionals $\mathcal{A}_1$ and $\mathcal{A}_2$:*

$$\mathcal{A}_1(t) \leqslant C^*\Big(E_0^\nu + \frac{1}{\nu^{2/3}}\,E_0^{1/3}\Big) + CE_0\,\mathcal{A}_1(t)^2, \tag{2.5}$$

$$\mathcal{A}_2(t) \leqslant C^*\Big(\frac{1}{\nu}\,E_0^{1/3} + E_0 + E_0^2 + (1 + \mathcal{A}_1(t)^2)\,\mathcal{A}_1(t)\Big). \tag{2.6}$$

Here C depends on μ and $\underline{\nu}$, and C^* depends on μ, $\underline{\nu}$ and (increasingly) on ρ^*.

The proof is given in Section 2.2.3. Let us point out that the computations in [14,16,17] depend heavily on the fact that the domain has finite measure. Lemmas 2.1 and 2.2 are the first to provide high regularity bounds for the solution (ρ, u) uniformly with respect to λ (large) in the whole space, with only bounded density.

Based on the above estimates, it turns out that the functionals $\mathcal{A}_1$ and $\mathcal{A}_2$ are under control (for large ν) as long as the density is upper-bounded. Therefore, the next step is devoted to estimating the upper bound of the density, whose proof is given in Section 2.2.4.

Lemma 2.3 (Density upper bound in terms of energies). *Assume (2.2). Then the following bounds hold true for the density:*

$$\|\rho - \tilde\rho\|_{L^\infty([0,t]\times\mathbb{R}^d)} \leqslant \|\rho_0 - \tilde\rho\|_{L^\infty(\mathbb{R}^d)} + \frac{C^*}{\nu^{1/3}} \times \begin{cases} (1 + E_0^{1/18})\,(E_0^{1/6} + \nu^{1/6}\mathcal{A}_1(t)^{1/6})\,(\mathcal{A}_1(t)^{1/3} + \mathcal{A}_2(t)^{1/3}), & d = 2,\\ (\mathcal{A}_1(t)^{1/2} + \mathcal{A}_2(t)^{1/2}), & d = 3. \end{cases} \tag{2.7}$$

Finally, we notice that for $d = 2$ there is a small factor $1/\nu$ (or its positive powers) before $\mathcal{A}_1(t)$ and $\mathcal{A}_2(t)$ in the estimates (2.3) and (2.7), while $\mathcal{A}_2(t)$ can be bounded by $\mathcal{A}_1(t)$ and ρ^* by (2.4). We can close the estimates in Lemmas 2.1, 2.2 and 2.3 by a bootstrap argument as in, e.g., [3, 16], which is not repeated here.

Lemma 2.4 (Global-in-time estimates under assumption (1.18) or (1.19)). *There exist c, depending only on μ and $\underline{\nu}$, and $\nu_0 \geqslant \underline{\nu}$, depending on μ, $\underline{\nu}$, E_0, $\|\nabla u_0\|_{L^2(\mathbb{R}^d)}$ and ρ_0^*, such that*

(1) *If $d = 2$ and $\nu \geqslant \nu_0$, then*

$$\mathcal{A}_1(t) + \mathcal{A}_2(t) \leqslant C_0^\nu \quad \text{and} \quad \|\rho - \tilde\rho\|_{L^\infty([0,t]\times\mathbb{R}^2)} \leqslant \|\rho_0 - \tilde\rho\|_{L^\infty(\mathbb{R}^2)} + (C_0^\nu)^{1/2}.$$

(2) *If* $d = 3$, $E_0^\nu E_0 \leqslant c$ *and* $\nu \geqslant \nu_0$, *then*

$$\mathcal{A}_1(t) + \mathcal{A}_2(t) \leqslant C_0^\nu \quad \textit{and} \quad \|\rho - \tilde{\rho}\|_{L^\infty([0,t]\times\mathbb{R}^3)} \leqslant \|\rho_0 - \tilde{\rho}\|_{L^\infty(\mathbb{R}^3)} + (C_0^\nu)^{1/2}.$$

Above, C_0^ν *depends on* μ, $\underline{\nu}$, ρ_0^* *and (superlinearly) on* E_0^ν.

2.1.2. A priori estimates for $\mathcal{A}_3$ and $\|\nabla u\|_{L^1_t L^\infty_x}$. Now we have Lemma 2.4, which gives the (uniform) bounds of functionals $\mathcal{A}_1$, $\mathcal{A}_2$ and ρ. We use the notation C_0 below to denote some time-independent constant depending on the initial data as follows:

$$C_0 = C_0(\mu, \underline{\nu}, m, d, p, \rho_0^*, E_0^\nu), \tag{2.8}$$

where m and p appear in the initial condition (1.4). C_0 may vary from lines to lines, and controls in particular $\mathcal{A}_1$, $\mathcal{A}_2$ and ρ. The next step is dedicated to translating these bounds into the tangential regularity estimates for the density, together with the Lipschitz norm of the velocity. As the tangential regularity $\mathcal{A}_3$ can be transported by Lipschitz continuous flow, we sketch the idea to show Lipschitz continuity of u as follows.

We first recall the following decomposition of the velocity gradient:

$$\begin{aligned}\nabla u &= \nabla \tilde{u} + \nabla u_G \\ &:= \Big(-\frac{1}{\nu}\mathcal{R}\mathcal{R}(-\Delta)^{-1}\operatorname{div}(\rho\dot{u}) - \frac{1}{\mu}\mathcal{R}\mathcal{R}(-\Delta)^{-1}\cdot\operatorname{curl}(\rho\dot{u})\Big) + \Big(\frac{1}{\nu}\mathcal{R}\mathcal{R}G\Big).\end{aligned} \tag{2.9}$$

where $\mathcal{R}_j = (\frac{1}{i}\partial_j)(-\Delta)^{-1/2}$, with $1 \leqslant j \leqslant d$, is the Riesz transform, and $G = P(\rho) - \widetilde{P}$. Indeed, we notice the following expression,

$$\Delta u^j = \partial_j \operatorname{div} u + \sum_{k=1}^{d} \partial_k \operatorname{curl}_{jk} u, \quad \text{for } j = 1, \dots, d, \tag{2.10}$$

with $\operatorname{curl}_{jk} u = \partial_k u^j - \partial_j u^k$, for $j, k = 1, \dots, d$, and from (1.13), (1.28) and (1.29), we have

$$\operatorname{div} u = \frac{1}{\nu}(F + G), \quad F = -(-\Delta)^{-1}\operatorname{div}(\rho\dot{u}) \quad \text{and} \quad \mu\operatorname{curl} u = -(-\Delta)^{-1}\operatorname{curl}(\rho\dot{u}). \tag{2.11}$$

Hence the velocity gradient can be expressed as in (2.9):

$$\begin{aligned}\nabla u &= -\nabla(-\Delta)^{-1}\Delta u = -\nabla(-\Delta)^{-1}\nabla\operatorname{div} u - \nabla(-\Delta)^{-1}\operatorname{div}(\operatorname{curl} u) \\ &= -\frac{1}{\nu}\nabla(-\Delta)^{-1}\nabla\big(F + G\big) - \frac{1}{\mu}\nabla(-\Delta)^{-1}\operatorname{div}(\mu\operatorname{curl} u) \\ &= \Big(-\frac{1}{\nu}\mathcal{R}\mathcal{R}(-\Delta)^{-1}\operatorname{div}(\rho\dot{u}) - \frac{1}{\mu}\mathcal{R}\mathcal{R}(-\Delta)^{-1}\cdot\operatorname{curl}(\rho\dot{u})\Big) + \Big(\frac{1}{\nu}\mathcal{R}\mathcal{R}G\Big).\end{aligned}$$

Thanks to the regularity of $\dot{u}$ provided by the functionals $\mathcal{A}_1$ and $\mathcal{A}_2$, we have that $\nabla\tilde{u} \in L^1((0,t), L^\infty(\mathbb{R}^d))$. Motivated by the pioneering work of Chemin [5, 6] and of Danchin, Fanelli, and Paicu [10], which show $\mathcal{R}\mathcal{R}G \in L^\infty(\mathbb{R}^d)$ provided with extra tangential regularity on G, the L^∞-bound for ∇u_G in our case relies on the following logarithmic inequality, which is simply the application of the Sobolev embedding $L^\infty(\mathbb{R}^d) \cap \dot{W}^{1,p}(\mathbb{R}^d) \subset \mathcal{C}^{1-d/p}(\mathbb{R}^d)$ to Theorem 7.40 of [1].

Proposition 2.5 ([1], L^∞-bound for double Riesz transforms provided with tangential regularity). *Let $\mathcal{X} = (X_\upsilon)_{1\leqslant\upsilon\leqslant m} \subset \mathbb{L}^{\infty,p}(\mathbb{R}^d)$, with $d < p < \infty$, be a* nondegenerate *family of $m \in \mathbb{N}^*$ vector fields as in Section 1.1. Let $1 \leqslant q < \infty$. Then there exists a constant $C = C(m, d, p, q) > 0$ such that for all $G \in L^q(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap \mathbb{L}^p_{\mathcal{X}}(\mathbb{R}^d)$, the following estimate holds true*:

$$\|\mathcal{R}\mathcal{R}G\|_{L^\infty(\mathbb{R}^d)} \leqslant C\|G\|_{L^q(\mathbb{R}^d)} + C\|G\|_{L^\infty(\mathbb{R}^d)}\Big(1 + \log\Big(e + \frac{\|G\|_{\mathbb{L}^p_{\mathcal{X}}(\mathbb{R}^d)}}{\|G\|_{L^\infty(\mathbb{R}^d)}}\Big)\Big). \tag{2.12}$$

With the aid of the above logarithmic estimate, we can propagate tangential regularity of density and achieve Lipschitz regularity of the velocity at the same time.

Lemma 2.6 (Tangential regularity for the density and Lipschitz continuity for the velocity). *Assume the initial condition (1.4) that $\rho_0 \in \mathbb{L}^p_{\mathcal{X}_0}(\mathbb{R}^d)$, where $\mathcal{X}_0 = (X_{0,\upsilon})_{1\leqslant\upsilon\leqslant m} \subset \mathbb{L}^{\infty,p}(\mathbb{R}^d)$ is a nondegenerate family of $m \in \mathbb{N}^*$ vectors fields, with $m \geqslant d-1$, with $2 < p < \infty$ if $d = 2$ or $3 < p < 6$ if $d = 3$.*

Then, the family of vector fields $\mathcal{X}(t) = (X_\upsilon(t))_{1\leqslant\upsilon\leqslant m}$, defined as solution of the Cauchy problem (1.8), is nondegenerate and $\mathcal{X}(t) \subset \mathbb{L}^{\infty,p}(\mathbb{R}^d)$. Moreover, we have $\rho(t) \in \mathbb{L}^p_{\mathcal{X}(t)}(\mathbb{R}^d)$, and the following bounds hold true:

$$\begin{cases} \mathcal{A}_3(t) \leqslant \mathcal{A}_3(0)\exp\Big(C_0\displaystyle\int_0^t \big[1 + \sqrt{t} + \|\nabla u(t')\|_{L^\infty(\mathbb{R}^d)}\big]\,dt'\Big), \\ \displaystyle\int_0^t \|\nabla u(t')\|_{L^\infty(\mathbb{R}^d)}\,dt' \leqslant C_0\Big(1 + \frac{\mathcal{A}_3(0)}{I(\mathcal{X}_0)}\Big)\exp(C_0 t). \end{cases} \tag{2.13}$$

The proof of the above lemma is the object of Section 2.2.5.

Remark 2.7 (Improved time regularity). We have the following improved time regularity, which is required for the uniqueness result, see, e.g., equation (4.31) in [10]: for some $t_0 > 0$,

$$\int_0^{t_0} \sigma(t')^s\,\|\nabla u(t')\|^2_{L^\infty(\mathbb{R}^d)}\,dt' < \infty,$$

where $s = 4/9$ if $d = 2$ and $s = 1/2$ if $d = 3$. Indeed, we apply Hölder's inequality with respect to the time variable to (2.62) in the proof in Section 2.2.5 to obtain (noticing (2.63) and (2.65))

$$\int_0^t \|\nabla u_G(t')\|^2_{L^\infty(\mathbb{R}^d)}\,dt' \leq C_0\, t\Big(1 + \frac{\mathcal{A}_3(0)}{I(\mathcal{X}_0)} + t + \int_0^t \|\nabla u(t')\|_{L^\infty(\mathbb{R}^d)}\,dt'\Big)^2$$

and similarly as in the proof of (2.64) and (2.66), we have

$$\int_0^t \sigma^{4/9}\,\|\nabla\tilde{u}\|^2_{L^\infty(\mathbb{R}^2)} \leqslant C_0(1 + t^{1/3}) \quad \text{and} \quad \int_0^t \sqrt{\sigma}\,\|\nabla\tilde{u}\|^2_{L^\infty(\mathbb{R}^3)} \leqslant C_0.$$

To complete the proof of Theorem 1.6, we need to construct an approximate sequence $(\rho^\delta, u^\delta)_\delta$ globally defined in time that converges to (ρ, u), the unique solution of (1.1). Once this is done, we will have obtained a sequence $(\rho^{(\nu)}, u^{(\nu)})$ of solutions to (1.1), and the last step will be to justify that this sequence converges to some (ϱ, v) that solves the inhomogeneous incompressible model. This is the purpose of Section 2.3, and, as we will see, the local solutions constructed in [10] cannot serve as building blocks. Thus, we will need to establish local well-posedness for the system (1.1) in Section B.

2.2. Proofs

In this subsection, we give the detailed proofs of Lemmas 2.1, 2.2, 2.3 and 2.6. Before that, we recall some basic facts, which will be used freely in the proofs below.

2.2.1. Basic facts. Under the assumption (2.2), we have the $L^\infty((0,t),L^q(\mathbb{R}^d))$ estimate for the pressure fluctuation term $G(t,x) = \big(P(\rho)\big)(t,x) - \widetilde{P}$ given in (1.25):

$$\|G\|_{L^\infty([0,t],L^q(\mathbb{R}^d))} \leqslant C^*(E_0)^{1/q}, \quad \text{with} \quad q\in[2,\infty], \tag{2.14}$$

where C^* depends on q and ρ^*. Here, the case $q=\infty$ follows straightforwardly from the definition. Recall also the estimate (1.27) for G by F: for any $l \geqslant 1$,

$$\frac{1}{\nu}\|G\|^{l+1}_{L^{l+1}((0,t)\times\mathbb{R}^d)} \leqslant C^*(l)E_0 + \frac{C}{\nu}\|F\|^{l+1}_{L^{l+1}((0,t)\times\mathbb{R}^d)}, \quad \forall t\in(0,\infty), \tag{2.15}$$

where we estimated $\|H_l(\rho_0)\|_{L^1(\mathbb{R}^d)}$ by $C^*(l)E_0$. Recall also the relations (2.10):

$$\Delta u^j = \partial_j \operatorname{div} u + \sum_{k=1}^d \partial_k \operatorname{curl}_{jk} u, \quad j=1,\dots,d,$$

and (2.11) between $\operatorname{div} u$, F, G, $\rho\dot{u}$ and $\operatorname{curl} u$:

$$\operatorname{div} u = \frac{1}{\nu}(F+G), \quad F = -(-\Delta)^{-1}\operatorname{div}(\rho\dot{u}), \quad \mu\operatorname{curl} u = -(-\Delta)^{-1}\operatorname{curl}(\rho\dot{u}). \tag{2.16}$$

By use of the $L^q(\mathbb{R}^d)$, $q\in(1,\infty)$, $d\geqslant 2$-boundedness of Riesz operators, the following estimates follow immediately:

$$\|\nabla u\|_{L^q(\mathbb{R}^d)} \leqslant C(q,d)\big(\|\operatorname{div} u\|_{L^q(\mathbb{R}^d)} + \|\operatorname{curl} u\|_{L^q(\mathbb{R}^d)}\big), \tag{2.17}$$

$$\|\nabla F\|_{L^q(\mathbb{R}^d)} + \mu\|\nabla\operatorname{curl} u\|_{L^q(\mathbb{R}^d)} \leqslant C(q,d)\|\rho\dot{u}\|_{L^q(\mathbb{R}^d)}. \tag{2.18}$$

We now recall the compensated integrability result by Coifman, Lions, Meyer and Semmes [8] in dimension two.

Proposition 2.8 (Coifman–Lions–Meyer–Semmes' estimate for $d=2$). *Consider two function $v,w\in\dot{H}^1(\mathbb{R}^2;\mathbb{R})$, and define*

$$g = \det\begin{pmatrix}\partial_1 v & \partial_2 v\\ \partial_1 w & \partial_2 w\end{pmatrix}.$$

Then g belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$, whence for all $f\in \mathrm{BMO}(\mathbb{R}^2)$, we have the estimate

$$\left|\int_{\mathbb{R}^2} f(x)g(x)\,dx\right| \leqslant \|f\|_{\mathrm{BMO}(\mathbb{R}^2)}\|\nabla v\|_{L^2(\mathbb{R}^2)}\|\nabla w\|_{L^2(\mathbb{R}^2)}.$$

In particular, since $\dot{H}^1(\mathbb{R}^2)\hookrightarrow \mathrm{BMO}(\mathbb{R}^2)$, for all $f\in\dot{H}^1(\mathbb{R}^2)$, we have

$$\left|\int_{\mathbb{R}^2} f(x)g(x)\,dx\right| \leqslant \|\nabla f\|_{L^2(\mathbb{R}^2)}\|\nabla v\|_{L^2(\mathbb{R}^2)}\|\nabla w\|_{L^2(\mathbb{R}^2)}.$$

It helps controlling the integrals appearing in the estimation of energies, for example $\int_{\mathbb{R}^2} F\,\det(\nabla u)\,dx$. The term $\det(\nabla u)$ arises naturally in the computation of products in dimension two, for instance,

$$\nabla u^j\cdot\nabla u^k\partial_k u^j=\operatorname{div}u\{|\nabla u|^2-\det(\nabla u)\},\quad \nabla u^l\cdot\partial_l u=(\operatorname{div}u)^2-2\det(\nabla u). \tag{2.19}$$

Here and in the following, we use Einstein's summation convention for repeated indices, unless otherwise claimed.

2.2.2. Proof of Lemma 2.1 for $d=2$. This paragraph is devoted to obtaining bounds for the functionals $\mathcal{A}_1$ and $\mathcal{A}_2$ as defined in (1.15) for $d=2$, provided with bounded density function (2.2). The constants in the following estimates may depend on the viscosity coefficient μ and the lower bound $\underline{\nu}$ for ν, while not on the viscosity coefficient ν which will be chosen to be big.

Proof of (2.3). The functional $\mathcal{A}_1$ arises while using $\dot u$ as a test functional in the weak formulation of the momentum equation $(1.1)_2$. By doing so, one obtains the following equality:

$$\begin{aligned}\mathcal{A}_1(t)=&\frac{\mu}{2}\|\nabla u_0\|^2_{L^2(\mathbb{R}^d)}+\frac{\mu+\lambda}{2}\|\operatorname{div}u_0\|^2_{L^2(\mathbb{R}^d)}-\mu\int_0^t\int_{\mathbb{R}^d}\nabla u^j\cdot\nabla u^k\partial_k u^j\\&+\frac{\mu}{2}\int_0^t\int_{\mathbb{R}^d}|\nabla u|^2\operatorname{div}u+\frac{\mu+\lambda}{2}\int_0^t\int_{\mathbb{R}^d}(\operatorname{div}u)^3\\&-(\mu+\lambda)\int_0^t\int_{\mathbb{R}^d}\operatorname{div}u\nabla u^l\cdot\partial_l u+\int_{\mathbb{R}^d}\operatorname{div}u(s)G(s)\big|_{s=0}^{s=t}\\&+\int_0^t\int_{\mathbb{R}^d}\nabla u^l\cdot\partial_l u\,G+\int_0^t\int_{\mathbb{R}^d}(\rho P'(\rho)-P(\rho)+\widetilde{P})(\operatorname{div}u)^2.\end{aligned}\tag{2.20}$$

Step 1. *Reformulation of the energy equality.*

In the following lines, we will reformulate the terms appearing in the right-hand side above by use of (2.16) and (2.19).

By (2.16) and (2.19), the sum of the third and the fourth terms on the right-hand side of (2.20) can be reduced as follows:

$$\begin{aligned}\mu\int_0^t\int_{\mathbb{R}^2}\operatorname{div}u\Big(\det(\nabla u)-\frac12|\nabla u|^2\Big)=&\frac{\mu}{\nu}\int_0^t\int_{\mathbb{R}^2}F\det(\nabla u)-\frac{\mu}{2\nu}\int_0^t\int_{\mathbb{R}^2}F|\nabla u|^2\\&+\frac{\mu}{\nu}\int_0^t\int_{\mathbb{R}^2}G\Big(\det(\nabla u)-\frac12|\nabla u|^2\Big),\end{aligned}\tag{2.21}$$

and similarly, the sum of the sixth and the eighth terms of (2.20) reads

$$\begin{aligned}&-\frac{\mu+\lambda}{\nu}\int_0^t\int_{\mathbb{R}^2}F\nabla u^l\cdot\partial_l u+\frac{\mu}{\nu}\int_0^t\int_{\mathbb{R}^2}\nabla u^l\cdot\partial_l uG\\&\qquad=2\frac{\mu+\lambda}{\nu}\int_0^t\int_{\mathbb{R}^2}F\det(\nabla u)-\frac{\mu+\lambda}{\nu}\int_0^t\int_{\mathbb{R}^2}F(\operatorname{div}u)^2\\&\qquad\quad+\frac{\mu}{\nu}\int_0^t\int_{\mathbb{R}^2}\nabla u^l\cdot\partial_l uG.\end{aligned}\tag{2.22}$$

Now we pack the fifth term of (2.20) and the middle term in the above (2.22) and use (2.16) to get

$$\frac{\mu+\lambda}{2}\int_0^t\int_{\mathbb{R}^2}(\operatorname{div}u)^3-\frac{\mu+\lambda}{\nu}\int_0^t\int_{\mathbb{R}^2}F(\operatorname{div}u)^2 = -\frac{\mu+\lambda}{2\nu^2}\int_0^t\int_{\mathbb{R}^2}(F^2-G^2)\operatorname{div}u. \tag{2.23}$$

Step 2. *Estimates for the integrals in terms of* E_0, E_0^ν *and* $L^4_{t,x}$*-norms of* $(\nabla u, G, F)$. We are ready to estimate all the terms above.

- With the help of Proposition 2.8 and (2.18), the first terms of the right-hand side of (2.21) and (2.22) can be estimated as follows:

$$\begin{aligned}\Big|\frac{3\mu+2\lambda}{\nu}\int_0^t\int_{\mathbb{R}^2}F\det(\nabla u)\Big| &\leqslant C\int_0^t\|\nabla F\|_{L^2(\mathbb{R}^2)}\|\nabla u\|^2_{L^2(\mathbb{R}^2)}\\ &= C\int_0^t\|\rho\dot u\|_{L^2(\mathbb{R}^2)}\|\nabla u\|^2_{L^2(\mathbb{R}^2)}\\ &\leqslant \eta\int_0^t\|\sqrt{\rho}\,\dot u\|^2_{L^2(\mathbb{R}^2)}+\frac{C\rho^*}{4\eta}\int_0^t\|\nabla u\|^4_{L^2(\mathbb{R}^2)},\end{aligned} \tag{2.24}$$

by Young's inequality, for some $\eta>0$ small enough to be determined later.

- By the energy balance (2.1) and the upper bound ρ^* for the density (2.2), the last term of (2.20) and the terms involving the pressure deviation G in (2.21) and(2.22) can be bounded as follows:

$$\begin{aligned}\Big|\int_0^t\int_{\mathbb{R}^d}(\rho P'(\rho)-P(\rho)+\widetilde{P})(\operatorname{div}u)^2\Big|+\frac{\mu}{\nu}\Big|\int_0^t\int_{\mathbb{R}^2}G\Big(\det(\nabla u)-\frac{1}{2}|\nabla u|^2\Big)\Big|&\\ +\frac{\mu}{\nu}\Big|\int_0^t\int_{\mathbb{R}^2}\nabla u^l\partial_l u G\Big|\leqslant\frac{C^*}{\nu}E_0.&\end{aligned} \tag{2.25}$$

- Next, the middle term of (2.21) is:

$$\begin{aligned}\frac{\mu}{2\nu}\Big|\int_0^t\int_{\mathbb{R}^2}F|\nabla u|^2\Big| &\leqslant \frac{C}{\nu}\int_0^t\|F\|_{L^4(\mathbb{R}^2)}\|\nabla u\|_{L^4(\mathbb{R}^2)}\|\nabla u\|_{L^2(\mathbb{R}^2)}\\ &\leqslant C\int_0^t\Big(\frac{1}{\nu^{5/2}}\|F\|^4_{L^4(\mathbb{R}^2)}+\frac{1}{\nu^{3/2}}\|\nabla u\|^4_{L^4(\mathbb{R}^2)}\Big)+C\int_0^t\|\nabla u\|^2_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{2.26}$$

- The term in (2.23) can be estimated as follows:

$$\begin{aligned}\frac{\mu+\lambda}{2\nu^2}\Big|\int_0^t\int_{\mathbb{R}^2}(F^2-G^2)\operatorname{div}u\Big| &\leqslant \frac{C}{\nu^3}\int_0^t\big(\|F\|^4_{L^4(\mathbb{R}^2)}+\|G\|^4_{L^4(\mathbb{R}^2)}\big)\\ &\quad +C\nu\int_0^t\|\operatorname{div}u\|^2_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{2.27}$$

- It only remains the first term in the last line of (2.20), which can be bounded as follows:

$$\Big| \int_{\mathbb{R}^d} \operatorname{div} u(s)\, G(s) \Big|_{s=0}^{s=t} \Big| \leqslant \eta\nu\, \|\operatorname{div} u(t)\|^2_{L^2(\mathbb{R}^2)} + \frac{C}{4\eta\nu}\, \|G(t)\|^2_{L^2(\mathbb{R}^2)} + \frac{C}{\nu}\, \|G_0\|^2_{L^2(\mathbb{R}^2)} + C\nu\, \|\operatorname{div} u_0\|^2_{L^2(\mathbb{R}^2)},$$

with the second term of the right-hand side controlled by the initial energy as in (2.14).

We combine all of these estimates and we choose η small in order to obtain the following:

$$\mathcal{A}_1(t) \leqslant C\Big(1 + \frac{C^*}{\nu}\Big) E_0^\nu + C\rho^* \int_0^t \|\nabla u\|^4_{L^2(\mathbb{R}^2)} + \frac{C}{\nu^{3/2}} \int_0^t \Big(\|\nabla u\|^4_{L^4(\mathbb{R}^2)} + \frac{1}{\nu}\, \|F\|^4_{L^4(\mathbb{R}^2)} + \frac{1}{\nu^{3/2}}\, \|G\|^4_{L^4(\mathbb{R}^2)} \Big),$$

where E_0^ν is given in (1.16). Hence Grönwall's lemma yields

$$\mathcal{A}_1(t) \leqslant \Big[C^* E_0^\nu + \frac{C}{\nu^{3/2}} \int_0^t \Big(\|\nabla u\|^4_{L^4(\mathbb{R}^2)} + \frac{1}{\nu}\, \|F\|^4_{L^4(\mathbb{R}^2)} + \frac{1}{\nu^{3/2}}\, \|G\|^4_{L^4(\mathbb{R}^2)} \Big) \Big] \exp(C^* E_0). \tag{2.28}$$

Step 3. *Final estimates.*

The next step is devoted to obtaining estimate for the $L^4((0,t)\times\mathbb{R}^2)$ norm of the velocity gradient ∇u, the pressure deviation G and the effective flux F.

- L^4-estimate for G. Recall (2.15) with $l = 3$:

$$\frac{1}{\nu}\, \|G\|^4_{L^4((0,t)\times\mathbb{R}^2)} \leqslant C^* E_0 + \frac{C}{\nu} \|F\|^4_{L^4((0,t)\times\mathbb{R}^2)}. \tag{2.29}$$

- L^4-estimate for F. The $L^4((0,t)\times\mathbb{R}^2)$-norm of the effective flux F follows from Gagliardo–Nirenberg's inequality,

$$\|f\|^2_{L^4(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)} \|\nabla f\|_{L^2(\mathbb{R}^2)},$$

and from (2.18):

$$\|F\|_{L^4(\mathbb{R}^2)} \leqslant C\|F\|^{1/2}_{L^2(\mathbb{R}^2)}\, \|\nabla F\|^{1/2}_{L^2(\mathbb{R}^2)} \leqslant C\|F\|^{1/2}_{L^2(\mathbb{R}^2)}\, \|\rho\dot{u}\|^{1/2}_{L^2(\mathbb{R}^2)},$$

which can be bounded further, by virtue of (2.16) and the definition of $E(t)$ and $\mathcal{A}_1(t)$, as follows:

$$\|F\|^4_{L^4((0,t)\times\mathbb{R}^2)} \leqslant C \int_0^t (\nu\|\operatorname{div} u\|_{L^2(\mathbb{R}^2)} + \|G\|_{L^2(\mathbb{R}^2)})^2\, \|\rho\dot{u}\|^2_{L^2(\mathbb{R}^2)} \leqslant C^*(\nu\mathcal{A}_1(t) + E_0)\, \mathcal{A}_1(t).$$

- L^4-estimate for ∇u. Similar as above for F, we have an L^4-estimate for $\operatorname{curl} u$:

$$\|\operatorname{curl} u\|_{L^4((0,t)\times\mathbb{R}^2)} \leqslant C\,\|\operatorname{curl} u\|^{1/2}_{L^\infty((0,t);L^2(\mathbb{R}^2))}\, \|\rho\dot{u}\|^{1/2}_{L^2((0,t)\times\mathbb{R}^2)} \leqslant C\mathcal{A}_1(t)^{1/2}.$$

Hence, by use of (2.16)–(2.17), the following inequality holds true:

$$\begin{aligned}\|\nabla u\|^4_{L^4((0,t)\times\mathbb{R}^2)} &\leqslant C\big(\|\operatorname{div} u\|^4_{L^4((0,t)\times\mathbb{R}^2)} + \|\operatorname{curl} u\|^4_{L^4((0,t)\times\mathbb{R}^2)}\big)\\ &\leqslant \frac{C}{\nu^4}\big(\|F\|^4_{L^4((0,t)\times\mathbb{R}^2)} + \|G\|^4_{L^4((0,t)\times\mathbb{R}^2)}\big) + C\|\operatorname{curl} u\|^4_{L^4((0,t)\times\mathbb{R}^2)}\\ &\leqslant \frac{C^*}{\nu^3} E_0 + C^*\mathcal{A}_1(t)(E_0+\mathcal{A}_1(t)).\end{aligned}$$

Finally, we go back to (2.28) and we have (2.3). ■

Proof of (2.4). We turn to providing a bound for the second functional $\mathcal{A}_2$ for $d = 2$. For this purpose, by (2.16), we rewrite the momentum equation $(1.1)_2$ as follows:

$$\rho\dot{u} = \mu\Delta u + \frac{\mu+\lambda}{\nu}\nabla F - \frac{\mu}{\nu}\nabla G. \tag{2.30}$$

We apply the operator $\partial_t \cdot + \operatorname{div}(\cdot u)$ to (2.30) and we obtain the following equation for the material derivative of the velocity:

$$\begin{aligned}&\partial_t(\rho\dot{u}^j) + \operatorname{div}(\rho\dot{u}^j u) - \mu\Delta\dot{u}^j - \frac{\mu+\lambda}{\nu}\partial_j\dot{F}\\ &\quad= -\mu\,\partial_k(\nabla u^j\cdot\partial_k u) + \mu\,\partial_k(\partial_k u^j \operatorname{div} u) - \mu\operatorname{div}(\partial_k u^j\partial_k u)\\ &\qquad+\frac{\mu+\lambda}{\nu}\partial_j(F\operatorname{div} u) - \frac{\mu+\lambda}{\nu}\operatorname{div}(F\partial_j u)\\ &\qquad+\frac{\mu}{\nu}\partial_j\big((\rho P'(\rho) - P(\rho) + \widetilde{P})\operatorname{div} u\big) + \frac{\mu}{\nu}\operatorname{div}(\partial_j u G),\quad j=1,\dots,d.\end{aligned}\tag{2.31}$$

Step 1. *Formulation of the energy equality.*

To obtain the functional $\mathcal{A}_2$, it suffices to multiply the equation above by $\sigma\dot{u}^j$, with $\sigma = \sigma(t) = \min\{1,t\}$, sum up j, and integrate it in time and space. The most delicate term is

$$-\frac{\mu+\lambda}{\nu}\partial_j\dot{F}$$

on the left-hand side of (2.31), which gives

$$-\frac{\mu+\lambda}{\nu}\int_0^t\int_{\mathbb{R}^2}\sigma\dot{u}^j\partial_j\dot{F} = \frac{\mu+\lambda}{\nu}\int_0^t\sigma\int_{\mathbb{R}^2}\dot{F}\operatorname{div}\dot{u}.$$

We first focus on this integral for a while. Applying material derivative to (2.16) gives

$$\operatorname{div}\dot{u} = \frac{1}{\nu}(\dot{F} - \rho P'(\rho)\operatorname{div} u) + \nabla u^k\cdot\partial_k u, \tag{2.32}$$

and hence

$$\begin{aligned}-\frac{\mu+\lambda}{\nu}\int_0^t\int_{\mathbb{R}^2}\sigma\dot{u}^j\partial_j\dot{F} &= \frac{\mu+\lambda}{\nu^2}\int_0^t\sigma\|\dot{F}\|^2_{L^2(\mathbb{R}^2)} + \frac{\mu+\lambda}{\nu}\int_0^t\sigma\int_{\mathbb{R}^2}\dot{F}\,\nabla u^k\cdot\partial_k u\\ &\quad-\frac{\mu+\lambda}{\nu^2}\int_0^t\sigma\int_{\mathbb{R}^2}\dot{F}\rho P'(\rho)\operatorname{div} u.\end{aligned}\tag{2.33}$$

To conclude, testing (2.31) by $\sigma\dot{u}$ implies

$$\frac{1}{2}\sigma(t)\,\|\sqrt{\rho}\,\dot{u}\|^2_{L^2(\mathbb{R}^2)}+\mu\int_0^t\sigma\|\nabla\dot{u}\|^2_{L^2(\mathbb{R}^2)}+\frac{\mu+\lambda}{\nu^2}\int_0^t\sigma\|\dot{F}\|^2_{L^2(\mathbb{R}^2)}$$
$$=\frac{1}{2}\int_0^{\sigma(t)}\|\sqrt{\rho}\,\dot{u}\|^2_{L^2(\mathbb{R}^2)}+\sum_{k=1}^4 I_k, \tag{2.34}$$

where

$$\begin{aligned}
I_1&=-\frac{\mu+\lambda}{\nu}\int_0^t\sigma\int_{\mathbb{R}^2}\dot{F}\,\nabla u^k\cdot\partial_k u+\frac{\mu+\lambda}{\nu^2}\int_0^t\sigma\int_{\mathbb{R}^2}\dot{F}\rho P'(\rho)\operatorname{div}u,\\
I_2&=\mu\int_0^t\sigma\int_{\mathbb{R}^2}\big(\nabla u^j\cdot\partial_k u\partial_k\dot{u}^j-\partial_k u^j\operatorname{div}u\partial_k\dot{u}^j+\partial_k u^j\partial_k u\cdot\nabla\dot{u}^j\big),\\
I_3&=\frac{\mu+\lambda}{\nu}\int_0^t\sigma\int_{\mathbb{R}^2}\big(-F\operatorname{div}u\operatorname{div}\dot{u}+F\partial_j u^k\partial_k\dot{u}^j\big),\\
I_4&=-\frac{\mu}{\nu}\int_0^t\sigma\int_{\mathbb{R}^2}\big(\operatorname{div}u\operatorname{div}\dot{u}(\rho P'(\rho)-P(\rho)+\widetilde{P})+\partial_j u^k\partial_k\dot{u}^j G\big).
\end{aligned}$$

Step 2. *Estimate for* I_1.

We focus first on the first integral in I_1, which can be reformulated by integration by parts (noticing $\sigma(0)=0$) as

$$\begin{aligned}
\int_0^t\sigma\int_{\mathbb{R}^2}\dot{F}\,\nabla u^k\cdot\partial_k u&=\sigma(t)\int_{\mathbb{R}^2}F(t)\nabla u^k\cdot\partial_k u(t)-\int_0^{\sigma(t)}\int_{\mathbb{R}^2}F\,\nabla u^k\cdot\partial_k u\\
&\quad-2\int_0^t\sigma\int_{\mathbb{R}^2}F\,\partial_k u\cdot\nabla\dot{u}^k+2\int_0^t\sigma\int_{\mathbb{R}^2}F\,\partial_k u\cdot\nabla u^l\partial_l u^k\\
&\quad-\int_0^t\sigma\int_{\mathbb{R}^2}F\operatorname{div}u\nabla u^k\cdot\partial_k u.
\end{aligned}\tag{2.35}$$

The first term of the right-hand side in (2.35) above can be estimated similarly as for the derivation of (2.24) and (2.27), using the equality

$$\nabla u^l\cdot\partial_l u=(\operatorname{div}u)^2-2\det(\nabla u)$$

in (2.19) as follows:

$$\begin{aligned}
\Big|\sigma(t)\int_{\mathbb{R}^2}F(t)(\nabla u^k\cdot\partial_k u)(t)\Big|&\leqslant C\sigma(t)\,\|\rho\dot{u}\|_{L^2(\mathbb{R}^2)}\,\|\nabla u\|^2_{L^2(\mathbb{R}^2)}\\
&\quad+\sigma(t)\,\frac{1}{\nu^2}\Big|\int_{\mathbb{R}^2}F(t)(F(t)+G(t))^2\Big|\\
&\leqslant\eta\sigma(t)\,\|\sqrt{\rho}\dot{u}\|^2_{L^2(\mathbb{R}^2)}+\sigma(t)\,\frac{C^2\rho^*}{4\eta}\,\|\nabla u\|^4_{L^2(\mathbb{R}^2)}\\
&\quad+\sigma(t)\,\frac{C}{\nu^2}\big(\|F(t)\|^3_{L^3(\mathbb{R}^2)}+\|G(t)\|^3_{L^3(\mathbb{R}^2)}\big).
\end{aligned}$$

Exactly as in the derivation of (2.24) and (2.27), the second integral of the right-hand side of (2.35) can be estimated as follows:

$$\Big|\int_0^{\sigma(t)}\int_{\mathbb{R}^2} F\nabla u^k\cdot\partial_k u\Big| \leqslant CE_0 + C\int_0^{\sigma(t)}\|\sqrt{\rho}\,\dot u\|^2_{L^2(\mathbb{R}^2)} + C\rho^*\int_0^{\sigma(t)}\|\nabla u\|^4_{L^2(\mathbb{R}^2)} + \frac{C}{\nu^3}\int_0^{\sigma(t)}\big(\|F\|^4_{L^4(\mathbb{R}^2)} + \|G\|^4_{L^4(\mathbb{R}^2)}\big).$$

In order to estimate the third term of the right-hand side of (2.35), we write

$$\partial_j u^k\partial_k\dot u^j = \operatorname{div} u\operatorname{div}\dot u - (\partial_1 u^1\partial_2\dot u^2 - \partial_2 u^1\partial_1\dot u^2) - (\partial_2 u^2\partial_1\dot u^1 - \partial_1 u^2\partial_2\dot u^1) \tag{2.36}$$

in such a way that, after making use of the compensated result Proposition 2.8 and Young's inequality, we have

$$\Big|\int_0^t\sigma\int_{\mathbb{R}^2} F\partial_k u\cdot\nabla\dot u^k\Big| \leqslant \eta\int_0^t\sigma\|\nabla\dot u\|^2_{L^2(\mathbb{R}^2)} + \frac{C\rho^*}{\eta}\int_0^t\sigma\|\sqrt{\rho}\,\dot u\|^2_{L^2(\mathbb{R}^2)}\|\nabla u\|^2_{L^2(\mathbb{R}^2)} + \frac{C}{\eta}\int_0^t\Big(\frac{1}{\nu^2}\|F\|^4_{L^4(\mathbb{R}^2)} + \frac{1}{\nu^2}\|G\|^4_{L^4(\mathbb{R}^2)}\Big). \tag{2.37}$$

Thanks to (2.16) and the equality

$$\partial_k u^l\nabla u^k\cdot\partial_l u = \operatorname{div} u\{(\operatorname{div} u)^2 - 3\det(\nabla u)\}$$

in (2.19), we have the following estimate for the last two terms of (2.35):

$$2\Big|\int_0^t\sigma\int_{\mathbb{R}^2} F\partial_k u\cdot\nabla u^l\partial_l u^k\Big| + \Big|\int_0^t\sigma\int_{\mathbb{R}^2} F\operatorname{div} u\nabla u^k\cdot\partial_k u\Big| \leqslant C\int_0^t\sigma\Big(\|\nabla u\|^4_{L^4(\mathbb{R}^2)} + \frac{1}{\nu^2}\|F\|^4_{L^4(\mathbb{R}^2)} + \frac{1}{\nu^2}\|G\|^4_{L^4(\mathbb{R}^2)}\Big).$$

Finally, using Hölder's and Young's inequalities, the second integral in I_1 can be estimated as follows:

$$\frac{\mu+\lambda}{\nu^2}\Big|\int_0^t\sigma\int_{\mathbb{R}^2}\dot F\rho P'(\rho)\operatorname{div} u\Big| \leqslant \eta\frac{\mu+\lambda}{\nu^2}\int_0^t\sigma\|\dot F\|^2_{L^2(\mathbb{R}^2)} + \frac{C^*}{\nu^2\eta}E_0.$$

Gathering the above computations, we obtain the following estimate for I_1:

$$\begin{aligned}|I_1| &\leqslant C\Big(1+\frac{C^*}{\nu^2\eta}\Big)E_0 + \eta\sigma(t)\|\sqrt{\rho}\dot u\|^2_{L^2(\mathbb{R}^2)} + \eta\int_0^t\sigma\|\nabla\dot u\|^2_{L^2(\mathbb{R}^2)}\\ &+ \eta\frac{\mu+\lambda}{\nu^2}\int_0^t\sigma\|\dot F\|^2_{L^2(\mathbb{R}^2)} + C\int_0^{\sigma(t)}\|\sqrt{\rho}\,\dot u\|^2_{L^2(\mathbb{R}^2)} + \sigma(t)\frac{C^*}{\eta}\|\nabla u\|^4_{L^2(\mathbb{R}^2)}\\ &+ \frac{C\rho^*}{\eta}\int_0^t\sigma\|\sqrt{\rho}\,\dot u\|^2_{L^2(\mathbb{R}^2)}\|\nabla u\|^2_{L^2(\mathbb{R}^2)} + C\rho^*\int_0^{\sigma(t)}\|\nabla u\|^4_{L^2(\mathbb{R}^2)}\\ &+ \sigma(t)\frac{C}{\nu^2}\big(\|F(t)\|^3_{L^3(\mathbb{R}^2)} + \|G(t)\|^3_{L^3(\mathbb{R}^2)}\big)\\ &+ C\int_0^t\sigma\Big(\|\nabla u\|^4_{L^4(\mathbb{R}^2)} + \frac{1}{\eta\nu^2}\|F\|^4_{L^4(\mathbb{R}^2)} + \frac{1}{\eta\nu^2}\|G\|^4_{L^4(\mathbb{R}^2)}\Big).\end{aligned}$$

Step 3 *Final estimates.*

We now turn to the estimate of the last terms $\sum_{k=2}^{4} I_k$ in (2.34). By Young's inequality, it is straightforward to get

$$|I_2| \leqslant \eta \int_0^t \sigma \|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)}^2 + \frac{C}{\eta} \int_0^t \sigma \|\nabla u\|_{L^4(\mathbb{R}^2)}^4.$$

Similar as in Step 2, we have

$$\begin{aligned}|I_3| &\leqslant \eta \int_0^t \sigma \|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)}^2 + \frac{C\rho^*}{\eta} \int_0^t \sigma \|\sqrt{\rho}\,\dot{u}\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u\|_{L^2(\mathbb{R}^2)}^2 \\ &\quad + \frac{C}{\nu^2\eta} \int_0^t \sigma \left(\|F\|_{L^4(\mathbb{R}^2)}^4 + \|G\|_{L^4(\mathbb{R}^2)}^4\right), \\ |I_4| &\leqslant \eta \int_0^t \sigma \|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)}^2 + \frac{CC^*}{\nu^2\eta} E_0.\end{aligned}$$

Summing up, we have for η small enough that

$$\begin{aligned}\mathcal{A}_2(t) &\leqslant C\Big(1 + \frac{C^*}{\nu^2}\Big) E_0 + C^*\left(1 + E_0 + \mathcal{A}_1(t)\right)\mathcal{A}_1(t) \\ &\quad + \sigma(t)\,\frac{C}{\nu^2}\left(\|F(t)\|_{L^3(\mathbb{R}^2)}^3 + \|G(t)\|_{L^3(\mathbb{R}^2)}^3\right) \\ &\quad + C\int_0^t \sigma\Big(\|\nabla u\|_{L^4(\mathbb{R}^2)}^4 + \frac{1}{\nu^2}\|F\|_{L^4(\mathbb{R}^2)}^4 + \frac{1}{\nu^2}\|G\|_{L^4(\mathbb{R}^2)}^4\Big). \end{aligned} \tag{2.38}$$

Recalling Step 3 in the proof of (2.3) above, we get similar L^4-estimates with the time weight σ. We have the following, similar as (2.29):

$$\frac{1}{2\nu}\int_0^t \sigma \|G\|_{L^4(\mathbb{R}^2)}^4 \leqslant C^* E_0 + \frac{C}{\nu}\int_0^t \sigma \|F\|_{L^4(\mathbb{R}^2)}^4,$$

which implies that

$$\begin{aligned}&\int_0^t \sigma\Big(\|\nabla u\|_{L^4(\mathbb{R}^2)}^4 + \frac{1}{\nu^2}\|F\|_{L^4(\mathbb{R}^2)}^4 + \frac{1}{\nu^2}\|G\|_{L^4(\mathbb{R}^2)}^4\Big) \\ &\qquad\qquad \leqslant \frac{C^*}{\nu^3} E_0 + C^*(E_0 + \mathcal{A}_1(t))\,\mathcal{A}_1(t). \end{aligned} \tag{2.39}$$

On the other hand, we have (2.14):

$$\frac{1}{\nu^2}\|G(t)\|_{L^3(\mathbb{R}^2)}^3 \leqslant \frac{C^*}{\nu^2} E_0,$$

and hence the following, thanks to Gagliardo–Nirenberg inequality and (2.16)–(2.18):

$$\begin{aligned}\frac{\sigma(t)}{\nu^2}\|F(t)\|_{L^3(\mathbb{R}^2)}^3 &\leqslant C\,\frac{\sigma(t)}{\nu^2}\|\nabla F(t)\|_{L^2(\mathbb{R}^2)}\|F(t)\|_{L^2(\mathbb{R}^2)}^2 \\ &\leqslant \sigma(t)\|\rho\dot{u}\|_{L^2(\mathbb{R}^2)}\Big(\frac{1}{\nu^2}\|G\|_{L^2(\mathbb{R}^2)}^2 + \|\operatorname{div} u\|_{L^2(\mathbb{R}^2)}^2\Big) \\ &\leqslant \eta\sigma(t)\|\sqrt{\rho}\,\dot{u}\|_{L^2(\mathbb{R}^2)}^2 + \frac{C^*}{\eta\nu^2}\Big(\frac{1}{\nu^2}E_0^2 + (\mathcal{A}_1(t))^2\Big). \end{aligned} \tag{2.40}$$

We finally combine (2.38), (2.39) and (2.40), we choose η small, and we derive (2.4). ∎

2.2.3. Proof of Lemma 2.2. This section is devoted to obtaining bounds for the functionals $\mathcal{A}_1$ and $\mathcal{A}_2$ as defined in (1.15), for $d = 3$. The proof is similar as that of Lemma 2.1, and we adapt the estimates in three dimensions, for instance, the $L^4(\mathbb{R}^2)$-norm is replaced by the $L^6(\mathbb{R}^3)$-norm below. Since Proposition 2.8 does not hold in three dimension anymore, we will simply use the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ in the estimates.

Proof of (2.5). We recall that the first functional appears while using $\dot{u}$ as a test function in the weak formulation of $(1.1)_2$. By doing so, we obtain again (2.20):

$$\begin{aligned}\mathcal{A}_1(t) = {}& \frac{\mu}{2}\|\nabla u_0\|^2_{L^2(\mathbb{R}^3)} + \frac{\mu+\lambda}{2}\|\operatorname{div} u_0\|^2_{L^2(\mathbb{R}^3)} - \mu\int_0^t\int_{\mathbb{R}^3}\nabla u^j\cdot\nabla u^k\partial_k u^j \\ &+ \frac{\mu}{2}\int_0^t\int_{\mathbb{R}^3}|\nabla u|^2\operatorname{div} u + \frac{\mu+\lambda}{2}\int_0^t\int_{\mathbb{R}^3}(\operatorname{div} u)^3 - (\mu+\lambda)\int_0^t\int_{\mathbb{R}^3}\operatorname{div} u\nabla u^l\partial_l u \\ &+ \int_0^t\int_{\mathbb{R}^3}\nabla u^l\partial_l u G + \int_{\mathbb{R}^3}\operatorname{div} u(s)\,G(s)\Big|_{s=0}^{s=t} \\ &+ \int_0^t\int_{\mathbb{R}^3}(\rho P'(\rho) - P(\rho) + \widetilde{P})(\operatorname{div} u)^2. \end{aligned} \tag{2.41}$$

Step 1. *Estimates in terms of* E_0^ν *and* $L^2_tL^6_x$*-norms of* $(\nabla u, G, F)$.

With the help of Hölder's inequality, the third and fourth terms on the right-hand side above can be straightforwardly estimated as follows:

$$\Big| -\mu\int_0^t\int_{\mathbb{R}^3}\nabla u^j\cdot\nabla u^k\partial_k u^j + \frac{\mu}{2}\int_0^t\int_{\mathbb{R}^3}|\nabla u|^2\operatorname{div} u\Big| \leqslant C\|\nabla u\|^3_{L^3((0,t)\times\mathbb{R}^3)}.$$

Similarly, together with the relation between $\operatorname{div} u$ and F and G, as well as Hölder's inequality, the fifth, sixth and seventh terms in (2.41) can be bounded by

$$C\int_0^t\|\operatorname{div} u\|_{L^2(\mathbb{R}^3)}\|\operatorname{div} u\|_{L^3(\mathbb{R}^3)}\|(F,G)\|_{L^6(\mathbb{R}^3)} + C\int_0^t\|\nabla u\|_{L^2(\mathbb{R}^3)}\|\nabla u\|_{L^3(\mathbb{R}^3)}\Big\|\Big(F,\frac{1}{\nu}G\Big)\Big\|_{L^6(\mathbb{R}^3)}$$

which is, by virtue of the interpolation inequality

$$\|f\|_{L^3(\mathbb{R}^3)} \lesssim \|f\|^{1/2}_{L^2(\mathbb{R}^3)}\|f\|^{1/2}_{L^6(\mathbb{R}^3)},$$

bounded by

$$C\int_0^t\Big(\|\sqrt{\nu}\operatorname{div} u\|^{3/2}_{L^2(\mathbb{R}^3)}\frac{1}{\nu^{5/4}}\|(F,G)\|^{3/2}_{L^6(\mathbb{R}^3)} + \|\nabla u\|^{3/2}_{L^2(\mathbb{R}^3)}\|\nabla u\|^{1/2}_{L^6(\mathbb{R}^3)}\Big\|\Big(F,\frac{1}{\nu}G\Big)\Big\|_{L^6(\mathbb{R}^3)}\Big).$$

This can be further estimated by Young's inequality by the following, with some small constant $\eta > 0$:

$$\eta\int_0^t\Big\|\Big(\nabla u, F, \frac{1}{\nu^{5/6}}G\Big)\Big\|^2_{L^6(\mathbb{R}^3)} + \frac{C}{\eta}\int_0^t\|(\sqrt{\nu}\operatorname{div} u, \nabla u)\|^6_{L^2(\mathbb{R}^3)}.$$

By the same argument, the last two terms in (2.41) can be bounded by

$$\eta\nu\|\operatorname{div} u(t)\|_{L^2(\mathbb{R}^3)}^2+\frac{C}{\eta}\|G(t)\|_{L^2(\mathbb{R}^3)}^2+\|\operatorname{div} u_0\|_{L^2(\mathbb{R}^3)}\|G_0\|_{L^2(\mathbb{R}^3)} + C^*\|\operatorname{div} u\|_{L^2((0,t)\times\mathbb{R}^3)}^2\leqslant \eta\nu\|\operatorname{div} u(t)\|_{L^2(\mathbb{R}^3)}^2+\Big(\frac{C^*}{\eta}+\frac{C^*}{\sqrt{\nu}}\Big)E_0^\nu.$$

Summing up, and by further applying the interpolation inequality and then Young's inequality to $\|\nabla u\|_{L^3((0,t)\times\mathbb{R}^3)}^3$, we obtain the following:

$$\begin{aligned}\mathcal{A}_1(t)\leqslant \frac{C^*}{\eta}E_0^\nu+\eta\Big(\nu\|\operatorname{div} u(t)\|_{L^2(\mathbb{R}^3)}^2+\int_0^t\Big\|\Big(\nabla u, F, \frac{1}{\nu^{5/6}}G\Big)\Big\|_{L^6(\mathbb{R}^3)}^2\Big)\\ +\frac{C}{\eta}\int_0^t\|(\sqrt{\nu}\operatorname{div} u,\nabla u)\|_{L^2(\mathbb{R}^3)}^6,\end{aligned}\tag{2.42}$$

with some small parameter $\eta\leqslant 1$ to be determined below.

Step 2. *Final estimates.*

We now turn to the estimate of the $L^2((0,t);L^6(\mathbb{R}^3))$-norm $(\nabla u, G, F)$, similar as in Step 3 in the proof of (2.3).

- *$L^2_tL^6_x$-estimate for G.*

Recall (1.26) with $l=5$:

$$\frac{d}{dt}\|H_5(\rho)\|_{L^1(\mathbb{R}^3)}+\frac{1}{\nu}\|G\|_{L^6(\mathbb{R}^3)}^6\leqslant\frac{1}{\nu}\|G\|_{L^6(\mathbb{R}^3)}^5\|F\|_{L^6(\mathbb{R}^3)}.$$

We define the time-dependent function

$$h(t):=\|H_5(\rho)\|_{L^1(\mathbb{R}^3)}^{1/3}.$$

Thanks to the equivalence between $\|H_5(\rho)\|_{L^1(\mathbb{R}^3)}$ and $\|G\|_{L^6(\mathbb{R}^3)}^6$, we have

$$\frac{1}{C^*}h\leqslant\|G\|_{L^6(\mathbb{R}^3)}^2\leqslant C^*h,$$

and so,

$$3\frac{d}{dt}h+\frac{1}{\nu C^*}h\leqslant\frac{1}{\nu}h^{1/2}\|F\|_{L^6(\mathbb{R}^3)},$$

which together with Young's inequality yields

$$3\sup_{[0,t]}h+\frac{1}{2\nu C^*}\int_0^t h\leqslant 3h(0)+2\frac{C^*}{\nu}\int_0^t\|F\|_{L^6(\mathbb{R}^3)}^2.$$

Finally, we find the following estimate for G in terms of E_0 and F:

$$\frac{1}{\nu}\int_0^t\|G\|_{L^6(\mathbb{R}^3)}^2\leqslant C^*E_0^{1/3}+\frac{C^*}{\nu}\int_0^t\|F\|_{L^6(\mathbb{R}^3)}^2.\tag{2.43}$$

- *$L^2_t L^6_x$-estimate for F.*

We use the Sobolev embedding

$$\|g\|_{L^6(\mathbb{R}^3)} \lesssim \|\nabla g\|_{L^2(\mathbb{R}^3)}$$

to bound the L^6_x-norm of F by (2.18):

$$\int_0^t \|F\|^2_{L^6(\mathbb{R}^3)} \leqslant C \int_0^t \|\nabla F\|^2_{L^2(\mathbb{R}^3)} = C \int_0^t \|\rho \dot{u}\|^2_{L^2(\mathbb{R}^3)}.$$

- *$L^2_t L^6_x$-estimate for ∇u.*

Similarly, by use of (2.16)–(2.17)–(2.18), the following inequality holds true:

$$\begin{aligned}\int_0^t \|\nabla u\|^2_{L^6(\mathbb{R}^3)} &\leqslant C\Big(\int_0^t \|\mathrm{div}\, u\|^2_{L^6(\mathbb{R}^3)} + \int_0^t \|\mathrm{curl}\, u\|^2_{L^6(\mathbb{R}^3)}\Big)\\ &\leqslant \frac{C}{\nu^2}\int_0^t \|(F,G)\|^2_{L^6(\mathbb{R}^3)} + C\int_0^t \|\rho \dot{u}\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Finally, (2.5) follows from (2.42) when we choose η small enough. ■

Proof of (2.6). Here we derive estimate for the functional $\mathcal{A}_2$ as defined in (1.15) for $d = 3$. We recall that it appears while rewriting the equation on the form (2.30), next applying the operator $\partial_t \cdot + \mathrm{div}(\cdot u)$ in order to obtain (2.31) which we test with the material derivative of the velocity $\dot{u}$. By doing so, we obtain (2.34):

$$\begin{aligned}&\frac{1}{2}\sigma(t)\|\sqrt{\rho}\,\dot{u}\|^2_{L^2(\mathbb{R}^3)} + \mu\int_0^t \sigma\|\nabla \dot{u}\|^2_{L^2(\mathbb{R}^3)} + \frac{\mu+\lambda}{\nu^2}\int_0^t \sigma\|\dot{F}\|^2_{L^2(\mathbb{R}^3)}\\ &\qquad = \frac{1}{2}\int_0^{\sigma(t)} \|\sqrt{\rho}\,\dot{u}\|^2_{L^2(\mathbb{R}^3)} + \sum_{k=1}^4 I_k,\end{aligned} \tag{2.44}$$

where I_k, $k = 1, 2, 3, 4$, are given as in (2.34), with $\mathbb{R}^2$ replaced by $\mathbb{R}^3$.

Step 1. *Estimates for I_1.*

By the identity (2.35) and Hölder's inequality, we achieve the estimates

$$\begin{aligned}\Big|\int_0^t \sigma \int_{\mathbb{R}^3} \dot{F}\,\nabla u^k \cdot \partial_k u\Big| &\lesssim \sigma(t)\,\|F\|_{L^6(\mathbb{R}^3)}\,\|\nabla u\|_{L^2(\mathbb{R}^3)}\,\|\nabla u\|_{L^3(\mathbb{R}^3)}\\ &\quad + \int_0^{\sigma(t)} \|F\|_{L^6(\mathbb{R}^3)}\,\|\nabla u\|_{L^2(\mathbb{R}^3)}\,\|\nabla u\|_{L^3(\mathbb{R}^3)}\\ &\quad + \int_0^t \sigma\,\|F\|_{L^6(\mathbb{R}^3)}\,\|\nabla u\|_{L^3(\mathbb{R}^3)}\,\|\nabla \dot{u}\|_{L^2(\mathbb{R}^3)}\\ &\quad + \int_0^t \sigma\,\|F\|_{L^6(\mathbb{R}^3)}\,\|\nabla u\|^2_{L^6(\mathbb{R}^3)}\,\|\nabla u\|_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Similar as in the proof of (2.5) above, we use the interpolation

$$\|\nabla u\|_{L^3(\mathbb{R}^3)} \lesssim \|\nabla u\|^{1/2}_{L^2(\mathbb{R}^3)}\,\|\nabla u\|^{1/2}_{L^6(\mathbb{R}^3)}$$

and Young's inequality to derive (noticing $\sigma(t) \leqslant 1$)

$$\begin{aligned}
\Big|\int_0^t \sigma \int_{\mathbb{R}^3} \dot{F}\, \nabla u^k \cdot \partial_k u\Big| &\leqslant C\big(\sqrt{\sigma(t)}\|F\|_{L^6(\mathbb{R}^3)}\big)\, \|\nabla u\|_{L^2(\mathbb{R}^3)}^{3/2}\big(\sqrt{\sigma(t)}\|\nabla u\|_{L^6(\mathbb{R}^3)}\big)^{1/2} \\
&\quad + \int_0^{\sigma(t)} \|F\|_{L^6(\mathbb{R}^3)}\|\nabla u\|_{L^6(\mathbb{R}^3)}^{1/2}\|\nabla u\|_{L^2(\mathbb{R}^3)}^{3/2} \\
&\quad + \int_0^t \sigma\|F\|_{L^6(\mathbb{R}^3)}\|\nabla u\|_{L^6(\mathbb{R}^3)}^2\|\nabla u\|_{L^2(\mathbb{R}^3)} \\
&\quad + \int_0^t \|\sqrt{\sigma} F\|_{L^6(\mathbb{R}^3)}\|\nabla u\|_{L^6(\mathbb{R}^3)}^{1/2}\|\nabla u\|_{L^2(\mathbb{R}^3)}^{1/2}\|\sqrt{\sigma}\nabla \dot{u}\|_{L^2(\mathbb{R}^3)} \\
&\leqslant \eta\Big(\sigma(t)\|(\nabla u, F)\|_{L^6(\mathbb{R}^3)}^2 + \int_0^t \sigma\,\|\nabla \dot{u}\|_{L^2(\mathbb{R}^3)}^2\Big) + \frac{C}{\eta}\,\|\nabla u\|_{L^2(\mathbb{R}^3)}^6 \\
&\quad + \frac{C}{\eta}\int_0^t \sigma\big(\|F\|_{L^6(\mathbb{R}^3)}^2\|\nabla u\|_{L^6(\mathbb{R}^3)} + \|F\|_{L^6(\mathbb{R}^3)}\|\nabla u\|_{L^6(\mathbb{R}^3)}^2\big)\|\nabla u\|_{L^2(\mathbb{R}^3)} \\
&\quad + \Big(\int_0^{\sigma(t)} \|F\|_{L^6(\mathbb{R}^3)}^2\Big)^{1/2}\Big(\int_0^{\sigma(t)} \big(\|\nabla u\|_{L^6(\mathbb{R}^3)}^2 + \|\nabla u\|_{L^2(\mathbb{R}^3)}^6\big)\Big)^{1/2}.
\end{aligned}$$

By the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and (2.16)–(2.17)–(2.18)–(2.43), we have

$$\begin{aligned}
&\|(F, \operatorname{curl} u)\|_{L^6(\mathbb{R}^3)} \leqslant C\rho^*\|\sqrt{\rho}\,\dot{u}\|_{L^2(\mathbb{R}^3)}, \\
&\frac{1}{\nu}\int_0^t \|G\|_{L^6(\mathbb{R}^3)}^2 \leqslant C^* E_0^{1/3} + \frac{C^*}{\nu}\int_0^t \|\sqrt{\rho}\,\dot{u}\|_{L^2(\mathbb{R}^3)}^2 \leqslant C^* E_0^{1/3} + \frac{C^*}{\nu}\mathcal{A}_1(t),
\end{aligned}$$

and

$$\|\nabla u\|_{L^6(\mathbb{R}^3)} \leqslant C(\|\operatorname{div} u\|_{L^6(\mathbb{R}^3)} + \|\operatorname{curl} u\|_{L^6(\mathbb{R}^3)}) \leqslant C\rho^*\|\sqrt{\rho}\,\dot{u}\|_{L^2(\mathbb{R}^3)} + \frac{C}{\nu}\|G\|_{L^6(\mathbb{R}^3)},$$

which implies

$$\begin{aligned}
&\sigma(t)\|(\nabla u, F)\|_{L^6(\mathbb{R}^3)}^2 \leqslant C^*\mathcal{A}_2(t) + \frac{C}{\nu^2}\, E_0^{1/3}, \\
&\int_0^t \sigma\|F\|_{L^6(\mathbb{R}^3)}\|(F, \nabla u)\|_{L^6(\mathbb{R}^3)}^2\|\nabla u\|_{L^2(\mathbb{R}^3)} \\
&\qquad\qquad \leqslant C^*\mathcal{A}_1(t)^{1/2}\mathcal{A}_2(t)^{1/2}\big(\mathcal{A}_1(t) + \frac{C^*}{\nu}\, E_0^{1/3}\big),
\end{aligned}$$

and

$$\begin{aligned}
&\Big(\int_0^{\sigma(t)} \|F\|_{L^6(\mathbb{R}^3)}^2\Big)^{1/2}\Big(\int_0^{\sigma(t)} \big(\|\nabla u\|_{L^6(\mathbb{R}^3)}^2 + \|\nabla u\|_{L^2(\mathbb{R}^3)}^6\big)\Big)^{1/2} \\
&\qquad\qquad \leqslant C^*\big(\mathcal{A}_1(\sigma(t))\big)^{1/2}\Big(\mathcal{A}_1(\sigma(t)) + \frac{1}{\nu}\, E_0^{1/3} + E_0\,\mathcal{A}_1(\sigma(t))^2\Big)^{1/2}.
\end{aligned}$$

It holds by Young's inequality that

$$\frac{\mu+\lambda}{\nu^2}\Big|\int_0^t \sigma\int_{\mathbb{R}^3} \dot{F}\rho P'(\rho)\operatorname{div} u\Big| \leqslant \eta\,\frac{\mu+\lambda}{2\nu^2}\int_0^t \sigma\|\dot{F}\|_{L^2(\mathbb{R}^3)}^2 + \frac{C^*}{\nu^2\eta}\, E_0.$$

Gathering all of these computations and using Young's inequality, we have

$$|I_1| \leqslant C^*\Big(\eta\,\mathcal{A}_2(t) + \frac{1}{\nu}\,E_0^{1/3} + \frac{1}{\eta^2}\,\mathcal{A}_1(t)^3 + \frac{1}{\eta}\,\mathcal{A}_1(t)\,\frac{1}{\nu^2}\,E_0^{2/3} + \mathcal{A}_1(\sigma(t)) + E_0\,\mathcal{A}_1(\sigma(t))^{3/2}\Big).$$

Step 2. *Final estimates.*

Recalling the definitions of I_k, $k = 2, 3, 4$, given in (2.34), we can estimate them similarly as in Step 1 as follows:

$$\begin{aligned}
&|I_2 + I_3 + I_4| \\
&\leqslant C\int_0^t \sigma\|\nabla\dot{u}\|_{L^2(\mathbb{R}^3)}\|(\nabla u, F)\|_{L^6(\mathbb{R}^3)}\|\nabla u\|_{L^3(\mathbb{R}^3)} + C^*\int_0^t \sigma\|\nabla\dot{u}\|_{L^2(\mathbb{R}^3)}\|\nabla u\|_{L^2(\mathbb{R}^3)} \\
&\leqslant \eta\int_0^t \sigma\|\nabla\dot{u}\|^2_{L^2(\mathbb{R}^3)} + \frac{C^*}{\eta}\int_0^t \big(\sigma\|(\nabla u, F)\|^3_{L^6(\mathbb{R}^3)}\|\nabla u\|_{L^2(\mathbb{R}^3)} + \|\nabla u\|^2_{L^2(\mathbb{R}^3)}\big) \\
&\leqslant \eta\,\mathcal{A}_2(t) + \frac{C^*}{\eta}\,\mathcal{A}_1(t)^{1/2}\Big(\mathcal{A}_2(t)^{1/2}\,\mathcal{A}_1(t) + \frac{E_0^{2/3}}{\nu^2} + \frac{E_0^{1/6}}{\nu^3}\,\mathcal{A}_1(t)\Big) + \frac{C^*}{\eta}\,E_0.
\end{aligned}$$

Gathering all of these computations, we obtain, after choosing η small enough,

$$\begin{aligned}
\mathcal{A}_2(t) \leqslant C^*\Big(\frac{1}{\nu}\,E_0^{1/3} + E_0 + \frac{E_0^{2/3}}{\nu^2}\,\mathcal{A}_1(t)^{1/2} + \Big(1 + \frac{1}{\nu^2}\,E_0^{2/3}\Big)\,\mathcal{A}_1(t)\Big) \\
+ C^*\mathcal{A}_1(t)^{3/2}\Big(E_0 + \frac{E_0^{1/6}}{\nu^3} + \mathcal{A}_1(t)^{3/2}\Big).
\end{aligned}$$

A further application of Young's inequality yields (2.6). ■

2.2.4. Proof of Lemma 2.3. In this paragraph, we derive a priori estimate for the upper bound of the density in terms of $\mathcal{A}_1$ and $\mathcal{A}_2$, under the assumption $\sup_{t,x}\rho(t,x) \leqslant \rho^*$. The basic facts in Section 2.2.1 will be used freely.

Proof. With the help of the expression of $\operatorname{div} u$ in (2.16), we begin by rewriting the mass equation $(1.1)_1$ in terms of F and G as follows:

$$\partial_t\rho + u\cdot\nabla\rho + \frac{\rho}{\nu}\,G = -\frac{\rho}{\nu}\,F.$$

Due to the fact that the pressure is an increasing function of the density and that $G = P(\rho) - \widetilde{P}$, the above equation yields

$$\partial_t|\rho - \tilde{\rho}| + u\cdot\nabla|\rho - \tilde{\rho}| + \frac{\rho}{\nu}\,|G| = -\frac{\rho}{\nu}\,\operatorname{sgn}(\rho - \tilde{\rho})F. \tag{2.45}$$

This yields immediately the following L^∞ estimate for the density, which we use on the short time interval $[0, \sigma(t)]$:

$$\sup_{[0,\sigma(t)]}\|\rho - \tilde{\rho}\|_{L^\infty(\mathbb{R}^d)} \leqslant \|\rho_0 - \tilde{\rho}\|_{L^\infty(\mathbb{R}^d)} + \frac{\rho^*}{\nu}\int_0^{\sigma(t)}\|F(s)\|_{L^\infty(\mathbb{R}^d)}\,ds. \tag{2.46}$$

For larger time, we would like to improve the L^1_t-norm for $\|F\|_{L^\infty_x}$ into L^3_t-norm, which requires less decay rate in time. From (2.45), we have

$$\frac{1}{3}\left(\partial_t|\rho-\tilde\rho|^3+u\cdot\nabla|\rho-\tilde\rho|^3\right)+\frac{\rho}{\nu}\,|G|\,|\rho-\tilde\rho|^2=-\frac{\rho}{\nu}\operatorname{sgn}(\rho-\tilde\rho)\,F|\rho-\tilde\rho|^2. \tag{2.47}$$

Also, since the pressure is an increasing function of the density such that

$$|G|\,|\rho-\tilde\rho|^2\sim_{C^*}|\rho-\tilde\rho|^3,$$

we derive from Young's inequality the following estimate on larger time interval $[\sigma(t),t]$:

$$\sup_{[\sigma(t),t]}\|\rho-\tilde\rho\|^3_{L^\infty(\mathbb{R}^d)}\leqslant\|\rho(\sigma(t))-\tilde\rho\|^3_{L^\infty(\mathbb{R}^d)}+\frac{C^*}{\nu}\int_{\sigma(t)}^t\|F(s)\|^3_{L^\infty(\mathbb{R}^d)}\,ds. \tag{2.48}$$

Gathering estimates (2.46) and (2.48), we have

$$\begin{aligned}\sup_{[0,t]}\|\rho-\tilde\rho\|_{L^\infty(\mathbb{R}^d)}\leqslant\|\rho_0-\tilde\rho\|_{L^\infty(\mathbb{R}^d)}&+\frac{C^*}{\nu}\int_0^{\sigma(t)}\|F(s)\|_{L^\infty(\mathbb{R}^d)}\,ds\\&+\frac{C^*}{\nu^{1/3}}\Big(\int_{\sigma(t)}^t\|F(s)\|^3_{L^\infty(\mathbb{R}^d)}ds\Big)^{1/3}.\end{aligned} \tag{2.49}$$

It only remains to estimate the norm of the effective flux F in terms of the functionals $\mathcal{A}_1$ and $\mathcal{A}_2$, and to do so, we distinguish two cases, according to the dimension.

Case $d=2$.

We recall that the effective flux is given by

$$F=-(-\Delta)^{-1}\operatorname{div}(\rho\dot u)=\nu\operatorname{div}u-G,\quad G=P(\rho)-\widetilde{P}. \tag{2.50}$$

The interpolation inequality yields the following estimate on the small interval $[0,\sigma(t)]$:

$$\begin{aligned}\int_0^{\sigma(t)}\|F\|_{L^\infty(\mathbb{R}^2)}&\leqslant C\int_0^{\sigma(t)}\|F\|^{1/3}_{L^2(\mathbb{R}^2)}\,\|\nabla F\|^{2/3}_{L^4(\mathbb{R}^2)}\\&\leqslant\int_0^{\sigma(t)}\|(G,\nu\operatorname{div}u)\|^{1/3}_{L^2(\mathbb{R}^2)}\,\|\rho\dot u\|^{2/3}_{L^4(\mathbb{R}^2)}\\&\leqslant\Big(C^*E_0^{1/6}+\nu^{1/6}\mathcal{A}_1(t)^{1/6}\Big)\int_0^{\sigma(t)}\|\rho\dot u\|^{2/3}_{L^4(\mathbb{R}^2)}.\end{aligned}$$

Since the density contains vacuum states, we are not allowed to bound the last factor in the above inequality solely by the Gagliardo–Nirenberg inequality. On $\mathbb{T}^2$, we have at our disposal a logarithmic interpolation inequality (see [16, 17, 21]) which is not valid in the whole space. To address this issue, we prove in Lemma A.1 an interpolation inequality that will allow us to take into account the vacuum state. Thus, from Lemma A.1, we have

$$\begin{aligned}\int_0^{\sigma(t)}\|\rho\dot u\|^{2/3}_{L^4(\mathbb{R}^2)}&\leqslant C^*\int_0^{\sigma(t)}\|\sqrt{\rho}\,\dot u\|^{1/3}_{L^2(\mathbb{R}^2)}\|\nabla\dot u\|^{1/3}_{L^2(\mathbb{R}^2)}\\&\quad+C^*E_0^{1/18}\int_0^{\sigma(t)}\|\sqrt{\rho}\,\dot u\|^{2/9}_{L^2(\mathbb{R}^2)}\,\|\nabla\dot u\|^{4/9}_{L^2(\mathbb{R}^2)}\\&\leqslant C^*\mathcal{A}_1(t)^{1/6}\mathcal{A}_2(t)^{1/6}+C^*E_0^{1/18}\,\mathcal{A}_1(t)^{1/9}\,\mathcal{A}_2(t)^{2/9},\end{aligned}$$

and therefore,

$$\int_0^{\sigma(t)} \|F\|_{L^\infty(\mathbb{R}^2)} \leqslant C^*\big(1+E_0^{1/18}\big)\big(E_0^{1/6}+\nu^{1/6}\mathcal{A}_1(t)^{1/6}\big)\big(\mathcal{A}_1(t)^{1/3}+\mathcal{A}_2(t)^{1/3}\big). \tag{2.51}$$

Similarly, the interpolation inequality and Lemma A.1 yield, on the larger time interval $[\sigma(t), t]$,

$$\begin{aligned}\int_{\sigma(t)}^t \|F\|^3_{L^\infty(\mathbb{R}^2)} &\leqslant \int_{\sigma(t)}^t \|(G, \nu \operatorname{div} u)\|_{L^2(\mathbb{R}^2)} \|\rho\dot u\|^2_{L^4(\mathbb{R}^2)} \\ &\leqslant C^*\big(C^*E_0^{1/2} + \nu^{1/2}\mathcal{A}_1(t)^{1/2}\big) \int_0^t \|\sqrt{\rho}\,\dot u\|_{L^2(\mathbb{R}^2)} \|\nabla \dot u\|_{L^2(\mathbb{R}^2)} \\ &\quad + C^*\big(C^*E_0^{1/2} + \nu^{1/2}\mathcal{A}_1(t)^{1/2}\big)\, E_0^{1/6} \int_0^t \|\sqrt{\rho}\,\dot u\|^{2/3}_{L^2(\mathbb{R}^2)} \|\nabla \dot u\|^{4/3}_{L^2(\mathbb{R}^2)} \\ &\leqslant C^*\big(C^*E_0^{1/2} + \nu^{1/2}\mathcal{A}_1(t)^{1/2}\big)\big(\mathcal{A}_1(t)^{1/2}\mathcal{A}_2(t)^{1/2} + E_0^{1/6}\mathcal{A}_1(t)^{1/3}\mathcal{A}_2(t)^{2/3}\big).\end{aligned}$$

Hence,

$$\int_{\sigma(t)}^t \|F\|^3_{L^\infty(\mathbb{R}^2)} \leqslant C^*(1 + E_0^{1/6})(C^*E_0^{1/2} + \nu^{1/2}\mathcal{A}_1(t)^{1/2})(\mathcal{A}_1(t) + \mathcal{A}_2(t)). \tag{2.52}$$

Finally, (2.49), (2.51) and (2.52) lead to

$$\begin{aligned}\sup_{[0,t]}\|\rho - \tilde\rho\|_{L^\infty(\mathbb{R}^2)} &\leqslant \|\rho_0 - \tilde\rho\|_{L^\infty(\mathbb{R}^2)} \\ &\quad + \frac{C^*}{\nu^{1/3}}\big(1+E_0^{1/18}\big)\big(E_0^{1/6}+\nu^{1/6}\mathcal{A}_1(t)^{1/6}\big)\big(\mathcal{A}_1(t)^{1/3}+\mathcal{A}_2(t)^{1/3}\big).\end{aligned}$$

Case $d = 3$.

From the expression of the effective flux (2.50), we have by the Gagliardo–Nirenberg inequality,

$$\begin{aligned}\int_0^{\sigma(t)} \|F\|_{L^\infty(\mathbb{R}^3)} &\leqslant C \int_0^{\sigma(t)} \|F\|^{1/2}_{L^6(\mathbb{R}^3)} \|\nabla F\|^{1/2}_{L^6(\mathbb{R}^3)} \\ &\leqslant C \int_0^{\sigma(t)} \|\rho\dot u\|^{1/2}_{L^2(\mathbb{R}^3)} \|\rho\dot u\|^{1/2}_{L^6(\mathbb{R}^3)} \leqslant C^*\mathcal{A}_1(t)^{1/4}\mathcal{A}_2(t)^{1/4},\end{aligned}$$

and

$$\int_{\sigma(t)}^t \|F\|^3_{L^\infty(\mathbb{R}^3)} \leqslant C \int_{\sigma(t)}^t \|\rho\dot u\|^{3/2}_{L^2(\mathbb{R}^3)} \|\rho\dot u\|^{3/2}_{L^6(\mathbb{R}^3)} \leqslant C^*\mathcal{A}_1(t)^{1/4}\mathcal{A}_2(t)^{5/4}.$$

Finally, we get

$$\sup_{[0,t]}\|\rho - \tilde\rho\|_{L^\infty(\mathbb{R}^3)} \leqslant \|\rho_0 - \tilde\rho\|_{L^\infty(\mathbb{R}^3)} + \frac{C^*}{\nu^{1/3}}\big(\mathcal{A}_1(t)^{1/2} + \mathcal{A}_2(t)^{1/2}\big),$$

and this ends the proof of Lemma 2.3. ∎

2.2.5. Proof of Lemma 2.6. In this section, we use the (time-independent) bound C_0 for $\mathcal{A}_1(t)$, $\mathcal{A}_2(t)$ and $\rho(t)$ to show the propagation of tangential regularity of the density, along with the Lipschitz bound on the velocity.

We recall that the family of vector fields $\mathcal{X}(t) = (X_\upsilon(t))_{1\leqslant\upsilon\leqslant m}$ is defined as solution of (1.8):

$$\begin{cases} \partial_t X_\upsilon + u\cdot\nabla X_\upsilon = \partial_{X_\upsilon} u = (X_\upsilon\cdot\nabla)u, \\ X_{\upsilon|t=0} = X_{0,\upsilon}, \end{cases} \tag{2.53}$$

and we can estimate the norms of

$$\begin{aligned} \|\mathcal{X}(t)\|_{\mathbb{L}^{\infty,p}(\mathbb{R}^d)} &= \sup_{1\leqslant\upsilon\leqslant m} \|X_\upsilon(t)\|_{\mathbb{L}^{\infty,p}(\mathbb{R}^d)} \\ &= \sup_{1\leqslant\upsilon\leqslant m} \big(\|X_\upsilon(t)\|_{L^\infty(\mathbb{R}^d)} + \|\nabla X_\upsilon(t)\|_{L^p(\mathbb{R}^d)}\big) \end{aligned}$$

by use of the Lipschitz norm of the velocity. On the other side, these norms will help to bound the Lipschitz norm of the velocity, and in particular, the pressure-related part ∇u_G in the decomposition (2.9) of ∇u:

$$\begin{aligned} \nabla u &= \nabla\tilde{u} + \nabla u_G \\ &:= \Big(-\frac{1}{\nu}\mathcal{R}\mathcal{R}(-\Delta)^{-1}\operatorname{div}(\rho\dot{u}) - \frac{1}{\mu}\mathcal{R}\mathcal{R}(-\Delta)^{-1}\cdot\operatorname{curl}(\rho\dot{u})\Big) + \Big(\frac{1}{\nu}\mathcal{R}\mathcal{R}G\Big). \end{aligned} \tag{2.54}$$

Finally, we will obtain the estimates for

$$\mathcal{A}_3(t) = \|\mathcal{X}(t)\|_{\mathbb{L}^{\infty,p}(\mathbb{R}^d)} + \sup_{1\leqslant\upsilon\leqslant m}\|\operatorname{div}(\rho X_\upsilon)(t)\|_{L^p(\mathbb{R}^d)}$$

by Grönwall's inequality.

Proof. Step 1. *Preliminary estimates for* $\|\mathcal{X}\|_{\mathbb{L}^{\infty,p}(\mathbb{R}^d)}$.

From (2.53), we deduce easily that

$$\|X_\upsilon(t)\|_{L^\infty(\mathbb{R}^d)} \leqslant \|X_{0,\upsilon}\|_{L^\infty} + \int_0^t \|X_\upsilon(s)\|_{L^\infty(\mathbb{R}^d)}\,\|\nabla u(s)\|_{L^\infty(\mathbb{R}^d)}\,ds. \tag{2.55}$$

We now take derivatives in (2.53) and we obtain

$$\partial_t\partial_k X_\upsilon^j + (u\cdot\nabla)\partial_k X_\upsilon^j = \partial_k X_\upsilon\cdot\nabla u^j - \partial_k u\cdot\nabla X_\upsilon^j + \partial_{X_\upsilon}\partial_k u^j. \tag{2.56}$$

We take the trace in the above equality and we make use of the expression of the divergence of the velocity $\operatorname{div} u = \frac{1}{\nu}\big(F+G\big)$, in order to obtain the following equation for $\operatorname{div} X_\upsilon$:

$$\partial_t(\operatorname{div} X_\upsilon) + u\cdot\nabla\operatorname{div} X_\upsilon = \frac{1}{\nu}\partial_{X_\upsilon}G + \frac{1}{\nu}\partial_{X_\upsilon}F.$$

Hence, it is straightforward to show the following:

$$\begin{aligned} \|\operatorname{div} X_\upsilon(t)\|_{L^p(\mathbb{R}^d)} &\leqslant \|\operatorname{div} X_{0,\upsilon}\|_{L^p(\mathbb{R}^d)} \\ &\quad + \frac{1}{\nu}\int_0^t \|\partial_{X_\upsilon}G\|_{L^p(\mathbb{R}^d)} + \frac{1}{\nu}\int_0^t \|X_\upsilon\|_{L^\infty(\mathbb{R}^d)}\,\|\nabla F\|_{L^p(\mathbb{R}^d)} \\ &\quad + \int_0^t \|\operatorname{div} u(s)\|_{L^\infty(\mathbb{R}^d)}\,\|\operatorname{div} X_\upsilon(s)\|_{L^p(\mathbb{R}^d)}\,ds. \end{aligned} \tag{2.57}$$

Taking the antisymmetric part in (2.56), we get the following equation for curl X_υ:

$$\begin{aligned}&\partial_t(\partial_k X_\upsilon^j - \partial_j X_\upsilon^k) + u\cdot\nabla(\partial_k X_\upsilon^j - \partial_j X_\upsilon^k)\\&\qquad = \partial_k X_\upsilon\cdot\nabla u^j - \partial_j X_\upsilon\cdot\nabla u^k + \partial_j u\cdot\nabla X_\upsilon^k - \partial_k u\cdot\nabla X_\upsilon^j + \partial_{X_\upsilon}(\partial_k u^j - \partial_j u^k),\end{aligned}$$

from which we deduce easily the following:

$$\begin{aligned}\|\mathrm{curl}\, X_\upsilon(t)\|_{L^p(\mathbb{R}^d)} \leqslant \|\mathrm{curl}\, X_{0,\upsilon}\|_{L^p(\mathbb{R}^d)} + \int_0^t \|\nabla X_\upsilon\|_{L^p(\mathbb{R}^d)}\|\nabla u\|_{L^\infty(\mathbb{R}^d)}\\ + \int_0^t \|X_\upsilon\|_{L^\infty(\mathbb{R}^d)}\|\nabla\,\mathrm{curl}\, u\|_{L^p(\mathbb{R}^d)}.\end{aligned}\tag{2.58}$$

By combining (2.57) and (2.58), we obtain the following estimate for the vector field gradient:

$$\begin{aligned}\|\nabla X_\upsilon(t)\|_{L^p(\mathbb{R}^d)} \leqslant \|\nabla X_{0,\upsilon}\|_{L^p(\mathbb{R}^d)} + \frac{1}{\nu}\int_0^t \|\partial_{X_\upsilon} G\|_{L^p(\mathbb{R}^d)}\\ + \int_0^t \|\nabla X_\upsilon(s)\|_{L^p(\mathbb{R}^d)}\|\nabla u(s)\|_{L^\infty(\mathbb{R}^d)}\,ds\\ + \int_0^t \|X_\upsilon(s)\|_{L^\infty(\mathbb{R}^d)}\Big\|\Big(\frac{1}{\nu}\nabla F(s), \nabla\,\mathrm{curl}\, u(s)\Big)\Big\|_{L^p(\mathbb{R}^d)}\,ds.\end{aligned}\tag{2.59}$$

Step 2. Estimates for $\mathcal{A}_3(t) = \|\mathcal{X}(t)\|_{\mathbb{L}^{\infty,p}(\mathbb{R}^d)} + \sup_{1\leqslant\upsilon\leqslant m}\|\mathrm{div}(\rho X_\upsilon)(t)\|_{L^p(\mathbb{R}^d)}$.

In order to estimate the $L^p(\mathbb{R}^d)$ norm of $\mathrm{div}(\rho X_\upsilon)$, we combine the equation $(1.1)_1$ on the density and the equation (2.53) on the vector field X_υ in order to obtain

$$\partial_t(\mathrm{div}(\rho X_\upsilon)) + \mathrm{div}(u\,\mathrm{div}(\rho X_\upsilon)) = 0,$$

from which we deduce the following estimates:

$$\begin{aligned}\|\mathrm{div}(\rho X_\upsilon)(t)\|_{L^p(\mathbb{R}^d)} \leqslant \|\mathrm{div}(\rho_0 X_{0,\upsilon})\|_{L^p(\mathbb{R}^d)}\\ + \int_0^t \|\mathrm{div}\, u(s)\|_{L^\infty(\mathbb{R}^d)}\|\mathrm{div}(\rho X_\upsilon)(s)\|_{L^p(\mathbb{R}^d)}\,ds.\end{aligned}\tag{2.60}$$

Consequently, we can estimate

$$\partial_{X_\upsilon} G = P'(\rho)\,\mathrm{div}(\rho X_\upsilon) - \rho\, P'(\rho)\,\mathrm{div}\, X_\upsilon$$

by

$$\|\partial_{X_\upsilon} G\|_{L^p(\mathbb{R}^d)} \leqslant C_0\big(\|\mathrm{div}(\rho X_\upsilon)\|_{L^p(\mathbb{R}^d)} + \|\mathrm{div}\, X_\upsilon\|_{L^p(\mathbb{R}^d)}\big).$$

We combine (2.55), (2.59) and (2.60) together with (2.18) to get

$$\mathcal{A}_3(t) \leqslant \mathcal{A}_3(0) + C_0\int_0^t \mathcal{A}_3(s)\Big(\frac{1}{\nu} + \|\nabla u(s)\|_{L^\infty(\mathbb{R}^d)} + \|\rho\dot{u}(s)\|_{L^p(\mathbb{R}^d)}\Big)\,ds.$$

Grönwall's lemma yields

$$\mathcal{A}_3(t) \leqslant \mathcal{A}_3(0)\exp\Big[C_0\int_0^t \big(1 + \|\nabla u(s)\|_{L^\infty(\mathbb{R}^d)} + \|\rho\dot{u}(s)\|_{L^p(\mathbb{R}^d)}\big)\,ds\Big].\tag{2.61}$$

Step 3. *Estimates for* $\|\nabla u_G\|_{L^1_t L^\infty_x}$.

Recall the second part, $\nabla u_G = \frac{1}{\nu}\mathcal{R}\mathcal{R}G$, in (2.54). We use Proposition 2.5 to bound it as follows:

$$\|\nabla u_G(t)\|_{L^\infty(\mathbb{R}^d)} \leqslant \frac{C}{\nu}\,\|G(t)\|_{L^2(\mathbb{R}^d)} + \frac{C}{\nu}\,\|G(t)\|_{L^\infty(\mathbb{R}^d)}\Big[1+\log\Big(e+\frac{\|G(t)\|_{\mathbb{L}^p_{\mathcal{X}(t)}(\mathbb{R}^d)}}{\|G(t)\|_{L^\infty(\mathbb{R}^d)}}\Big)\Big].$$

Now we focus on the estimate for

$$\|G(t)\|_{\mathbb{L}^p_{\mathcal{X}(t)}(\mathbb{R}^d)} = \frac{1}{I(\mathcal{X}(t))}\Big(\|G(t)\|_{L^\infty(\mathbb{R}^d)}\|\mathcal{X}(t)\|_{\mathbb{L}^{\infty,p}(\mathbb{R}^d)} + \sup_{1\leqslant\upsilon\leqslant m}\|\partial_{X_\upsilon}G(t)\|_{L^p(\mathbb{R}^d)}\Big) \leqslant \frac{C_0}{I(\mathcal{X}(t))}\,\mathcal{A}_3(t).$$

It is well known that the denominator $I(\mathcal{X}(t))$ has a positive lower bound as follows, see, e.g., equation (4.3) in [9]:

$$I(\mathcal{X}(t)) \geqslant I(\mathcal{X}_0)\exp\Big(-C\int_0^t \|\nabla u(s)\|_{L^\infty(\mathbb{R}^d)}\,ds\Big).$$

We hence have

$$\|\nabla u_G(t)\|_{L^\infty(\mathbb{R}^d)} \leqslant C_0\Big[1+\frac{\mathcal{A}_3(0)}{I(\mathcal{X}_0)}+\int_0^t\big(1+\|\rho\dot{u}\|_{L^p(\mathbb{R}^d)}+\|\nabla u\|_{L^\infty(\mathbb{R}^d)}\big)(s)\,ds\Big]. \tag{2.62}$$

Step 4. *Final estimates.*

We continue with the estimates for $\|\rho\dot{u}\|_{L^1_t L^p(\mathbb{R}^d)}$ and $\|\nabla\tilde{u}\|_{L^1_t L^\infty(\mathbb{R}^d)}$, taking the dimension into account.

Case $d=2$.

As in the proof of Lemma 2.3, by using our interpolation inequality in Lemma A.1, we obtain

$$\int_0^t \|\rho\dot{u}\|_{L^p(\mathbb{R}^2)} \leqslant C_0\int_0^t\Big[\|\sqrt{\rho}\dot{u}\|^{2/p}_{L^2(\mathbb{R}^2)}\|\nabla\dot{u}\|^{1/p'-1/p}_{L^2(\mathbb{R}^2)} + E_0^{\frac{1}{p}(1-p'/2)}\|\sqrt{\rho}\dot{u}\|^{p'/p}_{L^2(\mathbb{R}^2)}\|\nabla\dot{u}\|^{1-p'/p}_{L^2(\mathbb{R}^2)}\Big],$$

where the first integral of the right-hand side above is bounded as

$$\begin{aligned}\int_0^t \|\sqrt{\rho}\dot{u}\|^{2/p}_{L^2(\mathbb{R}^2)}\|\nabla\dot{u}\|^{1/p'-1/p}_{L^2(\mathbb{R}^2)} &= \int_0^t \|\sqrt{\rho}\dot{u}\|^{2/p}_{L^2(\mathbb{R}^2)}\big(\sigma\|\nabla\dot{u}\|^2_{L^2(\mathbb{R}^2)}\big)^{1/2-1/p}\sigma^{1/p-1/2}\\ &\leqslant \Big(\int_0^t\|\sqrt{\rho}\dot{u}\|^2_{L^2(\mathbb{R}^2)}\Big)^{1/p}\Big(\int_0^t\sigma\|\nabla\dot{u}\|^2_{L^2(\mathbb{R}^2)}\Big)^{1/2-1/p}\Big(\int_0^t\sigma^{2/p-1}\Big)^{1/2}\\ &\leqslant C(p)(1+\sqrt{t})\,\mathcal{A}_1(t)^{1/p}\,\mathcal{A}_2(t)^{1/2-1/p},\end{aligned}$$

and the second integral of the right-hand side above is bounded as

$$\begin{aligned}\int_0^t \|\sqrt{\rho}\dot{u}\|_{L^2(\mathbb{R}^2)}^{p'/p} \|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)}^{1-p'/p} \\ &= \int_0^t \big(\|\sqrt{\rho}\dot{u}\|_{L^2(\mathbb{R}^2)}^2\big)^{p'/(2p)} \big(\sigma\|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)}^2\big)^{1/2-p'/(2p)} \sigma^{p'/(2p)-1/2} \\ &\leqslant \Big(\int_0^t \|\sqrt{\rho}\dot{u}\|_{L^2(\mathbb{R}^2)}^2\Big)^{p'/(2p)} \Big(\int_0^t \sigma\|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)}^2\Big)^{1/2-p'/(2p)} \Big(\int_0^t \sigma^{p'/p-1}\Big)^{1/2} \\ &\leqslant C(p)(1+\sqrt{t})\,\mathcal{A}_1(t)^{p'/(2p)}\,\mathcal{A}_2(t)^{1/2-p'/(2p)}.\end{aligned}$$

In sum, for all $2 < p < \infty$,

$$\int_0^t \|\rho\dot{u}\|_{L^p(\mathbb{R}^2)} \leqslant C_0(1+\sqrt{t}). \tag{2.63}$$

Now following the computations leading to (2.51), it is straightforward to obtain

$$\begin{aligned}\int_0^t \|\nabla\tilde{u}\|_{L^\infty(\mathbb{R}^2)} ds &\leqslant \frac{1}{\nu}\int_0^t \|\mathcal{R}\mathcal{R}(-\Delta)^{-1}\operatorname{div}(\rho\dot{u})\|_{L^\infty(\mathbb{R}^2)} \\ &\quad + \frac{1}{\mu}\int_0^t \|\mathcal{R}\mathcal{R}(-\Delta)^{-1}\operatorname{curl}_{jk}(\rho\dot{u})\|_{L^\infty(\mathbb{R}^2)} \\ &\leqslant C^*\Big(\frac{1}{\nu^{5/6}}+1\Big)(1+t^{2/3})(1+E_0^{1/18})(A_1(t)^{1/2}+\mathcal{A}_2(t)^{1/2}) \\ &\leqslant C_0(1+t^{2/3}).\end{aligned} \tag{2.64}$$

We plug (2.63) into (2.62), sum (2.62) and (2.64) up, and finally use Grönwall's inequality to get the estimate $(2.13)_2$ for $d = 2$. The estimate $(2.13)_1$ for $\mathcal{A}_3(t)$ follows correspondingly from (2.61).

Case $d = 3$.

Similarly as in the proof of (2.63), for $3 < p < 6$ we interpolate the $L^p(\mathbb{R}^3)$ norm of $\sqrt{\rho}\dot{u}$ between $L^2(\mathbb{R}^3)$ and $L^6(\mathbb{R}^3)$, and then we make use of the embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ to derive

$$\begin{aligned}\int_0^t \|\rho\dot{u}\|_{L^p(\mathbb{R}^3)} &\leqslant C_0\int_0^t \|\sqrt{\rho}\,\dot{u}\|_{L^2(\mathbb{R}^3)}^{3/p-1/2} \big(\sigma\|\nabla\dot{u}\|_{L^2(\mathbb{R}^3)}^2\big)^{3/4-3/(2p)} \sigma^{3/(2p)-3/4} \\ &\leqslant C_0(1+\sqrt{t}).\end{aligned} \tag{2.65}$$

Thanks to the interpolation inequality and the Sobolev embedding, we have

$$\int_0^t \|\nabla\tilde{u}\|_{L^\infty(\mathbb{R}^3)} \leqslant \int_0^t \|\rho\dot{u}\|_{L^2(\mathbb{R}^3)}^{1/2}\|\rho\dot{u}\|_{L^6(\mathbb{R}^3)}^{1/2} \leqslant C_0(1+\sqrt{t}). \tag{2.66}$$

As above, we plug (2.65) into (2.62), sum (2.62) and (2.66) up, and finally use Grönwall's inequality to get the estimate $(2.13)_2$ for $d = 3$. The estimate $(2.13)_1$ for $\mathcal{A}_3(t)$ follows from (2.61) in dimension three. This ends the proof of Lemma 2.6. ∎

2.3. Proof of Theorem 1.6 and Corollary 1.10

This section is devoted to the final step in the proof of the main result, Theorem 1.6. We recall that we are considering the Cauchy problem associated with equations (1.1) and with initial data (1.2) satisfying (1.3) and (1.4).

Usually, the sequence of initial data $(\rho_0^\delta, u_0^\delta)$ is obtained by mollifying (ρ_0, u_0) with a smooth kernel. This regularization procedure has the unfortunate effect of destroying the density's structure. As observed in [29,50], the most effective approach is to construct the approximate solutions in a class that is very close to the limit. From this point of view, the local result obtained by Danchin, Fanelli, Paicu in [10] should be appropriate. However, the argument of the maximum regularity of the heat equation requires the density to be a small perturbation of a constant state, even for the local solution. We are therefore led to prove the local well-posedness of equations (1.1) stated in Theorem 1.4 in Section B.

Proof of Theorem 1.6. In order to apply the local-in-time well-posedness results in Theorem 1.4, we consider a sequence of initial densities $(\rho_0^\delta)_\delta$ satisfying: for all $0 < \delta < 1$,

$$\rho_0^\delta \geqslant \delta, \quad \rho_0^\delta - \tilde{\rho} \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap \mathbb{L}^p_{\mathcal{X}_0}(\mathbb{R}^d) \tag{2.67}$$

such that

$$\rho_0^\delta - \tilde{\rho} \xrightarrow{\delta \to 0} \rho_0 - \tilde{\rho} \quad \text{ in } L^2(\mathbb{R}^d).$$

The construction of the sequence of initial velocities $(u_0^\delta)_\delta \subset H^1(\mathbb{R}^d)$, converging strongly to u_0 in $H^1(\mathbb{R}^d)$ and satisfying the compatibility condition

$$\operatorname{div}\{2\mu \mathbb{D} u_0^\delta + (\lambda \operatorname{div} u_0^\delta - P(\rho_0^\delta) + \widetilde{P})\} \in L^2(\mathbb{R}^d) \tag{2.68}$$

can be found in Section 3.5 of [50]. Now we can apply Theorem 1.4 to get the existence of a unique solution (ρ^δ, u^δ) that satisfies

$$\begin{cases} \partial_t \rho^\delta + \operatorname{div}(\rho^\delta u^\delta) = 0, \\ \partial_t(\rho^\delta u^\delta) + \operatorname{div}(\rho^\delta u^\delta \otimes u^\delta) + \nabla P(\rho^\delta) = \mu \Delta u^\delta + (\mu + \lambda) \nabla \operatorname{div} u^\delta, \end{cases} \tag{2.69}$$

with initial data

$$(\rho^\delta)_{|t=0} = \rho_0^\delta \quad \text{ and } \quad (u^\delta)_{|t=0} = u_0^\delta.$$

The solution is defined up to a maximal time T^δ, and enjoys the regularity outlined in Theorem 1.4, which is sufficient for the computations performed in the preceding sections to be meaningful, leading to Lemmas 2.4 and 2.6. In particular, all the conditions outlined in the blow-up criterion (1.9) are satisfied, implying that $T^\delta = +\infty$. Finally, employing classical arguments involving compact embedding, the Aubin–Lions lemma and leveraging the regularity of the effective flux, one can establish the strong convergence of a subsequence of (ρ^δ, u^δ) to (ρ, u) satisfying the regularity in Theorem 1.6. Furthermore, given the improved-in-time Lipschitz bound of the velocity field in Remark 2.7, a change of variables into Lagrangian coordinates ensures the uniqueness of such a solution. We refer for example to [10] for the computations. ∎

Proof of Corollary 1.10. At this level, we obtain a sequence $(\rho^{(\nu)}, u^{(\nu)})_{\nu \geq \underline{\nu}}$ satisfying

$$(2.70) \qquad \begin{cases} \partial_t \rho^{(\nu)} + \operatorname{div}(\rho^{(\nu)} u^{(\nu)}) = 0, \\ \partial_t(\rho^{(\nu)} u^{(\nu)}) + \operatorname{div}(\rho^{(\nu)} u^{(\nu)} \otimes u^{(\nu)}) - \nabla F^{(\nu)} - \mu \Delta u^{(\nu)} \\ \qquad\qquad = -\dfrac{\mu}{\nu} \nabla F^{(\nu)} - \dfrac{\mu}{\nu} \nabla P(\rho^{(\nu)}), \end{cases}$$

with initial data (1.2) satisfying (1.3) and (1.4) and $\operatorname{div} u_0 = 0$. Above, the effective flux $F^{(\nu)}$ solves the following elliptic equation:

$$\Delta F^{(\nu)} = \operatorname{div}(\rho^{(\nu)} \dot{u}^{(\nu)}).$$

Given that $(\rho^{(\nu)})_\nu$ is bounded in $L^\infty((0,\infty) \times \mathbb{R}^d)$ and $(\sqrt{\rho^{(\nu)}} \dot{u}^{(\nu)})$ is bounded in $L^2((0,\infty) \times \mathbb{R}^d)$, it follows that $(-\nabla F^{(\nu)})_\nu$ is bounded in $L^2((0,\infty) \times \mathbb{R}^d)$, resulting in weak convergence (up to a subsequence) to some $\nabla \Pi \in L^2((0,\infty) \times \mathbb{R}^d)$. Obviously, the right-hand side of the second equation in (2.70) converges strongly to zero in $L^\infty((0,\infty), \dot{H}^{-1}(\mathbb{R}^d))$, given that $(\nu^{-1/2} F^{(\nu)})$ is bounded in $L^\infty((0,\infty), L^2(\mathbb{R}^d))$ and $(P(\rho^{(\nu)}) - \widetilde{P})_\nu$ is bounded in $L^\infty((0,\infty), L^2(\mathbb{R}^d))$.

The regularity of the sequence $(u^{(\nu)})$ ensures that, up to a subsequence, $(u^{(\nu)})$ converges strongly in $L^2_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^d)$ to some function $v \in L^\infty((0,\infty), H^1(\mathbb{R}^d))$. Furthermore, since the sequence $(\rho^{(\nu)})_\nu$ is bounded in $L^\infty((0,\infty) \times \mathbb{R}^d)$, it converges weakly-* to some $\varrho \in L^\infty((0,\infty) \times \mathbb{R}^d)$. Additionally, the sequence $(\operatorname{div} u^{(\nu)})_\nu$ converges strongly to zero in $L^\infty((0,\infty), L^2(\mathbb{R}^d))$, since from the bound of the functional $\mathcal{A}_1$, the sequence $(\nu \|\operatorname{div} u^{(\nu)}\|^2_{L^2(\mathbb{R}^d)})_\nu$ is bounded. These convergences, together with the Aubin–Lions lemma, are sufficient to pass to the limit in (2.70) and to establish that (ϱ, v) solves the incompressible model (1.22). The uniqueness result for (1.22) in [44] and the uniform bounds in Theorem 1.6 imply the convergence of the whole sequence $(\rho^{(\nu)}, u^{(\nu)})_\nu$ (instead of some subsequence). This completes the proof of Corollary 1.10. ∎

A. Interpolation inequality

Lemma A.1 (Density-weighted interpolation inequality). *Let $v \in \dot{H}^1(\mathbb{R}^2)$, and let $\rho \geq 0$ be such that $\sqrt{\rho} v \in L^2(\mathbb{R}^2)$ and $\rho - \tilde{\rho} \in L^p(\mathbb{R}^2)$ for some $1 < p < \infty$, with $\tilde{\rho} > 0$. Then $v \in L^2(\mathbb{R}^2)$ and there exists a constant $C > 0$, depending only on $\tilde{\rho}$ and p, such that the following estimate holds true*:

$$(A.1) \qquad \|v\|_{L^2(\mathbb{R}^2)} \leq C \left(\|\rho - \tilde{\rho}\|^{p/2}_{L^p(\mathbb{R}^2)} \|\nabla v\|_{L^2(\mathbb{R}^2)} + \|\sqrt{\rho} v\|_{L^2(\mathbb{R}^2)} \right).$$

Moreover, for all $2 < q < \infty$, we have

$$(A.2) \qquad \begin{aligned} \|\rho^{q'/(2q)} v\|_{L^q(\mathbb{R}^2)} \leq C \Big(& \|\sqrt{\rho} v\|^{2/q}_{L^2(\mathbb{R}^2)} \|\nabla v\|^{1/q' - 1/q}_{L^2(\mathbb{R}^2)} \\ & + \|\rho - \tilde{\rho}\|^{\frac{p}{q}(1 - q'/2)}_{L^p(\mathbb{R}^2)} \|\sqrt{\rho} v\|^{q'/q}_{L^2(\mathbb{R}^2)} \|\nabla v\|^{1 - q'/q}_{L^2(\mathbb{R}^2)} \Big). \end{aligned}$$

Proof. In the first step of the proof, we emulate the approach in Proposition A.6 of [44] by expressing

$$(A.3) \qquad \tilde{\rho} |v|^2 = (\tilde{\rho} - \rho) |v|^2 + \rho |v|^2.$$

Due to the assumption that $\sqrt{\rho}v \in L^2(\mathbb{R}^2)$, we only need to compute the integral of the first term of the right-hand above. With the help of interpolation, and the Hölder and Young inequalities, we have

$$\begin{aligned}\int_{\mathbb{R}^2} (\tilde\rho - \rho)|v|^2 &\leqslant \|\rho - \tilde\rho\|_{L^p(\mathbb{R}^2)} \|v\|^2_{L^{2p'}(\mathbb{R}^2)} \\ &\leqslant C_p \|\rho - \tilde\rho\|_{L^p(\mathbb{R}^2)} \|v\|^{2/p'}_{L^2(\mathbb{R}^2)} \|\nabla v\|^{2/p}_{L^2(\mathbb{R}^2)} \\ &\leqslant \frac{1}{2}\tilde\rho \|v\|^2_{L^2(\mathbb{R}^2)} + C_{p,\tilde\rho}\|\rho - \tilde\rho\|^p_{L^p(\mathbb{R}^2)} \|\nabla v\|^2_{L^2(\mathbb{R}^2)},\end{aligned}$$

and (A.1) just follows. Next, for all $2 < q < \infty$, the Hölder and Gagliardo–Nirenberg inequalities yield

$$\begin{aligned}\|\rho^{q'/(2q)} v\|_{L^q(\mathbb{R}^2)} &\leqslant \|\sqrt{\rho} v\|^{q'/q}_{L^2(\mathbb{R}^2)} \|v\|^{1-q'/q}_{L^{2q}(\mathbb{R}^2)} \\ &\leqslant C\|\sqrt{\rho} v\|^{q'/q}_{L^2(\mathbb{R}^2)} \|\nabla v\|^{1/q'-1/q}_{L^2(\mathbb{R}^2)} \|v\|^{1/q-q'/q^2}_{L^2(\mathbb{R}^2)} \\ &\leqslant C\|\sqrt{\rho} v\|^{q'/q}_{L^2(\mathbb{R}^2)} \|\nabla v\|^{1/q'-1/q}_{L^2(\mathbb{R}^2)} \|v\|^{2/q-q'/q}_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Hence (A.2) holds true while replacing the $L^2(\mathbb{R}^2)$ norm of the velocity by (A.1). ∎

B. Local well-posedness

In this section, we prove the local well-posedness result in Theorem 1.4 of the Navier–Stokes equations for a compressible fluid with an initial density having tangential regularity. Our method relies on a change of variables into Lagrangian coordinates, followed by the study of the linearized system and the full nonlinear system, in a similar way as in [49]. In particular, we do not require the density to be a small perturbation around a constant state in $L^\infty(\mathbb{R}^d)$.

More precisely, we consider the Cauchy problem of the system (1.1):

$$\text{(B.1)} \qquad \begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u, \end{cases}$$

equipped with initial data (1.2):

$$\text{(B.2)} \qquad \rho_{|t=0} = \rho_0 \quad \text{and} \quad (\rho u)_{|t=0} = \rho_0 u_0,$$

satisfying (1.3), (1.4) and (1.6):

$$\text{(B.3)} \qquad \begin{cases} 0 < \underline{\rho} \leqslant \rho_0(x), \quad \rho_0 - \tilde\rho \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap \mathbb{L}^p_{\mathcal{X}_0}(\mathbb{R}^d), \\ u_0 \in H^1(\mathbb{R}^d), \quad \mu\Delta u_0 + (\mu+\lambda)\nabla\operatorname{div} u_0 - \nabla P(\rho_0) \in L^2(\mathbb{R}^d). \end{cases}$$

Above, $\tilde\rho > 0$ is a constant and $\mathcal{X}_0 = (X_{0,\upsilon})_{1\leqslant\upsilon\leqslant m} \subset \mathbb{L}^{\infty,p}(\mathbb{R}^d)$, $d < p < \infty$, is a nondegenerate family of $m \in \mathbb{N}^*$ vectors fields, with $m \geqslant d-1$.

Here, we present the main lines of the proof of Theorem 1.4; detailed computations can be found in [49], in the more involved case of density-dependent viscosity.

Step 1. *Lagrangian coordinates.*

Let $0 < T \leqslant \infty$, let u be a Lipschitz vector field such that $\nabla u \in L^1((0,T), L^\infty(\mathbb{R}^d))$, and let us consider its flow map $\mathcal{X}$ given by

$$\mathcal{X}_{\bar{u}}(t,y) = y + \int_0^t u(\tau, \mathcal{X}_{\bar{u}}(\tau,y))\,d\tau =: y + \int_0^t \bar{u}(\tau,y)\,d\tau,$$

where, hereafter, for all $g = g(t,x)$, we define $\bar{g} = g(t,y)$ by

$$\bar{g}(t,y) = g(t, \mathcal{X}(t,y)).$$

By performing this change of variables, the equations (B.1) reads

$$\text{(B.4)} \qquad \begin{cases} \partial_t(\bar{\rho} J_{\bar{u}}) = 0, \\ \rho_0 \partial_t \bar{u} = \operatorname{div}\big(\operatorname{Adj}(D\mathcal{X}_{\bar{u}})\{2\mu \mathbb{D}_{A_{\bar{u}}}\bar{u} + (\lambda \operatorname{div}_{A_{\bar{u}}}\bar{u} - P(\bar{\rho}) + \widetilde{P})\}\big), \end{cases}$$

where

$$J_{\bar{u}} = \det(D\mathcal{X}_{\bar{u}}), \quad A_{\bar{u}} = (D\mathcal{X}_{\bar{u}})^{-1}, \quad \operatorname{div}_{A_{\bar{u}}} w = A_{\bar{u}}^T : \nabla w = Dw : A_{\bar{u}},$$
$$2\mathbb{D}_{A_{\bar{u}}} w = Dw \cdot A_{\bar{u}} + A_{\bar{u}}^T \cdot \nabla w.$$

Step 2. *Well-posedness of the linearized system.*

Motivated by (B.4), we are led to consider the linear system

$$\text{(B.5)} \qquad \begin{cases} \rho_0\,\partial_t v - \mu\,\Delta v - (\mu+\lambda)\,\nabla \operatorname{div} v = \operatorname{div} f, \\ v_{|t=0} = v_0, \end{cases}$$

where the source term f and the initial datum v_0 belong to the following space:

$$Y_T := \big\{(f, v_0) \in L^\infty((0,T), L^2(\mathbb{R}^d)) \times H^1(\mathbb{R}^d) \colon f, \partial_t f, \sigma \partial_{tt} f \in L^2((0,T)\times\mathbb{R}^d);$$
$$\sqrt{\sigma}\,\partial_t f \in L^\infty((0,T), L^2(\mathbb{R}^d)); \mu\,\Delta v_0 + (\mu+\lambda)\,\nabla \operatorname{div} v_0 + \operatorname{div} f_{|t=0} \in L^2(\mathbb{R}^d)\big\}.$$

The solution of the linearized system (B.5) is constructed in the following space:

$$Z_T := \big\{v \in L^\infty((0,T), H^1(\mathbb{R}^d)) : \nabla v, \partial_t v, \nabla \partial_t v, \sqrt{\sigma}\partial_{tt} v, \sigma \nabla \partial_{tt} v \in L^2((0,T)\times\mathbb{R}^d)$$
$$\partial_t v, \sqrt{\sigma}\nabla\partial_t v, \sigma \partial_{tt} v \in L^\infty((0,T), L^2(\mathbb{R}^d))\big\}.$$

It is straightforward to check that for $T < \infty$, every $v \in Z_T$ satisfies

$$v \in \mathcal{C}([0,T], H^1(\mathbb{R}^d)) \quad \text{and} \quad \partial_t v \in \mathcal{C}((0,T], L^2(\mathbb{R}^d)).$$

The well-posedness result for the linearized system (B.5) reads as follows.

Proposition B.1. *Let $0 < T \leqslant \infty$. For all $(f, v_0) \in Y_T$, there exists a unique solution $v \in Z_T$ of the Cauchy problem (B.5). Moreover, the following estimates holds true for v.*

(1) *Basic energy estimates:*

$$\sup_{[0,T]} \|(\sqrt{\rho_0}\, v, \sqrt{\rho_0}\, \partial_t v, \nabla v)\|^2_{L^2(\mathbb{R}^d)} + \int_0^T \|(\nabla v, \sqrt{\rho_0}\, \partial_t v, \nabla \partial_t v)\|^2_{L^2(\mathbb{R}^d)}$$
$$\lesssim \|(\sqrt{\rho_0}\, v_0, \sqrt{\rho_0}\, \partial_t v_{|t=0}, \nabla v_0)\|^2_{L^2(\mathbb{R}^d)} + \sup_{[0,T]} \|f\|^2_{L^2(\mathbb{R}^d)} + \int_0^T \|(f, \partial_t f)\|^2_{L^2(\mathbb{R}^d)}. \tag{B.6}$$

(2) *Higher order energy estimates:*

$$\sup_{[0,T]} \|(\sqrt{\sigma}\, \nabla \partial_t v, \sigma \sqrt{\rho_0}\, \partial_{tt} v)\|^2_{L^2(\mathbb{R}^d)} + \int_0^T \|(\sqrt{\sigma \rho_0}\, \partial_{tt} v, \sigma \nabla(\partial_{tt} v))\|^2_{L^2(\mathbb{R}^d)}$$
$$\lesssim \int_0^T \|\nabla \partial_t v\|^2_{L^2(\mathbb{R}^d)} + \sup_{[0,T]} \sigma \|\partial_t f\|^2_{L^2(\mathbb{R}^d)} + \int_0^T \sigma^2 \|\partial_{tt} f\|^2_{L^2(\mathbb{R}^d)}. \tag{B.7}$$

The constant appearing in the above estimates does not depend on the upper or lower bound of the density ρ_0.

The proof of Proposition B.1 is not part of the classical theory of parabolic systems due to the roughness of the density. However, it can be achieved by a regularization process followed by a compactness argument. We refer, for example, to Theorem 3.1 in [49] for the derivation of estimates (B.6) and (B.7).

Step 3. *Further estimates of the linearized system.*

By interpolating the estimates (B.6) and (B.7), we observe that the following estimates hold true for the velocity gradient and its time derivative.

Corollary B.2. *The following estimates hold true.*

(1) *Assuming that $f \in L^r((0,T), L^p(\mathbb{R}^d))$ for $2 \leqslant r \leqslant \infty$ and $2 < p < \infty$ if $d = 2$ or $2 < p \leqslant 6$ if $d = 3$, we have*

$$\|\nabla v\|^2_{L^r((0,T),L^p(\mathbb{R}^d))} \lesssim \|(f, v_0)\|^2_{Y_T} + \|f\|^2_{L^r((0,T),L^p(\mathbb{R}^d))}. \tag{B.8}$$

The same estimate holds also true, if $d = 3$, for $6 < p < \infty$, and $2 \leqslant r \leqslant 4p/(p-6)$.

(2) *For all $2 \leqslant r \leqslant \infty$ and $2 < p < \infty$ if $d = 2$ and $2 < p \leqslant 6$ if $d = 3$, we have*

$$\|\sigma^s \nabla \partial_t v\|^2_{L^r((0,T),L^p(\mathbb{R}^d))} \lesssim \|(f, v_0)\|^2_{Y_T} + \|\sigma^s \partial_t f\|^2_{L^r((0,T),L^p(\mathbb{R}^d))} \tag{B.9}$$

where

$$s = \begin{cases} 1 - 1/p - 1/r & \text{if } d = 2, \\ 5/4 - 3/(2p) - 1/r & \text{if } d = 3. \end{cases}$$

For $d = 3$, the same estimate also holds true for all $6 < p < \infty$ and $2 \leqslant r \leqslant 4p/(p-6)$.

(3) *Let $\mathcal{X}_0 = (X_{0,v})_{1 \leqslant v \leqslant m} \subset \mathbb{L}^{\infty,p}(\mathbb{R}^d)$, $d < p < \infty$, be a nondegenerate family of $m \in \mathbb{N}^*$ vectors fields, with $m \geqslant d - 1$.*

(a) *Assume that* $f \in L^r((0,T), L^\infty(\mathbb{R}^d) \cap \mathbb{L}^p_{\mathcal{X}_0}(\mathbb{R}^d))$*, with* $2 \leqslant r \leqslant 8$ *if* $d = 2$ *and* $2 \leqslant r \leqslant 32/9$ *if* $d = 3$*. Then* $\nabla v \in L^r((0,T), L^\infty(\mathbb{R}^d))$ *and the following estimate holds true*:

$$\|\nabla v\|^2_{L^r((0,T),L^\infty(\mathbb{R}^d))} \lesssim \|(f,v_0)\|^2_{Y_T} + \|f\|^2_{L^r((0,T),L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)\cap \mathbb{L}^p_{\mathcal{X}_0}(\mathbb{R}^d))}. \tag{B.10}$$

(b) *Assume* $f \in L^r((0,T), L^\infty(\mathbb{R}^d) \cap \mathbb{L}^p_{\mathcal{X}_0}(\mathbb{R}^d))$*, with* $2 < p < \infty$*,* $2 \leqslant r \leqslant 2p/(p-2)$ *if* $d = 2$ *and* $3 < p \leqslant 6$*,* $2 \leqslant r \leqslant 4p/(3p-6)$ *if* $d = 3$*. Then we have*

$$\|\partial_{\mathcal{X}_0} \nabla v\|^2_{L^r((0,T),L^p(\mathbb{R}^d))} \lesssim \|\mathcal{X}_0\|^2_{L^\infty(\mathbb{R}^d)} \|(f,v_0)\|^2_{Y_T} + \|\partial_{\mathcal{X}_0} f\|^2_{L^r((0,T),L^p(\mathbb{R}^d))} + \|\nabla \mathcal{X}_0\|^2_{L^p(\mathbb{R}^d)} \|f\|^2_{L^r((0,T),L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)\cap \mathbb{L}^p_{\mathcal{X}_0}(\mathbb{R}^d))}. \tag{B.11}$$

Let $d = 3$ *and* $6 < p < \infty$*,* $2 \leqslant r \leqslant \infty$*. If* $\sigma^s f \in L^r((0,T), L^\infty(\mathbb{R}^3) \cap \mathbb{L}^p_{\mathcal{X}_0}(\mathbb{R}^3))$ *and* $\sigma^{3/4-1/r} \partial_t f \in L^3(\mathbb{R}^3)$*, with*

$$s = \frac{3}{4} - \frac{1}{r} - \frac{3}{2p},$$

then, from (2)*, we have* $\sigma^{3/4-1/r} \nabla \partial_t v \in L^3(\mathbb{R}^3)$ *and*

$$\begin{aligned} \|\sigma^s \partial_{\mathcal{X}_0} \nabla v\|^2_{L^r((0,T),L^p(\mathbb{R}^3))} &\lesssim \|\sigma^s \partial_{\mathcal{X}_0} f\|^2_{L^r((0,T),L^p(\mathbb{R}^3))} \\ &+ \|\mathcal{X}_0\|^2_{L^\infty(\mathbb{R}^3)} \left(\|(f,v_0)\|^2_{Y_T} + \|\sigma^{3/4-1/r} \partial_t f\|^2_{L^r((0,T),L^3(\mathbb{R}^3))}\right) \\ &+ \|\nabla \mathcal{X}_0\|^2_{L^p(\mathbb{R}^3)} \|\sigma^s f\|^2_{L^r((0,T),L^2(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)\cap \mathbb{L}^p_{\mathcal{X}_0}(\mathbb{R}^3))}. \end{aligned} \tag{B.12}$$

The constant appearing in the above estimates depends on both the lower and upper bounds of the density.

Indeed, all the estimates in Corollary B.2 are based on the following expression of the velocity gradient:

$$\begin{aligned} \nabla v = \nabla \mathcal{P} v + \nabla \mathcal{Q} v = &-\frac{1}{\mu}(-\Delta)^{-1} \nabla \mathcal{P}(\rho_0 \partial_t v) - \frac{1}{\nu}(-\Delta)^{-1} \nabla \mathcal{Q}(\rho_0 \partial_t v) \\ &+ \frac{1}{\mu}(-\Delta)^{-1} \nabla \mathcal{P} \operatorname{div} f + \frac{1}{\nu}(-\Delta)^{-1} \nabla \mathcal{Q} \operatorname{div} f. \end{aligned} \tag{B.13}$$

The first two terms associated with $\partial_t v$ exhibit regularity due to the regularity of $\partial_t v$. In particular, their $L^r((0,T), L^p(\mathbb{R}^d))$ norm estimates can be obtained by interpolating the estimate (B.6). The $L^r((0,T), L^p(\mathbb{R}^d))$ norm estimate for the last two terms in the expression of the velocity gradient (B.13) follows from the continuity of Riesz operators on $L^p(\mathbb{R}^d)$ for all $1 < p < \infty$. These computations lead to (B.8).

The estimate (B.10) is obtained similarly: The $L^r((0,T), L^\infty(\mathbb{R}^d))$ norm of the terms associated with $\partial_t v$ can be estimated by interpolating the estimate (B.6), while the norm of the remaining terms is obtained using Proposition 2.5 since Riesz operators fail to be continuous on $L^\infty(\mathbb{R}^d)$.

To derive the estimate (B.9), we take time derivative of (B.13) and apply the continuity of Riesz transforms on $L^p(\mathbb{R}^d)$, $1 < p < \infty$, to obtain norms for the terms associated with $\partial_t f$. The norm of the first two terms, associated with $\partial_{tt} v$, can be obtained by interpolating estimate (B.7).

For the last estimates (B.11) and (B.12), we take the derivative along $\mathcal{X}_0$ in (B.13) and we obtain

$$\begin{aligned}\partial_{\mathcal{X}_0} \nabla v = &-\frac{1}{\mu} \mathcal{X}_0 \cdot \nabla(-\Delta)^{-1} \nabla \mathcal{P}(\rho_0 \partial_t v) - \frac{1}{\nu} \mathcal{X}_0 \cdot \nabla(-\Delta)^{-1} \nabla \mathcal{Q}(\rho_0 \partial_t v) \\ &+ \frac{1}{\mu} \partial_{\mathcal{X}_0} (-\Delta)^{-1} \nabla \mathcal{P} \operatorname{div} f + \frac{1}{\nu} \partial_{\mathcal{X}_0} (-\Delta)^{-1} \nabla \mathcal{Q} \operatorname{div} f.\end{aligned}$$

Once again, the norms of the first two terms are obtained using Hölder's inequality and by interpolating estimates (B.6) and (B.7). For the remaining terms, we use Lemma A.1 of [10]. This completes this step of the study of the linear system (B.5).

Step 4. *Final conclusion.*

Once we conclude the study of the linear system associated with (B.4), the next step is to define a map that is contracting for some small time $T > 0$, such that it admits a unique fixed point, which serves as a solution to (B.1) after reverting to Eulerian coordinates. With Proposition B.1 and Corollary B.2 in mind, we can verify that the unique solution of the full nonlinear system can be constructed in

$$\widetilde{Z}_T := \left\{ v \in Z_T \colon \nabla v \in L^2((0,T), L^\infty(\mathbb{R}^d) \cap \mathbb{L}^p_{\mathcal{X}_0}(\mathbb{R}^d)) \right\}$$

for $2 < p < \infty$ if $d = 2$ and $3 < p \leqslant 6$ if $d = 3$ by following the steps outlined in Section 4 of [49]. The only argument we need to specify is contained in the following lemmas.

Lemma B.3. *Let v be a vector field satisfying that $\nabla v \in L^1((0,t), L^\infty(\mathbb{R}^d))$ and, for some $t > 0$, $\partial_{\mathcal{X}_0} \nabla v \in L^1((0,t), L^p(\mathbb{R}^d))$. Assuming that*

$$V := \int_0^t \left[\|\nabla v\|_{L^\infty(\mathbb{R}^d)} + \|\partial_{\mathcal{X}_0} \nabla v\|_{L^p(\mathbb{R}^d)} \right] < 1, \tag{B.14}$$

then there exists a constant $K = K(V)$ such that the following estimate holds true:

$$\|\partial_{\mathcal{X}_0} \operatorname{Adj}(D\mathcal{X}_v(t)), \partial_{\mathcal{X}_0} A_v(t), \partial_{\mathcal{X}_0} J_v^{\pm 1}(t)\|_{L^p(\mathbb{R}^d)} \leqslant K \|\partial_{\mathcal{X}_0} Dv\|_{L^1((0,t), L^p(\mathbb{R}^d))}.$$

Moreover, we have for all $Dw \in \mathbb{L}^p_{\mathcal{X}_0}(\mathbb{R}^d)$,

$$\begin{aligned}&\|\partial_{\mathcal{X}_0}(\operatorname{Adj}(D\mathcal{X}_v(t)) D_{A_v(t)} w) - \partial_{\mathcal{X}_0} Dw\|_{L^p(\mathbb{R}^d)} \\ &\quad + \|\partial_{\mathcal{X}_0}(\operatorname{Adj}(D\mathcal{X}_v(t)) \operatorname{div}_{A_v(t)} w) - \partial_{\mathcal{X}_0} \operatorname{div} w\|_{L^p(\mathbb{R}^d)} \\ &\quad \leqslant K \big(\|Dw\|_{L^\infty(\mathbb{R}^d)} + \|\partial_{\mathcal{X}_0} Dw\|_{L^p(\mathbb{R}^d)} \big) \int_0^t \big(\|Dv\|_{L^\infty(\mathbb{R}^d)} + \|\partial_{\mathcal{X}_0} Dv\|_{L^p(\mathbb{R}^d)} \big).\end{aligned}$$

Lemma B.4. *Let v_1 and v_2 two vector fields satisfying (B.14), with $V_1, V_2 < 1$, and let $\delta v := v_2 - v_1$. Then, there exists a constant $K = K(V_1, V_2)$ such that the following esti-*

mate holds true:

$$\begin{aligned}
&\|\big(\partial_{\mathcal{X}_0} A_{v_2}(t) - \partial_{\mathcal{X}_0} A_{v_1}(t), \partial_{\mathcal{X}_0} \operatorname{Adj}(D\mathcal{X}_{v_2}(t)) - \partial_{\mathcal{X}_0} \operatorname{Adj}(D\mathcal{X}_{v_1}(t))\|_{L^p(\mathbb{R}^d)} \\
&\quad + \|\partial_{\mathcal{X}_0} J_{v_2}^{\pm 1}(t) - \partial_{\mathcal{X}_0} J_{v_1}^{\pm 1}(t)\big)\|_{L^p(\mathbb{R}^d)} \\
&\quad \leqslant K \int_0^t \big(\|D\delta v\|_{L^\infty(\mathbb{R}^d)} + \|\partial_{\mathcal{X}_0} D\delta v\|_{L^p(\mathbb{R}^d)}\big).
\end{aligned}$$

The particular case of $3 < p \leqslant 6$ for $d = 3$ is sufficient for constructing blocks for the global solution in Theorem 1.6. For $6 < p < \infty$ in three dimensions, the fixed point theorem may be applied in a closed subset of the following space:

$$\begin{aligned}
\widetilde{Z}_T := \big\{v \in Z_T \ : \ &\sigma^{3/4-1/r}\, \nabla\partial_t v \in L^r((0,T), L^3(\mathbb{R}^3)); \\
&\sigma^{3/4-1/r-3/(2p)}\, \nabla v \in L^r((0,T), L^\infty(\mathbb{R}^3) \cap \mathbb{L}^p_{\mathcal{X}_0}(\mathbb{R}^3))\big\}.
\end{aligned}$$

This ends the sketchy proof of Theorem 1.4.

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