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On various Carleson-type geometric lemmas and uniform rectifiability in metric spaces: Part 1

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Abstract. We introduce new flatness coefficients, which we shall call *ι-numbers*, for Ahlfors k-regular sets in metric spaces ($k \in \mathbb{N}$). Using these coefficients for $k = 1$, we characterize uniform 1-rectifiability in rather general metric spaces, completing earlier work by Hahlomaa and Schul. Our proof proceeds by quantifying an isometric embedding theorem due to Menger, and by an abstract argument that allows to pass from a local covering by continua to a global covering by 1-regular connected sets.

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1. Introduction

This note is intended as a contribution to a broad program aimed at extending the theory of *quantitative* (or uniform) *rectifiability*, pioneered by David and Semmes in Euclidean spaces [20, 21], to other metric spaces. Recent research in this direction concerns different classes of sets, depending on the ambient metric space:

(1) quantitatively rectifiable sets modelled on *Euclidean* spaces, such as 1-regular curves, or sets with big pieces of (bi-)Lipschitz images of Euclidean sets,

(2) sets that are quantitatively rectifiable by specific types of *non-Euclidean* Lipschitz graphs, for instance in Heisenberg groups and parabolic spaces.

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Here we focus on direction (1). We introduce new quantitative coefficients called *ι-numbers*. We characterize uniform 1-rectifiability in rather general metric spaces (Theorem 1.4) by a Carleson-type summability condition for ι-numbers. The proof uses results that we believe to be of independent interest and which we will explain in more detail in the subsequent paragraphs.

1.1. From β-numbers in Euclidean spaces to ι-numbers in metric spaces

Uniformly k-rectifiable sets in $\mathbb{R}^n$ ($k, n \in \mathbb{N}$, $1 \le k < n$) can be characterized in many equivalent ways, for instance as k-regular sets with big pieces of Lipschitz images of subsets of $\mathbb{R}^k$, or by means of a geometric lemma for Jones $\beta_{q,\mathcal{V}_k}$*-numbers* which quantifies the approximability of the set by k-dimensional planes. Here q is allowed to be any number $1 \le q < 2k/(k-2)$ if $k \ge 2$ and $1 \le q \le \infty$ if $k = 1$, recall Section 1.4 in Part I of [21]. By "k-regular" we mean sets that satisfy the Ahlfors s-regularity condition (2.4) for $s = k$. For the purpose of this introduction, we say that a k-regular set E in Euclidean space $\mathbb{R}^n$ satisfies the 2-*geometric lemma with respect to* $\beta_{q,\mathcal{V}_k}$, denoted $E \in \mathrm{GLem}(\beta_{q,\mathcal{V}_k}, 2)$, if there is a constant $M \ge 0$ such that

$$\int_{B_R(x_0)\cap E} \int_0^R \beta_{q,\mathcal{V}_k}(B_r(x) \cap E)^2 \, \frac{dr}{r} \, \mathrm{d}\mathcal{H}^k(x) \le MR^k \quad x_0 \in E,\ R > 0, \tag{1.1}$$

where the coefficients

$$\beta_{q,\mathcal{V}_k}(B_r(x) \cap E) := \inf_{V\in\mathcal{V}_k} \left(⨍_{B_r(x)\cap E} \left[\frac{\mathsf{d}(y, V)}{\operatorname{diam}(B_r(x) \cap E)} \right]^q \mathrm{d}\mathcal{H}^k(y) \right)^{1/q}, \quad q \in (0, \infty), \tag{1.2}$$

quantify in a scale-invariant and L^q-based way how well the set E is approximated by k-planes $V \in \mathcal{V}_k$ at $x \in E$ and scale $r > 0$ in the Euclidean distance. The number "2" in the definition of $\mathrm{GLem}(\beta_{q,\mathcal{V}_k}, 2)$ corresponds to the exponent "2" in the expression (1.1).

We study another family of quantitative coefficients that we call *ι-numbers*. They are well-suited for generalizations to metric spaces. Roughly speaking, ι-numbers measure "flatness" of a set using mappings into model spaces, rather than using the metric distance from approximating sets. They can be used to formulate a geometric lemma analogous to (1.1); see Definition 2.14 for a very general definition of geometric lemmas, which we state in terms of systems of Christ–David dyadic cubes. Roughly speaking, the symbol $\mathrm{GLem}(h, p, M)$ denotes a Carleson measure condition in the spirit of (1.1) with β-numbers replaced by other coefficients given by h, and the integrability exponent "2" replaced by "p".

For $k \in \mathbb{N}$ and a k-regular set E in a metric space (X, d), and for $q \in (0, \infty)$, we denote by $\iota_{q,k}(B_r(x) \cap E)$ the number

$$\inf_{\|\cdot\|} \inf_{f:B_r(x)\cap E\to\mathbb{R}^k} \left(⨍_{B_r(x)\cap E} ⨍_{B_r(x)\cap E} \left[\frac{|\mathsf{d}(y,z) - \|f(y) - f(z)\||}{\operatorname{diam}(B_r(x) \cap E)} \right]^q \mathrm{d}\mathcal{H}^k(y)\, \mathrm{d}\mathcal{H}^k(z) \right)^{1/q}. \tag{1.3}$$

Here the first infimum is taken over all norms on $\mathbb{R}^k$, and the functions f in the second infimum are assumed to be *Borel*.

For illustration, suppose that the double integral in (1.3) vanishes for some $\|\cdot\|$ and f. Then, because $\|\cdot\|$ is bi-Lipschitz equivalent to the Euclidean norm on $\mathbb{R}^k$, up to $\mathcal{H}^k$ measure zero, $B_r(x) \cap E$ is bi-Lipschitz equivalent to a subset of Euclidean $\mathbb{R}^k$. If also the ambient space (X, d) is the Euclidean space $\mathbb{R}^n$, actually much more is true. Since f arises as an isometric embedding from a positive measure subset of $(\mathbb{R}^k, \|\cdot\|)$ into (strictly convex) Euclidean space, one can show that in fact $B_r(x) \cap E$ must be essentially contained in a k-plane, see Section 3.1.2 of [32]. A refinement of this observation is stated in Proposition 2.36.

For the development of a meaningful theory in metric spaces, it is crucial to allow all possible norms $\|\cdot\|$ on $\mathbb{R}^k$ in (1.3), not just the Euclidean norm. This is similar in spirit to the use of norms in the definition of the *Gromov–Hausdorff bilateral weak geometric lemma* (BWLG) in Definition 3.1.5 of [8], and in both cases the norms are allowed to depend on the point x and the scale r. In the recent breakthrough [8], Bate, Hyde, and Schul characterized, in arbitrary metric spaces, k-regular sets with big pieces of Lipschitz images of $\mathbb{R}^k$ as those k-regular sets that satisfy a Gromov–Hausdorff BWGL, or some other equivalent conditions inspired by Euclidean quantitative rectifiability. Not contained in their characterization is, quoting the authors, "a condition on the square summability of some suitable variant of the Jones β-number", while they observed that generalizing the main result in [3] could be a first step in this direction. Finding a suitable Carleson summability condition in this generality is a well-known problem to which Schul alluded already in [57]. We do not claim to obtain a solution of this problem for $k > 1$, but we hope that the present paper could serve as a motivation to investigate characterizations of geometric lemmas for ι-coefficients. Here we show that the validity of a geometric lemma for the $\iota_{1,1}$-numbers defined in (1.3) for $k = q = 1$ is indeed equivalent to uniform 1-rectifiability in rather general metric spaces. In a companion paper [32], for arbitrary $k \in \mathbb{N}$, we study a variant of the ι-numbers that are tailored specifically to Euclidean spaces and we prove that these $\iota_{1,\mathcal{V}_k}$*-numbers* can be used to characterize uniformly k-rectifiable sets in $\mathbb{R}^n$ for any $k \geq 1$.

1.2. From Menger curvature to ι-numbers in dimension 1

We explain some ideas behind the characterization of uniformly 1-rectifiable sets by means of $\iota_{1,1}$-numbers.

Recall that $\iota_{1,1}(B_r(x) \cap E) = 0$ implies the existence of an isometric embedding from $(B_r(x) \cap E, \mathsf{d})$, up to a $\mathcal{H}^1$ null set, into $\mathbb{R}$. (Here we may without loss of generality assume that the target space $\mathbb{R}$ is equipped with the Euclidean norm). Menger [50] proved criteria for isometric embeddability of metric spaces into Euclidean $\mathbb{R}^k$. The case $k = 1$ of one of his results can be stated as follows, see [27]. If (X, d) is a space with at least five points such that the triangular excess vanishes for any triple of points in X, that is, every such triple embeds isometrically into $\mathbb{R}$, then the whole space X embeds isometrically into $\mathbb{R}$.

In Theorem 4.27, we obtain a quantitative version of Menger's result that applies to metric spaces where the triangular excess of point triples is not necessarily identically zero, but sufficiently small. In particular, we give a condition under which such spaces embed into $(\mathbb{R}, |\cdot|)$ by an almost isometry. With a further refinement of the arguments, we obtain also an integral version of this statement (see Theorem 4.19) which can then be applied to relate $\iota_{1,1}$-numbers to the *metric β-numbers* known from the literature. The latter are quantitative coefficients defined in terms of triangular excess, see (2.22). We call them *κ-numbers* in this note to emphasize the connection with Menger curvature, see (2.20) and Example 2.21.

Building on Theorem 4.19 and heavily on earlier work of Hahlomaa [35] and Schul [55, 57, 58], we obtain the following characterization.

Theorem 1.4 (Characterizations of uniform 1-rectifiability). *Let* (X, d) *be a complete, doubling, and quasiconvex metric space. The following conditions are quantitatively equivalent for a* 1*-regular set* E *in* (X, d):

(1) E *is contained in a closed and connected* 1*-regular set,*

(2) E *has big pieces of Lipschitz images of subsets of* $\mathbb{R}$*,*

(3) E *has big pieces of bi-Lipschitz images of subsets of* $\mathbb{R}$*,*

(4) E *satisfies the geometric lemma* $\mathrm{GLem}(\kappa, 1)$*,*

(5) E *satisfies the geometric lemma* $\mathrm{GLem}(\iota_{1,1}, 1)$.

The equivalence of the conditions (2), (3), and (4) was proven by Schul in [57] using also earlier work by Hahlomaa and himself. Compared to these results, the novelty in our Theorem 1.4 is the equivalence of the other conditions with property (5). The implication from (4) to (1) is also new for quasiconvex spaces. For bounded sets in *geodesic* spaces, it was stated by Schul in Theorem 3.11 of [56], attributed to Hahlomaa, see also Theorem 1.5 in [55]. Our proof of the implication "(4) $\Rightarrow$ (1)" (formulated as Corollary 3.20) is directly based on one of Hahlomaa's published results (Theorem 1.1 in [35]), coupled with an abstract argument that allows to pass from a local covering by continua to a global covering by 1-regular connected sets, see Corollary 3.2 and Corollary 3.14. This proof strategy works in quasiconvex spaces and thus makes the result applicable for instance in the first Heisenberg group $\mathbb{H}^1$ equipped with the Korányi distance $d_{\mathbb{H}^1}$. We discuss such an application below in Theorem 3.22.

Adapting Menger's ideas, we were able to show that conditions (4) and (5) are not only equivalent at the level of geometric lemmas, but the coefficients κ and $\iota_{1,1}$ are comparable on neighborhoods of individual dyadic cubes. For more detailed statements, see Corollary 4.6 and Theorem 4.17, in particular (4.18), later in this note. By the work of Bate, Hyde, and Schul [8] in the 1-regular case, the conditions in Theorem 1.4 are further equivalent to E satisfying a (Gromov–Hausdorff) bilateral weak geometric lemma or any of the other conditions stated in Theorem B of [8].

1.3. Relation to previous work

This note has been motivated by several lines of research, which we briefly sketch here. The results are too numerous for an exhaustive list, but we hope that the interested reader will find some directions for further reading. We also refer to the survey [49] by Mattila for more information.

1.3.1. Euclidean-type (quantitative) rectifiability. The qualitative theory of Federer-type rectifiability in metric spaces (using Lipschitz images of subsets of Euclidean spaces) [1, 7, 41] and the already well-established quantitative theory in Euclidean space [20, 21] motivated the recent work by Bate, Hyde, and Schul [8]. This provides several equivalent characterizations of sets that are quantitatively rectifiable modelled on Euclidean spaces. Pivotal examples from the literature where this notion of uniform rectifiability is well-suited are low-dimensional sets in Heisenberg groups [13,25,29,37], and subsets of regular curves in metric spaces [35,55]. For the case of 1-dimensional sets in metric spaces, there is also a growing body of literature concerned with the travelling salesman theorems and quantitative methods for the study of *qualitatively* rectifiable sets [4–6, 31, 46, 47]. Li introduced and used in [45] *stratified β-numbers* to characterize subsets of Carnot groups that are contained in rectifiable curves. Coefficients of this type certainly seem promising to study also uniform rectifiability for low-dimensional sets in Carnot groups. On the other hand, they are defined specifically for the setting of stratified Lie groups, while an advantage of the κ- and the ι-coefficients is their versatility. The κ-numbers are tailored to 1-dimensional sets, but higher-dimensional variants have been considered in [3, 58] for images of Lipschitz functions $f : [0,1]^k \to (\mathsf{X}, \mathsf{d})$. Various coefficients related to Menger curvatures have also been used to characterize higher-dimensional (uniform) rectifiability in Euclidean spaces [33, 42, 43, 51]. Investigating connections between ι-numbers and higher-dimensional variants of κ-numbers could be an interesting topic for future research.

1.3.2. Other notions of quantitative rectifiability. Motivated by specific PDEs, quantitative theories of rectifiability have also been developed in settings where the natural building blocks are different from Lipschitz images of subsets of $\mathbb{R}^k$. This applies for instance to quantitative rectifiability for 1-codimensional sets in parabolic spaces [11, 38, 39, 44] (where *regular parabolic Lipschitz graphs* are used) and sub-Riemannian Heisenberg groups [12, 14, 15, 30] (where *intrinsic Lipschitz graphs* are studied), and is not directly related to the present paper.

1.3.3. Axiomatic results in metric spaces. In addition to the mentioned papers, which concern specific model spaces, there are also a results available that deal with concepts related to rectifiability and quantitative rectifiability in rather abstract, axiomatic settings [10, 23, 26]. While [10, 26] are motivated by applications to parabolic spaces and Heisenberg groups, respectively, the main ingredients in both cases are abstract metric space results. The paper [26] contains a sufficient criterion for a metric space to admit a big bi-Lipschitz piece of a model space. Unfortunately, the assumptions of the theorem are stronger than the information we can deduce from the validity of a geometric lemma for ι-numbers. In [10], the authors provide a general framework for the study of corona decompositions and geometric lemmas in metric spaces, and we follow their notation to a large extent. However, the main results in [10] do not seem to have direct applications in our setting, which concerns ι-numbers defined through mappings, rather than β-numbers defined through approximating sets.

1.3.4. Approximate isometries. The definition of ι-numbers is inspired by the **b**-numbers studied by the second-named author in [59]. The coefficients employed in the present paper differ from the **b**-numbers in two crucial ways: first, they are L^q-based, $q \in [1, \infty)$, instead of L^∞-based, and second, they can be defined in arbitrary metric spaces. The **b**-numbers in [59] are defined via approximate isometries. A geometric lemma for ι-numbers heuristically still yields many almost isometric mappings from the given set to model spaces, but in general it remains an open question if and how this information can be used to build big pieces of Euclidean bi-Lipschitz images inside the set.

Structure of the paper. Section 2 contains preliminaries. In particular, we collect various facts about geometric lemmas that will be used here and in the sequel [32]. In the main part of the paper (Section 3 and Section 4), we discuss 1-regular sets in metric spaces. In Section 3.1, we give sufficient local conditions for the existence of global 1-regular covering continua for sets in metric spaces. In Section 3.2, we present an application (Corollary 3.20) where these local conditions are satisfied thanks to a result by Hahlomaa. As a corollary, we complete in Section 4 the proof of the characterization of uniform 1-rectifiability in metric spaces stated in Theorem 1.4. Appendix A contains technical results needed in Section 4, related to the quantification of Menger's theorem about isometric embeddings into $\mathbb{R}$.

2. Preliminaries

Notation. We write $A \lesssim B$ to denote the existence of an absolute constant $C \geq 1$ such that $A \leq CB$. The inequality $A \lesssim B \lesssim A$ is abbreviated to $A \sim B$. If the constant C is allowed to depend on a parameter "p", we indicate this by writing $A \lesssim_p B$. We denote the diameter of a set E in a metric space by $\mathrm{diam}(E)$ and use the convention that $\mathrm{diam}(E) = +\infty$ if E is unbounded.

2.1. Standard quantitative notions

Throughout this paper – and its sequel [32] –, we employ quantitative notions that are ubiquitous in the theory of uniform rectifiability in Euclidean spaces and that are increasingly applied in other metric spaces as well. The terminology used in Sections 2.1.1–2.1.2 follows closely the presentation in [10] in the case of Hausdorff measures $\mu = \mathcal{H}^s|_E$. Readers familiar with the standard terminology may wish to proceed directly to Section 2.2, where we introduce new quantitative coefficients.

Definition 2.1 (Quasiconvex metric space). A metric space (X, d) is called *quasiconvex* if there exists a constant $L \geq 1$, called *quasiconvexity constant*, such that every couple of points $x, y \in \mathrm{X}$ can be joined by a curve of length at most $L\mathsf{d}(x, y)$.

We denote by $B_r(x) = \{y \in \mathrm{X} : \mathsf{d}(x, y) < r\}$ the open ball with center x and radius r in a given metric space (X, d).

A metric space (X, d) is commonly said to be *doubling* if for all $x \in \mathrm{X}$ and $r > 0$ the ball $B_r(x)$ can be covered by the union of at most C balls of radius $r/2$, for some constant $C > 1$ independent of x and r. For our purposes, it will be more convenient to use the following, equivalent, condition:

Definition 2.2 (Doubling metric space). A metric space (X, d) is called *doubling* if there exists a constant $D \geq 1$, called *doubling constant*, such that, for every $\varepsilon \in (0, 1/2]$, every subset of X of diameter r in can be covered by $\leq \varepsilon^{-D}$ sets of diameter at most εr.

The covering in Definition 2.2 can also be taken to have uniformly bounded overlap, with multiplicity depending only on D.

2.1.1. Ahlfors regular sets and dyadic systems.

Definition 2.3 (s-regular sets). A set $E \subset (\mathrm{X}, \mathsf{d})$ with $\operatorname{diam}(E) > 0$ is said to be *s-regular*, $s > 0$, if it is closed and there exists $C \geq 1$, called *regularity constant*, such that

$$C^{-1} r^s \leq \mathcal{H}^s(B_r(x) \cap E) \leq C r^s, \quad x \in E, \ r \in (0, 2\operatorname{diam}(E)), \tag{2.4}$$

in which case we write $E \in \operatorname{Reg}_s(C)$. Furthermore, if only the first (respectively, the second) inequality in (2.4) is satisfied and E is not necessarily closed, we say that the set E is *lower* (respectively, *upper*) s-regular, and we write $E \in \operatorname{Reg}_s^-(C)$ (respectively, $E \in \operatorname{Reg}_s^+(C)$). Finally, we say that the metric space (X, d) is s-regular if the whole set X is an s-regular set with respect to d. We also use the term *Ahlfors regular* to denote the class of sets that are s-regular for some exponent s.

Up to replacing C by $2^s C$, the second inequality in (2.4) holds also for arbitrary $x \in \mathrm{X}$ and $r > 0$. Moreover, it can be checked from the definition that if $E_i \in \operatorname{Reg}_s(C_i)$ for $i \in \{1, 2\}$ are intersecting sets of a common ambient space, then $E_1 \cup E_2 \in \operatorname{Reg}_s(C)$ with a constant C that can be taken to depend only on the regularity constants C_1 and C_2 of the two initial sets.

Regular sets in metric spaces admit systems of generalized dyadic cubes. For k-regular sets in $\mathbb{R}^n$, the existence of such systems was proven by David in Section B.3 of [17] and in [18]. More generally, Christ constructed dyadic cube systems for spaces of homogeneous type in Theorem 11 of [16], see also [40] and references therein for variants of this construction. We use the version for Ahlfors regular sets in metric spaces as stated in Lemma 2.5 of [10], see also Section 5.5 of [22], but we include a separate notational convention for bounded sets following the comment on p. 22 of [22]. If the regular set E is bounded, we define $\mathbb{J} := \{j \in \mathbb{Z} \colon j \geq n\}$, where $n \in \mathbb{Z}$ is such that $2^{-n} \leq \operatorname{diam}(E) < 2^{-n+1}$; otherwise, we denote $\mathbb{J} := \mathbb{Z}$.

Theorem and Definition 2.5 (Dyadic systems [16, 17]). *For any $s > 0$ and $C \geq 1$, there exists a constant $c_0 \in (0, 1)$ such that in an arbitrary metric space, every set $E \in \operatorname{Reg}_s(C)$ admits a* system of dyadic cubes *$\Delta = \bigcup_{j \in \mathbb{J}} \Delta_j$, where Δ_j is a family of pairwise disjoint Borel sets $Q \subset E$* (cubes) *satisfying*

(1) *$E = \bigcup_{Q \in \Delta_j} Q$ for each $j \in \mathbb{J}$,*

(2) *for $i, j \in \mathbb{J}$ with $i \leq j$, if $Q \in \Delta_i$ and $Q' \in \Delta_j$, then either $Q' \subset Q$ or $Q \cap Q' = \emptyset$,*

(3) *for $j \in \mathbb{J}$, $Q' \in \Delta_j$ and $i < j$ with $i \in \mathbb{J}$, there is a unique $Q \in \Delta_i$* (ancestor) *such that $Q' \subset Q$,*

(4) *for $j \in \mathbb{J}$ and $Q \in \Delta_j$, it holds* $\operatorname{diam}(Q) \leq c_0^{-1} 2^{-j}$,

(5) *for $j \in \mathbb{J}$ and $Q \in \Delta_j$, there is a point $x_Q \in E$* (center) *such that $B_{c_0 2^{-j}}(x_Q) \cap E \subset Q$.*

For $j \in \mathbb{J}$ and $Q \in \Delta_j$, we denote $\ell(Q) := 2^{-j}$ and refer to this as the side length *of the cube. We also define*

$$\Delta_{Q_0} := \{Q \in \Delta : Q \subset Q_0\}, \quad Q_0 \in \Delta,$$

and for a given constant $K > 1$, we set

$$KQ := \{x \in E : \operatorname{dist}(x, Q) \leq (K-1) \operatorname{diam}(Q)\}.$$

Following are additional comments and notations about a system of dyadic cubes that will be useful in the sequel. If $E \in \mathrm{Reg}_s(C)$ and Δ is a dyadic system on E, then for $Q \in \Delta$ and $K > 1$, the set KQ is simply the intersection of E with the closed $(K-1)\,\mathrm{diam}(Q)$ neighborhood of Q, and

$$KQ \subset B_{K\,\mathrm{diam}(Q)}(x_Q) \cap E. \tag{2.6}$$

Moreover, it follows from conditions (4)–(5) that

$$B_{c_0\ell(Q)}(x_Q) \cap E \subset Q \subset B_{c_0^{-1}\ell(Q)}(x_Q) \cap E, \quad Q \in \Delta. \tag{2.7}$$

Since $E \in \mathrm{Reg}_s(C)$, this and (4) imply that

$$\begin{aligned} C^{-1}(c_0\,\ell(Q))^s &\le \mathcal{H}^s(Q) \le C(c_0^{-1}\ell(Q))^s, \\ C^{-2/s}\,c_0\,\ell(Q) &\le \mathrm{diam}(Q) \le c_0^{-1}\,\ell(Q). \end{aligned} \tag{2.8}$$

Combining the second estimate in (2.8) and condition (1) we can infer the existence of a constant $K = K(s, C) > 1$ such that the following holds for all $z \in E$ and $0 < R < \mathrm{diam} E$. If $j \in \mathbb{J}$ is such that $2^{-j} \le R < 2^{-j+1}$, then there exists $Q \in \Delta_j$ such that

$$E \cap B(z, R) \subset KQ. \tag{2.9}$$

For every $Q \in \Delta_{j_0}$ and $j \in \mathbb{N} \cup \{0\}$, we define the j-th descendants of Q by

$$F_j(Q) := \{Q' \in \Delta_{j+j_0} : Q' \subset Q\}. \tag{2.10}$$

It is easy to deduce from the first part of (2.8), and observing that the cubes in $F_i(Q)$ are pairwise disjoint, that

$$\mathrm{card}(F_j(Q)) \le c\,2^{s\cdot j}, \tag{2.11}$$

for some constant c depending only on s and C. Similarly, using again (2.8), for all $K \ge 1$ and all $Q \in \Delta_j$, $j \in \mathbb{J}$, there exist cubes $Q_1, \dots, Q_m \in \Delta_j$, not necessarily distinct, such that

$$KQ \subset \bigcup_{i=1}^{m} Q_i \subset K_0 Q \tag{2.12}$$

where $m \in \mathbb{N}$ and $K_0 > 1$ are constants depending only on s, C and K.

Finally, we note that combining (1) and (2) in Definition 2.5, it follows that

$$\sum_{Q' \in F_j(Q)} \mathcal{H}^s(Q') = \mathcal{H}^s(Q), \quad Q \in \Delta,\ j \in \mathbb{N} \cup \{0\}. \tag{2.13}$$

2.1.2. Geometric lemmas for various coefficient functions. Throughout the paper, we will encounter various coefficients that measure how well an s-regular set E satisfies a certain property at the scale and location of a given dyadic cube Q. We are mainly concerned with the question whether E fulfills a Carleson-type summability condition in the spirit of a *geometric lemma* for the given set of coefficients. We first introduce the notation for discussing these questions in a unified framework.

We let $\mathcal{B}(\mathrm{X})$ be the Borel σ-algebra of a metric space (X, d). For a closed set $E \subset \mathrm{X}$, the family $\{B \cap E : B \in \mathcal{B}(\mathrm{X})\}$ coincides with the Borel σ-algebra on E with respect to the topology induced by the metric $\mathsf{d}|_E$. We denote by $\mathcal{D}_s(E)$ the family of bounded Borel sets in E that have positive $\mathcal{H}^s$ measure. In particular, if E is s-regular and Δ a dyadic system on E, then $\Delta \subset \mathcal{D}_s(E)$ and also $KQ \in \mathcal{D}_s(E)$ for every $Q \in \Delta$ and $K > 1$.

Definition 2.14 (Geometric lemma). Given $p \in (0, \infty)$, $s > 0$, an s-regular set E in a metric space, $\mu := \mathcal{H}^s \lfloor_E$ and a function $h\colon \mathcal{D}_s(E) \to [0, 1]$, we say that E satisfies the *p-geometric lemma with respect to h*, and write $E \in \mathrm{GLem}(h, p)$, if there exists a constant M such that, for every dyadic system Δ on E, we have

$$\sum_{Q \in \Delta_{Q_0}} h(2Q)^p \mu(Q) \le M\mu(Q_0), \quad Q_0 \in \Delta. \tag{2.15}$$

In this case, we also write $E \in \mathrm{GLem}(h, p, M)$.

In practice, the function h will often arise as $h(S) := H(S \cap E)$, where H depends on the regularity exponent s of E, but is defined for a larger class of Borel sets of the ambient space X, see Examples 2.17–2.21 and Definition 2.31.

Remark 2.16. For many functions h of interest, and in particular for all the relevant ones appearing in this note, the condition "$E \in \mathrm{GLem}(\gamma, p)$" is equivalent to requiring (2.15) for a *specific* dyadic system $\Delta(E)$ on E, rather than for *all* possible such systems. See Lemma 2.23 and Remark 2.30 (or Remark 2.28 in [10]). A related statement for multi-resolution families (instead of systems of dyadic cubes) is Lemma B.1 in [9].

We next give two examples of geometric lemmas that have appeared in the literature. In the first one, the coefficient function h is a generalization of the classical β-numbers from Jones' traveling salesman theorem.

Example 2.17 (β-numbers). We recall the definition of the usual L^q-based β-numbers, that appeared already in (1.2):

$$\beta_{q,\mathcal{V}_k}(B_r(x) \cap E) := \inf_{V \in \mathcal{V}_k} \Big(\fint_{B_r(x)\cap E} \Big[\frac{\mathsf{d}(y, V)}{\mathrm{diam}(B_r(x) \cap E)}\Big]^q \,\mathrm{d}\mathcal{H}^k(y)\Big)^{1/q}, \quad q \in (0, \infty),$$

where E is a k-regular subset of $\mathbb{R}^n$ and $\mathcal{V}_k$ is the family of k-dimensional affine planes. We now generalize this notion to an arbitrary metric space (X, d) by considering, instead of planes, a general family $\mathcal{A}$ of (non-empty) subsets of X such that each point of X is contained in at least one element $A \in \mathcal{A}$. For $q \in (0, \infty]$, $s > 0$, a closed set $E \subset \mathrm{X}$ of locally finite $\mathcal{H}^s$-measure, and $\mu := \mathcal{H}^s \lfloor_E$, we then define for every $A \in \mathcal{A}$,

$$\beta_{q,A}(S) := \begin{cases} \Big(\frac{1}{\mu(S)} \int_S \Big[\frac{\mathsf{d}(y,A)}{\mathrm{diam}(S)}\Big]^q \,\mathrm{d}\mu(y)\Big)^{1/q}, & q \in (0, \infty), \\ \sup_{y \in S} \frac{d(y,A)}{\mathrm{diam}(S)}, & q = \infty, \end{cases} \quad \text{for } S \in \mathcal{D}_s(E)$$

and

$$\beta_{q,\mathcal{A}}(S) := \inf_{A \in \mathcal{A}} \beta_{q,A}(S). \tag{2.18}$$

This definition is typically applied if S is a "surface ball" $B_r(x) \cap E$ or a set of the form KQ for a dyadic cube Q on an s-regular set E. The definition of $\beta_{q,\mathcal{A}}(S)$, however, makes sense more generally, whenever S is a Borel set with $0 < \operatorname{diam}(S) < \infty$, and we will occasionally apply it in this sense. If E is s-regular, the condition $E \in \mathrm{GLem}(\beta_{q,\mathcal{A}}, p)$ is equivalent to the condition "$E \in \mathrm{GLem}(\mathcal{A}, p, q)$" stated in Definition 2.16 of [10]. The additional assumption on $\mathcal{A}$ is imposed to ensure that the function $\beta_{q,\mathcal{A}}$ takes values in $[0, 1]$, which will be convenient in the following. In practice, milder assumptions would often suffice. Finally, $\beta_{q,\mathcal{A}}$ of course depends on μ (respectively on the set E), but since this dependence will always be clear from the context, we do not indicate it in our notation.

The next example arises from the study of (quantitative) 1-rectifiability in metric spaces, and it involves exclusively 1-dimensional Hausdorff measures. The relevant coefficients can be thought of as 1-dimensional *metric β-numbers*, and they appeared with different notations in the literature. For the purpose of this paper, we will refer to them as *κ-numbers*, where κ is indicative of the connection to Menger curvature. Before stating the definition, we introduce some notation.

Given a metric space (X, d) and three points $x_1, x_2, x_3 \in \mathrm{X}$, we define the *triangular excess*

$$\begin{aligned} \partial(\{x_1, x_2, x_3\}) &:= \inf_{\sigma \in S_3} \left\{ \partial_1(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) \right\} \\ &:= \inf_{\sigma \in S_3} \left\{ \mathsf{d}(x_{\sigma(1)}, x_{\sigma(2)}) + \mathsf{d}(x_{\sigma(2)}, x_{\sigma(3)}) - \mathsf{d}(x_{\sigma(1)}, x_{\sigma(3)}) \right\}, \end{aligned} \tag{2.19}$$

where S_3 is the group of permutations of $\{1, 2, 3\}$. Note that $\partial(\{x_1, x_2, x_2\})$ depends only on the set $\{x_1, x_2, x_2\}$, while $\partial_1(x_1, x_2, x_3)$ takes into account also the order. If the three points lie at comparable distance from each other, then their triangular excess is related to their *Menger curvature* $c(x_1, x_2, x_3)$, as indicated by formula (2.20). Let $\{x_1', x_2', x_3'\}$ be the image of $\{x_1, x_2, x_3\}$ under an isometric embedding of the triple into the Euclidean plane. If x_1', x_2', x_3' are colinear, we define $c(x_1, x_2, x_3) = 0$, otherwise $c(x_1, x_2, x_3) = 1/R$, where R is the radius of the unique circle passing through x_1', x_2' and x_3'. With this definition, if (x_1, x_2, x_3) belongs to

$$\mathcal{F} := \{(x_1, x_2, x_3) \in E \times E \times E : \mathsf{d}(x_i, x_j) \le A\mathsf{d}(x_k, x_l), \text{ for all } i, j, k, l \in \{1,2,3\}, k \neq l\}$$

for some constant $A > 0$, then

$$c^2(x_1, x_2, x_3) \operatorname{diam}\{x_1, x_2, x_3\}^3 \sim_A \partial(\{x_1, x_2, x_3\}). \tag{2.20}$$

see Remark 2.3 in [56] or Remark 1.2 in [55].

Example 2.21 (κ-numbers). For $s > 0$ and a closed set $E \subset \mathrm{X}$ of locally finite $\mathcal{H}^1$-measure, and with $\mu := \mathcal{H}^1 \lfloor_E$, we define

$$\kappa(S) := \frac{1}{\mu(S)^3} \int_S \int_S \int_S \frac{\partial(\{x_1, x_2, x_3\})}{\operatorname{diam}(S)} \,\mathrm{d}\mu(x_1)\,\mathrm{d}\mu(x_2)\,\mathrm{d}\mu(x_3), \tag{2.22}$$

for $S \in \mathcal{D}_s(E)$. Conditions in the spirit of geometric lemmas for the coefficients κ and related rectifiability results have appeared in work of Hahlomaa and Schul [34–36,55–57].

For many applications, it is irrelevant whether the condition (2.15) in the definition of the geometric lemma is stated with constant "2" on the left-hand side or with any other constant $K > 1$. This is the case for the coefficient functions in Examples 2.17 and 2.21 (but also later in Definition 2.31), as the following result shows (see also Remark 2.30 below). This is akin to the situation in the case of Jones' traveling salesman theorem in $\mathbb{R}^n$, see Corollary 2.3 in [6].

Lemma 2.23 (Different neighborhoods of cubes). *Let $E \in \mathrm{Reg}_s(C)$, $\mu := \mathcal{H}^s\lfloor_E$, and let $h\colon \mathcal{D}_s(E) \to [0,1]$ be a function with the following property. For every $N \geq 1$, there exists a constant C_N such that the following monotonicity condition holds for all $A, B \in \mathcal{D}_s(E)$:*

$$A \subset B \text{ and } \mathrm{diam}(B)^s \leq N\mu(A) \implies h(A) \leq C_N h(B). \tag{2.24}$$

Then, for every $K > K_0 \geq 1$, there exists a constant $m = m(K_0, K, s, C)$ such that if Δ is a dyadic system on E and $Q_0 \in \Delta$, then

$$\sum_{Q \in \Delta_{Q_0}} h(KQ)^p\, \mu(Q) \lesssim \sum_{Q \in \Delta_{\hat{Q}_0}} h(K_0 Q)^p\, \mu(Q), \tag{2.25}$$

where the implicit constant depends only on K_0, K, s, C and p, while $\hat{Q}_0 \in \Delta$ is such that $Q_0 \subset \hat{Q}_0$ and $\ell(\hat{Q}_0) \leq 2^m \ell(Q)$. In particular, the validity of $\mathrm{GLem}(h, p)$ *does not depend on the choice of dyadic system.*

Proof. We assume first that E is unbounded. Let $K_0 \geq 1$ and fix a constant $K > K_0$. Then there exists $m = m(K_0, K, s, C) \in \mathbb{N}$ such that for every $Q \in \Delta$, there is a unique ancestor $\hat{Q} \in \Delta$ with

$$Q \subset \hat{Q}, \quad \ell(\hat{Q}) = 2^m \ell(Q), \quad \text{and} \quad KQ \subset K_0 \hat{Q}. \tag{2.26}$$

Indeed, for arbitrary $m \in \mathbb{N}$, there clearly exists $\hat{Q} \in \Delta$ satisfying the first two conditions in (2.26), and for such $\hat{Q}$, it follows for all $y \in KQ$ that

$$\begin{aligned} \mathrm{dist}(y, \hat{Q}) &\leq \mathrm{dist}(y, Q) \leq (K-1)\,\mathrm{diam}(Q) \leq (K-1)\, c_0^{-1}\, \ell(Q) = (K-1)\, c_0^{-1}\, 2^{-m}\, \ell(\hat{Q}) \\ &\overset{(2.8)}{\leq} (K-1)\, C^{2/s}\, c_0^{-2}\, 2^{-m}\, \mathrm{diam}(\hat{Q}), \end{aligned}$$

which shows that m can be made large enough, depending only on s, C (also via c_0), K_0 and K, such that also the third condition in (2.26) is satisfied. Once m is fixed, $\hat{Q}$ is uniquely determined according to condition (3) in Theorem and Definition 2.5.

Since the coefficient function h has the monotonicity property (2.24), and since $KQ \subset K_0\hat{Q}$ and $\mathrm{diam}(K_0\hat{Q})^s \lesssim_{s,C,K_0,K} \mathcal{H}^s(Q)$ by (2.26) and (2.6), it follows that

$$h(KQ) \lesssim_{s,C,K,K_0} h(K_0\hat{Q}). \tag{2.27}$$

From (2.26) and (2.27), we then easily deduce for every $Q_0 \in \Delta$ that

$$\begin{aligned} \sum_{Q \in \Delta_{Q_0}} h(KQ)^p \mu(Q) &\lesssim \sum_{Q \in \Delta_{Q_0}} h(K_0\hat{Q})^p \mu(Q) \leq \sum_{Q \in \Delta_{Q_0}} h(K_0\hat{Q})^p \mu(\hat{Q}) \\ &\lesssim \sum_{Q \in \Delta_{\hat{Q}_0}} h(K_0 Q)^p \mu(Q), \end{aligned}$$

where the implicit constants may depend on all the parameters s, C, K, K_0, and p. In the last inequality, we used the fact that for every $Q' \in \Delta$, there are only $\lesssim_{m,s,C}$ many $Q \in \Delta_{Q'}$ such that $\ell(Q') = 2^m \ell(Q)$ (see (2.11)).

If E is bounded, then there still exists a constant $m = m(K_0, K, s, C)$ such that (2.26) holds, but only for $Q \in \Delta_j$ with $j \geq n + m$ (recall that we have defined Δ_j for $j \geq n$, where $2^{-n} \leq \operatorname{diam}(E) < 2^{-n+1}$). If $Q_0 \in \Delta_j$ for some $j < n + m$, then we can conclude exactly as we did in the case of unbounded sets E. If, on the other hand, $Q_0 \in \Delta_j$ for some $j > n + m$, then we define $\hat{Q}_0$ to be the unique cube in Δ_n with $\hat{Q}_0 \supseteq Q_0$. Then we split the relevant sum as follows:

$$\sum_{Q \in \Delta_{Q_0}} h(KQ)^p \mu(Q) = \sum_{Q \in \Delta_{Q_0} \cap [\bigcup_{j \geq n+m} \Delta_j]} h(KQ)^p \mu(Q) + \sum_{Q \in \Delta_{Q_0} \cap [\bigcup_{j < n+m} \Delta_j]} h(KQ)^p \mu(Q). \tag{2.28}$$

The first sum on the right-hand side can be bounded by $\sum_{Q \in \Delta_{\hat{Q}_0}} h(K_0 Q)^p \mu(Q)$ using the same computations as in the case of unbounded sets E, observing that $Q \in \Delta_{Q_0} \cap [\bigcup_{j \geq n+m} \Delta_j]$ implies $\hat{Q} \subset \hat{Q}_0$ with the new definition of $\hat{Q}$. It remains to control the second sum on the right-hand side of (2.28). Since $|h|$ is assumed to take values in $[0, 1]$, we can simply bound this by $\sum_{Q \in \Delta_{Q_0} \cap [\bigcup_{j < n+m} \Delta_j]} \mu(Q)$ and use that the sum runs over at most m generations. Coupled with (2.13), this yields the claim.

To show that $\operatorname{GLem}(h, p)$ is independent of the dyadic system, fix two dyadic systems Δ and $\tilde{\Delta}$, and assume that $\operatorname{GLem}(h, p)$ holds for $\tilde{\Delta}$. For all $Q \in \Delta_j$, there exists $\tilde{Q} \in \tilde{\Delta}_j$ such that $2Q \subset k\tilde{Q}$, where $k > 1$ is a constant depending only on s and C. Fix any $Q_0 \in \Delta$ and note that for all $Q \in \Delta_{Q_0}$, it holds $\tilde{Q} \subset k_0 Q_0 \subset k_0 k \tilde{Q}_0$, with $k_0 > 1$ a constant depending only on s and C. Hence using (2.24) we obtain that

$$\sum_{Q \in \Delta_{Q_0}} h(2Q)^p \mu(Q) \lesssim_{s,C} \sum_{Q \in \Delta_{Q_0}} h(k\tilde{Q})^p \mu(Q) \lesssim_{s,C} \sum_{Q' \in \tilde{\Delta},\, Q' \subset k_0 k \tilde{Q}_0} h(kQ')^p \mu(Q'), \tag{2.29}$$

where similarly as above, we used that for all $Q' \in \Delta$ there exists $\lesssim_{s,C}$ cubes $Q \in \Delta$ such that $\tilde{Q} = Q'$. From inequality (2.29) applying (2.12) and then (2.25), we easily obtain that $\operatorname{GLem}(h, p)$ holds for Δ. ∎

Remark 2.30. The coefficients of both Example 2.17 and Example 2.21 satisfy assumption (2.24) of the above lemma. To see this, let $E \in \operatorname{Reg}_s(C)$ and $A, B \in \mathcal{D}_s(E)$ satisfy $A \subset B$ and $\operatorname{diam}(B)^s \leq N\mu(A)$ for some constant $N \geq 1$. Then by the s-regularity of E, we have

$$\frac{1}{\mu(A)} \leq \frac{N}{\operatorname{diam}(B)^s} \leq \frac{CN}{\mu(B)} \quad \text{and} \quad \operatorname{diam}(A) \geq C^{-1/s} \mu(A)^{1/s} \geq (NC)^{-1/s} \operatorname{diam}(B).$$

This clearly shows the validity of (2.24) for $h \in \{\beta_{q,\mathcal{A}}, \kappa\}$.

2.2. New coefficients to measure flatness in metric spaces

We introduce new coefficients to measure flatness of a set in a metric space. Here, *flatness* means roughly speaking the existence of approximate isometric embeddings of the set into $\mathbb{R}^k$, in an L^q-sense. The natural measures to use in this definition are k-dimensional Hausdorff measures, for integer-valued k.

Definition 2.31. For $k \in \mathbb{N}$, a closed set $E \subset \mathrm{X}$ of locally finite $\mathcal{H}^k$-measure, and $\mu := \mathcal{H}^k\lfloor_E$, we define

$$\iota_{q,k}(S) := \inf_{\|\cdot\| \text{ norm on } \mathbb{R}^k} \inf_{f:S\to\mathbb{R}^k} \Big(\frac{1}{\mu(S)^2}\int_S\int_S \Big[\frac{|\mathsf{d}(x,y)-\|f(x)-f(y)\||}{\operatorname{diam}(S)}\Big]^q \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\Big)^{1/q}, \tag{2.32}$$

for $S \in \mathcal{D}_s(E)$, where the functions f in the second infimum are assumed to be Borel. Moreover, we define the number $\iota_{q,k,\mathrm{Eucl}}(S)$ by considering in the infimum (2.32) only the Euclidean norm $\|\cdot\|_{\mathrm{Eucl}}$ (which we often denote simply by $|\cdot|$).

The numbers $\iota_{q,k}$ can be interpreted as an *L^q-unilateral version of the Gromov–Hausdorff β-numbers* from Definition 3.1.3 in [8]. The function $\iota_{q,k}$ associated to a k-regular set E clearly enjoys the monotonicity property required in Lemma 2.23 (see Remark 2.30) and hence, again, for the purpose of this paper, the constant "2" in the definition of "$E \in \mathrm{GLem}(\iota_{q,k}, p)$" could be replaced by any constant $K_0 > 1$, and the validity of $\mathrm{GLem}(\iota_{q,k}, p)$ is independent of the choice of dyadic system.

Remark 2.33. In a companion paper, [32], we consider a variant of the ι-numbers that is more suitable for comparison with the β-numbers from Example 2.17 in Euclidean spaces. In particular, we define coefficients $\iota_{1,\mathcal{V}_k}$ using orthogonal projections onto k-dimensional affine planes and we prove that k-regular set $E \subset \mathbb{R}^n$ is uniformly k-rectifiable if and only if $E \in \mathrm{GLem}(\iota_{1,\mathcal{V}_k}, 1)$. We refer to [32] for the definition of $\iota_{1,\mathcal{V}_k}$, but for illustrative purposes, we mention that for subsets of Euclidean spaces, already the numbers $\iota_{q,k,\mathrm{Eucl}}$ and $\iota_{q,k}$ are related to affine k-planes, as Propositions 2.34 and 2.36 below show (for the definition of the numbers $\beta_{q,\mathcal{V}_k}$, recall (1.2) or Example 2.17).

Proposition 2.34. *Let $E \in \mathrm{Reg}_k(C)$ be a k-regular subset of $\mathbb{R}^n$, where $k \in \mathbb{N}$ and $k < n$, and let Δ be a system of dyadic cubes for E. Then for all $q \in [1,\infty)$ and all $Q \in \Delta$, it holds*

$$\beta^2_{q,\mathcal{V}_k}(2Q) \le \beta^2_{2q,\mathcal{V}_k}(2Q) \le \tilde{C}\,\iota_{q,k,\mathrm{Eucl}}(2Q), \tag{2.35}$$

where $\tilde{C}$ is a constant depending only on k, p and C.

Proposition 2.36. *Let $E \in \mathrm{Reg}_k(C)$ be a k-regular subset of $\mathbb{R}^n$, where $k \in \mathbb{N}$ and $k \le n$, and let Δ be a system of dyadic cubes for E. Suppose that for some $Q \in \Delta$ and $q \in [1,\infty)$, it holds*

$$\iota_{q,k}(Q) = 0.$$

Then $\iota_{q,k,\mathrm{Eucl}}(Q) = \beta_{q,\mathcal{V}_k}(Q) = 0$. In particular, up to a $\mathcal{H}^k$-zero measure set, Q is contained in a k-dimensional plane.

Propositions 2.34 and 2.36 are proven in Section 3.1 of [32].

3. Sufficient conditions for the existence of regular covering curves

This section aims at clarifying several technical points regarding the question when a 1-regular set E in a complete, doubling and quasiconvex metric space is contained in a regular curve Γ_0. This discussion is partially motivated by an application in Theorem 1.3 of [13] about 1-dimensional singular integral operators in the Heisenberg group, and it will be employed also in Section 4 to state and characterize a notion of *uniform* 1*-rectifiability* in a large class of metric spaces.

In Section 3.1, we explain how the existence of the covering curve Γ_0 for a set E can be derived from certain quantitative *local* information on E. We will first prove the *bounded* case in Theorem 3.1 and Corollary 3.2 and then in Section 3.1.1 we present a covering lemma that allows to extend this result to the *unbounded* case. Finally, in Section 3.2, these results are applied to construct a covering by a 1-regular connected set, based on integral bounds on the Menger curvature. This proves and generalizes to quasiconvex metric spaces a result first stated in [56]. Finally, we review an application of this result in the Heisenberg group.

3.1. Construction of 1-regular covering continua

In this section, we consider conditions for a bounded set to be contained in a 1-regular *curve*. Actually it will be sufficient to show that the set is contained in a *closed connected* 1*-regular set of finite length*. Indeed, if (X, d) is a complete metric space, and $\Gamma \subset \mathrm{X}$ a closed connected subset of finite $\mathcal{H}^1$ measure, then Γ is automatically a compact Lipschitz *curve* by Lemma 2.8 in [4], see also Lemmas 2.2–2.3 in [55]. Moreover, if we assume in addition that Γ is a 1-regular set, then under the previous assumptions, it will be automatically a 1*-regular curve* (in the sense of Section 1.1 of [55]) by Lemma 2.3 in [55].

Theorem 3.1 (From local covering by continua to global 1-regular covering). *Let (X, d) be a complete, doubling, and quasiconvex metric space. Assume that $K \subset X$ is a bounded set with the following property. There exists a constant $C > 0$ such that, for every $x \in K$ and $0 < r \leq \mathrm{diam}(K)$, there is a closed connected set $\Gamma_{x,r} \subset X$ with $\Gamma_{x,r} \supseteq K \cap B_r(x)$ and $\mathcal{H}^1(\Gamma_{x,r}) \leq Cr$. Then there exists a (closed) connected set $\Gamma_0 \in \mathrm{Reg}_1(C_0)$, with C_0 depending only on the doubling and quasiconvexity constants of (X, d) and on C, such that*

$$\Gamma_0 \supseteq K \quad \textit{and} \quad \mathcal{H}^1(\Gamma_0) \leq C\,\mathrm{diam}(K).$$

Moreover, the set Γ_0 may be chosen such that

$$\mathcal{H}^1(\Gamma_0) = \min\{\mathcal{H}^1(\Gamma) : \Gamma \textit{ closed and connected, and } \Gamma \supseteq K\}.$$

From Theorem 3.1 above and the extension property of Lipschitz maps, we immediately infer the following result.

Corollary 3.2 (From local covering by Lipschitz images to global 1-regular covering). *Let (X, d) be a complete, doubling, and quasiconvex metric space. Assume that $K \subset X$ is a bounded set with the following property. There exists a constant $C > 0$ such that, for every $x \in K$ and $0 < r \leq \mathrm{diam}(K)$, there are a set $A_{x,r} \subset [0, 1]$ and a surjective Cr-Lipschitz map $f_{x,r}\colon A_{x,r} \to K \cap B_r(x)$. Then there exists a (closed) connected set $\Gamma_0 \in \mathrm{Reg}_1(C_0)$,* *with C_0 depending only on the doubling and quasiconvexity constants of* (X,d) *and on* C*, such that*

$$\Gamma_0 \supseteq K \quad \textit{and} \quad \mathcal{H}^1(\Gamma_0) \le C'\mathrm{diam}(K),$$

where C' depends only on C and the quasiconvexity constant of (X,d)*. Moreover, the set Γ_0 may be chosen such that*

$$\mathcal{H}^1(\Gamma_0) = \min\{\mathcal{H}^1(\Gamma) : \Gamma \textit{ closed and connected, and } \Gamma \supseteq K\}.$$

Proof of Corollary 3.2 using Theorem 3.1. Let (X,d) and K be as in the statement of Corollary 3.2. Since (X,d) is complete and quasiconvex, the pair $(\mathbb{R},\mathrm{X})$ has the Lipschitz extension property, see for instance [48]. In particular, there exists a constant $C' \ge 1$, depending only on the given $C > 0$ and the quasiconvexity constant of (X,d), such that every Cr-Lipschitz map $f_{x,r}\colon A_{x,r} \to K \cap B_r(x)$ as in the assumptions of the corollary can be extended to a $C'r$-Lipschitz map $F_{x,r} : ([0,1],|\cdot|) \to (\mathrm{X},\mathsf{d})$ with $F_{x,r}|_{A_{x,r}} = f_{x,r}$. Then K satisfies the assumptions of Theorem 3.1 with $\Gamma_{x,r} = F_{x,r}([0,1])$ and constant C'. Thus Corollary 3.2 follows from Theorem 3.1. ∎

Actually Theorem 3.1, and hence also Corollary 3.2, hold without the boundedness assumption on K. We will prove this in the next Subsection 3.1.1 (see in particular Corollary 3.14), since it will be first necessary to prove independently the bounded case.

The proof of Theorem 3.1 is based on an idea which we learned from Tuomas Orponen in the case $(\mathrm{X},\mathsf{d}) = (\mathbb{R}^2,|\cdot|)$, see Exercise 1.6 in [52]. Similar ideas have been used in [19], p. 197 ff. The strategy is to show that there exists a closed connected set Γ_0 *of minimal $\mathcal{H}^1$ measure* containing K, and then to prove that this Γ_0 must in fact be 1-regular. Before explaining the details, we list the key ingredients needed to run the argument in metric spaces. We will apply the following result, which follows from versions of Blaschke's theorem and Gołąb's theorem in metric spaces. For comments and a proof of Gołąb's theorem in this setting, see also p. 840 and p. 846 of [53].

Theorem 3.3 (Existence result; Theorem 4.4.20 in [2]). *Let* (X,d) *be a proper metric space. Suppose that $K \subset X$ is non-empty, and assume that there exists a closed connected set Γ in X such that $\mathcal{H}^1(\Gamma) < \infty$ and $\Gamma \supseteq K$. Then the minimum problem*

$$\min\{\mathcal{H}^1(\Gamma) : \Gamma \textit{ closed and connected, and } \Gamma \supseteq K\}$$

has a solution.

We recall that a *proper* metric space is one in which all closed balls are compact, and that every proper metric space is complete, and every complete and doubling metric space is proper.

Theorem 3.3 will be used to show the existence of a minimal-length covering continuum Γ_0 for the set K in Theorem 3.1. To prove 1-regularity of Γ_0, we will argue by contradiction and construct a covering continuum Γ_0' of smaller length in the hypothetical scenario that 1-regularity of Γ_0 fails. This construction of Γ_0' proceeds by locally modifying Γ_0 using the assumption on K. The only step which is trickier in our setting than in the case $\mathrm{X} = \mathbb{R}^2$ is to maintain connectedness of the modified set. In $\mathbb{R}^2$, this can be achieved simply by adding a suitable circle (and possibly a line segment). Our construction instead uses the following observation.

Proposition 3.4 (Short paths connecting points in quasiconvex doubling spaces). *Assume that (X, d) is a quasiconvex metric space which is doubling with constant $D \geq 1$. Then there exists a constant $\lambda > 0$ (depending on the doubling and quasiconvexity constants) such that for all $x \in X$ and $r > 0$, the following holds. Every finite set $P \subset \overline{B_r(x)}$ is contained in a closed connected set Γ with*

$$\mathcal{H}^1(\Gamma) \leq \lambda r \operatorname{card}(P)^{(2D-1)/2D}. \tag{3.5}$$

Proof. Without loss of generality, we may assume that $\operatorname{card}(P) \geq 2^{2D}$. Since (X, d) is doubling with constant $D \geq 1$, the ball $B_r(x)$ can be covered by balls

$$B_j = B_{r \operatorname{card}(P)^{-1/2D}}(x_j), \quad j = 1, \dots, N,$$

with bounded cardinality and overlap:

$$N \lesssim \operatorname{card}(P)^{1/2} \quad \text{and} \quad \sup_{y \in \mathrm{X}} \operatorname{card}(\{i \in \{1, \dots, N\} : y \in B_i\}) \lesssim_D 1, \tag{3.6}$$

recall Definition 2.2 and the subsequent comment. It will become clear at the end of the proof why the radii of the balls B_j were chosen as above. At this point, we just note that they are of the form εr, with $\varepsilon \in (0, 1/2]$.

Up to removing unnecessary balls, we can assume that $B_j \cap B_r(x) \neq \emptyset$ for all j. To proceed, we connect every x_j, $j = 2, \dots, N$, to x_1 by a curve of length $\lesssim r$, using the quasiconvexity of (X, d) and the fact that $\mathsf{d}(x_1, x_j) \leq 4r$. The union of these curves is a closed connected set Γ_0 with

$$\mathcal{H}^1(\Gamma_0) \lesssim rN \lesssim r \operatorname{card}(P)^{1/2}.$$

Second, for every $j = 1, \dots, N$, each point in $P \cap B_j$ can be connected to the center x_j of the ball B_j by a curve of length $\lesssim r \operatorname{card}(P)^{-1/2D}$, again thanks to quasiconvexity. Thus the points $P \cap B_j$ can be connected to x_j by a closed connected set Γ_j of total measure

$$\mathcal{H}^1(\Gamma_j) \lesssim \operatorname{card}(P \cap B_j)\, r \operatorname{card}(P)^{-1/2D}.$$

Moreover,

$$\mathcal{H}^1\Big(\bigcup_{j=1}^N \Gamma_j\Big) \leq \sum_{j=1}^N \mathcal{H}^1(\Gamma_j) \lesssim r \operatorname{card}(P)^{-1/(2D)} \sum_{j=1}^N \operatorname{card}(P \cap B_j) \lesssim_D r \operatorname{card}(P)^{1-1/(2D)},$$

where we have used the controlled overlap of the balls B_j, $j = 1, \dots, N$, in the last inequality, as quantified in (3.6). Finally, $\Gamma = \Gamma_0 \cup (\bigcup_{j=1}^n \Gamma_j)$ is a covering continuum for P with the desired property (3.5) since

$$\frac{1}{2} \leq 1 - \frac{1}{2D} \quad \text{for } D \geq 1. \qquad \blacksquare$$

We are now ready to prove the main theorem of this section.

Proof of Theorem 3.1. Let (X,d) and K be as in the statement of Theorem 3.1. Applying the assumption to a point $x \in K$ and $r = \operatorname{diam}(K)$, we find a closed connected set $\Gamma \supset K$ with $\mathcal{H}^1(\Gamma) \leq C \operatorname{diam}(K) < \infty$. Since every complete and doubling metric space is proper, we can the apply Theorem 3.3 to deduce that there exists a closed and connected set $\Gamma_0 \supseteq K$ with smallest $\mathcal{H}^1$ measure among all such sets. Being connected, the set Γ_0 is automatically *lower* 1-regular with a universal constant, that is,

$$\mathcal{H}^1(\Gamma_0 \cap B_r(x)) \gtrsim r, \quad x \in \Gamma_0,\ 0 < r \leq 2 \operatorname{diam}(\Gamma_0),$$

see for instance Lemma 4.4.5 in [2]. Hence it suffices to prove that Γ_0 is also *upper* 1-regular.

To this end, fix a constant $C_0 > 2C$. The precise value of C_0 will be determined later. Towards a contradiction, we assume that Γ_0 fails to be upper 1-regular with constant C_0, that is, $\Gamma_0 \notin \operatorname{Reg}_1^+(C_0)$. Thus there exists $x_0 \in \Gamma_0$ and $0 < r \leq \operatorname{diam}(\Gamma_0)$ such that

$$\mathcal{H}^1(\Gamma_0 \cap B_r(x_0)) > C_0 r. \tag{3.7}$$

(If upper Ahlfors regularity fails, it has to fail for a radius $r \leq \operatorname{diam}(\Gamma_0)$.) We want to work with the essentially largest radius with this property. To be more precise, we set

$$\begin{aligned} r_0 &:= \sup\{s \in [r, \operatorname{diam}(\Gamma_0)] : \mathcal{H}^1(\Gamma_0 \cap B_s(x_0)) > C_0 s\} \\ &= \sup\{s \in [r, \operatorname{diam}(\Gamma_0)] : q(s) > C_0\}, \end{aligned}$$

where

$$q(s) := \frac{\mathcal{H}^1(\Gamma_0 \cap B_s(x_0))}{s}, \quad s \in [r, \operatorname{diam}(\Gamma_0)].$$

Clearly, $q(r) > C_0$ and $q(s) < C_0$ for $s \in [\operatorname{diam}(\Gamma_0)/2, \operatorname{diam}(\Gamma_0)]$ since

$$\mathcal{H}^1(\Gamma_0 \cap B_s(x_0)) \leq \mathcal{H}^1(\Gamma_0) \leq 2\, \frac{C \operatorname{diam}(K)}{\operatorname{diam}(\Gamma_0)}\, \frac{1}{2} \operatorname{diam}(\Gamma_0) < C_0\, \frac{1}{2} \operatorname{diam}(\Gamma_0)$$

for all s, by the minimality property of Γ_0 and the existence of $\Gamma \supset K$ with $\mathcal{H}^1(\Gamma) \leq C \operatorname{diam}(K)$. It follows that $r \leq r_0 < \operatorname{diam}(\Gamma_0)/2$ (and thus $2r_0 < \operatorname{diam}(\Gamma_0)$).

We will now locally modify Γ_0 at $B_{r_0}(x_0)$ to construct a closed connected $\Gamma_0' \supseteq K$ with $\mathcal{H}^1(\Gamma_0') < \mathcal{H}^1(\Gamma_0)$. This will contradict the minimality property of Γ_0 and thus show that the counter assumption on the existence of a ball as in (3.7) cannot be true. Thus Γ_0 must in fact be upper 1-regular with constant C_0.

We now explain the modification of Γ_0 locally around $B_{r_0}(x_0)$. To ensure connectedness of the modified set Γ_0', we will apply Proposition 3.4. Hence we would like to consider a ball $B_\rho(x_0)$ with $\rho \sim r_0$ such that we have a suitable upper bound on $\operatorname{card}(\Gamma_0 \cap \partial B_\rho(x_0))$. For this purpose, we apply the coarea (Eilenberg) inequality (Theorem 1 in [28]) to the 1-Lipschitz function $f\colon (\mathsf{X},\mathsf{d}) \to \mathbb{R}$ given by $f(x) := \mathsf{d}(x, x_0)$. Then

$$\begin{aligned} \int^*_{[r_0,2r_0]} \mathcal{H}^0(\Gamma_0 \cap \partial B_s(x_0))\, ds &\leq \int^*_{\mathbb{R}} \mathcal{H}^0([\Gamma_0 \cap B_{2r_0}(x_0) \setminus B_{r_0}(x_0)] \cap f^{-1}(\{s\}))\, ds \\ &\leq \mathcal{H}^1(\Gamma_0 \cap B_{2r_0}(x_0) \setminus B_{r_0}(x_0)) \leq C_0 r_0, \end{aligned}$$

where we used the maximality property of r_0 in the last step. It follows that there must exists $\rho \in [r_0, 2r_0]$ such that

$$\mathcal{H}^0(\Gamma_0 \cap \partial B_\rho(x_0)) \leq C_0.$$

We next apply Proposition 3.4 to the point set

$$P := \Gamma_0 \cap \partial B_\rho(x_0).$$

Since $\rho < \mathrm{diam}(\Gamma_0)$, $x_0 \in \Gamma_0$ and Γ_0 is connected, P contains clearly at least one point. Proposition 3.4 allows us to find a closed connected set Γ_P in (X, d) with

$$\Gamma_P \supseteq P \quad \text{and} \quad \mathcal{H}^1(\Gamma_P) \leq \lambda\, \rho (C_0)^{(2D-1)/(2D)} = \lambda\, C_0^{-1/(2D)}\, C_0\, \rho, \tag{3.8}$$

where λ depends only on the doubling and quasiconvexity constants of (X, d).

The set $[\Gamma_0 \setminus B_\rho(x_0)] \cup \Gamma_P$ is connected by construction. If $K \cap B_\rho(x_0)$ is empty, there is nothing further to be done, but otherwise, we have to enlarge our continuum in order to cover $K \cap B_\rho(x_0)$ as well. The assumption of Theorem 3.1 allows us to find a (possibly empty) closed connected set $\Gamma_{x_0,\rho}$ such that

$$\Gamma_{x_0,\rho} \supseteq K \cap B_\rho(x_0) \quad \text{and} \quad \mathcal{H}^1(\Gamma_{x_0,\rho}) \leq 2C\rho. \tag{3.9}$$

By quasiconvexity, it is further possible to connect Γ_P and $\Gamma_{x_0,\rho}$ by a curve $\Gamma_{P,x_0,\rho}$ with

$$\mathcal{H}^1(\Gamma_{P,x_0,\rho}) \lesssim \rho. \tag{3.10}$$

If $\Gamma_{x_0,\rho} = \emptyset$, we simply put $\Gamma_{P,x_0,\rho} := \emptyset$. By construction, we know that

$$\mathcal{H}^1(\Gamma_0 \cap B_\rho(x_0)) \geq \mathcal{H}^1(\Gamma_0 \cap B_{r_0}(x_0)) = C_0\, r_0 \geq \frac{C_0}{2}\, \rho. \tag{3.11}$$

Hence it is clear that we can choose the constant C_0 large enough (depending only on the doubling and quasiconvexity constants of (X, d) and on C) such that the upper bounds for the $\mathcal{H}^1$ measure in (3.8)–(3.10) are each less than $C_0\rho/6$, so that

$$\mathcal{H}^1(\Gamma_P \cup \Gamma_{x_0,\rho} \cup \Gamma_{P,x_0,\rho}) < \frac{C_0}{2}\, \rho \leq \mathcal{H}^1(\Gamma_0 \cap B_\rho(x_0)). \tag{3.12}$$

Since

$$\Gamma_0' := [\Gamma_0 \setminus B_\rho(x_0)] \cup \Gamma_P \cup \Gamma_{x_0,\rho} \cup \Gamma_{P,x_0,\rho}$$

is a closed connected set of smaller $\mathcal{H}^1$ measure than Γ_0, we have reached a contradiction with the minimality property of Γ_0. Thus the counter assumption cannot hold and in fact $\Gamma_0 \in \mathrm{Reg}_1^+(C_0)$, and eventually, $\Gamma_0 \in \mathrm{Reg}_1(C_0')$, where C_0' the maximum of the lower and upper regularity constants of Γ_0. ∎

3.1.1. The unbounded case. Next we give a sufficient criterion ensuring that an unbounded set which can be locally covered by connected 1-regular sets, can be itself covered by a connected 1-regular set. The argument is inspired by [19], p. 202 ff.

Proposition 3.13. *Let (X, d) be a quasiconvex and doubling metric space and let $t_0 > 1$ be any constant. Let $E \subset X$ be a set such that for every $x \in E$ and $r > 0$ there exists a connected set $\Gamma_{x,r} \in \mathrm{Reg}_1(C)$, satisfying $B_r(x) \cap E \subset \Gamma_{x,r} \subset B_{t_0 r}(x)$ and where $C > 0$ is a constant independent of x and r. Then E is contained in a connected set $\Gamma_0 \in \mathrm{Reg}_1(\tilde{C})$, where $\tilde{C}$ is a constant depending only on t_0, C and the doubling and quasiconvexity constants of (X, d).*

First we observe that Proposition 3.13 allows immediately to improve the statements of Theorem 3.1 and Corollary 3.2 to the unbounded case.

Corollary 3.14. *Theorem 3.1 and Corollary 3.2 hold true also for unbounded sets K (if the respective assumptions are satisfied for all $0 < r < \infty$).*

Proof. We only need to remove the boundedness assumption from Theorem 3.1, then it would be automatically removed also from Corollary 3.2, which is deduced from it. Let $K \subset \mathrm{X}$ be a set satisfying the hypotheses of Theorem 3.1 with constant C, except that it is unbounded. For every $x \in K$ and $r > 0$, the set $K' := B_r(x) \cap K$ satisfies all the assumptions of Theorem 3.1, being a bounded subset of K. Hence by Theorem 3.1 (in the bounded case), we deduce that $B_r(x) \cap K$ is contained in a closed connected set $\Gamma_{x,r} \in \mathrm{Reg}_1(C_0)$, where C_0 is a constant depending only on C and the doubling and quasiconvexity constants of (X, d), and such that $\mathcal{H}^1(\Gamma_{x,r}) \leq 2Cr$. This and the 1-regularity show $\mathrm{diam}(\Gamma_{x,r}) \leq 2C_0 C r$, so that $\Gamma_{x,r} \subset B_{2C_0 C r}(x)$. Hence the assumptions of Proposition 3.13 with $E = K$ are satisfied taking $t_0 = 2CC_0$, which concludes the proof. ∎

Theorem 3.1, in the bounded case, will be needed for the proof of Proposition 3.13, which is why we did not prove that theorem directly in the full version. The main technical tool needed for the proof of Proposition 3.13 is the following covering lemma.

Lemma 3.15. *Let (X, d) be a doubling metric space, fix $x_0 \in X$ and let $t > 1$ be any constant. Then there exist a (countable) family of balls $\mathcal{B}$ with radius ≥ 1 and a constant $M \geq 2$ depending only on t and the doubling constant of (X, d) such that the following hold:*

(i) *the family $\mathcal{B}$ covers X and the covering $\{tB\}_{B \in \mathcal{B}}$ has multiplicity less than M,*

(ii) *for every $R \geq 1$*
$$\sum_{B \in \mathcal{B},\, tB \cap B_R(x_0) \neq \emptyset} r(B) \leq MR,$$
where $r(B)$ denotes the radius of B,

(iii) *for every $B_r(x) \in \mathcal{B}$, it holds that $d(x, x_0) \leq Mr$,*

(iv) *$\#\{B' \in \mathcal{B} : tB' \cap tB \neq \emptyset\} \leq M$, for every $B \in \mathcal{B}$.*

Proof. Fix $x_0 \in \mathrm{X}$, $t > 1$ and a constant $\lambda > 2t$. We construct the family $\mathcal{B}$ as union of families $\mathcal{B}_k$, $k \in \mathbb{N} \cup \{0\}$, of balls defined as follows. Set $\mathcal{B}_0 := \{B_1(x_0)\}$. For every $k \in \mathbb{N}$, denote $A_k := B_{\lambda^k}(x_0) \setminus B_{\lambda^{k-1}}(x_0)$ and consider a set $\mathcal{F}_k \subset A_k$ such that $d(x, y) \geq \lambda^{k-3}$ for every distinct $x, y \in \mathcal{F}_k$ and assume that $\mathcal{F}_k$ is a maximal set with this property. Set $\mathcal{B}_k := \{B_{\lambda^{k-3}}(x)\}_{x \in \mathcal{F}_k}$ and $\mathcal{B} := \bigcup_{k=0}^{\infty} \mathcal{B}_k$. By construction $\mathcal{B}$ is a covering of X and (iii) holds provided $M \geq \lambda^3$. Since the space (X, d) is doubling and $\mathrm{diam}(A_k) \leq 2\lambda^k$, the set A_k can be covered by $\leq 4^D \lambda^{3D}$ sets of diameter at most $\lambda^{k-3}/2$, where $D > 0$ is the doubling constant of (X, d) (recall Definition 2.2). Since each of these sets intersects at most one of the centers of the balls in $\mathcal{B}_k$, we deduce that

$$\#\mathcal{B}_k \leq 4^D \lambda^{3D}. \tag{3.16}$$

By construction the balls $\mathcal{B}_k$ cover the annulus A_k and again by the doubling property of (X, d), such covering has multiplicity $\leq M$, provided M is big enough depending only on the doubling constant of (X, d). Next we observe that, by the triangle inequality, for every $B \in \mathcal{B}_k$ and $x \in tB$, it holds that

$$\begin{aligned}\lambda^{k-2} < \lambda^{k-2}\Big(\lambda - \frac{t}{\lambda}\Big) \leq \lambda^{k-1} - t\lambda^{k-3} \leq d(x, x_0) \leq \lambda^k + t\lambda^{k-3}\\ \leq \lambda^{k+1}\Big(\frac{1}{\lambda} + \frac{t}{\lambda^4}\Big) < \lambda^{k+1},\end{aligned}$$

where in the first and last inequality we used that $\lambda > 2t > 2$. In particular, $tB \subset A_{k-1} \cup A_k \cup A_{k+1}$ for every $B \in \mathcal{B}_k$ and $k \geq 2$. These observations together with (3.16) show at once both (i) and (iv) for suitably chosen M depending only on D. It remains to show (ii). Fix $R \geq 1$ and $k \in \mathbb{N}$ such that $R \in [\lambda^{k-1}, \lambda^k)$. By what we just observed,

$$\{B \in \mathcal{B} : tB \cap B_R(x_0) \neq \emptyset\} \subset \{B \in \mathcal{B} : tB \cap B_{\lambda^k}(x_0) \neq \emptyset\} \subset \bigcup_{i=0}^{k+1} \mathcal{B}_i.$$

Therefore,

$$\begin{aligned}\sum_{B \in \mathcal{B},\, tB \cap B_R(x_0) \neq \emptyset} r(B) \leq \sum_{i=0}^{k+1} \sum_{B \in \mathcal{B}_i} r(B) \overset{(3.16)}{\leq} 4^D \lambda^{3D}\Big(1 + \sum_{i=1}^{k+1} \lambda^{i-3}\Big)\\ \leq 4^D \lambda^{3D}(1 + \lambda^{k-1}) \leq 4^D \lambda^{3D}\, 2R,\end{aligned}$$

which proves (ii) for suitable M. Combining what we said so far, M can be chosen large enough, depending only on t and the doubling constant of (X, d), such that the conditions (i)–(iv) hold. ■

The covering of balls given by Lemma 3.15 is useful thanks to the following fact.

Lemma 3.17. *Fix (X, d) a doubling metric space, $x_0 \in X$ and $t > 1$ a constant. Let $\mathcal{B}$ be a family of balls as given by Lemma 3.15 applied with t and x_0. Let $\{\Gamma_B\}_{B \in \mathcal{B}}$ be upper 1-regular sets $\Gamma_B \subset tB$, with $\Gamma_B \in \mathrm{Reg}_1^+(C)$, where $C > 0$ is a constant independent of B. Then $\Gamma := \bigcup_{B \in \mathcal{B}} \Gamma_B \in \mathrm{Reg}_1^+(\tilde{C})$, where $\tilde{C}$ depends only on C, t and the doubling constant of (X, d). Moreover, if Γ_B is closed for every $B \in \mathcal{B}$, then Γ is closed.*

Proof. Let $z \in \Gamma$ be arbitrary. Since $\mathcal{B}$ covers X, there exists at least one $B \in \mathcal{B}$ with $z \in B$. Denote by $r(B)$ the radius of B. Now fix $R > 0$ arbitrarily. If $R \leq (t-1)r(B)$, then $B_R(z) \subset tB$. By the assumption of the lemma, $\Gamma_{B'} \cap B_R(z) \subset tB' \cap B_R(z)$ for every $B' \in \mathcal{B}$. It follows that if a set $\Gamma_{B'}$ intersects $B_R(z)$, then necessarily $tB \cap tB' \neq \emptyset$. Therefore, by (iv) of Lemma 3.15, we have

$$\mathcal{H}^1(B_R(z) \cap \Gamma) = \mathcal{H}^1(B_R(z) \cap tB \cap \Gamma) \leq \sum_{B' \in \mathcal{B},\, tB \cap tB' \neq \emptyset} H^1(B_R(z) \cap \Gamma_{B'}) \leq MCR,$$

where $M \geq 1$ is the constant given by Lemma 3.15. If instead $R \geq (1-t)r(B)$, by (iii) of Lemma 3.15 and since $z \in B \in \mathcal{B}$, it holds

$$d(z, x_0) \leq r(B) + Mr(B) \leq (1-t)^{-1}(1+M)R.$$

Hence $B_R(z) \subset B_{c_t R}(x_0)$, where $c_t := (1-t)^{-1}(1+M) + 1$. Therefore, by (ii) of Lemma 3.15, we have

$$\begin{aligned}\mathcal{H}^1(B_R(z) \cap \Gamma) &\leq \mathcal{H}^1(B_{c_t R}(x_0) \cap \Gamma) \leq \sum_{B \in \mathcal{B}} \mathcal{H}^1(B_{c_t R}(x_0) \cap \Gamma_B) \\ &\leq \sum_{B \in \mathcal{B}, tB \cap B_{c_t R}(x_0) \neq \emptyset} \mathcal{H}^1(\Gamma_B) \leq Ct \sum_{B \in \mathcal{B}, tB \cap B_{c_t R}(x_0) \neq \emptyset} r(B) \leq Ctc_t \cdot MR,\end{aligned}$$

where we used again that $\Gamma_B \subset tB$.

Finally, assume that each Γ_B is closed and let $\{x_n\}_n \subset \Gamma$ be a converging sequence in (X, d). In particular, $\{x_n\}_n$ is bounded, and by (iii) in Lemma 3.15, we deduce that $\{x_n\}_n$ is contained in a finite union of closed sets Γ_B. This shows that x_n must converge to a point in some $\Gamma_B \subset \Gamma$, and so Γ is closed. ∎

We are now ready to prove Proposition 3.13.

Proof of Proposition 3.13. Fix a constant $t > t_0$ to be chosen later, depending only on t_0 and on the doubling and quasiconvexity constants of (X, d). Moreover, throughout the proof, $\tilde{C}$ denotes a constant whose value might change from line to line, but depending only on C, t_0 and the doubling and quasiconvexity constants of (X, d).

Let $\mathcal{B}$ be the family of balls given by Lemma 3.15 applied to the metric space (X, d) and with constant $t > 1$. For every ball $B = B_r(x) \in \mathcal{B}$, we build a connected set $\Gamma_B \subset tB$ as follows. If $B \cap E = \emptyset$, set $\Gamma_B = \emptyset$, otherwise fix $y \in B \cap E$. By assumption, there exists a connected set $\Gamma_B \in \mathrm{Reg}_1(C)$ such that

$$B \cap E \subset B_{2r}(y) \cap E \subset \Gamma_B \subset B_{2t_0 r}(y) \subset B_{2t_0 r + r}(x) \subset B_{tr}(x),$$

provided $t \geq 2t_0 + 1$. Define $\Gamma := \bigcup_{B \in \mathcal{B}} \Gamma_B$. Since the family $\mathcal{B}$ covers X, the set Γ contains E. Moreover, by Lemma 3.17 it follows that $\Gamma \in \mathrm{Reg}_1^+(\tilde{C})$ and that if each Γ_B is closed then also Γ is closed. However, Γ is not necessarily connected. To fix this, for every $B = B_r(x)$ we build an additional closed connected set Γ'_B as follows. First observe that by Theorem 3.1 and since (X, d) is quasiconvex we have that for every $x_1, x_2 \in \mathrm{X}$ there exists a 1-*regular* (closed) connected set $\gamma \in \mathrm{Reg}_1(\tilde{C})$ such that $x_1, x_2 \in \gamma$ and $\mathrm{diam}(\gamma) \leq L\mathsf{d}(x_1, x_2)$, where L depends only on the doubling and quasiconvexity constants of X.

We pass to the construction of Γ'_B. For every $B' = B_{r'}(x') \in \mathcal{B}$ such that $B' \cap B \neq \emptyset$ and $r' \leq r$ consider a 1-regular closed connected set γ containing x and x' (as above) and define Γ'_B as the union of all such sets. Moreover, if $E \cap B \neq \emptyset$, we add to Γ'_B also a 1-regular closed connected set γ containing x and $y \in E \cap B$, again as above. In particular, by (iv) of Lemma 3.15, Γ'_B is the union of at most an M-number of 1-regular (closed) and connected sets of type $\mathrm{Reg}_1(\tilde{C})$, all containing the point x, where $M > 0$ is a constant depending only on t_0 and the doubling and quasiconvexity constants of X (since it depends also on t). Therefore, up to modifying the value of $\tilde{C}$, $\Gamma'_B \in \mathrm{Reg}_s(\tilde{C})$ (note that Γ'_B is closed). Moreover, $\mathrm{diam}(\Gamma'_B) \leq Lr$. Finally, set $\Gamma' := \bigcup_{B \in \mathcal{B}} \Gamma'_B$.

Observe that by construction, up to choosing $t > L$, it holds $\Gamma'_B \subset tB$ for all $B \in \mathcal{B}$. Therefore by Lemma 3.17 we deduce that $\Gamma' \in \mathrm{Reg}^+_s(\tilde{C})$ (i.e., it is upper 1-regular) and that Γ' is closed. Hence $\Gamma_0 := \Gamma \cup \Gamma' \in \mathrm{Reg}^+_1(\tilde{C})$, Γ_0 contains E and Γ_0 is closed if Γ is closed (which is the case if all Γ_B are closed). It remains to show that Γ_0 is connected (lower 1-regularity would then also follow, see, e.g., Lemma 4.4.5 in [2]). Since each Γ_B is connected and intersects Γ', it is sufficient to show that Γ' is connected. Suppose by contradiction that $\Gamma' \subset U \cup V$, where U and V are two disjoint open subsets of X such that $U \cap \Gamma' \neq \emptyset \neq V \cap \Gamma'$. The centers of the balls in $\mathcal{B}$ must be contained in Γ'. Indeed, if $B = B_r(x) \in \mathcal{B}$ and $x \notin \Gamma'$, then by construction the open ball B does not intersect any (other) ball in $\mathcal{B}$, which contradicts the connectedness of X (since it is quasiconvex). Therefore the center of every ball in $\mathcal{B}$ must belong to either U or V. Moreover, since by construction each set Γ'_B is connected and contains the center of B, each V and U contain at least one center of a ball in $\mathcal{B}$. However, if two balls B' and B are centered in U and V, respectively, then $B \cap B' = \emptyset$, otherwise by construction both their centers belong to a connected set contained in $U \cup V$, which contradicts the fact that U and V are open and disjoint. This means that X is covered by two (non-empty) family of countable balls with the property that every ball in one family does not intersects any other ball in the other. This concludes the proof. ∎

3.2. Application in the context of Menger curvature

Corollary 3.2 allows to prove a version of Theorem 3.11 in [56] for metric spaces that are quasiconvex, doubling and complete (but not necessarily geodesic). The argument is based on the following result, proved in Theorem 1.1 in [35] (see the comment around (2.1) in [35]). Recall Section 2.1.2 for the definition of the Menger curvature $c(x_1, x_2, x_3)$.

Theorem 3.18 (Hahlomaa). *There exists a universal constant $K_0 > 1$ such that the following holds. Let (E, d) be a bounded 1-regular metric space with $E \in \mathrm{Reg}_1(C)$, and such that*

$$\mathbf{c}(E) := \int_{\mathcal{F}} c^2(x_1, x_2, x_3)\, \mathrm{d}\mathcal{H}^1(x_1)\, \mathrm{d}\mathcal{H}^1(x_2)\, \mathrm{d}\mathcal{H}^1(x_3) < +\infty,$$

where

$$\mathcal{F} := \{(x_1, x_2, x_3) \in E : d(x_i, x_j) \le K_0\, d(x_k, x_l),\ \text{for all } i, j, k, l \in \{1, 2, 3\},\ k \neq l\}.$$

Then there exist $A \subset [0, 1]$ and a Lipschitz surjective function $f : A \to E$ such that

$$\mathrm{Lip}(f) \le D(\mathbf{c}(E) + \mathrm{diam}(E)),$$

where D is a constant depending only on C.

We will also need the following elementary result that allows to localize the s-regularity condition.

Proposition 3.19 (Localizing s-regular sets, Lemma 2.2 in [10]). *Suppose $E \in \mathrm{Reg}_s(C)$ is a subset of a metric space (X, d). Then for all $x \in E$ and $r \in (0, \mathrm{diam}(E))$, there exists a set $E_{x,r} \in \mathrm{Reg}_s(\tilde{C})$, where $\tilde{C}$ is a constant depending only on s and C, such that*

$$B_r(x) \cap E \subset E_{x,r} \subset B_{3r}(x) \cap E.$$

In Lemma 2.2 of [10], only closed regular sets that are regular at arbitrarily large scales are considered; however, the same proof without changes works for the version that we reported above.

Corollary 3.20 (Integral Menger curvature condition). *There exists a universal constant $K_0 > 1$ such that the following holds. Let $E \in \mathrm{Reg}_1(C_E)$ be a subset of a complete, doubling and quasiconvex metric space (X, d) and suppose that there exists a constant $C > 0$ such that*

$$\int_{\mathcal{F}\cap(B_R(x))^3} c^2(x_1,x_2,x_3)\,\mathrm{d}\mathcal{H}^1(x_1)\,\mathrm{d}\mathcal{H}^1(x_2)\,\mathrm{d}\mathcal{H}^1(x_3) \le CR, \quad x \in E,\ R > 0, \tag{3.21}$$

where

$$\mathcal{F} := \{(x_1,x_2,x_3) \in E^3 : d(x_i,x_j) \le K_0\, d(x_k,x_l),\ \textit{for all } i,j,k,l \in \{1,2,3\},\ k \neq l\}.$$

Then E is contained in a closed connected set $\Gamma_0 \in \mathrm{Reg}_1(\tilde{C}_0)$ with $\mathcal{H}^1(\Gamma_0) \le \tilde{C}\,\mathrm{diam}(E)$, where $\tilde{C} \ge 1$ is a constant depending only on C, C_E and the quasiconvexity constant of (X,d), while $\tilde{C}_0$ is a constant that may additionally depend also on the doubling constant of (X,d).

Proof. We want to apply Corollary 3.2 (recall that by Corollary 3.14, it also holds for unbounded K, with the same statement). To build the sets $A_{x,r}$ and the maps $f_{x,r}$ required in its statement, we combine Theorem 3.18 and Proposition 3.19.

If E is bounded, for every $x \in E$ and $r = \mathrm{diam}(E)$, we can take directly $A_{x,r} = A$ and $f = f_{x,r}$ given by Theorem 3.18.

For E bounded or unbounded and $r \in (0, \mathrm{diam}(E))$, by Proposition 3.19 there exists a set $E_{x,r} \subset E$ with $E_{x,r} \in \mathrm{Reg}_1(\tilde{C}_E)$ satisfying

$$B_r(x) \cap E \subset E_{x,r} \subset B_{3r}(x) \cap E$$

and with $\tilde{C}_E$ a constant depending only C_E. We have

$$\begin{aligned}\int_{\mathcal{F}\cap(E_{x,r})^3} & c^2(x_1,x_2,x_3)\,\mathrm{d}\mathcal{H}^1(x_1)\,\mathrm{d}\mathcal{H}^1(x_2)\,\mathrm{d}\mathcal{H}^1(x_3) \\ &\le \int_{\mathcal{F}\cap(B_{3r}(x))^3} c^2(x_1,x_2,x_3)\,\mathrm{d}\mathcal{H}^1(x_1)\,\mathrm{d}\mathcal{H}^1(x_2)\,\mathrm{d}\mathcal{H}^1(x_3) \le 3Cr.\end{aligned}$$

Therefore we can apply Theorem 3.18 to the metric space $(E_{x,r}, d|_{E_{x,r}})$ to obtain a surjective Lipschitz map $\tilde{f}_{x,r}\colon \tilde{A}_{x,r} \to E_{x,r}$ for some $\tilde{A}_{x,r} \subset [0,1]$ and with

$$\mathrm{Lip}(\tilde{f}_{x,r}) \le D\,(3Cr + \mathrm{diam}(E_{x,r})) \le D(3Cr + 6r),$$

where D is a constant depending only on C_E. Hence $\mathrm{Lip}(\tilde{f}_{x,r}) \le \tilde{C} r$, where $\tilde{C}$ is constant depending only on C and C_E. Since $B_r(x) \cap E \subset E_{x,r}$, we immediately see that the set $A_{x,r} := \tilde{f}_{x,r}^{-1}(B_r(x) \cap E)$ and the map $f_{x,r} := \tilde{f}_{x,r}|_{Ax,r}$ have the required properties, and an application of Corollary 3.14 yields the result. ∎

Our initial motivation for Corollary 3.20 was to provide details for the first step in the proof of one of the main results in [13] concerning singular integral operators in the first *Heisenberg group* $\mathbb{H}^1$. The group $\mathbb{H}^1 = (\mathbb{R}^3, \cdot)$ is defined by the product

$$(x,t)\cdot(x',t') = (x_1 + x_1', x_2 + x_{2'}, t + t' + \omega(x,x')), \quad (x,t),(x',t') \in \mathbb{R}^2 \times \mathbb{R},$$

where

$$\omega(x,x') := \frac{1}{2}[x_1x_2' - x_2x_1'], \quad x,x' \in \mathbb{R}^2.$$

The left-invariant *Korányi metric* on $\mathbb{H}^1$ is defined by

$$d_{\mathbb{H}^1}(p,p') := \|p^{-1}\cdot p'\|_{\mathbb{H}^1}, \quad \text{where} \quad \|(x,t)\|_{\mathbb{H}^1} := \sqrt[4]{|x|^4 + 16t^2},$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^2$.

Theorem 1.3 in [13] states the following.

Theorem 3.22 (Chousionis, Li). *Let $K\colon \mathbb{H}^1 \setminus \{0\} \to [0,\infty)$ be defined by*

$$K(p) = \frac{\Omega(p)^2}{\|p\|_{\mathbb{H}^1}}, \qquad \text{where} \quad \Omega(x,y,t) := \frac{|(x,y)|^{1/2}}{\|(x,y,t)\|_{\mathbb{H}^1}},$$

and let $E \subset \mathbb{H}^1$ be a 1-regular set. If the truncated singular integrals

$$T_\varepsilon f(p) = \int_{E\setminus B(p,\varepsilon)} K(q^{-1}\cdot p) f(q)\,\mathrm{d}\mathcal{H}^1(q)$$

are uniformly bounded in $L^2(\mathcal{H}^1\lfloor E)$, then E is contained in a 1-regular curve.

The proof of this result in [13] starts with the observation that it suffices to show for some $\alpha > 0$ that

$$\iiint_{\Sigma(\alpha)\cap B(p,R)^3} c^2(p_1,p_2,p_3)\,\mathrm{d}\mathcal{H}^1(p_1)\,\mathrm{d}\mathcal{H}^1(p_2)\,\mathrm{d}\mathcal{H}^1(p_3) \lesssim R, \quad p \in E,\ R > 0,$$

where

$$\Sigma(\alpha) := \bigcup_{r>0}\{(p_1,p_2,p_3) \in E\times E\times E : \alpha r \le d_{\mathbb{H}^1}(p_i,p_j) \le r,\ i \ne j\}$$

and the Menger curvature c is computed with respect to $d_{\mathbb{H}^1}$. The authors refer to p. 123 of [35]; however, from this statement it was not immediately clear to us how to obtain the 1-*regular curve* containing E, whose existence is claimed in Theorem 3.22. A good indication is provided by Theorem 3.11 in [56], where Schul stated without proof a version of Hahlomaa's result [35] which he derived from the arguments in [35]. Corollary 3.20 is a generalization of Theorem 3.11 in [56]: it works not only for geodesic, but for quasiconvex spaces, and also for unbounded sets E. These relaxed assumptions are crucial for the application in the proof of Theorem 1.3 in [13], where $\mathbb{H}^1$ is endowed with the (non-geodesic but quasiconvex) Korányi distance $d_{\mathbb{H}^1}$. In addition to this greater flexibility, we believe that it is valuable to have all the details for the proof of Corollary 3.20 (or Theorem 3.11 in [56]) available as a combination of published work by Hahlomaa and the arguments we provide in this paper.

4. Uniformly 1-rectifiable subsets of metric spaces

The goal of this section is to characterize uniformly 1-rectifiable sets in terms of the ι-numbers introduced in Definition 2.31. What does uniform 1-rectifiability mean in this context? In Euclidean spaces, a 1-regular set is rightfully called *uniformly* 1*-rectifiable* if it is contained in a 1-regular curve, since this property is equivalent with many other notions of quantitative rectifiability that make sense also for higher-dimensional sets [20, 21]. The main result of this section, Theorem 4.17, together with Corollary 4.6, motivates an analogous definition of uniform 1-rectifiability in a large class of metric spaces. A combination of these results was stated as Theorem 1.4 in the introduction.

Corollary 4.6 is based on relations between regular curves, Lipschitz images and the κ-numbers from Example 2.21. While these connections are in essence due in one direction to Hahlomaa [35] and in the other direction to Schul [55], the novelty here is the construction of a closed and connected 1-regular *global* covering set based on the other equivalent characterizations. To make this implication rigorous in arbitrary complete, doubling, and quasiconvex metric spaces, we will need the results from Section 3, and in particular, we will apply Hahlomaa's result in the form of Corollary 3.20. The proof of Theorem 4.17 takes up most space in this section, and at the core of it lies Theorem 4.19, by which we can control the new ι-numbers from above by the more familiar κ-numbers.

4.1. Characterization following Hahlomaa and Schul

Theorem 4.1 (based on [35, 55]). *Let (X, d) be a complete, doubling, and quasiconvex metric space. Then a 1-regular set $E \subset X$ is contained in a 1-regular closed and connected set Γ_0 if and only if for all $z \in E$ and $R > 0$,*

$$\iiint_{(E\cap B_R(z))^3} \frac{\partial(\{x_1,x_2,x_3\})}{\operatorname{diam}\{x_1,x_2,x_3\}^3}\, \mathrm{d}\mathcal{H}^1(x_1)\, \mathrm{d}\mathcal{H}^1(x_2)\, \mathrm{d}\mathcal{H}^1(x_3) \le CR \tag{4.2}$$

for some constant $C \geq 1$.

Moreover, if E is bounded, then the statement also holds with "closed and connected set" replaced by "curve", and if (4.2) holds, we can choose $\Gamma_0 \in \mathrm{Reg}_1(\widetilde{C})$ such that $\operatorname{diam}(\Gamma_0) \le \widetilde{C} \operatorname{diam}(E)$, where $\widetilde{C}$ depends only on C, the 1-regularity constant of E and the doubling and quasiconvexity constants of (X, d). Conversely, if E is contained in a 1-regular closed and connected set Γ_0, then the constant C in (4.2) can be bounded in terms of the 1-regularity constant of Γ_0 and the quasiconvexity constant of (X, d).

Proof. We begin by discussing the first part of the theorem. Assume that (4.2) holds for a 1-regular set $E \subset \mathrm{X}$ with regularity constant C_E. The goal is to apply Corollary 3.20 to show that E is contained in a closed and connected 1-regular set Γ_0. To verify the assumptions for E, let $K_0 > 1$ be the universal constant from Corollary 3.20 and denote, as before,

$$\mathcal{F} = \{(x_1,x_2,x_3) \in E\times E\times E : d(x_i,x_j) \le K_0\, d(x_k,x_l),\ \text{for all } i,j,k,l \in \{1,2,3\}, k \ne l\}.$$

By making the domain of integration for the triple integral in (4.2) possibly smaller, we deduce from the assumption that, for $z \in E$ and $R > 0$,

$$\iiint_{\mathcal{F} \cap (B_R(z))^3} \frac{\partial(\{x_1, x_2, x_3\})}{\operatorname{diam}\{x_1, x_2, x_3\}^3} \,\mathrm{d}\mathcal{H}^1(x_1)\,\mathrm{d}\mathcal{H}^1(x_2)\,\mathrm{d}\mathcal{H}^1(x_3) \leq CR. \tag{4.3}$$

By the equivalence between triangular excess and Menger curvature stated in (2.20) for triples in $\mathcal{F}$, (4.3) yields

$$\iiint_{\mathcal{F} \cap (B_R(z))^3} c^2(x_1, x_2, x_3)\,\mathrm{d}\mathcal{H}^1(x_1)\,\mathrm{d}\mathcal{H}^1(x_2)\,\mathrm{d}\mathcal{H}^1(x_3) \lesssim_{K_0} CR, \quad z \in E,\ R > 0.$$

Then Corollary 3.20 shows that there exists a closed and connected 1-regular subset $\Gamma_0 \in \operatorname{Reg}_1(\widetilde{C})$ of (X, d) containing E for $\widetilde{C}$ as in the statement of the theorem. If E is bounded, then Corollary 3.20 yields further that $\mathcal{H}^1(\Gamma_0) \leq \widetilde{C}\operatorname{diam}(E) < \infty$. Since (X, d) is complete, such Γ_0 is automatically compact, and a posteriori, the trace of a 1-regular Lipschitz curve, recall the discussion at the beginning of Section 3.1.

The other implication was known before. Indeed, assume that E is contained in a closed and connected 1-regular subset Γ_0. Then Theorem 1.10 in [55] (see also Theorem 3.12 and (3.9) in [56]) shows that (4.2) holds with "E" replaced by "Γ_0". (The proof argument in Section 3 of [55] to deduce Theorem 1.10 from Theorem 1.8 therein seems to be formulated under the implicit assumption that the ambient space is geodesic, but the argument works analogously for quasiconvex spaces, with the implicit constant in (4.2) depending on the quasiconvexity constant of (X, d) in addition to the 1-regularity constant of Γ_0.)

As inequality (4.2) remains true if the domain of integration is replaced by a smaller set, (4.2) also holds for the ($\mathcal{H}^1$-measurable) $E \subset \Gamma_0$. ■

Theorem 4.1 gives a characterization for a 1-regular set E to be contained in a closed and connected 1-regular curve. The next result, Corollary 4.6, provides further justification for calling such sets *uniformly* 1-*rectifiable*. We recall the relevant terminology.

Definition 4.4. Let $k \in \mathbb{N}$. A k-regular set E in a metric space (X, d) has *big pieces of Lipschitz images* (BPLI) if there exist constants $c, L > 0$ such that for every $x \in E$ and $0 < r < \operatorname{diam}(E)$, there exists an L-Lipschitz function $f\colon A \to E$, where A is a subset of the Euclidean ball $B_r(0) \subset \mathbb{R}^k$, and $\mathcal{H}^k(f(A) \cap B_r(x)) \geq cr^k$.

Definition 4.5. Let $k \in \mathbb{N}$. A k-regular set E in a metric space (X, d) has *big pieces of bi-Lipschitz images* (BPBI) if there exist constants $c, L > 0$ such that for every $x \in E$ and $0 < r < \operatorname{diam}(E)$, there exists an L-bi-Lipschitz embedding $f\colon A \to E$, where A is a subset of the Euclidean ball $B_r(0) \subset \mathbb{R}^k$, and $\mathcal{H}^k(f(A) \cap B_r(x)) \geq cr^k$.

Corollary 4.6. *Let (X, d) be a complete, doubling, quasiconvex metric space. Let $E \subset X$ be* 1*-regular. The following conditions are equivalent*:

(1) *E is contained in a closed and connected* 1*-regular set Γ_0,*

(2) *E has* BPLI*,*

(3) *E has* BPBI*,*

(4) *$E \in \operatorname{GLem}(\kappa, 1)$, for κ as in Example* 2.21.

Moreover, if E is bounded, these conditions are further equivalent to E being contained in a 1*-regular* curve*. Also, if* (1) *holds, then* (2) *holds with* BPLI *constants depending only on the* 1*-regularity constants of E and Γ_0, as well as the doubling and quasiconvexity constants of* (X, d)*. Conversely, if* (4) *holds for* $E \in \mathrm{GLem}(\kappa, 1, M) \cap \mathrm{Reg}_1(C)$*, then* (1) *holds with* $\Gamma_0 \in \mathrm{Reg}_1(\widetilde{C})$*, where* $\widetilde{C}$ *depends only on* M*,* C*, and the doubling and quasiconvexity constants of* (X, d)*.*

We call a 1-regular set E *uniformly* 1*-rectifiable* if it satisfies one (and thus all) of the properties (1)–(4). The equivalence of (2)–(3) was established by Schul in Corollary 1.2 of [58], while Corollary 1.3 in [57] states a third equivalent condition that is very similar to (4), but formulated in terms of *multiresolution families* instead of dyadic systems as we used in the definition of the geometric lemma. However, the equivalence of these two versions of the Carleson condition is standard, see the beginning of the proof of Theorem 1.1 in [57]. For the convenience of the reader, we show how the implication (3) to (4) can be deduced from published results. This is the content of Proposition 4.7.

Proposition 4.7. *Let* (X, d) *be a complete, doubling, quasiconvex metric space and let* $E \subset X$ *be* 1*-regular. If E has* BPBI*, then* $E \in \mathrm{GLem}(\kappa, 1)$*, for* κ *as in Example* 2.21*.*

Proof. The proof consists of three steps. First we show that connected 1-regular sets in metric spaces satisfy $\mathrm{GLem}(\kappa, 1)$. This is essentially the content of Theorem 1.11 in [55], but the latter is formulated in terms of *multiresolution families* instead of dyadic systems. Indeed, Theorem 1.11 in [55] states that for every connected set $\Gamma \in \mathrm{Reg}_1(C)$ in a metric space and any associated multiresolution family, for $z \in \Gamma$ and $R > 0$,

$$\sum_{\substack{B \in \mathcal{G}^\Gamma \\ B \subset B_R(z)}} \int_B \int_B \int_B \partial(\{x_1, x_2, x_3\}) \,\mathrm{rad}(B)^{-3} \,\mathrm{d}\mathcal{H}^1|_\Gamma(x_1) \,\mathrm{d}\mathcal{H}^1|_\Gamma(x_2) \,\mathrm{d}\mathcal{H}^1|_\Gamma(x_3) \lesssim R, \tag{4.8}$$

where the implicit constant depends on C and the constant "A" in the definition of the multiresolution family $\mathcal{G}^\Gamma$. Recall that a multiresolution family is given by

$$\mathcal{G}^\Gamma := \{B_{A2^{-n}}(x) : x \in X_n^\Gamma,\ n \in \mathbb{N}\},$$

where $A > 1$ is a chosen constant and X_n^Γ is any 2^{-n}-net for Γ, i.e., a set of points in Γ with the properties that $\mathsf{d}(x, y) > 2^{-n}$ for all $x, y \in X_n^\Gamma$ (2^{-n}-separation) and for every $z \in \Gamma$ there exists $x \in X_n^\Gamma$ such that $\mathsf{d}(x, z) \le 2^{-n}$ (maximality).

In order to deduce that $\Gamma \in \mathrm{GLem}(\kappa, 1)$ in the sense of Definition 2.14, we take an arbitrary dyadic system Δ on Γ and fix a cube $Q_0 \in \Delta$ with $\ell(Q_0) = 2^{-j_0}$. We need to bound from above the expression

$$\begin{aligned}\sum_{Q \in \Delta_{Q_0}} \kappa(2Q)\mathcal{H}^1(Q) &= \sum_{j \ge j_0} \sum_{Q \in \Delta_{Q_0} \cap \Delta_j} \frac{\mathcal{H}^1(Q)}{\mathcal{H}^1(2Q)^3} \\ &\quad \times \int_{2Q} \int_{2Q} \int_{2Q} \frac{\partial(\{x_1, x_2, x_3\})}{\mathrm{diam}(2Q)} \,\mathrm{d}\mathcal{H}|_\Gamma^1(x_1) \,\mathrm{d}\mathcal{H}|_\Gamma^1(x_2) \,\mathrm{d}\mathcal{H}|_\Gamma^1(x_3).\end{aligned} \tag{4.9}$$

Now for a fixed $j \ge j_0$, the collection of "centers" $\{x_Q : Q \in \Delta_j\}$ (as in item (5) in Definition 2.5) is 2^{-n_j}-separated, where n_j is the smallest natural number such that $2^{-n_j} < c_0 2^{-j}$

with the constant $c_0 \in (0,1)$ from Definition 2.5. The collection is not necessarily *maximal*, but by adding points if necessary, one can enlarge it to a family $X^\Gamma_{n_j}$ as in the definition of multiresolution families. By construction, every $n \in \mathbb{N}$ appears as n_j for at most one index j. Moreover, by (2.6) and (2.8), we can choose a constant $A > 1$, depending only on the 1-regularity constant C, such that

$$2Q \subset B_{A2^{-n_j}}(x_Q) \cap \Gamma, \quad Q \in \Delta_j. \tag{4.10}$$

Associated to the given dyadic system Δ, we fix now a multiresolution family $\mathcal{G}^\Gamma$ with the chosen constant A and such that $\{x_Q : Q \in \Delta_j\} \subset X^\Gamma_{n_j}$. Moreover, there exists a constant K, depending only on C, such that

$$B_{A2^{-n_j}}(x_Q) \cap \Gamma \subset B_{K\mathrm{diam}(Q_0)}(x_{Q_0}) \cap \Gamma, \quad Q \in \Delta_{Q_0} \cap \Delta_j. \tag{4.11}$$

Indeed, for $z \in B_{A2^{-n_j}}(x_Q) \cap \Gamma$, we have

$$\mathrm{dist}(z, Q_0) \le \mathsf{d}(z, x_Q) < A2^{-n_j} < Ac_0 2^{-j} \overset{(2.8)}{\lesssim_C} \mathrm{diam}(Q_0),$$

which proves (4.11) for a suitable constant K. Then, by the construction of $\mathcal{G}^\Gamma$, the inclusions (4.10) and (4.11), and the property (2.8) of dyadic cubes, we can bound the expression in (4.9) from above as follows:

$$\begin{aligned}&\sum_{j \ge j_0} \sum_{Q \in \Delta_{Q_0} \cap \Delta_j} \frac{\mathcal{H}^1(Q)}{\mathcal{H}^1(2Q)^3} \int_{2Q}\int_{2Q}\int_{2Q} \frac{\partial(\{x_1,x_2,x_3\})}{\mathrm{diam}(2Q)} \, \mathrm{d}\mathcal{H}|^1_\Gamma(x_1)\, \mathrm{d}\mathcal{H}|^1_\Gamma(x_2)\, \mathrm{d}\mathcal{H}|^1_\Gamma(x_3)\\ &\lesssim_C \sum_{\substack{B \in \mathcal{G}^\Gamma \\ B \subset B_{K\mathrm{diam}(Q_0)}(x_{Q_0})}} \int_B\int_B\int_B \partial(\{x_1,x_2,x_3\})\, \mathrm{rad}(B)^{-3}\, \mathrm{d}\mathcal{H}^1|_\Gamma(x_1)\, \mathrm{d}\mathcal{H}^1|_\Gamma(x_2)\, \mathrm{d}\mathcal{H}^1|_\Gamma(x_3).\end{aligned} \tag{4.12}$$

Combined with Schul's result (4.8), the two inequalities (4.9) and (4.12) yield

$$\sum_{Q \in \Delta_{Q_0}} \kappa(2Q)\, \mathcal{H}^1(Q) \lesssim_C \mathcal{H}^1(Q_0),$$

and hence, as $Q_0 \in \Delta$ was arbitrary, $\Gamma \in \mathrm{GLem}(\kappa, 1, M)$, with M depending only on C.

In the second step of the proof of Proposition 4.7, we observe that the set E given in the statement has big pieces of connected 1-regular sets since it has BPBI (with some constants c and L). Indeed, for $x \in E$ and $0 < r < \mathrm{diam}(E)$, let $f(A)$ be the associated bi-Lipschitz piece as in the definition of BPBI. Then, for arbitrary $y \in f(A)$ and $s > 0$, the set $f^{-1}(f(A) \cap B_s(y))$ is contained in an interval of length $2Ls$ by the L-bi-Lipschitz property of f. Upon translating and rescaling by $2Ls$, we thus find a set $A_{y,s} \subset [0,1]$ and a $2L^2 s$-Lipschitz function from $A_{y,s}$ onto $f(A) \cap B_s(y)$. Since y and s were arbitrary, Corollary 3.2 then yields a connected set $\Gamma_0 \in \mathrm{Reg}_1(C_0)$ with C_0 depending only on doubling and quasiconvexity constants of (X, d) and on L such that $\Gamma_0 \supseteq f(A)$. Since $f(A)$ was chosen as in the definition of BPBI, we have in particular $\mathcal{H}^1(E \cap \Gamma_0 \cap B_r(x)) \ge cr$. Repeating the same argument for all x and r, we find that E has big pieces (as defined in Definition 2.11 in [10]) of connected 1-regular sets.

Finally, by the first two steps of the proof, we know that E has big pieces of sets that satisfy GLem$(\kappa, 1)$ with uniform constants. Then it follows from an abstract argument that E itself satisfies GLem$(\kappa, 1)$. This abstract argument is known as stability of geometric lemmas under the "big pieces functor" and was formulated in great generality in Proposition 2.23 of [10], which we can apply to conclude the proof. ∎

Our main contribution to Corollary 4.6 is the proof of "(4) ⇒ (1)", and it is precisely this implication which arises as a corollary of Theorem 4.1. To complete the circle of equivalent statements, we also discuss the implication "(1) ⇒ (2)".

Proof of Corollary 4.6. We assume first that (1) holds, that is, E is a 1-regular set contained in a closed and connected 1-regular set Γ_0, and we will deduce (2). We call the *data* of $(E, \Gamma_0, \mathrm{X})$ the collection of the 1-regularity constants of E and Γ_0, as well as the doubling and quasiconvexity constants of (X, d). We could essentially use Γ_0 to construct a big Lipschitz image in $E \cap B_r(x)$, for an arbitrarily given point $x \in E$ and $0 < r < \operatorname{diam}(E)$, but the localization argument will be simpler if we use curves given by Theorem 4.1. Let $C \geq 1$ be a large enough constant, to be chosen momentarily. Using the localization property for s-regular sets, stated in Proposition 3.19, we can find a 1-regular set $E_{x,r}$, with regularity constant depending only the regularity constant of E, such that

$$B_{r/3C}(x) \cap E \subset E_{x,r} \subset B_{r/C}(x) \cap E.$$

As a subset of E, the set $E_{x,r}$ is still covered by Γ_0. By Theorem 1.10 in [55], the connected 1-regular set Γ_0 satisfies

$$\iiint_{(\Gamma_0 \cap B_R(z))^3} \frac{\partial(\{x_1, x_2, x_3\})}{\operatorname{diam}\{x_1, x_2, x_3\}^3} \, d\mathcal{H}^1(x_1) \, d\mathcal{H}^1(x_2) \, d\mathcal{H}^1(x_3) \leq C_0 R,$$

where C_0 depends on the Ahlfors regularity constant of Γ_0. Since $E_{x,r} \subset \Gamma_0$, it follows that $E_{x,r}$ satisfies condition (4.2) in Theorem 4.1, with the same constant "C_0". Hence, by Theorem 4.1, there exists a 1-regular curve $\Gamma_{x,r} \supset E_{x,r}$ with $\Gamma_{x,r} \in \operatorname{Reg}_1(\tilde{C})$ and $\operatorname{diam}(\Gamma_{x,r}) \leq \tilde{C} \operatorname{diam}(E_{x,r}) (\leq 2\tilde{C} r/C)$, where $\tilde{C}$ depends only on the data of $(E, \Gamma_0, \mathrm{X})$. In particular, by choosing C large enough depending only on the data of $(E, \Gamma_0, \mathrm{X})$, we may assume that $\Gamma_{x,r} \subset B_r(x)$.

Now Lemma 2.8 in [4] and a straightforward reparametrization show that there exists an L-Lipschitz function $\gamma\colon [-r, r] \to \mathrm{X}$ with $\gamma([-r, r]) = \Gamma_{x,r}$ and L bounded in terms of the data of $(E, \Gamma_0, \mathrm{X})$. Moreover,

$$\mathcal{H}^1(\gamma([-r, r]) \cap E \cap B_r(x)) \geq \mathcal{H}^1(E_{x,r}) \geq \mathcal{H}^1(E \cap B_{r/3C}(x)) \sim_C r.$$

Repeating the same argument for every $x \in E$ and $0 < r < \operatorname{diam}(E)$ proves that E satisfies (2), that is, it has BPLI (with constants depending only on the data of $(E, \Gamma_0, \mathrm{X})$). Then E has also BPBI (condition (3)) by Corollary 1.2 in [58], and finally, satisfies the geometric lemma in condition (4) by Proposition 4.7.

In the converse direction, assume now that (4) holds for a set $E \in \operatorname{Reg}_1(C)$ in X. Fix a dyadic system Δ on E and consider arbitrary $z \in E$ and $0 < R < \operatorname{diam}(E)/2$. Then there exists $j_0 \in \mathbb{J}$ such that $B_R(z) \cap E \subset \bigcup_{i=1}^m Q_{0,i}$ with $Q_{0,i} \in \Delta_{j_0}$, $2^{-j_0-1} \leq R < 2^{-j_0}$

and m depending only on C. To see this, recall that $B_R(z) \cap E$ is covered by the union of the cubes $Q \in \Delta_{j_0}$. Now if $Q \cap B_R(z) \neq \emptyset$, then $Q \subset B_{\widetilde{C}R}(z) \cap E$ for a constant $\widetilde{C} = \widetilde{C}(C)$. Since $E \in \mathrm{Reg}_1(C)$, the elements in Δ_{j_0} are disjoint and thanks to the lower bound for the $\mathcal{H}^1$-measure of $Q \in \Delta_{j_0}$ stated in (2.8) we have that at most $m \lesssim_C 1$ elements $Q \in \Delta_{j_0}$ can intersect $B_R(z)$.

Finally, we will show that there exists a constant $K = K(C)$ such that, for every $Q_{0,i} \in \Delta_{j_0}$,

$$\iiint_{[E\cap B(z,R)]^3} \frac{\partial(\{x_1,x_2,x_3\})}{(\mathrm{diam}\{x_1,x_2,x_3\})^3}\, d\mathcal{H}^1(x_1)\, d\mathcal{H}^1(x_2)\, d\mathcal{H}^1(x_3) \lesssim_C \sum_{i=1}^{m} \sum_{Q\in\Delta_{Q_{0,i}}} \kappa(KQ)\mathcal{H}^1(Q). \tag{4.13}$$

This will allow us to verify the assumption of Theorem 4.1 via (4) ($E \in \mathrm{GLem}(\kappa, 1)$). Then we deduce that E is contained in a closed and connected 1-regular set Γ_0, thus (1) holds.

To conclude the proof, we verify inequality (4.13). We first decompose the domain of integration as $[E \cap B_R(z)]^3 = \bigcup_{j \geq j_0} A_j$, where

$$A_j := \{(x_1,x_2,x_3) \in [E \cap B_R(z)]^3 : 2^{-j} \leq \mathrm{diam}\{x_1,x_2,x_3\} < 2^{-j+1}\}. \tag{4.14}$$

It suffices to consider $j \geq j_0$, since $A_j \neq \emptyset$ implies that

$$2^{-j} \leq 2R < 2^{-j_0+1}. \tag{4.15}$$

Thus, if $(x_1,x_2,x_3) \in A_j$, then $\{x_1,x_2,x_3\} \subset B_r(x_3)$ for some $j \geq j_0$ with $2^{-j} \leq r < 2^{-j+1}$. Recalling the basic property of dyadic cubes stated in (2.9), there exists a constant $K = K(1, C) > 1$, depending only on the Ahlfors regularity constant of $E \in \mathrm{Reg}_1(C)$ such that there is $Q \in \Delta_j$ with $\{x_1,x_2,x_3\} \subset B(x_3, r) \subset KQ$ and $Q \subset Q_{0,i}$ for some $i \in \{1,\dots,m\}$. Thus

$$A_j \subset \bigcup_{i=1}^{m} \bigcup_{Q\in\Delta_{Q_{0,i}}\cap\Delta_j} (KQ)^3. \tag{4.16}$$

Therefore,

$$\begin{aligned}
&\iiint_{[E\cap B_R(z)]^3} \frac{\partial(\{x_1,x_2,x_3\})}{(\mathrm{diam}\{x_1,x_2,x_3\})^3}\, d\mathcal{H}^1(x_1)\, d\mathcal{H}^1(x_2)\, d\mathcal{H}^1(x_3)\\
&\overset{(4.16),(2.8)}{\lesssim_C} \sum_{j\geq j_0}\sum_{i=1}^{m} \sum_{Q\in\Delta_{Q_{0,i}}\cap\Delta_j} \iiint_{[KQ]^3} \frac{\partial(\{x_1,x_2,x_3\})}{\mathrm{diam}(KQ)^3}\, d\mathcal{H}^1(x_1)\, d\mathcal{H}^1(x_2)\, d\mathcal{H}^1(x_3)\\
&\overset{(2.8)}{\lesssim_C} \sum_{i=1}^{m} \sum_{Q\in\Delta_{0,i}} \kappa(KQ)\, \mathcal{H}^1(Q).
\end{aligned}$$

By the assumption that $E \in \mathrm{GLem}(\kappa, 1)$ and recalling Lemma 2.23 (see also Remark 2.30), this shows as desired that (4.2) holds for all $z \in E$ with a constant depending only on M, C and the doubling and quasiconvexity constants of (X, d), at least for $R < \mathrm{diam}(E)/2$. If E is unbounded, we are done. Otherwise, if E is bounded and $R \geq \mathrm{diam}(E)/2$, then we apply the preceding argument with $j_0 = n$, where n is the smallest integer in $\mathbb{J}$, and the arguments go through verbatim if we replace in (4.15) the bound "$2R$" by "$\mathrm{diam}(E)$". In any case, the assumption of Theorem 4.1 is satisfied for E, and the first part of the corollary follows.

The part concerning the covering of bounded sets E by 1-regular curves is also a consequence of Theorem 4.1. ■

4.2. Characterization using $\iota_{1,1}$-numbers

In this section, we complement Corollary 4.6 by providing a further equivalent characterization for uniform 1-rectifiability, now in terms of the ι-numbers from Definition 2.31.

Theorem 4.17. *Let (X, d) be a metric space and let $E \in \mathrm{Reg}_1(c_E)$. Then the following are equivalent*:

(1) $E \in \mathrm{GLem}(\kappa, 1)$,

(2) $E \in \mathrm{GLem}(\iota_{1,1}, 1)$.

In fact, if $\triangle$ is a system of Christ–David dyadic cubes on the 1-regular set E, then

$$3^{-1}\kappa(2Q) \leq \iota_{1,1}(2Q) \leq C\kappa(7Q), \quad Q \in \Delta, \tag{4.18}$$

where $C \geq 1$ is a constant depending only on c_E.

Moreover, if (X, d) is complete, quasiconvex, and doubling, and if one (and thus both) of conditions (1) and (2) hold, then E is uniformly 1-rectifiable.

For the implication (2) ⇒ (1) in Theorem 4.17, we will need the following result, the proof of which is postponed to the next subsection.

Theorem 4.19 (L^1-quantified Menger theorem). *Let (X, d) be a bounded and 1-regular metric space. Then there exists a Borel map $f : X \to \mathbb{R}$ such that*

$$\fint_X \fint_X \frac{||f(x) - f(y)| - \mathsf{d}(x, y)|}{\mathrm{diam}(X)} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \leq C \fint_X \fint_X \fint_X \frac{\partial(\{x, y, z\})}{\mathrm{diam}(X)} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(z), \tag{4.20}$$

where $\mu := \mathcal{H}^1$ and C is a constant depending only on the regularity constant of X.

Proof of Theorem 4.17. Once the equivalence of (1) and (2) is established for a set E in a complete, quasiconvex and doubling metric space, it follows from Corollary 4.6 that E with these properties is uniformly 1-rectifiable.

Thus we concentrate on the equivalence of (1) and (2), for which is enough to show the estimate (4.18). Let $\triangle$ be a system of Christ-David cubes on E, and fix $Q \in \triangle$. We first prove the following inequality:

$$\kappa(2Q) \leq 3\iota_{1,1}(2Q). \tag{4.21}$$

To this end, let $x_1, x_2, x_3 \in 2Q$ be arbitrary, and consider any function $f: 2Q \to \mathbb{R}$ (which we later take to be Borel) and any norm $\|\cdot\|$ on $\mathbb{R}$. Then there exists $\sigma \in S_3$ (depending on x_1, x_2, x_3) such that

$$f(x_{\sigma(1)}) \le f(x_{\sigma(2)}) \le f(x_{\sigma(3)})$$

and hence $\partial_1(f(x_{\sigma(1)}), f(x_{\sigma(2)}), f(x_{\sigma(3)})) = 0$, where $\partial_1(\cdot)$ is computed with respect to $\|\cdot\|$, which is a constant multiple of $|\cdot|$ (see (2.19) for the expressions of ∂_1 and ∂). Therefore,

$$\begin{aligned}
\frac{\partial(\{x_1,x_2,x_3\})}{\mathrm{diam}(2Q)} &\le \frac{\partial_1(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})}{\mathrm{diam}(2Q)} \\
&= \frac{\partial_1(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}) - \partial_1(f(x_{\sigma(1)}),f(x_{\sigma(2)}),f(x_{\sigma(3)}))}{\mathrm{diam}(2Q)} \\
&\le \frac{|d(x_{\sigma(1)},x_{\sigma(2)}) - \|f(x_{\sigma(1)})-f(x_{\sigma(2)})\||}{\mathrm{diam}(2Q)} + \frac{|d(x_{\sigma(2)},x_{\sigma(3)}) - \|f(x_{\sigma(2)})-f(x_{\sigma(3)})\||}{\mathrm{diam}(2Q)} \\
&\quad + \frac{|d(x_{\sigma(1)},x_{\sigma(3)}) - \|f(x_{\sigma(1)})-f(x_{\sigma(3)})\||}{\mathrm{diam}(2Q)}.
\end{aligned}$$

Note that the last expression is unchanged if we replace each "$\sigma(i)$" by "i". Hence integrating and taking the infimum over all Borel functions $f: 2Q \to \mathbb{R}$ and $\|\cdot\|$, using the definition of $\kappa(\cdot)$ and $\iota_{1,1}(\cdot)$, proves the inequality (4.21). In particular, (2) implies (1).

Next we prove the following opposite inequality

$$\iota_{1,1}(2Q) \le C\kappa(7Q), \tag{4.22}$$

where $C > 0$ is a constant depending only on c_E. Fix $Q \in \triangle$, $z \in Q$, and set $d(Q) := \mathrm{diam}(Q)$. Applying Proposition 3.19, we can find a 1-regular set $E_Q \subset E$ such that $B_{2d(Q)}(z) \cap E \subset E_Q \subset B_{6d(Q)}(z) \cap E$ and with a regularity constant depending only on c_E (if $2d(Q) \ge \mathrm{diam}(E)$, we simply take $E_Q = E$). In particular,

$$2Q \subset E_Q \subset 7Q$$

Then applying Theorem 4.19 to the metric space $(E_Q, \mathsf{d}|_{E_Q})$, we obtain a Borel map $f: E_Q \to \mathbb{R}$ satisfying

$$\begin{aligned}
&\fint\fint_{(E_Q)^2} ||f(x)-f(y)| - \mathsf{d}(x,y)|\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}\mathcal{H}^1(y) \\
&\qquad \le \tilde{C} \fint\fint\fint_{(E_Q)^3} \partial\{x,y,z\}\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}\mathcal{H}^1(y)\,\mathrm{d}\mathcal{H}^1(z),
\end{aligned} \tag{4.23}$$

where $\tilde{C}$ is a constant depending only on c_E. Moreover, by the 1-regularity of E and by the property of dyadic systems stated in (2.8), we have

$$c_E^{-1} c_0\, d(Q) \le \mathcal{H}^1(Q) \le \mathcal{H}^1(2Q) \le \mathcal{H}^1(E_Q) \le \mathcal{H}^1(7Q) \le 7c_E\, d(Q),$$

where c_0 is the constant, depending only on c_E, appearing in the definition of dyadic system. Therefore, using (4.23),

$$\begin{aligned}\frac{1}{\mathcal{H}^1(2Q)^2}\int\int_{(2Q)^2} &\frac{||f(x)-f(y)|-\mathsf{d}(x,y)|}{\operatorname{diam}(2Q)}\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}\mathcal{H}^1(y)\\ &\le c_0^{-2}\,7^2\,c_E^4 \fint\fint_{(E_Q)^2}\frac{||f(x)-f(y)|-\mathsf{d}(x,y)|}{\operatorname{diam}(2Q)}\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}\mathcal{H}^1(y)\\ &\le \tilde{C}\cdot c_0^{-2}\,7^2\,c_E^4\fint\fint\fint_{(E_Q)^3}\frac{\partial\{x,y,z\}}{\operatorname{diam}(2Q)}\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}\mathcal{H}^1(y)\,\mathrm{d}\mathcal{H}^1(z),\\ &\le C\fint\fint\fint_{(7Q)^3}\frac{\partial\{x,y,z\}}{\operatorname{diam}(7Q)}\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}\mathcal{H}^1(y)\,\mathrm{d}\mathcal{H}^1(z),\end{aligned}$$

where C is a constant depending only on c_E. This proves (4.22), which combined with Lemma 2.23 (whose assumption are satisfied for the coefficients κ by Remark 2.30) gives also the implication (1) $\Rightarrow$ (2). ∎

4.3. Constructing good maps into $\mathbb{R}$

The main goal of this subsection is to prove Theorem 4.19. This requires us to construct maps $f:\mathrm{X}\to\mathbb{R}$ with good properties using a suitable control for the triangular excess of point triples in X. We introduce some notation to make this precise. We say that a map between two metric spaces $f:(\mathrm{X}_1,\mathsf{d}_1)\to(\mathrm{X}_2,\mathsf{d}_2)$ is a *δ-isometry*, for some $\delta\ge 0$, if

$$|\mathsf{d}_2(f(x),f(y))-\mathsf{d}_1(x,y)|\le\delta,\quad x,y\in\mathrm{X}.$$

For every $S\subset(\mathrm{X},\mathsf{d})$, we define

$$\partial S:=\sup_{\{x,y,z\}\subset S}\partial(\{x,y,z\}).$$

Given three points x,y,z in a metric space (X,d), we write $[xyz]$ if

$$\mathsf{d}(x,y)+\mathsf{d}(y,z)-\mathsf{d}(x,z)=\partial(\{x,y,z\}),$$

which is in fact equivalent to

$$\mathsf{d}(x,z)\ge\max(\mathsf{d}(x,y),\mathsf{d}(y,z)).$$

Clearly, $[xyz]\Leftrightarrow[zyx]$, and moreover, at least one of the properties $[xyz]$, $[xzy]$, or $[zxy]$ always holds.

Definition 4.24 (Almost circular points). Let (X,d) be a metric space and fix a number $\eta\ge 0$. We say that four points $P_1,P_2,P_3,P_4\in\mathrm{X}$ are *η-circular* if

$$|\mathsf{d}(P_i,P_j)-\mathsf{d}(P_k,P_l)|\le\eta, \tag{4.25}$$

for any choice of (distinct) indices $i,j,k,l\in\{1,2,3,4\}$.

The name η-circular comes from the fact that any two couples of antipodal points in the sphere (of any dimension) give rise to four 0-circular points. The motivation behind the above definitions is the following result by Menger [50] (see also [27, 54]).

Theorem 4.26 (Menger). *Suppose that a metric space* (X, d) *satisfies* $\partial X = 0$*. Then either* X *contains only four points which are* 0*-circular, or* X *can be isometrically embedded in* $\mathbb{R}$.

Theorem 4.19 is a sort of L^1-quantified generalization of the above theorem in the case of 1-regular metric measure spaces. In fact, an L^∞-quantified version also holds for arbitrary metric spaces. This is a bit similar in spirit to Lemma 6.4 in [24], which concerns the construction of good maps locally from a *curve* intersected with a ball into $\mathbb{R}$.

Theorem 4.27 (L^∞-quantified Menger theorem). *Let* (X, d) *be a metric space such that* $\partial X \le \beta$*, with* $\beta \ge 0$*, and containing five points having pairwise distances strictly grater than* 30β*. Then there exists a map* $f : (X, \mathsf{d}) \to (\mathbb{R}, |\cdot|)$ *that is a* 40β*-isometry.*

Even if not needed in this note, we will prove this at the end of this section, as it follows easily from the preliminary results needed in the proof of Theorem 4.19. More precisely, both Theorem 4.19 and Theorem 4.27 are consequences of the following elementary technical lemmas. The first one says that given four points in a metric space, either they are almost circular or they can be embedded in $\mathbb{R}$ with an explicit almost isometry.

Lemma 4.28 (4-points lemma). *Let* (X, d) *be a metric space. Let* $P, Q, R, S \in X$ *and* $\beta \ge 0$ *be such that*

$$\partial\{P, Q, R, S\} \le \beta \quad \textit{and} \quad \mathsf{d}(P, Q) > 2\beta.$$

Then at least one of the following holds:

(i) *the map* $f : \{P, Q, R, S\} \to \mathbb{R}$*, defined by* $f(Q) := \mathsf{d}(P, Q)$ *and for* $x \in \{P, R, S\}$*, by*

$$f(x) := \begin{cases} -\mathsf{d}(x, P) & \text{if } \mathsf{d}(x, Q) \ge \max(\mathsf{d}(P, Q), \mathsf{d}(x, P)), \\ \mathsf{d}(x, P) & \text{otherwise,} \end{cases}$$

is a 2β*-isometry,*

(ii) *the points* P, Q, R, S *are* 2β*-circular.*

The second technical lemma essentially implies that if ∂X is small and X contains four almost circular points, then all the points in X must be close to those points. It is instructive to keep in mind the example where P_1, P_2, P_3, P_4 are given by two pairs of antipodal points on the circle S^1 equipped with the inner distance.

Lemma 4.29 (Attraction to circular points). *Let* (X, d) *be a metric space and let* $\beta \ge 0$*. Suppose that the points* $P_1, P_2, P_3, P_4 \in X$ *are* 4β*-circular,* $\partial\{P_1, P_2, P_3, P_4\} \le \beta$*, and* $\mathsf{d}(P_i, P_j) > 15\beta$ *for all* $i \ne j$.

Then for every $Q \in X$*, at least one of the following holds*:

(i) $\mathsf{d}(Q, P_i) \le 15\beta$ *for some* $i \in \{1, 2, 3, 4\}$,

(ii) $\partial\{P_1, P_2, P_3, P_4, Q\} > \beta$.

The proofs of Lemma 4.28 and Lemma 4.29 are elementary but rather tedious, and can be found in Appendix A. Assuming their validity, we now prove the main result of this subsection, Theorem 4.19. We will split the proof in several lemmas, but before that we fix some notations. From now on, (X, d) will be a bounded 1-regular metric space with regularity constant $C_{\mathrm{X}} \geq 1$, i.e., $\mathrm{X} \in \mathrm{Reg}_1(C_{\mathrm{X}})$. We also set $\mu := \mathscr{H}^1$, the 1-dimensional Hausdorff measure in (X, d) and, set

$$\beta := \fint_{\mathrm{X}} \fint_{\mathrm{X}} \fint_{\mathrm{X}} \frac{\partial\{x, y, z\}}{r} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(z), \tag{4.30}$$

where $r := \mathrm{diam}(\mathrm{X})$. Note that the map $\mathrm{X}^3 \ni (x, y, z) \mapsto \partial\{x, y, z\}$ is continuous as infimum of a finite number of continuous functions. Without loss of generality, we can assume that $\beta \leq \delta$, for some $\delta > 0$ small to be chosen later and depending only on C_{X}. Indeed, by taking $f : \mathrm{X} \to \mathbb{R}$ as $f \equiv 0$, we can always make the left-hand side of (4.20) less than or equal to one. Fix also a constant $C > 0$ big enough to be chosen later depending only on C_{X}.

We start by giving an upper bound on the number $\partial \mathrm{X} = \sup_{x,y,z \in \mathrm{X}} \partial\{x, y, z\}$.

Lemma 4.31. *It holds*

$$\partial X \leq \frac{r}{200}. \tag{4.32}$$

Proof. It suffices to consider the case $\partial \mathrm{X} > 0$. Set

$$\beta_\infty := \frac{1}{r} \, \partial \mathrm{X}.$$

Let $x_1, x_2, x_3 \in \mathrm{X}$ be such that

$$\frac{\partial\{x_1, x_2, x_3\}}{r} \geq \frac{\beta_\infty}{2}.$$

Then, setting

$$\zeta := \frac{r \, \beta_\infty}{12},$$

we have

$$\frac{\partial\{x, y, z\}}{r} \geq \frac{\beta_\infty}{4}, \quad \forall (x, y, z) \in B_\zeta(x_1) \times B_\zeta(x_2) \times B_\zeta(x_3).$$

Hence, by 1-regularity,

$$\beta \geq \mu(\mathrm{X})^{-3} \int_{B_\zeta(x_1) \times B_\zeta(x_2) \times B_\zeta(x_3)} r^{-1} \, \partial\{x, y, z\} \geq \tilde{C} \, \beta_\infty^4,$$

using again $\mu(\mathrm{X}) \leq C_{\mathrm{X}} \, r$, and where $\tilde{C} > 0$ is a constant depending only on C_{X}. This shows that

$$\beta_\infty \leq (\tilde{C}^{-1} \beta)^{1/4} \leq (\tilde{C}^{-1} \delta)^{1/4}. \tag{4.33}$$

Therefore, choosing δ small enough we have $\beta_\infty \leq 1/200$, which is (4.32). ■

Lemma 4.34. *There exist two points $P, Q \in X$ satisfying*

(i) $$\int_{X\times X} \frac{\partial\{P,x,y\}}{r}\, \mathrm{d}\mu(x)\mathrm{d}\mu(y) \le C\beta\mu(X)^2$$ *and* $$\int_{X\times X} \frac{\partial\{Q,x,y\}}{r}\, \mathrm{d}\mu(x)\mathrm{d}\mu(y) \le C\beta\mu(X)^2,$$

(ii) $$\int_X \frac{\partial\{P,Q,x\}}{r}\, \mathrm{d}\mu(x) \le C\beta\mu(X),$$

(iii) $\mathsf{d}(P,Q) \ge r/2$.

Proof. Define the sets

$$A_1 := \left\{P \in \mathrm{X} : \fint_{\mathrm{X}\times\mathrm{X}} r^{-1}\,\partial\{P,x,y\}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \le C\beta\right\} \subset \mathrm{X},$$
$$A_2 := \left\{(P,Q) \in \mathrm{X}\times\mathrm{X} : \fint_{\mathrm{X}} r^{-1}\,\partial\{P,Q,x\}\,\mathrm{d}\mu(x) \le C\beta\right\} \subset \mathrm{X}\times\mathrm{X}.$$

By the dominated convergence theorem, both A_1 and A_2 are closed sets. By (4.30) and the Markov inequality,

$$\mu(\mathrm{X}\setminus A_1) \le \frac{\mu(\mathrm{X})}{C} \quad \text{and} \quad (\mu\otimes\mu)((\mathrm{X}\times\mathrm{X})\setminus A_2) \le \frac{\mu(\mathrm{X})^2}{C}. \tag{4.35}$$

The first inequality above gives

$$(\mu\otimes\mu)(\mathrm{X}\times\mathrm{X}\setminus(A_1\times A_1)) \le 2\mu((\mathrm{X}\setminus A_1)\times\mathrm{X}) \le 2\,\frac{\mu(\mathrm{X})^2}{C}.$$

Additionally, by 1-regularity we have

$$(\mu\otimes\mu)(\{(x,y)\in\mathrm{X}\times\mathrm{X} : \mathsf{d}(x,y)\ge r/2\}) \ge C_{\mathrm{X}}^{-2}\,\mu(\mathrm{X})^2/2,$$

where we used that $\mu(\mathrm{X}) \le C_{\mathrm{X}} r$. Together with (4.35), this shows that if we choose C big enough, it holds

$$A_2 \cap (A_1\times A_1) \cap \{(x,y) : \mathsf{d}(x,y) \ge r/2\} \ne \emptyset$$

and any couple (P,Q) in this set has the three desired properties. ∎

From now on, we fix two points $P, Q \in \mathrm{X}$ as given by Lemma 4.34. We define the map $f : \mathrm{X} \to \mathbb{R}$ by imposing $f(P) := 0$, $f(Q) := \mathsf{d}(P,Q)$ and

$$f(x) := \begin{cases} -\mathsf{d}(x,P) & \text{if } [xPQ], \\ \mathsf{d}(x,P) & \text{otherwise}, \end{cases}$$

for every $x \notin \{P,Q\}$. Recall that $[xPQ]$ means $\mathsf{d}(x,Q) \ge \max\{\mathsf{d}(x,P), \mathsf{d}(P,Q)\}$.

Lemma 4.36. *The map f is Borel measurable and*

$$||f(x) - f(y)| - \mathsf{d}(x,y)| \le 2\min(\mathsf{d}(x,P), \mathsf{d}(y,P)), \quad x, y \in X. \tag{4.37}$$

Proof. The Borel measurability follows noting that the restriction of f to either the closed set $\{x : [xPQ] \text{ holds}\}$ or its complement is continuous (in fact, 1-Lipschitz). To show estimate (4.37), note that by the triangle inequality, $|\mathsf{d}(x, y) - \mathsf{d}(y, P)| \le \mathsf{d}(x, P)$, and that by definition, $|f(y)| = \mathsf{d}(y, P)$ and $|f(x)| = \mathsf{d}(x, P)$, hence

$$\begin{aligned}\big||f(x) - f(y)| - \mathsf{d}(x, y)\big| &= \big||f(x) - f(y)| - \mathsf{d}(x, y) + |f(y)| - |f(y)|\big| \\ &\le \big||f(x) - f(y)| - |f(y)|\big| + \big|-\mathsf{d}(x, y) + |f(y)|\big| \\ &\le |\mathsf{d}(x, y) - |f(y)|| + |f(x)| \le 2\mathsf{d}(x, P).\end{aligned}$$

Arguing in the same for y, we get (4.37). ∎

We are now ready to prove the main result of this section.

Proof of Theorem 4.19. Recall that our goal is to show that

$$\int_{\mathcal{S}} \frac{||f(x) - f(y)| - \mathsf{d}(x, y)|}{\operatorname{diam}(\mathrm{X})} \, d\mu(x) \, d\mu(y) \le \tilde{C} \beta \mu(\mathrm{X})^2 \tag{4.38}$$

holds with $\mathcal{S} = \mathrm{X} \times \mathrm{X}$ and for some constant $\tilde{C}$ depending only on C_{X}. We proceed by proving (4.38) for different sets $\mathcal{S}$ that partition $\mathrm{X} \times \mathrm{X}$. Define

$$\begin{aligned}&\mathcal{G} := \{(x, y) \in \mathrm{X} \times \mathrm{X} \; : ||f(x) - f(y)| - \mathsf{d}(x, y)| \le 5\,\partial\{x, y, P, Q\}\}, \\ &\mathcal{B} := \mathrm{X} \times \mathrm{X} \setminus \mathcal{G}.\end{aligned}$$

Estimate (4.38) holds with $\mathcal{S} = \mathcal{G}$. To see this, by definition of $\mathcal{G}$ we have

$$\int_{\mathcal{G}} r^{-1} ||f(x) - f(y)| - \mathsf{d}(x, y)| \, d\mu(x) \, d\mu(y) \le \int_{\mathcal{G}} 5r^{-1} \, \partial\{x, y, P, Q\} \, d\mu(x) \, d\mu(y).$$

From this and the obvious inequality

$$\partial\{x, y, P, Q\} \le \partial\{P, Q, x\} + \partial\{P, Q, y\} + \partial\{Q, x, y\} + \partial\{P, x, y\},$$

we obtain

$$\begin{aligned}&\int_{\mathcal{G}} r^{-1} ||f(x) - f(y)| - \mathsf{d}(x, y)| \, d\mu(x) \, d\mu(y) \\ &\qquad \le 5r^{-1} \int_{\mathcal{G}} \partial\{P, Q, x\} + \partial\{P, Q, y\} + \partial\{Q, x, y\} + \partial\{P, x, y\} \, d\mu(x) \, d\mu(y),\end{aligned}$$

from which plugging in the estimates (i) and (ii) of Lemma 4.34, which are satisfied by P and Q, we have

$$\begin{aligned}&\int_{\mathcal{G}} r^{-1} ||f(x) - f(y)| - \mathsf{d}(x, y)| \, d\mu(x) \, d\mu(y) \\ &\qquad \le 10\mu(\mathrm{X}) \int_{\mathrm{X}} r^{-1} \, \partial\{P, Q, x\} \, d\mu(x) + 5 \int_{\mathrm{X}\times\mathrm{X}} r^{-1} \, \partial\{P, x, y\} \, d\mu(x) \, d\mu(y) \\ &\qquad\quad + 5 \int_{\mathrm{X}\times\mathrm{X}} r^{-1} \, \partial\{Q, x, y\} \, d\mu(x) \, d\mu(y) \le 20 C \beta \mu(\mathrm{X})^2.\end{aligned}$$

Our goal is now show that (4.38) holds with $\mathcal{S} = \mathcal{B}$. Thanks to (4.32), we have that $\mathsf{d}(P,Q) > 2\partial\{x,y,P,Q\}$ for every $x,y \in \mathrm{X}$. Hence for every $x,y \in \mathcal{B}$, we can apply Lemma 4.28 to the points x,y,P,Q and get

$$\mathcal{B} \subset \{(x,y) \in \mathrm{X}\times\mathrm{X} : \text{the points } x,y,P,Q \text{ are } 2\partial\{x,y,P,Q\}\text{-circular}\}.$$

Indeed, the first case in Lemma 4.28 cannot happen by the definition of $\mathcal{B}$. We further divide $\mathcal{B}$ as

$$\mathcal{B}_1 := \{(x,y) \in \mathcal{B} : \mathsf{d}(\{x,y\},\{P,Q\}) \le 30\,\partial\{x,y,P,Q\}\} \quad \text{and} \quad \mathcal{B}_2 := \mathcal{B} \setminus \mathcal{B}_1.$$

Estimate (4.38) holds with $\mathcal{S} = \mathcal{B}_1$. To see this, let $x,y \in \mathcal{B}_1$. If $\mathsf{d}(x,P) < 30\,\partial\{x,y,P,Q\}$ or $\mathsf{d}(y,P) < 30\,\partial\{x,y,P,Q\}$, by (4.37) we have

$$||f(x)-f(y)| - \mathsf{d}(x,y)| \le 64\,\partial\{x,y,P,Q\}. \tag{4.39}$$

If instead $\mathsf{d}(x,Q) < 30\,\partial\{x,y,P,Q\}$ holds (or $\mathsf{d}(y,Q) < 30\,\partial\{x,y,P,Q\}$), by the $2\partial\{x,y,P,Q\}$-circularity, we have that $\mathsf{d}(y,P) < 32\partial\{x,y,P,Q\}$ (or that $\mathsf{d}(x,P) < 32\partial\{x,y,P,Q\}$), and so by (4.37) we get again (4.39). Hence using (4.39) and then estimate (ii) of Lemma 4.34, we have

$$\begin{aligned}\int_{\mathcal{B}_1} r^{-1}\,||f(x)-f(y)| - \mathsf{d}(x,y)|\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) &\le \int_{\mathcal{B}_1} \frac{64\,\partial\{x,y,P,Q\}}{r}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\ &\le 64\cdot 4\beta C\mu(\mathrm{X})^2.\end{aligned}$$

It remains to prove that (4.38) holds with $\mathcal{S} = \mathcal{B}_2$. This is the most difficult set to deal with, because the couples $(x,y) \in \mathcal{B}_2$ are circular and spread apart. To estimate their contribution, we will need to consider also the other points in X. For $x,y \in \mathcal{B}_2$, define the number $D(x,y)$ as the minimum distance of two points in $\{x,y,P,Q\}$. By the definition of $\mathcal{B}_2$, by (4.32) and by $2\partial\{x,y,P,Q\}$-circularity, it holds

$$D(x,y) > 30\,\partial\{x,y,P,Q\}. \tag{4.40}$$

We claim that

$$D(x,y) \le 15\beta_\infty r, \qquad x,y \in \mathcal{B}_2. \tag{4.41}$$

Indeed suppose this is not the case, i.e., $D(x,y) > 15\beta_\infty r = 15\,\partial\mathrm{X}$. Then by Lemma 4.29 applied to the whole X, with the points x,y,P,Q and with $\beta = \partial\mathrm{X}$ (note that x,y,P,Q are $4\partial\mathrm{X}$-circular because $\partial\{x,y,P,Q\} \le \partial\mathrm{X}$ and $x,y \in \mathcal{B}_2 \subset \mathcal{B}$), it must hold that

$$\mathsf{d}(z,\{P,Q,x,y\}) \le 15\,\partial\mathrm{X} = 15\,r\,\beta_\infty \overset{(4.33)}{\le} (\tilde{C}^{-1}\delta)^{1/4}\,15\,\mathrm{diam}(\mathrm{X}), \qquad z \in \mathrm{X}.$$

This however contradicts the 1-regularity of X, provided δ is small enough, hence (4.41) holds. Using (4.41), we can also conclude that

$$\mathsf{d}(P,Q) > D(x,y) \quad \text{and} \quad \mathsf{d}(x,y) > D(x,y). \tag{4.42}$$

Indeed, combining (4.32) with (4.41), we get that $\mathsf{d}(P,Q) > 3D(x,y)$. Then using that $\mathsf{d}(P,Q) > 3D(x,y)$ and the $2\partial\{x,y,P,Q\}$-circularity of x,y,P,Q, we get

$$\mathsf{d}(x,y) \geq \mathsf{d}(P,Q) - 2\partial\{x,y,P,Q\} \overset{(4.40)}{\geq} 3D(x,y) - D(x,y) \geq 2D(x,y) > 0,$$

which shows (4.42). For fixed $x, y \in \mathcal{B}_2$, we now consider all the points in X and divide them into two sets, the "attracted" and the "non-attracted" points:

$$\mathcal{A}(x,y) := \{z \in \mathrm{X} : \mathsf{d}(z,\{P,Q,x,y\}) \leq 15D(x,y)\},$$
$$\overline{\mathcal{A}}(x,y) := \mathrm{X} \setminus \mathcal{A}(x,y).$$

The set $\mathcal{A}(x,y)$ is small in measure. Indeed, by 1-regularity,

$$\mu(\mathcal{A}(x,y)) \leq 4C_{\mathrm{X}}\, 15D(x,y) \overset{(4.41)}{\leq} 900\, C_{\mathrm{X}} r\beta_\infty \overset{(4.33)}{\leq} 900\, C_{\mathrm{X}}^2 \mu(\mathrm{X})\, (\tilde{C}^{-1}\delta)^{1/4}.$$

Hence, assuming δ small enough, we have that $\mu(\overline{\mathcal{A}}(x,y)) \geq \mu(\mathrm{X})/2$. We now apply Lemma 4.29 with the points x, y, P, Q with $\beta := D(x,y)/20$ (note that these points are 4β-circular since they are $2\partial\{x,y,P,Q\}$-circular and $4\beta > 2\partial\{x,y,P,Q\}$ by (4.40)). By Lemma 4.29, we obtain that

$$\overline{\mathcal{A}}(x,y) \subset \{z \in \mathrm{X} : \partial\{z,x,y,P,Q\} > D(x,y)/20\}, \tag{4.43}$$

since the first option in Lemma 4.29 cannot happen by definition of $\overline{\mathcal{A}}(x,y)$. Because $\partial\{x,y,P,Q\} < D(x,y)/30$ (recall (4.40)), the inclusion (4.43) implies that

$$\partial\{z,x,y\} + \partial\{z,x,P\} + \partial\{z,x,Q\} + \partial\{z,y,P\} + \partial\{z,y,Q\} > \frac{D(x,y)}{20}, \tag{4.44}$$

for $z \in \bar{\mathcal{A}}(x,y)$. Note now that by (4.42) and the definition of $D(x,y)$ we have

$$D(x,y) = \min\big(\mathsf{d}(x,P), \mathsf{d}(x,Q), \mathsf{d}(y,P), \mathsf{d}(y,Q)\big).$$

Moreover, since x,y,P,Q are $D(x,y)$-circular (since $D(x,y) > 2\partial\{x,y,P,Q\}$ by (4.40)), we get that $\mathsf{d}(x,P) \leq \mathsf{d}(y,Q) + D(x,y)$ and $\mathsf{d}(y,P) \leq \mathsf{d}(x,Q) + D(x,y)$. Combining the last two observations we deduce that $\min(\mathsf{d}(x,P), \mathsf{d}(y,P)) \leq 2D(x,y)$. Hence, by (4.37),

$$||f(x) - f(y)| - \mathsf{d}(x,y)| \leq 4D(x,y), \quad x, y \in \mathcal{B}_2. \tag{4.45}$$

Recalling $\mu(\overline{\mathcal{A}}(x,y)) \geq \mu(X)/2$, we can finally estimate

$$\begin{aligned}
&\int_{(x,y)\in\mathcal{B}_2} r^{-1}\, ||f(x) - f(y)| - \mathsf{d}(x,y)|\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) \\
&= \int_{(x,y)\in\mathcal{B}_2} \mu(\bar{\mathcal{A}}(x,y))^{-1} \int_{z\in\bar{\mathcal{A}}(x,y)} r^{-1}\, ||f(x) - f(y)| - \mathsf{d}(x,y)|\, \mathrm{d}\mu(z)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) \\
&\overset{(4.45)}{\leq} 8\mu(\mathrm{X})^{-1} \int_{(x,y)\in\mathcal{B}_2} \int_{z\in\bar{\mathcal{A}}(x,y)} r^{-1} D(x,y)\, \mathrm{d}\mu(z)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) \\
&\overset{(4.44)}{\leq} \frac{8\cdot 20}{\mu(\mathrm{X})\, r} \int_{\mathrm{X}^3} \partial\{z,x,y\} + \partial\{z,x,P\} + \partial\{z,x,Q\} \\
&\qquad\qquad + \partial\{z,y,P\} + \partial\{z,y,Q\} \mathrm{d}\mu(z)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) \\
&\leq 8\cdot 20(4C+1)\,\beta\,\mu(\mathrm{X})^2,
\end{aligned}$$

where in the last inequality we used the definition of β in (4.30), and the property (i) of the points P, Q. This shows that (4.38) holds with $\mathcal{S} = \mathcal{B}_2$. Since we showed previously that (4.38) holds with $\mathcal{S} \in \{\mathcal{G}, \mathcal{B}_1\}$ and $\mathrm{X} \times \mathrm{X} = \mathcal{G} \cup \mathcal{B}_1 \cup \mathcal{B}_2$, the proof is concluded. ■

We conclude with the proof of Theorem 4.27 which, even if not used in the sequel, we believe to be interesting on its own.

Proof of Theorem 4.27. If $\operatorname{diam}(\mathrm{X}) \leq 40\beta$, the statement is trivial. Hence we can assume the existence of two points $P, Q \in \mathrm{X}$ such that $\mathsf{d}(P, Q) > 40\beta$. We now define the map $f: \mathrm{X} \to \mathbb{R}$ as follows. Set $f(P) := 0$, $f(Q) := \mathsf{d}(P, Q)$, and for any other point $x \in \mathrm{X}$,

$$f(x) := \begin{cases} -\mathsf{d}(x, P) & \text{if } \mathsf{d}(x, Q) \geq \max(\mathsf{d}(P, Q), \mathsf{d}(x, P)), \\ \mathsf{d}(x, P) & \text{otherwise.} \end{cases}$$

We need to prove that for every $x, y \in \mathrm{X}$, it holds

$$||f(y) - f(x)| - \mathsf{d}(x, y)| \leq 40\beta. \tag{4.46}$$

Fix $x, y \in \mathrm{X}$. If $||f(y) - f(x)| - \mathsf{d}(x, y)| \leq 5\beta$, there is nothing to prove, hence we can assume that $||f(x) - f(y)| - \mathsf{d}(x, y)| > 5\beta$. Hence from Lemma 4.28 we deduce that the points x, y, P, Q are 2β-circular. Observe that this implies that $\mathsf{d}(x, y) > 15\beta$. Suppose now that $\mathsf{d}(x, P) < 20\beta$; then by the triangle inequality, $|\mathsf{d}(x, y) - \mathsf{d}(y, P)| < 20\beta$, hence

$$\begin{aligned} ||f(y) - f(x)| - \mathsf{d}(x, y)| &= ||f(y) - f(x)| - \mathsf{d}(x, y) \pm |f(y)|| \\ &\leq |\mathsf{d}(x, y) - |f(y)|| + |f(x)| < 40\beta, \end{aligned}$$

because $|f(y)| = \mathsf{d}(y, P)$ and $|f(x)| = \mathsf{d}(x, P)$. The same holds if $\mathsf{d}(y, P) < 20\beta$. Hence we are left to prove (4.46) in the case $\mathsf{d}(x, P), \mathsf{d}(y, P) \geq 20\beta$, which thanks to 2β-circularity of $\{x, y, P, Q\}$ gives also that $\mathsf{d}(y, Q), \mathsf{d}(x, Q) > 15\beta$. Recall also that $\mathsf{d}(P, Q) \geq 40\beta$ and $\mathsf{d}(x, y) \geq 15\beta$. Then we can apply Lemma 4.29 and deduce that for every $z \in \mathrm{X}$, it holds that $\mathsf{d}(z, R) \leq 15\beta$ for some $R \in \{P, Q, x, y\}$ (note that the second alternative in Lemma 4.29 does not occur because $\partial \mathrm{X} \leq \beta$). This contradicts the fact that (X, d) contains five points at pairwise distance strictly greater than 30β and concludes the proof. ■

A. Almost circular points

This appendix contains the proofs of Lemmas 4.28 and 4.29 concerning almost circular points. We start by introducing short-hand notation that we will often use in the proofs of this section. Let a, b and c be real numbers. We write $a \sim_\varepsilon b$ to denote $|a - b| \leq \varepsilon$. This convention is used exclusively within this section, so that there should be no confusion with the notation introduced at the beginning of Section 2. Note that if $a \sim_\varepsilon b$ and $c \sim_{\varepsilon'} d$, then $a - c \sim_{\varepsilon+\varepsilon'} b - d$, and that $a \sim_\varepsilon b$ if and only if $a - c \sim_\varepsilon b - c$.

We also recall from Section 4.3 that we write $[xyz]$ for points x, y and z in a metric space (X, d) if

$$\mathsf{d}(x, y) + \mathsf{d}(y, z) - \mathsf{d}(x, z) = \partial(\{x, y, z\}),$$

which is equivalent to

$$\mathsf{d}(x,z) \geq \max(\mathsf{d}(x,y), \mathsf{d}(y,z)).$$

Moreover, $\partial S = \sup_{\{x,y,z\}\subset S} \partial(\{x,y,z\})$.

We start with a simple criterion to check that four points are almost circular (recall Definition 4.24).

Lemma A.1. *Let* (X, d) *be a metric space and let the points* $x_1, x_2, x_3, x_4 \in X$ *be such that* $\partial\{x_1,x_2,x_3,x_4\} \leq \delta$ *and*

$$[x_1x_2x_3],\ [x_2x_3x_4],\ [x_3x_4x_1],\ [x_4x_1x_2] \tag{A.2}$$

hold. Then the points x_1, x_2, x_3, x_4 *are* 2δ*-circular.*

Proof. According to our definitions, we have to check that $\mathsf{d}(x_i,x_j) \sim_{2\delta} \mathsf{d}(x_k,x_l)$ for any choice of distinct $i,j,k,l \in \{1,2,3,4\}$. The assumptions imply that

(i) $\mathsf{d}(x_1,x_3) \sim_\delta \mathsf{d}(x_1,x_2) + \mathsf{d}(x_2,x_3)$, (ii) $\mathsf{d}(x_2,x_4) \sim_\delta \mathsf{d}(x_2,x_3) + \mathsf{d}(x_3,x_4)$,
(iii) $\mathsf{d}(x_1,x_3) \sim_\delta \mathsf{d}(x_1,x_4) + \mathsf{d}(x_3,x_4)$, (iv) $\mathsf{d}(x_2,x_4) \sim_\delta \mathsf{d}(x_1,x_2) + \mathsf{d}(x_1,x_4)$.

Subtracting (i) and (iii) and subtracting (ii) and (iv), we get

$$\begin{aligned} &\mathsf{d}(x_1,x_2) + \mathsf{d}(x_2,x_3) \sim_{2\delta} \mathsf{d}(x_1,x_4) + \mathsf{d}(x_3,x_4),\\ &\mathsf{d}(x_2,x_3) + \mathsf{d}(x_3,x_4) \sim_{2\delta} \mathsf{d}(x_1,x_2) + \mathsf{d}(x_1,x_4). \end{aligned} \tag{A.3}$$

Subtracting the two in (A.3), we obtain $\mathsf{d}(x_1,x_2) - \mathsf{d}(x_3,x_4) \sim_{4\delta} \mathsf{d}(x_3,x_4) - \mathsf{d}(x_1,x_2)$, which gives $\mathsf{d}(x_1,x_2) \sim_{2\delta} \mathsf{d}(x_4,x_3)$. Switching the order of the second in (A.3) and subtracting again the two shows that $\mathsf{d}(x_2,x_3) - \mathsf{d}(x_1,x_4) \sim_{4\delta} \mathsf{d}(x_1,x_4) - \mathsf{d}(x_2,x_3)$, from which $\mathsf{d}(x_2,x_3) \sim_{2\delta} \mathsf{d}(x_1,x_4)$. Finally, summing (i) and (iii) and summing (ii) and (iv) gives

$$\begin{aligned} 2\mathsf{d}(x_1,x_3) &\sim_{2\delta} \mathsf{d}(x_1,x_2) + \mathsf{d}(x_2,x_3) + \mathsf{d}(x_1,x_4) + \mathsf{d}(x_3,x_4),\\ 2\mathsf{d}(x_2,x_4) &\sim_{2\delta} \mathsf{d}(x_1,x_2) + \mathsf{d}(x_2,x_3) + \mathsf{d}(x_1,x_4) + \mathsf{d}(x_3,x_4). \end{aligned}$$

Hence $2\mathsf{d}(x_1,x_3) \sim_{4\delta} 2\mathsf{d}(x_2,x_4)$, and so $\mathsf{d}(x_1,x_3) \sim_{2\delta} \mathsf{d}(x_2,x_4)$, which concludes the proof. ∎

Next we prove the 4-*points lemma*, which gives a quantitative condition for four points $\{P,Q,R,S\}$ to either admit an (explicitly given) 2β-isometry f into $\mathbb{R}$, or to be 2β-circular. (A similar conclusion was obtained under different assumptions in Lemma 2.2 of [34].)

Proof of Lemma 4.28. It is sufficient to prove the lemma for $\beta > 0$. As in the statement, we define $f\colon\{P,Q,R,S\} \to \mathbb{R}$ by $f(Q) := \mathsf{d}(P,Q)$, and for $x \in \{P,R,S\}$, by

$$f(x) := \begin{cases} -\mathsf{d}(x,P) & \text{if } \mathsf{d}(x,Q) \geq \max\{\mathsf{d}(P,Q), \mathsf{d}(x,P)\},\\ \mathsf{d}(x,P) & \text{otherwise.} \end{cases}$$

It is straightforward to see that f satisfies the rough isometry condition at least with respect to the points P and Q. Indeed, by definition, $||f(P) - f(x)| - \mathsf{d}(P,x)| = 0$ for every $x \in \{Q, R, S\}$. Next we show that $\mathsf{d}(x, Q) \sim_\beta |f(x) - f(Q)|$ for every $x \in \{P, R, S\}$. If $[xPQ]$, and so $f(x) = -\mathsf{d}(x, P)$, this is immediate because $\partial\{x, P, Q\} \leq \beta$, hence we assume $f(x) = \mathsf{d}(x, P)$. In this case, if we have $[xQP]$, then

$$||f(x) - f(Q)| - \mathsf{d}(x, Q)| = |\mathsf{d}(x, P) - \mathsf{d}(P, Q) - \mathsf{d}(x, Q)| \leq \beta.$$

If instead $[PxQ]$, then

$$||f(x) - f(Q)| - \mathsf{d}(x, Q)| = |\mathsf{d}(P, Q) - \mathsf{d}(P, x) - \mathsf{d}(x, Q)| \leq \beta.$$

Thanks to these observations, to show that $f : \{P, Q, R, S\} \to \mathbb{R}$ is a 2β-isometry, it is enough to show that

$$||f(R) - f(S)| - \mathsf{d}(R, S)| \leq 2\beta. \tag{A.4}$$

Hence to prove the lemma it is sufficient to show that either (A.4) holds or that the points $\{P, Q, R, S\}$ are 2β-circular. Throughout the proof, we will repeatedly use the following fact, often without mentioning it explicitly: if $[x_1x_2x_3]$ holds, then

$$\partial(\{x_1, x_2, x_3\}) \leq \beta \quad \iff \quad \mathsf{d}(x_1, x_2) + \mathsf{d}(x_2, x_3) \leq \mathsf{d}(x_1, x_3) + \beta.$$

It will be more convenient to name $x := R$ and $y := S$, to better distinguish these points from P and Q. Up to swapping x and y, we can assume that $f(x) \leq f(y)$, hence we need to consider only the following three cases:

(1) $f(x) = -\mathsf{d}(x, P)$ and $f(y) = -\mathsf{d}(y, P)$,
(2) $f(x) = -\mathsf{d}(x, P)$ and $f(y) = +\mathsf{d}(y, P)$,
(3) $f(x) = +\mathsf{d}(x, P)$ and $f(y) = +\mathsf{d}(y, P)$.

Case 1. $f(x) = -\mathsf{d}(x, P)$ and $f(y) = -\mathsf{d}(y, P)$.

This is equivalent to the validity of both $[xPQ]$ and $[yPQ]$. Using the assumption $f(x) \leq f(y)$ and the triangle inequality, we see that in this case, (A.4) is equivalent to

$$\mathsf{d}(x, y) + \mathsf{d}(y, P) \leq \mathsf{d}(x, P) + 2\beta. \tag{A.5}$$

We need to consider also the point Q. At least one of the conditions $[xQy]$, $[xyQ]$ and $[yxQ]$ holds. Suppose first that $[xQy]$ holds. Using the assumptions $\partial\{P, Q, R, S\} \leq \beta$ and $\mathsf{d}(P, Q) \geq 2\beta$, we find that

$$\begin{aligned} \mathsf{d}(x, y) + 4\beta &\leq [\mathsf{d}(x, P) + \mathsf{d}(P, Q)] + [\mathsf{d}(y, P) + \mathsf{d}(P, Q)] \\ &\overset{[xPQ],[yPQ]}{\leq} \mathsf{d}(x, Q) + \mathsf{d}(y, Q) + 2\beta. \end{aligned}$$

However, this leads to a contradiction, since then

$$\mathsf{d}(x, y) \leq \mathsf{d}(x, Q) + \mathsf{d}(y, Q) - 2\beta \overset{[xQy]}{\leq} \mathsf{d}(x, y) - \beta,$$

which is impossible (since $\beta > 0$). Thus $[xQy]$ in Case 1 cannot occur. If instead $[xyQ]$, we have

$$\begin{aligned} \mathsf{d}(x, P) + \mathsf{d}(P, Q) \geq \mathsf{d}(x, Q) &\overset{[xyQ]}{\geq} \mathsf{d}(x, y) + \mathsf{d}(y, Q) - \beta \\ &\overset{[yPQ]}{\geq} \mathsf{d}(x, y) + \mathsf{d}(y, P) + \mathsf{d}(P, Q) - 2\beta, \end{aligned}$$

which shows (A.5). Finally, if $[yxQ]$ holds, we have

$$\begin{aligned}\mathsf{d}(y,P)+\mathsf{d}(P,Q)\geq \mathsf{d}(x,Q) &\overset{[yxQ]}{\geq} \mathsf{d}(x,y)+\mathsf{d}(x,Q)-\beta\\ &\overset{[xPQ]}{\geq} \mathsf{d}(x,y)+\mathsf{d}(x,P)+\mathsf{d}(P,Q)-2\beta,\end{aligned}$$

which coupled with the assumption $\mathsf{d}(x,P)\geq \mathsf{d}(y,P)$ shows again (A.5).

Case 2. $f(x)=-\mathsf{d}(x,P)$ and $f(y)=\mathsf{d}(y,P)$.

This means that $[xPQ]$ holds, and $[yPQ]$ does not hold. In this case, (A.4) is equivalent to

$$\mathsf{d}(y,P)+\mathsf{d}(x,P)\leq \mathsf{d}(x,y)+2\beta. \tag{A.6}$$

Since $[yPQ]$ does not hold, at least one of the two options $[PyQ]$, $[PQy]$ must be valid, so we only need to prove that in these two sub-cases either (A.6) is true or that x, y, P, Q are 2β-circular.

Case 2a. $[PyQ]$ holds.

We have

$$\begin{aligned}\mathsf{d}(y,P)+\mathsf{d}(x,P) &\overset{[PyQ]}{\leq} \mathsf{d}(P,Q)-\mathsf{d}(y,Q)+\mathsf{d}(x,P)+\beta\\ &\overset{[xPQ]}{\leq} \mathsf{d}(x,Q)-\mathsf{d}(y,Q)+2\beta\leq \mathsf{d}(x,y)+2\beta,\end{aligned}$$

which yields (A.6) in this case.

Case 2b. $[PQy]$ holds.

If $[xPy]$, then (A.6) trivially holds true. If instead $[xyP]$, then

$$\begin{aligned}\mathsf{d}(P,Q) &\overset{[xPQ]}{\leq} \mathsf{d}(x,Q)-\mathsf{d}(x,P)+\beta \overset{[xyP]}{\leq} \mathsf{d}(x,Q)-\mathsf{d}(x,y)-\mathsf{d}(y,P)+2\beta\\ &\overset{[PQy]}{\leq} \mathsf{d}(x,Q)-\mathsf{d}(x,y)-\mathsf{d}(P,Q)-\mathsf{d}(Q,y)+3\beta\\ &\leq -\mathsf{d}(P,Q)+3\beta.\end{aligned}$$

Therefore, $\mathsf{d}(P,Q)\leq 3\beta/2$, which is impossible since $\mathsf{d}(P,Q)>2\beta$ by assumption.

Hence it remains to consider the case when $[Pxy]$ holds. We need to consider now also the point Q and the cases $[yQx]$, $[yxQ]$, and $[xyQ]$. If $[yQx]$, then

$$\begin{aligned}\mathsf{d}(x,y) &\overset{[yQx]}{\geq} \mathsf{d}(y,Q)+\mathsf{d}(Q,x)-\beta \overset{[xPQ]}{\geq} \mathsf{d}(y,Q)+\mathsf{d}(x,P)+\mathsf{d}(P,Q)-2\beta\\ &\geq \mathsf{d}(y,P)+\mathsf{d}(x,P)-2\beta,\end{aligned}$$

which implies (A.6) in this case. If instead $[yxQ]$, then

$$\begin{aligned}\mathsf{d}(P,Q) &\overset{[PQy]}{\leq} \mathsf{d}(P,y)-\mathsf{d}(Q,y)+\beta \overset{[yxQ]}{\leq} \mathsf{d}(P,y)-\mathsf{d}(y,x)-\mathsf{d}(x,Q)+2\beta\\ &\overset{[xPQ]}{\leq} \mathsf{d}(P,y)-\mathsf{d}(x,y)-\mathsf{d}(x,P)-\mathsf{d}(P,Q)+3\beta\\ &\leq -\mathsf{d}(P,Q)+3\beta.\end{aligned}$$

Therefore, analogously as in a previous case, $\mathsf{d}(P,Q)\leq 3\beta/2$, which is impossible since $\mathsf{d}(P,Q)>2\beta$ by assumption.

Hence we are left with the case when $[xyQ]$ holds. Summarizing the current assumptions, we are in the situation where $[PQy]$, $[xPQ]$, $[Qyx]$ and $[yxP]$ hold. Applying Lemma A.1, we obtain that x, y, P, Q are 2β-circular.

Case 3. $f(x) = \mathsf{d}(x, P)$ and $f(y) = \mathsf{d}(y, P)$.

That is, neither $[xPQ]$ nor $[yPQ]$ holds. Since $\mathsf{d}(x, P) = f(x) \le f(y) = \mathsf{d}(y, P)$ by assumption, the desired condition (A.4) simplifies in this case to

$$\mathsf{d}(x, y) + \mathsf{d}(x, P) \le \mathsf{d}(y, P) + 2\beta. \tag{A.7}$$

If $[yxP]$, then (A.7) holds true even with "2β" replaced by "β" on the right-hand side. In the following, we assume therefore that $[yxP]$ does *not* hold. Since $\mathsf{d}(y, P) \ge \mathsf{d}(x, P)$, the only way $[yxP]$ can fail is if $\mathsf{d}(x, y) > \mathsf{d}(y, P)$. In that case we have $[xPy]$, which we now add as a standing assumption to all the following sub-cases.

Case 3a. $[xQP]$ and $[yQP]$ hold.

This is similar to Case 1: using $\mathsf{d}(P, Q) \ge 2\beta$, we find that

$$\begin{aligned}\mathsf{d}(x, y) + 4\beta &\le [\mathsf{d}(x, Q) + \mathsf{d}(P, Q)] + [\mathsf{d}(y, Q) + \mathsf{d}(P, Q)] \\ &\overset{[xQP],[yQP]}{\le} \mathsf{d}(x, P) + \mathsf{d}(y, P) + 2\beta \overset{[xPy]}{\le} \mathsf{d}(x, y) + 3\beta,\end{aligned}$$

which is impossible (since $\beta > 0$). Thus the Case 3a cannot occur under the standing assumption that $[xPy]$.

Case 3b. Exactly one of $[xQP]$ and $[yQP]$ holds.

Assume first that $[xQP]$ holds and $[yQP]$ does *not* hold. Since in Case 3 also $[yPQ]$ does not hold, we must necessarily have that $[PyQ]$. This, together with the assumption $\mathsf{d}(x, P) \le \mathsf{d}(y, P)$, yields

$$\mathsf{d}(y, P) \overset{[PyQ]}{\le} \mathsf{d}(P, Q) \overset{[xQP]}{\le} \mathsf{d}(x, P) \le \mathsf{d}(y, P).$$

Therefore $\mathsf{d}(y, P) = \mathsf{d}(P, Q)$, which by $[PyQ]$ would imply that also $[yQP]$ holds, which is a contradiction. Thus, Case 3b can only occur if $[yQP]$ holds and $[xQP]$ does *not* hold.

Since also $[xPQ]$ does not hold in Case 3, we must necessarily have that $[PxQ]$. Then

$$\begin{aligned}\mathsf{d}(y, P) &\overset{[yQP]}{\ge} \mathsf{d}(y, Q) + \mathsf{d}(Q, P) - \beta \overset{[PxQ]}{\ge} \mathsf{d}(y, Q) + \mathsf{d}(P, x) + \mathsf{d}(x, Q) - 2\beta \\ &\ge \mathsf{d}(y, x) + \mathsf{d}(P, x) - 2\beta.\end{aligned}$$

This concludes the proof of (A.7) in Case 3b.

Case 3c. Neither $[xQP]$ nor $[yQP]$ holds.

As the assumptions in Case 3 also rule out the validity of $[xPQ]$ and $[yPQ]$, we must necessarily have that $[PxQ]$ and $[PyQ]$ in Case 3c. We also recall the standing assumption $[xPy]$ to which we reduced the discussion at the beginning of Case 3.

We need to consider also the points x, y and Q together, and distinguish the cases $[xyQ]$, $[yxQ]$, and $[xQy]$. If $[xyQ]$, then

$$\begin{aligned}\mathsf{d}(x, y) + \mathsf{d}(x, P) &\overset{[xyQ]}{\le} \mathsf{d}(x, Q) - \mathsf{d}(y, Q) + \mathsf{d}(x, P) + \beta \\ &\overset{[PxQ]}{\le} \mathsf{d}(P, Q) - \mathsf{d}(y, Q) + 2\beta \le \mathsf{d}(P, y) + \mathsf{d}(y, Q) - \mathsf{d}(y, Q) + 2\beta.\end{aligned}$$

Thus (A.7) holds true in this case. Next we assume $[yxQ]$ instead of $[xyQ]$. We apply an analogous argument as before, but use additionally the assumption $\mathsf{d}(x,P) \le \mathsf{d}(y,P)$. This yields

$$\begin{aligned}\mathsf{d}(x,y)+\mathsf{d}(x,P) \le \mathsf{d}(x,y)+\mathsf{d}(y,P) &\overset{[yxQ]}{\le} \mathsf{d}(y,Q)-\mathsf{d}(x,Q)+\mathsf{d}(y,P)+\beta \\ &\overset{[PyQ]}{\le} \mathsf{d}(P,Q)-\mathsf{d}(x,Q)+2\beta \le \mathsf{d}(P,x)+\mathsf{d}(x,Q)-\mathsf{d}(x,Q)+2\beta \\ &\le \mathsf{d}(P,y)+2\beta,\end{aligned}$$

which confirms (A.7) also in this case. It remains the case when $[xQy]$ holds. Summarizing, the current assumptions are $[xPy]$, $[PyQ]$, $[yQx]$ and $[QxP]$. We can apply Lemma A.1 and obtain that x, y, P, Q are 2β-circular. ∎

We now prove Lemma 4.29 concerning the attraction to circular points.

Proof of Lemma 4.29. It suffices to prove the statement for $\beta > 0$. Let $P_1, P_2, P_3, P_4 \in \mathrm{X}$ be four points as in the statement, i.e., $\partial\{P_1, P_2, P_2, P_4\} \le \beta$, $\mathsf{d}(P_i, P_j) > 15\beta$ for all $i \neq j$, and they are 4β-circular:

$$\mathsf{d}(P_i,P_j) \sim_{4\beta} \mathsf{d}(P_k,P_l), \tag{A.8}$$

for any choice of (distinct) indices $1 \le i, j, k, l \le 4$.

To conclude the proof of the statement of the lemma, it is sufficient to prove that if $\partial\{P_1, P_2, P_2, P_4, Q\} \le \beta$, then

$$\mathsf{d}(Q,\{P_1,P_2,P_3,P_4\}) \le 15\beta, \quad Q \in \mathrm{X}. \tag{A.9}$$

We argue by contradiction, that is, we assume that $\partial\{P_1, P_2, P_2, P_4, Q\} \le \beta$ and that there exists $Q \in \mathrm{X}$ such that $\mathsf{d}(Q, P_i) > 15\beta$ for every $i \in \{1,2,3,4\}$. We make the following claim.

Claim. *For every choice of (pairwise distinct) indices $i, j, k \in \{1,2,3,4\}$, there exists a 2β-isometry $f\colon\{Q, P_i, P_j, P_k\} \to \mathbb{R}$.*

This claim will be applied in “Case 2” later in the proof. To prove the claim, assume towards a contradiction that its statement is not true for some choice of $i, j, k \in \{1,2,3,4\}$. Then, since $\mathsf{d}(Q,P_i) \ge 2\beta$ for every $i = 1,2,3,4$, from Lemma 4.28 we must have that the points Q, P_i, P_j, P_k are 2β-circular. This implies that

$$\mathsf{d}(Q,P_i) \sim_{2\beta} \mathsf{d}(P_j,P_k), \quad \mathsf{d}(Q,P_j) \sim_{2\beta} \mathsf{d}(P_i,P_k) \quad \text{and} \quad \mathsf{d}(Q,P_k) \sim_{2\beta} \mathsf{d}(P_i,P_j),$$

which combined with (A.8) gives

$$\mathsf{d}(Q,P_i)\sim_{6\beta}\mathsf{d}(P_l,P_i), \;\; \mathsf{d}(Q,P_j)\sim_{6\beta}\mathsf{d}(P_l,P_j) \;\; \text{and} \;\; \mathsf{d}(Q,P_k)\sim_{6\beta}\mathsf{d}(P_l,P_k), \tag{A.10}$$

where $\{l\} = \{1,2,3,4\} \setminus \{i,j,k\}$. Up to reordering the indices i, j, k, we can assume that $\mathsf{d}(P_l,P_i) \ge \max\{\mathsf{d}(P_l,P_j), \mathsf{d}(P_l,P_k)\}$. From this last inequality and $\mathsf{d}(P_i,P_j) \sim_{4\beta} \mathsf{d}(P_l,P_k)$ (recall (A.8)), we obtain $\mathsf{d}(P_i,P_j) \le \mathsf{d}(P_l,P_i) + 4\beta$. In fact, this inequality can be improved to $\mathsf{d}(P_i, P_j) \le \mathsf{d}(P_l, P_i)$. Assume towards a contradiction that $\mathsf{d}(P_l, P_i) < \mathsf{d}(P_i, P_j)$. Then, under the current assumptions, $\partial\{P_l, P_i, P_j\} \le \beta$ would imply that

$$\mathsf{d}(P_l, P_j) + \mathsf{d}(P_l, P_i) \le \mathsf{d}(P_i, P_j) + \beta \le \mathsf{d}(P_l, P_i) + 5\beta,$$

which contradicts the initial assumption $\mathsf{d}(P_l, P_j) > 15\beta$. Thus we know that in fact,

$$\max\{\mathsf{d}(P_i, P_j), \mathsf{d}(P_l, P_j)\} \le \mathsf{d}(P_l, P_i),$$

from which $\partial\{P_l, P_i, P_j\} \le \beta$ implies that

$$\mathsf{d}(P_l, P_i) \sim_\beta \mathsf{d}(P_l, P_j) + \mathsf{d}(P_j, P_i). \tag{A.11}$$

Similarly, since $\mathsf{d}(Q, P_i) \sim_{6\beta} \mathsf{d}(P_l, P_i)$, $\mathsf{d}(Q, P_l) > 15\beta$, $\mathsf{d}(P_l, P_i) > 15\beta$, and we are assuming $\partial\{P_i, P_l, Q\} \le \beta$, we can check that the only possibility is

$$\mathsf{d}(Q, P_l) \sim_\beta \mathsf{d}(Q, P_i) + \mathsf{d}(P_i, P_l). \tag{A.12}$$

Indeed, if the maximum of $\{\mathsf{d}(Q, P_i), \mathsf{d}(P_i, P_l), \mathsf{d}(Q, P_l)\}$ was achieved by $\mathsf{d}(Q, P_i)$ or $\mathsf{d}(P_l, P_i)$, then using $\mathsf{d}(Q, P_i) \sim_{6\beta} \mathsf{d}(P_l, P_i)$ and $\partial\{P_i, P_l, Q\} \le \beta$ would lead to a contradiction with $\mathsf{d}(Q, P_l) > 15\beta$. Hence $\mathsf{d}(Q, P_l) \ge \max\{\mathsf{d}(Q, P_i), \mathsf{d}(P_i, P_l)\}$, which implies (A.12). Combining (A.12) and (A.10) we obtain

$$\begin{aligned} 2\mathsf{d}(P_i, P_l) - 7\beta &\overset{\text{(A.10)}}{\le} \mathsf{d}(Q, P_i) + \mathsf{d}(P_i, P_l) - \beta \overset{\text{(A.12)}}{\le} \mathsf{d}(Q, P_l) \le \mathsf{d}(Q, P_j) + \mathsf{d}(P_j, P_l) \\ &\overset{\text{(A.10)}}{\le} 2\mathsf{d}(P_l, P_j) + 6\beta, \end{aligned}$$

which contradicts (A.11), since $\beta > 0$ and $\mathsf{d}(P_i, P_j) \ge 15\beta$. This concludes the proof of the above claim on thus the existence of 2β-isometry $f\colon\{Q, P_i, P_j, P_k\} \to \mathbb{R}$ for all (pairwise distinct) $i, j, k \in \{1, 2, 3, 4\}$.

We now return to the proof of (A.9) by contradiction. Up to relabelling, we can also assume that $\mathsf{d}(P_1, P_3) = \max_{1\le i,j\le 4} \mathsf{d}(P_i, P_j)$. Then, Since $\partial\{P_1, P_2, P_3, P_4\} \le \beta$, we have $[P_1 P_4 P_3]$, $[P_1 P_2 P_3]$ and

$$\mathsf{d}(P_1, P_3) \sim_\beta \mathsf{d}(P_1, P_2) + \mathsf{d}(P_2, P_3), \quad \mathsf{d}(P_1, P_3) \sim_\beta \mathsf{d}(P_1, P_4) + \mathsf{d}(P_4, P_3). \tag{A.13}$$

Moreover, we must have

$$\mathsf{d}(P_2, P_4) \sim_\beta \mathsf{d}(P_2, P_3) + \mathsf{d}(P_3, P_4), \quad \mathsf{d}(P_2, P_4) \sim_\beta \mathsf{d}(P_2, P_1) + \mathsf{d}(P_1, P_4), \tag{A.14}$$

as can be easily deduced from (A.13), recalling also that $\mathsf{d}(P_2, P_4) \sim_{4\beta} \mathsf{d}(P_1, P_3)$ (by 4β-circularity), that $\mathsf{d}(P_1, P_3) = \max_{1\le i,j\le 4} \mathsf{d}(P_i, P_j)$ and that $\mathsf{d}(P_i, P_j) > 15\beta$. For example if we had instead $\mathsf{d}(P_2, P_3) \sim_\beta \mathsf{d}(P_2, P_4) + \mathsf{d}(P_3, P_4)$, it would imply that $\mathsf{d}(P_2, P_3) > \mathsf{d}(P_1, P_3)$, which is false.

Consider now the points P_1, Q and P_3. There are three possible cases: $[P_1 P_3 Q]$, $[P_1 Q P_3]$, and $[P_3 P_1 Q]$. However, since the assumptions on P_1 and P_3 are symmetric, up to swapping P_1 and P_3 we can distinguish only two cases: $[P_1 P_3 Q]$ or $[P_1 Q P_3]$.

Case 1. $[P_1 P_3 Q]$ holds.

Then

$$\mathsf{d}(P_1, P_4) + \mathsf{d}(P_4, P_3) + \mathsf{d}(P_3, Q) - 2\beta \overset{\text{(A.13)}}{\leq} \mathsf{d}(P_1, P_3) + \mathsf{d}(P_3, Q) - \beta \overset{[P_1 P_3 Q]}{\leq} \mathsf{d}(P_1, Q) \leq \mathsf{d}(P_1, P_4) + \mathsf{d}(P_4, Q),$$

which shows that

$$\mathsf{d}(P_4, Q) \sim_{2\beta} \mathsf{d}(P_4, P_3) + \mathsf{d}(P_3, Q). \tag{A.15}$$

Analogously, exchanging P_4 with P_2 (again using (A.13)), we can show that

$$\mathsf{d}(P_2, Q) \sim_{2\beta} \mathsf{d}(P_2, P_3) + \mathsf{d}(P_3, Q). \tag{A.16}$$

However, the above two relations will lead to a contradiction, recalling $\partial\{P_2, Q, P_4\} \leq \beta$. Indeed, up to exchanging P_2 and P_4, we can assume that either $[P_2 Q P_4]$ or $[P_2 P_4 Q]$ holds. If $[P_2 Q P_4]$ holds, then

$$\begin{aligned}\mathsf{d}(P_2, P_4) &\overset{[P_2 Q P_4]}{\geq} \mathsf{d}(P_2, Q) + \mathsf{d}(Q, P_4) - \beta \\ &\overset{\text{(A.15)},\text{(A.16)}}{\geq} \mathsf{d}(P_2, P_3) + \mathsf{d}(P_4, P_3) + 2\mathsf{d}(P_3, Q) - 5\beta \\ &\geq \mathsf{d}(P_2, P_4) + 2\mathsf{d}(P_3, Q) - 5\beta.\end{aligned}$$

The above however contradicts $\mathsf{d}(P_3, Q) > 15\beta$. Suppose instead $[P_2 P_4 Q]$. Then

$$\mathsf{d}(P_2, P_4) \overset{[P_2 P_4 Q]}{\leq} \mathsf{d}(P_2, Q) - \mathsf{d}(P_4, Q) + \beta \overset{\text{(A.15)}}{\leq} \mathsf{d}(P_2, P_3) - \mathsf{d}(P_4, P_3) + 3\beta < \mathsf{d}(P_2, P_4),$$

where in the second step we used also the triangle inequality and in the last inequality we used that $\mathsf{d}(P_4, P_3) \geq 15\beta$ and that $\mathsf{d}(P_2, P_4) \geq \mathsf{d}(P_2, P_3)$ (which comes from the first part of (A.14) and $\mathsf{d}(P_3, P_4) > 15\beta$). This is a contradiction which shows that Case 1 cannot happen.

Case 2. $[P_1 Q P_3]$ holds.

Recall that by the claim (A.9), we have the existence of maps $f : \{Q, P_1, P_2, P_3\} \to \mathbb{R}$ and $g : \{Q, P_1, P_3, P_4\} \to \mathbb{R}$ that are 2β-isometries. From the fact that $\mathsf{d}(P_1, P_3) \geq \mathsf{d}(P_i, P_j)$ for all i, j and that $\mathsf{d}(P_i, P_j) \geq 15\beta$ for all distinct i, j (part of the initial assumptions), the point $f(P_2)$ must lie in the interval with endpoints $f(P_1), f(P_3)$. Similarly, from the assumption $[P_1 Q P_3]$ (and again from $\mathsf{d}(Q, P_i) \geq 15\beta$ for every $i = 1, 2, 3, 4$), it follows that also $f(Q)$ must lie in the same interval. Hence, up to replacing f with $-f$ and swapping the labels of P_1 and P_3 (observe that the current assumptions are symmetric in P_1 and P_3), we can assume that

$$f(P_1) \leq f(Q) \leq f(P_2) \leq f(P_3). \tag{A.17}$$

Analogously, the point $g(P_4)$ must lie in the interval with endpoints $g(P_1), g(P_3)$, and up to replacing g with $-g$, we can also assume that

$$g(P_1) \leq g(P_4) \leq g(P_3).$$

There remain two possibilities for the position of $g(Q)$:

$$g(P_1) \le g(Q) \le g(P_4) \quad \text{or} \quad g(P_4) \le g(Q) \le g(P_3). \tag{A.18}$$

Suppose that $g(P_1) \le g(Q) \le g(P_4)$ holds. As f and g are 2β-isometries (and since $\mathsf{d}(P_i, P_j) \ge 15\beta$, $\mathsf{d}(Q, P_i) \ge 15\beta$ for every $i, j = 1, 2, 3, 4$, $i \ne j$), from this and (A.17) we deduce that

$$\mathsf{d}(P_1, P_2) \sim_\beta \mathsf{d}(P_1, Q) + \mathsf{d}(Q, P_2) \quad \text{and} \quad \mathsf{d}(P_1, P_4) \sim_\beta \mathsf{d}(P_1, Q) + \mathsf{d}(Q, P_4).$$

Therefore,

$$\begin{aligned}\mathsf{d}(P_2, P_4) &\le \mathsf{d}(P_2, Q) + \mathsf{d}(Q, P_4) \le \mathsf{d}(P_1, P_2) + \mathsf{d}(P_1, P_4) - 2\mathsf{d}(P_1, Q) + 2\beta \\ &\overset{(A.14)}{\le} \mathsf{d}(P_2, P_4) - 2\mathsf{d}(P_1, Q) + 3\beta,\end{aligned}$$

which contradicts $\mathsf{d}(P_1, Q) \ge 15\beta > 0$. If instead $g(P_4) \le g(Q) \le g(P_3)$ is satisfied, we have

$$\mathsf{d}(P_1, P_2) \sim_\beta \mathsf{d}(P_1, Q) + \mathsf{d}(Q, P_2) \quad \text{and} \quad \mathsf{d}(P_4, P_3) \sim_\beta \mathsf{d}(P_4, Q) + \mathsf{d}(Q, P_3).$$

Therefore

$$\begin{aligned}\mathsf{d}(P_1, P_3) + \mathsf{d}(P_2, P_4) &\le \mathsf{d}(P_1, Q) + \mathsf{d}(Q, P_3) + \mathsf{d}(P_2, Q) + \mathsf{d}(Q, P_4) \\ &\le \mathsf{d}(P_1, P_2) + \mathsf{d}(P_4, P_3) + 2\beta,\end{aligned}$$

from which using the first of both (A.13) and (A.14) on the left-hand side, we deduce

$$\mathsf{d}(P_1, P_2) + 2\mathsf{d}(P_2, P_3) + \mathsf{d}(P_3, P_4) - 2\beta \le \mathsf{d}(P_1, P_2) + \mathsf{d}(P_4, P_3) + 2\beta.$$

This however gives $\mathsf{d}(P_2, P_3) \le 2\beta$, which is a contradiction because by assumption $\mathsf{d}(P_2, P_3) \ge 15\beta > 0$. ∎

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