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Regularity for solutions of non-uniformly elliptic equations in non-divergence form

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Abstract. We prove an Aleksandrov–Bakelman–Pucci estimate for non-uniformly elliptic equations in non-divergence form. Moreover, we investigate the local behavior of solutions of such equations by proving local boundedness and a weak Harnack inequality. Here we impose an integrability assumption on ellipticity representing degeneracy or singularity, instead of specifying the particular structure of ellipticity.

1. Introduction

In this paper, we study regularity properties for solutions of non-uniformly elliptic equations in non-divergence form. To illustrate the issue, let us begin with the simplest example: a second-order, linear elliptic equation in non-divergence form:

$$a_{ij} D_{ij} u = f \quad \text{in } B_1, \tag{1.1}$$

where the coefficient matrix $a = (a_{ij})_{1 \le i,j \le n}$ and the nonhomogeneous term f are measurable. In order to capture the ellipticity of a, we introduce

$$\lambda(x) := \inf_{\xi \in \mathbb{R}^n} \frac{\xi \cdot a(x)\xi}{|\xi|^2} \quad \text{and} \quad \Lambda(x) := \sup_{\xi \in \mathbb{R}^n} \frac{\xi \cdot a(x)\xi}{|\xi|^2}. \tag{1.2}$$

In particular, we say $a = (a_{ij})$ is *uniformly elliptic* if there exist ellipticity constants $0 < \lambda_0 \le \Lambda_0 < \infty$ such that

$$\lambda_0 \le \lambda(x) \le \Lambda(x) \le \Lambda_0.$$

The regularity theory of (possibly nonlinear) uniformly elliptic operators in non-divergence form is by now classical; we refer to the comprehensive books [9,25] and references therein. In particular, Aleksandrov [1], Bakelman [4] and Pucci [42] independently proved a maximum principle: if $u \in C(\overline{B_1}) \cap W^{2,n}_{\mathrm{loc}}(B_1)$ is a strong subsolution of (1.1), then there exists a constant $C = C(n, \lambda_0, \Lambda_0) > 0$ such that

$$\sup_{B_1} u \le \sup_{\partial B_1} u^+ + C \|f^-\|_{L^n(\Gamma^+(u^+))},$$

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where $\Gamma^+(u^+)$ is the upper contact set of $u^+ = \max\{u, 0\}$; see Section 2 for the precise definition. The ABP maximum principle has become a fundamental tool in establishing local estimates for the associated equations, such as local boundedness, weak Harnack inequalities and interior Hölder estimates.

The goal of this paper is to develop the ABP maximum principle and to derive interior a priori estimates for solutions of *non-uniformly*, nonlinear elliptic equations. In our framework, the ellipticity functions $1/\lambda$ and Λ are not necessarily bounded, but they satisfy some integrability conditions. To be precise, we let B_1 be a unit ball in $\mathbb{R}^n$ and define two measurable functions $\lambda, \Lambda\colon B_1 \to [0,\infty]$ such that $\lambda \le \Lambda$,

$$1/\lambda \in L^p(B_1) \quad \text{and} \quad \Lambda \in L^q(B_1). \tag{1.3}$$

It is noteworthy that the uniformly elliptic case corresponds to the choice $p = q = \infty$. Moreover, we define the following generalized versions of the *Pucci extremal operators*:

$$\mathcal{M}^+_{\lambda,\Lambda}(M)(x) := \Lambda(x) \sum_{e_i \ge 0} e_i(M) + \lambda(x) \sum_{e_i < 0} e_i(M),$$
$$\mathcal{M}^-_{\lambda,\Lambda}(M)(x) := \lambda(x) \sum_{e_i \ge 0} e_i(M) + \Lambda(x) \sum_{e_i < 0} e_i(M),$$

where $x \in B_1$, $M \in \mathcal{S}^n := \{M \mid M \text{ is an } n \times n \text{ real symmetric matrix}\}$ and the $e_i(M)$ are the eigenvalues of M. For constant ellipticity λ_0 and Λ_0, it reduces to the classical Pucci extremal operators; see [9, 14] for instance.

Throughout the paper, we assume that a pair (p, q) satisfies

$$\frac{1}{p} + \frac{1}{q} \le \frac{1}{n}, \tag{1.4}$$

and we set the constants $\theta, \tau \in [n, \infty]$ to satisfy

$$\frac{1}{\theta} = \frac{1}{n} - \frac{1}{p} - \frac{1}{q} \quad \text{and} \quad \frac{1}{\tau} = \frac{1}{n} - \frac{1}{p}.$$

Then we are concerned with an L^θ-strong solution u of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u)(x) \ge f(x) \quad \text{or} \quad \mathcal{M}^-_{\lambda,\Lambda}(D^2u)(x) \le f(x)$$

for a nonhomogeneous term $f \in L^\tau(B_1)$; see Section 2 for details.

We begin with the Aleksandrov–Bakelman–Pucci estimates for L^θ-strong subsolutions. Several corollaries of Theorem 1.1 are discussed at the end of Section 3.

Theorem 1.1 (ABP estimates). *Let $f \in L^\tau(B_1)$ and suppose that $u \in W^{2,\theta}(B_1)$ is an L^θ-strong solution of*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \ge f \quad \text{in } B_1.$$

Then there exists a universal constant $C = C(n) > 0$ such that

$$\sup_{B_1} u \le \sup_{\partial B_1} u^+ + C\left(\int_{\Gamma^+(u^+)} \left(\frac{f^-(x)}{\lambda(x)}\right)^n \mathrm{d}x\right)^{1/n}.$$

After the celebrated works by Aleksandrov, Bakelman and Pucci, the ABP maximum principle has been widely studied in different contexts. Just to name a few, the ABP estimate, concerning uniformly elliptic/parabolic equations in non-divergence form, was achieved for

(i) viscosity solutions of fully nonlinear elliptic equations [7, 8];

(ii) strong solutions of linear parabolic equations [36, 45];

(iii) viscosity solutions of fully nonlinear parabolic equations [46];

(iv) L^p-viscosity solutions of fully nonlinear elliptic/parabolic equations [10, 15];

(v) viscosity solutions of fully nonlinear elliptic equations with gradient growth terms [34, 35].

We refer to [6, 19] for improvements of the ABP estimates in other directions. On the other hand, non-uniformly elliptic equations with particular structure have been considered relatively recently by several authors in various circumstances: [3, 18, 28] when an operator is given by $|Du|^\gamma \mathcal{M}^{\pm}_{\lambda_0,\Lambda_0}(D^2u)$ with $\gamma > -1$, [2] for p-Laplace equations and the mean curvature flow, and [29, 40] for elliptic equations that hold only where the gradient is large. In this paper, we concentrate on analyzing non-uniformly elliptic equations whose degeneracy and singularity are implicitly encoded in the integrability of $1/\lambda$ and Λ.

We next move our attention to local estimates for solutions of Pucci extremal operators. We first show a local boundedness result for strong subsolutions.

Theorem 1.2 (Local boundedness). *Let $f \in L^\tau(B_1)$. Suppose that $u \in W^{2,\theta}_{\mathrm{loc}}(B_1)$ is an L^θ-strong solution of*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \geq f \quad \text{in } B_1.$$

Then for $0 < t \leq n$, we have

$$\sup_{B_{1/2}} u \leq C\left(\left\|(u^+)^{t/n}\frac{\Lambda}{\lambda}\right\|^{n/t}_{L^n(B_1)} + \left\|\frac{f^-}{\lambda}\right\|_{L^n(B_1)}\right)$$

for a universal constant $C = C(n,t) > 0$.

In particular, for $t > 0$, there exists $C = C(n, t, \|1/\lambda\|_{L^p(B_1)}, \|\Lambda\|_{L^q(B_1)}) > 0$ such that

$$\sup_{B_{1/2}} u \leq C\left(\|u^+\|_{L^{\theta t/n}(B_1)} + \left\|\frac{f^-}{\lambda}\right\|_{L^n(B_1)}\right).$$

We also prove a weak Harnack inequality for viscosity supersolutions under a stronger assumption on (p,q).

Theorem 1.3 (Weak Harnack inequality). *Let $f \in L^\tau(B_1)$ and assume that*

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2n}.$$

Moreover, suppose that $u \in W^{2,\theta}_{\mathrm{loc}}(B_1)$ is an L^θ-strong solution of

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq f \quad \text{in } B_1.$$

If u is nonnegative in B_1, then we have

$$\|u\|_{L^t(B_{1/2})} \leq C\left(\inf_{B_{1/2}} u + \left\|\frac{f}{\lambda}\right\|_{L^n(B_1)}\right)$$

for some positive constants t and C which depend only on n, $\|1/\lambda\|_{L^p(B_1)}$ and $\|\Lambda\|_{L^q(B_1)}$.

As consequences of Theorem 1.2 and Theorem 1.3, we provide a Harnack inequality and a version of interior Hölder estimates of strong solutions in Section 4.

We now describe two simple, but interesting observations regarding our main theorems:

(i) For $n = 1$ and $\gamma > 0$, let us consider a linear operator $Lu = |x|^\gamma u_{xx}$ in $B_1 = (-1, 1)$. We then claim that $u(x) = |x|$ is a C-viscosity solution of $Lu = 0$ in B_1; see Definition 2.4 for the definition of C-viscosity solutions. Indeed, for $x_0 \in B_1 \setminus \{0\}$, then $u_{xx}(x_0) = 0$ and so $Lu(x_0) = 0$ in the classical sense. For $x_0 = 0$, if we let $\varphi \in C^2(B_1)$ be a test function such that $u - \varphi$ has a local maximum (or minimum) at 0, then

$$L\varphi(0) = |x|^\gamma\, \varphi_{xx}|_{x=0} = 0.$$

Therefore, we conclude that u is a viscosity solution of $Lu = 0$ in B_1.

On the other hand, if we choose ellipticity functions $\lambda(x) = \Lambda(x) = |x|^\gamma$, then it immediately follows that a viscosity solution u of $Lu = 0$ in B_1 satisfies

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \geq 0 \quad \text{and} \quad \mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq 0 \quad \text{in } B_1.$$

Moreover, it is easy to see that $\Lambda \in L^\infty(B_1)$ and $1/\lambda \in L^p(B_1)$ for any $p < 1/\gamma$, while u does not enjoy the (weak) minimum principle in B_1. Hence, even though we impose stronger integrability conditions on $1/\lambda$ and Λ than (1.3), Theorem 1.1 does not hold for general "viscosity solution" u. In other words, this example shows that the "strong solution" condition on u is essential in our framework.

(ii) For $n = 2$, we consider a linear operator $Lu = 2u_{xx} + y^2 u_{yy}$ in $B_1 = \{(x, y) \mid x^2 + y^2 < 1\}$. Then ellipticity functions are given by $\lambda(x, y) = y^2$ and $\Lambda(x, y) = 2$, where $1/\lambda = |y|^{-2} \notin L^1(B_1)$. It follows from a direct calculation that $u(x, y) = y^2 \cos x$ is a classical (or strong) solution of $Lu = 0$ in B_1. Since $u(0, 0) = 0 = \min_{\partial B_1} u$, u does not satisfy the strong maximum principle and the weak Harnack inequality. In short, this example guarantees the necessity of (a version of) integrability criteria on $1/\lambda$ and Λ in Theorem 1.3. Nevertheless, the optimality of our assumption on (p, q) is not satisfied by this example, and it remains an interesting open problem.

Let us finally discuss similar consequences for linear non-uniformly elliptic equations in divergence form. In particular, as a variational counterpart of (1.1), the authors of [5,44] considered a weak solution u of

$$-D_j(a_{ij} D_i u) = 0 \quad \text{in } B_1,$$

where the ellipticity of a is measured by λ and Λ defined in (1.2). In [44], Trudinger established interior estimates such as local boundedness, Harnack inequality and a version of Hölder regularity for weak solutions, provided that $1/\lambda \in L^p(B_1)$ and $\Lambda \in L^q(B_1)$, with

$$\frac{1}{p} + \frac{1}{q} < \frac{2}{n}.$$

Recently, Bella and Schäffner [5] improved the result by replacing the condition with

$$\frac{1}{p} + \frac{1}{q} < \frac{2}{n-1},$$

and proved that this condition is indeed sharp. The strategy of both papers mainly relied on a modification of the Moser iteration method, which is not available for operators in non-divergence form. We also refer to [16, 41] for related results.

The paper is organized as follows. In Section 2, we summarize several notations which will be used throughout the paper. Section 3 is devoted to the proof of Theorem 1.1 by adopting sequential approximation techniques. Finally, in Section 4, we investigate local behaviors of strong solutions: local boundedness for subsolutions and a weak Harnack inequality for supersolutions.

2. Preliminaries

We first introduce a concept of L^θ-strong solutions for the Pucci extremal operators $\mathcal{M}^{\pm}_{\lambda,\Lambda}$.

Definition 2.1 (L^θ-strong solutions). Let $f \in L^\tau_{\mathrm{loc}}(B_1)$. A function $u \in W^{2,\theta}_{\mathrm{loc}}(B_1)$ is an L^θ-strong solution of $\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \ge f$ in B_1 if

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) := \Lambda(x) \sum_{e_i \ge 0} e_i(D^2u(x)) + \lambda(x) \sum_{e_i < 0} e_i(D^2u(x)) \ge f(x) \quad \text{a.e. in } B_1,$$

where the $e_i(M)$ are the eigenvalues of $M \in \mathcal{S}^n$.

Similarly, a function $u \in W^{2,\theta}_{\mathrm{loc}}(B_1)$ is an L^θ-strong solution of $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \le f$ in B_1 if

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) := \lambda(x) \sum_{e_i \ge 0} e_i(D^2u(x)) + \Lambda(x) \sum_{e_i < 0} e_i(D^2u(x)) \le f(x) \quad \text{a.e. in } B_1.$$

Remark 2.2. The constants θ and τ are chosen to satisfy that $(\Lambda/\lambda)D^2u$ and f/λ are contained in L^n-space. If $1/\lambda$ and Λ further belong to L^∞-space, then it corresponds to the uniformly elliptic setting with $p = q = \infty$ and $\theta = \tau = n$. In this case, Definition 2.1 coincides with the definition of L^n-strong solutions given in [10].

We provide now a few simple properties of $\mathcal{M}^\pm = \mathcal{M}^\pm_{\lambda,\Lambda}$.

Lemma 2.3. *Let $M, N \in \mathcal{S}^n$. Then the following hold a.e.*

(i) $\mathcal{M}^-(M) \le \mathcal{M}^+(M)$.

(ii) $\mathcal{M}^-(M) = -\mathcal{M}^+(-M)$.

(iii) $\mathcal{M}^\pm(\alpha M) = \alpha \mathcal{M}^\pm(M)$ *if* $\alpha \ge 0$.

(iv) $\mathcal{M}^+(M) + \mathcal{M}^-(N) \le \mathcal{M}^+(M+N) \le \mathcal{M}^+(M) + \mathcal{M}^+(N)$.

For later uses, we also define C-viscosity solutions when λ, Λ and f are continuous; see [9, 14], for instance.

Definition 2.4 (C-viscosity solutions). Let $\lambda, \Lambda, f \in C(B_1)$ with $0 \le \lambda(x) \le \Lambda(x)$ for $x \in B_1$. A function $u \in C(B_1)$ is a C-viscosity solution of $\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \ge f$ in B_1 if for all $\varphi \in C^2(B_1)$ and any point $x_0 \in B_1$ at which $u - \varphi$ has a local maximum, one has

$$\mathcal{M}^+_{\lambda(x_0),\Lambda(x_0)}(D^2\varphi(x_0)) \ge f(x_0).$$

In a similar way, a function $u \in C(B_1)$ is a C-viscosity solution of $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \le f$ in B_1 if for all $\varphi \in C^2(B_1)$ and any point $x_0 \in B_1$ at which $u - \varphi$ has a local minimum, one has

$$\mathcal{M}^-_{\lambda(x_0),\Lambda(x_0)}(D^2\varphi(x_0)) \le f(x_0).$$

The upper contact set Γ^+ will be used for the proof of ABP estimates.

Definition 2.5. For a function $u\colon \Omega \to \mathbb{R}$ and $r > 0$, the *upper contact sets* of u are defined by

$$\Gamma^+(u) = \Gamma^+(u,\Omega) = \{x \in \Omega \mid \exists p \in \mathbb{R}^n \text{ such that } u(y) \le u(x) + \langle p, y - x\rangle, \forall y \in \Omega\},$$
$$\Gamma^+_r(u) = \Gamma^+_r(u,\Omega) = \{x \in \Omega \mid \exists p \in \overline{B_r(0)} \text{ such that } u(y) \le u(x) + \langle p, y - x\rangle, \forall y \in \Omega\}.$$

For sets $A_1, A_2, \ldots$, we define

$$\limsup_{j\to\infty} A_j := \bigcap_{n=1}^{\infty} \bigcup_{k>n} A_k.$$

Lemma 2.6 (Lemma A.1 in [10]). *Let u_j, $j = 1, 2, \ldots$, be functions defined on sets Ω_j, where Ω_j are open and increase to Ω; that is, $\Omega_j \subset \Omega_{j+1}$ and $\bigcup_j \Omega_j = \Omega$. Let u_j converge uniformly to a continuous function u on each Ω_j. Then*

(i) $\limsup_{j\to\infty} \Gamma^+(u_j, \Omega_j) \subset \Gamma^+(u,\Omega)$.
(ii) $\limsup_{j\to\infty} |\Gamma^+(u_j, \Omega_j)| \le |\Gamma^+(u,\Omega)|$.
(iii) $\limsup_{j\to\infty} \Gamma^+_r(u_j, \Omega_j) \subset \Gamma^+_r(u,\Omega)$.

We finally state the cube decomposition lemma, which shall be appropriate for our purposes in Section 4.

Lemma 2.7 (Lemma 9.23 in [25]). *Let K_0 be a cube in $\mathbb{R}^n$, $w \in L^1(K_0)$, and set*

$$D_k = \{x \in K_0 \mid w(x) \le k\} \quad \text{for } k \in \mathbb{R}.$$

Suppose that there exist constants $\delta \in (0, 1)$ and $C > 0$ such that

$$\sup_{K_0 \cap K_{3r}(z)} (w - k) \le C,$$

whenever k and $K = K_r(z) \subset K_0$ satisfy

$$|D_k \cap K| \ge \delta|K|.$$

Then it follows that, for all k,

$$\sup_{K_0} (w - k) \le C\Big(1 + \frac{\log(|D_k|/|K_0|)}{\log \delta}\Big).$$

2.1. Applications

In this section, we present concrete examples of degenerate or singular equations in non-divergence form, which are contained in our framework.

(i) *Issacs equations.*

Issacs equations, which naturally arise in probability theory [22] (stochastic control and differential games), are given by

$$\inf_{\alpha}\sup_{\beta}(A_{\alpha\beta}(x)D^2u(x)) = f \quad \text{in } B_1,$$

where $A_{\alpha\beta}(\cdot)$ (for any α and β in index sets) are matrices satisfying

$$\lambda(x)I_n \le A_{\alpha\beta}(x) \le \Lambda(x)I_n,$$

with $1/\lambda \in L^p(B_1)$ and $\Lambda \in L^q(B_1)$. We note that linear elliptic operators with ellipticity λ and Λ, and the Pucci extremal operators $\mathcal{M}^{\pm}_{\lambda,\Lambda}$, can be understood as special cases of Issacs operators.

(ii) *Monge–Ampère equations.*

The Monge–Ampère equation, which appears from the prescribed Gaussian curvature equation [21] (or "Minkowski problem"), is a fully nonlinear, degenerate elliptic equation given by

$$\det D^2u = f \quad \text{in } B_1.$$

It has important applications in convex geometry and optimal transportation. For simplicity, we consider an equation

$$G(D^2u) := \log\det D^2u = \log f. \tag{2.1}$$

Then we have $G_{ij} = u^{ij}$, where u^{ij} denotes the inverse of the Hessian matrix D^2u. Thus, if we denote by λ and Λ the ellipticity functions defined in (1.2) for the coefficient matrix (u^{ij}), then we observe that $1/\Lambda$ and $1/\lambda$ are the smallest and largest eigenvalue of D^2u, respectively. Since

u is convex if and only if (2.1) is degenerate elliptic, and

u is uniformly convex if and only if (2.1) is uniformly elliptic,

we can interpret the integrability assumptions (1.3) on $1/\lambda$ and Λ as some "intermediate" convexity on u. In other words, there exist two measurable functions $h, H\colon B_1 \to [0,\infty]$ such that

$$h(x)I_n \le D^2u(x) \le H(x)I_n,$$

with

$$h^{-1} \in L^q \quad \text{and} \quad H \in L^p.$$

We point out that [11, 37] developed a Harnack inequality for solutions of the linearized Monge–Ampère equations. Later, [38] extended this result under relaxed assumption on the convexity, which partially overlaps with ours. More precisely, Maldonado [38] dealt with linear degenerate/singular equations, whose coefficient matrix has a specific structure given by $(D^2\varphi)^{-1}$; see the structural conditions in [38] for details.

Further interior and boundary regularity results on a class of Monge–Ampère equations can be found in [12, 20] for the uniformly elliptic setting, and in [26, 27, 43] for the degenerate elliptic setting.

(iii) *Equations with particular degeneracy/singularity.*

In [17], the authors employed the partial Legendre transform to convert the two-dimensional Monge–Ampère equation

$$\det D^2 u = |x|^{\alpha} \quad \text{for } \alpha > 0$$

into the linear equation

$$v_{xx} + |x|^{\alpha} v_{yy} = 0 \quad \text{in } B_1. \tag{2.2}$$

Then the pair (p, q) corresponding to the ellipticity functions given by $\lambda(x, y) = |x|^{\alpha}$ and $\Lambda(x, y) = 1$ satisfy the structural condition (1.4) when $\alpha < 1/2$. In fact, (2.2) is a particular example of (degenerate elliptic) Grushin operators; see [23, 24, 39] for related results.

Moreover, a similar type of equation can be found in an extension problem related to the fractional Laplacian [13]. To be precise, the solution u of the degenerate/singular equations

$$\begin{cases} \Delta_x u + z^{(2s-1)/s} u_{zz} = 0 & \text{in } \mathbb{R}^n \times [0, \infty), \\ u = f & \text{on } \mathbb{R}^n \times \{0\}, \end{cases}$$

satisfies

$$(-\Delta)^s f(x) = -C(n, s)\, u_z(x, 0)$$

for $s \in (0, 1)$. We note that u solves the equation in the (unbounded) half space $\mathbb{R}^n \times [0, \infty)$. It is easy to check that the ellipticity functions of this problem satisfy the integrability conditions (1.3) when $(n+1)/(2n+3) < s < (n+1)/(2n+1)$. We refer to [33] for related examples.

3. ABP estimates

In order to prove Theorem 1.1, we are going to provide a version of Proposition 2.12 in [10] (ABP estimates for continuous coefficients and C-viscosity solutions) and of Theorem 4.6 in [47] (the existence of L^n-strong solutions for Dirichlet problems). It is noteworthy that an additional approximation technique is required to control the ellipticity functions λ and Λ, which are not necessarily bounded in L^{∞}.

Lemma 3.1. *Let $f \in C(B_1)$. Assume $\lambda, \Lambda \colon B_1 \to (0, \infty)$ and $1/\lambda, \Lambda \in C(\overline{B_1})$. Moreover, suppose that $u \in C(\overline{B_1})$ is a C-viscosity solution of*

$$\mathcal{M}^{+}_{\lambda,\Lambda}(D^2 u) \geq f \quad \text{in } B_1.$$

Then there exists a universal constant $C = C(n) > 0$ such that

$$\sup_{B_1} u \leq \sup_{\partial B_1} u^{+} + C \Big(\int_{\Gamma^{+}(u^{+})} \Big(\frac{f^{-}(x)}{\lambda(x)} \Big)^n \, dx \Big)^{1/n}.$$

Proof. We will follow the proof provided in Appendix A of [10]. We begin by assuming that $u \in C^2(B_1) \cap C(\overline{B_1})$ and later remove this assumption via approximations. We set

$$r_0 = \frac{\sup_{B_1} u - \sup_{\partial B_1} u^+}{2}.$$

For $r < r_0$, let $p \in B_r$ and let $\hat{x} \in \overline{B_1}$ be a maximum point of $u(\cdot) - \langle p, \cdot \rangle$, so that

$$u(\hat{x}) - \langle p, \hat{x} \rangle \geq u(x) - \langle p, x \rangle \quad \text{or} \quad u(x) - u(\hat{x}) \leq \langle p, x - \hat{x} \rangle$$

for any $x \in \overline{B_1}$. It follows that

$$\sup_{B_1} u - u(\hat{x}) \leq 2|p| \leq 2r < 2r_0 = \sup_{B_1} u - \sup_{\partial B_1} u^+,$$

and then $2(r_0 - r) + \sup_{\partial B_1} u^+ < u(\hat{x})$. In particular, we have $\hat{x} \in B_1$ and $u(\hat{x}) > 0$. Since $Du(\hat{x}) = p$ and $D^2 u(\hat{x}) \leq 0$, we conclude that for $0 < r < r_0$, $\Gamma_r^+(u^+)$ is a compact subset of B_1 and

$$B_r = B_r(0) = Du(\Gamma_r^+(u^+)) \quad \text{and} \quad D^2 u(x) \geq 0 \quad \text{on } \Gamma_r^+(u^+) \subset \{u > 0\}.$$

We now employ the change of variables $p = Du(x)$ to obtain

$$\int_{B_r} \mathrm{d}p \leq \int_{\Gamma_r^+(u^+)} |\det D^2 u| \, \mathrm{d}x \leq \int_{\Gamma_r^+(u^+)} \Big(\frac{-\mathrm{tr} D^2 u}{n} \Big)^n \, \mathrm{d}x. \tag{3.1}$$

Since $\mathcal{M}^+_{\lambda,\Lambda}(D^2 u)(x) \geq f(x)$ and $D^2 u \leq 0$ on $\Gamma^+(u^+)$, we have

$$\lambda \mathrm{tr}(D^2 u) \geq f(x) \quad \text{on } \Gamma^+(u^+)$$

and (3.1) implies

$$r^n |B_1| = \int_{B_r} \mathrm{d}p \leq \frac{1}{n^n} \int_{\Gamma^+(u^+)} \Big(\frac{f^-(x)}{\lambda(x)} \Big)^n \, \mathrm{d}x.$$

Since λ, Λ and f are continuous, the general case follows from the standard approximation argument as in Appendix A of [10]; see the remark below for more comments. ∎

Remark 3.2 (Sup-convolutions). In order to regularize u in the proof of Lemma 3.1, one needs to deal with the sup–convolution of u together with a mollification. In fact, given $u \in C(\overline{\Omega})$ and $\varepsilon > 0$, the *sup-convolution* of u is defined by

$$u^\varepsilon(x) := \sup_{y \in \Omega} \Big(u(y) - \frac{1}{2\varepsilon} |x - y|^2 \Big).$$

Then the sup-convolution u^ε satisfies the following useful properties (see [30–32] for details):

(i) u^ε is Lipschitz continuous on Ω, and $u^\varepsilon \to u$ uniformly on Ω as $\varepsilon \to 0$.

(ii) u^ε is semiconvex; more precisely, there exists a measurable function $M : \Omega \to \mathcal{S}^n$ such that for a.e. $x \in \Omega$,

$$u^\varepsilon(y) = u^\varepsilon(x) + \langle Du^\varepsilon(x), y - x \rangle + \frac{1}{2} \langle M(x)(y - x), y - x \rangle + o(|y - x|^2)$$

and

$$M(x) \geq -\tfrac{1}{\varepsilon} I.$$

(iii) If u^ε_η is a standard mollification of u^ε, then $D^2 u^\varepsilon_\eta \geq -(1/\varepsilon)I$ and

$$D^2 u^\varepsilon_\eta(x) \to M(x) \quad \text{a.e. in } \Omega \quad \text{as } \eta \to 0.$$

(iv) Let f and F be continuous. If u is a C-viscosity solution of

$$F(D^2 u, x) \geq f(x) \quad \text{in } \Omega,$$

then u^ε is a C-viscosity solution of

$$F^\varepsilon(D^2 u^\varepsilon, x) \geq f^\varepsilon(x) \quad \text{a.e. in } \Omega_{2(\varepsilon\|u\|_{L^\infty(\Omega)})^{1/2}},$$

where $\Omega_\delta := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \delta\}$ for $\delta > 0$, and

$$F^\varepsilon(N, x) := \sup_{|x-y| \leq 2(\varepsilon\|u\|_\infty)^{1/2}} F(N, y) \quad \text{and} \quad f^\varepsilon(x) := \inf_{|x-y| \leq 2(\varepsilon\|u\|_\infty)^{1/2}} f(y).$$

An inf-convolution v_ε, which can be defined in an analogous way, satisfies similar properties.

Lemma 3.3. *Let $f \in L^n(B_1)$, $\psi \in C(\partial B_1)$ and assume that $1/\lambda, \Lambda \in C(\overline{B_1})$. Then there exists an L^n-strong solution $u \in C(\overline{B_1}) \cap W^{2,n}_{\text{loc}}(B_1)$ of*

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2 u) = f & \text{in } B_1, \\ u = \psi & \text{on } \partial B_1. \end{cases} \tag{3.2}$$

Moreover, u satisfies the uniform estimate

$$\|u\|_{L^\infty(B_1)} \leq \|\psi\|_{L^\infty(\partial B_1)} + C\|f/\lambda\|_{L^n(B_1)}. \tag{3.3}$$

We note that the lemma still holds if we replace the operator $\mathcal{M}^+_{\lambda,\lambda}$ with $\mathcal{M}^-_{\lambda,\lambda}$.

Proof. Due to the continuity of λ and Λ in $\overline{B_1}$, we observe that

$$0 < \lambda_0 = \min_{B_1} \lambda \leq \max_{B_1} \Lambda = \Lambda_0 < \infty.$$

Thus, we can understand the first equation of (3.2) as

$$F(D^2 u, x) = f \quad \text{in } B_1,$$

where $F(N, x) := \mathcal{M}_{\lambda(x),\Lambda(x)}(N)$ is a (λ_0, Λ_0)-elliptic operator. Then the existence of an L^n-strong solution u follows from Theorem 4.6 in [47]; a similar existence result in a different setting can be found in Corollary 3.10 of [10]. Moreover, the uniform L^∞-estimate (3.3) can be obtained by applying Lemma 3.1 for $\pm u$. ∎

We are now ready to prove the first main theorem.

Proof of Theorem 1.1. We employ several regularization techniques; more precisely, we approximate the ellipticity functions λ and Λ, and then the forcing term f. For simplicity, we may omit "a.e." if no confusion occurs.

(i) *Approximation of* λ *and* Λ.

We first define truncated ellipticity functions

$$\lambda_j^0 := (\lambda \wedge j) \vee j^{-1} \quad \text{and} \quad \Lambda_j^0 := (\Lambda \wedge j) \vee j^{-1},$$

which satisfy

$$\|1/\lambda - 1/\lambda_j^0\|_p \to 0, \quad \|\Lambda - \Lambda_j^0\|_q \to 0 \quad \text{and} \quad j^{-1} \leq \lambda_j^0 \leq \Lambda_j^0 \leq j.$$

Since $C(\overline{B_1})$ is dense in $L^p(B_1)$ for any $p \in [1, \infty)$, we can take two sequences of functions $\{\lambda_j\}_{j=1}^{\infty} \subset C(\overline{B_1})$ and $\{\Lambda_j\}_{j=1}^{\infty} \subset C(\overline{B_1})$ such that

$$(2j)^{-1} \leq \lambda_j \leq \Lambda_j \leq 2j, \quad \|1/\lambda_j - 1/\lambda_j^0\|_p < j^{-1} \quad \text{and} \quad \|\Lambda_j - \Lambda_j^0\|_q < j^{-1}.$$

In particular, we have

$$\|1/\lambda_j - 1/\lambda\|_p \to 0 \quad \text{and} \quad \|\Lambda_j - \Lambda\|_q \to 0. \tag{3.4}$$

We now would like to find the inequality satisfied by u, in terms of Pucci extremal operators with 'good' ellipticity λ_j and Λ_j. Indeed, since $u \in W^{2,\theta}(B_1)$ satisfies

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = \Lambda(x) \sum_{e_i>0} e_i(D^2u(x)) + \lambda(x) \sum_{e_i<0} e_i(D^2u(x)) \geq f(x),$$

we observe that

$$\begin{aligned}\mathcal{M}^+_{\lambda_j,\Lambda_j}(D^2u) &= \Lambda_j \sum_{e_i>0} e_i(D^2u) + \lambda_j \sum_{e_i<0} e_i(D^2u) \\ &= \Lambda \sum_{e_i>0} e_i(D^2u) + (\Lambda_j - \Lambda) \sum_{e_i>0} e_i(D^2u) \\ &\quad + \lambda \sum_{e_i<0} e_i(D^2u) + (\lambda_j - \lambda) \sum_{e_i<0} e_i(D^2u) =: f_j.\end{aligned}$$

By recalling that $f \in L^\tau(B_1)$, $\Lambda \in L^q(B_1)$ and $D^2u \in L^\theta(B_1)$, it turns out that

$$f_j = f + (\Lambda_j - \Lambda) \sum_{e_i>0} e_i(D^2u) + (\lambda_j - \lambda) \sum_{e_i<0} e_i(D^2u) \in L^n(B_1).$$

(ii) *Approximation of* f_j.

For fixed $j \in \mathbb{N}$, let $\{f_{j,k}\}_{k=1}^{\infty} \subset C^\infty(B_1)$ be a sequence of smooth functions such that

$$\|f_{j,k} - f_j\|_n \to 0 \quad \text{as } k \to \infty. \tag{3.5}$$

Then we let $\psi_{j,k} \in W^{2,n}_{\text{loc}}(B_1) \cap C(\overline{B_1})$ solve

$$\begin{cases} \mathcal{M}^-_{\lambda_j,\Lambda_j}(D^2\psi_{j,k}) = f_{j,k} - f_j & \text{in } B_1 \\ \psi_{j,k} = 0 & \text{on } \partial B_1, \end{cases}$$

whose existence is guaranteed by Lemma 3.3. From the estimate (3.3),

$$\|\psi_{j,k}\|_\infty \leq C\|(f_{j,k} - f_j)/\lambda_j\|_n,$$

where the constant $C > 0$ is independent of $k \in \mathbb{N}$. Therefore, it immediately follows that

$$\|\psi_{j,k}\|_\infty \to 0 \quad \text{as } k \to \infty. \tag{3.6}$$

(iii) *Conclusion. ABP estimates.*

If we set $w := u + \psi_{j,k} - \|\psi_{j,k}\|_\infty$, then we observe that

$$\begin{aligned}\mathcal{M}^+_{\lambda_j,\Lambda_j}(D^2w) &\geq \mathcal{M}^+_{\lambda_j,\Lambda_j}(D^2u) + \mathcal{M}^-_{\lambda_j,\Lambda_j}(D^2\psi_{j,k})\\ &\geq f_j + (f_{j,k} - f_j) = f_{j,k}.\end{aligned}$$

Since λ_j, Λ_j and $f_{j,k}$ are regularized enough so that Lemma 3.1 is applicable, we have

$$\sup_{B_1} w \leq \sup_{\partial B_1} w^+ + C\Big(\int_{\Gamma^+(w^+)} \Big(\frac{f^-_{j,k}(x)}{\lambda_j(x)}\Big)^n \,\mathrm{d}x\Big)^{1/n}.$$

By letting $k \to \infty$ together with (3.5), (3.6) and Lemma 2.6, we deduce

$$\sup_{B_1} u \leq \sup_{\partial B_1} u^+ + C\Big(\int_{\Gamma^+(u^+)} \Big(\frac{f^-_{j}(x)}{\lambda_j(x)}\Big)^n \,\mathrm{d}x\Big)^{1/n}.$$

Moreover, by applying Hölder's inequality, we obtain

$$\begin{aligned}&\Big\|\frac{f_j^-}{\lambda_j} - \frac{f^-}{\lambda}\Big\|_n \leq \Big\|\frac{f_j}{\lambda_j} - \frac{f}{\lambda}\Big\|_n\\ &\quad\leq \Big\|\Big(\frac{1}{\lambda} - \frac{1}{\lambda_j}\Big)\lambda\sum_{e_i<0} e_i(D^2u)\Big\|_n + \Big\|\frac{1}{\lambda_j}(\Lambda_j - \Lambda)\sum_{e_i>0} e_i(D^2u)\Big\|_n + \Big\|\frac{f}{\lambda_j} - \frac{f}{\lambda}\Big\|_n\\ &\quad\leq \Big\|\frac{1}{\lambda} - \frac{1}{\lambda_j}\Big\|_p \|\Lambda\|_q \|D^2u\|_\theta + \Big\|\frac{1}{\lambda_j}\Big\|_p \|\Lambda_j - \Lambda\|_q \|D^2u\|_\theta + \Big\|\frac{1}{\lambda_j} - \frac{1}{\lambda}\Big\|_p \|f\|_\tau.\end{aligned}$$

Therefore, by passing the limit $j \to \infty$ together with (3.4), we finally conclude that

$$\sup_{B_1} u \leq \sup_{\partial B_1} u^+ + C\Big(\int_{\Gamma^+(u^+)} \Big(\frac{f^-(x)}{\lambda(x)}\Big)^n \,\mathrm{d}x\Big)^{1/n}$$

as desired. ■

Remark 3.4. Although we only deal with elliptic equations in the present paper, we expect that our method for deriving the ABP estimates can be extended to parabolic equations with some modifications. For example, one may follow the proof of Theorem 2 in [2] to prove the parabolic counterpart of Lemma 3.1.

Corollary 3.5. *Let $f \in L^\tau(B_1)$. Suppose that $u \in W^{2,\theta}(B_1)$ is an L^θ-strong solution of*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq f \quad \text{in } B_1.$$

Then there exists a universal constant $C = C(n) > 0$ such that

$$\sup_{B_1} u^- \leq \sup_{\partial B_1} u^- + C\Big(\int_{\Gamma^+(u^-)} \Big(\frac{f^+(x)}{\lambda(x)}\Big)^n \,\mathrm{d}x\Big)^{1/n}.$$

Proof. The conclusion immediately follows by considering $-u$ instead of u in the proof of Theorem 1.1. ■

Corollary 3.6. *Let $\lambda, \Lambda, f \in C(B_1)$ with $0 \le \lambda \le \Lambda$ in B_1. Suppose that $u \in W^{2,\theta}_{\mathrm{loc}}(B_1)$ is an L^θ-strong solution of $\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \ge f$ in B_1. Then u is also a C-viscosity solution of $\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \ge f$ in B_1.*

Proof. Since $\theta \ge n$, we have $u \in C(B_1)$. We assume by contradiction that for some $\varphi \in C^2(B_1)$, $u - \varphi$ has a (strict) local maximum at $x_0 \in B_1$ and

$$\mathcal{M}^+_{\lambda(x_0),\Lambda(x_0)}(D^2\varphi(x_0)) < f(x_0).$$

By the continuity of λ, Λ and f, we have

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2\varphi) < f$$

near x_0. On the other hand, we observe from Lemma 2.3 that

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2(u-\varphi)) \ge \mathcal{M}^+_{\lambda,\Lambda}(D^2u) - \mathcal{M}^+_{\lambda,\Lambda}(D^2\varphi) > 0 \quad \text{a.e. in } B_\eta(x_0) \text{ for some } \eta > 0.$$

We now apply Theorem 1.1 in $B_\eta(x_0)$ to conclude that

$$(u-\varphi)(x_0) \le \sup_{\partial B_\eta(x_0)} (u-\varphi),$$

which leads to a contradiction. ∎

We say a measurable function $F\colon \mathcal{S}^n \times B_1 \to \mathbb{R}$ is $(\lambda(\cdot), \Lambda(\cdot))$-elliptic if

$$\mathcal{M}^-_{\lambda,\Lambda}(N)(x) \le F(M+N,x) - F(M,x) \le \mathcal{M}^+_{\lambda,\Lambda}(N)(x)$$

for any $M, N \in \mathcal{S}^n$ and $x \in B_1$ a.e.. We note that the Pucci extremal operators $\mathcal{M}^\pm_{\lambda,\Lambda}$ are $(\lambda(\cdot), \Lambda(\cdot))$-elliptic. The notion of L^θ-strong solution defined in Definition 2.1 can be easily extended to such fully nonlinear operators F.

Corollary 3.7 (Comparison principle). *Let $f \in L^\tau(B_1)$ and let F be $(\lambda(\cdot), \Lambda(\cdot))$-elliptic. Suppose that $u, v \in W^{2,\theta}(B_1)$ are, respectively, L^θ-strong subsolution and supersolution of $F(D^2w, x) = f(x)$ in B_1. If $u \le v$ on ∂B_1, then $u \le v$ in B_1.*

Proof. By the definition of $(\lambda(\cdot), \Lambda(\cdot))$-ellipticity, we have

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2(u-v))(x) \ge F(D^2u, x) - F(D^2v, x) \ge 0.$$

The desired result follows from Theorem 1.1. ∎

4. Local estimates

In this section, we utilize the ABP maximum principle (Theorem 1.1) to obtain interior a priori estimates of L^θ-strong solutions of non-uniformly elliptic Pucci extremal operators. We refer to Theorems 9.20 and 9.22 in [25] for local boundedness and weak Harnack inequality for strong solutions of uniformly elliptic linear equations.

We begin with the local boundedness for L^θ-strong subsolutions.

Proof of Theorem 1.2. For simplicity, we omit "a.e." if no confusion occurs. For $\beta \geq 2$ to be determined later, we define an auxiliary function η by

$$\eta(x) = (1 - |x|^2)^{\beta}. \tag{4.1}$$

Then a direct calculation shows

$$\begin{aligned} D_i \eta &= -2\beta\, x_i (1 - |x|^2)^{\beta - 1}, \\ D_{ij} \eta &= -2\beta\, \delta_{ij} (1 - |x|^2)^{\beta-1} + 4\beta(\beta - 1)\, x_i\, x_j\, (1 - |x|^2)^{\beta-2}. \end{aligned}$$

By setting $v = \eta u$, we have

$$\begin{aligned} \mathcal{M}^+_{\lambda,\Lambda}(D^2 v) &= \mathcal{M}^+_{\lambda,\Lambda}(\eta D^2 u + Du \otimes D\eta + D\eta \otimes Du + u D^2 \eta) \\ &\geq \mathcal{M}^+_{\lambda,\Lambda}(\eta D^2 u) + \mathcal{M}^-_{\lambda,\Lambda}(Du \otimes D\eta + D\eta \otimes Du + u D^2 \eta) \\ &=: I_1 + I_2, \end{aligned}$$

where we write $(x \otimes y)_{ij} = x_i y_j$ for $x, y \in \mathbb{R}^n$. We first obtain that

$$I_1 = \eta\, \mathcal{M}^+_{\lambda,\Lambda}(D^2 u) \geq \eta f \geq -f^-.$$

On the other hand, it follows from the definition of the upper contact set that for any $x \in \Gamma^+(v^+) = \Gamma^+(v^+, B_1)$, $v(x)$ is nonnegative and

$$|Dv(x)| \leq \frac{v(x)}{1 - |x|}.$$

Thus we have

$$|Du| = \frac{1}{\eta}\, |Dv - u D\eta| \leq \frac{1}{\eta}\Big(\frac{v}{1 - |x|} + u|D\eta|\Big) \leq 2(1 + \beta)\, \eta^{-1/\beta} u \quad \text{ on } \Gamma^+(v^+).$$

Therefore, we utilize the estimates

$$\begin{aligned} |Du|\,|D\eta| &\leq 4\beta(1 + \beta)\, \eta^{-2/\beta} v \leq 8\beta^2\, \eta^{-2/\beta} v, \\ u|D_{ij}\eta| &\leq (2\beta \eta^{1/\beta} + 4\beta(\beta - 1))\, \eta^{-2/\beta} v \leq 4\beta^2\, \eta^{-2/\beta} v \end{aligned}$$

to derive

$$I_2 \geq -20\,\Lambda\, n^2 \beta^2\, \eta^{-2/\beta} v \quad \text{ on } \Gamma^+(v^+).$$

We now apply the ABP estimates (Theorem 1.1) to derive

$$\begin{aligned} \sup_{B_1} v &\leq C\Big(\int_{\Gamma^+(v^+)} \Big[\Big(\frac{\beta^2 \Lambda(x)\, \eta^{-2/\beta}(x) v^+(x)}{\lambda(x)}\Big)^n + \Big(\frac{f^-(x)}{\lambda(x)}\Big)^n\Big]\, \mathrm{d}x\Big)^{1/n} \\ &\leq C\Big(\Big(\sup_{B_1} v^+\Big)^{1-2/\beta}\, \|(u^+)^{2/\beta} \Lambda/\lambda\|_n + \|f^-/\lambda\|_n\Big). \end{aligned} \tag{4.2}$$

Here we choose $\beta = 2n/t$ (≥ 2). Then an application of Young's inequality

$$ab \leq \frac{a^s}{s} + \frac{b^{s'}}{s'}$$

for

$$s=(1-t/n)^{-1},\quad s'=(1-1/s)^{-1},\quad a=\Big(s\varepsilon\sup_{B_1}v^+\Big)^{1/s}\quad\text{and}\quad b=(s\varepsilon)^{-1/s}$$

gives

$$\Big(\sup_{B_1}v^+\Big)^{1-t/n}\le\varepsilon\sup_{B_1}v^+ + c_{n,t}\,\varepsilon^{1-n/t}\quad\text{for any }\varepsilon>0. \tag{4.3}$$

In particular, (4.2) and (4.3), together with the choice

$$\varepsilon=\frac{1}{2C}\,\|(u^+)^{t/n}\Lambda/\lambda\|_n^{-1},$$

yield that

$$\sup_{B_{1/2}}u\le C\big(\|(u^+)^{t/n}\Lambda/\lambda\|_n^{n/t}+\|f^-/\lambda\|_n\big).$$

Finally, an application of Hölder's inequality concludes that

$$\sup_{B_{1/2}}u\le C\big(\|1/\lambda\|_p^{n/t}\|\Lambda\|_q^{n/t}\|u^+\|_{\theta t/n}+\|f^-/\lambda\|_n\big).$$ ■

We now move our attention to the weak Harnack inequality for L^θ-strong supersolutions.

Proof of Theorem 1.3. For $\varepsilon>0$, we set

$$\bar u=u+\varepsilon+\|f/\lambda\|_n,$$
$$w=-\log\bar u,\quad v=\eta w\quad\text{and}\quad g=f/\bar u,$$

where η is the auxiliary function defined by (4.1), with $\beta\ge 2$ to be determined later. It is easily checked that

$$D_iw=-\bar u^{-1}D_i\bar u,$$
$$D_{ij}w=\bar u^{-2}D_i\bar uD_j\bar u-\bar u^{-1}D_{ij}\bar u=D_iwD_jw-\bar u^{-1}D_{ij}u.$$

Then a direct calculation yields that

$$\begin{aligned}\mathcal{M}^+_{\lambda,\Lambda}(D^2v)&\\ &=\mathcal{M}^+_{\lambda,\Lambda}(\eta D^2w+Dw\otimes D\eta+D\eta\otimes Dw+wD^2\eta)\\ &=\mathcal{M}^+_{\lambda,\Lambda}(-\eta\bar u^{-1}D^2u+\eta Dw\otimes Dw+Dw\otimes D\eta+D\eta\otimes Dw+wD^2\eta)\\ &\ge\mathcal{M}^+_{\lambda,\Lambda}(-\eta\bar u^{-1}D^2u)+\mathcal{M}^-_{\lambda,\Lambda}(\eta Dw\otimes Dw+Dw\otimes D\eta+D\eta\otimes Dw)\\ &\quad+\mathcal{M}^-_{\lambda,\Lambda}(wD^2\eta)\\ &\ge-g\eta+\mathcal{M}^-_{\lambda,\Lambda}(\eta Dw\otimes Dw+Dw\otimes D\eta+D\eta\otimes Dw)+\mathcal{M}^-_{\lambda,\Lambda}(wD^2\eta).\end{aligned} \tag{4.4}$$

(i) We first prove the following Cauchy–Schwarz inequality for matrices:

$$\pm(Dw\otimes D\eta+D\eta\otimes Dw)\le\eta^{-1}D\eta\otimes D\eta+\eta Dw\otimes Dw.$$

It can be written in an equivalent form: for any $a \in \mathbb{R}^n$,

$$|\langle (Dw \otimes D\eta + D\eta \otimes Dw)a, a\rangle| \le \langle (\eta^{-1} D\eta \otimes D\eta + \eta Dw \otimes Dw)a, a\rangle.$$

Indeed, this inequality follows from the following simple observation:

$$\langle (b \otimes c)a, a\rangle = [(b \otimes c)a]_i \, a_i = (b \otimes c)_{ij} \, a_j a_i = a_i \, b_i \, a_j \, c_j = \langle a, b\rangle \langle a, c\rangle$$

for any $a, b, c \in \mathbb{R}^n$.

(ii) We control the term $\eta^{-1}|D\eta|^2$ as

$$\eta^{-1}|D\eta|^2 \le 4\beta^2 \eta^{1-2/\beta}.$$

(iii) The eigenvalues of $D^2\eta$ are

$$\begin{aligned} 4\beta(\beta-1)(1-|x|^2)^{\beta-2}|x|^2 - 2\beta(1-|x|^2)^{\beta-1} &\quad \text{with multiplicity } 1,\\ -2\beta(1-|x|^2)^{\beta-1} &\quad \text{with multiplicity } n-1. \end{aligned}$$

Let $\alpha := 1/(3n)$. We note that the first eigenvalue becomes nonnegative if

$$\alpha \le |x| \le 1 \quad \text{and} \quad \beta \ge 1 + 1/(2\alpha^2).$$

Therefore, for $\alpha \le |x| \le 1$ and $\beta \ge 1 + 1/(2\alpha^2)$, we obtain

$$\begin{aligned} &\mathcal{M}^-_{\lambda,\Lambda}(D^2\eta)\\ &\quad = \lambda\,[4\beta(\beta-1)(1-|x|^2)^{\beta-2}|x|^2 - 2\beta(1-|x|^2)^{\beta-1}] - \Lambda(n-1)\,[2\beta(1-|x|^2)^{\beta-1}]\\ &\quad = \lambda\,[4\beta(\beta-1)(1-|x|^2)^{\beta-2}|x|^2] - (\lambda + (n-1)\Lambda)\,[2\beta(1-|x|^2)^{\beta-1}]\\ &\quad = 2\beta(1-|x|^2)^{\beta-2}\,[2\lambda(\beta-1)|x|^2 - (\lambda + (n-1)\Lambda)(1-|x|^2)]. \end{aligned}$$

On the other hand, if $|x| \le \alpha$, then

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2\eta) \ge -\Lambda n\,[2\beta(1-|x|^2)^{\beta-1}].$$

By plugging the previous estimates obtained in (i), (ii) and (iii) into (4.4), we have

$$\begin{aligned} \mathcal{M}^+_{\lambda,\Lambda}(D^2v) &\ge -g\eta - \eta^{-1}\mathcal{M}^+_{\lambda,\Lambda}(D\eta \otimes D\eta) + \mathcal{M}^-_{\lambda,\Lambda}(wD^2\eta)\\ &= -g\eta - \eta^{-1}\Lambda\,|D\eta|^2 + \mathcal{M}^-_{\lambda,\Lambda}(wD^2\eta)\\ &\ge -|g| - 4\beta^2\Lambda - \frac{2\Lambda n\beta}{1-\alpha^2}\,v^+\,\mathbf{1}_{\{|x|\le\alpha\}} + 2\beta(1-|x|^2)^{-2}\,[2\lambda(\beta-1)|x|^2\\ &\quad - (\lambda + (n-1)\Lambda)(1-|x|^2)]\,v^+\,\mathbf{1}_{\{|x|\ge\alpha\}}\\ &=: \widetilde{f} \end{aligned}$$

on $\Gamma^+(v^+)$. We now apply the ABP estimates (Corollary 3.5) to derive

$$\sup_{B_1} v \le C\Big(\int_{\Gamma^+(v^+)} \Big(\frac{\widetilde{f}^-(x)}{\lambda(x)}\Big)^n \,\mathrm{d}x\Big)^{1/n}.$$

Therefore, by recalling that $\|g/\lambda\|_n \le 1$, we obtain

$$\sup_{B_1} v \le C + C\beta^2\|\Lambda/\lambda\|_n + C\beta\|\Lambda/\lambda\|_n \cdot \sup_{B_1} v \cdot |\{|x| \le \alpha\} \cap \{v > 0\}|^{1/n} + C \sup_{B_1} v\Big(\int_{\alpha\le|x|\le 1} \Big[\Big(\frac{\Lambda}{\lambda} - \frac{\beta}{1-|x|^2}\Big)_+ \frac{\beta}{1-|x|^2}\Big]^n \,\mathrm{d}x\Big)^{1/n}.$$

Since $1/p + 1/q < 1/(2n)$, an application of Hölder's inequality yields that

$$\int_{\alpha\le|x|\le 1} \Big[\Big(\frac{\Lambda}{\lambda} - \frac{\beta}{1-|x|^2}\Big)_+ \frac{\beta}{1-|x|^2}\Big]^n \,\mathrm{d}x \le \int_{\{\alpha\le|x|\le 1\}\cap U_\beta} \Big(\frac{\Lambda}{\lambda}\Big)^{2n} \mathrm{d}x \le \|1/\lambda\|_p^{2n}\,\|\Lambda\|_q^{2n}\,|U_\beta|^{1-(2n)/p-(2n)/q},$$

where

$$U_\beta := \Big\{|x| \le 1 : \frac{\Lambda(x)}{\lambda(x)} \ge \frac{\beta}{1-|x|^2}\Big\}.$$

We also have the following inequality:

$$|U_\beta| \le \Big|\Big\{\frac{\Lambda}{\lambda} \ge \beta\Big\}\Big| \le \beta^{-2n} \int |\Lambda/\lambda|^{2n}.$$

Hence, there exists a constant $\beta > 0$, which depends only on $\|1/\lambda\|_p$, $\|\Lambda\|_q$ and n, such that

$$\int_{\alpha\le|x|\le 1} \Big[\Big(\frac{\Lambda}{\lambda} - \frac{\beta}{1-|x|^2}\Big)_+ \frac{\beta}{1-|x|^2}\Big]^n \,\mathrm{d}x \le \frac{1}{(2C)^n}.$$

By combining all estimates above, we conclude that

$$\sup_{B_1} v \le C + C \sup_{B_1} v \cdot |\{|x| \le \alpha\} \cap \{v > 0\}|^{1/n}.$$

In order to finish the proof, we would like to exploit the cube decomposition lemma (Lemma 2.7). For this purpose, let us define $K_R(z)$ to be the open cube, parallel to the coordinate axes, with centre z and the side length $2R$. Since $B_\alpha \subset K_\alpha(0) \subset\subset B_1$ for $\alpha = 1/(3n)$, we have

$$\sup_{B_1} v \le C\Big(1 + \sup_{B_1} v^+|K_\alpha^+|^{1/n}\Big),$$

where $K_\alpha^+ := \{x \in K_\alpha \mid v > 0\}$. Hence, whenever

$$\frac{|K_\alpha^+|}{|K_\alpha|} \le \theta := [2(2\alpha)^n C]^{-1},$$

we obtain

$$\sup_{B_1} v \le 2C.$$

We point out that

(i) this procedure is stable under the transformation $x \to \alpha(x - z)/r$ for $B_{r/\alpha}(z) \subset B_1(0)$;

(ii) we can repeat this argument for $w - k$ instead of w for arbitrary $k \in \mathbb{R}$.

Thus, by applying Lemma 2.7 with $\delta = 1-\theta$, $K_0 = K_\alpha(0)$ and $\alpha = 1/(3n)$, we obtain

$$\sup_{K_0}(w-k) \le C\Big(1 + \frac{\log(|D_k|/|K_0|)}{\log\delta}\Big),$$

where $D_k = \{x \in K_0 \mid w(x) \le k\}$. In other words, if we write

$$\mu_t = |\{x \in K_0 \mid \bar{u} > t\}|, \quad \text{with } t = e^{-k},$$

then

$$\mu_t \le C\Big(\inf_{K_0}\bar{u}/t\Big)^{\kappa},$$

where C and κ are positive universal constants. By recalling Lemma 9.7 in [25], we obtain

$$\int_{B_\alpha}\bar{u}^t \le C\Big(\inf_{B_\alpha}\bar{u}\Big)^t,$$

for $t = \kappa/2$. The desired weak Harnack inequality follows by letting $\varepsilon \to 0$, together with the covering and scaling argument. ■

We remark that if u is a strong solution of $F(D^2u, x) = f(x)$ for a $(\lambda(\cdot), \Lambda(\cdot))$-elliptic operator F with $F(0, x) = 0$, then u is contained in the (*extended*) *Pucci class*, i.e., u satisfies

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \ge -|f| \quad \text{and} \quad \mathcal{M}^-_{\lambda,\Lambda}(D^2u) \le |f|.$$

Indeed, the following corollaries hold for a wide class of functions: not only solutions of degenerate/singular fully nonlinear equations, but also functions in the (extended) Pucci class.

Corollary 4.1 (Harnack inequality). *Let $f \in L^\tau(B_1)$. Assume that*

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2n}.$$

Moreover, suppose that $u \in W^{2,\theta}_{\mathrm{loc}}(B_1)$ be an nonnegative L^θ-strong solution of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \ge -|f| \quad \textit{and} \quad \mathcal{M}^-_{\lambda,\Lambda}(D^2u) \le |f| \quad \textit{in } B_1.$$

Then there exists a constant $C > 0$, depending only on $\|1/\lambda\|_p$, $\|\Lambda\|_q$ and n, such that

$$\sup_{B_{1/2}} u \le C\Big(\inf_{B_{1/2}} u + \|f/\lambda\|_{L^n(B_1)}\Big).$$

Proof. The Harnack inequality immediately follows from the local boundedness (Theorem 1.2) and the weak Harnack inequality (Theorem 1.3). ■

In the uniformly elliptic framework, an application of the Harnack inequality (Corollary 4.1) in an iterative manner yields a priori Hölder estimates of solutions. Nevertheless, due to the dependence of the constant C on $\|1/\lambda\|_p$ and $\|\Lambda\|_q$, this argument is in general not valid anymore in the non-uniformly elliptic situation. Instead, we have the following large-scale Hölder continuity, as in Theorem 5.1 of [44] and Corollary 4.2 of [5]. We set

$$\hat{\lambda}(r) := \sup_{r\le R\le 1}\Big(\fint_{B_R}\lambda^{-p}\Big)^{1/p} \quad \text{and} \quad \hat{\Lambda}(r) := \sup_{r\le R\le 1}\Big(\fint_{B_R}\Lambda^{q}\Big)^{1/q} \quad \text{for } r \in (0,1).$$

Corollary 4.2 (Hölder continuity "on large scales"). *Let $f \in L^\tau(B_1)$. Assume that*

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2n}$$

and that $u \in W^{2,\theta}_{\mathrm{loc}}(B_1)$ be an L^θ-strong solution of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 u) \geq -|f| \quad \text{and} \quad \mathcal{M}^-_{\lambda,\Lambda}(D^2 u) \leq |f| \quad \text{in } B_1.$$

Moreover, suppose that $\widehat{\lambda}(r_1) < \infty$ and $\widehat{\Lambda}(r_1) < \infty$ for some $0 < r_1 < 1/4$.

Then for all $r \in [r_1, 1/2]$, we have

$$\operatorname*{osc}_{B_r} u \leq C r^\alpha \big(\|u\|_{L^\infty(B_1)} + \|f/\lambda\|_{L^n(B_1)}\big),$$

where C and α are positive constants depending only on n, $\widehat{\lambda}(r_1)$ and $\widehat{\Lambda}(r_1)$.

In particular, if we further assume that $\widehat{\lambda}(0+) < \infty$ and $\widehat{\Lambda}(0+) < \infty$, then the classical Hölder continuity of u holds.

Proof. For a scaled function $u_r(x) := u(rx)$, we observe that u_r is an L^θ-strong solution of

$$\mathcal{M}^+_{\lambda_r,\Lambda_r}(D^2 u_r) \geq -|f_r| \quad \text{and} \quad \mathcal{M}^-_{\lambda_r,\Lambda_r}(D^2 u_r) \leq |f_r| \quad \text{in } B_1,$$

where

$$\lambda_r(x) := \lambda(rx), \quad \Lambda_r(x) := \Lambda(rx) \quad \text{and} \quad f_r(x) := r^2 f(rx).$$

Moreover, we have

$$\|\Lambda_r\|_{L^q(B_1)} = \Big(\int_{B_1} \Lambda^q(rx)\,dx\Big)^{1/q} = \Big(\fint_{B_r} \Lambda^q(y)\,dy\Big)^{1/q}$$

and

$$\|f_r/\lambda_r\|_{L^n(B_1)} = r\|f/\lambda\|_{L^n(B_r)} \leq r\|f/\lambda\|_{L^n(B_1)}.$$

Therefore, the desired oscillation control follows from the standard iteration and scaling argument; see, for instance, Lemma 8.23 in [25]. ∎

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